MAT-free reflection arrangements

Michael Cuntz
Institut für Algebra, Zahlentheorie und Diskrete Mathematik
Fakultät für Mathematik und Physik, Leibniz Universität Hannover
D-30167 Hannover, Germany
cuntz@math.uni-hannover.de

Paul Mücksch
Fakultät für Mathematik, Ruhr-Universität Bochum
D-44780 Bochum, Germany
paul.muecksch@rub.de

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Abstract

We introduce the class of MAT-free hyperplane arrangements which is based on the Multiple Addition Theorem by Abe, Barakat, Cuntz, Hoge, and Terao. We also investigate the closely related class of MAT2-free arrangements based on a recent generalization of the Multiple Addition Theorem by Abe and Terao. We give classifications of the irreducible complex reflection arrangements which are MAT-free respectively MAT2-free. Furthermore, we ask some questions concerning relations to other classes of free arrangements.

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1 Introduction

A hyperplane arrangement $\mathcal{A}$ is a finite set of hyperplanes in a finite dimensional vector space $V \cong \mathbb{K}^\ell$ where $\mathbb{K}$ is some field. The intersection lattice $L(\mathcal{A})$ of $\mathcal{A}$ encodes its combinatorial properties. It is a main theme in the study of hyperplane arrangements to link algebraic properties of $\mathcal{A}$ with the combinatorics of $L(\mathcal{A})$.

The algebraic property of *freeness* of a hyperplane arrangement $\mathcal{A}$ was first studied by Saito [Sai80] and Terao [Ter80a]. In fact, it turns out that freeness of $\mathcal{A}$ imposes strong combinatorial constraints on $L(\mathcal{A})$: by Terao's Factorization Theorem [OT92, Thm. 4.137] its characteristic polynomial factors over the integers. Conversely, sufficiently strong conditions on $L(\mathcal{A})$ imply the freeness of $\mathcal{A}$. One of the main tools to derive such conditions is Terao's Addition-Deletion Theorem 8. It motivates the class of *inductively free* arrangements (see Definition 9). In this class the freeness of $\mathcal{A}$ is combinatorial, i.e. it is completely determined by $L(\mathcal{A})$ (cf. Definition 5). Recently, a remarkable generalization of the Addition-Deletion theorem was obtained by Abe. His Division Theorem [Abe16, Thm. 1.1] motivates the class of *divisionally free* arrangements. In this class freeness is a combinatorial property too.

Despite having these useful tools at hand, it is still a major open problem, known as Terao's Conjecture, whether in general the freeness of $\mathcal{A}$ actually depends only on $L(\mathcal{A})$, provided the field $\mathbb{K}$ is fixed (see [Zie90] for a counterexample when one fixes $L(\mathcal{A})$ but changes the field). We should also mention at this point the very recent results by Abe further examining Addition-Deletion constructions together with divisional freeness [Abe18b], [Abe18a].

A variation of the addition part of the Addition-Deletion theorem 8 was obtained by Abe, Barakat, Cuntz, Hoge, and Terao in [ABC$^+$16]: the Multiple Addition Theorem 12 (MAT for short). Using this theorem, the authors gave a new uniform proof of the Kostant-Macdonald-Shapiro-Steinberg formula for the exponents of a Weyl group. In the same way the Addition-Theorem defines the class of inductively free arrangements, it is now natural to consider the class $\mathfrak{MF}$ of those free arrangements, called *MAT-free*, which can be build inductively using the MAT (Definition 13). It is not hard to see (Lemma 18) that MAT-freeness only depends on $L(\mathcal{A})$. In this paper, we investigate classes of MAT-free arrangements beyond the classes considered in [ABC$^+$16].

Complex reflection groups (classified by Shephard and Todd [ST54]) play an important role in the study of hyperplane arrangements: many interesting examples and counterexamples are related or derived from the reflection arrangement $\mathcal{A}(W)$ of a complex reflection group W. It was proven by Terao [Ter80b] that reflection arrangements are always free. There has been a series of investigations dealing with reflection arrangements and their connection to the aforementioned combinatorial classes of free arrangements (e.g. [BC12], [HR15], [Abe16]). Therefore, it is natural to study reflection arrangements in conjunction with the new class of MAT-free arrangements.

