The Infinitesimal Torelli Theorem for Irregular Varieties

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Abstract

In this thesis we prove the infinitesimal Torelli theorem for certain classes of irregular varieties. Given a compact Kähler manifold, the infinitesimal Torelli problem asks whether the differential of the period map of a Kuranishi family is injective. Unlike the classical Torelli theorem for curves, there is a negative answer for example for hyperelliptic curves of genus greater than 2. Nevertheless, the infinitesimal Torelli theorem holds for many other classes of manifolds. Following Green's proof for sufficiently ample hypersurfaces in arbitrary varieties, we prove it for smooth ample hypersurfaces and more generally complete intersections in general abelian varieties by reducing it to showing the surjectivity of certain multiplication maps of vector bundles on the ambient abelian variety. Then we derive numerical conditions for such multiplication maps to be surjective giving an effective bound on Green's result in this particular case. We also investigate the more general case of irregular varieties with globally generated cotangent bundle which do not embed into their Albanese varieties.

Key words: infinitesimal Torelli theorem, period map, irregular varieties, abelian varieties, ample divisors, projective normality.

Kurzzusammenfassung

In dieser Doktorarbeit wird der infinitesimale Torelli-Satz für gewisse Klassen von irregulären Varietäten bewiesen. Das infinitesimale Torelli-Problem für eine kompakte Kähler-Mannigfaltigkeit fragt, ob das Differential der Periodenabbildung einer Kuranishi-Familie injektiv ist. Im Gegensatz zum klassischen Torelli-Satz für Kurven ist die Antwort z.B. für hyperelliptische Kurven von Geschlecht größer als 2 negativ. Trotzdem gilt der infinitesimale Torelli-Satz für viele Klassen von Mannigfaltigkeiten. Dem Beweis von Green für glatte ausreichend ample Hyperflächen in beliebigen Varietäten folgend, zeigen wir den infinitesimalen Torelli-Satz für glatte ample Hyperflächen und allgemeiner vollständige Durchschnitte in allgemeinen abelschen Varietäten, indem wir ihn auf die Surjektivität gewisser Multiplikationsabbildungen von Vektorbündeln auf der umgebenden abelschen Varietät reduzieren. Anschließend leiten wir numerische Bedinungen für die Surjektivität solcher Multiplikationsabbildungen her und erhalten somit eine effektive Abschätzung für Greens Ergebnis in diesem speziellen Fall. Außerdem untersuchen wir den allgemeineren Fall von irregulären Varietäten mit global erzeugtem Kotangentialbündel, welche nicht in ihre Ablanese-Varietäten eingebettet sind.

Schlagwörter: infinitesimaler Torelli-Satz, Periodenabbildung, irreguläre Varietäten, abelsche Varietäten, ample Divisoren, projektive Normalität.

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Contents

Introduction

Torelli-type theorems, broadly speaking, ask whether we can tell complex varieties apart by means of their Hodge structure. There are many different Torelli-type theorems. The main focus of this thesis is the infinitesimal Torelli theorem. Given a family of compact Kähler manifolds $\phi\colon \mathcal{X} \to B$ with fibers $X_b := \phi^{-1}(b)$, the singular cohomology groups of the fibers with complex coefficients as well as the Hodge numbers are constant. The Hodge decomposition on the other hand varies so a family of compact Kähler manifolds induces what is called a variation of Hodge structure. There is a period domain $\mathcal{D}$ parametrizing the set of Hodge structures and thus one can define the period map by sending a point $b \in B$ to the Hodge filtration of its fiber. Griffiths showed this map to be holomorphic (see [Gri68a], [Gri68b] and [Gri70]) allowing us to study its differential. The infinitesimal Torelli theorem asks whether the period map of a Kuranishi family is an immersion, that is to say whether its differential is injective. By the works of Griffiths there is a cohomological interpretation of the differential of the period map. To prove the infinitesimal Torelli theorem for a Kähler manifold X of dimension n it is enough to show injectivity of the map

$$H^1(X, T_X) \to \mathrm{Hom}(H^0(X, \omega_X), H^1(X, \Omega_X^{n-1}))$$

given by the cup product and the interior product. For a curve C it follows easily from a classical result by Max Noether (see for example [ACGH85, p. 117]) that the infinitesimal Torelli theorem holds if and only if C has genus $g(C) \leq 2$ or $g(C) > 2$ and C is non-hyperelliptic. That means that in this case very ampleness of the canonical sheaf is equivalent to the infinitesimal Torelli theorem. For surfaces, however, Garra and Zucconi show that for any $n \geq 5$ there exists a generically smooth $n + 9$ dimensional irreducible component of the moduli space of algebraic surfaces of general type such that for a general element of it the infinitesimal Torelli theorem fails (see [GZ08]). Thus finding classes of manifolds that satisfy the infinitesimal Torelli theorem is still an open problem.

While the infinitesimal Torelli theorem is an interesting problem in and of itself, it also sometimes plays a role in proving other types of Torelli theorems. For example in the case of K3 surfaces or more generally Calabi-Yau manifolds, proving the infinitesimal Torelli theorem (which is easy because the canonical bundle is trivial) is an essential step in proving the global Torelli theorem (see for example [BHPVdV04, Chapter VIII]).

In Section 1.1 we recall the basics of Hodge theory in order to be able to state the above more precisely. Since we will study irregular varieties, i.e. varieties admitting morphisms to non-trivial abelian varieties, in Section 1.3 we collect some important facts about abelian varieties. Then we will consider specifically hypersurfaces in abelian varieties and compute their Hodge numbers in Section 1.4. In Section 1.5 we will discuss the classical Torelli theorem for curves. Finally, in Section 1.6 we will discuss the infinitesimal Torelli theorem for higher-dimensional varieties. It holds for projective varieties with trivial canonical bundle – and thus in particular for abelian varieties – and Reider proved it for irregular surfaces of general type with almost very ample cotangent bundle that have no irrational pencils (see [Rei88]). Outside of this not much is known for irregular varieties. The condition that the cotangent bundle be almost very ample in particular implies that it is globally generated which means that the Albanese map is an immersion. It thus seems natural to study subvarieties of abelian varieties. In this case the Albanese map is not just an immersion but in addition injective and thus an embedding. In [Gri68a] and [Gri68b] Griffiths proved that the infinitesimal Torelli theorem holds for smooth hypersurfaces of high degree in projective space and in [Gre85] Green generalized this to *sufficiently ample* smooth hypersurface sections in arbitrary smooth complete algebraic varieties by reducing the problem to showing surjectivity of the multiplication map of global sections of certain line bundles on the ambient variety. He does not give a bound on the required ampleness, however.

We will apply Green's result in the particular case of hypersurfaces (or more generally complete intersections) in abelian varieties and show that in all but the last step of the proof ampleness is sufficient. If $X \subset A$ is an ample hypersurface in a g-dimensional abelian variety A and $L = \mathcal{O}_A(X)$ then in the final step one needs to prove that the multiplication map

$$H^0(A, L) \otimes H^0(A, L^{g-1}) \to H^0(A, L^g)$$

is surjective. For this ampleness alone is not sufficient. Chapter 2 is therefore devoted to studying multiplication maps of line bundles on abelian varieties in more detail.

In Section 2.1 we briefly derive a necessary condition for surjectivity to hold by comparing dimensions before discussing the related concept of projective normality in Section 2.2. Projective normality of a polarized abelian variety (A, L) is equivalent to the surjectivity of the multiplication map $S^2H^0(A, L) \to H^0(A, L^2)$ which in fact implies the infinitesimal Torelli theorem for a smooth hypersurface section of L. It is well known that L defines a projectively normal embedding if it is at least a third power of another line bundle or if it is basepoint-free and a square of another line bundle. Therefore we are primarily interested in primitive line bundles.

For abelian surfaces projective normality is well understood. In Section 2.3 we give a brief overview of the known results. Comparing them to the known results for the infinitesimal Torelli theorem for curves gives us some interesting examples.

In Section 2.4 we will generalize methods from [Iye03] using theta functions and theta groups to prove the following theorem

Theorem A *Let L be a line bundle on a simple abelian variety A of dimension g. For any $k \in \mathbb{N}$, if $h^0(A, L) > \left(\frac{k+1}{k}\right)^g \cdot g!$, the multiplication map*

$$H^0(A, L) \otimes H^0(A, L^k) \to H^0(A, L^{k+1})$$

is surjective.

It requires the ambient abelian variety to be simple. We will discuss briefly why this assumption is necessary in Section 2.5.

In [HT11] Hwang and To give a bound for projective normality to hold using methods from the study of local positivity. In Section 2.6 we will use their methods to prove the following theorem which holds for a general polarized abelian variety.

Theorem B *Let (A, L) be a general polarized abelian variety of dimension $g \geq 2$. If $h^0(A, L) \geq \frac{1}{2g!}\left(\frac{4g(k+1)}{k}\right)^g$ then the multiplication map*

$$H^0(A, L) \otimes H^0(A, L^k) \to H^0(A, L^{k+1})$$

is surjective.

In the last section of Chapter 2 we will discuss how one could obtain a theorem for an arbitrary abelian variety using Nadel vanishing.

Finally in Chapter 3 we will prove the infinitesimal Torelli theorem for certain classes of irregular varieties. Before applying our results about surjectivity of multiplication maps, in Section 3.1 we give a direct proof of the infinitesimal Torelli theorem for hypersurfaces inducing principal polarizations:

Theorem C *Let A be an abelian variety of dimension $g \geq 3$ and let $X \subset A$ be a smooth hypersurface defining a principal polarization. Then the infinitesimal Torelli theorem holds for X.*

Then in Section 3.2 we apply Green's proof of the infinitesimal Torelli theorem for sufficiently ample hypersurfaces in arbitrary varieties to the case of hypersurfaces in abelian varieties. We show that all steps in the proof aside from showing surjectivity of the multiplication map work for hypersurfaces that are merely ample and then use Theorem A to obtain the following theorem:

Theorem D *Let X be a smooth hypersurface on a simple g-dimensional abelian variety A. If $h^0(A, \mathcal{O}(X)) > \left(\frac{g}{g-1}\right)^g \cdot g!$, then the infinitesimal Torelli theorem holds for X.*

Similarly applying Theorem B we obtain a theorem for a smooth hypersurface in a general abelian variety:

Theorem E *Let $X \subset A$ be a smooth hypersurface in a general abelian variety. If $h^0(A, \mathcal{O}_A(X)) \geq \frac{1}{2g!}\left(\frac{4g^2}{g-1}\right)^g$ then the infinitesimal Torelli theorem holds for X.*

In Section 3.3 we generalize this to complete intersections. The relevant multiplication map is one of vector bundles but decomposing it into a direct sum of multiplication maps of line bundles allows us to apply our results from Chapter 2 to obtain the following theorem.

Theorem F *Let $X = D_1 \cap \ldots \cap D_c$ be a complete intersection of ample divisors D_i on a g-dimensional simple abelian variety A. Then the infinitesimal Torelli theorem holds for X if one of the following holds:*
(i) $h^0(A, \mathcal{O}_A(\sum_{i=1}^c D_i)) > 2^g g!$.
(ii) $h^0(A, \mathcal{O}_A(D_i)) > \left(1 + \frac{1}{\lceil \frac{g-1}{c} \rceil}\right)^g g!$ *for all* $i \in \{1, \ldots, c\}$.

Again, we can also apply Theorem B to obtain the following theorem for general complete intersections in general abelian varieties.

Theorem G *Let $X = D_1 \cap \ldots \cap D_c$ be a complete intersection of general ample divisors D_i on a general g-dimensional abelian variety A. Then the infinitesimal Torelli theorem holds for X if one of the following holds:*
(i) $h^0(A, \mathcal{O}_A(\sum_{i=1}^c D_i)) > \frac{1}{2g!}(8g)^g$.
(ii) $h^0(A, \mathcal{O}_A(D_i)) > \frac{1}{2g!}\left(\frac{4g(\lceil \frac{g-1}{c} \rceil + 1)}{\lceil \frac{g-1}{c} \rceil}\right)^g g!$ *for all* $i \in \{1, \ldots, c\}$.

Finally, in Section 3.4 we investigate what happens if the cotangent bundle of X is globally generated – and thus the Albanese map is an immersion – but the image of X is a singular divisor in its Albanese variety.

In Chapter 4 we give an overview of the remaining open questions and how one could obtain more general results.

1 Preliminaries

In this chapter we will recall some definitions necessary to discuss the infinitesimal Torelli theorem in more detail. In the first two sections we will recall the basics of Hodge theory as well as some important facts about abelian varieties. In Section 1.4 we will compute the Hodge numbers of a smooth hypersurface of a given polarization type in an abelian variety. In 1.5 we briefly recall the classical Torelli theorem for curves and finally in the last section we will discuss the infinitesimal Torelli theorem and give an overview of some of the known results.

1.1 Hodge theory

In this section we recall the basics of Hodge theory which gives us a convenient language to talk about Torelli-type theorems. The reader can consult [Voi07] for further details.

Let X be an n-dimensional riemannian manifold. Denote by $\mathcal{A}_X^k$ the sheaf of smooth k-forms on X with differential $d\colon \mathcal{A}_X^k \to \mathcal{A}_X^{k+1}$ given by the exterior derivative. There is an adjoint operator $\delta\colon \mathcal{A}_X^k \to \mathcal{A}_X^{k-1}$ defined by $\delta = (-1)^{kn+n+1}\star d\star$ where $\star$ denotes the Hodge star which depends on the chosen metric. From this we can define the *Laplacian* $\Delta = d\delta + \delta d$. This allows us to define the set of *harmonic forms* of degree k with respect to Δ by $\mathcal{H}^k(X) = \{\alpha \in \mathcal{A}_X^k \mid \Delta\alpha = 0\}$. If X is compact there is a natural isomorphism

$$\mathcal{H}^k(X) \cong H^k(X, \mathbb{R})$$

(see [Voi07, Theorem 5.23]).

Now let X be an n-dimensional complex manifold with a hermitian metric and denote by $\mathcal{A}_X^k$ the sheaf of smooth complex valued k-forms. It can be decomposed into a direct sum of sheaves of (p,q)-forms with $p+q=k$, i.e. we have a decomposition

$$\mathcal{A}^k = \bigoplus_{p+q=k} \mathcal{A}_X^{p,q}. \tag{1.1}$$

The differential $d\colon \mathcal{A}_X^k \to \mathcal{A}_X^{k+1}$ maps $\mathcal{A}_X^{p,q}$ to $\mathcal{A}_X^{p+1,q} \oplus \mathcal{A}_X^{p,q+1}$. Thus we can define the so called *Dolbeault operators* $\partial\colon \mathcal{A}_X^{p,q} \to \mathcal{A}_X^{p+1,q}$ and $\overline{\partial}\colon \mathcal{A}_X^{p,q} \to \mathcal{A}_X^{p,q+1}$ as the composition of d with the projection to $\mathcal{A}_X^{p+1,q}$ and $\mathcal{A}_X^{p,q+1}$, respectively, so that we have $d = \partial + \overline{\partial}$. Now, completely analogously to the above, we can define

Laplacians Δ_∂, $\Delta_{\overline{\partial}}$ and Δ_d with respect to ∂, $\overline{\partial}$ and d, respectively, and thus we can define harmonic forms with respect to these Laplacians. Let $\mathcal{H}^k(X)$ denote the space of harmonic forms with respect to Δ_d. As in the real case, if X is compact there is a natural isomorphism

$$\mathcal{H}^k(X) \cong H^k(X, \mathbb{C}). \tag{1.2}$$

However, in general the different Laplacians are not necessarily related and Δ_d does not preserve the bidegree of a (p,q)-form which means that the decomposition in (1.1) of k-forms above does not necessarily induce a decomposition of harmonic k-forms. In order for things to behave nicely we need another condition. If h is a hermitian metric on X, the imaginary part of h defines a 2-form (in fact a $(1,1)$-form) ω. If this form is closed, i.e. if $d\omega = 0$, we call it a *Kähler form* and we call X a *Kähler manifold*. For Kähler manifolds we have in fact

$$\Delta_\partial = \Delta_{\overline{\partial}} = \frac{1}{2}\Delta_d$$

(see [Voi07, Theorem 6.7]). From this it is easy to see that Δ_d preserves the bidegree of a (p,q)-form and consequently that if a k-form is harmonic with respect to Δ_d, so are its (p,q)-components (see [Voi07, Theorem 6.9]). Therefore, we obtain a decomposition

$$\mathcal{H}^k(X) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(X)$$

of harmonic k-forms into harmonic (p,q)-forms. Finally using the isomorphism in (1.2) we obtain a decomposition

$$H^k(X, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} = H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$$

called the *Hodge decomposition*. This decomposition is in fact independent of the choice of the Kähler metric ω. For simplicity we will often omit the X and simply write $H^{p,q}$ if the manifold is clear from the context. We have an identification $H^{p,q} \cong H^q(X, \Omega_X^p)$ with the sheaf cohomology of the sheaf of holomorphic p-forms Ω_X^p and a symmetry $H^{p,q} = \overline{H^{q,p}}$ where $\overline{H^{q,p}}$ denotes the complex conjugate of $H^{q,p}$. The dimensions $h^{p,q} := \dim H^{p,q}$ are called the Hodge numbers of X and from the above symmetry property it is clear that $h^{p,q} = h^{q,p}$. Furthermore, by Serre duality we have $H^q(X, \Omega_X^p)^\vee \cong H^{n-q}(X, (\Omega_X^p)^\vee \otimes \omega_X) = H^{n-q}(X, \Omega_X^{n-p})$ which gives $h^{p,q} = h^{n-p,n-q}$. It is common to arrange the Hodge numbers in a diamond shape called the *Hodge diamond.*

$$\begin{array}{ccccccccc} & & & & h^{n,n} & & & & \\ & & & h^{n,n-1} & & h^{n-1,n} & & & \\ & & \iddots & & \vdots & & \ddots & & \\ & h^{n,1} & & & & & & h^{1,n} & \\ h^{n,0} & & h^{n-1,1} & & \cdots & & h^{1,n-1} & & h^{0,n} \\ & h^{n-1,0} & & & & & & h^{0,n-1} & \\ & & \ddots & & \vdots & & \iddots & & \\ & & & h^{1,0} & & h^{0,1} & & & \\ & & & & h^{0,0} & & & & \end{array}$$

The Kähler form ω induces a bilinear map

$$Q\colon H^k(X,\mathbb{Z}) \times H^k(X,\mathbb{Z}) \to \mathbb{Z}$$

by

$$Q(\alpha,\beta) = \int_X \alpha \wedge \beta \wedge \omega^{n-k}.$$

This bilinear map Q is symmetric if k is even and alternating otherwise. It satisfies the following two properties called the *Hodge-Riemann bilinear relations* (see [Voi07, §6.3.2]):

(1) $Q(\alpha,\beta) = 0$ for $\alpha \in H^{p,q}, \beta \in H^{p',q'}$ with $p \neq q'$,

(2) $(-1)^{\frac{k(k-1)}{2}} i^{p-q-k} Q(\alpha,\overline{\alpha}) > 0$ for any non-zero α of type (p,q) in the kernel of the map

$$\begin{aligned} H^k(X,\mathbb{R}) &\to H^{2n-k+2}(X,\mathbb{R}) \\ \alpha &\mapsto \alpha \wedge \omega^{n-k+1} \end{aligned}$$

where ω is the Kähler form of X.

The Hodge decomposition motivates the following more abstract definition.

Definition 1.1.1 A *Hodge structure of weight* k consists of a finitely generated free abelian group, $H_{\mathbb{Z}}$ along with a decomposition $H_{\mathbb{C}} = \bigoplus_{p+q=k} H^{p,q}$ of its complexification $H_{\mathbb{C}} = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ into complex vector subspaces satisfying the symmetry property $H^{p,q} = \overline{H^{q,p}}$. A Hodge structure of weight k together with a non-degenerate integral bilinear form Q on $H_{\mathbb{Z}}$ which is symmetric if k is even and alternating otherwise, and satisfies the Hodge-Riemann bilinear relations (1) and (2) above, is called a *polarized Hodge structure of weight* k.

The cohomology groups $H^k(X,\mathbb{C})$ as well as the Hodge numbers are constant in families but the Hodge decomposition varies. A family of Kähler manifolds thus defines what is called a variation of Hodge structures. It would seem natural to define a space parametrizing Hodge structures and define a map that sends a point in the base space of the family to the Hodge decomposition of its fiber and study the properties of this map. However, it turns out that in order for this map to behave nicely we have to pass from the Hodge decomposition to a different object. We can define the *Hodge filtration* on the cohomology of X from its Hodge decomposition. It is the decreasing filtration

$$H^k(X,\mathbb{C}) = F^0H^k(X,\mathbb{C}) \supset F^1H^k(X,\mathbb{C}) \supset \ldots \supset F^kH^k(X,\mathbb{C}) \supset \{0\} \tag{1.3}$$

given by the direct sum

$$F^pH^k(X,\mathbb{C}) = \bigoplus_{r\geq p} H^{r,k-r}(X).$$

This is essentially equivalent to the Hodge decomposition as for every p we have

$$H^k(X,\mathbb{C}) = F^pH^k(X,\mathbb{C}) \oplus \overline{F^{k-p+1}H^k(X,\mathbb{C})} \tag{1.4}$$

and the Hodge decomposition is then determined by

$$H^{p,q}(X) = F^pH^k(X,\mathbb{C}) \cap \overline{F^qH^k(X,\mathbb{C})} \tag{1.5}$$

meaning one can recover the Hodge decomposition from the Hodge filtration.

The set of decreasing filtrations as in (1.3) is parametrized by the flag variety determined by the dimensions of the vector spaces in the decreasing filtration. The open subset $\mathcal{D}$ of this flag variety of filtrations satisfying the condition in (1.4) is called the *period domain* and it has the structure of a complex manifold.

The important advantage of the Hodge filtration is that unlike the Hodge decomposition it varies holomorphically in families which allows us to define a nicely behaved period map. We will come back to this map in Section 1.5.

1.2 Vanishing theorems

In this section we will briefly collect some important vanishing theorems that will be useful in later sections.

Theorem 1.2.1 (Kodaira vanishing) *Let X be a compact Kähler manifold, let L be an ample line bundle on X and let ω_X denote the canonical bundle of X, then*

$$H^i(X,\omega_X \otimes L) = 0 \textit{ for } i > 0.$$

A generalization of this which will be useful in Section 1.4 is the following:

Theorem 1.2.2 (Nakano vanishing) *Let X be a compact Kähler manifold, let L be an ample line bundle on X and let Ω_X^p denote the sheaf of holomorphic p-forms on X, then*

$$H^q(X, \Omega_X^p \otimes L) = 0 \text{ for } p + q > \dim(X).$$

Sometimes it will be useful to have vanishing theorems for line bundles that are not ample but only big and nef. The following theorem is a generalization of Kodaira vanishing in this direction:

Theorem 1.2.3 (Kawamata-Viehweg vanishing) *Let X be a compact Kähler manifold, let L be a big and nef line bundle on X and let ω_X denote the canonical bundle of X, then*

$$H^i(X, \omega_X \otimes L) = 0 \text{ for } i > 0.$$

Finally, in Section 2.7 we will encounter Nadel vanishing which is a vanishing theorem for multiplier ideal sheaves and generalizes Kawamata-Viehweg vanishing.

