

# L-equivalence for degree five elliptic curves, elliptic fibrations and K3 surfaces

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## ABSTRACT

We construct non-trivial L-equivalence between curves of genus one and degree five, and between elliptic surfaces of multisection index five. These results give the first examples of L-equivalence for curves (necessarily over non-algebraically closed fields) and provide a new bit of evidence for the conjectural relationship between L-equivalence and derived equivalence.

The proof of the L-equivalence for curves is based on Kuznetsov's Homological Projective Duality for  $\text{Gr}(2, 5)$ , and L-equivalence is extended from genus one curves to elliptic surfaces using the Ogg–Shafarevich theory of twisting for elliptic surfaces.

Finally, we apply our results to K3 surfaces and investigate when the two elliptic L-equivalent K3 surfaces we construct are isomorphic, using Neron–Severi lattices, moduli spaces of sheaves and derived equivalence. The most interesting case is that of elliptic K3 surfaces of polarization degree ten and multisection index five, where the resulting L-equivalence is new.

## 1. Introduction

### 1.1. The Grothendieck ring of varieties and L-equivalence

Recall that the Grothendieck ring of varieties  $K_0(\text{Var}/k)$  is generated as an abelian group by isomorphism classes  $[X]$  of schemes of finite type  $X/k$  modulo the scissor relations

$$[X] = [U] + [Z]$$

for every closed  $Z \subset X$  with open complement  $U = X \setminus Z$ . The product structure on  $K_0(\text{Var}/k)$  is induced by product of schemes. We write  $\mathbb{L} \in K_0(\text{Var}/k)$  for the class of the affine line  $[\mathbb{A}^1]$ .

The concept of L-equivalence stems from the recently discovered fact that  $\mathbb{L}$  is a zero-divisor [4]. Specifically, for Calabi–Yau threefolds  $X, Y$  in the so-called Pfaffian–Grassmannian correspondence, the classes satisfy  $[X] \neq [Y]$  and

$$\mathbb{L}^n \cdot ([X] - [Y]) = 0, \tag{1.1}$$

where one can take any  $n \geq 6$  [4, 24]. Following [22], we say that smooth projective connected varieties  $X$  and  $Y$  are *L-equivalent* if equation (1.1) holds for some  $n \geq 1$ , and we say that  $X$  and  $Y$  are *non-trivially L-equivalent* if in addition  $[X] \neq [Y]$ . If  $X$  and  $Y$  are not covered by rational curves and  $X$  and  $Y$  are not birational, then an L-equivalence between them is automatically non-trivial (see, for example, [22, Proposition 2.2]).

There are at least two important reasons why one would want to study L-equivalence. First, it seems to be closely related to derived equivalence [17, 20, 22]. As an evidence for this,

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the classes of derived categories of L-equivalent varieties in the Bondal–Larsen–Lunts ring of triangulated categories [3] are equal, and since for Calabi–Yau varieties the derived categories are indecomposable, it is very likely that non-trivially L-equivalent Calabi–Yau varieties are actually derived equivalent (see [17, 22] for an extended discussion of this relationship). In fact, all currently known examples of pairs of non-trivially L-equivalent varieties are known to be derived equivalent. These examples include K3 surfaces [14, 17, 19, 22], Calabi–Yau threefolds [4, 5, 18], Calabi–Yau fivefolds [23] and Hilbert schemes of points on K3 surfaces [29].

The second reason to study L-equivalence is the relation to rationality problems, specifically to that of cubic fourfolds. Namely, the approach of [12] can be used to show that very general cubic fourfolds are not rational as soon as one has sufficient control over the L-equivalence relation.

In this paper, we study L-equivalence for genus one curves and elliptic surfaces, in particular for elliptic K3 surfaces.

### 1.2. Genus one curves

We work over a field of characteristic zero. Let  $X$  be a genus one curve with a line bundle of degree  $d$ . For every  $k$  coprime to  $d$ , we can consider the Jacobian  $Y = \text{Jac}^k(X)$  which is a fine moduli space parametrizing degree  $k$  line bundles on  $X$ . Of course, if  $X$  has a rational point, then all Jacobians  $\text{Jac}^k(X)$  are isomorphic to  $X$ , however, in general this is not the case, and  $X$  and  $Y$  are typically different torsors over the same elliptic curve  $E = \text{Jac}^0(X)$ .

**THEOREM 1.1** [1]. *If  $k$  and  $d$  are coprime, then genus one curves  $X$  and  $\text{Jac}^k(X)$  are derived equivalent, and furthermore, every smooth projective variety  $Y$  derived equivalent to  $X$  will be of the form  $Y = \text{Jac}^k(X)$  for some  $k$  coprime to  $d$ .*

In light of a conjectural relation between L-equivalence and derived equivalence, we may ask the following:

**QUESTION 1.2.** When are genus one curves  $X$  and  $Y = \text{Jac}^k(X)$  L-equivalent?

Due to the periodicity relations  $\text{Jac}^{k+d}(X) \simeq \text{Jac}^k(X)$ ,  $\text{Jac}^{-k}(X) \simeq \text{Jac}^k(X)$  and the isomorphism  $X \simeq \text{Jac}^1(X)$ , the first non-trivial test case is  $d = 5$ . Furthermore, in the  $d = 5$  case the only non-trivial coprime Jacobian is  $Y = \text{Jac}^2(X) \simeq \text{Jac}^3(X)$ . Our first main result is the following:

**THEOREM 1.3** (see Theorem 2.9). *If  $X$  is a genus one curve with a line bundle of degree five and  $Y = \text{Jac}^2(X)$ , then  $X$  and  $Y$  are L-equivalent, and in general this L-equivalence is non-trivial.*

More precisely, we show that (1.1) holds for  $X$  and  $Y$  when  $n \geq 4$  (and does not hold for  $n = 0$ ). This is the first existing construction of non-trivial L-equivalence for curves, as all the previous constructions were for K3 surfaces or Calabi–Yau varieties of higher dimension.

As it is often the case with proving L-equivalence we relate the geometry of  $X$  and  $Y$  to Homological Projective Duality of Kuznetsov [21]. Specifically, as one of the steps in the proof of the theorem above we prove the following:

**PROPOSITION 1.4** (see Proposition 2.8 for the precise statement). *If  $X$  is a genus one curve with a line bundle of degree five and  $Y = \text{Jac}^2(X)$ , then  $X$  and  $Y$  are homologically projectively dual codimension 5 linear sections of  $\text{Gr}(2, 5)$ .*

We note the interplay between the moduli space geometry and the Homological Projective Duality geometry, in particular either of the two approaches can be used to show derived equivalence of  $X$  and  $Y = \text{Jac}^2(X)$ . If one starts with the  $Y = \text{Jac}^2(X)$  description, derived equivalence follows from Theorem 1.1 and if one starts with the Homological Projective Duality description of Proposition 1.4, derived equivalence follows from [21].

To generalize our work and to construct L-equivalence of genus one curves in degrees  $d > 5$ , it seems necessary to study explicit geometry of the moduli space of curves of genus one and degree  $d$ . This geometry is well understood for each  $2 \leq d \leq 5$  in terms of double covers of  $\mathbb{P}^1$  branched in four points, cubics in  $\mathbb{P}^2$ , intersections of two quadrics in  $\mathbb{P}^3$ , and one-dimensional linear sections of  $\text{Gr}(2, 5)$ , respectively, however, no such explicit description seems to be known for  $d > 5$ . We note that the same explicit geometry of genus one curves of degree five as linear sections of  $\text{Gr}(2, 5)$  that we rely on in this work has been used to study the average size of 5-Selmer groups and the average ranks of elliptic curves [2].

