Electron. Commun. Probab. **18** (2013), no. 24, 1–13.
DOI: 10.1214/ECP.v18-2392
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

The Yamada-Watanabe theorem for mild solutions to stochastic partial differential equations

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Abstract

We prove the Yamada-Watanabe Theorem for semilinear stochastic partial differential equations with path-dependent coefficients. The so-called "method of the moving frame" allows us to reduce the proof to the Yamada-Watanabe Theorem for stochastic differential equations in infinite dimensions.

Keywords: stochastic partial differential equation ; mild solution ; martingale solution ; pathwise uniqueness.
AMS MSC 2010: 60H15 ; 60H10.
Submitted to ECP on October 24, 2012, final version accepted on March 26, 2013.

1 Introduction

The goal of the present paper is to establish the Yamada-Watanabe Theorem – which originates from the paper [17] – for mild solutions to semilinear stochastic partial differential equations (SPDEs)

$$dX(t) = (AX(t) + \alpha(t, X))dt + \sigma(t, X)dW(t) \tag{1.1}$$

in the spirit of [2, 12, 6] with path-dependent coefficients. More precisely, denoting by H the state space of (1.1), we will prove the following result (see, e.g. [9] for the finite dimensional case):

Theorem 1.1. *The SPDE (1.1) has a unique mild solution if and only if both of the following two conditions are satisfied:*

1. *For each probability measure μ on $(H, \mathcal{B}(H))$ there exists a martingale solution (X, W) to (1.1) such that μ is the distribution of $X(0)$.*
2. *Pathwise uniqueness for (1.1) holds.*

The precise conditions on A, α and σ, under which Theorem 1.1 holds true, are stated in Assumptions 2.2 and 3.1 below. So far, the following two versions of the Yamada-Watanabe Theorem in infinite dimensions are known in the literature:

- For SPDEs of the type (1.1) with state-dependent coefficients $\alpha(t, X(t))$ and $\sigma(t, X(t))$; see [11].
- For stochastic evolution equations in the framework of the variational approach; see [13].

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We will divide the proof of Theorem 1.1 into two steps:

1. First, we show that we can reduce the proof to Hilbert space valued SDEs
$$dY_t = \bar{\alpha}(t, Y)dt + \bar{\sigma}(t, Y)dW_t. \tag{1.2}$$
This is due to the "method of the moving frame", which has been presented in [5], see also [16].
2. For Hilbert space valued SDEs (1.2) however, the Yamada-Watanabe Theorem is a consequence of [13].

The remainder of this paper is organized as follows: In Section 2 we present the general framework, in Section 3 we provide the proof of Theorem 1.1, and in Section 4 we show an example illustrating Theorem 1.1.

2 Framework and definitions

In this section, we prepare the required framework and definitions. The framework is similar to that in [13] and we refer to this paper for further details.

Let H be a separable Hilbert space and let $(S_t)_{t \geq 0}$ be a C_0-semigroup on H with infinitesimal generator $A : \mathcal{D}(A) \subset H \to H$. The path space

$$\mathbb{W}(H) := C(\mathbb{R}_+; H)$$

is the space of all continuous functions from $\mathbb{R}_+$ to H. Equipped with the metric

$$\rho(w_1, w_2) := \sum_{k=1}^{\infty} 2^{-k} \Big(\sup_{t \in [0,k]} \| w_1(t) - w_2(t) \| \wedge 1 \Big), \tag{2.1}$$

the path space $(\mathbb{W}(H), \rho)$ is a Polish space. Furthermore, we define the subspace

$$\mathbb{W}_0(H) := \{ w \in \mathbb{W}(H) : w(0) = 0 \}$$

consisting of all functions from the path space $\mathbb{W}(H)$ starting in zero. For $t \in \mathbb{R}_+$ we denote by $\mathcal{B}_t(\mathbb{W}(H))$ the σ-algebra generated by all maps $\mathbb{W}(H) \to H$, $w \mapsto w(s)$ for $s \in [0, t]$. Let $\mathcal{C}(H)$ be the collection of all cylinder sets of the form

$$\{ w \in \mathbb{W}(H) : w(t_1) \in B_1, \dots, w(t_n) \in B_n \} \tag{2.2}$$

with $t_1, \dots, t_n \in \mathbb{R}_+$ and $B_1, \dots, B_n \in \mathcal{B}(H)$ for some $n \in \mathbb{N}$, and let $\mathcal{C}'(H)$ be the collection of all cylinder sets of the form

$$\{ w \in \mathbb{W}(H) : (w(t_1), \dots, w(t_n)) \in B \} \tag{2.3}$$

for $t_1, \dots, t_n \in \mathbb{R}_+$ and $B \in \mathcal{B}(H)^{\otimes n}$ for some $n \in \mathbb{N}$. Similarly, for $t \in \mathbb{R}_+$ let $\mathcal{C}_t(H)$ be the collection of all cylinder sets of the form (2.2) with $t_1, \dots, t_n \in [0, t]$ and $B_1, \dots, B_n \in \mathcal{B}(H)$ for some $n \in \mathbb{N}$, and let $\mathcal{C}'_t(H)$ be the collection of all cylinder sets of the form (2.3) for $t_1, \dots, t_n \in [0, t]$ and $B \in \mathcal{B}(H)^{\otimes n}$ for some $n \in \mathbb{N}$.

Lemma 2.1. *The following statements are true:*

1. *We have* $\mathcal{B}(\mathbb{W}(H)) = \sigma(\mathcal{C}(H)) = \sigma(\mathcal{C}'(H))$.
2. *We have* $\mathcal{B}_t(\mathbb{W}(H)) = \sigma(\mathcal{C}_t(H)) = \sigma(\mathcal{C}'_t(H))$ *for each* $t \in \mathbb{R}_+$.

Proof. We can argue as in the finite dimensional case, see e.g. [14, Section 2.II]. □

Let U be another separable Hilbert space and let $L_2(U,H)$ denote the space of all Hilbert-Schmidt operators from U to H equipped with the Hilbert-Schmidt norm. Let $\alpha : \mathbb{R}_+ \times \mathbb{W}(H) \to H$ and $\sigma : \mathbb{R}_+ \times \mathbb{W}(H) \to L_2(U,H)$ be mappings.