1.3 Abelian varieties

Since we will be studying irregular varieties we should recall some basic facts about abelian varieties. Most of the following can be found in [BL04].

Definition 1.3.1 A g-dimensional *complex torus* is the quotient of a g-dimensional complex vector space V by a full rank sublattice Λ of V. An *abelian variety* is a complex torus that is projective.

A projective embedding of course arises from an ample line bundle so we need to know some facts about line bundles on complex tori. Let $T = V/\Lambda$ be a complex torus of dimension g, and let L be a line bundle on T then the first Chern class $c_1(L)$ of L can be considered as a hermitian form

$$H : V \times V \to \mathbb{C}.$$

The imaginary part $E = \mathrm{im}H$ then defines an alternating form that is integral on the lattice Λ. By the elementary divisor theorem there exists a basis of Λ such that E is given by the matrix

$$\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}$$

where $D = \mathrm{diag}(d_1, \ldots, d_g)$ is a diagonal matrix with entries $d_i \geq 0$ such that $d_i | d_{i+1}$ for all $i \in \{1, \ldots, g-1\}$. The vector $(d_1, \ldots, d_g)$ is called the *type of the line bundle*. We will often work with tensor powers of line bundles. Instead of $L^{\otimes k}$ we will simply write L^k to denote the k-th tensor power. Since this could be confused with the self-intersection number we will write the latter in terms of the first Chern class, i.e. we will write $c_1(L)^g$ instead of L^g for the self-intersection. If L is of type $(d_1, \dots, d_g)$ then for any $k \in \mathbb{N}$, since $c_1(L^k) = kc_1(L)$, L^k is of type $(kd_1, \dots, kd_g)$. A line bundle of type $(1, d_2, \dots, d_g)$, i.e. a line bundle that is not a non-trivial tensor power of another line bundle, is called a *primitive*.

The most important case for us is when the hermitian form $H = c_1(L)$ is positive definite which corresponds to the case that the line bundle L is ample. Some power of L then induces a projective embedding meaning that T is an abelian variety and we will denote it by $T = A$ from now on. In this case we call H a polarization on A. It is also common to call L itself the polarization. The tuple (A, H) or (A, L) is called a *polarized abelian variety*. The *type of the polarization* is simply the type of L. A polarization of type $(1, \dots, 1)$ is called a *principal polarization*.

A line bundle L also induces a map

$$\phi_L \colon A \to \mathrm{Pic}^0(A) = \widehat{A}, x \mapsto t_x^* L \otimes L^{-1}$$

to the dual abelian variety $\widehat{A}$ where $t_x \colon A \to A$ denotes translation by x. By the Theorem of the Square this map is a homomorphism. Its kernel is denoted by $K(L)$. For L ample $K(L)$ is finite and ϕ_L thus an isogeny. Since any ample line bundle induces such an isogeny to the dual abelian variety and since any isogeny to the dual abelian variety whose analytic representation is a positive definite hermitian form is induced by an ample line bundle (see [BL04, Theorem 2.5.5]), it is also common to refer to this isogeny ϕ_L as the polarization.

The type of L can be read off from the kernel $K(L)$. One can choose a (non-canonical) decomposition $V = V_1 \oplus V_2$ into real subspaces V_1, V_2 which are maximally isotropic with respect to the alternating form E. This decomposition induces a decomposition $K(L) = K(L)_1 \oplus K(L)_2$. If L is of type $(d_1, \dots, d_g)$ then $K(L)_i \cong \bigoplus \mathbb{Z}/d_j\mathbb{Z}$ for $i = 1, 2$. In particular $K(L)$ is trivial, and thus ϕ_L is an isomorphism if and only if L induces a principal polarization.

If L is not principal we can consider the quotient map $\pi \colon A \to A/K(L)_1$. By the isogeny theorem $A/K(L)_1$ carries a principal polarization M such that $L = \pi^* M$. Since M is a principal polarization ϕ_M is an isomorphism $A/K(L)_1 \cong \mathrm{Pic}^0(A/K(L)_1)$. Then the pullback map $\pi^* \colon \mathrm{Pic}^0(A/K(L)_1) \to \mathrm{Pic}^0(A)$ is essentially given by taking the quotient by the remaining factor $K(L)_2$ so that $\phi_L = \pi^* \circ \phi_M \circ \pi$.

For us the cohomology of a line bundle is of particular importance. An important theorem is the following:

Theorem 1.3.2 (Geometric Riemann-Roch) *For any line bundle L on a g-dimensional abelian variety A we have*

$$\chi(L) = \sum_{k=0}^{g} (-1)^k h^k(X, L) = \frac{c_1(L)^g}{g!}.$$

See e.g. [BL04, Theorem 3.6.3] for a proof. For an ample line bundle of type $(d_1, \dots, d_g)$ we have that $H^0(A, L)$ has dimension $d_1 \cdots d_g$. It is easy to see (for example by Kodaira vanishing) that all higher cohomology groups vanish. And thus in this case the geometric Riemann-Roch theorem says that

$$h^0(A, L) = \frac{c_1(L)^g}{g!}. \tag{1.6}$$

In particular this implies that for any $k \in \mathbb{N}$ we have

$$h^0(A, L^k) = k^g h^0(A, L). \tag{1.7}$$

This also means that any statement about the dimension of the vector space of global sections of an ample line bundle can be rephrased in terms of its top self-intersection number.

The objects we will be studying are irregular varieties. The irregularity of a compact complex manifold X is the Hodge number $h^{0,1} = h^1(X, \mathcal{O}_X)$ and is usually denoted by $q(X)$ or just q if it is clear from context which manifold it refers to. For a Kähler manifold this is the same as the Hodge number $h^{1,0} = h^0(X, \Omega^1_X)$. For a smooth curve C this coincides with the genus $g = g(C)$ and one can associate a g-dimensional torus J_C to C. There are two ways to define it. Algebraically it is defined as the connected component of the trivial line bundle in the Picard group, i.e. $J_C := \mathrm{Pic}^0(C)$. Analytically we define it as follows. The first homology group of classes of cycles can be embedded into the dual of the space of holomorphic 1-forms by integrating along some representative of the homology classes. More precisely we have an inclusion

$$\begin{aligned} \iota\colon H_1(C, \mathbb{Z}) &\to H^0(C, \Omega^1_C)^\vee \\ [\gamma] &\mapsto \left(\omega \mapsto \int_\gamma \omega\right). \end{aligned}$$

The injectivity of this map follows from the Hodge decomposition

$$H^1(C, \mathbb{C}) = H^0(C, \Omega^1_C) \oplus \overline{H^0(C, \Omega^1_C)}$$

because ι is the composition

$$H_1(C, \mathbb{Z}) \to H^1(C, \mathbb{C})^\vee = H^0(C, \Omega^1_C)^\vee \oplus \overline{H^0(C, \Omega^1_C)^\vee} \xrightarrow{\pi_1} H^1(C, \Omega^1_C)^\vee$$

where the first map comes from Poincaré duality and π_1 denotes the projection onto the first factor. The image of $[\gamma] \in H_1(C, \mathbb{Z})$ in $H^0(C, \Omega^1_C)^\vee \oplus \overline{H^0(C, \Omega^1_C)^\vee}$ must be invariant under complex conjugation which means it is the sum of an element in $H^1(C, \Omega^1_C)$ and its complex conjugate so the injectivity of ι follows because by the universal coefficient theorem the map $H_1(C,\mathbb{Z}) \to H_1(C,\mathbb{C}) = H^1(C,\mathbb{C})^\vee$ is injective.

The image of the $H_1(C,\mathbb{Z})$ under this embedding is therefore a lattice in the vector space $H^0(C,\Omega^1_C)^\vee$ and thus we can define $J_C \coloneqq H^0(C,\Omega^1_C)^\vee/H_1(C,\mathbb{Z})$.

Proposition 1.3.3 The analytic and algebraic Jacobian are isomorphic.

Proof: Consider the short exact exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_C \to \mathcal{O}_C^* \to 0.$$

Taking cohomology gives the long exact sequence

$$H^1(C,\mathbb{Z}) \to H^1(C,\mathcal{O}_C) \xrightarrow{\varphi} H^1(C,\mathcal{O}_C^*) \xrightarrow{\psi} H^2(C,\mathbb{Z}).$$

Here ψ is simply the degree map after composing with the isomorphism $H^2(C,\mathbb{Z}) \cong \mathbb{Z}$ given by the fundamental class. The first term is isomorphic to $H_1(C,\mathbb{Z})$ by Poincaré duality, the second term is isomorphic to $H^0(C,\Omega^1_C)^\vee$ by Serre duality, the third term is simply the Picard group $\operatorname{Pic}(C)$ and the last term is isomorphic to $\mathbb{Z}$ again by Poincaré duality. Thus the image of φ is isomorphic to the analytic Jacobian $H^0(C,\Omega^1_C)^\vee/H_1(C,\mathbb{Z})$ and the kernel of ψ is isomorphic to $\operatorname{Pic}^0(C)$ and we are done because of exactness. □

From the analytic description it is easy to see that the Jacobian of a genus g curve is a g-dimensional torus and from the algebraic description we see that it is in fact projective and thus an abelian variety.

In order to properly state the Torelli theorem in Section 1.5 we should take a closer look how the projective embedding arises. The cup product form Q on $H^1(C,\mathbb{Z})$ can be interpreted as an element $q \in H^{1,1}(J_C) \cap H^2(J_C,\mathbb{Z})$ so by Lefschetz's theorem on $(1,1)$-classes it is the first Chern class of an ample divisor Θ_C defined up to translation called the *theta divisor*. Since the cup product is unimodular this polarization is a principal polarization.

One can define a map from C to its Jacobian called the *Abel-Jacobi map* by picking some base point $p \in C$ and integrating along a path from p:

$$\begin{aligned} u_p \colon C &\to J_C \\ q &\mapsto \left(\omega \mapsto \int_p^q \omega\right). \end{aligned}$$

Alternatively, using the algebraic description of the Jacobian this map becomes

$$\begin{aligned} u_p \colon C &\to J_C \\ q &\mapsto \mathcal{O}_C(q-p). \end{aligned}$$

While this map depends on the choice of base point p, if we choose a different base point p' the images of u_p and $u_{p'}$ are related by a translation. Namely we have

$$u_{p'}(q) = \mathcal{O}_C(q - p') = \mathcal{O}_C(q - p' + p - p) = u_p(q) + u_{p'}(p).$$

Completely analogously, if X is a complex manifold of irregularity $q(X) = q$ we can associate a q-dimensional abelian variety called the Albanese variety to X:

$$\mathrm{Alb}(X) = H^0(X, \Omega^1_X)^\vee / H_1(X, \mathbb{Z}).$$

As in the case of curves there is a morphism $a\colon X \to \mathrm{Alb}(X)$ called the *Albanese morphism* again depending on a choice of base point p. The Albanese variety has a universal property. Given any compact complex torus A, for any morphism $X \to A$ sending the base point p to 0 we have a diagram

$$\begin{array}{ccc} X & \longrightarrow & A \\ & {\scriptstyle a}\searrow \quad \nearrow{\scriptstyle \exists !} & \\ & \mathrm{Alb}(X) & \end{array}$$

such that the map $\mathrm{Alb}(X) \to A$ is a morphism of abelian varieties. That is to say any morphism from X to a compact torus factors uniquely through the Albanese of X. Now since the irregularity of X is the dimension of its Albanese variety we can think of irregular varieties as the varieties admitting a non-constant morphism to an abelian variety. This morphism will be an important tool to prove the infinitesimal Torelli theorem for some classes of such varieties.

1.4 Hodge numbers of hypersurfaces in abelian varieties

In this section we will compute some Hodge numbers that will become useful for relating our results to the ones in [Rei88] and [Pet88]. We start by deriving the Hodge diamond of an abelian variety of dimension g. Since the cotangent bundle is trivial of rank g we can identify $\Omega^p_A \cong \mathcal{O}_A^{\oplus \binom{g}{p}}$ and therefore, using Hodge symmetry twice, we obtain

$$h^{p,q}(A) = h^q(\Omega^p_A) = h^q(\mathcal{O}_A)^{\oplus \binom{g}{p}} = h^0(\Omega^q_A)^{\oplus \binom{g}{p}} = h^0(\mathcal{O}_A)^{\oplus \binom{g}{p}\binom{g}{q}} = \binom{g}{p}\binom{g}{q}.$$

Now let $X \subset A$ be a smooth ample hypersurface. We write $n := g - 1$ for the dimension of X. Let $L = \mathcal{O}_A(X)$ and write $d := h^0(A, L)$. Note that $L|_X \cong N_{X/A} \cong \omega_X$ where $N_{X/A}$ denotes the normal bundle of X in A.

We will simply write $h^{p,q}$ for $h^{p,q}(X)$. First we calculate the Hodge numbers $h^{i,n} = h^{n,i}$. Consider the exact sequence

$$0 \to \mathcal{O}_A \to L \to \omega_X \to 0.$$

Taking cohomology gives

$$\cdots \to H^i(A, L) \to H^i(X, \omega_X) \to H^{i+1}(A, \mathcal{O}_A) \to H^{i+1}(A, L) \to \cdots .$$

For $i > 0$ the terms on the outside vanish so we get an isomorphism $H^i(X, \omega_X) \cong H^{i+1}(A, \mathcal{O}_A)$. For $i = 0$ we get

$$0 \to H^0(A, \mathcal{O}_A) \to H^0(A, L) \to H^0(X, \omega_X) \to H^1(A, \mathcal{O}_A) \to H^1(A, L)$$

and the last term vanishes so we get $h^0(X, \omega_X) = h^0(A, L) + h^1(A, \mathcal{O}_A) - h^0(A, \mathcal{O}_A) = d + g - 1$. Summarizing we have

$$h^{i,n} = h^{n,i} = \begin{cases} d + g - 1 & \text{for } i = 0, \\ \binom{g}{i+1} & \text{for } i > 0. \end{cases}$$

Using Serre duality this gives us the Hodge numbers $h^{i,0} = h^{0,i}$ as well.

Now consider the conormal sequence

$$0 \to N_{X/A}^{\vee} \to \Omega_A^1|_X \to \Omega_X^1 \to 0.$$

Using the identifications $N_{X/A} \cong \omega_X$ and $\Omega_A^1 \cong \mathcal{O}_A^{\oplus g}$ this becomes

$$0 \to \omega_X^{\vee} \to \mathcal{O}_X^{\oplus g} \to \Omega_X^1 \to 0. \tag{1.8}$$

For any short exact sequence of vector bundles

$$0 \to L \to E \to F \to 0$$

where L has rank 1 (i.e. it is a line bundle) taking wedge products induces a short exact sequence

$$0 \to L \otimes \bigwedge^{k-1} E \to \bigwedge^{k} E \to \bigwedge^{k} F \to\to 0.$$

Applying this to (1.8) and tensoring with ω_X yields

$$0 \to \Omega_X^p \to \omega_X^{\oplus \binom{g}{p+1}} \to \Omega_X^{p+1} \otimes \omega_X \to 0.$$

Taking cohomology gives

$$\cdots \to H^{q-1}(\Omega_X^{p+1} \otimes \omega_X) \to H^q(\Omega_X^p) \to H^q(\omega_X)^{\oplus \binom{g}{p+1}} \to H^q(\Omega_X^{p+1} \otimes \omega_X) \to \cdots .$$

For $p+q>n$ the terms on the outside vanish by Nakano vanishing so we have $H^q(\Omega_X^p) \cong H^q(\omega_X)^{\oplus\binom{g}{p+1}}$ and thus $h^q(\Omega_X^p) = \binom{g}{p+1}\binom{g}{q+1}$. Note that the special case $q=0$ is ruled out by $p+q>n$. Now using Serre duality it is easy to see that $h^q(\Omega_X^p) = \binom{g}{p}\binom{g}{q}$ for $p+q<n$.

Now all that remains are the Hodge numbers $h^{p,q}$ with $p+q=n$, i.e. the middle row of the Hodge diamond. If we fix p we determined $h^{p,q}$ for all but one value of q, namely $q=n-p$. Therefore, we can use the Euler characteristic to obtain the remaining number. On the one hand we have

$$\chi(\Omega_X^p) = \sum_{i=0}^{n}(-1)^i h^{p,i}$$

which gives us

$$\begin{aligned} h^{p,q} &= (-1)^q\chi(\Omega_X^p) + \sum_{i=0,i\neq q}^{n}(-1)^{i+q}h^{p,i} \\ &= (-1)^q\chi(\Omega_X^p) + (-1)^{q-1}\left(\binom{g}{p}\sum_{i=0}^{q-1}(-1)^i\binom{g}{i} + \binom{g}{p+1}\sum_{i=q+1}^{n}(-1)^i\binom{g}{i+1}\right) \end{aligned}$$

and simplyfing the binomial coefficients using the identity

$$\sum_{i=0}^{k}(-1)^i\binom{n}{i} = (-1)^k\binom{n+1}{k}$$

we obtain

$$h^{p,q} = (-1)^q\chi(\Omega_X^p) + \frac{g-1}{g}\binom{g}{p}\binom{g}{p+1}. \tag{1.9}$$

On the other hand, from the exact sequence

$$0 \to \Omega_X^{p-1}\otimes\omega_X^\vee \to \mathcal{O}_X^{\oplus\binom{g}{p}} \to \Omega_X^p \to 0$$

we get

$$\chi(\Omega_X^p) = \binom{g}{p}\chi(\mathcal{O}_X) - \chi(\Omega_X^{p-1}\otimes\omega_X^\vee) \tag{1.10}$$

Now take the sequence

$$0 \to \Omega_X^{p-2}\otimes\omega_X^\vee \to \mathcal{O}_X^{\oplus\binom{g}{p-1}} \to \Omega_X^{p-1} \to 0$$

and tensor it with $\omega_X^\vee$ to obtain

$$0 \to \Omega_X^{p-2}\otimes\omega_X^{\otimes -2} \to (\omega_X^\vee)^{\oplus\binom{g}{p-1}} \to \Omega_X^{p-1}\otimes\omega_X^\vee \to 0$$

which yields

$$\chi(\Omega_X^{p-1} \otimes \omega_X^\vee) = \binom{g}{p-1}\chi(\omega_X^\vee) - \chi(\Omega_X^{p-2} \otimes \omega_X^{\otimes -2})$$

and plugging this into (1.10) gives

$$\chi(\Omega_X^p) = \binom{g}{p}\chi(\mathcal{O}_X) - \binom{g}{p-1}\chi(\omega_X^\vee) + \chi(\Omega_X^{p-2} \otimes \omega_X^{-2}).$$

Applying this successively, each time replacing the exact sequence with the one for one less wedge power and tensoring with one more tensor power of $\omega_X^\vee$ we finally obtain

$$\chi(\Omega_X^p) = \sum_{k=0}^{p}(-1)^i\binom{g}{p-k}\chi(\omega_X^{-k}). \tag{1.11}$$

Next we calculate the Euler characteristic of ω_X^{-k}. By Serre duality we have $h^i(\omega_X^{-k}) = h^{n-i}(\omega_X^{k+1})$ which means that $\chi(\omega_X^{-k}) = (-1)^{g-1}\chi(\omega_X^{k+1})$. Tensoring the standard restriction sequence of X with L^{k+1} gives

$$0 \to L^k \to L^{k+1} \to \omega_X^{k+1} \to 0.$$

Again we take Euler characteristics. Since L^k is an ample line bundle on an abelian variety its Euler characteristic is equal to $h^0(A, L^k)$ and by the geometric Riemann-Roch theorem (see Theorem 1.3.2) this is equal to $k^g h(A, L) = k^g d$ and similarly for L^{k+1}. Since Euler characteristics are additive in short exact sequences we get

$$\chi(\omega_X^{-k}) = (-1)^{g-1}\chi(\omega_X^{k+1}) = (-1)^{g-1}((k+1)^g - k^g)d.$$

Plugging this into (1.11) gives us

$$\begin{aligned}\chi(\Omega_X^p) &= \left((-1)^{g-1}\sum_{k=0}^{p}(-1)^k\binom{g}{p-k}((k+1)^g - k^g)\right)d \\ &= \left((-1)^{g-1}((-1)^p(p+1)^g + \sum_{k=0}^{p-1}(-1)^k\binom{g}{p-k}(k+1)^g)\right)d\end{aligned}$$

$$\begin{aligned}\chi(\Omega_X^p) &= \left((-1)^{g-1}\sum_{k=0}^{p}(-1)^k\binom{g}{p-k}((k+1)^g - k^g)\right)d \\ &= \left((-1)^{g-1-p}\sum_{k=0}^{p}(-1)^k\binom{g}{k}((p-k+1)^g - (p-k)^g)\right)d \\ &= \left((-1)^{g-1-p}\sum_{k=0}^{p}(-1)^k\binom{g+1}{k}((p-k+1)^g\right)d.\end{aligned}$$

The sum is precisely the Eulerian number $\left\langle {g \atop p} \right\rangle$, i.e. the number of permutations of the numbers $1, \ldots, g$ where exactly p elements are greater than the previous element. Finally plugging this into (1.9) gives us our final formula

$$h^{p,n-p} = \left\langle {g \atop p} \right\rangle d + \frac{g-1}{g} \binom{g}{p} \binom{g}{p+1}$$

where $d = h^0(A, L)$ is the dimension of the space of global sections of L.

With this we have fully determined the Hodge diamond of a smooth ample hypersurface in an abelian variety of any dimension. For example a smooth ample curve in an abelian surface has the Hodge diamond

$$\begin{array}{ccc} & 1 & \\ d+1 & & d+1 \\ & 1 & \end{array} ,$$

for a smooth ample surface in an abelian threefold we have

$$\begin{array}{ccccc} & & 1 & & \\ & 3 & & 3 & \\ d+2 & & 4d+6 & & d+2 \\ & 3 & & 3 & \\ & & 1 & & \end{array}$$

and for a smooth ample threefold in an abelian fourfold we have

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 4 & & 4 & & \\ & 6 & & 16 & & 6 & \\ d+3 & & 11d+18 & & 11d+18 & & d+3 \\ & 6 & & 16 & & 6 & \\ & & 4 & & 4 & & \\ & & & 1 & & & \end{array}$$

1.5 Torelli theorems

Before we define the period map and talk about the infinitesimal Torelli theorem we will recall the Classical Torelli theorem for curves.

Theorem 1.5.1 (Clasical Torelli Theorem) *Let C, C' be two smooth genus g curves such that their polarized Jacobians are isomorphic, i.e. $(J_C, \Theta_C) \cong (J_{C'}, \Theta_{C'})$, then C and C' are isomorphic.*

A proof can for example be found in [And58].

By the analytic construction of the Jacobian and the relation of the theta divisor to the cup product form Q this is really saying that we can recover a curve C from its polarized Hodge structure $(H^1(C,\mathbb{Z}),Q)$.

It can also be rephrased in terms of the Torelli map. Let $\mathcal{M}_g$ denote the coarse moduli space of smooth genus g curves. It is a quasi-projective variety of dimension $3g-3$ for $g \geq 2$. Let further $\mathcal{A}_g$ denote the coarse moduli space of principally polarized abelian varieties of dimension g. It is a quasi-projective variety of dimension $\frac{1}{2}g(g+1)$.

Corollary 1.5.2 The Torelli map $\mathcal{M}_g \to \mathcal{A}_g$ sending the isomorphism class $[C]$ of a genus g curve C to the class $[(J_C,\Theta_C)]$ of its polarized Jacobian is injective.