### 1.3. Elliptic surfaces

Let  $k$  be an algebraically closed field of characteristic zero. We work with elliptic surfaces without a section; by a *multisection index* of such a surface we mean the minimal fiber degree of a multisection. Our second main result is:

**THEOREM 1.5** (see Theorem 3.2). *If  $X \rightarrow C$  is an elliptic surface of multisection index five and  $Y = \text{Jac}^2(X/C)$ , then  $X$  and  $Y$  are L-equivalent, and in general this L-equivalence is non-trivial.*

We note that the derived equivalence of  $X$  and  $Y$  had been proved by Bridgeland [6].

We also investigate the case of elliptic K3 surfaces in detail, and answer the question when the L-equivalence constructed in Theorem 1.5 is in fact non-trivial. Here we take  $k = \mathbb{C}$ .

L-equivalence for K3 surfaces is one of the central open questions in the field. As a general structural result, it is proved by Efimov [9] that every L-equivalence class of K3 surfaces contains only finitely many isomorphism classes in it. Previously known cases when non-trivial L-equivalence of derived equivalent K3 surfaces has been constructed are K3 surfaces of degrees eight and two and Picard rank two [22], K3 surfaces of degree twelve and Picard rank one [14, 17], and K3 surfaces of degree two and Picard rank two [19].

Let  $X \rightarrow \mathbb{P}^1$  be an elliptic K3 surface of multisection index five, then  $Y = \text{Jac}^2(X/\mathbb{P}^1)$  is also an elliptic K3 surface (see, for example, [16, Proposition 11.4.5]), and by Theorem 1.5 these K3 surfaces are L-equivalent. The next Proposition explains when  $X$  and  $Y$  are not isomorphic.

**PROPOSITION 1.6** (see Proposition 3.10). *Let  $X \rightarrow \mathbb{P}^1$  be an elliptic K3 surface of Picard rank two, and multisection index five, with a polarization of degree  $2d$ , and let  $Y = \text{Jac}^2(X/\mathbb{P}^1)$ .*

- (1) *If  $d \equiv 2 \pmod{5}$  or  $d \equiv 3 \pmod{5}$ , then  $X$  and  $Y$  are isomorphic.*
- (2) *If  $d \equiv 1 \pmod{5}$  or  $d \equiv 4 \pmod{5}$ , then  $X$  and  $Y$  are not isomorphic.*
- (3) *If  $d \equiv 0 \pmod{5}$ , and  $X$  is very general in moduli, then  $X$  and  $Y$  are not isomorphic.*

We note that for every  $d$  such K3 surfaces exist and form an 18-dimensional irreducible subvariety in the moduli space of degree  $2d$  polarized K3 surfaces. Such elliptic K3 surfaces may have more than one elliptic fibrations (in fact a Picard rank two elliptic K3 has always one or two elliptic fibrations), and by an isomorphism of elliptic K3 surfaces we mean an isomorphism of K3 surfaces, regardless of the elliptic fibration structure. The above Proposition is proved by analyzing lattice theory of the corresponding K3 surfaces, along the lines of [31, 32].

Explicitly, the case (2) of the Proposition covers elliptic K3 surfaces of degrees twelve ( $d \equiv 1 \pmod{5}$ ) and eight ( $d \equiv 4 \pmod{5}$ ), considered previously in [14, 17] and [22] respectively.

The K3 surfaces in case (3) can be geometrically described as intersections of  $\text{Gr}(2, 5)$ , three hyperplanes and a quadric in  $\mathbb{P}^9$ , and containing an elliptic quintic curve (see Example 3.7). This is a genuinely new instance of non-trivial L-equivalence between K3 surfaces.

## 2. Dual elliptic quintics

In this section, we work over a field  $k$  of characteristic zero.

### 2.1. Hyperplane sections of the Grassmannian

We recall some standard facts about the Grassmannian  $\text{Gr}(2, 5)$  and its smooth and singular hyperplane sections.

Let  $V$  be a five-dimensional vector space; we consider the Plücker embedding  $\text{Gr}(2, V) \subset \mathbb{P}(\Lambda^2(V)) \simeq \mathbb{P}^9$  and the hyperplane sections  $D_\theta := \text{Gr}(2, V) \cap H_\theta$ , parametrized by points of the dual projective space  $[\theta] \in \mathbb{P}(\Lambda^2(V^\vee))$ , where  $\theta \in \Lambda^2(V^\vee)$  is a non-zero two-form.-

By a kernel of a two-form  $\theta \in \Lambda^2(V^\vee)$  we mean the subspace

$$\text{Ker}(\theta) = \{v \in V : \theta(v \wedge u) = 0 \text{ for all } u \in V\}.$$

For a non-zero form there are two cases.

(1) General case:  $\text{Ker}(\theta)$  is one-dimensional. Then  $\theta$  can be written as  $x_1 \wedge x_2 + x_3 \wedge x_4$  for some basis in  $V$ .

(2) Special case:  $\text{Ker}(\theta)$  is three-dimensional. Then  $\theta$  is decomposable and can be written as  $x_1 \wedge x_2$  in some basis. In other words,  $[\theta] \in \text{Gr}(2, V^\vee) \subset \mathbb{P}(\Lambda^2(V^\vee))$ .

It is well known that the two Grassmannians  $\text{Gr}(2, V)$  and  $\text{Gr}(2, V^\vee)$  are projectively dual in their Plücker embeddings. More precisely, we have the following well-known result:

LEMMA 2.1. *Let  $A \subset \mathbb{P}(\Lambda^2(V^\vee))$  be a linear subspace, and consider its orthogonal subspace*

$$A^\perp = \{p \in \Lambda^2(V) : \theta(p) = 0 \text{ for all } \theta \in A\} \subset \Lambda^2(V).$$

*Then  $[U] \in \text{Gr}(2, V)$  is a singular point of  $X_A := \text{Gr}(2, V) \cap \mathbb{P}(A^\perp)$  if and only if for every  $\theta \in A$ ,  $\theta(U) = 0$  and for some  $\theta_0 \in A$ ,  $U \subset \text{Ker}(\theta_0)$ .*

*In particular, the hyperplane section  $D_\theta$  is singular if and only if  $\theta \in \text{Gr}(2, V^\vee)$ , and in this case the singular locus of  $D_\theta$  is isomorphic to  $\mathbb{P}^2$ .*

*Proof.* The projective tangent space to  $\text{Gr}(2, V)$  at a point  $[U]$  is  $\mathbb{P}(U \wedge V) \subset \mathbb{P}(\Lambda^2(V))$ , and it follows that the hyperplane  $H_\theta$  is tangent to  $\text{Gr}(2, V)$  if and only if  $\theta|_{U \wedge V} = 0$ , that is  $U \subset \text{Ker}(\theta)$ . Thus, if  $\text{Ker}(\theta)$  is one-dimensional,  $D_\theta$  is smooth, and if  $\theta \in \text{Gr}(2, V^\vee)$  so that the  $\text{Ker}(\theta)$  is three-dimensional,  $D_\theta$  is singular along  $\text{Gr}(2, \text{Ker}(\theta)) \simeq \mathbb{P}^2$ .