Our main result is the following.

Theorem 1. *Except for the arrangement $\mathcal{A}(G_{32})$, an irreducible reflection arrangement is MAT-free if and only if it is inductively free. The arrangement $\mathcal{A}(G_{32})$ is inductively free but not MAT-free. Thus every reflection arrangement is MAT-free except the reflection arrangements of the imprimitive reflection groups $G(e,e,\ell)$, $e > 2$, $\ell > 2$ and of the reflection groups*

$$G_{24}, G_{27}, G_{29}, G_{31}, G_{32}, G_{33}, G_{34}.$$

A further generalization of the MAT 12 was very recently obtained by Abe and Terao [AT19]: the Multiple Addition Theorem 2 14 (MAT2 for short). Again, one might consider the inductively defined class of arrangements which can be build from the empty arrangement using this more general tool, i.e. the class $\mathfrak{MF}'$ of *MAT2-free* arrangements (Definition 15). By definition, this class contains the class of MAT-free arrangements. Regarding reflection arrangements we have the following:

Theorem 2. *Let $\mathcal{A} = \mathcal{A}(W)$ be an irreducible reflection arrangement. Then $\mathcal{A}$ is MAT2-free if and only if it is MAT-free.*

In contrast to (irreducible) reflection arrangements, in general the class of MAT-free arrangements is properly contained in the class of MAT2-free arrangements (see Proposition 28).

Based on our classification of MAT-free (MAT2-free) reflection arrangements and other known examples ([ABC$^+$16], [CRS19]) we arrive at the following question:

Question 3. Is every MAT-free (MAT2-free) arrangement inductively free?

In [CRS19] the authors proved that all ideal subarrangements of a Weyl arrangement are inductively free by extensive computer calculations. A positive answer to Question 3 would directly imply their result and yield a uniform proof (cf. [CRS19, Rem. 1.5(d)]).

Looking at the class of divisionally free arrangements which properly contains the class of inductively free arrangements [Abe16, Thm. 4.4] a further natural question is:

Question 4. Is every MAT-free (MAT2-free) arrangement divisionally free?

This article is organized as follows: in Section 2 we briefly recall some notions and results about hyperplane arrangements and free arrangements used throughout our exposition. In Section 3 we give an alternative characterization of MAT-freeness and two easy necessary conditions for MAT/MAT2-freeness. Furthermore, we comment on the relation of the two classes $\mathfrak{MF}$ and $\mathfrak{MF}'$ and on the product construction. Section 4 and Section 5 contain the proofs of Theorem 1 and Theorem 2. In the last Section 6 we comment on Question 3 and further problems connected with MAT-freeness.

2 Hyperplane arrangements and free arrangements

Let $\mathcal{A}$ be a hyperplane arrangement in $V \cong \mathbb{K}^\ell$ where $\mathbb{K}$ is some field. If $\mathcal{A}$ is empty, then it is denoted by Φ_ℓ.

The *intersection lattice* $L(\mathcal{A})$ of $\mathcal{A}$ consists of all intersections of elements of $\mathcal{A}$ including V as the empty intersection. Indeed, with the partial order by reverse inclusion $L(\mathcal{A})$ is a geometric lattice [OT92, Lem. 2.3]. The *rank* $\operatorname{rk}(\mathcal{A})$ of $\mathcal{A}$ is defined as the codimension of the intersection of all hyperplanes in $\mathcal{A}$.

If $x_1, \ldots, x_\ell$ is a basis of V^*, to explicitly give a hyperplane we use the notation $(a_1, \ldots, a_\ell)^\perp := \ker(a_1x_1 + \cdots + a_\ell x_\ell)$.