A question one might ask is whether the Torelli map is an immersion. It turns out that in general this is not the case.

Theorem 1.5.3 (Infinitesimal Torelli theorem for curves) *Let* $\tau\colon \mathcal{M}_g \to \mathcal{A}_g$ *be the Torelli map.*

(i) If $g = 2$*, then* τ *is injective on tangent spaces.*

(ii) If $g \geq 3$*, then* τ *is injective on tangent spaces at* $[C]$ *if and only if* C *is non-hyperelliptic.*

Proof: Let C be a curve of genus $g \geq 2$. The tangent space to the deformation space of C in $\mathcal{M}_g$ can be identified with $H^1(C,T_C)$, the tangent space to the deformation space of its Jacobian can be identified with $S^2H^1(C,\mathcal{O}_C)$. Therefore, the differential of the Torelli map at C is injective if and only if the map

$$H^1(C,T_C) \to S^2H^1(C,\mathcal{O}_C)$$

is injective. Using Serre duality this is equivalent to the surjectivity of the multiplication map

$$S^2H^0(C,\omega_C) \to H^0(C,\omega_C^2).$$

It is a classical result by Max Noether (see for example [ACGH85, p. 117]) that this map is surjective if and only if $g = 1,2$ or $g \geq 3$ and C is not hyperelliptic. □

It should be noted, however, that the restriction of the Torelli map to the hyperelliptic locus is in fact an immersion. That is to say, if one only considers deformations within the hyperelliptic locus, the Torelli map is injective on tangent vectors.

1.6 The infinitesimal Torelli theorem

It is this latter question – whether the period map is an immersion – that we would like to generalize to other classes of varieties. Consider a family of compact Kähler manifolds $\phi\colon \mathcal{X} \to B$, i.e. a proper holomorphic submersion of complex manifolds with Kähler fibers. Denote by X_b the fiber $\phi^{-1}(b)$ of ϕ over $b \in B$ and fix $0 \in B$. Write $X = X_0$. Ehresmann's theorem ensures that in some neighborhood U of 0 there are well defined isomorphisms of the cohomology groups $H^k(X_b, \mathbb{Z}) \cong H^k(X_0, \mathbb{Z})$ (see for example [Voi07, Theorem 9.3]). While these isomorphisms preserve the Hodge numbers, they will in general not preserve the Hodge structure. The Hodge decomposition of $H^k(X_b, \mathbb{C})$ varies continuously with b so we can define the *period map*. For given k and p the p-th piece of the period map with respect to the k-th cohomology group is defined by

$$\mathcal{P}^{p,k}\colon U \to \mathrm{Grass}(b^{p,k}, H^k(X,\mathbb{C})), b \mapsto F^pH^k(X_b,\mathbb{C})$$

where $F^pH^k(X_b,\mathbb{C})$ denotes the p-th step of the Hodge filtration and $b^{p,k} = \dim F^pH^k(X_b,\mathbb{C})$ (note that this is independent of b as all X_b have the same Hodge numbers). In his seminal papers [Gri68a], [Gri68b] and [Gri70] Griffiths studied this map. He showed that it is holomorphic so we can consider its differential at $0 \in B$

$$d\mathcal{P}^{p,k}\colon T_{B,0} \to \mathrm{Hom}(F^pH^k(X,\mathbb{C}), H^k(X,\mathbb{C})/F^pH^k(X,\mathbb{C})).$$

Now we have an exact sequence

$$0 \to T_X \to T_{\mathcal{X}}|_X \to \phi^*T_B|_X \to 0$$

and the cokernel $\phi^*T_B|_X$ is in fact trivial of fiber $T_{B,0}$. Thus the long exact cohomology sequence induces a map

$$\rho\colon T_{B,0} = H^0(X, \phi^*T_B|_X) \to H^1(X, T_X)$$

called the *Kodaira-Spencer map* of the family $\phi\colon \mathcal{X} \to B$ at 0. Griffiths showed that $d\mathcal{P}^{p,k}$ is the composition of the Kodaira-Spencer map with the map

$$H^1(X,T_X) \to \mathrm{Hom}(H^{k-p}(X,\Omega_X^p), H^{k-p+1}(X,\Omega_X^{p-1}))$$

given by the cup product and the interior product (see [Gri68a], [Gri68b] and [Gri70]).

There exists a semi-universal deformation of X, i.e. there is a family $\phi\colon \mathcal{X} \to B$ with $X = \phi^{-1}(b_0)$ such that any other family $\phi'\colon \mathcal{X}' \to B'$ with b_0' is obtained as the fiber product of ϕ over B and a suitable morphism $f\colon B' \to B$ with $f(b_0') = b_0$

(see e.g. [MK06] or [Kur65]). In other words there is a cartesian square

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{X} \\ \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \phi} \\ B' & \xrightarrow{f} & B. \end{array}$$

Such a semi-universal deformation is also called a *Kuranishi family* of X. Note that the Kuranishi family is unique up to non-unique isomorphism. Now we say that the infinitesimal Torelli theorem (ITT in the following) holds for a compact Kähler manifold X if the period map $\mathcal{P}^{n,n}$ of a Kuranishi family of X is an immersion. Since the Kodaira-Spencer map is an isomorphism for a Kuranishi family, in order to prove the ITT we need to show injectivity of the map

$$H^1(X, T_X) \to \mathrm{Hom}(H^0(X, \omega_X), H^1(X, \Omega_X^{n-1})).$$

For a curve C this is exactly what we discussed in the previous section so the ITT holds if and only if C has genus $g(C) \leq 2$ or $g(C) > 2$ and C is non-hyperelliptic. That is to say that in this case very ampleness of the canonical sheaf is equivalent to the ITT. For surfaces, however, Garra and Zucconi show that for any $n \geq 5$ there exists a generically smooth $n+9$ dimensional irreducible component of the moduli space of algebraic surfaces such that for a general element of it the canonical sheaf is very ample but the ITT fails (see [GZ08]). Thus finding classes of objects that satisfy the ITT is still an open problem.

We collect here some of the known results. Note that the ITT used to be called the local Torelli theorem in the literature of the past. The ITT holds for

- hypersurfaces in projective space with ample canonical bundle (see [Gri68a] and [Gri68b]),
- complete intersections in projective space with ample canonical bundle (see [Pet75]),
- cyclic covers of complete intersections in projective space with ample canonicle bundle (see [Kiĭ78]),
- varieties with trivial canonical bundle (see [Gri68a] and [Gri68b]).

The last bullet point is particularly easy to see as for a variety X with $\omega_X \cong \mathcal{O}_X$ we have $T_X \cong T_X \otimes \omega_X \cong \Omega_X^{n-1}$ and the map

$$H^1(X, T_X) \to \mathrm{Hom}(H^0(X, \omega_X), H^1(X, \Omega_X^{n-1}))$$

is an isomorphism.

For irregular surfaces there already are some results with respect to the ITT. First we need to give a few definitions.

Definition 1.6.1 A *pencil* on a surface S is a morphism with connected fibers $S \to C$ where C is a smooth curve. If $g(C) \geq 1$ it is called an *irrational pencil.*

Definition 1.6.2 A vector bundle $\mathcal{E}$ on a manifold X is said to be *globally generated* if the evaluation map $H^0(X, \mathcal{E}) \otimes \mathcal{O}_X \to \mathcal{E}$ is surjective.

Dualizing the evaluation map and taking the projectivization yields a map

$$\varphi\colon \mathbb{P}(\mathcal{E}^\vee) \to \mathbb{P}(H^0(X, \mathcal{E}))^\vee.$$

Definition 1.6.3 A vector bundle $\mathcal{E}$ on a projective variety X of dimension n is called *almost very ample* if it is globally generated and the map φ from above has degree one onto its image and contracts at most finitely many $(n-1)$-dimensional varieties.

Reider proves the following theorem.

Theorem 1.6.4 ([Rei88]) *Let S be an irregular surface of general type with the following properties:*
(a) S has no irrational pencils;
(b) the cotangent bundle Ω^1_S is almost very ample.
Then the ITT holds for S.

Peters extends the arguments to irregular varieties of higher dimensions.

Theorem 1.6.5 ([Pet88]) *Let X be a projective variety of dimension n. If*
(a) for all $1 \leq d \leq n-1$, there is no rational map from X onto a variety Y of dimension d with $h^d(Y, \mathcal{O}_Y) \geq d+1$,
(b) $H^{n-q}(\Omega_X^{q+1} \otimes \omega_X) = 0$ for $q = 1, \ldots n-1$,
(c)

$$\left.\begin{array}{lll} \sum_{i=1}^{n-1} h^{i,n-i} & \geq 2 \sum_{i=1}^{n-2} h^{i,n-1-i} & \text{for } n \text{ even} \\ \sum_{i=1}^{n} h^{i,n-i} & \geq \sum_{i=1}^{n-1} h^{i,n-1-i} & \text{for } n \text{ odd} \end{array}\right\}$$

and
(d) the cotangent bundle Ω^1_X is almost very ample,
then the ITT holds for X.

Note that by Nakano vanishing condition (b) holds in particular if the canonical bundle ω_X is ample. In the case of threefolds Peters then proves that the inequality of Hodge numbers in (c) is always satisfied to obtain the following theorem.

Theorem 1.6.6 ([Pet88]) *Let X be an irregular threefold with the following properties:*

(a) X admits no rational map onto a curve of genus greater than or equal to 2 or onto a surface of geometric genus greater than or equal to 3;

(b) $H^2(X, \Omega_X^2 \otimes \omega_X) = 0$;

(c) the cotangent bundle Ω_X^1 is almost very ample.

Then the ITT holds for X.

Remark 1.6.7 The condition that the cotangent bundle be almost very ample implies in particular that it is globally generated. In other words the evaluation map eval: $H^0(X, \Omega_X^1) \otimes \mathcal{O}_X \to \Omega_X^1$ is surjective. Its dual, the differential of the Albanese map $a: X \to A = \mathrm{Alb}(X)$, is thus an injective morphism of vector bundles and its cokernel $N_a := (a^*T_A)/T_X$ is a vector bundle of rank $q(X) - \dim(X)$. Therefore, as soon as Ω_X^1 is globally generated we have a short exact sequence of vector bundles

$$0 \to T_X \to H^0(\Omega_X^1) \otimes \mathcal{O}_X \to N_a \to 0. \tag{1.12}$$

This is in particular the case, when a is an embedding and N_a is simply the normal bundle $N_{X/A}$.

For higher-dimensional irregular varieties it is not clear when the inequality on Hodge numbers in Theorem 1.6.5 holds. From our calculations in Section 1.4 it is easy to see that it is always satisfied in the case of smooth ample hypersurfaces in abelian varieties. However, for an irregular variety of dimension n and irregularity q, $\mathbb{P}(T_X)$ has dimension $2n-1$ and $\mathbb{P}(H^0(\Omega_X^1))$ has dimension $q-1$. Therefore condition (d) in Theorem 1.6.5 implies that $2n-1 \leq q-1$ or equivalently $q \geq 2n$. Consequently for a hypersurface in an abelian variety condition (d) is never satisfied. Debarre proves that if A is a g-dimensional abelian variety then the intersection of at least $\lceil g/2 \rceil$ sufficiently ample general hypersurfaces has ample cotangent bundle ([Deb05, Theorem 7] and [Deb05, Theorem 8]). However, it is not clear when the cotangent bundle is almost very ample.

In [Gre85] Green generalizes the results of Griffiths for hypersurfaces in projective space of high degree to *sufficiently ample* hypersurfaces X in arbitrary varieties Y. He does so by showing that under the hypothesis that X is sufficiently ample, the ITT follows from the surjectivity of the multiplication map

$$H^0(X, \omega_X) \otimes H^0(X, L^{n-1} \otimes \omega_X) \to H^0(X, L^{n-1} \otimes \omega_X^2).$$

This step mainly uses the existence of the short exact sequence (1.12). In a second step he shows, under the assumption that X is sufficiently ample, that the surjectivity of the multiplication map on X above follows from the surjectivity of the multiplication map

$$H^0(Y, L \otimes \omega_Y) \otimes H^0(Y, L^n \otimes \omega_Y) \to H^0(Y, L^{n+1} \otimes \omega_Y^2)$$

on the ambient variety Y. Finally he shows, again under the assumption that X is sufficiently ample, that this map is in fact surjective.

Since Reider's result requires that the cotangent bundle be globally generated and by Remark 1.6.7 this is not so much stronger than having a subvariety of an abelian variety, we will use Green's methods to obtain the ITT in this case. The first two steps turn out to work even when X is simply ample. The last step, showing the surjectivity of

$$H^0(A, L) \otimes H^0(A, L^n) \to H^0(A, L^{n+1}),$$

is easy when L is a power of another line bundle but quite difficult for primitive line bundles. In the next section we will study such multiplication maps.

2 Surjectivity of the multiplication map

This section is devoted to the study of multiplication maps of global sections of line bundles on abelian varieties. It will be useful to have notation for more general multiplication maps. Let X be a variety and let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on X. Write

$$\mu_{X,\mathcal{F},\mathcal{G}} \colon H^0(X,\mathcal{F}) \otimes H^0(X,\mathcal{G}) \to H^0(X,\mathcal{F}\otimes\mathcal{G})$$

for the corresponding multiplication map of sections. We will also sometimes denote the image of $\mu_{X,\mathcal{F},\mathcal{G}}$ by $H^0(X,\mathcal{F}).H^0(X,\mathcal{G})$ and thus the map is surjective if $H^0(X,\mathcal{F}).H^0(X,\mathcal{G}) = H^0(X,\mathcal{F}\otimes\mathcal{G})$. If the variety is clear from the context we will suppress it in the notation and simply write $\mu_{\mathcal{F},\mathcal{G}}$. The most important type of multiplication map for us will be multiplication of sections of different powers of a line bundle, i.e. μ_{L^a,L^b}. When L is clear from the context we will simply write $\mu_{a,b}$. Finally one of the line bundles will often be a power of the other, i.e. we have a multiplication map $\mu_{1,k}$. In this case we will simply write μ_k. These are the multiplication maps we will be studying in this chapter but it will prove useful to have more general notation once we investigate complete intersections and irregular varieties that do not embed into their Albanese smoothly.

2.1 Necessary condition for surjectivity

Let (A,L) be a polarized g-dimensional abelian variety. For any $k \in \mathbb{N}$ let

$$\mu_k \colon H^0(A,L) \otimes H^0(A,L^k) \to H^0(A,L^{k+1})$$

be the multiplication map of sections. In particular μ_{g-1} is the relevant map for the ITT as remarked at the end of the last section. Comparing dimensions of domain and range for μ_k we first investigate briefly when this map fails to be surjective for purely dimensional reasons giving us a necessary condition for surjectivity. Then we relate this to the known results for the ITT in the case of curves.

Since $c_1(L^k) = kc_1(L)$ for any $k \in \mathbb{N}$ and by the geometric Riemann-Roch theorem $h^0(A,L) = \frac{c_1(L)^g}{g!}$ we get that, if μ_k is surjective, the inequality

$$k^g h^0(A,L)^2 \geq (k+1)^g h^0(A,L)$$

holds. Or equivalently surjectivity of μ_k implies that

$$h^0(A,L) \geq \left(\frac{k+1}{k}\right)^g.$$

In particular we see that for $g \geq 2$ the map μ_{g-1} cannot be surjective unless

$$h^0(A, L) \geq \left(\frac{g}{g-1}\right)^g.$$

In fact for large g the value on the right hand side converges to e. Since failure of μ_{g-1} to be surjective does not imply failure of the ITT, this does not say much, but it does confirm that ampleness of L alone is not sufficient for this approach as μ_{g-1} fails to be surjective for example for a principal polarization. This immediately gives us an example where the ITT holds despite μ_{g-1} not being surjective. A curve C on an abelian surface defining a principal polarization will thus provide such an example. Indeed such a curve has genus 2, hence as discussed in Section 1.5 the ITT holds, while $h^0(A, \mathcal{O}_A(C)) = 1 \not\geq 2^2 = 4$.

2.2 Projective normality

The multiplication maps μ_k for abelian varieties have in fact been studied in particular in the context of projective normality. Projective normality can be defined for any projective variety and therefore in particular for abelian varieties. We use the definitions given in [BL04].

Definition 2.2.1 A projective variety $X \subset \mathbb{P}^N$ is called *projectively normal* in $\mathbb{P}^N$ if its homogeneous coordinate ring is an integrally closed domain. A line bundle L on X is called *normally generated* if it is very ample and X is projectively normal under the associated projective embedding.

It can be shown that X is projectively normal in $\mathbb{P}^N$ if and only if the natural restriction map $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(k)) \to H^0(X, \mathcal{O}_X(k))$ is surjective for every $k \geq 1$ (see [Har77, II Ex 5.14]). Now if the embedding $\varphi\colon X \hookrightarrow \mathbb{P}^N$ is given by the complete linear series of a line bundle $L \cong \varphi^*\mathcal{O}_{\mathbb{P}^N}(1)$ we have canonical isomorphisms

$$H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(k)) \cong S^k H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1)) \cong S^k H^0(X, L)$$

where S^k denotes the k-th symmetric power and thus the restriction map is identified with the multiplication map

$$\rho_k\colon S^k H^0(X, L) \to H^0(X, L^k).$$

By [Mum70, p. 38] surjectivity of the maps ρ_k for all $k \geq 2$ already implies that L is very ample and thus L is normally generated if and only if ρ_k is surjective for all $k \geq 2$. Note that the map ρ_1 is simply the identity and thus automatically surjective. Also note that in the case of abelian varieties surjectivity of ρ_2 in fact implies surjectivity of ρ_k for $k \geq 3$ so that projective normality in this case is equivalent to the surjectivity of ρ_2 (see e.g. [Iye03]).

Using this we can relate projective normality and the surjectivity of μ_k.

Lemma 2.2.2 ([BL04]) For an ample line bundle L on a projective variety X the following conditions are equivalent:

(i) L is normally generated.

(ii) The map $\mu_k\colon H^0(X,L)\otimes H^0(X,L^k)\to H^0(X,L^{k+1})$ is surjective for all $k\geq 1$.

In particular, as we will see in Section 3.2, for abelian varieties L being normally generated implies the ITT for any smooth element of the linear system $|L|$.

Before we progress and establish numerical criteria for the surjectivity of μ_k we will gather some of the known results about projective normality.

The following results are due to Koizumi and Ohbuchi.

Theorem 2.2.3 ([Koi76], [Ohb88]) *Let L be an ample line bundle on an abelian variety A. Then*

(i) L^k is normally generated for $k\geq 3$,

(ii) L^2 is normally generated if L is basepoint-free.

A proof can be found in [BL04, §7]. It relies on an explicit formula for the multiplication maps in terms of theta functions. This explicit formula exists for all ample line bundles but assuming that L is a square of an ample line bundle the formula takes on a simplified form which makes it possible to show surjectivity. Note that if we do not need surjectivity of all multiplication maps μ_k but only of μ_{g-1} we can in fact get rid of the condition that L be basepoint free in the second part of the theorem. The only place this condition plays a role is in [BL04, Proposition 7.2.3] where the surjectivity of μ_1 is shown for the square of an ample line bundle. That means that the condition on basepoints is not necessary to prove the ITT for $g\geq 3$ using the results on projective normality. The question remains what happens for line bundles that are not powers of other line bundles, i.e. line bundles of type $(1,d_2,\dots,d_g)$.

2.3 Abelian surfaces with primitive polarization

While studying multiplication maps on abelian surfaces will not yield anything new with regard to the ITT - it is fully understood for curves - we will still include the known results for completeness and because it provides some interesting examples.

Projective normality is well understood for abelian surfaces with very ample primitive polarization, as the following theorem shows:

Theorem 2.3.1 ([Laz90], [FG04]) *Let (A,L) be a polarized abelian surface of type $(1,d)$ such that L is very ample. Then the embedding $A\hookrightarrow\mathbb{P}(H^0(A,L))^\vee\cong\mathbb{P}^{d-1}$ is projectively normal if and only if $d\geq 7$.*

Lazarsfeld's paper is hard to find but [Ago17] summarizes the main ideas of the proof. It is easy to see that the embedding cannot be projectively normal for $d \leq 6$. For $d \leq 4$ the line bundle is not even very ample. For $d = 5$ and $d = 6$ the dimension of $S^2H^0(A, L)$ is 15 and 21 respectively whereas the dimension of $H^0(A, L^2)$ is 20 and 24 respectively so the maps $\rho_2\colon S^2H^0(A, L) \to H^0(A, L^2)$ cannot be surjective.

Since Theorem 2.3.1 requires L to be very ample one can use the projective embedding $A \hookrightarrow \mathbb{P}(H^0(A, L)^\vee) \cong \mathbb{P}^{d-1}$ it defines. Consider the restriction sequence

$$0 \to \mathcal{I}_{A/\mathbb{P}^{d-1}} \to \mathcal{O}_{\mathbb{P}^{d-1}} \to \mathcal{O}_A \to 0.$$

Twisting by $\mathcal{O}_{\mathbb{P}^{d-1}}(2)$ and taking cohomology gives

$$\begin{aligned} 0 &\to H^0(\mathcal{I}_{A/\mathbb{P}^{d-1}}(2)) \to H^0(\mathcal{O}_{\mathbb{P}^{d-1}}(2)) \to H^0(\mathcal{O}_A(2)) \\ &\to H^1(\mathcal{I}_{A/\mathbb{P}^{d-1}}(2)) \to 0. \end{aligned}$$

In fact $H^0(\mathcal{O}_{\mathbb{P}^{d-1}}(2)) \cong S^2H^0(A, L)$ and $H^0(\mathcal{O}_A(2)) \cong H^0(A, L^2)$, so the map in the middle is ρ_2. Let $I = H^0(\mathcal{I}_{A/\mathbb{P}^{d-1}}(2))$ and let $U = H^1(\mathcal{I}_{A/\mathbb{P}^{d-1}}(2))$. Thus we have

$$0 \to I \to S^2H^0(A, L) \xrightarrow{\rho_2} H^0(A, L^2) \to U \to 0$$

and ρ_2 is surjective if the cokernel U vanishes. The idea to prove Theorem 2.3.1 is to put lower and upper bounds on the dimension of U and derive a contradiction.

We know that for curves the ITT holds if and only if the curve is non-hyperelliptic or the genus g is less than or equal to 2. By [BO19, Theorem 2.8] for any smooth hyperelliptic curve C embedded in an abelian surface A, the genus $g(C)$ is $2, 3, 4$ or 5 and A is polarized of type $(1, g(C)-1)$. The only case that does not match up with Theorem 2.3.1 is $g(C) = 2$. By the above, A is then principally polarized. In this case the ITT holds but μ_1 cannot be surjective for purely dimensional reasons. This is not a contradiction as failure of μ_1 to be surjective does not imply failure of the ITT. Here we also see that projective normality really is stronger than the ITT in our case. By the above a smooth element of a polarization of type $(1, 5)$ or $(1, 6)$ on an abelian surface is a non-hyperelliptic curve and thus satisfies the ITT but by Theorem 2.3.1 it does not define a projetively normal embedding.

2.4 The case of simple abelian varieties

In this section we will derive a numerical condition for the surjectivity of μ_k in terms of the dimension of the vector space $H^0(A, L)$ that works in any dimension under the condition that the abelian variety A is simple. This is the main technical result in our preprint [Blo19].