More generally, if  $\theta_1, \dots, \theta_k$  form a basis of  $A$ , and  $[U] \in X_A = \text{Gr}(2, V) \cap H_{\theta_1} \cap \dots \cap H_{\theta_k}$  so that all  $\theta_i$  vanish on  $U$ , then the projective tangent space to  $[U]$  at  $X_A$  is

$$\mathbb{P}(U \wedge V) \cap H_{\theta_1} \cap \dots \cap H_{\theta_k} \subset \mathbb{P}(\Lambda^2(V)),$$

and this intersection is not transverse if and only if  $\theta_1, \dots, \theta_k$  are linearly dependent when restricted to  $\mathbb{P}(U \wedge V)$ , which is equivalent to existence of a non-zero form  $\theta \in A$  vanishing on  $U \wedge V$ , or equivalently  $U \subset \text{Ker}(\theta)$ . □

LEMMA 2.2. *The class in the Grothendieck ring of the Grassmannian is*

$$[\text{Gr}(2, V)] = 1 + \mathbb{L} + 2\mathbb{L}^2 + 2\mathbb{L}^3 + 2\mathbb{L}^4 + \mathbb{L}^5 + \mathbb{L}^6$$

and the classes of its smooth and singular hyperplane sections  $D_\theta = \text{Gr}(2, V) \cap H_\theta$  are given by

$$[D_\theta] = \begin{cases} 1 + \mathbb{L} + 2\mathbb{L}^2 + 2\mathbb{L}^3 + \mathbb{L}^4 + \mathbb{L}^5, & \theta \notin \text{Gr}(2, V^\vee) \\ 1 + \mathbb{L} + 2\mathbb{L}^2 + 2\mathbb{L}^3 + 2\mathbb{L}^4 + \mathbb{L}^5, & \theta \in \text{Gr}(2, V^\vee) \end{cases}$$

*Proof.* The computation for  $\text{Gr}(2, 5)$  is standard: it is a variety with an affine cell decomposition whose cells are parametrized by Young diagrams fitting into a  $3 \times 2$  rectangle, the codimension of a cell given by the number of blocks in the diagram.

The formula for the classes of  $D_\theta$  is proved as in [5, Lemma 7.2]. The paper [5] is written over an algebraically closed field of characteristic zero, however, the proof of [5, Lemma 7.2] only uses the fact that  $k$  is algebraically closed when stating that the class of the three-dimensional quadric in the Grothendieck ring of varieties is  $1 + \mathbb{L} + \mathbb{L}^2 + \mathbb{L}^3$ .

Without assuming  $k$  to be algebraically closed this is not true in general, but a particular quadric appearing is the Lagrangian Grassmannian  $\text{LG}(2, 4)$  of a non-degenerate symplectic form. Since symplectic forms over an arbitrary field can be taken into a standard form, an easy computation shows that  $\text{LG}(2, 4)$  is a *split* quadric, that is it contains a maximal isotropic subspace defined over the ground field, hence its class is indeed  $1 + \mathbb{L} + \mathbb{L}^2 + \mathbb{L}^3$ , see, for example, [22, Example 2.8], and the proof of [5, Lemma 2.9] goes through without any further modification. □

**PROPOSITION 2.3.** *For any locally closed subset  $S \subset \mathbb{P}(\Lambda^2(V^\vee))$ , consider the universal hyperplane section of  $\text{Gr}(2, V)$ :*

$$\mathcal{H}_S := \{([U] \in \text{Gr}(2, V), [\theta] \in S) : [U] \in H_\theta\} \subset \text{Gr}(2, V) \times S.$$

Then we have

$$[\mathcal{H}_S] = [S](1 + \mathbb{L} + 2\mathbb{L}^2 + 2\mathbb{L}^3 + \mathbb{L}^4 + \mathbb{L}^5) + \mathbb{L}^4 \cdot [S \cap \text{Gr}(2, V^\vee)].$$

*Proof.* Presenting  $S$  as  $(S \setminus \text{Gr}(2, V^\vee)) \cup (S \cap \text{Gr}(2, V^\vee))$ , we see that it suffices to show the statement when either  $S \subset \mathbb{P}(\Lambda^2(V^\vee)) \setminus \text{Gr}(2, V^\vee)$  or  $S \subset \text{Gr}(2, V^\vee)$ .

Let  $S \subset \text{Gr}(2, V^\vee)$ . The family of kernels  $\text{Ker}(\theta)$ ,  $\theta \in S$  forms a locally free sheaf of rank three over  $\mathcal{H}_S$ , and considering the relative position of the fibers of this sheaf with respect to the fibers of the tautological bundle coming from  $\text{Gr}(2, V)$  allows to repeat the proof of Lemma 2.2 and to deduce that

$$[\mathcal{H}_S] = [S](1 + \mathbb{L} + 2\mathbb{L}^2 + 2\mathbb{L}^3 + 2\mathbb{L}^4 + \mathbb{L}^5),$$

which is what we had to prove in this case.

The other case is proved analogously. □

We need one more result regarding incidence rank one sheaves on hyperplane sections of Grassmannians. Let  $V$  be an  $n$ -dimensional space, and let  $D \subset \text{Gr}(k, n)$  be the Schubert divisor  $\sigma_{1,0,\dots,0}$  corresponding to a fixed  $(n - k)$ -dimensional linear subspace  $W \subset V$ , that is

$$D := \{[U] \in \text{Gr}(k, n) : \dim(U \cap W) \geq 1\} \subset \text{Gr}(k, n).$$

See [11, §14.7; 13, §1.5] for the basic properties of the Schubert cycles  $\sigma_{a_1, \dots, a_k}$ .

Consider the resolution  $\tilde{D} \rightarrow D$  defined as

$$\tilde{D} := \{([U], [l]) \in \text{Gr}(k, n) \times \mathbb{P}(W) : l \subset U \cap W\}. \tag{2.1}$$

Then  $\tilde{D}$  is a Grassmannian bundle over  $\mathbb{P}(W)$ . We write  $h$  for the hyperplane section on  $\mathbb{P}(W)$ , as well as for its class on  $\tilde{D}$ , and we write  $H$  for the hyperplane section on  $\text{Gr}(k, V) \subset \mathbb{P}(\Lambda^k(V))$  and its class on  $\tilde{D}$ .

LEMMA 2.4. *The  $H$ -degree of the  $c_1(\mathcal{O}(h)) \in \text{Pic}(\tilde{D})$  is equal to the degree of the Schubert cycle  $\sigma_{2,0,\dots,0}$  on  $\text{Gr}(k, n)$ , that is*

$$c_1(\mathcal{O}(h)) \cdot H^{k(n-k)-2} = \sigma_{2,0,\dots,0} \cdot H^{k(n-k)-2}.$$

*Proof.* A codimension one linear subspace  $W' \subset W$  gives rise to an irreducible divisor representing  $c_1(\mathcal{O}(h))$ :

$$Z = \{([U], [l]) \in \tilde{D} : l \subset U \cap W'\} \subset \tilde{D},$$

and this divisor maps birationally onto its image

$$\{[U] \in \text{Gr}(k, n) : \dim(U \cap W') \geq 1\} \subset D \subset \text{Gr}(k, n).$$

This subvariety represents the class  $\sigma_{2,0,\dots,0}$  in the Chow groups of the Grassmannian, and it follows that  $H$ -degree of  $c_1(\mathcal{O}(h))$  is equal to the  $H$ -degree of  $\sigma_{2,0,\dots,0}$ .  $\square$

## 2.2. Elliptic quintics, Jacobians and duality

DEFINITION 2.5. An elliptic quintic is a smooth projective genus one curve which admits a line bundle of degree five.

By Riemann–Roch theorem a degree five line bundle  $\mathcal{L}$  on an elliptic quintic  $X$  is very ample and defines an embedding  $X \subset \mathbb{P}H^0(X, \mathcal{L})^\vee = \mathbb{P}^4$ .

LEMMA 2.6. *Let  $V$  be a five-dimensional  $k$ -vector space, and  $A \subset \Lambda^2(V^\vee)$  be a five-dimensional subspace. If  $X = \text{Gr}(2, V) \cap \mathbb{P}(A^\perp)$  is a transverse intersection, then  $X$  is an elliptic quintic and every elliptic quintic is obtained in this way.*

*Proof.* The first claim follows from the adjunction formula, while the second one is a classical fact known as existence of a Pfaffian representation for an elliptic quintic, see [10] for a modern exposition.  $\square$

For any smooth projective curve  $X$  and an integer  $k \in \mathbb{Z}$ , we consider the degree  $k$  Jacobian  $\text{Jac}^k(X)$ , defined as the moduli space of degree  $k$  line bundles on  $X$ . If  $X$  is an elliptic quintic, then by tensoring with the degree five line bundle and by dualizing we obtain the isomorphisms

$$\text{Jac}^{k+5}(X) \simeq \text{Jac}^k(X), \quad \text{Jac}^{-k}(X) \simeq \text{Jac}^k(X).$$

Thus in this case all Jacobians are isomorphic to one of the

$$E := \text{Jac}^0(X), \quad X = \text{Jac}^1(X) \simeq \text{Jac}^4(X), \quad Y = \text{Jac}^2(X) \simeq \text{Jac}^3(X).$$

Here  $E$  is an elliptic curve, that is a genus one curve with a rational point and  $X$  and  $Y$  are  $E$ -torsors.  $E$ -torsors are parametrized by the Weil–Chatelet group  $H^1(k, E)$  [30, X.3]. If  $[X] \in H^1(k, E)$  is the class of the torsor  $X$ , it is well known that for any  $k \in \mathbb{Z}$ ,  $d \cdot [X] = [\text{Jac}^k(X)]$  (see, for example, [16, Remark 11.5.2]).