Assumption 2.2. *We suppose that the following conditions are satisfied:*

1. *α is $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{W}(H))/\mathcal{B}(H)$-measurable such that for each $t \in \mathbb{R}_+$ the mapping $\alpha(t,\bullet)$ is $\mathcal{B}_t(\mathbb{W}(H))/\mathcal{B}(H)$-measurable.*
2. *σ is $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{W}(H))/\mathcal{B}(L_2(U,H))$-measurable such that for each $t \in \mathbb{R}_+$ the mapping $\sigma(t,\bullet)$ is $\mathcal{B}_t(\mathbb{W}(H))/\mathcal{B}(L_2(U,H))$-measurable.*

We call a filtered probability space $\mathbb{B} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions a *stochastic basis*. In the sequel, we shall use the abbreviation $\mathbb{B}$ for a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, and the abbreviation $\mathbb{B}'$ for another stochastic basis $(\Omega', \mathcal{F}', (\mathcal{F}'_t)_{t\geq 0}, \mathbb{P}')$. For a sequence $(\beta_k)_{k\in\mathbb{N}}$ of independent Wiener processes we call the sequence

$$W = (\beta_k)_{k\in\mathbb{N}}$$

a *standard $\mathbb{R}^\infty$-Wiener process*.

Definition 2.3. *A pair (X,W), where X is an adapted process with paths in $\mathbb{W}(H)$ and W is a standard $\mathbb{R}^\infty$-Wiener process on a stochastic basis $\mathbb{B}$ is called a* martingale solution *to (1.1), if we have $\mathbb{P}$–almost surely*

$$\int_0^t \|\alpha(s,X)\| ds + \int_0^t \|\sigma(s,X)\|^2_{L_2(U,H)} ds < \infty \quad \text{for all } t \geq 0$$

and $\mathbb{P}$–almost surely it holds

$$X(t) = S_t X(0) + \int_0^t S_{t-s}\alpha(s,X)ds + \int_0^t S_{t-s}\sigma(s,X)dW(s), \quad t \geq 0.$$

Remark 2.4. *In finite dimensions, a pair (X,W) as in Definition 2.3 is called a* weak solution. *As in [2, Chapter 8], we use the term* martingale solution *in order to avoid ambiguities with the concept of a* weak solution *to (1.1), which means that for each $\zeta \in \mathcal{D}(A^*)$ we have $\mathbb{P}$–almost surely*

$$\langle \zeta, X(t)\rangle = \langle \zeta, X(0)\rangle + \int_0^t \big(\langle A^*\xi, X(s)\rangle + \langle \zeta, \alpha(s,X)\rangle\big)ds + \int_0^t \langle \zeta, \sigma(s,X)\rangle dW(s)$$

for all $t \geq 0$. Sometimes, the latter concept is also called an analytically weak solution, *see [12].*

Remark 2.5. *By the measurability conditions from Assumption 2.2, the processes $\alpha(\bullet,X)$ and $\sigma(\bullet,X)$ from Definition 2.3 are adapted.*

Remark 2.6. *The stochastic integral from Definition 2.3 is defined as*

$$\int_0^t S_{t-s}\sigma(s,X)dW(s) := \int_0^t S_{t-s}\sigma(s,X) \circ J^{-1} d\bar{W}(s), \quad t \geq 0,$$

where $J : U \to \bar{U}$ is a one-to-one Hilbert Schmidt operator into another Hilbert space $\bar{U}$, and

$$\bar{W} := \sum_{k=1}^\infty \beta_k J e_k,$$

where $(e_k)_{k\in\mathbb{N}}$ denotes an orthonormal basis of U, is an $\bar{U}$-valued trace class Wiener process with covariance operator $Q = JJ^*$. Further details about this topic can be found in [12, Section 2.5].

Definition 2.7. *We say that* weak uniqueness *holds for (1.1), if for two martingale solutions* (X,W) *and* (X',W') *on stochastic bases* $\mathbb{B}$ *and* $\mathbb{B}'$ *with*

$$\mathbb{P}^{X(0)} = (\mathbb{P}')^{X'(0)}$$

as measures on $(H, \mathcal{B}(H))$*, we have*

$$\mathbb{P}^X = (\mathbb{P}')^{X'}$$

as measures on $(\mathbb{W}(H), \mathcal{B}(\mathbb{W}(H)))$*.*

Definition 2.8. *We say that* pathwise uniqueness *holds for (1.1), if for two martingale solutions* (X,W) *and* (X',W) *on the same stochastic basis* $\mathbb{B}$ *and with the same* $\mathbb{R}^\infty$*-Wiener process* W *such that* $\mathbb{P}(X(0) = X'(0)) = 1$ *we have* $X = X'$ *up to indistinguishability.*

Definition 2.9. *Let* $\hat{\mathcal{E}}(H)$ *be the set of maps* $F : H \times \mathbb{W}_0(\bar{U}) \to \mathbb{W}(H)$ *such that for every probability measure* μ *on* $(H, \mathcal{B}(H))$ *there exists a map*

$$F_\mu : H \times \mathbb{W}_0(\bar{U}) \to \mathbb{W}(H),$$

which is $\overline{\mathcal{B}(H) \otimes \mathcal{B}(\mathbb{W}_0(\bar{U}))}^{\mu\otimes\mathbb{P}^Q} / \mathcal{B}(\mathbb{W}(H))$*-measurable, such that for* μ*–almost all* $x \in H$ *we have*

$$F(x,w) = F_\mu(x,w) \quad \text{for } \mathbb{P}^Q\text{–almost all } w \in \mathbb{W}_0(\bar{U}).$$

Here $\overline{\mathcal{B}(H) \otimes \mathcal{B}(\mathbb{W}_0(\bar{U}))}^{\mu\otimes\mathbb{P}^Q}$ *denotes the completion of* $\mathcal{B}(H)\otimes\mathcal{B}(\mathbb{W}_0(\bar{U}))$ *with respect to* $\mu\otimes\mathbb{P}^Q$*, and* $\mathbb{P}^Q$ *denotes the distribution of the* Q*-Wiener process* $\bar{W}$ *on* $(\mathbb{W}_0(\bar{U}), \mathcal{B}(\mathbb{W}_0(\bar{U})))$*. Of course,* F_μ *is* $\mu \otimes \mathbb{P}^Q$*–almost everywhere uniquely determined.*