Recall that an abelian variety is simple if the only abelian subvarieties of A are $\{0\}$ and A itself. Our aim is to generalize the techniques used by Iyer in the following theorem.

Theorem 2.4.1 ([Iye03]) *Let L be an ample line bundle on a g-dimensional simple abelian variety A. If $h^0(A, L) > 2^g \cdot g!$, then L gives a projectively normal embedding, for all $g \geq 1$.*

It does not seem that the condition that A be simple can easily be removed, even if we want to prove surjectivity of μ_k for $k > 1$. However, we can improve the bound in that case.

In the following let (B, M) be a principally polarized abelian variety with $\theta \in H^0(B, M)$ the unique (up to a scalar) section. Write $B = V/\Lambda$ and let $\Lambda = \Lambda_1 \oplus \Lambda_2$ be a decomposition for M. Fix $k \in \mathbb{N}$. There is a natural action on $H^0(B, M^k)$ by the theta group $\mathcal{G}(M^k) = \{(b, \varphi) \mid b \in K(M^k),\ \varphi\colon t_b^* M^k \xrightarrow{\cong} M^k\}$. We can choose compatible isomorphisms $\varphi_b\colon t_b^* M^k \to M^k$ for $b \in K(M^k)_1$ so that for any $b, b' \in K(M^k)_1$ we have $\varphi_b(t_{b'}^* \varphi_{b'}) = \varphi_{b'}(t_b^* \varphi_b)$. That means that the action of $\mathcal{G}(M^k)$ induces an action of $K(M^k)_1$. For our purpose we want to find a section $\widetilde{\theta} \in H^0(B, M^k)$ that is invariant under this action. Consider the isogeny

$$\varphi\colon B \to B' = B/K(M^k)_1$$

and let M' be a line bundle on B' such that $\varphi^* M' = M^k$. Since M' is a principal polarization there is a unique (again up to a scalar) section $\theta' \in H^0(B', M')$. We can take $\widetilde{\theta} = \varphi^* \theta'$ since clearly for any $\lambda \in K(M^k)_1$ we have $t_\lambda^* \widetilde{\theta} = t_\lambda^* \varphi^* \theta' = \varphi^* t_{\varphi(\lambda)}^* \theta' = \varphi^* \theta' = \widetilde{\theta}$ for any $\lambda \in K(M^k)_1$. Abusing notation a little we will also write θ and $\widetilde{\theta}$ for the associated theta divisors.

Using the Theorem of the Square we see that for any $b \in B$

$$\begin{aligned} t_{kb}^* M \otimes t_{-b}^* M^k &\cong t_b^* M^k \otimes M^{-k+1} \otimes t_{-b}^* M^k \\ &\cong M^k \otimes M^k \otimes M^{-k+1} \\ &\cong M^{k+1} \end{aligned}$$

so the divisor $t_{kb}^* \theta + t_{-b}^* \widetilde{\theta}$ is an element of the linear system $|(k+1)\theta|$ thus we have a morphism

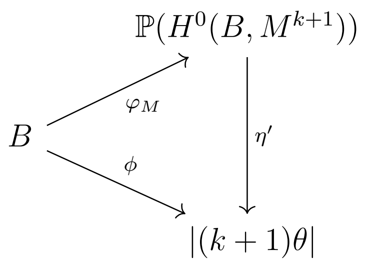
$$\begin{aligned} \phi\colon B &\to |(k+1)\theta| \\ b &\mapsto t_{kb}^* \theta + t_{-b}^* \widetilde{\theta}. \end{aligned}$$

The following proposition is a generalization of a result by Wirtinger that can be found in [Mum74, p. 335].

Proposition 2.4.2 For any $k \in \mathbb{N}$ there is a nondegenerate bilinear map $\eta \colon H^0(B, M^{k+1}) \otimes H^0(B, M^{k+1}) \to \mathbb{C}$ such that if η induces the isomorphism

$$\eta' \colon \mathbb{P}(H^0(B, M^{k+1})) \xrightarrow{\cong} \mathbb{P}(H^0(B, M^{k+1})^\vee) = |(k+1)\theta|$$

then the diagram



commutes.

Proof: Consider the morphism

$$\begin{aligned} s \colon B \times B &\to B \times B \\ (x, y) &\mapsto (x + ky, x - y). \end{aligned}$$

We now have an isomorphism

$$s^*(p_1^* M \otimes p_2^* M^k) \cong p_1^* M^{k+1} \otimes p_2^* M^{k(k+1)}.$$

To see this it suffices to compare the first Chern class and the semicharacters of both line bundles (see [BL04, Lemma 7.1.1] for the case $k = 1$). For any $m \in \mathbb{N}$ and $\alpha \in K(M^m)_1$ we will write $\theta_\alpha^m = \varphi_\alpha(t_\alpha^* \theta^m)$ with $\varphi_\alpha \colon t_\alpha^* M^m \to M^m$ the compatibly chosen isomorphisms from before so that $\{\theta_\alpha^m \mid \alpha \in K(M^m)_1\}$ defines a basis for $H^0(B, M^m)$. Now we can write

$$s^*(p_1^* \theta \otimes p_2^* \tilde{\theta}) = \sum_{\substack{\alpha \in K(M^{k+1})_1 \\ \beta \in K(M^{k(k+1)})_1}} c_{\alpha\beta} p_1^* \theta_\alpha^{k+1} \otimes p_2^* \theta_\beta^{k(k+1)}. \tag{2.1}$$

We want to obtain dependencies between the coefficients $c_{\alpha\beta}$ to see that they are determined by a square matrix which we will use to define η. Consider the pullback of equation (2.1) by $t_{(0,-\gamma)}$ with $\gamma \in K(M^k)_1$. On the left hand side, since

$$\begin{aligned} s(t_{(0,-\gamma)}(x, y)) &= s(x, y - \gamma) \\ &= (x + k(y - \gamma), x - (y - \gamma)) \\ &= (x + ky - k\gamma, x - y + \gamma) \\ &= t_{(0,\gamma)}(s(x, y)), \end{aligned}$$

we get

$$\begin{aligned} t^*_{(0,-\gamma)} s^*(p_1^*\theta \otimes p_2^*\tilde{\theta}) &= s^* t^*_{(0,\gamma)}(p_1^*\theta \otimes p_2^*\tilde{\theta}) \\ &= s^*(p_1^*\theta \otimes p_2^* t^*_\gamma \tilde{\theta}) \\ &= s^*(p_1^*\theta \otimes p_2^*\widetilde{\theta}). \end{aligned}$$

Here, we obtain the last line because we chose $\widetilde{\theta}$ such that it is invariant under translation by $\gamma \in K(M^k)_1$. On the right hand side we have

$$t^*_{(0,-\gamma)} \left(\sum_{\substack{\alpha \in K(M^{k+1})_1 \\ \beta \in K(M^{k(k+1)})_1}} c_{\alpha\beta} p_1^* \theta_\alpha^{k+1} \otimes p_2^* \theta_\beta^{k(k+1)} \right) \\ = \sum_{\substack{\alpha \in K(M^{k+1})_1 \\ \beta \in K(M^{k(k+1)})_1}} c_{\alpha\beta} p_1^* \theta_\alpha^{k+1} \otimes p_2^* t^*_{-\gamma} \theta_\beta^{k(k+1)}.$$

The pullbacks on the right hand side permute the basis elements, comparing coefficients gives $c_{\alpha\beta} = c_{\alpha,\beta-\gamma}$.

Now because $\gcd(k, k+1) = 1$, the exact sequence

$$0 \to K(M^k)_1 \to K(M^{k(k+1)})_1 \to K(M^{k+1})_1 \to 0$$

splits and thus $K(M^{k(k+1)})_1 \cong K(M^k)_1 \oplus K(M^{k+1})_1$. Therefore, for any $\beta \in K(M^{k(k+1)})_1$ there is exactly one $\gamma \in K(M^k)_1$ such that $\beta - \gamma \in K(M^{k+1})_1$, namely γ is the k-torsion part of β. Ultimately this means that we can choose representatives $\alpha, \beta \in K(M^{k+1})_1$ so that the matrix $(c_{\alpha\beta})$ is determined by $\alpha, \beta \in K(M^{k+1})_1$.

We still need to show that $\det(c_{\alpha\beta}) \neq 0$. If the determinant were zero, the element $s^*(p_1^*\theta \otimes p_2^*\tilde{\theta})$ would be contained in a proper subspace $W_1 \otimes W_2$ with $W_1 \subsetneq H^0(B, M^{k+1})$ of $H^0(B, M^{k+1}) \otimes H^0(B, M^{k(k+1)})$. However, translation by an element $b \in K(M^{k+1})$ acts on $H^0(B, M^{k+1})$ and since $K(M^{k+1}) \subset K(M^{k(k+1)})$ the group of $(k+1)$-torsion points B_{k+1} of B induces an action of $\Delta(B_{k+1}) = \{(b,b) \mid b \in B_{k+1}\}$ on $H^0(B, M^{k+1}) \otimes H^0(B, M^{k(k+1)})$. The element $s^*(p_1^*\theta \otimes p_2^*\widetilde{\theta})$ is invariant under this action. By [Mum66, Theorem 2] the action of the theta group $\mathcal{G}(M^{k+1})$ is irreducible. From this it follows that the action of $K(M^{k+1})$ on $H^0(B, M^{k+1})$ is irreducible, so $s^*(p_1^*\theta \otimes p_2^*\widetilde{\theta})$ cannot lie in such a proper subspace. We conclude that $\det(c_{\alpha\beta}) \neq 0$ so $\eta(\theta_\alpha^{k+1}, \theta_\beta^{k+1}) \coloneqq c_{\alpha\beta}$ defines the desired bilinear map η.

The equation (2.1) can be expressed as

$$\theta(u + nv)\tilde{\theta}(u - v) = \sum_{\alpha,\beta \in K(M^{k+1})_1} c_{\alpha\beta} \theta_\alpha^{k+1}(u) \theta_\beta^{k(k+1)}(v) \text{ for any } u, v \in B.$$

For each $v \in B$ this implies that u is in the support of the divisor $t_{nv}^*\theta + t_{-v}^*\widetilde{\theta}$ if and only if it is a zero of $\sum c_{\alpha\beta}\theta_\beta^{k(k+1)}(v)\theta_\alpha^{k+1}$ which gives that $\phi(v) = \eta'(\varphi_M(v))$.□

With this generalized version of the proposition we can use the following proposition from Iyer's paper without modification.

Proposition 2.4.3 ([Iye03]) Let L be an ample line bundle on a g-dimensional simple abelian variety A. Let G be a finite subgroup of A with $|G| > h^0(A, L) \cdot g!$. Then the image of G under the rational map $\varphi_L \colon A \to \mathbb{P}(H^0(A, L))$ generates $\mathbb{P}(H^0(A, L))$.

Unfortunately the assumption that A be simple seems to be crucial for the proof that Iyer gives. In Section 2.5 we will investigate what goes wrong for non-simple abelian varieties.

With this we can prove the following theorem.

Theorem 2.4.4 *Let L be a line bundle on a simple abelian variety A of dimension g. For any $k \in \mathbb{N}$, if $h^0(A, L) > \left(\frac{k+1}{k}\right)^g \cdot g!$, the multiplication map*

$$\mu_k \colon H^0(A, L) \otimes H^0(A, L^k) \to H^0(A, L^{k+1})$$

is surjective.

Proof: Write $A = V/\Lambda$. Given a decomposition $V = V_1 \oplus V_2$ of L, let $H = K(L)_1$ and consider the isogeny

$$\pi \colon A \to B = A/H.$$

There is a principal polarization M on B such that $\pi^* M = L$. The character group $\widehat{H} := \operatorname{Hom}(H, \mathbb{C}^*)$ is a subgroup of $\operatorname{Pic}^0(B)$ so a character $\alpha \in \widehat{H}$ corresponds to a degree 0 line bundle on B also denoted by α. We have a decomposition $\pi_*\mathcal{O}_A = \bigoplus_{\alpha\in\widehat{H}} \alpha$. This gives us

$$\begin{aligned} \pi_* L &= \pi_*(\mathcal{O}_A \otimes L) \\ &= \pi_*(\mathcal{O}_A \otimes \pi^* M) \\ &= \pi_*\mathcal{O}_A \otimes M && \text{(projection formula)} \\ &= \bigoplus_{\alpha\in\widehat{H}} M \otimes \alpha. \end{aligned}$$

More generally, for any $m \in \mathbb{N}$, $\pi_* L^m = \bigoplus_{\alpha\in\widehat{H}} M^m \otimes \alpha$. Consequently

$$H^0(A, L^m) \cong \bigoplus_{\alpha\in\widehat{H}} H^0(B, M^m \otimes \alpha).$$

for any $m \in \mathbb{N}$. However, given a power of L we take the larger subgroup $K(L^k)_1$ and get a finer decomposition. We will do that specifically for the second factor of μ_k. Analogously to before, let $G = K(L^k)_1$ and consider the isogeny

$$\pi' \colon A \to B' = A/G.$$

Once again B' is principally polarized say with polarization M' and $L^k = \pi'^* M'$. With the same arguments as above we can decompose

$$H^0(A, L^k) \cong \bigoplus_{\alpha \in \widehat{G}} H^0(B, M' \otimes \alpha).$$

Due to our choices of subgroups $H = K(L)_1 = kK(L^k)_1$ is a subgroup of G so that these decompositions are compatible. This gives $\widehat{H} = \phi_M(\pi(K(L)_2))$ and $\widehat{G} = \phi M'(\pi'(K(L^k)_2))$. The following diagram summarizes the situation

$$\begin{array}{ccccc} & & \overset{\pi'}{\frown} & & \\ A & \xrightarrow{\pi} & B & \xrightarrow{\varphi} & B' \\ \downarrow{\scriptstyle \phi_L} & & \downarrow{\scriptstyle \phi_M} & & \downarrow{\scriptstyle \phi_{M'}} \\ \operatorname{Pic}^0(A) & \xleftarrow[\pi^*]{} & \operatorname{Pic}^0(B) & \xleftarrow[\varphi^*]{} & \operatorname{Pic}^0(B'). \end{array}$$

Note that the second square does not commute but that we have instead $\varphi^* \circ \phi_{M'} \circ \varphi = k \cdot \phi_M$.

Now we can write our multiplication map as

$$\mu_k \colon \bigoplus_{\alpha \in \widehat{H}, \beta \in \widehat{G}} H^0(B, M \otimes \alpha) \otimes H^0(B', M' \otimes \beta) \overset{1 \otimes \varphi^*}{\to} \bigoplus_{\gamma \in \widehat{H}} H^0(B, M^{k+1} \otimes \gamma).$$

We can decompose $\mu_k = \bigoplus_{\gamma \in \widehat{H}} \mu_{k,\gamma}$ where

$$\mu_{k,\gamma} \colon \bigoplus_{\beta \in \widehat{G}} H^0(B, M \otimes \gamma \otimes \varphi^* \beta) \otimes H^0(B', M' \otimes \beta^{-1}) \to H^0(B, M^{k+1} \otimes \gamma).$$

Now since ϕ_M is an isomorphism we can take $H' := \phi_M^{-1}(\widehat{H}) = \pi(K(L)_2)$, $G' := \phi_{M'}^{-1}(\widehat{G}) = \pi'(K(L^k)_2)$ and $\widetilde{G} := \varphi^{-1}(G') \cap \pi(K(L^k)_2)$. Taking $c \in H'$ such that $\gamma = \phi_M((k+1)c) = \phi_{M^{k+1}}(c)$ and writing out the definitions of ϕ_M and $\phi_{M'}$, we obtain

$$\mu_{k,\gamma} \colon \bigoplus_{\substack{b' \in G', b \in \widetilde{G} \\ \varphi(b) = b'}} H^0(B, t^*_{(k+1)c+kb} M) \otimes H^0(B', t^*_{-b'} M') \to H^0(B, t^*_c M^{k+1}).$$

The difference between this and the proof in [Iye03] is that we are now taking the sum over the much larger group G'. Let θ be the unique theta divisor of $|M|$ and $\widetilde{\theta} \in |M^{k+1}|$ the pullback along φ of the unique theta divisor θ' in $|M'|$. We see that $\mu_{k,\gamma}$ is surjective if the linear system $|t_c^* M^{k+1}|$ is generated by divisors of the form $t^*_{(k+1)c+kb}\theta + t^*_{-b}\widetilde{\theta} = t^*_c(t^*_{k(c+b)}\theta + t^*_{-(c+b)}\widetilde{\theta})$ with $b \in \widetilde{G}$. By Proposition 2.4.2 it is thus surjective if the image of $\widetilde{G}$ under $\phi_c := t_c^* \circ \phi$ generates $|t_c^* M^{k+1}|$ or equivalently if the image of $\widetilde{G}$ under ϕ generates $|M^{k+1}|$. Now by assumption we have $|\widetilde{G}| = h^0(A, L^k) = k^g \cdot h^0(A, L) > (k+1)^g \cdot g! = h^0(B, M^{k+1}) \cdot g!$ and thus we can apply Proposition 2.4.3 to finish the proof. □

This approach only uses the cardinality of the group $K(L)_1$ but does not make use of its group structure which determines the type of the polarization. In [FG05] Fuentes García further develops these ideas and improves Iyer's bound for threefolds and fourfolds with polarizations of specific types. He conjectures that a general g-dimensional polarized abelian variety (A, L) with polarization of type $(1, \dots, 1, d)$ is projectively normal if $d > 2^{g+1} - 1$. In his recent preprint [Ito20] Ito proves this conjecture using the basepoint-freeness threshold introduced in [JP20].

2.5 Non-simple abelian varieties

The condition that A be simple or some weaker version of it is necessary, at least for projective normality. We can construct polarized abelian varieties (A, L) with $h^0(A, L)$ arbitrarily large that are not projectively normal.

Consider the product abelian variety $A = C \times B$ where $(C, \mathcal{O}_C(2p))$ with $p \in C$ a point on C is a (2)-polarized elliptic curve and (B, L) is a $(g-1)$-dimensional polarized abelian variety with polarization of type $(d_2, \dots, d_g)$ with all d_i are odd, e.g. an odd power of a principal polarization. Denote by $\pi_1 \colon A \to C$ and $\pi_2 \colon A \to B$ the projection maps. Now A carries the product polarization $\pi_1^*\mathcal{O}_C(2p) \otimes \pi_2^* L$ which must be primitive because $\gcd(2, d_i) = 1$ for all i but it cannot be normally generated no matter how large the d_i are as the restriction to C is only basepoint free but not very ample.

One would expect that for each abelian subvariety B a numerical condition on the sections of the restriction $L|_B$ implying projective normality can be derived. Indeed, in the case of abelian threefolds Lozovanu proved such a theorem. We will discuss it in some more detail in Section 2.7.

In order to understand precisely where simplicity is needed for Theorem 2.4.4 we need to examine Iyer's proof of Proposition 2.4.3. We give a brief sketch here.

What Iyer proves is actually the equivalent proposition.

Proposition 2.5.1 ([Iye03]) Let D be an ample divisor on a g-dimensional simple abelian variety A. Let G be a finite subgroup of A that is contained in D, then $|G| \leq D^g$.

In the proof Iyer considers all the translates $D+h$ for $h \in G$. One can reduce to the case that these divisors are all linearly equivalent. Then let $Y = \bigcap_{h\in G}(D+h)$ so that it is invariant under translation by elements in G and let $s = \dim(Y)$. It can be shown that there are $g-s$ divisors $D_j \in |D|$ intersecting properly such that $Y = \bigcap_{j=1}^{g-s} D_j$. The degree of Y is now simply the intersection number $[Y].[D^s]$ and it is clear that $\deg(Y) \leq D^g$. In particular, if the dimension of Y is zero, since G is contained in Y we have that $|G| \leq D^g$.

Now assume that $s > 0$. Let $Y = Y_1 \cup \ldots \cup Y_r$ be a decomosition of Y into irreducible components such that $\dim(Y_1) = \dim(Y) = s$. Since Y is G-invariant by construction we have that $\bigcup_{h\in G}(Y_1+h) \subset Y$ and thus $\sum_{h\in G/G_{Y_1}} \deg(Y_1+h) \leq \deg(Y)$ where $G_{Y_1} = \{h \in G \mid Y_1 + h = Y_1\}$ is the subgroup of G stabilizing Y_1. From this we obtain an inequality $|G/G_{Y_1}| \deg(Y_1) \leq \deg(Y) \leq D^g$ or equivalently $|G| \leq \frac{|G_{Y_1}|}{\deg(Y_1)} D^g$ and it remains to show that $|G_{Y_1}| \leq \deg(Y_1)$. To do so Iyer uses the fact that G_{Y_1} is contained in the stabilizer $\mathrm{Stab}(Y_1) = \{a \in A \mid Y_1 + a = Y_1\}$ and shows that $\deg(\mathrm{Stab}(Y_1)) \leq \deg(Y_1)$. Finally since A is simple $\mathrm{Stab}(Y_1)$ must be zero-dimensional and thus $|G_{Y_1}| \leq \deg(\mathrm{Stab}(Y_1)) \leq \deg(Y_1)$.

The crucial point is that the stabilizer $\mathrm{Stab}(Y_1)$ is zero-dimensional as this allows us to compare its degree with the cardinality of G_{Y_1}. Therefore, as long as one could ensure that this is the case, Theorem 2.4.4 would hold for non-simple abelian varieties as well. Unfortunately it seems very hard to control this stabilizer.

2.6 The case of general abelian varieties

The following theorem by Hwang and To works for abelian varieties of arbitrary dimension. However, it comes with the caveat that it only holds for a general abelian variety.

Theorem 2.6.1 ([HT11]) *A general g-dimensional polarized abelian variety (A, L) of a given polarization type and satisfying $h^0(A, L) \geq \frac{8^g}{2} \cdot \frac{g^g}{g!}$ is projectively normal.*

This bound is worse than Iyer's up to $g = 23$ but it is better asymptotically as can easily be seen using Stirling's formula.

Consider the product $A \times A$ with projection maps $\pi_i\colon A \times A \to A$ and let $\Delta = \{(a, a) \in A \times A\}$ be the diagonal in $A \times A$. We recall that projective normality is equivalent to the surjectivity of μ_1. Hwang and To deduce the surjectivity of μ_1 from the vanishing of $H^1(A \times A, \mathcal{I}_\Delta \otimes \pi_1^* L \otimes p_2^* L)$. This is a very common and useful technique for showing surjectivity of multiplication maps so we make it more precise with the following slightly more general lemma.

Lemma 2.6.2 Let X be a smooth complex manifold, let E_1, E_2 be vector bundles on X and let L be a line bundle on X. The multiplication map

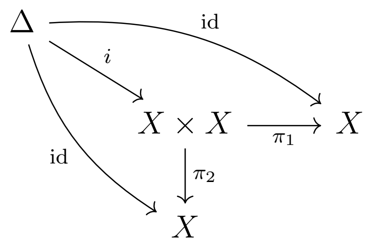
$$H^0(X, E_1 \otimes L^a) \otimes H^0(X, E_2 \otimes L^b) \to H^0(Y, E_1 \otimes E_2 \otimes L^{a+b})$$

is surjective if $H^1(X \times X, \mathcal{I}_\Delta \otimes \pi_1^*(E_1 \otimes L^a)) \otimes \pi_2^*(E_2 \otimes L^b)) = 0$ where $\Delta = \{(x, x) \in X \times X\}$ is the diagonal on $X \times X$ and $\pi_i \colon X \times X \to X$ denotes the projection onto the i-th factor.