In particular, we see that since  $X$  has degree five, then the order of  $[X]$  equals 5 unless  $X$  has a rational point in which case  $[X] = 0$ . Let  $Y = \text{Jac}^2(X)$ , then  $X \simeq \text{Jac}^2(Y) \simeq \text{Jac}^3(Y)$ . We call  $X$  and  $Y$  the *dual elliptic quintics*. It is clear that if  $X$  has a rational point, which is always the case when the base field  $k$  is algebraically closed, then  $X$  and  $Y$  are isomorphic.

The following result shows what happens when  $X$  has no rational points.

LEMMA 2.7. *If  $X$  has no rational points and the  $j$ -invariant satisfies  $j(E) \neq 1728$ , then  $X$  and  $Y$  are not isomorphic.*

*Proof.* The dual elliptic quintics  $X$  and  $Y$  give rise to elements  $[X], [Y] \in H^1(k, E)$  of order five, and  $[Y] = 2[X]$ .

The classes  $[X], [Y]$  correspond to isomorphic genus one curves if and only if  $[Y]$  lies in the  $\text{Aut}(E)$ -orbit of  $[X]$  in  $H^1(k, E)$  [30, Exercise 10.4].

If we assume that for an automorphism  $\sigma \in \text{Aut}(E)$ , we have  $\sigma([X]) = [Y] = 2[X]$ , the action of  $\sigma$  on  $H^1(k, E)$  preserves the subgroup  $\mathbb{Z}/5$  generated by  $[X]$  and we get a surjective group homomorphism  $\langle \sigma \rangle \rightarrow (\mathbb{Z}/5)^* \simeq \mathbb{Z}/4$ . In particular, the order of  $\sigma$  should be a multiple of 4. On the other hand, since  $j(E) \neq 1728$  and  $\text{char}(k) = 0$ , we have  $\text{Aut}(E) = \mathbb{Z}/2$  or  $\text{Aut}(E) = \mathbb{Z}/6$ , and no such  $\sigma$  exists.

Thus,  $X$  and  $Y$  are not isomorphic. □

We now explain duality between elliptic quintics in terms of projective duality.

PROPOSITION 2.8. *Let  $V$  be a five-dimensional  $k$ -vector space and let  $A \subset \Lambda^2(V^\vee)$  be a five-dimensional subspace. We consider the Grassmannian  $\text{Gr}(2, V) \subset \mathbb{P}(\Lambda^2(V))$  and the dual Grassmannian  $\text{Gr}(2, V^\vee) \subset \mathbb{P}(\Lambda^2(V^\vee))$ . For a five-dimensional linear subspace  $A \subset \Lambda^2(V^\vee)$ , let*

$$\begin{aligned} X &:= \text{Gr}(2, V) \cap \mathbb{P}(A^\perp) \\ Y &:= \text{Gr}(2, V^\vee) \cap \mathbb{P}(A). \end{aligned}$$

*Assume that  $X$  is a smooth transverse intersection, so that  $X$  is a genus one curve. Then  $Y$  is also a smooth transverse intersection and  $X$  and  $Y$  are dual elliptic quintics, that is we have*

$$Y \simeq \text{Jac}^3(X), \quad X \simeq \text{Jac}^2(Y).$$

*Proof.* By [7, Proposition 2.24], if  $X$  is a smooth transverse intersection, then the same is true for  $Y$ .

We recall the construction of the universal line bundle  $\mathcal{M}$  on  $X \times Y$  which is used to prove derived equivalence of  $X$  and  $Y$  in [21]. At each point  $([U], [\theta]) \in X \times Y$ , we consider the vector space  $\mathcal{M}_{[U], \theta} := U \cap \text{Ker}(\theta)$ . Let us show that this space is one-dimensional. On the one hand, we have  $\theta(U) = 0$  so that  $U$  cannot have trivial intersection with  $\text{Ker}(\theta)$ , otherwise dimension of  $\text{Ker}(\theta)$  would be greater than 3. On the other hand,  $U$  cannot be contained in  $\text{Ker}(\theta)$ , otherwise  $[U]$  would be a singular point of  $X$  by Lemma 2.1.

Thus  $\mathcal{M}$ , considered as a sheaf given by the kernel of

$$p_1^*(\mathcal{U}|_X) \oplus p_2^*(\mathcal{K}|_Y) \rightarrow V \otimes \mathcal{O}_{X \times Y}$$

on  $X \times Y$ , where  $p_1, p_2$  are the projections from  $X \times Y$  on the two factors,  $\mathcal{U} \subset V \otimes \mathcal{O}_{\text{Gr}(2, V)}$  is the tautological rank two subbundle on  $\text{Gr}(2, V)$  and  $\mathcal{K} \subset V \otimes \mathcal{O}_{\text{Gr}(2, V^\vee)}$  is the rank three subbundle of kernels of 2-forms, is a locally free sheaf of rank one.

We now compute the bidegree of  $\mathcal{M}$ . For any  $\theta \in Y$ , since  $X$  does not intersect the singular locus of  $D_\theta$  (otherwise  $X$  would have been singular),  $X$  is isomorphic to its preimage in the resolution  $\widetilde{D}_\theta$  defined by (2.1).

It follows from the construction of  $\mathcal{M}$  that the restriction  $\mathcal{M}|_{X \times \theta}$  is isomorphic to the restriction of the line bundle  $\mathcal{O}(-h)$  from  $\widetilde{D}_\theta$  to  $X$ , and thus by Lemma 2.4 the degree of  $\mathcal{M}|_{X \times \theta}$  is equal up to sign to the degree of  $\sigma_{2,0}$  in  $\text{Gr}(2, 5)$ . The latter degree is equal to 3, as can be computed using the Pieri formula [11, §14.7; 13, §1.5].

The vector bundle  $\mathcal{M}$  determines a morphism  $f : Y \rightarrow \text{Jac}^{-3}(X)$ . We claim that  $f$  is an isomorphism; we first check this over the algebraic closure of  $k$ . Over  $\bar{k}$ , the Fourier–Mukai transform defined by  $\mathcal{M}$  is a derived equivalence between  $X$  and  $Y$  by [21, Section 4.1, Section 6.1]. Derived equivalence implies that  $\text{Ext}^*(\mathcal{M}|_{X \times \theta}, \mathcal{M}_{X \times \theta'}) = \text{Ext}^*(\mathcal{O}_\theta, \mathcal{O}_{\theta'}) = 0$  for  $\theta \neq \theta'$ , so that  $f$  is injective on  $\bar{k}$ -points. Since the domain of  $f$  is projective and the codomain is irreducible, and both have the same dimension,  $f$  is in fact bijective on  $\bar{k}$ -points. Finally, since  $\text{char}(k) = 0$  by assumption and  $\text{Jac}^{-3}(X)$  is normal,  $f_{\bar{k}}$  is an isomorphism by Zariski’s main theorem. Galois descent implies that  $f$  is an isomorphism too. Thus, we see that

$$Y \simeq \text{Jac}^{-3}(X) \simeq \text{Jac}^3(X),$$

where the second isomorphism is dualization.