Definition 5. Let $\mathfrak{C}$ be a class of arrangements and let $\mathcal{A} \in \mathfrak{C}$. If for all arrangements $\mathcal{B}$ with $L(\mathcal{B}) \cong L(\mathcal{A})$, (where $\mathcal{A}$ and $\mathcal{B}$ do not have to be defined over the same field), we have $\mathcal{B} \in \mathfrak{C}$, then the class $\mathfrak{C}$ is called *combinatorial.*

If $\mathfrak{C}$ is a combinatorial class of arrangements such that every arrangement in $\mathfrak{C}$ is free than $\mathcal{A} \in \mathfrak{C}$ is called *combinatorially free.*

For $X \in L(\mathcal{A})$ the *localization* $\mathcal{A}_X$ of $\mathcal{A}$ at X is defined by:

$$\mathcal{A}_X := \{H \in \mathcal{A} \mid X \subseteq H\},$$

and the *restriction* $\mathcal{A}^X$ of $\mathcal{A}$ to X is defined by:

$$\mathcal{A}^X := \{X \cap H \mid H \in \mathcal{A} \setminus \mathcal{A}_X\}.$$

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two arrangements in V_1 respectively V_2. Then their *product* $\mathcal{A}_1 \times \mathcal{A}_2$ is defined as the arrangement in $V = V_1 \oplus V_2$ consisting of the following hyperplanes:

$$\mathcal{A}_1 \times \mathcal{A}_2 := \{H_1 \oplus V_2 \mid H_1 \in \mathcal{A}_1\} \cup \{V_1 \oplus H_2 \mid H_2 \in \mathcal{A}_2\}.$$

We note the following facts about products (cf. [OT92, Ch. 2]):

- $|\mathcal{A}_1 \times \mathcal{A}_2| = |\mathcal{A}_1| + |\mathcal{A}_2|$.
- $L(\mathcal{A}_1 \times \mathcal{A}_2) = \{X_1 \oplus X_2 \mid X_1 \in L(\mathcal{A}_1) \text{ and } X_2 \in L(\mathcal{A}_2)\}$.
- $(\mathcal{A}_1 \times \mathcal{A}_2)^X = \mathcal{A}_1^{X_1} \times \mathcal{A}_2^{X_2}$ if $X = X_1 \oplus X_2$ with $X_i \in L(\mathcal{A}_i)$.

Let $S = S(V^*)$ be the symmetric algebra of the dual space. We fix a basis $x_1, \ldots, x_\ell$ for V^* and identify S with the polynomial ring $\mathbb{K}[x_1, \ldots, x_\ell]$. The algebra S is equipped with the grading by polynomial degree: $S = \bigoplus_{p \in \mathbb{Z}} S_p$, where S_p is the set of homogeneous polynomials of degree p ($S_p = \{0\}$ for $p < 0$).

A $\mathbb{K}$-linear map $\theta : S \to S$ which satisfies $\theta(fg) = \theta(f)g + f\theta(g)$ is called a $\mathbb{K}$-*derivation*. Let $\mathrm{Der}(S)$ be the S-module of $\mathbb{K}$-derivations of S. It is a free S-module with basis $D_1, \ldots, D_\ell$ where D_i is the partial derivation $\partial/\partial x_i$. We say that $\theta \in \mathrm{Der}(S)$ is *homogeneous of polynomial degree* p provided $\theta = \sum_{i=1}^{\ell} f_i D_i$ with $f_i \in S_p$ for each $1 \leqslant i \leqslant \ell$. In this case we write $\mathrm{pdeg}\,\theta = p$. We obtain a $\mathbb{Z}$-grading for the S-module $\mathrm{Der}(S)$: $\mathrm{Der}(S) = \bigoplus_{p \in \mathbb{Z}} \mathrm{Der}(S)_p$.

Definition 6. For $H \in \mathcal{A}$ we fix $\alpha_H \in V^*$ with $H = \ker(\alpha_H)$. The *module of $\mathcal{A}$-derivations* is defined by

$$D(\mathcal{A}) := \{\theta \in \mathrm{Der}(S) \mid \theta(\alpha_H) \in \alpha_H S \text{ for all } H \in \mathcal{A}\}.$$

We say that $\mathcal{A}$ is *free* if the module of $\mathcal{A}$-derivations is a free S-module.