Proof: We have the commutative diagram

$$\begin{array}{ccc} H^0(X \times X, \pi_1^*(E_1 \otimes L^a) \otimes \pi_2^*(E_2 \otimes L^b)) & \to & H^0(\Delta, \pi_1^*(E_1 \otimes L^a) \otimes \pi_2^*(E_2 \otimes L^b)) \\ \downarrow \cong & & \downarrow \cong \\ H^0(X, E_1 \otimes L^a) \otimes H^0(X, E_2 \otimes L^b) & \longrightarrow & H^0(X, E_1 \otimes E_2 \otimes L^{a+b}). \end{array}$$

The horizontal map on the bottom is the one we are interested in, the one on the top is simply restriction to the diagonal. The isomorphism on the left is obtained by the Künneth formula. Finally the isomorphism on the right comes from the isomorphism $\Delta \cong X$. More precisely we have a diagram



such that $\pi_j \circ i = \mathrm{id}$ for $j = 1, 2$. Restriction to the diagonal is simply the pullback along the inclusion map i and since tensor products commute with pullbacks we get the identification on the right.

Now clearly it is enough to show that the horizontal map on the top is surjective. For this consider the restriction sequence

$$0 \to \mathcal{I}_\Delta \to \mathcal{O}_{Y \times Y} \to \mathcal{O}_\Delta \to 0.$$

Tensoring with the appropriate vector bundles and taking the long exact cohomology sequence we see that it is enough to show that

$$H^1(X \times X, \mathcal{I}_\Delta \otimes \pi_1^*(E_1 \otimes L^a) \otimes \pi_2^*(E_2 \otimes L^b)) = 0. \qquad \square$$

Given two vector bundles L and M on A we write $L \boxtimes M$ for $\pi_1^* L \otimes \pi_2^* M$. Now let $\widetilde{A \times A} \xrightarrow{\pi} A \times A$ be the blowup of $A \times A$ along the diagonal Δ and let E denote the exceptional divisor. For convenience we write $\mathcal{L} = \pi^*(L \boxtimes L)$. By [Laz04a, Lemma 4.3.16.] we have an identification

$$\begin{aligned} H^1(A \times A, \mathcal{I}_\Delta \otimes L \boxtimes L) &= H^1(\widetilde{A \times A}, \mathcal{L} \otimes \mathcal{O}_{\widetilde{A \times A}}(-E)) \\ &= H^1(\widetilde{A \times A}, K_{\widetilde{A \times A}} \otimes \mathcal{L} \otimes \mathcal{O}_{\widetilde{A \times A}}(-gE)) \end{aligned}$$

where the second equality comes from the fact that the relative canonical divisor of the blowup of a smooth subvariety of codimension c is simply $(c-1)E$. Hwang and To then show that under the numerical assumption in Theorem 2.6.1 the line bundle $\mathcal{L} \otimes \mathcal{O}(-gE)$ is big and nef and $H^1(\widetilde{A \times A}, K_{\widetilde{A \times A}} \otimes \mathcal{L} \otimes \mathcal{O}(-gE))$ thus vanishes by Kawamata-Viehweg vanishing. Since we want to show μ_k for $k > 1$ to be surjective we can adapt the same methods to improve the bound.

Let (A, L) be a polarized abelian variety. Because A is a complex torus we can write $A = V/\Lambda$. There exists a unique translation-invariant Kähler form ω on A such that $c_1(L) = [\omega] \in H^2(A, \mathbb{Z})$. The real part of ω defines an inner product $\langle\ ,\ \rangle_L$ on V. Let $||\ ||_L$ be the associated norm. The *Buser-Sarnak invariant* $m(A, L)$ of (A, L) is defined as square of the length of the shortest non-zero lattice vector in Λ with respect to the inner product $\langle\ ,\ \rangle_L$, i. e.

$$m(A, L) \coloneqq \min_{\lambda \in \Lambda \setminus \{0\}} ||\lambda||_L^2 .$$

More generally, we can consider a g-dimensional compact complex torus $T = V/\Lambda$ with translation-invariant Kähler form, inner product and norm as above. Let S be a k-dimensional compact complex subtorus where $0 \le k < g$. Such a subtorus is given by the quotient of a k dimensional subvectorspace $F \cong \mathbb{C}^k$ of V by a sublattice $\Lambda_S \subset \Lambda$ of rank $2k$ such that $\Lambda_S = \Lambda \cap F$. Now let $F^\perp$ denote the orthogonal complement of F with respect to $\langle\ ,\ \rangle_L$, and let $q_F \colon V \to F$ and $q_{F^\perp} \colon V \to F^\perp$ denote the associated unitary projection map. The *relative Buser-Sarnak invariant* is then defined to be

$$m(T, S, \omega) \coloneqq \min_{\lambda \in \Lambda \setminus \Lambda_S} ||q_{F^\perp}(\lambda)||_L^2 .$$

Clearly we have $m(A, L) = m(A, \{0\}, c_1(L))$.

Now consider the $2g$-dimensional abelian variety $A \times A$ and denote by $\pi_i \colon A \times A \to A$ the projection onto the i-th factor. Clearly for any $k \in \mathbb{N}$, $L \boxtimes L^k$ defines an ample line bundle on $A \times A$. The associated translation-invariant Kähler form on $A \times A$ is given by $\omega_{A \times A} \coloneqq \pi_1^*\omega + k\pi_2^*\omega$. The diagonal

$$\Delta = \{(x, y) \in A \times A \mid x = y\}$$

is a g-dimensional subvariety. We want to relate the relative Buser-Sarnak invariant $m(A \times A, \Delta, \omega_{A \times A})$ to $m(A, L)$. Write $A \times A = (V \times V)/(\Lambda \times \Lambda)$ and denote by

$\langle\ ,\ \rangle_{L\boxtimes L^k}$ and $||\ ||_{L\boxtimes L^k}$ the inner product and norm respectively on $V\times V$ associated to $\omega_{A\times A}$. Because $\omega_{A\times A}=\pi_1^*\omega+k\pi_2^*\omega$ we have that

$$\langle\ ,\ \rangle_{L\boxtimes L^k}=\pi_1^*\langle\ ,\ \rangle_L+k\pi_2^*\langle\ ,\ \rangle_L$$

as well as

$$||\ ||^2_{L\boxtimes L^k}=\pi_1^*||\ ||^2_L+k\pi_2^*||\ ||^2_L\,.$$

The diagonal Δ is isomorphic to F/Λ_Δ where $F=\{(z,z)\mid z\in V\}\subset V\times V$ and $\Lambda_\Delta=\{(\lambda,\lambda)\mid \lambda\in\Lambda\}\subset\Lambda\times\Lambda$. Let $F^\perp$ denote the orthogonal complement of F in $V\times V$ with respect to $\langle\ ,\ \rangle_{L\boxtimes L^k}$. Finally let $q_F\colon V\times V\to F$ and $q_{F^\perp}\colon V\times V\to F^\perp$ denote the corresponding unitary projection map.

Lemma 2.6.3 The unitary projection map $q_{F^\perp}$ is given by

$$q_{F^\perp}(\lambda_1,\lambda_2)=\left(\frac{k(\lambda_1-\lambda_2)}{k+1},\frac{\lambda_2-\lambda_1}{k+1}\right).$$

Proof: First we check that $(\lambda_1,\lambda_2)\in F^\perp$ if and only if for every $\lambda\in\Lambda$

$$\begin{aligned}0&=\langle(\lambda_1,\lambda_2),(\lambda,\lambda)\rangle_{L\boxtimes L^k}\\&=\langle\lambda_1,\lambda\rangle_L+k\langle\lambda_2,\lambda\rangle_L\\&=\langle\lambda_1+k\lambda_2,\lambda\rangle_L\,.\end{aligned}$$

That is to say $\lambda_1=-k\lambda_2$. It is easy to see that the image of $q_{F^\perp}$ is exactly $F^\perp$ and its null space is F. We still need to check that the map is actually a projection map, i.e. that $q^2_{F^\perp}=q_{F^\perp}$. We have

$$\begin{aligned}q^2_{F^\perp}(\lambda_1,\lambda_2)&=q_{F^\perp}\left(\frac{k(\lambda_1-\lambda_2)}{k+1},\frac{\lambda_2-\lambda_1}{k+1}\right)\\&=\left(\frac{k\left(\frac{k(\lambda_1-\lambda_2)}{k+1}-\frac{\lambda_2-\lambda_1}{k+1}\right)}{k+1},\frac{\frac{\lambda_2-\lambda_1}{k+1}-\frac{k(\lambda_1-\lambda_2)}{k+1}}{k+1}\right)\\&=\left(\frac{k\frac{(k+1)(\lambda_1-\lambda_2)}{k+1}}{k+1},\frac{\frac{(k+1)(\lambda_2-\lambda_1)}{k+1}}{k+1}\right)\\&=q_{F^\perp}(\lambda_1,\lambda_2).\end{aligned}$$

□

Lemma 2.6.4 We have

$$m(A\times A,\Delta,\omega_{A\times A})=\frac{k}{k+1}m(A,L).$$

PROOF: As shown in the previous lemma, the unitary projection map is given by

$$q_{F^\perp}(\lambda_1,\lambda_2) = \left(\frac{k(\lambda_1-\lambda_2)}{k+1}, \frac{\lambda_2-\lambda_1}{k+1}\right).$$

and therefore we have

$$\begin{aligned} ||q_{F^\perp}(\lambda_1,\lambda_2)||^2_{L\boxtimes L^k} &= \left|\left|\frac{k(\lambda_1-\lambda_2)}{k+1}\right|\right|^2_L + k\left|\left|\frac{\lambda_2-\lambda_1}{k+1}\right|\right|^2_L \\ &= \frac{(k^2+k)\,||\lambda_1-\lambda_2||}{(k+1)^2} \\ &= \frac{k\,||\lambda_1-\lambda_2||_L}{k+1} \end{aligned}$$

and finally

$$\begin{aligned} m(A\times A,\Delta,\omega_{A\times A}) &= \min_{(\lambda_1,\lambda_2)\in\Lambda\times\lambda\setminus\Lambda_\Delta} ||q_{F^\perp}(\lambda_1,\lambda_2)||^2_{L\boxtimes L^k} \\ &= \frac{k}{k+1}\min_{\lambda\in\Lambda\setminus\{0\}}||\lambda||^2_L \\ &= \frac{k}{k+1}m(A,L). \end{aligned}$$ □

The following definition is not essential at this point but it allows us to phrase certain results in a concise way and is commonly used by people working in this area.

Definition 2.6.5 Let X be an irreducible projective variety, let L be a nef line bundle, let $\mathcal{I}\subseteq\mathcal{O}_X$ be an ideal sheaf and let $\mu\colon X'=\mathrm{Bl}_{\mathcal{I}}X\to X$ be the blowup of X along $\mathcal{I}$ with exceptional divisor E. The *Seshadri constant* of L along $\mathcal{I}$ is defined as

$$\varepsilon(X,L,\mathcal{I}) \coloneqq \sup\{\varepsilon\in\mathbb{R}\mid \mu^*L\otimes\mathcal{O}_{X'}(-\varepsilon E)\text{ is nef on }X'\}.$$

While we will be interested in the Seshadri constant of $L\boxtimes L^k$ along the ideal sheaf of the diagonal on $A\times A$, the most commonly studied Seshadri constants are in fact Seshadri constants of points. In the case of abelian varieties, since we can translate points, the Seshadri constant is the same for all points so it is common to simply write $\varepsilon(A,L)$. Now let $\pi:\widetilde{A\times A}\to A\times A$ be the blowup of $A\times A$ along the diagonal Δ and let $E=\pi^{-1}(\Delta)$ be the associated exceptional divisor.

Proposition 2.6.6 Let (A,L) be a polarized abelian variety of dimension g and let $\mathcal{L}=\pi^*(L\boxtimes L^k)$. Then $\mathcal{L}\otimes\mathcal{O}_{\widetilde{A\times A}}(-\alpha E)$ is nef on $\widetilde{A\times A}$ for all $\alpha\in\mathbb{R}$ with $0\le\alpha\le\frac{k\pi}{4(k+1)}m(A,L)$. In other words

$$\varepsilon(A\times A, L\boxtimes L^k,\mathcal{I}_\Delta)\ge\frac{k\pi}{4(k+1)}m(A,L).$$

PROOF: The proof is analogous to [HT11, Proposition 3.2] but using Lemma 2.6.4. □

Lemma 2.6.7 Let $(A, L), g, E$ and $\mathcal{L}$ be as in Proposition 2.6.6. If $\mathcal{L} \otimes \mathcal{O}_{\widetilde{A\times A}}(-gE)$ is nef and $c_1(L)^g > \left(\frac{(k+1)g}{k}\right)^g$, then $\mathcal{L} \otimes \mathcal{O}_{\widetilde{A\times A}}(-gE)$ is big.

PROOF: We calculate the top self-intersection of $\mathcal{L} \otimes \mathcal{O}_{\widetilde{A\times A}}(-gE)$. First we see that

$$\begin{aligned} c_1(\mathcal{L})^{2g} &= c_1(L \boxtimes L^k)^{2g} \\ &= \sum_{l=0}^{2g} \binom{2g}{l} c_1(L)^l (k c_1(L))^{2g-l} \\ &= k^g \frac{(2g)!}{g! \cdot g!} c_1(L)^g \cdot c_1(L)^g \end{aligned}$$

as for all $l \neq g$ either $l > g$ or $2g - l > g$ and thus all the terms except for the middle one vanish. Now we have

$$\begin{aligned} c_1(\mathcal{L} \otimes \mathcal{O}_{\widetilde{A\times A}}(-gE))^{2g} &= \sum_{l=0}^{2g} \binom{2g}{l} c_1(\mathcal{L})^{2g-l} c_1(\mathcal{O}_{\widetilde{A\times A}}(-gE))^l \\ &= \sum_{l=0}^{2g} (-1)^l g^l \binom{2g}{l} c_1(\mathcal{L})^{2g-l} c_1(\mathcal{O}_{\widetilde{A\times A}}(E))^l. \end{aligned}$$

We consider the summands separately. The term for $l = 0$ is simply $c_1(\mathcal{L})^{2g}$ which we have calculated above. Both $\mathcal{L}$ and $\mathcal{O}_{\widetilde{A\times A}}(E)$ are line bundles on $\widetilde{A \times A}$. For $g > 1$ we can remove one copy of $\mathcal{O}_{\widetilde{A\times A}}(E)$ and replace both line bundles by their restriction to E. Locally there is an identification $E \cong \Delta \times \mathbb{P}^{g-1}$. Let $\sigma\colon E \to \mathbb{P}^{g-1}$ and $\eta\colon E \to \Delta \cong A$ denote the projections, then we have $\mathcal{O}_{\widetilde{A\times A}}(E)|_E = \sigma^*\mathcal{O}_{\mathbb{P}^{g-1}}(-1)$ and $\mathcal{L}|_E = \eta^*(L \otimes L^k) = \eta^*(L^{k+1})$. Thus we get

$$\begin{aligned} c_1(\mathcal{L})^{2g-l} c_1(\mathcal{O}_{\widetilde{A\times A}}(E))^l &= c_1(\mathcal{L}|_E)^{2g-l} c_1(\mathcal{O}(E)|_E)^{l-1} \\ &= ((k+1)c_1(L))^{2g-l} c_1(\mathcal{O}_{\mathbb{P}^{g-1}}(-1))^{l-1} \\ &= (-1)^{l-1}(k+1)^{2g-l} c_1(L)^{2g-l} c_1(\mathcal{O}_{\mathbb{P}^{g-1}}(1))^{l-1}. \end{aligned}$$

When $l < g$ the first Chern class vanishes and when $l > g$ the second Chern class vanishes. The only terms that remain are thus $l = 0$ and $l = g$. Therefore we have

$$\begin{aligned} c_1(\mathcal{L} \otimes \mathcal{O}(-gE))^g &= k^g \frac{(2g)!}{g! \cdot g!} c_1(L)^g \cdot c_1(L)^g - ((k+1)g)^g \frac{(2g)!}{g! \cdot g!} c_1(L)^g \\ &= \frac{(2g)!}{g! \cdot g!} c_1(L)^g (k^g c_1(L)^g - ((k+1)g)^g). \end{aligned}$$

This number is positive if and only if $c_1(L)^g > \left(\frac{(k+1)g}{k}\right)^g$. Combining this with the well-known fact that a nef line bundle whose top self-intersection is positive is big proves the statement. □

With this we can prove the following theorem.

Theorem 2.6.8 *Let (A,L) be a general polarized abelian variety of dimension $g \geq 2$ and of type $(d_1,\dots,d_g)$. If $h^0(A,L) = \prod_{i=1}^g d_i \geq \frac{1}{2g!}\left(\frac{4g(k+1)}{k}\right)^g$ then the multiplication map μ_k is surjective.*

Proof: Let $\mathcal{A}_{(d_1,\dots,d_g)}$ be the moduli space of g-dimensional polarized abelian varieties of type $(d_1,\dots,d_g)$ and suppose $\prod_{i=1}^g d_i \geq \frac{1}{2g!}\left(\frac{4g(k+1)}{k}\right)^g$. Now by [Bau98, Theorem A] there exists some polarized abelian variety $(A_0,L_0) \in \mathcal{A}_{(d_1,\dots,d_g)}$ such that $m(A_0,L_0) = \frac{1}{\pi}\sqrt[g]{2c_1(L_0)^g}$. By the geometric Riemann-Roch theorem we have $h^0(A_0,L_0) = \frac{c_1(L_0)^g}{g!}$ and thus

$$\begin{aligned} m(A_0,L_0) &= \frac{1}{\pi}\sqrt[g]{2c_1(L_0)^g} \\ &= \frac{1}{\pi}\sqrt[g]{2g!h^0(A_0,L_0)} \\ &\geq \frac{4g(k+1)}{k\pi} \end{aligned}$$

or equivalently

$$\frac{k\pi}{4(k+1)}m(A_0,L_0) \geq g.$$

Therefore, by Proposition 2.6.6

$$\varepsilon(A\times A, L\boxtimes L^k, \mathcal{I}_\Delta) \geq g$$

and thus $\mathcal{L}_0 \otimes \mathcal{O}_{\widetilde{A_0\times A_0}}(-gE_0)$ is nef. We see immediately that

$$\begin{aligned} c_1(L_0)^g &= h^0(A_0,L_0)g! \\ &\geq \frac{1}{2}\left(\frac{4g(k+1)}{k}\right)^g \\ &= \frac{4^g}{2}\cdot\left(\frac{g(k+1)}{k}\right)^g \\ &> \left(\frac{(k+1)g}{k}\right)^g \end{aligned}$$

so by Lemma 2.6.7 $(L_0 \boxtimes L_0^k) \otimes \mathcal{O}_{\widetilde{A_0 \times A_0}}(-gE_0)$ is big. Kawamata-Viehweg vanishing now implies the vanishing of $H^1(A_0 \times A_0, \mathcal{I}_\Delta \otimes (L_0 \boxtimes L_0^k))$ which implies that μ_k is surjective. The existence of some $(A_0, L_0) \in \mathcal{A}_{(d_1,\dots,d_g)}$ for which μ_k is surjective then implies that it holds for a general $(A, L) \in \mathcal{A}_{(d_1,\dots,d_g)}$. □

This is true regardless of whether L is primitive or not but obviously in the latter case we already have much stronger results. The important part is that this gives a criterion for a general hypersurface in a general abelian variety with primitive polarization. However, it would be preferable to have a criterion that works for any abelian variety.

2.7 Approach using multiplier ideals and vanishing

We have now obtained two different results concerning the surjectivity of the multiplication map μ_k. Theorem 2.4.4 requires the abelian variety to be simple and Theorem 2.6.8 requires it to be general. It would be preferable to have a theorem that holds for any abelian variety. As we have seen in Section 2.5, the condition that A be simple seems to be crucial in the proof of Theorem 2.4.4. The methods from the previous section seem more likely to yield results that work for any abelian variety.

A commonly studied generalization of projective normality are the properties (N_p) introduced by Green and Lazarsfeld in [GL86]. The precise definition of these properties can also be found in [Laz04a, Definition 1.8.50]. For our purposes it is enough to know that the property (N_0) holds for a line bundle if and only if it is normally generated.

In [LPP11] a criterion for L to satisfy (N_p) is given. Namely it states that L satisfies (N_p) if

$$\varepsilon(A, L) > (p + 2)g.$$

As above, the case for $p = 0$ follows from the vanishing of $H^1(A \times A, \mathcal{I}_\Delta \otimes (L \boxtimes L))$. Lazarsfeld, Pareschi and Popa deduce this directly from Nadel vanishing by realizing $\mathcal{I}_\Delta$ as the multiplier ideal associated to a suitable $\mathbb{Q}$-divisor on $A \times A$. The existence of such a divisor is guaranteed by the same criterion as above in Hwang's and To's proof (implying a bound on the Seshadri constant). Thus if we could derive another criterion for the existence of such a divisor we could possibly remove the condition that (A, L) be general. We have been considering L as a line bundle so far but in this context it will be more practical to use additive notation occasionally.

We recall some basic facts about multiplier ideals. For more details refer to [Laz04b, Chapter 9].

Definition 2.7.1 Let X be a smooth complex variety and let D be a $\mathbb{Q}$-divisor on X. A *log resolution* of D (or of the pair (X, D)) is a projective birational mapping

$$\mu\colon X' \to X$$

with X' non-singular such that if E denotes the exceptional locus of μ, $\mu^*D + E$ is a divisor with simple normal crossing support.

Definition 2.7.2 Let X be a smooth complex variety, let D be an effective $\mathbb{Q}$-divisor on X and let $\mu\colon X' \to X$ be a log resolution of (X, D). The *multiplier ideal sheaf* associated to D is defined as

$$\mathcal{J}(D) = \mu_*\mathcal{O}_{X'}(K_{X'/X} - \lfloor \mu^*D \rfloor)$$

where $K_{X'/X} = K_{X'} - \mu^*K_X$ is the relative canonical divisor.

Note that the multiplier ideal sheaf does not depend on the choice of resolution.

Theorem 2.7.3 (Nadel vanishing) *Let X be a smooth projective variety, let D be any $\mathbb{Q}$-divisor on X and let L be any integral divisor such that $L - D$ is big and nef. Then*

$$H^i(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{J}(D)) = 0 \textit{ for } i > 0.$$

Using this we can rephrase the proof from the previous section in terms of Nadel vanishing. The techniques used here will be useful later when we consider irregular varieties that do not embed into their Albanese variety.

Lemma 2.7.4 Let X be a smooth complex variety of dimension n, let L be an ample line bundle on X and let $Z \subset X$ be a smooth subvariety of codimension c. Then for any $a, b \geq 1$, if $\varepsilon(X, L, \mathcal{I}_Z) > \frac{b+c-1}{a}$,

$$H^1(X, L^a \otimes \mathcal{I}_Z^b) = 0.$$

PROOF: In order to show the vanishing of $H^1(X, L^a \otimes \mathcal{I}_Z^b)$ using Nadel vanishing we need to find a $\mathbb{Q}$-divisor D satisfying two conditions:
(1) $\mathcal{I}_Z^b$ must be the multiplier ideal $\mathcal{J}(D)$ associated to D and
(2) $aL - D$ must be big and nef.
An easy way to construct such a $\mathbb{Q}$-divisor is to use [Laz04b, Example 9.3.5]. We will blow up X along Z, construct a suitable divisor on the blowup, and push it forward. Let $\mu\colon X' \to X$ denote this blowup and let E denote the exceptional divisor. Since we blow up along Z the strict transform of a divisor on X' containing Z will be a divisor on X intersecting E. Moreover, the local components of the divisor on X will be smooth along Z if and only if its strict transform intersects E transversally.