Finally,  $X \simeq \text{Jac}^2(Y)$  follows by symmetry by repeating the last part of the above argument with the roles of  $X$  and  $Y$  switched, as the degree of the Schubert cycle  $\sigma_{2,0,0}$  on  $\text{Gr}(3, 5)$  is equal to 2. □

We now deduce L-equivalence of the dual elliptic quintics from their projective duality construction.

**THEOREM 2.9.** *Let  $X$  and  $Y$  be smooth projective dual elliptic quintics. Then  $X$  and  $Y$  are L-equivalent, more precisely we have*

$$\mathbb{L}^4([X] - [Y]) = 0,$$

and in general  $[X] \neq [Y]$ .

*Proof.* By Lemma 2.6 and Proposition 2.8, there exists a five-dimensional subspace  $A \subset \Lambda^2(V^\vee)$  such that

$$X \simeq \text{Gr}(2, V) \cap \mathbb{P}(A^\perp)$$

$$Y \simeq \text{Gr}(2, V^\vee) \cap \mathbb{P}(A).$$

We consider the universal hyperplane section  $\mathcal{H} \subset \text{Gr}(2, V) \times \mathbb{P}(A)$ :

$$\mathcal{H} := \{U \in \text{Gr}(2, V), \theta \in \mathbb{P}(A) : \theta(U) = 0\}$$

and compute its class in the Grothendieck ring of varieties in two ways.

We apply Proposition 2.3 to  $S := \mathbb{P}(A) \subset \mathbb{P}(\Lambda^2(V^\vee))$  to obtain

$$[\mathcal{H}] = [\mathbb{P}^4](1 + \mathbb{L} + 2\mathbb{L}^2 + 2\mathbb{L}^3 + \mathbb{L}^4 + \mathbb{L}^5) + \mathbb{L}^4 \cdot [Y]. \tag{2.2}$$

On the other hand, the morphism  $\mathcal{H} \rightarrow \text{Gr}(2, V)$  is Zariski locally trivial over locally closed subset  $\text{Gr}(2, V) \setminus X$  and  $X$  with fibers  $\mathbb{P}^3$  and  $\mathbb{P}^4$ , respectively, so that we have

$$[\mathcal{H}] = [\text{Gr}(2, 5)][\mathbb{P}^3] + \mathbb{L}^4 \cdot [X]. \tag{2.3}$$

We compare (2.2) and (2.3). An easy computation shows that both  $[\mathbb{P}^4](1 + \mathbb{L} + 2\mathbb{L}^2 + 2\mathbb{L}^3 + \mathbb{L}^4 + \mathbb{L}^5)$  and  $[\text{Gr}(2, 5)][\mathbb{P}^3]$  are equal to

$$\mathbb{L}^9 + 2\mathbb{L}^8 + 4\mathbb{L}^7 + 6\mathbb{L}^6 + 7\mathbb{L}^5 + 7\mathbb{L}^4 + 6\mathbb{L}^3 + 4\mathbb{L}^2 + 2\mathbb{L} + 1$$

(for  $[\text{Gr}(2, 5)]$  see Lemma 2.2). Thus, (2.2) and (2.3) together give

$$\mathbb{L}^4 \cdot ([X] - [Y]) = 0.$$

Finally  $X$  and  $Y$  are in general not isomorphic by Lemma 2.7, and since  $X$  and  $Y$  are not uniruled, the standard argument shows that  $[X] \neq [Y]$  [22, Proposition 2.2]. □



### 3. Elliptic surfaces of index five

In this section,  $k$  is an algebraically closed field of characteristic zero, and we assume  $k = \mathbb{C}$  when discussing Hodge lattices of K3 surfaces.

#### 3.1. $L$ -equivalence of elliptic surfaces

We refer to [8, Chapter 2] for general discussion of elliptic surfaces and their Jacobians. We recall the basic concepts. By an *elliptic surface* we mean a smooth projective surface  $X$  with a morphism  $\pi : X \rightarrow C$  to a smooth projective curve  $C$  such that the general fiber of  $\pi$  is a genus one curve. We always assume that  $X$  is relatively minimal, that is the fibers of  $\pi$  do not contain  $(-1)$ -curves.

We do not assume that  $\pi$  admits a section. By the *index* of an elliptic surface we mean the minimal positive degree of a multisection of  $\pi$ .

For every  $k \in \mathbb{Z}$ , one can consider the relative Jacobian  $Y = \text{Jac}^k(X/C)$ ;  $Y$  is another elliptic surface over the same base curve  $C$  defined as the unique minimal regular model with the generic fiber  $\text{Jac}^k(X_{k(C)})$ . As in the genus one curve case, if  $X$  admits a section, then all Jacobians  $\text{Jac}^k(X/C)$  are isomorphic to  $X$  over  $C$ .

**LEMMA 3.1.** *If  $X \rightarrow C$  is an elliptic surface and  $Y = \text{Jac}^k(X/C)$ , then for every point  $c \in C$ , the reduced fibers  $(X_c)_{\text{red}}$  and  $(Y_c)_{\text{red}}$  are isomorphic.*

*Proof.* This follows from [8, Chapter 2, Proposition 1 and 2]. □

We now consider the case when the multisection index of an elliptic surface  $X \rightarrow C$  is equal to 5, and analogously to the genus one curve case we call  $X$  and  $Y = \text{Jac}^2(X/C) \simeq \text{Jac}^3(X/C)$  the dual elliptic fibrations.

**THEOREM 3.2.** *Let  $X \rightarrow C$  be an elliptic fibration of index five over an algebraically closed field of characteristic zero, and let  $Y = \text{Jac}^2(X/C)$ . Then  $X$  and  $Y$  are  $L$ -equivalent, more precisely we have*

$$\mathbb{L}^4([X] - [Y]) = 0.$$

*Proof.* Let  $X_{k(C)}$ ,  $Y_{k(C)}$  be the generic fibers of  $X$  and  $Y$ . By Theorem 2.9, we have  $\mathbb{L}^4([X_{k(C)}] - [Y_{k(C)}]) = 0$  in  $K_0(\text{Var}/k(C))$ . Therefore, by [27, Proposition 3.4], there exists a non-empty open set  $U \subset C$  such that

$$\mathbb{L}^4([X_U] - [Y_U]) = 0,$$

in  $K_0(\text{Var}/k)$ , where  $X_U$ ,  $Y_U$  are preimages of  $U$  in  $X$  and  $Y$ , respectively. Let  $C \setminus U = \{c_1, \dots, c_n\}$ , then  $X_{c_i}$  and  $Y_{c_i}$  are isomorphic for each  $i$  by Lemma 3.1. In particular,  $\mathbb{L}^4([X_{c_i}] - [Y_{c_i}]) = 0$ ; summing everything together we obtain the desired  $L$ -equivalence statement. □

In the next section, we show that elliptic K3 surfaces of index five and Picard rank two provide examples when  $X$  and  $Y$  are not isomorphic, see Proposition 3.10, so that  $[X] \neq [Y]$  (see, for example, [22, Proposition 2.8]).

#### 3.2. Elliptic K3 surfaces of Picard rank two

We consider elliptic K3 surfaces over  $k = \mathbb{C}$ . Recall that for a K3 surface  $\text{NS}(X) \simeq \text{Pic}(X)$  is a free finitely generated abelian group whose rank is called the Picard rank of  $X$ . Intersection pairing gives  $\text{NS}(X)$  a structure of a lattice. See [16, Chapter 14] for an introduction to lattices. We write  $U$  for the hyperbolic plane, and  $N(X)$  for the extended Neron–Severi lattice

$N(X) = U \oplus \text{NS}(X)$  under the Mukai pairing. We say that two indefinite lattices have the same genus if they have the same rank, signature and discriminant groups.