If $\mathcal{A}$ is a free arrangement we may choose a homogeneous basis $\{\theta_1, \ldots, \theta_\ell\}$ for $D(\mathcal{A})$. Then the polynomial degrees of the θ_i are called the *exponents* of $\mathcal{A}$ and they are uniquely determined by $\mathcal{A}$, [OT92, Def. 4.25]. We write $\exp(\mathcal{A}) := (\mathrm{pdeg}\,\theta_1, \ldots, \mathrm{pdeg}\,\theta_\ell)$. Note that the empty arrangement Φ_ℓ is free with $\exp(\Phi_\ell) = (0, \ldots, 0) \in \mathbb{Z}^\ell$. If $d_1, \ldots, d_\ell \in \mathbb{Z}$ with $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_\ell$ we write $(d_1, \ldots, d_\ell)_\leqslant$.

The notion of freeness is compatible with products of arrangements:

Proposition 7 ([OT92, Prop. 4.28]). *Let $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ be a product of two arrangements. Then $\mathcal{A}$ is free if and only if both $\mathcal{A}_1$ and $\mathcal{A}_2$ are free. In this case if* $\exp(\mathcal{A}_i) = (d_1^i, \dots, d_{\ell_i}^i)$ *for $i = 1, 2$ then*

$$\exp(\mathcal{A}) = (d_1^1, \dots, d_{\ell_1}^1, d_1^2, \dots, d_{\ell_2}^2).$$

The following theorem provides a useful tool to prove the freeness of arrangements.

Theorem 8 (Addition-Deletion [OT92, Thm. 4.51]). *Let $\mathcal{A}$ be a hyperplane arrangement and $H_0 \in \mathcal{A}$. We call $(\mathcal{A}, \mathcal{A}' = \mathcal{A} \setminus \{H_0\}, \mathcal{A}'' = \mathcal{A}^{H_0})$ a* triple of arrangements. *Any two of the following statements imply the third:*

1. *$\mathcal{A}$ is free with* $\exp(\mathcal{A}) = (b_1, \dots, b_{l-1}, b_\ell)$,
2. *$\mathcal{A}'$ is free with* $\exp(\mathcal{A}') = (b_1, \dots, b_{\ell-1}, b_\ell - 1)$,
3. *$\mathcal{A}''$ is free with* $\exp(\mathcal{A}'') = (b_1, \dots, b_{\ell-1})$.

The preceding theorem motivates the following definition.

Definition 9 ([OT92, Def. 4.53]). The class $\mathfrak{IF}$ of *inductively free* arrangements is the smallest class of arrangements which satisfies

1. the empty arrangement Φ_ℓ of rank ℓ is in $\mathfrak{IF}$ for $\ell \geqslant 0$,
2. if there exists a hyperplane $H_0 \in \mathcal{A}$ such that $\mathcal{A}'' \in \mathfrak{IF}$, $\mathcal{A}' \in \mathfrak{IF}$, and $\exp(\mathcal{A}'') \subset \exp(\mathcal{A}')$, then $\mathcal{A}$ also belongs to $\mathfrak{IF}$.

Here $(\mathcal{A}, \mathcal{A}', \mathcal{A}'') = (\mathcal{A}, \mathcal{A} \setminus \{H_0\}, \mathcal{A}^{H_0})$ is a triple as in Theorem 8.

The class $\mathfrak{IF}$ is easily seen to be combinatorial [CH15, Lem. 2.5].

The following result was a major step in the investigation of freeness properties for reflection arrangements.

Theorem 10 ([HR15, Thm. 1.1], [BC12, Thm. 5.14]). *For W a finite complex reflection group, the reflection arrangement $\mathcal{A}(W)$ is inductively free if and only if W does not admit an irreducible factor isomorphic to a monomial group $G(r, r, \ell)$ for $r, \ell \geqslant 3$, G_{24}, G_{27}, G_{29}, G_{31}, G_{33}, or G_{34}.*

Definition 11 (cf. [AT16]). Let $\mathcal{A}$ be an arrangement with $|\mathcal{A}| = n$. We say that $\mathcal{A}$ has a *free filtration* if there are subarrangements

$$\emptyset = \mathcal{A}_0 \subsetneq \mathcal{A}_1 \subsetneq \cdots \subsetneq \mathcal{A}_{n-1} \subsetneq \mathcal{A}_n = \mathcal{A}$$

such that $|\mathcal{A}_i| = i$ and $\mathcal{A}_i$ is free for all $1 \leqslant i \leqslant n$.