Definition 2.10. *A martingale solution* (X,W) *to (1.1) on a stochastic basis* $\mathbb{B}$ *is called a* mild solution *if there exists a mapping* $F \in \hat{\mathcal{E}}(H)$ *such that the following conditions are satisfied:*

1. *For all* $x \in H$ *and* $t \in \mathbb{R}_+$ *the mapping*

$$\mathbb{W}_0(\bar{U}) \to \mathbb{W}(H), \quad w \mapsto F(x,w)$$

   is $\overline{\mathcal{B}_t(\mathbb{W}_0(\bar{U}))}^{\mathbb{P}^Q} / \mathcal{B}_t(\mathbb{W}(H))$*-measurable, where* $\overline{\mathcal{B}_t(\mathbb{W}_0(\bar{U}))}^{\mathbb{P}^Q}$ *denotes the completion with respect to* $\mathbb{P}^Q$ *in* $\mathcal{B}(\mathbb{W}_0(\bar{U}))$*.*
2. *We have up to indistinguishability*

$$X = F_{\mathbb{P}^{X(0)}}(X(0), \bar{W}).$$

Definition 2.11. *We say that the SPDE (1.1) has a* unique mild solution *if there exists a mapping* $F \in \hat{\mathcal{E}}(H)$ *such that:*

1. *For all* $x \in H$ *and* $t \in \mathbb{R}_+$ *the mapping*

$$\mathbb{W}_0(\bar{U}) \to \mathbb{W}(H), \quad w \mapsto F(x,w)$$

   is $\overline{\mathcal{B}_t(\mathbb{W}_0(\bar{U}))}^{\mathbb{P}^Q} / \mathcal{B}_t(\mathbb{W}(H))$*-measurable, where* $\overline{\mathcal{B}_t(\mathbb{W}_0(\bar{U}))}^{\mathbb{P}^Q}$ *denotes the completion with respect to* $\mathbb{P}^Q$ *in* $\mathcal{B}(\mathbb{W}_0(\bar{U}))$*.*

2. *For every standard $\mathbb{R}^\infty$-Wiener process W on a stochastic basis $\mathbb{B}$ and any $\mathcal{F}_0$-measurable random variable $\xi : \Omega \to H$ the pair (X, W), where $X := F(\xi, \bar{W})$, is a martingale solution to (1.1) with $\mathbb{P}(X(0) = \xi) = 1$.*
3. *For any martingale solution (X, W) to (1.1) we have up to indistinguishability*

$$X = F_{\mathbb{P}^{X(0)}}(X(0), \bar{W}).$$

Remark 2.12. *For $A = 0$ the SPDE (1.1) becomes a SDE, and in this case we speak about a* strong solution (unique strong solution)*, if the conditions from Definition 2.10 (Definition 2.11) are fulfilled.*

3 Proof of Theorem 1.1

In this section, we shall provide the proof of Theorem 1.1. The general framework is that of Section 2. In particular, we suppose that the coefficients α and σ satisfy Assumption 2.2. As mentioned in Section 1, we shall utilize the "method of the moving frame" from [5]. For this, we require the following assumption on the semigroup $(S_t)_{t \geq 0}$.

Assumption 3.1. *We suppose that there exist another separable Hilbert space $\mathcal{H}$, a C_0-group $(U_t)_{t \in \mathbb{R}}$ on $\mathcal{H}$ and continuous linear operators $\ell \in L(H, \mathcal{H})$, $\pi \in L(\mathcal{H}, H)$ such ℓ is injective, we have $\operatorname{rg}(\pi) = H$ and $\ker(\pi) = \operatorname{rg}(\ell)^\perp$, and the diagram*

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{U_t} & \mathcal{H} \\ \uparrow \ell & & \downarrow \pi \\ H & \xrightarrow{S_t} & H \end{array}$$

commutes for every $t \in \mathbb{R}_+$, that is

$$\pi U_t \ell = S_t \quad \text{for all } t \in \mathbb{R}_+. \tag{3.1}$$

Remark 3.2. *According to [5, Prop. 8.7], this assumption is satisfied if the semigroup $(S_t)_{t \geq 0}$ is* pseudo-contractive *(one also uses the notion* quasi-contractive*), that is, there is a constant $\omega \in \mathbb{R}$ such that*

$$\|S_t\| \leq e^{\omega t} \quad \text{for all } t \geq 0.$$

This result relies on the Szőkefalvi-Nagy theorem on unitary dilations (see e.g. [15, Thm. I.8.1], or [3, Sec. 7.2]). In the spirit of [15], the group $(U_t)_{t \in \mathbb{R}}$ is called a dilation *of the semigroup $(S_t)_{t \geq 0}$.*

Remark 3.3. *The Szőkefalvi-Nagy theorem was also utilized in [8, 7] in order to establish results concerning stochastic convolution integrals.*

In the sequel, for some closed subspace $K \subset H$ we denote by Π_K the orthogonal projection on K.

Lemma 3.4. *The following statements are true:*

1. *We have $\pi \ell = \operatorname{Id}|_H$.*
2. *We have $\ell \pi = \Pi_{\operatorname{rg}(\ell)}$ and $\ell \pi|_{\operatorname{rg}(\ell)} = \operatorname{Id}|_{\operatorname{rg}(\ell)}$.*

Proof. The first statement follows from (3.1) with $t = 0$. For the second statement, note that $\operatorname{rg}(\ell)$ is closed, because ℓ is injective. Moreover, by Assumption 3.1 we have $\operatorname{rg}(\ell\pi) = \operatorname{rg}(\ell)$ and $\ker(\ell\pi) = \ker(\pi) = \operatorname{rg}(\ell)^\perp$, showing that $\ell\pi$ is the orthogonal projection on the closed subspace $\operatorname{rg}(\ell)$. Consequently, we also have $\ell\pi|_{\operatorname{rg}(\ell)} = \operatorname{Id}|_{\operatorname{rg}(\ell)}$. □