We only consider projective K3 surfaces, that is the ones admitting a polarization. We think of polarization as a class of an ample divisor in  $\text{NS}(X)$ . Since by degree reasons the class of a polarization is linearly independent to the class of the fiber of an elliptic fibration, the minimal Picard rank of an elliptic K3 surface is equal to 2. Good references about such K3 surfaces are papers of Stellari [31] and van Geemen [32], and [16, Chapter 11].

LEMMA 3.3 (32, Remark 4.2). *Let  $X$  be an elliptic K3 surface of index  $t > 0$  and of Picard rank two. Let  $F \in \text{NS}(X)$  be the class of the fiber. Then there exists a polarization  $H$  such that  $H \cdot F = t$ , and  $H, F$  form a basis of  $\text{NS}(X)$ .*

*Proof.* Let us first show that  $F \in \text{NS}(X)$  is a primitive class. Indeed, if  $F = mC$ , for  $m \geq 1$ , then  $C$  will be an effective divisor contained in a fiber. Since we assume that Picard rank of  $X$  is two, all fibers are irreducible, and  $m = 1$ .

Since  $F$  is a primitive class, there exists  $D \in \text{NS}(X)$  such that  $D, F$  form a basis of  $\text{NS}(X)$ . Up to replacing  $D$  by  $-D$  we may assume that  $D \cdot F = t$ . A simple computation shows that the only possible  $(-2)$ -classes in  $\text{NS}(X)$  are given by  $\pm(D + \frac{2-D^2}{2t}F)$ , hence there is at most one  $(-2)$ -curve in  $X$ .

We consider  $H = D + nF$ . It is clear that

$$H^2 = D^2 + 2nt > 0$$

for  $n \gg 0$ . If  $C$  is a  $(-2)$ -curve, then

$$H \cdot C = D \cdot C + nF \cdot C > 0$$

for  $n \gg 0$  since  $C$  is not in any fiber (otherwise the Picard rank of  $X$  would be at least three). Hence,  $H$  is ample for  $n \gg 0$  by [16, Proposition 2.1.4].  $\square$

For a pair of integers  $t > 0$  and  $d \in \mathbb{Z}$ , we consider a rank two lattice  $\Lambda_{t,d}$  with basis  $H, F$  and pairing defined by

$$\begin{pmatrix} 2d & t \\ t & 0. \end{pmatrix} \tag{3.1}$$

There always exist projective K3 surfaces with  $\text{NS}(X) \simeq \Lambda_{t,d}$  [16, Corollary 14.3.1]. Any such K3 surface is elliptic because  $\text{NS}(X)$  contains a square-zero class [16, Proposition 11.1.3]. Furthermore since the embedding of  $\Lambda_{t,d}$  into a K3 lattice is unique up to isomorphism by [16, Corollary 14.3.1] the locus of these K3 surfaces is an irreducible locally closed subset of dimension 18 in the moduli space of all degree  $2d$  polarized K3 surfaces.

Note that  $t$  is a well-defined invariant of  $\Lambda_{t,d}$ , as the discriminant of (3.1) is  $-t^2$ . The following result describes the complete set of invariants of  $\Lambda_{t,d}$  in the case when  $t$  is an odd prime.

PROPOSITION 3.4 (van Geemen, Stellari). *Let  $t > 0$  be an odd prime, and let  $d, d' \in \mathbb{Z}$ .*

- (1)  $\Lambda_{t,d}$  is isomorphic to  $\Lambda_{t,d'}$  if and only if  $d \equiv d' \pmod{t}$  or  $dd' \equiv 1 \pmod{t}$ .
- (2)  $O(\Lambda_{t,d}) = \{\pm 1\}$  if  $d \not\equiv \pm 1 \pmod{t}$  and  $O(\Lambda_{t,d}) = \mathbb{Z}/2 \times \mathbb{Z}/2 = \{\pm 1, \pm J\}$ , where  $J$  is the isometry swapping the two isotropic classes if  $d \equiv \pm 1 \pmod{t}$ .
- (3) The discriminant group  $A_{t,d} = \Lambda_{t,d}^*/\Lambda_{t,d}$  is  $(\mathbb{Z}/t)^2$  if  $t$  divides  $d$ , and for  $\text{gcd}(d, t) = 1$ , it is  $A_{t,d} = \mathbb{Z}/t^2$  with the square of the generator given by  $\frac{-2d}{t^2}$ .
- (4)  $\Lambda_{t,d}, \Lambda_{t,d'}$  are in the same genus if and only if  $d' \equiv k^2d \pmod{t}$  for some integer  $k$  coprime to  $t$ .

*Proof.* (1) is [32, Proposition 3.7]. and (2) is [32, Lemma 4.6]. The result in (3) is easy for  $t|d$  as we can assume  $d = 0$ . For  $\gcd(d, t) = 1$ , (3) is the computation in the proof of [31, Lemma 3.2 (ii)]. (4) is [31, Lemma 3.2 (ii)].  $\square$

EXAMPLE 3.5. If  $t = 5$ , then there are four isomorphism classes of lattices  $\Lambda_{5,d}$ :

$$\Lambda_{5,0}, \Lambda_{5,1}, \Lambda_{5,2} \simeq \Lambda_{5,3}, \Lambda_{5,4}.$$

The discriminant group  $A_{t,d} = \Lambda_{t,d}^*/\Lambda_{t,d}$  for  $\Lambda_{5,0}$  is  $\mathbb{Z}/5 \oplus \mathbb{Z}/5$ , and it is  $\mathbb{Z}/25$  in the other cases.

The lattices  $\Lambda_{5,1}$  and  $\Lambda_{5,4}$  are in the same genus, whereas the other lattices have only one isomorphism class in each genus.

Finally, the lattices  $\Lambda_{5,1}, \Lambda_{5,4}, \Lambda_{5,5}$  admit an isometry  $J$  permuting the two isotropic classes, and the isometry group is  $\mathbb{Z}/2 \times \mathbb{Z}/2 = \{\pm 1\} \times \{\pm J\}$ , whereas the lattice  $\Lambda_{5,2} \simeq \Lambda_{5,3}$  has the isometry group  $\mathbb{Z}/2 = \{\pm 1\}$ .

Explicitly one can get a K3 surface with  $\text{NS}(X) = \Lambda_{t,d}$  by taking a general K3 surface containing a degree  $t$  elliptic curve.

EXAMPLE 3.6. A very general degree eight K3 surface  $X \subset \mathbb{P}^5$  which contains a normal rational curve  $C$  of degree three, has  $H^2 = 8, C \cdot H = 3, C^2 = -2$ , so that

$$\text{NS}(X) \simeq \begin{pmatrix} 8 & 3 \\ 3 & -2 \end{pmatrix}.$$

Such a K3 surface admits an elliptic fibration provided by the pencil  $F = H - C$ , which consists of the residual elliptic quintics in the hyperplane sections of  $X$  through  $C$  and it is easy to compute that we have

$$\text{NS}(X) \simeq \Lambda_{5,4}.$$

We note that  $X$  admits a unique elliptic fibration [32, 4.7].

EXAMPLE 3.7. A general degree ten K3 surface  $X$  is a complete intersection of a Grassmannian  $\text{Gr}(2, 5) \subset \mathbb{P}^9$  with three hyperplanes and a quadric [26, Corollary 0.3].

As soon as  $X$  contains a normal elliptic quintic curve  $F \subset \mathbb{P}^4$ , it will admit an elliptic fibration of index five, and generically we have

$$\text{NS}(X) \simeq \begin{pmatrix} 10 & 5 \\ 5 & 0 \end{pmatrix}$$

in the basis  $H, F$ . In fact, if we write  $F' = H - F$ , we see that  $\text{NS}(X)$  is isomorphic to  $\Lambda_{5,0}$ . We note that  $F'$  gives rise to a second elliptic fibration structure on  $X$ , cf. [32, 4.7].

We prepare to address the question when  $\text{Jac}^k(X/\mathbb{P}^1)$  and  $X$  are isomorphic.