Very recently, Abe [Abe18a] introduced the class $\mathfrak{AF}$ of *additionally free* arrangements. Arrangements in $\mathfrak{AF}$ are by definition exactly the arrangements admitting a free filtration. Furthermore, it is a direct consequence of [Abe18a, Thm. 1.4] that the class $\mathfrak{AF}$ is combinatorial.

3 Multiple Addition Theorem

The following theorem presented in [ABC$^+$16] is a variant of the addition part ((2) and (3) imply (1)) of Theorem 8.

Theorem 12 (Multiple Addition Theorem (MAT))**.** *Let $\mathcal{A}'$ be a free arrangement with* $\exp(\mathcal{A}') = (d_1, \dots, d_\ell)_\leqslant$ *and $1 \leqslant p \leqslant \ell$ the multiplicity of the highest exponent, i.e.,*

$$d_{\ell-p} < d_{\ell-p+1} = \cdots = d_\ell =: d.$$

Let $H_1, \dots, H_q$ be hyperplanes with $H_i \notin \mathcal{A}'$ for $i = 1, \dots, q$. Define

$$\mathcal{A}''_j := (\mathcal{A}' \cup \{H_j\})^{H_j} = \{H \cap H_j \mid H \in \mathcal{A}'\}, \quad j = 1, \dots, q.$$

Assume that the following three conditions are satisfied:

(1) $X := H_1 \cap \cdots \cap H_q$ is q-codimensional.

(2) $X \not\subseteq \bigcup_{H \in \mathcal{A}'} H$.

(3) $|\mathcal{A}'| - |\mathcal{A}''_j| = d$ for $1 \leqslant j \leqslant q$.

Then $q \leqslant p$ and $\mathcal{A} := \mathcal{A}' \cup \{H_1, \dots, H_q\}$ is free with $\exp(\mathcal{A}) = (d_1, \dots, d_{\ell-q}, d+1, \dots, d+1)_\leqslant$.

Note that in contrast to Theorem 8 no freeness condition on the restriction is needed to conclude the freeness of $\mathcal{A}$ in Theorem 12. The MAT motivates the following definition.

Definition 13. The class $\mathfrak{MF}$ of *MAT-free* arrangements is the smallest class of arrangements subject to

(i) Φ_ℓ belongs to $\mathfrak{MF}$, for every $\ell \geqslant 0$;

(ii) if $\mathcal{A}' \in \mathfrak{MF}$ with $\exp(\mathcal{A}') = (d_1, \dots, d_\ell)_\leqslant$ and $1 \leqslant p \leqslant \ell$ the multiplicity of the highest exponent $d = d_\ell$, and if $H_1, \dots, H_q$, $q \leqslant p$ are hyperplanes with $H_i \notin \mathcal{A}'$ for $i = 1, \dots, q$ such that:

  (1) $X := H_1 \cap \cdots \cap H_q$ is q-codimensional,
  (2) $X \not\subseteq \bigcup_{H \in \mathcal{A}'} H$,
  (3) $|\mathcal{A}'| - |(\mathcal{A}' \cup \{H_j\})^{H_j}| = d$, for $1 \leqslant j \leqslant q$,

then $\mathcal{A} := \mathcal{A}' \cup \{H_1, \dots, H_q\}$ also belongs to $\mathfrak{MF}$ and has exponents $\exp(\mathcal{A}) = (d_1, \dots, d_{\ell-q}, d+1, \dots, d+1)_\leqslant$.