Now, we introduce several mappings, namely

$$\begin{aligned} &\Gamma : \mathbb{W}(\mathcal{H}) \to \mathbb{W}(H), \quad \Gamma(w) := \pi U(w - \Pi_{\mathrm{rg}(\ell)^\perp} w(0)), \\ &a : \mathbb{R}_+ \times \mathbb{W}(H) \to \mathcal{H}, \quad a(t,w) := U_{-t}\ell\alpha(t,w), \\ &b : \mathbb{R}_+ \times \mathbb{W}(H) \to L_2(U,\mathcal{H}), \quad b(t,w) := U_{-t}\ell\sigma(t,w), \\ &\bar{\alpha} : \mathbb{R}_+ \times \mathbb{W}(\mathcal{H}) \to \mathcal{H}, \quad \bar{\alpha}(t,w) := a(t,\Gamma(w)), \\ &\bar{\sigma} : \mathbb{R}_+ \times \mathbb{W}(\mathcal{H}) \to \mathcal{H}, \quad \bar{\sigma}(t,w) := b(t,\Gamma(w)). \end{aligned} \tag{3.2}$$

Lemma 3.5. *The following statements are true:*

1. *The mapping Γ is $\mathcal{B}(\mathbb{W}(\mathcal{H}))/\mathcal{B}(\mathbb{W}(H))$-measurable.*
2. *The mapping Γ is $\mathcal{B}_t(\mathbb{W}(\mathcal{H}))/\mathcal{B}_t(\mathbb{W}(H))$-measurable for each $t \in \mathbb{R}_+$.*

Proof. Let $C \in \mathcal{C}(H)$ be a cylinder set of the form

$$C = \{w \in \mathbb{W}(H) : w(t_1) \in B_1, \dots, w(t_n) \in B_n\}$$

with $t_1, \dots, t_n \in \mathbb{R}_+$ and $B_1, \dots, B_n \in \mathcal{B}(H)$ for some $n \in \mathbb{N}$. Then we have

$$\Gamma^{-1}(C) = \bigcap_{k=1}^{n} \{w \in \mathbb{W}(\mathcal{H}) : w(t_k) - \Pi_{\mathrm{rg}(\ell)^\perp} w(0) \in (\pi U_t)^{-1}(B_k)\} \in \mathcal{C}'(H).$$

By Lemma 2.1, the mapping Γ is $\mathcal{B}_t(\mathbb{W}(\mathcal{H}))/\mathcal{B}_t(\mathbb{W}(H))$-measurable, showing the first statement. The second statement is proven analogously. □

Lemma 3.6. *The following statements are true:*

1. *$\bar{\alpha}$ is $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{W}(\mathcal{H}))/\mathcal{B}(\mathcal{H})$-measurable and for each $t \in \mathbb{R}_+$ the mapping $\bar{\alpha}(t, \bullet)$ is $\mathcal{B}_t(\mathbb{W}(\mathcal{H}))/\mathcal{B}(\mathcal{H})$-measurable.*
2. *$\bar{\sigma}$ is $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{W}(\mathcal{H}))/\mathcal{B}(L_2(U,\mathcal{H}))$-measurable and for each $t \in \mathbb{R}_+$ the mapping $\bar{\sigma}(t, \bullet)$ is $\mathcal{B}_t(\mathbb{W}(\mathcal{H}))/\mathcal{B}(L_2(U,\mathcal{H}))$-measurable.*

Proof. Note that the mapping

$$\mathbb{R}_+ \times H \to \mathcal{H}, \quad (t,h) \mapsto U_{-t}\ell h$$

is continuous, and hence $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(H)/\mathcal{B}(\mathcal{H})$-measurable. Therefore, the claimed measurability properties of $\bar{\alpha}$ and $\bar{\sigma}$ follow from Lemma 3.5 and Assumption 2.2. □

By virtue of Lemma 3.6, we may apply the Yamada-Watanabe Theorem from [13], and obtain:

Theorem 3.7. *The SDE (1.2) has a unique strong solution if and only if both of the following two conditions are satisfied:*

1. *For each probability measure ν on $(\mathcal{H}, \mathcal{B}(\mathcal{H}))$ there exists a martingale solution (Y, W) to (1.2) such that ν is the distribution of $Y(0)$.*
2. *Pathwise uniqueness for (1.2) holds.*

Now, our idea for the proof of Theorem 1.1 is as follows: The proof that the existence of a unique mild solution to the SPDE (1.1) implies the two conditions from Theorem 1.1 is straightforward and can be provided as in [13]. For the proof of the converse implication, we will first show that the conditions from Theorem 1.1 imply the conditions from Theorem 3.7, see Propositions 3.13 and 3.14. Then, we will apply Theorem 3.7, which gives us the existence of a unique strong solution to the SDE (1.2), and finally, we will prove that this implies the existence of a unique mild solution to the SPDE (1.1), see Proposition 3.16. For the following four results (Lemma 3.8 to Corollary 3.11), we fix a stochastic basis $\mathbb{B} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$.

Lemma 3.8. *Let $\eta : \Omega \to \mathcal{H}$ be a $\mathcal{F}_0$-measurable random variable, let (X,W) be a martingale solution to (1.1) with $X(0) = \pi\eta$, and set*

$$Y := \eta + \int_0^\bullet a(s,X)ds + \int_0^\bullet b(s,X)dW(s).$$

Then (Y,W) is a martingale solution to (1.2) with $Y(0) = \eta$, and we have $X = \Gamma(Y)$ up to indistinguishability.