LEMMA 3.8. *If  $X$  is a K3 surface with  $\text{NS}(X) = \Lambda_{t,d}$ , and  $\gcd(t, k) = 1$ , then  $\text{NS}(\text{Jac}^k(X/\mathbb{P}^1)) = \Lambda_{t,d \cdot k^2}$  for any elliptic fibration on  $X$ .*

*Proof.* Let  $N(X) = U \oplus \text{NS}(X)$  be the extended Neron–Severi lattice and let  $e_1, e_2$  be a basis of  $U$  consisting of two isotropic vectors with  $e_1 \cdot e_2 = -1$ . Then  $v = F + ke_2 \in N(X)$  is the Mukai vector giving rise to the moduli space  $Y = \text{Jac}^k(X/\mathbb{P}^1)$  [16, Example 16.2.4]. Using [25, Theorem 1.4] we have

$$\text{NS}(Y) = v^\perp/v.$$

Explicitly, we have

$$v^\perp = \langle F, e_2, kH + te_1 \rangle = \langle v, e_2, kH + te_1 \rangle,$$

so that

$$v^\perp/v = \langle e_2, kH + te_1 \rangle,$$

and the intersection form on this lattice is isomorphic to  $\Lambda_{t,d,k^2}$ . □

We need the following result, which describes the group of Hodge isometries of the transcendental lattice for a sufficiently general K3 surface. This group is important for studying derived equivalence between K3 surfaces. In particular, it appears in the counting formula for the number of Fourier-Mukai partners [15, Theorem 2.3]. In the proof, we follow the strategy of [15, Proposition B.1] (see also [28, Lemma 4.1]).

LEMMA 3.9. *If  $X$  has Picard rank  $\rho < 20$  and  $X$  is very general in the moduli space of K3 surfaces polarized by a fixed sublattice  $\text{NS}(X)$  of the K3 lattice, then the group of Hodge isometries of the transcendental lattice  $T_X$  is  $\{\pm 1\}$ .*

*Proof.* By the Torelli theorem for K3 surfaces, a (marked) K3 surface polarized by  $\text{NS}(X)$  is determined by a holomorphic 2-form  $\sigma_X \in T_X \otimes \mathbb{C}$ , considered up to scalar. Since the choice of the form  $\sigma$  is given by the condition

$$\sigma_X^2 = 0 \quad \text{and} \quad \sigma_X \bar{\sigma}_X > 0, \tag{3.2}$$

and for a very general choice of  $\sigma_X$  satisfying (3.2),  $\sigma_X^\perp$  in  $T_X \otimes \mathbb{C}$  contains no non-trivial integral class, we conclude that the moduli of (marked) K3 surface polarized by  $\text{NS}(X)$  has dimension  $\text{rk}(T_X) - 2$ .

Let us fix an isometry  $g$  of  $T_X$ , and assume that  $g$  induces a Hodge isometry of  $T_X$  for the K3 surface  $X$  corresponding to  $\sigma_X$ . We use [15, Proposition B.1]. For any choice of  $\sigma_X$ , the group of Hodge isometries of  $T_X$  is a finite cyclic group of even order  $2m$ , and without loss of generality we may assume that  $g$  is a generator of this group. Furthermore in this case  $g$  acts on  $\sigma_X$  via multiplication by a primitive  $2m$ -th root of unity. Finally,  $T_X \otimes \mathbb{C}$  decomposes into a direct sum of eigenspaces of  $g$  as

$$T_X \otimes \mathbb{C} = \bigoplus_{\xi} V_{\xi}, \tag{3.3}$$

where  $\xi$  runs over all primitive  $2m$ -th roots of unity, and the dimension of each eigenspace  $V_{\xi}$  is  $\text{rk} T_X / \varphi(2m)$  with  $\varphi(-)$  being the Euler function (see [15, Steps 4, 5 in the proof of Proposition B.1]). Since  $\sigma_X$  is an eigenvector for  $g$ , we have  $\sigma_X \in V_{\xi}$  for some  $\xi$ . It follows that the moduli of such K3 surfaces has dimension at most  $\text{rk}(T_X) / \varphi(2m) - 1$ .

By assumption  $\rho < 20$ , so that we have  $\text{rk}(T_X) > 2$ . If  $m > 1$ , then  $\varphi(2m) \geq 2$  and

$$\text{rk}(T_X) / \varphi(2m) - 1 \leq \text{rk}(T_X) / 2 - 1 < \text{rk}(T_X) - 2,$$

where the right-hand side is the dimension of the moduli space of K3 surfaces polarized by  $\text{NS}(X)$  and the left-hand side is the dimension of the closed subvariety in the moduli where  $g$  becomes the generator for the group of Hodge isometries. This means that unless  $g = \pm 1$ ,  $g$  is not a Hodge isometry of  $T_X$  of a general K3 surface in the moduli.

Since the group of isometries of  $T_X$  is countable, very general choices of  $\sigma_X$  would give K3 surfaces with the group of Hodge isometries of  $T_X$  equal to  $\{\pm 1\}$ . □

We now consider the multisection index five case. According to Lemma 3.3, an elliptic K3 surface with Picard rank two will have Neron–Severi lattice isomorphic to one of the  $\Lambda_{5,d}$ , where  $d$  is considered modulo 5. See Example 3.5 for more details about these lattices.

PROPOSITION 3.10. *Let  $X$  be an elliptic K3 surface with  $\text{NS}(X) = \Lambda_{5,d}$ , and let  $Y = \text{Jac}^2(X/\mathbb{P}^1)$ .*

- (1) *If  $d = 2$  or  $d = 3$ , then  $X$  and  $Y$  are isomorphic.*
- (2) *If  $d = 1$  or  $d = 4$ , then  $X$  and  $Y$  are not isomorphic.*
- (3) *If  $d = 0$ , and  $X$  is very general in moduli, then  $X$  and  $Y$  are not isomorphic.*

*Proof.* (1) It suffices to show that  $X$  does not have non-trivial Fourier–Mukai partners. We note that by Proposition 3.4 (1) and (4),  $\Lambda_{5,2} \simeq \Lambda_{5,3}$  is the only isometry class of a lattice in its genus. Hence, the counting formula for Fourier–Mukai partners [15, Theorem 2.3] has only one term and since by Proposition 3.4 (2) the orthogonal group  $O(\Lambda_{5,2})$  consists of  $\pm 1$ , this term is equal to 1.

(2) By Lemma 3.8, taking  $\text{Jac}^2$  interchanges the Neron–Severi lattices  $\Lambda_{5,1}$  and  $\Lambda_{5,4}$ , and since these lattices are not isomorphic,  $X$  and  $Y$  are not isomorphic.

(3) If  $X$  and  $Y$  are isomorphic, then the Fourier–Mukai transform  $\Phi : \mathcal{D}^b(X) \simeq \mathcal{D}^b(\text{Jac}^2(X/\mathbb{P}^1))$  corresponding to the moduli space  $\text{Jac}^2(X/C)$  on  $X$  induces a Hodge isometry of  $H^*(X, \mathbb{Z})$  taking one Mukai vector to the other [16, Section 16.3].

Consider the extended Neron–Severi lattice  $N(X) = U \oplus \text{NS}(X)$ , where we choose a basis  $e_1, e_2$  for  $U$  consisting of two isotropic vectors satisfying  $e_1 \cdot e_2 = -1$ . The action of  $\Phi$  takes  $e_1$  (Mukai vector for moduli space  $X$  on  $X$ ) to  $F + 2e_2$  (Mukai vector for moduli space  $Y$  on  $X$ ).

We note that one such isometry  $g_0 \in O(N(X))$  is

$$\begin{aligned} e_1 &\mapsto 2e_2 + F \\ e_2 &\mapsto -2e_1 - H \\ H &\mapsto 2H + 5e_1 \\ F &\mapsto -2F - 5e_2 \end{aligned} \tag{3.4}$$

and any other isometry  $g$  mapping  $e_1$  to  $F + 2e_2$  will have the form

$$g = g_0 \cdot h,$$

where  $h \in O(N(X), e_1)$  is an isometry of  $N(X)$  fixing  $e_1$ .