The pullback μ^*L is never ample as its intersection number with curves contained in the exceptional divisor is zero. It is, however, still big and nef and by [Laz04a, Example 2.2.19] the divisor $\mu^*L - \varepsilon E$ is then ample for $\varepsilon > 0$ small. Thus for $N \gg 0$ Bertini's Theorem allows us to find a smooth irreducible divisor $\widetilde{D}$ in the linear system $|N(\mu^*L - \varepsilon E)|$ that only meets E transversally.

Since Z has codimension c the projectivized normal bundle $\mu|_E \colon E \to Z$ is a locally trivial fibration with $\mathbb{P}^{c-1}$ fibers. Suppose $E_z := \mu^{-1}(z)$ is the fiber of some general point $z \in Z$. Let C be any $\mathbb{P}^1$ in E_z, then we have $\widetilde{D}.C = N(\mu^*L - \varepsilon E).C = N(\mu^*L.C) - \varepsilon N(E.C) = \varepsilon N$. We will then take $D = \lambda \cdot \mu_* \widetilde{D}$ for some $\lambda \in \mathbb{Q}$ chosen such that both conditions from above are satisfied.

For condition (1) we have that $\mathcal{J}(D) = \mu_*(K_{X'/X} - \lfloor \mu^* D \rfloor)$ since blowing up Z gives a log resolution of (X, D). The relative canonical divisor of a single blowup is simply $(c-1)E$ where c is the codimension of the subvariety we blew up, i.e.

$$K_{X'/X} = (c-1)E.$$

The total transform μ^*D is $\lambda\widetilde{D} + \lambda\varepsilon N E$ (the multiplicity of the exceptional divisor is simply the intersection number $\widetilde{D}.C$ from above). And for $\lambda \in (0,1)$ we thus get $\lfloor \mu^*D \rfloor = \lfloor \lambda \varepsilon N \rfloor E$. Putting things together we obtain

$$\mathcal{J}(D) = \mu_*((c - 1 - \lfloor \lambda \varepsilon N \rfloor)E).$$

If our choice of N was sufficiently large we can thus choose $\lambda = \frac{b+c-1}{\varepsilon N} \in (0,1)$ to obtain $\mathcal{J}(D) = \mu_* \mathcal{O}_{X'}(-bE) = \mathcal{I}_Z^b$.

We must now check that our choice of λ is compatible with condition (2). Since μ is proper and surjective $L^a - D$ is big and nef if and only if $\mu^*L^a - \mu^*D = \mu^*L^a - \lambda\mu^*\mu_*\widetilde{D}$ is big and nef. Since $\widetilde{D} \in |N(\mu^*L - \varepsilon E)|$ we have that $\mu^*\mu_*\widetilde{D} = \widetilde{D} + \varepsilon N E \in |N\mu^*L|$. Therefore, $\mu^*L^a - \mu^*D$ is an element in the linear system $|(a - \lambda N)\mu^*L|$ and is thus big and nef if and only if $a - \lambda N > 0$. With our previous choice of λ this gives us $a - \frac{b+c-1}{\varepsilon} > 0$ or equivalently $\varepsilon > \frac{b+c-1}{a}$. Above we needed that $\mu^*L - \varepsilon E$ is ample. The supremum over all $\varepsilon > 0$ such that this holds is precisely the Seshadri constant $\varepsilon(X, L, \mathcal{I}_Z)$ of D along Z which by assumption is greater than $\frac{b+c-1}{a}$ so we are done. □

Applying this lemma to the case where we want to show that $H^1(A \times A, L \boxtimes L^k \otimes \mathcal{I}_\Delta)$ vanishes we see that we need $\varepsilon(A \times A, L \boxtimes L^k, \mathcal{I}_\Delta) > g$ which we can obtain from the relationship of the Seshadri constant and the Buser-Sarnak invariant as in [HT11].

In [LPP11] Lazarsfeld, Pareschi and Popa give another method to obtain a suitable $\mathbb{Q}$-divisor on $A \times A$ by realizing it as the pullback of a suitable divisor on

A via the difference map. We want to construct an effective $\mathbb{Q}$-divisor E on $A \times A$ such that its multiplier ideal coincides with the ideal sheaf of the diagonal, i.e.

$$\mathcal{J}(E) = \mathcal{I}_\Delta,$$

and such that $L \boxtimes L(-E)$ is ample or at least big and nef. For that purpose, if we had an effective $\mathbb{Q}$-divisor E_0 on A such that

$$E_0 \equiv_{\text{num}} \frac{1-c}{2} L$$

for some $0 < c \ll 1$ and

$$\mathcal{J}(E_0) = \mathcal{I}_0,$$

we could pull it back via the difference map

$$\delta \colon A \times A \to A,\ (x, y) \mapsto x - y,$$

i.e. set $E = \delta^* E_0$. Then, since forming multiplier ideals commutes with pullbacks under smooth morphisms, we have

$$\mathcal{J}(E) = \mathcal{J}(\delta^* E_0) = \delta^* \mathcal{J}(E_0) = \delta^* \mathcal{I}_0 = \mathcal{I}_\Delta.$$

Furthermore, denote by

$$\alpha \colon A \times A \to A,\ (x, y) \mapsto x + y,$$

the addition map. Let $\mathcal{P}$ denote the Poincaré bundle on $A \times \widehat{A}$. If $\phi_L \colon A \to \widehat{A}$ denotes the isogeny from A to its dual induced by L write

$$P = (\mathrm{id} \times \phi_L)^* \mathcal{P}.$$

We need a lemma first.

Lemma 2.7.5 The following identities hold:
(i) $\alpha^* L \cong (L \boxtimes L) \otimes P$;
(ii) $\delta^* L \cong (L \boxtimes L) \otimes P^{-1}$.

PROOF: Both identities follow from the seesaw principle (see e.g. [Mum08, p. 78]). □

With this lemma we have

$$\begin{aligned}
\delta^* L \otimes \alpha^* L &\cong (L \boxtimes (-1)^* L) \otimes P^{-1} \otimes (L \boxtimes L) \otimes P \\
&\cong p_1^* L \otimes (-p_2)^* L \otimes p_1^* L \otimes p_2^* L \\
&\cong p_1^* L^2 \otimes p_2^* (L \otimes (-1)^* L) \\
&\cong L^2 \boxtimes (L \otimes (-1)^* L),
\end{aligned}$$

which means that

$$\begin{aligned} L \boxtimes L \otimes \mathcal{O}(-E) &\equiv_{\text{num}} (L \boxtimes L) \otimes \left(-\frac{1-c}{2}\right) (\delta^* L) \\ &\cong (L \boxtimes L) \left(-\frac{1-c}{2}\right) \otimes (L^{-2} \boxtimes L^{-2} \otimes \alpha^* L^{-1}) \\ &\cong L \boxtimes L \otimes (L \boxtimes L)^{-1} \otimes c(L \boxtimes L) \otimes \left(\frac{1-c}{2}\right) \alpha^* L \\ &\cong c(L \boxtimes L) \otimes \left(\frac{1-c}{2}\right) \alpha^* L \end{aligned}$$

which is ample. Now a similar argument as in Lemma 2.7.4, $\varepsilon(A, L) > 2g$ implies the existence of a suitable E_0 (see also [LPP11, Lemma 1.2]).

We would like to apply these methods to $L \boxtimes L^k$. Unfortunately the lower bound on the Seshadri constant does not improve due to the presence of a primitive factor, meaning we gain nothing by studying μ_k instead of projective normality.

In [Loz18] Lozovanu studies abelian threefolds. He shows that if the zero locus of $\mathcal{J}(E_0)$ is zero-dimensional it can be cut down to I_0. Then he procceds to construct such an E_0 under numerical conditions on the self-intersection of L as well as the restriction of L to abelian subvarieties.

Theorem 2.7.6 ([Loz18]) *Let (A, L) be a polarized abelian threefold such that $h^0(A, L) > 78$. Assume the following conditions:*

(i) For any abelian surface $S \subseteq A$ one has $h^0(S, L|_S) > 4$.

(ii) For any elliptic curve $E \subseteq A$ one has $h^0(E, L|_E) > 4$.

Then L gives a projectively normal embedding of A.

Note that Lozovanu actually proves a more general result about (A, L) satisfying the property (N_p).

In fact if we are only interested in (N_0) this condition can be weakened. Assume that $\mathcal{J}(E_0) = \mathcal{I}_{\{0\} \cup Y}$ for some higher-dimensional subvariety $Y \subset A$ such that $0 \notin Y$. Again setting $E := \delta^* E_0$ we get

$$\mathcal{J}(E) = \mathcal{I}_{\Delta \cup Z}$$

where $Z \subset A \times A$ is a subvariety such that $Z \cap \Delta = \emptyset$. The ampleness required for Nadel vanishing does not depend on what $\mathcal{J}(E)$ is, only that $E \equiv_{\text{num}} \frac{1-c}{2} L$ so just as above we can conclude that $H^1(A \times A, \mathcal{I}_{\Delta \cup Z} \otimes L \boxtimes L) = 0$. Now get an inclusion of ideal sheaves $\mathcal{I}_{\Delta \cup Z} \hookrightarrow \mathcal{I}_\Delta$ that fits into the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{I}_{\Delta \cup Z} & \longrightarrow & \mathcal{O}_{A \times A} & \longrightarrow & \mathcal{O}_\Delta \oplus \mathcal{O}_Z & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{I}_\Delta & \longrightarrow & \mathcal{O}_{A \times A} & \longrightarrow & \mathcal{O}_\Delta & \longrightarrow & 0. \end{array}$$

Tensoring with $L \boxtimes L$ and taking cohomology yields

$$\begin{array}{ccccc} H^0((\mathcal{O}_\Delta \oplus \mathcal{O}_Z) \otimes L \boxtimes L) & \longrightarrow & H^1(\mathcal{I}_{\Delta \cup Z} \otimes L \boxtimes L) = 0 & & \\ \downarrow\!\!\!\downarrow & & \downarrow & & \\ H^0(\mathcal{O}_\Delta \otimes L \boxtimes L) & \longrightarrow & H^1(\mathcal{I}_\Delta \otimes L \boxtimes L) & \longrightarrow & H^1(L \boxtimes L) = 0. \end{array}$$

Since the expression on the top right is zero, the composition of the horizontal map on the top and the vertical map on the right must be the zero map. Therefore, the composition of the vertical map on the left and the first horizontal map on the bottom must also be the zero map and since the vertical map is a surjection it follows that the first map on the bottom is the zero map. As the object after it is zero it is a surjection so $H^1(A \times A, \mathcal{I}_\Delta \otimes L \boxtimes L) = 0$.

3 The infinitesimal Torelli theorem for certain classes of irregular varieties

In this chapter we will finally prove the ITT in some cases. First we will consider principal polarizations. The results about surjectivity of the multiplication map from Chapter 2 will not help us here but we can give a direct proof. In Section 3.2 we will apply Green's proof of the ITT for sufficiently ample hypersurfaces in arbitrary varieties to the case of hypersurfaces in abelian varieties and put a numerical bound on the required ampleness using the results obtained in Chapter 2. In Section 3.3 we will generalize this to complete intersections in abelian varieties. Finally in Section 3.4 we will investigate what happens for irregular varieties with globally generated cotangent bundles whose Albanese image is a mildly singular divisor.

3.1 Principal polarizations

We cannot use the methods from Chapter 2 to show the ITT for principal polarizations. The multiplication map μ_{g-1} cannot be surjective for dimensional reasons. We can check the ITT directly. Let $X \subset A$ be a hypersurface in an abelian variety $A = V/\Lambda$ of dimension $g = n+1 \geq 3$. To simplify notation we use the fact that the tangent bundle of A is trivial of fiber V and that by the adjunction formula the normal bundle $N_{X/A}$ is isomorphic to ω_X. The usual normal bundle sequence then takes the form

$$0 \to T_X \to V \otimes \mathcal{O}_X \to \omega_X \to 0.$$

Tensoring it with $\omega_X \otimes H^0(\omega_X)^\vee$ yields

$$0 \to \Omega_X^{n-1} \otimes H^0(\omega_X)^\vee \to V \otimes \omega_X \otimes H^0(\omega_X)^\vee \to \omega_X^2 \otimes H^0(\omega_X)^\vee \to 0.$$

Taking cohomology for both sequences we obtain the following commutative diagram

$$\begin{array}{ccc} H^1(X, T_X) & \xrightarrow{\varphi_1} & V \otimes H^1(X, \mathcal{O}_X) \\ \downarrow{\scriptstyle d\mathcal{P}^{n,n}} & & \downarrow{\scriptstyle \varphi_2} \\ H^0(X, \omega_X)^\vee \otimes H^1(X, \Omega_X^{n-1}) & \longrightarrow & V \otimes H^0(\omega_X)^\vee \otimes H^1(\omega_X) \end{array} \tag{3.1}$$

where the horizontal maps come from the exact sequences, the left vertical map is the highest piece of the differential of the period map and the vertical map on the right is induced by the cup product and the interior product.

Lemma 3.1.1 If $\mathcal{O}_A(X)$ is a principal polarization then the map φ_1 from above is injective.

PROOF: The long exact sequence in cohomology of the tangent bundle sequence gives

$$H^0(X, T_X) \to V \to H^0(X, \omega_X) \xrightarrow{\alpha} H^1(X, T_X) \xrightarrow{\varphi_1} V \otimes H^1(X, \mathcal{O}_X).$$

By Serre duality we have $H^0(T_X) \cong H^n(\Omega^1_X \otimes \omega_X)$ which vanishes by Nakano vanishing. Now the long exact sequence in cohomology of the restriction sequence is

$$0 \to H^0(A, \mathcal{O}_A) \to H^0(A, \mathcal{O}_A(X)) \to H^0(X, \omega_X) \to H^1(A, \mathcal{O}_A) \to 0$$

so that $h^0(\omega_X) = h^0(\mathcal{O}_A(X)) + g - 1$ and if $\mathcal{O}_A(X)$ is a principal polarization this gives us $h^0(\omega_X) = g$. Therefore, α must be the zero map and thus φ_1 is injective.□

Lemma 3.1.2 If $\mathcal{O}_A(X)$ is a principal polarization then the map φ_2 from above is injective.

PROOF: The map is simply the identity on the factor V so we will consider the map

$$H^1(X, \mathcal{O}_X) \to H^0(X, \omega_X)^\vee \otimes H^1(X, \omega_X).$$

Since $g \geq 3$ using the standard restriction sequence as we have seen in Section 1.4 we can identify $H^1(X, \mathcal{O}_X) \cong H^1(A, \mathcal{O}_A)$. Furthermore, we can identify $H^1(X, \omega_X) \cong H^2(A, \mathcal{O}_A) \cong \bigwedge^2 H^1(A, \mathcal{O}_A)$ and because the polarization is principal we can also identify $H^0(X, \omega_X)^\vee \cong H^1(A, \mathcal{O}_A)^\vee$. We thus have a commutative diagram

$$\begin{array}{ccc} H^0(X, \omega_X)^\vee \otimes H^1(X, \omega_X) & \xrightarrow{\cong} & H^1(A, \mathcal{O}_A)^\vee \otimes \bigwedge^2 H^1(A, \mathcal{O}_A) \\ \varphi \uparrow & & \uparrow \\ H^1(X, \mathcal{O}_X) & \xrightarrow{\cong} & H^1(A, \mathcal{O}_A) \end{array}$$

where the vertical map on the right is induced by the cup product. Since this map is injective, so is φ. □

Using these two lemmas we immediately obtain the following theorem.

Theorem 3.1.3 *Let A be an abelian variety of dimension $g \geq 3$ and let $X \subset A$ be a smooth hypersurface defining a principal polarization. Then the ITT holds for X.*

PROOF: By Lemma 3.1.1 and 3.1.2 the maps φ_1 and φ_2 in (3.1) are injective. It follows directly that $d\mathcal{P}^{n,n}$ must be injective as well. □

3.2 Hypersurfaces in abelian varieties

In this section we consider the simplest case, when the Albanese morphism embeds X as a hypersurface into an abelian variety. The contents of this section have been summarized in the preprint [Blo19].

In [Gre85] Green proves the ITT for sufficiently ample hypersurfaces in arbitrary varieties. We should make more precise what this means.

Definition 3.2.1 Let Y be a projective variety. A property is said to hold for *sufficiently ample* line bundles L on Y if there exists an ample line bundle L_0 on Y such that the property holds for all line bundles L on Y such that $L \otimes L_0^{-1}$ is ample.

Green shows the following theorem.

Theorem 3.2.2 *Let Y be a smooth complete algebraic variety of dimension $n+1 \geq 2$. Then for a sufficiently ample line bundle L on Y, the infinitesimal Torelli theorem holds for any smooth X in the linear system $|L|$.*

Specifically he starts with the short exact sequence

$$0 \to T_X \to T_Y|_X \to N_{X/Y} \to 0$$

and, under the assumption that $L := \mathcal{O}_Y(X)$ is sufficiently ample, deduces the following diagram

$$\begin{array}{ccc} H^0(X, \omega_X) \otimes H^1(X, \Omega_X^{n-1})^\vee & \longrightarrow & H^1(X, T_X)^\vee \\ \uparrow & & \uparrow \\ H^0(Y, L \otimes \omega_Y) \otimes H^0(Y, L^n \otimes \omega_Y) & \longrightarrow & H^0(Y, L^{n+1} \otimes \omega_Y^2) \end{array}$$

where the map on the top is the dual of the differential of the period map and the map on the bottom is multiplication of sections. Since the map on the right is surjective, surjectivity of the multiplication map on the bottom then implies the ITT. Green then proves the ITT by proving surjectivity of this map under the assumption that L is sufficiently ample.

Now let $Y = A$ be an abelian variety and let $X \subset A$ be a hypersurface on A. Many of the proofs simplify since abelian varieties have trivial tangent bundle (and thus trivial cotangent and canonical bundle). This will allow us to remove the condition that $L := \mathcal{O}_A(X)$ be sufficiently ample and replace it by it simply being ample in all but the last step of the proof. For the last step, showing that the multiplication map above is actually surjective, we will use the results we obtained in Chapter 2.

Take the short exact sequence

$$0 \to T_X \to T_A|_X \to N_{X/A} \to 0$$

and dualize it to obtain

$$0 \to N_{X/A}^{\vee} \to \Omega_A^1|_X \to \Omega_X^1 \to 0.$$

Now for every $k \geq 1$, taking the k-th wedge power, we obtain a long exact sequence

$$0 \to S^k N_{X/A} \to \Omega_A^1 \otimes S^{k-1} N_{X/A} \to \dots \Omega_A^{k-1} \otimes N_{X/A} \to \Omega_A^k|_X \to \Omega_X^k \to 0.$$

Using spectral sequences obtained from this long exact sequence, in a more general case for any ambient variety and any codimension Green shows the following two lemmas.

Lemma 3.2.3 ([Gre85, Lemma 1.5]) Let Y be a compact Kähler manifold and $X \subset Y$ a complex submanifold of dimension n. If

$$H^i(X, \Omega_Y^j \otimes S^m N_{X/Y}^{\vee}) = 0 \text{ for all } 1 \leq i < n, 0 \leq j \leq n, 1 \leq m \leq n$$

then

$$H^{p,q}(X) \cong H^q(X, \Omega_Y^p|_X) \text{ if } p + q < n$$

and there is a short exact sequence

$$0 \to \frac{H^{n-p}(X, \Omega_X^p)}{\text{im } H^{d-p}(X, \Omega_Y^{n-1}|_X)} \to \left(\frac{H^0(X, S^p N_{X/Y} \otimes \omega_X^2)}{\text{im } H^0(X, S^{p-1} N_{X/Y} \otimes T_Y \otimes \omega_X^2)} \right)^{\vee}$$
$$\to \ker(H^{n+1-p}(X, \Omega_Y^p|_X) \to H^{n+1-p}(X, \Omega_X^p)) \to 0.$$

Lemma 3.2.4 ([Gre85, Lemma 1.10]) Let Y be a compact Kähler manifold and $X \subset Y$ a complex submanifold of dimension n. If

$$H^i(X, \Omega_Y^j \otimes S^m N_{X/Y}^{\vee} \otimes \omega_X^{-1}) = 0 \text{ for all } 1 \leq i < n, 1 \leq j \leq n, 1 \leq m \leq n-2$$

then

$$H^1(X, T_X) \cong \left(\frac{H^0(X, S^{n-1} N_{X/Y} \otimes \omega_X^2)}{\text{im} H^0(X, S^{n-2} N_{X/Y} \otimes T_Y \otimes \omega_X^2)} \right)^{\vee}.$$

With this we can show the following.

Lemma 3.2.5 Let A be an abelian variety of dimension $n+1$ and let L be an ample line bundle on A. If X is a smooth element of the linear system $|L|$, then

$$H^1(X, T_X) \cong \left(\frac{H^0(X, L^{n-1} \otimes \omega_X^2)}{\text{im } H^0(X, L^{n-2} \otimes \omega_X^2 \otimes \mathcal{O}_X^{\oplus n+1})} \right)^{\vee} \tag{3.2}$$

and there is a short exact sequence

$$0 \to \frac{H^1(X, \Omega_X^{n-1})}{\text{im } H^1\left(X, \mathcal{O}_X^{\oplus\binom{n+1}{2}}\right)} \to \left(\frac{H^0(X, L^{n-1} \otimes \omega_X^2)}{\text{im } H^0(X, L^{n-2} \otimes \omega_X^2 \otimes \mathcal{O}_X^{\oplus n+1})}\right)^\vee$$
$$\to \ker\left(H^2\left(X, \mathcal{O}_X^{\oplus\binom{n+1}{2}}\right) \to H^2(X, \omega_X^{n-1})\right) \to 0. \tag{3.3}$$

Proof: First we observe that due to the canonical bundle of A being trivial, the adjunction formula gives an isomorphism $\omega_X \cong L|_X \cong N_{X/A}$. Using this and once again the fact that the cotangent bundle of A is trivial we see that the conditions of Lemma 3.2.3 and Lemma 3.2.4 are in fact the same. Now consider the restriction sequence

$$0 \to \mathcal{O}_A(-X) \to \mathcal{O}_A \to \mathcal{O}_X \to 0.$$

Tensoring it with L^{m+1}, we obtain a long exact sequence in cohomology

$$\ldots \to H^i(A, L^{m+1}) \to H^i(X, L|_X^{m+1}) \to H^{i+1}(A, L^m) \to \ldots.$$

If L is ample the left and right terms vanish for $i > 0$, meaning the middle term must also be zero. Using Serre duality as well as the isomorphisms above we get

$$\begin{aligned} 0 &= H^i(X, L|_X^{m+1}) \\ &= H^{g-1-i}(X, L^{-m-1} \otimes \omega_X)^\vee \\ &= H^{n-i}(X, S^m N_{X/A}^\vee)^\vee. \end{aligned}$$

Dualizing again we see that the condition of Lemma 3.2.3 and Lemma 3.2.4 is satisfied and thus we obtain the short exact sequence (3.3) as well as the isomorphism (3.2). □

Corollary 3.2.6 Let A be an abelian variety of dimension $n+1$ and let L be an ample line bundle on A. If X is a smooth element of the linear system $|L|$, then for $n \geq 1$ there exists a commutative diagram

$$\begin{array}{ccc} H^0(X, \omega_X) \otimes H^1(X, \Omega_X^{n-1})^\vee & \xrightarrow{(d\mathcal{P}^{n,n})^\vee} & H^1(X, T_X)^\vee \\ \uparrow{\scriptstyle \varphi_1} & & \twoheaduparrow{\scriptstyle \varphi_2} \\ H^0(X, \omega_X) \otimes H^0(X, L^{n-1} \otimes \omega_X) & \xrightarrow{\mu_{X,n}} & H^0(X, L^{n-1} \otimes \omega_X^2). \end{array} \tag{3.4}$$

Proof: The map φ_1 is induced by (3.3) (dualizing the second arrow), φ_2 is induced by (3.2). The commutativity of (3.4) is clear as multiplication commutes with all the differentials of the spectral sequences. □

To prove the infinitesimal Torelli theorem, we need to show that the derivative of the period map is injective or equivalently that its dual is surjective. Using the commutative diagram (3.4) we see that it is enough to show that the multiplication map on the bottom is surjective.