Abe and Terao [AT19] proved the following generalization of Theorem 12:

Theorem 14 (Multiple Addition Theorem 2 (MAT2), [AT19, Thm. 1.4])**.** *Assume that* $\mathcal{A}'$ *is a free arrangement with* $\exp(\mathcal{A}') = (d_1, d_2, \ldots, d_\ell)_\leqslant$*. Let*

$$t := \begin{cases} \min\{i \mid d_i \neq 0\} & \text{if } \mathcal{A}' \neq \Phi_\ell \\ 0 & \text{if } \mathcal{A}' = \Phi_\ell \end{cases}.$$

For $H_s, \ldots, H_\ell \notin \mathcal{A}$ *with* $s > t$*, define* $\mathcal{A}''_j := (\mathcal{A}' \cup \{H_j\})^{H_j}$*,* $\mathcal{A} := \mathcal{A}' \cup \{H_s, \ldots, H_\ell\}$ *and assume the following conditions:*

(1) $X := \bigcap_{i=s}^{\ell} H_i$ *is* $(\ell - s + 1)$*-codimensional,*

(2) $X \not\subset \bigcup_{K \in \mathcal{A}'} K$*, and*

(3) $|\mathcal{A}'| - |\mathcal{A}''_j| = d_j$ *for* $j = s, \ldots, \ell$*.*

Then $\mathcal{A}$ *is free with exponents* $(d_1, d_2, \ldots, d_{s-1}, d_s + 1, \ldots, d_\ell + 1)_\leqslant$*. Moreover, there is a basis* $\theta_1, \theta_2, \ldots, \theta_{s-1}, \eta_s, \ldots, \eta_\ell$ *for* $D(\mathcal{A}')$ *such that* $\deg \theta_i = d_i$*,* $\deg \eta_j = d_j$*,* $\theta_i \in D(\mathcal{A})$ *and* $\eta_j \in D(\mathcal{A} \setminus \{H_j\})$ *for all* i *and* j*.*

This in turn motivates:

Definition 15. The class $\mathfrak{MF}'$ of *MAT2-free* arrangements is the smallest class of arrangements subject to

(i) Φ_ℓ belongs to $\mathfrak{MF}'$, for every $\ell \geqslant 0$;

(ii) if $\mathcal{A}' \in \mathfrak{MF}'$ with $\exp(\mathcal{A}') = (d_1, d_2, \ldots, d_\ell)_\leqslant$ and if $H_s, \ldots, H_\ell$ are hyperplanes with $H_i \notin \mathcal{A}'$ for $i = s, \ldots, \ell$, where

$$s > \begin{cases} \min\{i \mid d_i \neq 0\} & \text{if } \mathcal{A}' \neq \Phi_\ell \\ 0 & \text{if } \mathcal{A}' = \Phi_\ell \end{cases},$$

and with

(1) $X := H_s \cap \cdots \cap H_\ell$ is $(\ell - s + 1)$-codimensional,
(2) $X \nsubseteq \bigcup_{H \in \mathcal{A}'} H$,
(3) $|\mathcal{A}'| - |(\mathcal{A}' \cup \{H_j\})^{H_j}| = d_j$ for $s \leqslant j \leqslant \ell$,

then $\mathcal{A} := \mathcal{A}' \cup \{H_s, \ldots, H_\ell\}$ also belongs to $\mathfrak{MF}'$ and has exponents $\exp(\mathcal{A}) = (d_1, \ldots, d_{s-1}, d_s + 1, \ldots, d_\ell + 1)_\leqslant$.

We note the following:

Remark 16. 1. We have $\mathfrak{MF} \subseteq \mathfrak{MF}'$.

2. If $\mathcal{A}$ is a free arrangement with $\exp(\mathcal{A}) = (0, \ldots, 0, 1, \ldots, 1, d, \ldots, d)_\leqslant$, i.e. $\mathcal{A}$ has only two distinct exponents $\neq 0$, then it is clear from the definitions that $\mathcal{A}$ is MAT2-free if and only if $\mathcal{A}$ is MAT-free.

Example 17. 1. If $\mathrm{rk}(\mathcal{A}) = 2$ then $\mathcal{A}$ is MAT-free and therefore MAT2-free too.