We now consider the action of  $g$  on the discriminant group  $N(X)^*/N(X) \simeq \text{NS}(X)^*/\text{NS}(X) \simeq A_{5,0} = \mathbb{Z}/5 \oplus \mathbb{Z}/5$  generated by  $\frac{1}{5}H, \frac{1}{5}F$  (cf. Proposition 3.4 (3)). Since we assume that the action of  $g$  is induced by a Hodge isometry of  $H^*(X, \mathbb{Z})$ , the action of  $g$  on the discriminant group is the same as the action induced by a Hodge isometry of  $T_X$ . By Lemma 3.9 for general  $X$  this action on the discriminant group is  $\pm 1$ .

We note that the action of  $O(N(X), e_1)$  on the discriminant group factors through  $O(e_1^\perp/e_1) = O(\text{NS}(X))$ , so by [32, Lemma 4.6] its action is given by one of the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

On the other hand, we see from (3.4) that the action of  $\pm g_0$  on  $A_{5,0}$  does not belong to the subgroup above. Therefore, there is no element  $g \in O(N(X))$  which maps  $e_1$  to  $F + 2e_2$  and is induced by a Hodge isometry of  $H^*(X, \mathbb{Z})$ .  $\square$

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### References

1. B. ANTIEAU, D. KRASHEN and M. WARD, ‘Derived categories of torsors for abelian schemes’, *Adv. Math.* 306 (2017) 1–23.
2. M. BHARGAVA and A. SHANKAR, ‘The average size of the 5-Selmer group of elliptic curves is 6, and the average rank is less than 1’, Preprint, 2013, arXiv:1312.7859.
3. A. I. BONDAL, M. LARSEN and V. A. LUNTS, ‘Grothendieck ring of pretriangulated categories’, *Int. Math. Res. Not.* (2004) 1461–1495.
4. L. A. BORISOV, ‘The class of the affine line is a zero divisor in the Grothendieck ring’, *J. Algebraic Geom.* 27 (2018) 203–209.
5. L. A. BORISOV, A. CĂLDĂRARU and A. PERRY, ‘Intersections of two Grassmannians in  $\mathbb{P}^9$ ’, *J. Reine Angew. Math.* 760 (2020) 133–162.
6. T. BRIDGELAND, ‘Fourier-Mukai transforms for elliptic surfaces’, *J. reine angew. Math.* 498 (1998) 115–133.
7. O. DEBARRE and A. KUZNETSOV, ‘Gushel–Mukai varieties: classification and birationalities’, *Algebr. Geom.* 5 (2018) 15–76.
8. I. DOLGACHEV, ‘Algebraic surfaces with  $q = p_g = 0$ ’, *Algebraic surfaces*, Centro Internazionale Matematico Estivo Summer School 76 (ed. G. Tomassini; Springer, Berlin–Heidelberg, 2010) 97–215.
9. A. I. EFIMOV, ‘Some remarks on L-equivalence of algebraic varieties’, *Selecta Math. (N. S.)* 24 (2018) 3753–3762.
10. T. FISHER, ‘Invariant theory for the elliptic normal quintic, I. Twists of  $X(5)$ ’, *Math. Ann.* 356 (2013) 589–616.
11. W. FULTON, *Intersection theory*, 2nd edn, *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics 2* (Springer, Berlin, 1998).
12. S. GALKIN and E. SHINDER, ‘The Fano variety of lines and rationality problem for a cubic hypersurface’, Preprint, 2014, arXiv:1405.5154.
13. P. GRIFFITHS and J. HARRIS, *Principles of algebraic geometry*, Pure and Applied Mathematics (Wiley–Interscience, New York, NY, 1978).
14. B. HASSETT and K.-W. LAI, ‘Cremona transformations and derived equivalences of K3 surfaces’, *Compos. Math.* 154 (2018) 1508–1533.
15. S. HOSONO, B. H. LIAN, K. OGUIISO and S.-T. YAU, ‘Fourier-Mukai number of a K3 surface’, *Algebraic structures and moduli spaces*, CRM Proceedings & Lecture Notes 38 (eds J. Hurtubise and E. Markman; American Mathematical Society, Providence, RI, 2004) 177–192.
16. D. HUYBRECHTS, *Lectures on K3 surfaces*, Cambridge Studies in Advanced Mathematics 158 (Cambridge University Press, Cambridge, 2016).
17. A. ITO, M. MIURA, S. OKAWA and K. UEDA, ‘Derived equivalence and Grothendieck ring of varieties: the case of K3 surfaces of degree 12 and abelian varieties’, Preprint, 2016, arXiv:1612.08497.
18. A. ITO, M. MIURA, S. OKAWA and K. UEDA, ‘The class of the affine line is a zero divisor in the Grothendieck ring: via G2-Grassmannians’, *J. Algebraic Geom.* 28 (2019) 245–250.
19. G. KAPUSTKA, M. KAPUSTKA and R. MOSCHETTI, ‘Equivalence of K3 surfaces from Verra threefolds’, Preprint, 2017, arXiv:1712.06958.
20. Y. KAWAMATA, ‘Birational geometry and derived categories’, *Surv. Differ. Geom.* 22 (2018) 291–317.
21. A. G. KUZNETSOV, ‘Hyperplane sections and derived categories’, *Izv. Ross. Akad. Nauk Ser. Mat.* 70 (2006) 23–128; translation in *Izv. Math.* 70 (2006) 447–547.
22. A. KUZNETSOV and E. SHINDER, ‘Grothendieck ring of varieties, D- and L-equivalence, and families of quadrics’, *Selecta Math. (N. S.)* 24 (2018) 3475–3500.
23. L. MANIVEL, ‘Double spinor Calabi-Yau varieties’, *Épjournal Geom. Algébrique* 3 (2019), Art. 2, 14 pp.
24. N. MARTIN, ‘The class of the affine line is a zero divisor in the Grothendieck ring: an improvement’, *C. R. Math. Acad. Sci. Paris* 354 (2016) 936–939.
25. S. MUKAI, ‘On the moduli space of bundles on K3 surfaces, I’, *Vector bundles on algebraic varieties*, Tata Institute of Fundamental Research Studies in Mathematics 11 (Tata Institute of Fundamental Research, Bombay, 1987) 341–413.
26. S. MUKAI, ‘Curves, K3 surfaces and Fano 3-folds of genus  $\leq 10$ ’, *Algebraic geometry and commutative algebra*, vol. I (eds H. Hijikata et al.; Kinokuniya, Tokyo, 1988) 357–377.

27. J. NICAISE and J. SEBAG, ‘The Grothendieck ring of varieties’, *Motivic integration and its interactions with model theory and non-Archimedean geometry*, vol. I, London Mathematical Society Lecture Note Series 383 (eds R. Cluckers, J. Nicaise and J. Sebag; Cambridge University Press, Cambridge, 2011) 145–188.
28. K. OGUIISO, ‘K3 surfaces via almost-primes’, *Math. Res. Lett.* 9 (2002) 47–63.
29. S. OKAWA, ‘An example of birationally inequivalent projective symplectic varieties which are D-equivalent and L-equivalent’, Preprint, 2018, arXiv:1801.09385.
30. J. H. SILVERMAN, *The arithmetic of elliptic curves*, Graduate Texts in Mathematics 106 (Springer, New York, 1986).
31. P. STELLARI, ‘Some remarks about the FM-partners of K3 surfaces with Picard numbers 1 and 2’, *Geom. Dedicata* 108 (2004) 1–13.
32. B. VAN GEEMEN, ‘Some remarks on Brauer groups of K3 surfaces’, *Adv. Math.* 197 (2005) 222–247.

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