Remark 3.2.7 Due to the isomorphism $L|_X \cong \omega_X$ this multiplication map could be written solely in terms of the canonical bundle of X. Thus if ω_X is normally generated, then the ITT holds for X. Meaning while very ampleness of the canonical bundle is not sufficient for the ITT in higher dimensions, at least in the case of hypersurfaces in abelian varieties normal generation of the canonical bundle is in fact sufficient.

We want to relate this map to a multiplication map in cohomology on A. We proceed analogously to [Gre85, Lemma 1.24], again replacing the condition that L be sufficiently ample.

Lemma 3.2.8 ([Gre85, Lemma 1.24]) Let A be an abelian variety of dimension $n+1$ with $n \geq 1$ and let L be an ample line bundle on A. Then the infinitesimal Torelli theorem holds for any smooth $X \in |L|$ if the multiplication map

$$H^0(A, L) \otimes H^0(A, L^n) \stackrel{\mu_{A,n}}{\to} H^0(A, L^{n+1})$$

is surjective.

Proof: As discussed above, Lemma 3.2.5 shows that the multiplication map on the bottom of diagram (3.4) being surjective implies the ITT. Again we use the restriction sequence from above tensored with L^{n+1} to obtain the long exact sequence

$$\ldots \to H^0(A, L^{n+1}) \stackrel{\tau}{\to} H^0(X, L|_X^{n+1}) \to H^1(A, L^{n+1}) \to \ldots.$$

The last term vanishes for L ample which implies that τ is surjective. Thus it is clear from the commutative diagram

$$\begin{array}{ccc} H^0(X, L|_X) \otimes H^0(X, L|_X^n) & \xrightarrow{\mu_{X,n}} & H^0(X, L|_X^{n+1}) \\ \uparrow & & \tau \uparrow \\ H^0(A, L) \otimes H^0(A, L^n) & \xrightarrow{\mu_{A,n}} & H^0(A, L^{n+1}) \end{array}$$

that surjectivity of $\mu_{A,n}$ implies surjectivity of $\mu_{X,n}$. □

To summarize the results of this section so far we have a commutative diagram

$$\begin{array}{ccc} H^0(X,\omega_X)\otimes H^1(X,\Omega_X^{n-1})^\vee & \xrightarrow{(dP^{n,n})^\vee} & H^1(X,T_X)^\vee \\ \varphi_1\uparrow & & \varphi_2\uparrow \\ H^0(X,L|_X)\otimes H^0(X,L|_X^n) & \xrightarrow{\mu_{X,n}} & H^0(X,L|_X^{n+1}) \\ \uparrow & & \uparrow \\ H^0(A,L)\otimes H^0(A,L^n) & \xrightarrow{\mu_{A,n}} & H^0(A,L^{n+1}). \end{array}$$

Due to the surjectivity of the two vertical maps on the right, surjectivity of $(dP^{n,n})^\vee$, and thus the ITT, follows from the surjectivity of the multiplication map $\mu_{A,n}$.

Using Lemma 3.2.8 as well as the results of Chapter 2 we obtain the ITT as an immediate corollary. Applying Theorem 2.4.4 we obtain the following theorem.

Theorem 3.2.9 *Let X be a hypersurface on a simple g-dimensional abelian variety A. If $h^0(A,\mathcal{O}(X)) > \left(\frac{g}{g-1}\right)^g \cdot g!$, then the infinitesimal Torelli theorem holds for X.*

For the case $g=2$ this gives the condition $h^0(A,L) > 8$ which does not tell us anything new. In fact we already know that a line bundle on an abelian surface with $h^0(A,L) = 7$ (necessarily of type (1,7)) is even projectively normal (see [MS01] or [Ago17]). However, for higher dimensions our result directly improves the bound in [Iye03]. For $g=3$ for example, a line bundle with $h^0(A,L) > 20$ gives a hypersurface for which the infinitesimal Torelli theorem holds whereas Iyer's sufficient condition for projective normality is $h^0(A,L) > 48$.

Corollary 3.2.10 Let $S \subset A$ be a smooth complex projective surface that embeds into its Albanese A as a hypersurface. If S has geometric genus $p_g > 22$ and A is simple then the ITT holds for S.

Proof: Consider the exact sequence

$$0 \to \mathcal{O}_A \to \mathcal{O}_A(S) \to \mathcal{O}_S(S) \to 0.$$

By adjunction we have $\omega_S \cong \mathcal{O}_S(S)$ so taking cohomology and comparing dimensions gives

$$h^0(\mathcal{O}_A(S)) = p_g + 1 - 3 > 20$$

so we can apply Theorem 3.2.9. □

Applying Theorem 2.6.8 instead we obtain the follwing theorem.

Theorem 3.2.11 *Let $X \subset A$ be a smooth hypersurface in a general abelian variety. If $h^0(A,\mathcal{O}_A(X)) \geq \frac{1}{2g!}\left(\frac{4g^2}{g-1}\right)^g$ then the ITT holds for X.*

3.3 Complete intersections in abelian varieties

In this section we will generalize the results of the previous section to the case where X embeds into its Albanese as a complete intersection of smooth ample divisors. Similar to the case of hypersurfaces we will show that the ITT follows from the surjectivity of a multiplication map. In this case it will be a multiplication map of vector bundles. However, decomposing it into a direct sum of multiplication maps of line bundles allows us to apply the results from Chapter 2 here as well.

Let $X = D_1 \cap \ldots \cap D_c \subset A$ be the complete intersection of c ample divisors on a g-dimensional abelian variety A so that X has codimension c. We will also write $n = g - c$ for the dimension of X. In the case where X was a hypersurface we had the identification $\omega_X \cong N_{X/A} \cong \mathcal{O}_X(X)$. Now the canonical and normal bundle are distinct. On the one hand we have the standard resolution for complete intersections

$$\begin{aligned} 0 \to \mathcal{O}_A\left(-\sum_{i=1}^{c} D_i\right) &\overset{\varphi_c}{\to} \ldots \\ \to \bigoplus_{i<j} \mathcal{O}_A(-D_i - D_j) \overset{\varphi_2}{\to} \bigoplus_{i=1}^{c} \mathcal{O}_A(-D_i) &\overset{\varphi_1}{\to} \mathcal{O}_A \to \mathcal{O}_X \to 0 \end{aligned} \tag{3.5}$$

where the map φ_1 is given by multiplication with f_i vanishing on D_i. The image of φ_1 is the ideal sheaf $\mathcal{I}_X$ and since $\varphi_2|_X \equiv 0$ is the zero map, restricting to X yields an isomorphism $\mathcal{I}_X \otimes \mathcal{O}_X \cong \bigoplus_i \mathcal{O}_X(-D_j)$. Finally since $N_{X/A}^\vee \cong \mathcal{I}_X/\mathcal{I}_X^2 \cong \mathcal{I}_X \otimes \mathcal{O}_X$ we obtain

$$N_{X/A} \cong \bigoplus_{i=1}^{c} \mathcal{O}_X(D_i).$$

On the other hand for the canonical bundle the adjunction formula gives us $\omega_X \cong \det(N_{X/A})$ and thus

$$\omega_X \cong \mathcal{O}_X\left(\sum_{i=1}^{c} D_i\right).$$

Lemma 3.3.1 Let $X \subset A$ be a complete intersection of ample divisors in an abelian variety with $\dim(X) = n$. Then there exists a commutative diagram

$$\begin{array}{ccc} H^0(X,\omega_X) \otimes H^1(X,\Omega_X^{n-1})^\vee & \longrightarrow & H^1(X,T_X)^\vee \\ \uparrow & & \uparrow \\ H^0(X,\omega_X) \otimes H^0(X,S^{n-1}N_{X/A} \otimes \omega_X) & \longrightarrow & H^0(X,S^{n-1}N_{X/A} \otimes \omega_X^{\otimes 2}). \end{array}$$

PROOF: We will deduce the diagram from Lemma 3.2.5. In order to do so we need to check that the conditions in Lemma 3.2.3 and Lemma 3.2.4 are satisfied, i.e. we need to check that

$$H^i(X, S^m N_{X/A}^\vee) = 0 \text{ for all } 1 \le i < n, 1 \le m \le n$$

and

$$H^i(X, S^m N_{X/A}^\vee \otimes \omega_X^{-1}) = 0 \text{ for all } 1 \le i < n, 1 \le m \le n-2.$$

Using our identification of the normal bundle above we see that

$$H^i(X, S^m N_{X/A}^\vee) = \bigoplus_{m_1+\ldots+m_r=m} H^i\left(X, \mathcal{O}_X\left(\sum_{i=1}^c -m_i D_i\right)\right)$$

and it is enough to show that the individual summands vanish. Fix one of these summands. Consider the long exact sequence (3.5) and split it into short exact sequences

$$0 \to \mathcal{I}_{k+1} \to \bigoplus_{i_1<\ldots<i_k} \mathcal{O}_A\left(\sum_{j=1}^k -D_{i_j}\right) \to \mathcal{I}_k \to 0$$

where $\mathcal{I}_k \cong \operatorname{im}(\varphi_k) = \ker(\varphi_{k-1})$. In particular we have $\mathcal{I}_1 = \mathcal{I}_X$ and $\mathcal{I}_c = \mathcal{O}_A\left(-\sum_{i=1}^c D_i\right)$. Twisting with $\mathcal{O}_A\left(\sum_{m_1+\ldots+m_c=m} -m_i D_i\right)$ and taking the long exact sequence in cohomology gives an isomorphism

$$H^i\left(A, \mathcal{I}_k\left(\sum_{i=1}^c -m_i D_i\right)\right) \cong H^{i+1}\left(A, \mathcal{I}_{k+1}\left(\sum_{i=1}^c -m_i D_i\right)\right)$$

for $i < g-1$. Applying this successively we obtain

$$H^i\left(X, \mathcal{O}_X\left(\sum_{i=1}^c -m_i D_i\right)\right) \cong H^{i+c}\left(A, \mathcal{O}_A\left(\sum_{i=1}^c -(m_i+1) D_i\right)\right) = 0$$

for $i < g - c = \dim(X)$. Similarly we can write

$$H^i(X, S^m N_{X/A}^\vee \otimes \omega_X) = \bigoplus_{m_1+\ldots+m_c=m} H^i\left(X, \mathcal{O}_X\left(\sum_{i=1}^c -(m_i+1) D_i\right)\right)$$

and use the same argument to show the vanishing of the summands. This allows us to apply Lemma 3.2.5 to obtain the desired diagram. □

Again we complete the diagram by a multiplication map of sections on A.

Lemma 3.3.2 Let $m_1, \dots, m_r \geq 1$. The restriction map

$$H^0\left(A, \mathcal{O}_A\left(\sum_{i=1}^{c}(m_i D_i)\right)\right) \to H^0\left(X, \mathcal{O}_X\left(\sum_{i=1}^{c}(m_i D_i)\right)\right)$$

is surjective.

Proof: Using the sequence of the complete intersection and splitting it into short exact sequences we again obtain an isomorphism $H^1(X, \mathcal{I}_X(\sum m_i D_i)) \cong H^{1+r}(A, \mathcal{O}_A(\sum(m_i - 1)D_i)) = 0$. □

In the following, given a vector $\mathbf{m} = (m_1, \dots, m_c)$, denote the line bundle $\mathcal{O}_A(\sum_{i=1}^{c} m_i D_i)$ by $L_{\mathbf{m}}$. Note that using this notation we have $\omega_X \cong L_{\mathbf{1}}|_X$ where $\mathbf{1} = (1, \dots, 1)$. Using the above we obtain

$$\begin{array}{ccc}
H^0(X, \omega_X) \otimes H^1(X, \Omega^1_X)^\vee & \longrightarrow & H^1(X, T_X)^\vee \\
\uparrow & & \twoheaduparrow \\
H^0(X, \omega_X) \otimes H^0(X, S^{d-1}N_{X/A} \otimes \omega_X) & \longrightarrow & H^0(X, S^{d-1}N_{X/A} \otimes \omega_X^{\otimes 2}) \\
\uparrow & & \twoheaduparrow \\
\bigoplus H^0(A, L_{\mathbf{1}}) \otimes H^0(A, L_{\mathbf{d}} \otimes L_{\mathbf{1}}) & \longrightarrow & \bigoplus H^0(A, L_{\mathbf{d}} \otimes L_{\mathbf{1}} \otimes L_{\mathbf{1}}).
\end{array} \tag{3.6}$$

Once again the ITT for X follows from the surjectivity of the map on the bottom which in the present case is a direct sum of multiplication maps. We have only studied multiplication maps of sections of a line bundle and some power of it. Since many of the maps occurring are not of this type we need to be able to relate them to those multiplication maps we understand. In particular we show that tensoring one of the factors by an ample line bundle preserves surjectivity. We first need some lemmas. Recall that for line bundles L and M on A we denote the multiplication map

$$H^0(A, L) \otimes H^0(A, M) \to H^0(A, L \otimes M)$$

by $\mu_{L,M}$ and that we denote the image of this map by $H^0(A, L).H^0(A, M)$.

Lemma 3.3.3 Let L and M be ample line bundles on an abelian variety A. For every nonempty open subset $U \subset \operatorname{Pic}^0(A)$ we have

$$\sum_{\alpha \in U} H^0(A, L \otimes \alpha).H^0(A, M \otimes \alpha^{-1}) = H^0(A, L \otimes M).$$

Proof: [BL04, Lemma 7.3.3.] □

Lemma 3.3.4 Let L and M be ample line bundles on an abelian variety A. If the multiplication map

$$H^0(A,L) \otimes H^0(A,M) \to H^0(A, L \otimes M)$$

is surjective, then for α in some nonempty open subset $U \subset \mathrm{Pic}^0(A)$ the multiplication map

$$H^0(A,L) \otimes H^0(A, M \otimes \alpha) \to H^0(A, L \otimes M \otimes \alpha)$$

is surjective as well.

Proof: Consider the product $A \times \widehat{A}$ and the projection maps $p_A \colon A \times \widehat{A} \to A$ and $p_{\widehat{A}} \colon A \times \widehat{A} \to \widehat{A}$. Let $\mathcal{F}_1 = p_{\widehat{A}*} p_A^* L$ and $\mathcal{F}_2 = p_{\widehat{A}*}(p_A^* M \otimes \mathcal{P})$ where $\mathcal{P}$ denotes the Poincaré bundle on $A \times \widehat{A}$. The fibers $\mathcal{F}_1(\alpha) = H^0(A, L)$ and $\mathcal{F}_2(\alpha) = H^0(A, M \otimes \alpha)$ have constant dimension for all $\alpha \in \widehat{A}$ and therefore $\mathcal{F}_1$ and $\mathcal{F}_2$ are vector bundles. Consider the map

$$\mu \colon \mathcal{F}_1 \otimes \mathcal{F}_2 \to p_{\widehat{A}*}(p_A^*(L \otimes M) \otimes \mathcal{P}).$$

The induced map on the zero fiber is

$$\mu(0) \colon H^0(A,L) \otimes H^0(A,M) \to H^0(A, L \otimes M)$$

which by assumption is surjective. By semi-continuity the map $\mu(\alpha)$ is thus also surjective α in some open subset of $\widehat{A}$. □

With the above lemma we can show the following.

Proposition 3.3.5 Let L, M and N be ample line bundles on an abelian variety A. If $\mu_{L,M}$ is surjective then $\mu_{L,M \otimes N}$ is surjective as well.

Proof: By Lemma 3.3.3 we can write

$$H^0(L).H^0(M \otimes N) = \sum_{\alpha \in U} H^0(L).H^0(M \otimes \alpha).H^0(N \otimes \alpha^{-1})$$

for any non-empty open subset $U \subset \mathrm{Pic}^0(A)$. Now since $\mu_{L,M}$ is surjective if we have chosen U such that Lemma 3.3.4 can be applied we can write

$$H^0(L).H^0(M \otimes \alpha) = H^0(L \otimes M \otimes \alpha).$$

Putting these things together we obtain

$$H^0(L).H^0(M \otimes N) = \sum_{\alpha \in U} H^0(L \otimes M \otimes \alpha).H^0(N \otimes \alpha^{-1})$$

and finally appliying Lemma 3.3.3 again we obtain

$$H^0(L).H^0(M \otimes N) = H^0(L \otimes M \otimes N).$$

□

Theorem 3.3.6 *Let $X = D_1 \cap \ldots \cap D_c$ be a complete intersection of ample divisors D_i on a g-dimensional abelian variety A, let $L_i = \mathcal{O}_A(D_i)$ denote the corresponding line bundles, let $L = L_{\mathbf{1}}$ and let $m = \lceil \frac{g-1}{c} \rceil$. The ITT holds for X if one of the following holds:*

(i) $\mu_{L,L}$ is surjective, i.e. L is normally generated.

(ii) The multiplication maps $\mu_{L_i^m, L_i}$ are surjective for all $i \in \{1, \ldots, c\}$.

Proof: From the previous discussion it suffices to show the surjectivity of the multiplication maps on the bottom of diagram 3.6. If we fix one of these maps there are different ways to decompose the line bundles that occur. For (i) the map is simply $\mu_{L \otimes L_{\mathbf{d}}, L}$. By Proposition 3.3.5 this map is surjective if $\mu_{L,L}$ is, i.e. if L is normally generated.

For (ii) we find the largest occurring coefficient on the left hand side, say this is $d_j + 1$. Now the map can be written as $\mu_{L_j^{d_j+1} \otimes M, L_j \otimes N}$ where M and N are the appropriate tensor products of the $\mathcal{O}_A(D_i)$. We can again apply Proposition 3.3.5 to see that this is surjective if $\mu_{L_j^{d_j+1}, L_i}$ surjective. Now since every possible $d+1$-th power occurs, there will be multiple summands such that D_i has the largest coefficient. Since surjectivity of μ_n implies surjectivity of μ_m for all $m \geq n$ we just need to consider the smallest value the largest coefficient d_i can take. Since the sum of the d_i is $g - r - 1$ we have that the sum of all the coefficients on the left hand side ist $g - 1$. Therefore, the smallest value occurring for the largest coefficient is $\lceil \frac{g-1}{c} \rceil$. □

Finally using what we learned about multiplication maps in Chapter 2 we obtain the following theorems as immediate corollaries.

Theorem 3.3.7 *Let $X = D_1 \cap \ldots \cap D_c$ be a complete intersection of ample divisors D_i on a g-dimensional simple abelian variety A. Then the ITT holds for X if one of the following holds:*

(i) $h^0(A, \mathcal{O}_A(\sum_{i=1}^c D_i)) > 2^g g!$;

(ii) $h^0(A, \mathcal{O}_A(D_i)) > \left(1 + \frac{1}{\lceil \frac{g-1}{c} \rceil}\right)^g g!$ for all $i \in \{1, \ldots, c\}$.

Proof: Just as above the first condition is Iyer's result applied to $\mathcal{O}_A(\sum D_i)$ and the second condition is Theorem 2.4.4 applied to $\mathcal{O}_A(D_i)$ for all $\{1, \ldots, c\}$. □

For example for a surface $S = D_1 \cap D_2 \subset A$ that is a complete intersection of smooth ample divisors D_1 and D_2 in a simple abelian fourfold A, the ITT holds if $h^0(A, D_i) > 121$ for $i = 1, 2$.

Theorem 3.3.8 *Let $X = D_1 \cap \ldots \cap D_c$ be a complete intersection of general ample divisors D_i on a general g-dimensional abelian variety A. Then the ITT holds for X if one of the following holds:*

(i) $h^0(A, \mathcal{O}_A(\sum_{i=1}^c D_i)) > \frac{1}{2g!}(8g)^g$;

(ii) $h^0(A, \mathcal{O}_A(D_i) > \frac{1}{2g!}\left(\frac{4g(\lceil\frac{g-1}{c}\rceil+1)}{\lceil\frac{g-1}{c}\rceil}\right)^g g!$ *for all* $i \in \{1, \dots, c\}$.

PROOF: The first condition is Hwang's and To's result applied to the line bundle $\mathcal{O}_A(\sum D_i)$ and the second condition is Theorem 2.6.8 applied to the line bundles $\mathcal{O}_A(D_i)$ for all $\{1, \dots, c\}$. □

3.4 Irregular varieties with globally generated cotangent bundle

In this section we will investigate what happens for irregular varieties which do not embed into their Albanese variety. Most of Section 3.2 carries over as long as we assume the cotangent bundle to be globally generated.

Let X be a smooth projective irregular variety of dimension n. Assume that X is of maximal Albanese dimension, in particular that means that the irregularity q is at least n. The differential $da\colon T_X \to a^*T_A$ is an injective morphism of sheaves but not necessarily of vector bundles. Denote its cokernel by $N_a \coloneqq (a^*T_A)/T_X$. Now there is still an exact sequence

$$0 \to T_X \xrightarrow{da} H^0(X, \Omega_X^1)^\vee \otimes \mathcal{O}_X \to N_a \to 0.$$

Since N_a is not necessarily a vector bundle, dualizing this sequence yields

$$0 \to N_a^\vee \to H^0(X, \Omega_X^1) \otimes \mathcal{O}_X \to \Omega_X^1 \to \mathcal{E}xt^1_{\mathcal{O}_X}(N_a^\vee, \mathcal{O}_X) \to 0. \tag{3.7}$$

Note that $T_A = H^0(X, \Omega_X^1)^\vee \otimes \mathcal{O}_A$. The dual of da is the evaluation map

$$\mathrm{ev}\colon H^0(X, \Omega_X^1) \otimes \mathcal{O}_X \to \Omega_X^1$$

so da is everywhere pointwise injective (and hence N_a is locally free) if and only if Ω_X^1 is globally generated.