Proof. By the definition of Y we have $Y(0) = \eta$. Moreover, since (X,W) is a martingale solution to (1.1) with $X(0) = \pi\eta$, by identity (3.1), Lemma 3.4 and definitions (3.2) we obtain $\mathbb{P}$–almost surely

$$\begin{aligned}
X(t) &= S_t\pi\eta + \int_0^t S_{t-s}\alpha(s,X)ds + \int_0^t S_{t-s}\sigma(s,X)dW(s) \\
&= \pi U_t\bigg(\ell\pi\eta + \int_0^t U_{-s}\ell\alpha(s,X)ds + \int_0^t U_{-s}\ell\sigma(s,X)dW(s)\bigg) \\
&= \pi U_t\bigg(\Pi_{\mathrm{rg}(\ell)}\eta + \int_0^t a(s,X)ds + \int_0^t b(s,X)dW(s)\bigg) \\
&= \pi U_t\bigg(\eta + \int_0^t a(s,X)ds + \int_0^t b(s,X)dW(s) - \Pi_{\mathrm{rg}(\ell)^\perp}\eta\bigg) \\
&= \pi U_t(Y(t) - \Pi_{\mathrm{rg}(\ell)^\perp}Y(0)) = \Gamma(Y)(t) \quad \text{for all } t \in \mathbb{R}_+,
\end{aligned}$$

showing that $X = \Gamma(Y)$ up to indistinguishability, and therefore, by (3.2) we obtain up to indistinguishability

$$\begin{aligned}
Y &= \eta + \int_0^\bullet a(s,X)ds + \int_0^\bullet b(s,X)dW(s) \\
&= \eta + \int_0^\bullet a(s,\Gamma(Y))ds + \int_0^\bullet b(s,\Gamma(Y))dW(s) \\
&= \eta + \int_0^\bullet \bar{\alpha}(s,Y)ds + \int_0^\bullet \bar{\sigma}(s,Y)dW(s),
\end{aligned}$$

proving that (Y,W) is a martingale solution to (1.2) with $Y(0) = \eta$. □

Corollary 3.9. *Let $\xi : \Omega \to H$ be a $\mathcal{F}_0$-measurable random variable, let (X,W) be a martingale solution to (1.1) with $X(0) = \xi$, and set*

$$Y := \ell\xi + \int_0^\bullet a(s,X)ds + \int_0^\bullet b(s,X)dW(s).$$

Then (Y,W) is a martingale solution to (1.2) with $Y(0) = \ell\xi$, and we have $X = \Gamma(Y)$ up to indistinguishability.

Proof. Setting $\eta := \ell\xi$, this follows from Lemmas 3.4 and 3.8. □

Lemma 3.10. *Let $\eta : \Omega \to \mathcal{H}$ be a $\mathcal{F}_0$-measurable random variable, let (Y,W) be a martingale solution to (1.2) with $Y(0) = \eta$, and set $X := \Gamma(Y)$. Then (X,W) is a martingale solution to (1.1) with $X(0) = \pi\eta$, and we have up to indistinguishability*

$$Y = \eta + \int_0^\bullet a(s,X)ds + \int_0^\bullet b(s,X)dW(s).$$

Proof. Since (Y,W) is a martingale solution to (1.2) with $Y(0)=\eta$, by definitions (3.2), Lemma 3.4 and identity (3.1) we obtain $\mathbb{P}$–almost surely

$$
\begin{aligned}
X(t) = \Gamma(Y)(t) &= \pi U_t(Y(t) - \Pi_{\mathrm{rg}(\ell)^\perp} Y(0)) \\
&= \pi U_t\bigg(\eta + \int_0^\bullet \bar{\alpha}(s,Y)ds + \int_0^\bullet \bar{\sigma}(s,Y)dW(s) - \Pi_{\mathrm{rg}(\ell)^\perp}\eta\bigg) \\
&= \pi U_t\bigg(\Pi_{\mathrm{rg}(\ell)}\eta + \int_0^\bullet a(s,\Gamma(Y))ds + \int_0^\bullet b(s,\Gamma(Y))dW(s)\bigg) \\
&= \pi U_t\bigg(\ell\pi\eta + \int_0^\bullet U_{-s}\ell\alpha(s,X)ds + \int_0^\bullet U_{-s}\ell\sigma(s,X)dW(s)\bigg) \\
&= S_t\pi\eta + \int_0^t S_{t-s}\alpha(s,X)ds + \int_0^t S_{t-s}\sigma(s,X)dW(s) \quad \text{for all } t \in \mathbb{R}_+,
\end{aligned}
$$

Therefore, (X,W) is a martingale solution to (1.1) with $X(0)=\pi\eta$. Moreover, by definitions (3.2) we get up to indistinguishability

$$
\begin{aligned}
Y &= \eta + \int_0^\bullet \bar{\alpha}(s,Y)ds + \int_0^\bullet \bar{\sigma}(s,Y)dW(s) \\
&= \eta + \int_0^\bullet a(s,\Gamma(Y))ds + \int_0^\bullet b(s,\Gamma(Y))dW(s) \\
&= \eta + \int_0^\bullet a(s,X)ds + \int_0^\bullet b(s,X)dW(s),
\end{aligned}
$$

finishing the proof. □

Corollary 3.11. *Let $\xi : \Omega \to H$ be a $\mathcal{F}_0$-measurable random variable, let (Y,W) be a martingale solution to (1.2) with $Y(0)=\ell\xi$, and set $X := \Gamma(Y)$. Then (X,W) is a martingale solution to (1.1) with $X(0)=\xi$, and we have up to indistinguishability*

$$
Y = \ell\xi + \int_0^\bullet a(s,X)ds + \int_0^\bullet b(s,X)dW(s).
$$

Proof. Setting $\eta := \ell\xi$, this follows from Lemmas 3.4 and 3.10. □

The following auxiliary result provides us with a standard extension which we require for the proof of Proposition 3.13.

Lemma 3.12. *Let (X',W') be a martingale solution to (1.1) on a stochastic basis $\mathbb{B}'$ and let ν be a probability measure on $(\mathcal{H},\mathcal{B}(\mathcal{H}))$. Then, there exist a stochastic basis $\mathbb{B}$, a martingale solution (X,W) to (1.1) on $\mathbb{B}$ such that the distributions of $X(0)$ and $X'(0)$ coincide, and a $\mathcal{F}_0$-measurable random variable $\eta : \Omega \to \mathcal{H}$ such that ν is the distribution of η.*

Proof. We define the stochastic basis $\mathbb{B}$ as

$$
\begin{aligned}
\Omega &:= \Omega' \times \mathcal{H}, \\
\mathcal{F} &:= \overline{\mathcal{F}' \otimes \mathcal{B}(\mathcal{H})}^{\mathbb{P}'\otimes\nu}, \\
\mathcal{F}_t &:= \bigcap_{\epsilon>0} \sigma(\mathcal{F}'_{t+\epsilon} \otimes \mathcal{B}(\mathcal{H}), \mathcal{N}), \quad t \geq 0, \\
\mathbb{P} &:= \mathbb{P}' \otimes \nu,
\end{aligned}
$$

where $\mathcal{N}$ denotes all $\mathbb{P}'\otimes\nu$–nullsets in $\mathcal{F}'\otimes\mathcal{B}(\mathcal{H})$. Then the random variable

$$
\nu : \Omega \to \mathcal{H}, \quad \eta(\omega',h) := h
$$

is $\mathcal{F}_0$-measurable and has the distribution ν. We define the H-valued processes

$$X(\omega', h) := X'(\omega') \quad \text{and} \quad W(\omega', h) := W'(\omega').$$

Then W is a standard $\mathbb{R}^\infty$-Wiener process, because W' is a standard $\mathbb{R}^\infty$-Wiener process. The independence of the increments with respect to the new filtration $(\mathcal{F}_t)_{t\geq 0}$ is shown as in the proof of [12, Prop. 2.1.13]. Moreover, the distributions of $X(0)$ and $X'(0)$ coincide, and the pair (X, W) is a martingale solution to (1.1), because (X', W') is a martingale solution to (1.1). □

Proposition 3.13. *Suppose for each probability measure μ on $(H, \mathcal{B}(H))$ there exists a martingale solution (X, W) to (1.1) such that μ is the distribution of $X(0)$. Then, for each probability measure ν on $(\mathcal{H}, \mathcal{B}(\mathcal{H}))$ there exists a martingale solution (Y, W) to (1.2) such that ν is the distribution of $Y(0)$.*

Proof. Let ν be a probability measure on $(\mathcal{H}, \mathcal{B}(\mathcal{H}))$. Then the image measure $\mu := \nu^\pi$ is a probability measure on $(H, \mathcal{B}(H))$. By assumption, there exists a martingale solution (X', W') to (1.1) on a stochastic basis $\mathbb{B}'$ such that μ is the distribution of $X'(0)$. According to Lemma 3.12, there exist a stochastic basis $\mathbb{B}$, a martingale solution (X, W) on $\mathbb{B}$ such that μ is the distributions of $X(0)$, and a $\mathcal{F}_0$-measurable random variable $\eta : \Omega \to \mathcal{H}$ such that ν is the distribution of η. We set

$$Y := \eta + \int_0^\bullet a(s, X)ds + \int_0^\bullet b(s, X)dW(s).$$

By Lemma 3.8, the pair (Y, W) is a martingale solution to (1.1) with $Y(0) = \eta$. □

Proposition 3.14. *If pathwise uniqueness for (1.1) holds, then pathwise uniqueness for (1.2) holds, too.*

Proof. Let (Y, W) and (Y', W) be two martingale solutions to (1.2) on the same stochastic basis $\mathbb{B}$ such that $\mathbb{P}(Y(0) = Y'(0)) = 1$. We set $X := \Gamma(Y)$ and $Y' := \Gamma(Y')$. By Lemma 3.10, the pairs (X, W) and (X', W) are two martingale solutions to (1.1) with $X(0) = \pi Y(0)$ and $X'(0) = \pi Y'(0)$, and we have up to indistinguishability

$$\begin{aligned} Y &= Y(0) + \int_0^\bullet a(s, X)ds + \int_0^\bullet b(s, X)dW(s), \\ Y' &= Y'(0) + \int_0^\bullet a(s, X')ds + \int_0^\bullet b(s, X')dW(s). \end{aligned}$$

This gives us

$$\mathbb{P}(X(0) = X'(0)) = \mathbb{P}(\pi Y(0) = \pi Y'(0)) = 1.$$

Since pathwise uniqueness for (1.1) holds, we deduce that $X = X'$ up to indistinguishability. This implies up to indistinguishability

$$\begin{aligned} Y &= Y(0) + \int_0^\bullet a(s, X)ds + \int_0^\bullet b(s, X)dW(s) \\ &= Y'(0) + \int_0^\bullet a(s, X')ds + \int_0^\bullet b(s, X')dW(s) = Y', \end{aligned}$$

proving that pathwise uniqueness for (1.2) holds. □

The following auxiliary result is required for the proof of Proposition 3.16.

Lemma 3.15. *Let ν be an arbitrary probability measure on $(H,\mathcal{B}(H))$. We define the image measure $\nu := \mu^\ell$ on $(\mathcal{H},\mathcal{B}(\mathcal{H}))$. Then the mapping*

$$(\ell,\mathrm{Id}) : H \times \mathbb{W}_0(\bar{U}) \to \mathcal{H} \times \mathbb{W}_0(\bar{U})$$

is $\overline{\mathcal{B}(H)\otimes\mathcal{B}(\mathbb{W}_0(\bar{U}))}^{\mu\otimes\mathbb{P}^Q} / \overline{\mathcal{B}(\mathcal{H})\otimes\mathcal{B}(\mathbb{W}_0(\bar{U}))}^{\nu\otimes\mathbb{P}^Q}$*-measurable.*

Proof. Let $B\cup N \in \overline{\mathcal{B}(\mathcal{H})\otimes\mathcal{B}(\mathbb{W}_0(\bar{U}))}^{\nu\otimes\mathbb{P}^Q}$ be an arbitrary measurable set with a Borel set $B\in\mathcal{B}(\mathcal{H})\otimes\mathcal{B}(\mathbb{W}_0(\bar{U}))$ and a $\nu\otimes\mathbb{P}^Q$–nullset $N\subset\mathcal{H}\times\mathbb{W}_0(\bar{U})$. Then we have

$$(\ell,\mathrm{Id})^{-1}(B)\in\mathcal{B}(H)\otimes\mathcal{B}(\mathbb{W}_0(\bar{U})),$$

because (ℓ,Id) is $\mathcal{B}(H)\otimes\mathcal{B}(\mathbb{W}_0(\bar{U}))/\mathcal{B}(\mathcal{H})\otimes\mathcal{B}(\mathbb{W}_0(\bar{U}))$-measurable. For arbitrary Borel sets $C\in\mathcal{B}(H)$ and $D\in\mathcal{B}(\mathbb{W}_0(\bar{U}))$ we have