In the following we always assume that Ω_X^1 is globally generated so that (3.7) becomes the short exact sequence

$$0 \to N_a^\vee \to H^0(X, \Omega_X^1) \otimes \mathcal{O}_X \to \Omega_X^1 \to 0.$$

We proceed exactly like in [Gre85]. From the above short exact sequence we obtain for every $k \geq 1$ a long exact sequence

$$0 \to S^k N_a^\vee \to \Omega_A^1 \otimes S^{k-1} N_a^\vee \to \cdots \to \Omega_A^{k-1} \otimes N_a^\vee \to \Omega_A^k \otimes \mathcal{O}_X \to \Omega_X^k \to 0.$$

Using this we can prove analogous lemmas as in the case that X embeds as a subvariety into A. The proofs are essentially the same as in [Gre85] but replacing the normal bundle $N_{X/A}$ by N_a.

Lemma 3.4.1 Let X be a smooth n-dimensional projective variety of maximal Albanese dimension and irregularity $q(X) \geq n+1$ with Ω^1_X is globally generated and let $a\colon X \to A = \mathrm{Alb}(A)$ be its Albanese morphism. If $H^i(X, S^m N_a^\vee) = 0$ for $0 \leq i < n$ and $1 \leq m \leq n$ then for any $k \geq 1$ there is a short exact sequence

$$\begin{aligned} 0 \to & \frac{H^0(X, \Omega^p_X)}{\mathrm{im}(H^{n-p}(X, \Omega^p_A \otimes \mathcal{O}_X))} \\ & \to \left(\frac{H^0(X, S^p N_a \otimes \omega_X)}{\mathrm{im}(H^0(X, S^{p-1} N_a \otimes T_X \otimes \omega_X))} \right)^\vee \\ & \to \ker(H^{n+1-p}(X, \Omega^p_A \otimes \mathcal{O}_X) \to H^{n+1-p}(X, \Omega^p_X)) \to 0. \end{aligned}$$

Lemma 3.4.2 Let X be a smooth n-dimensional projective variety of maximal Albanese dimension and of irregularity $q(X) \geq n+1$ and let $a\colon X \to A = \mathrm{Alb}(A)$ be its Albanese morphism. If $H^i(X, S^m N_a^\vee \otimes \omega_X^\vee) = 0$ for $0 \leq i < n$ and $1 \leq m \leq n-2$ then

$$H^1(X, T_X)^\vee \cong \left(\frac{H^0(X, S^{n-1} N_a^\vee \otimes \omega_X^2)}{\mathrm{im} H^0(X, S^{n-2} N_a^\vee \otimes T_A \otimes \omega_X^2)} \right)^\vee.$$

As before, in order to ensure that the vanishing conditions in the previous two lemmas hold, we will restrict to the case where $q = n+1$ so that X maps to a divisor in A and N_a is of rank 1 i.e. a line bundle. From the adjunction formula we have $N_a \cong \omega_X$.

Lemma 3.4.3 Let X be a smooth n-dimensional projective variety of maximal Albanese dimension and of irregularity $q(X) = n+1$ with Ω^1_X globally generated and ω_X ample, let $a\colon X \to A = \mathrm{Alb}(X)$ be its Albanese morphism. We have an isomorphism

$$H^1(X, T_X)^\vee \cong \left(\frac{H^0(X, N_a^{n-1} \otimes \omega_X^2)}{\mathrm{im} H^0(X, N_a^{n-2} \otimes T_A \otimes \omega_X^2)} \right)^\vee \tag{3.8}$$

and there is a short exact sequence

$$\begin{aligned} 0 \to \left(\frac{H^1(X, \Omega^{n-1}_X)}{\mathrm{im} H^1(X, \Omega^{n-1}_A \otimes \mathcal{O}_X)} \right) \to \left(\frac{H^0(X, N_a^{n-1} \otimes \omega_X)}{\mathrm{im} H^0(X, N_a^{n-2} \otimes T_A \otimes \omega_X^2)} \right)^\vee \\ \to (\ker(H^2(X, \Omega^{n-1}_A \otimes \mathcal{O}_X)) \to H^2(X, \Omega^{n-1}_X)) \to 0. \end{aligned}$$

Proof: We want to apply the two previous lemmas so we need to make sure that their hypotheses are satisfied, i.e. we need $H^i(X, N_a^{-m}) = 0$ for $0 \leq i < n$ and $1 \leq m \leq n$. Since $\omega_X \cong N_a$ is ample this follows from Kodaira vanishing. $\square$

As before we immediately obtain the following corollary.

Corollary 3.4.4 Let X be a smooth n-dimensional projective variety of maximal Albanese dimension and of irregularity $q(X) = n + 1$ with ω_X ample, let $a\colon X \to A = \mathrm{Alb}(A)$ be its Albanese morphism. Then there is a commutative diagram

$$\begin{array}{ccc} H^0(X,\omega_X) \otimes H^1(X,\Omega_X^{n-1})^\vee & \longrightarrow & H^1(X,T_X)^\vee \\ \uparrow & & \uparrow \\ H^0(X,\omega_X) \otimes H^0(X,N_a^{n-1} \otimes \omega_X) & \longrightarrow & H^0(X,N_a^{n-1} \otimes \omega_X^2). \end{array}$$

We would like to complete this commutative diagram in the same way as before. At this point we should investigate what N_a actually looks like. In the case that X embeds into A it was of course isomorphic to $a^*\mathcal{O}_A(X)$ so we were able to pass from a multiplication map on cohomology of X involving N_a to a multiplication map on cohomology of A involving $\mathcal{O}_A(X)$. Now since the image of X under the Albanese map is singular the situation is slightly more complicated. In the following we always assume that the Albanese morphism is birational onto its image.

Lemma 3.4.5 Let X be a smooth n-dimensional projective variety of irregularity $q(X) = n+1$ with Ω_X^1 globally generated, let $a\colon X \to A = \mathrm{Alb}(A)$ be its Albanese morphism so that $D := a(X)$ is a divisor in A with singular locus $\mathrm{Sing(D)} = \Sigma$. Then there exists a divisor H on X such that $\mathcal{O}_X(-H) = \mathcal{I}_\Sigma\mathcal{O}_X$ where $\mathcal{I}_\Sigma\mathcal{O}_X$ denotes the image of the map $a^*\mathcal{I}_\Sigma \to \mathcal{O}_X$ and the normal bundle N_a is isomorphic to $a^*\mathcal{O}_A(D) \otimes \mathcal{O}_X(-H)$.

Proof: Unlike in the case that X is embedded in A, $a^*\mathcal{O}_A(-D)$ is now only a subsheaf of the conormal sheaf $N_a^\vee$. We want to study how they differ. The vector bundles $a^*\Omega_A^1 = H^0(X,\Omega_X^1) \otimes \mathcal{O}_X$ and Ω_X^1 are of rank g and $g-1$ respectively so $N_a^\vee$ is of rank 1. Given some point $x \in X$ let V be a neighborhood of $a(x)$ in A and let $U \subseteq a^{-1}(V)$ be a neighborhood of x. Let $x_1, \ldots, x_{g-1}$ and $y_1, \ldots, y_g$ be local coordinates around x and and $a(x)$, and let $y_i = a_i(x_1, ..., x_{g-1})$ be the local expressions of a. We can then write $\Omega_A^1|_V = \mathcal{O}_V\langle dy_1, \ldots, dy_g\rangle$ and $\Omega_X^1|_U = \mathcal{O}_U\langle dx_1, \ldots, dx_{g-1}\rangle$. The evaluation map is then given by the Jacobian matrix, sending dy_i to $da_i = \sum_{j=1}^{g-1} \frac{\partial a_i}{\partial x_j} dx_j$. Now the image of a generator of $N_a^\vee|_U$ generates the kernel of the evaluation map. Write it as a linear combination $\sum_{i=1}^g g_i dy_i$. Because Ω_X^1 is globally generated the g_i cannot all vanish simultaneously at any point. On the other hand if the defining equation of the divisor D is given by $f = 0$ then we know that $\sum_{i=1}^g a^* \frac{\partial f}{\partial y_i} dy_i$ lies in the kernel of the evaluation map so it must be a multiple of $\sum_{i=1}^g g_i dy_i$. Let $h(x) = \gcd\left(\frac{\partial f}{\partial y_i}(a(x))\right)$, then $g_i(x) = \frac{\frac{\partial f}{\partial y_i}(a(x))}{h(x)}$. The divisor $H := \{h = 0\}$ is supported exactly on the points of X that map to the singular locus Σ of D and the map $a^*\mathcal{O}_A(-D) \to N_A^\vee$ is now simply multiplication by h and thus $N_a^\vee$ is isomorphic to $a^*\mathcal{O}_A(-D) \otimes \mathcal{O}_X(H)$. Dualizing yields $N_a \cong a^*\mathcal{O}_A(D) \otimes \mathcal{O}_X(-H)$. Finally, $\mathcal{O}_X(-H) = \mathcal{I}_\Sigma \mathcal{O}_X$ follows from the fact that the g_i do not have any common zeros and thus the ideal generated by the pullbacks along a of the partial derivatives of f is equal to that generated by the greatest common divisor h. □

Using the projection formula, for every $m \geq 1$ we obtain

$$a_* N_a^m = \mathcal{O}_A(mD) \otimes a_*\mathcal{O}_X(-mH).$$

We also have a map

$$a^*\mathcal{I}_\Sigma \twoheadrightarrow \mathcal{O}_X(-H).$$

Pushing forward and using the natural map $\mathcal{I}_\Sigma \to a_*a^*\mathcal{I}_\Sigma$ we obtain

$$\begin{array}{ccc} \mathcal{I}_\Sigma & \longrightarrow & a_*\mathcal{O}_X(-H) \\ & \searrow \quad \nearrow & \\ & a_*a^*\mathcal{I}_\Sigma & \end{array}$$

which for every $m \in \mathbb{N}$ induces a map

$$\mathcal{I}_\Sigma^m \to a_*\mathcal{O}_X(-mH).$$

Lemma 3.4.6 If D is a normal crossing divisor the map $\mathcal{I}_\Sigma^m \to a_*\mathcal{O}_X(-mH)$ is surjective.

PROOF: Let $p \in A$ and let $V \subset A$ be a neighborhood of p. There exist local coordinates $y_1, \dots, y_g$ in V such that we can write D as the zero locus of $f = y_1 \cdots y_r$. For each $j \in \{1, \dots r\}$ we have a local component $y_j = 0$ for which we can consider the preimage in X. Let U_j denote this preimage. On each local component we can pull back our coordinates. For a fixed j write $x_i := a^*y_i$. On U_j we have $x_j = 0$. For the pullbacks of the partial derivatives we get $a^*\frac{\partial f}{\partial y_i} = x_1 \cdots \widehat{x_i} \cdots x_r$ so the only one that does not vanish is $a^*\frac{\partial f}{\partial y_j} = x_1 \cdots \widehat{x_j} \cdots x_r$. Thus the local equation of H in the j-th component is $x_1 \cdots \widehat{x_j} \cdots x_r$. A section s of $a_*\mathcal{O}_X(-mH)$ on V is given by a tuple $(s_1, \dots, s_r)$ where each s_i is a section of $\mathcal{O}_X(-mH)$ on U_i so we can write $s_i(x) = \alpha_i(x) \cdot (x_1, \dots \widehat{x_i} \cdots x_r)^m$. We need to find a section b of $\mathcal{I}_\Sigma^m$ that maps to s. Such a section can be written as a linear combination of powers of the partial derivatives

$$b(y) = \sum_{e_1 + \ldots + e_r = m} \beta_{e_1,\ldots,e_r}(y) \prod_{i=1}^{r} (y_1 \cdots \widehat{y_i} \cdots y_r)^{e_i}.$$

In the component $y_j = 0$ all the terms except for the m-th power of the derivative with respect to y_j vanish so we are left with

$$b(y) = \beta_j(y) \cdot (y_1 \cdots \widehat{y_j} \cdots y_r)^m.$$

Pulling back we get $a^*b(x) = \beta_j(a(x)) \cdot (x_1, \dots \widehat{x_j} \cdots x_r)^m$. So if we choose β_j such that $a^*\beta_j = \alpha_j$ for all j, e.g. setting $\beta_j(y_1, \dots, y_g) = \alpha(y_1, \dots \widehat{y_j}, \dots y_g)$, b maps to s. □

Lemma 3.4.7 If D is a normal crossing divisor with at most double points, the kernel of the map $\mathcal{I}_\Sigma^m \to a_*\mathcal{O}_X(-H)$ is given by $\mathcal{O}_A(-D) \otimes \mathcal{I}_\Sigma^{m-2}$.

Proof: Let b be a section of $\mathcal{I}_\Sigma^m$ as in the previous lemma. As above it can be written as a linear combination

$$b(y) = \sum_{e_1+e_2=m} \beta_{e_1,e_2}(y) y_1^{e_1} y_2^{e_2}.$$

Suppose b is an element in the kernel. That means b is a multiple of y_1y_2. For the pure powers of the partial derivatives this only means that the coefficient needs to be a multiple of the missing coordinate but does not give an additional condition. However, for a mixed term of the form $y_1^{e_1}y_2^{e_2}$ such that the exponents e_i add up to m factoring out y_1y_2 means removing two partial derivatives and thus the powers of the remaining terms sum up to $m-2$. □

This gives us a short exact sequence

$$0 \to \mathcal{O}_A(-D) \otimes \mathcal{I}_\Sigma^{m-2} \to \mathcal{I}_\Sigma^m \to a_*\mathcal{O}_X(-mH) \to 0.$$

Tensoring with $\mathcal{O}_A(mD)$ gives

$$0 \to \mathcal{O}_A((m-1)D) \otimes \mathcal{I}_\Sigma^{m-2} \to \mathcal{I}_\Sigma^m \otimes \mathcal{O}_A(mD) \to a_*N_a \to 0$$

which induces a long exact sequence in cohomology

$$H^0(A, \mathcal{O}_A(mD) \otimes \mathcal{I}_\Sigma^m) \to H^0(X, N_a) \to H^1(A, \mathcal{O}_A((m-1)D) \otimes \mathcal{I}_\Sigma^{m-2}). \quad (3.9)$$

Lemma 3.4.8 If D is a normal crossing divisor on an abelian variety A that is locally the intersection of two components so that the singular locus $\Sigma := \mathrm{Sing}(D)$ is given by the intersection of the two components and thus itself smooth, and if $\varepsilon(A, D, \mathcal{I}_\Sigma) > 1$ then the pullback map

$$H^0(A, \mathcal{O}_A(mD) \otimes \mathcal{I}_\Sigma^m) \to H^0(X, N_a^m)$$

is surjective.

PROOF: By the sequence (3.9) obtained from Lemma 3.4.6 and Lemma 3.4.7 it suffices to show the vanishing of $H^1(A, \mathcal{O}_A((m-1)D) \otimes \mathcal{I}_\Sigma^{m-2})$. Since by assumption Σ is smooth and $\varepsilon(A, D, \mathcal{I}_\Sigma) > 1$, this follows from Lemma 2.7.4. □

Corollary 3.4.9 Let A be a simple abelian surface and let D be a normal crossing divisor, then the pullback map

$$H^0(A, \mathcal{I}_\Sigma^m \otimes \mathcal{O}_A(mD)) \to H^0(X, N_a)$$

is surjective.

PROOF: By the previous lemma we just need to check that $\varepsilon(A, D, \mathcal{I}_\Sigma) > 1$. In this case $\varepsilon(A, D, \mathcal{I}_\Sigma)$ is the regular Seshadri constant at a point. It is well-known that the Seshadri constant of a line bundle at a point on an abelian variety is the same for all points and greater than or equal to 1 (see e.g. [BL04, Proposition 15.4.1]). If A is furthermore simple then the inequality is strict (see e.g. [BL04, Theorem 15.4.2]). □

Thus under the assumption that $\varepsilon(A, D, \mathcal{I}_\Sigma) > 1$ or that A is a simple abelian surface we can complete the diagram like this:

$$\begin{array}{ccc}
H^0(X, N_a) \otimes H^1(X, \Omega_X^{n-1})^\vee & \longrightarrow & H^1(X, T_X)^\vee \\
\uparrow & & \uparrow\!\!\uparrow \\
H^0(X, N_a) \otimes H^0(X, N_a^n) & \longrightarrow & H^0(X, N_a^{n+1}) \\
\uparrow\!\!\uparrow & & \uparrow \\
H^0(A, \mathcal{O}_A(D) \otimes \mathcal{I}_\Sigma) \otimes H^0(A, \mathcal{O}_A(nD) \otimes \mathcal{I}_\Sigma^n) & \longrightarrow & H^0(A, \mathcal{O}_A((n+1)D) \otimes \mathcal{I}_\Sigma^{n+1}).
\end{array}$$

Now we have again reduced the ITT for X to the surjectivity of a multiplication map on its Albanese variety. However, now we are not only considering line bundles anymore but also ideal sheaves. It seems unlikely that the methods of Section 2.4 can be adapted to this situation. Instead we try to apply Nadel vanishing again. The problem we encounter there is that $\mathcal{O}_A(dD) \otimes \mathcal{I}_\Sigma^m$ is not locally free and thus tensoring a short exact sequence with it is not necessarily left-exact. Once again we blow up Σ in order to turn the ideal sheaf associated to it into a line bundle. Let $\mu\colon A' \to A$ denote the blowup, let $E = \mu^{-1}(\Sigma)$ denote the exceptional divisor and let $\widetilde{D}$ denote the total transform of D. Again under the assumptions from Lemma 3.4.6 and Lemma 3.4.7, for any $m \geq 1$ we have $H^0(A', \mathcal{O}_{A'}(m(\widetilde{D} - E))) \cong H^0(A, \mu_*\mathcal{O}_{A'}(m(\widetilde{D} - E)))$ and since by the projection formula we have

$$\mu_*(\mu^*\mathcal{O}_A(mD) \otimes \mathcal{O}_{A'}(-mE)) \cong \mathcal{O}_A(mD) \otimes \mu_*\mathcal{O}_{A'}(-mE) \cong \mathcal{O}_A(mD) \otimes \mathcal{I}_\Sigma^m$$

and thus we have an isomorphism

$$H^0(A, \mathcal{O}_A(mD) \otimes \mathcal{I}_\Sigma^m) \cong H^0(A', \mathcal{O}_{A'}(m(\widetilde{D} - E))).$$

In fact if D' is the strict transform of D under the blowup we have $\widetilde{D} = \mu^* D = D' + 2E$ because D has two components which meet in the singular locus Σ. Thus $\mathcal{O}_{A'}(m(\widetilde{D} - E)) = \mathcal{O}_{A'}(m(D' + E))$ and we need to study the surjectivity of the multiplication map

$$H^0(A', L) \otimes H^0(A', L^n) \to H^0(A', L^{n+1})$$

where A' is the blowup of a smooth codimension 2 subvariety of an abelian variety and L is an ample line bundle.

Studying such multiplication maps is significantly more difficult than in the case of abelian varieties. We can use Lemma 2.7.4 to give a numerical condition on the ampleness of L in terms of Seshadri constant but unlike in the case of abelian varieties we have no way to express these in terms of the dimension of the space of global sections.

4 Conclusion and outlook

We have proved the ITT in some cases. In Section 3.2 and Section 3.3 we have given a sufficient numerical condition for it to hold for smooth hypersurfaces and more generally complete intersections in abelian varieties by reducing the ITT to the surjectivity of the multiplication map of a line bundle $L = \mathcal{O}_A(X)$ on the ambient abelian variety. Theorem 2.4.4 requires the ambient abelian variety to be simple and Theorem 2.6.8 requires it to be general. A question that remains is whether one can derive a numerical condition for the surjectivity of the multiplication map for any abelian variety. In Section 2.5 we pointed out that a numerical condition on the number of sections of L alone cannot be sufficient and that it seems likely that one would need numerical conditions on the restriction of L to any abelian subvariety as well.

Question A Given a non-simple abelian variety A of dimension g and an ample line bundle L on A. For a given $k \in \mathbb{N}$, can we give numerical conditions on $L|_B$ for any abelian subvariety $B \subset A$ that ensure that the multiplication map μ_k is surjective? More specifically, can we give numerical conditions that ensure that μ_{g-1} is surjective, thus proving the ITT for any smooth section in $|L|$?

As discussed in Section 2.7 a promising approach to prove such a theorem is to construct a suitable $\mathbb{Q}$-divisor on the product variety $A \times A$ and use Nadel vanishing as in [LPP11]. Indeed in [Loz18] Lozovanu proves a theorem in this spirit for abelian threefolds and it seems likely that similar methods will yield a theorem in arbitrary dimension. One caveat is that in order to construct a suitable $\mathbb{Q}$-divisor on $A \times A$ one first constructs a suitable $\mathbb{Q}$-divisor on A and then pulls it back via the difference map. While doing so the presence of a primitive factor in the multiplication map μ_k prevents us from improving the bounds compared to showing surjectivity of μ_1. Thus in order to prove the ITT for a smooth hypersurface section in any abelian variety using this approach it seems one would actually need to show projective normality.

In Section 3.4 we extended the methods from the case of hypersurfaces and complete intersections in abelian varieties to the case of irregular varieties of maximal Albanese dimension with globally generated cotangent bundle and ample canonical bundle. In this case the Albanese morphism is still injective on tangent spaces but not necessarily an embedding. We showed that if the image under the Albanese morphism is a mildly singular divisor, the ITT still follows from the surjectivity of a multiplication map on the Albanese variety involving ideal sheaves.

Question B Given an abelian variety A and a mildly singular divisor D on A with singular locus $\mathrm{Sing}(D) = \Sigma$. For a given $k \in \mathbb{N}$, can we give numerical conditions that ensure that the multiplication map

$$H^0(A, \mathcal{O}_A(D) \otimes \mathcal{I}_\Sigma) \otimes H^0(A, \mathcal{O}_A(kD) \otimes \mathcal{I}_\Sigma^k) \to H^0(A, \mathcal{O}_A((k+1)D) \otimes \mathcal{I}_\Sigma^{k+1})$$

is surjective?

One way to deal with the presence of ideal sheaves is to consider the blowup along them.

Question C Let A be an abelian variety and let $\mu \colon A' \to A$ be the blowup of a smooth codimension 2 subvariety of A. Given a line bundle L on A' can we give numerical conditions ensuring the surjectivity of the multiplication map

$$H^0(A', L) \otimes H^0(A', L^k) \to H^0(A', L^{k+1})$$

for a given $k \in \mathbb{N}$?

Recall that as an intermediate step to prove the ITT we reduced it to showing the surjectivity of the multiplication of the canonical bundle of the variety itself before pulling it back to a multiplication map on its Albanese variety. In the case of hypersurfaces and complete intersections in abelian varieties this was very helpful as multiplication maps on abelian varieties are much better understood than on irregular varieties. It allowed us to prove theorems even for line bundles which are not powers of other line bundles. For irregular varieties which do not embed into their Albanese variety we either have to deal with multiplication maps involving ideal sheaves or we blow up, in wich case the ambient variety is no longer abelian. Thus showing surjectivity is much more complicated and it may be a better strategy to work with the multiplication map of the canonical bundle on X directly.

Question D Given a smooth projective n-dimensional variety X of maximal Albanese dimension and of irregularity $q(X) = n + 1$ with Ω^1_X globally generated and ω_X ample. For a given $k \in \mathbb{N}$, when is the multiplication map

$$H^0(X, \omega_X) \otimes H^0(X, \omega_X^k) \to H^0(X, \omega_X^{k+1})$$

surjective?

While projective normality of pluricanonical series has in fact been studied (see e.g. [MR19]), it is unclear whether the methods used can be applied when only one factor of the multiplication map is a higher power of the canonical bundle.

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