$$\begin{aligned}(\mu\otimes\mathbb{P}^Q)^{(\ell,\mathrm{Id})}(C\times D) &= (\mu\otimes\mathbb{P}^Q)((\ell,\mathrm{Id})^{-1}(C\times D)) = (\mu\otimes\mathbb{P}^Q)(\ell^{-1}(C)\times D)\\ &= \mu(\ell^{-1}(C))\cdot\mathbb{P}^Q(D) = \mu^\ell(C)\cdot\mathbb{P}^Q(D) = \nu(C)\cdot\mathbb{P}^Q(D) = (\nu\otimes\mathbb{P}^Q)(C\times D),\end{aligned}$$

showing that

$$(\mu\otimes\mathbb{P}^Q)^{(\ell,\mathrm{Id})} = \nu\otimes\mathbb{P}^Q.$$

There exists a set $N'\in\mathcal{B}(\mathcal{H})\otimes\mathcal{B}(\mathbb{W}_0(\bar{U}))$ satisfying $N\subset N'$ and $(\nu\otimes\mathbb{P}^Q)(N')=0$. We obtain

$$(\mu\otimes\mathbb{P}^Q)((\ell,\mathrm{Id})^{-1}(N')) = (\mu\otimes\mathbb{P}^Q)^{(\ell,\mathrm{Id})}(N') = (\nu\otimes\mathbb{P}^Q)(N') = 0,$$

showing that $(\ell,\mathrm{Id})^{-1}(N)$ is a $\mu\otimes\mathbb{P}^Q$–nullset. Consequently, we have

$$(\ell,\mathrm{Id})^{-1}(B\cup N) = (\ell,\mathrm{Id})^{-1}(B)\cup(\ell,\mathrm{Id})^{-1}(N)\in\overline{\mathcal{B}(H)\otimes\mathcal{B}(\mathbb{W}_0(\bar{U}))}^{\mu\otimes\mathbb{P}^Q},$$

proving that (ℓ,Id) is $\overline{\mathcal{B}(H)\otimes\mathcal{B}(\mathbb{W}_0(\bar{U}))}^{\mu\otimes\mathbb{P}^Q} / \overline{\mathcal{B}(\mathcal{H})\otimes\mathcal{B}(\mathbb{W}_0(\bar{U}))}^{\nu\otimes\mathbb{P}^Q}$-measurable. □

Proposition 3.16. *If the SDE (1.2) has a unique strong solution, then the SPDE (1.1) has a unique mild solution.*

Proof. Suppose the SDE (1.2) has a unique mild solution. Then, there exists a mapping $G\in\hat{\mathcal{E}}(\mathcal{H})$ such that the three conditions from Definition 2.11 are fulfilled. In detail, the following conditions are satisfied:

- $G:\mathcal{H}\times\mathbb{W}_0(\bar{U})\to\mathbb{W}(\mathcal{H})$ is a mapping such that for every probability measure ν on $(\mathcal{H},\mathcal{B}(\mathcal{H}))$ there exists a map

$$G_\nu:\mathcal{H}\times\mathbb{W}_0(\bar{U})\to\mathbb{W}(\mathcal{H}),$$

  which is $\overline{\mathcal{B}(\mathcal{H})\otimes\mathcal{B}(\mathbb{W}_0(\bar{U}))}^{\nu\otimes\mathbb{P}^Q}/\mathcal{B}(\mathbb{W}(\mathcal{H}))$-measurable, such that for ν–almost all $y\in\mathcal{H}$ we have

$$G(y,w) = G_\nu(y,w) \quad \text{for } \mathbb{P}^Q\text{–almost all } w\in\mathbb{W}_0(\bar{U}). \tag{3.3}$$

- For all $y\in\mathcal{H}$ and $t\in\mathbb{R}_+$ the mapping

$$\mathbb{W}_0(\bar{U})\to\mathbb{W}(\mathcal{H}),\quad w\mapsto G(y,w)$$

  is $\overline{\mathcal{B}_t(\mathbb{W}_0(\bar{U}))}^{\mathbb{P}^Q}/\mathcal{B}_t(\mathbb{W}(\mathcal{H}))$-measurable, where $\overline{\mathcal{B}_t(\mathbb{W}_0(\bar{U}))}^{\mathbb{P}^Q}$ denotes the completion with respect to $\mathbb{P}^Q$ in $\mathcal{B}(\mathbb{W}_0(\bar{U}))$.

- For every standard $\mathbb{R}^\infty$-Wiener process W on a stochastic basis $\mathbb{B}$ and any $\mathcal{F}_0$-measurable random variable $\eta : \Omega \to \mathcal{H}$ the pair (Y, W), where $Y := G(\eta, \bar{W})$, is a martingale solution to (1.2) with $\mathbb{P}(Y(0) = \eta) = 1$.
- For any martingale solution (Y, W) to (1.2) we have up to indistinguishability

$$Y = G_{\mathbb{P}^{Y(0)}}(Y(0), \bar{W}).$$

We define the mapping

$$F : H \times \mathbb{W}_0(\bar{U}) \to \mathbb{W}(H), \quad F(x, w) := \Gamma(G(\ell x, w)),$$

which is $\overline{\mathcal{B}(H) \otimes \mathcal{B}(\mathbb{W}_0(\bar{U}))}^{\mu \otimes \mathbb{P}^Q} / \mathcal{B}(\mathbb{W}(H))$-measurable by virtue of Lemmas 3.5 and 3.15. Let us prove that $F \in \hat{\mathcal{E}}(H)$. For this purpose, let μ be an arbitrary probability measure on $(H, \mathcal{B}(H))$. We define the image measure $\nu := \mu^\ell$. Then ν is a probability measure on $(\mathcal{H}, \mathcal{B}(\mathcal{H}))$. Furthermore, we define the mapping

$$F_\mu : H \times \mathbb{W}_0(\bar{U}) \to \mathbb{W}(H), \quad F_\mu(x, w) := \Gamma(G_\nu(\ell x, w)).$$

There is a ν–nullset $N \subset \mathcal{H}$ such that for all $y \in N^c$ identity (3.3) is satisfied. The set $\ell^{-1}(N) \subset H$ is a μ–nullset. Indeed, there is a set $N' \in \mathcal{B}(\mathcal{H})$ satisfying $N \subset N'$ and $\nu(N') = 0$. We obtain

$$\mu(\ell^{-1}(N')) = \mu^\ell(N') = \nu(N') = 0,$$

showing that $\ell^{-1}(N) \subset H$ is a μ–nullset. Let $x \in \ell^{-1}(N)^c = \ell^{-1}(N^c)$ be arbitrary. Then we have $\ell x \in N^c$, and hence

$$F(x, w) = \Gamma(G(\ell x, w)) = \Gamma(G_\nu(\ell x, w)) = F_\mu(x, w)$$

for $\mathbb{P}^Q$–almost all $w \in \mathbb{W}_0(\bar{U})$. Consequently, we have $F \in \hat{\mathcal{E}}(H)$.

Now, we shall prove that the mapping F satisfies the three conditions from Definition 2.11. For all $x \in H$ and $t \in \mathbb{R}_+$ the mapping

$$\mathbb{W}_0(\bar{U}) \to \mathbb{W}(H), \quad w \mapsto F(x, w)$$

is $\overline{\mathcal{B}_t(\mathbb{W}_0(\bar{U}))}^{\mathbb{P}^Q} / \mathcal{B}_t(\mathbb{W}(H))$-measurable due to Lemma 3.5.

Let W be a standard $\mathbb{R}^\infty$-Wiener process on a stochastic basis $\mathbb{B}$, and let $\xi : \Omega \to H$ be a $\mathcal{F}_0$-measurable random variable. Then the pair (Y, W), where $Y := G(\ell\xi, \bar{W})$, is a martingale solution to (1.2) with $\mathbb{P}(Y(0) = \ell\xi) = 1$. By Corollary 3.11, the pair (X, W), where $X := F(\xi, \bar{W}) = \Gamma(Y)$, is a martingale solution to (1.1) with $\mathbb{P}(X(0) = \xi) = 1$.

Finally, let (X, W) be a martingale solution to (1.1) and set

$$Y := \ell X(0) + \int_0^\bullet a(s, X) ds + \int_0^\bullet b(s, X) dW(s).$$

By Corollary 3.9, the pair (Y, W) is a martingale solution to (1.1) with $\mathbb{P}(Y(0) = \ell X(0)) = 1$, and we have $X = \Gamma(Y)$ up to indistinguishability. Denoting by ν the distribution of $Y(0)$, we have up to indistinguishability

$$Y = G_\nu(Y(0), \bar{W}).$$

Furthermore, denoting by μ the distribution of $X(0)$, we obtain

$$\nu = \mathbb{P}^{Y(0)} = \mathbb{P}^{\ell X(0)} = (\mathbb{P}^{X(0)})^\ell = \mu^\ell.$$

We deduce that up to indistinguishability

$$\begin{aligned} X &= \Gamma(Y) = \Gamma(G_\nu(Y(0),\bar{W})) \\ &= \Gamma(G_\nu(\ell X(0),\bar{W})) = F_\mu(X(0),\bar{W}). \end{aligned}$$

Consequently, the mapping F fulfills the three conditions from Definition 2.11, proving that the SPDE (1.1) has a unique mild solution. □

Now, the proof of Theorem 1.1 is a direct consequence: If the SPDE (1.1) has a unique mild solution, then arguing as in [13] shows that the two conditions from Theorem 1.1 are fulfilled. Conversely, if these two conditions are satisfied, then combining Propositions 3.13, 3.14, Theorem 3.7 and Proposition 3.16 shows that the SPDE (1.1) has a unique mild solution.

4 An example

In this section, we shall illustrate Theorem 1.1 and consider SPDEs of the type

$$dX(t) = (AX(t) + B(t,X(t)) + F(t,X(t)))dt + \sqrt{Q}dW_t, \tag{4.1}$$

which have been studied in [1], with a Hölder continuous mapping B. We fix a finite time horizon $T > 0$, an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of H and suppose (as in [1, Section 1.1]) that the following conditions are satisfied:

- A is selfadjoint, with compact resolvent, and there is a non-decreasing sequence $(\alpha_n)_{n\in\mathbb{N}} \subset (0,\infty)$ such that $Ae_n = -\alpha_n e_n$ for all $n \in \mathbb{N}$.
- For the mapping $B : [0,T]\times H \to H$ there exist constants $L_B, M_B > 0$ and $\alpha \in (0,1]$ such that

$$\begin{gathered} \|B(t,x) - B(t,y)\| \le L_B \|x-y\|^\alpha \quad \text{for all } x,y \in H \text{ and } t \in [0,T], \\ \|B(t,x)\| \le M_B \quad \text{for all } x \in H \text{ and } t \in [0,T]. \end{gathered}$$

- For the mapping $F : [0,T] \times H \to H$ there exists a constant $L_F > 0$ such that

$$\|F(t,x) - F(t,y)\| \le L_F \|x-y\| \quad \text{for all } x,y \in H \text{ and } t \in [0,T].$$

- $Q : H \to H$ is a nonnegative, selfadjoint, bounded operator such that $\operatorname{Tr} Q < \infty$ or $\sum_{n\in\mathbb{N}} \frac{\|B_n\|_\alpha}{\alpha_n} < \infty$, where $B_n = \langle B, e_n \rangle$ and

$$\|B_n\|_\alpha = \sup_{\substack{t\in[0,T] \\ x\in H}} \|B_n(t,x)\| + \sup_{\substack{t\in[0,T] \\ x,y\in H \text{ with } x\neq y}} \frac{\|B_n(t,x) - B_n(t,y)\|}{\|x-y\|^\alpha}.$$

- $Q_t := \int_0^t S_s Q S_s^* ds$ is a trace class operator for each $t > 0$.
- $S_t(H) \subset Q_t^{1/2}(H)$ for each $t > 0$.
- We have $\int_0^T \|Q_t^{-1/2} S_t\|^{1+\theta} dt < \infty$ for some $\theta \ge \max\{\alpha, 1-\alpha\}$.

Furthermore, in order to ensure the existence of martingale solutions, we suppose that S_t is a compact operator for each $t > 0$. Then, as indicated in [1], strong existence holds true. Indeed, by [6, Theorem 3.14] we have the existence of martingale solutions, and by [1, Theorem 7] pathwise uniqueness holds true. Hence, according to Theorem 1.1, the SPDE (4.1) has a unique mild solution.

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Acknowledgments. The author is grateful to an anonymous referee for valuable comments.