2. Every ideal subarrangement of a Weyl arrangement is MAT-free and therefore also MAT2-free, [ABC+16].

Lemma 18. *The classes* $\mathfrak{MF}$ *and* $\mathfrak{MF}'$ *are combinatorial.*

Proof. The class of all empty arrangements is combinatorial and contained in $\mathfrak{MF}$. Let $\mathcal{A} \in \mathfrak{MF}$ ($\mathcal{A} \in \mathfrak{MF}'$). Since conditions (1)–(3) in Definition 13 (respectively Definition 15) only depend on $L(\mathcal{A})$ the claim follows. See also [AT19, Thm. 5.1]. □

If an arrangement $\mathcal{A}$ is MAT-free, the MAT-steps yield a partition of $\mathcal{A}$ whose dual partition gives the exponents of $\mathcal{A}$. Vice versa, the existence of such a partition suffices for the MAT-freeness of the arrangement:

Lemma 19. *Let $\mathcal{A}$ be an ℓ-arrangement. Then $\mathcal{A}$ is MAT-free if and only if there exists a partition $\pi = (\pi_1 | \cdots | \pi_n)$ of $\mathcal{A}$ where for all $0 \leqslant k \leqslant n-1$,*

(1) $\mathrm{rk}(\pi_{k+1}) = |\pi_{k+1}|$,

(2) $\cap_{H \in \pi_{k+1}} H = X_{k+1} \nsubseteq \bigcup_{H' \in \mathcal{A}_k} H'$ *where* $\mathcal{A}_k = \bigcup_{i=1}^k \pi_i$,

(3) $|\mathcal{A}_k| - |(\mathcal{A}_k \cup \{H\})^H| = k$ *for all* $H \in \pi_{k+1}$.

In this case $\mathcal{A}$ has exponents $\exp(\mathcal{A}) = (d_1, \ldots, d_\ell)_\leqslant$ *with* $d_i = |\{k \mid |\pi_k| \geqslant \ell - i + 1\}|$.

Proof. This is immediate from the definition. □

Definition 20. If π is a partition as in Lemma 19 then π is called an *MAT-partition* for $\mathcal{A}$.

If we have chosen a linear ordering $\mathcal{A} = \{H_1, \ldots, H_m\}$ of the hyperplanes in $\mathcal{A}$, to specify the partition π, we give the corresponding ordered set partition of $[m] = \{1, \ldots, m\}$.

Example 21. Supersolvable arrangements, a proper subclass of inductively free arrangements [OT92, Thm. 4.58], are not necessarily MAT2-free: an easy calculation shows that the arrangement denoted $\mathcal{A}(10, 1)$ in [Grü09] is supersolvable but not MAT2-free. In particular $\mathcal{A}(10, 1)$ is neither MAT-free.

Restrictions of MAT2-free (MAT-free) arrangements are not necessarily MAT2-free (MAT-free):

Example 22. Let $\mathcal{A} = \mathcal{A}(E_6)$ be the Weyl arrangement of the Weyl group of type E_6. Then $\mathcal{A}$ is MAT-free by Example 17(2). Let $H \in \mathcal{A}$. A simple calculation (with the computer) shows that $\mathcal{A}^H$ is not MAT2-free.

We have two simple necessary conditions for MAT-freeness respectively MAT2-freeness. The first one is:

Lemma 23. *Let $\mathcal{A}$ be a non-empty MAT2-free arrangement with exponents* $\exp(\mathcal{A}) = (d_1, \ldots, d_\ell)_\leqslant$. *Then there is an $H \in \mathcal{A}$ such that $|\mathcal{A}| - |\mathcal{A}^H| = d_\ell$. In particular, the same holds, if $\mathcal{A}$ is MAT-free.*

Proof. By definition there are $H_q, \ldots, H_\ell \in \mathcal{A}$, $2 \leqslant q$ such that $\mathcal{A}' := \mathcal{A} \setminus \{H_q, \ldots, H_\ell\}$ is MAT2-free. Furthermore by condition (1) the hyperplanes $H_q, \ldots, H_\ell$ are linearly independent. Let $H := H_\ell$. By condition (2), we have $X = \cap_{i=q}^{\ell} H_i \nsubseteq \cup_{H' \in \mathcal{A}'} H'$ and thus $|\mathcal{A}^H| = |(\mathcal{A}' \cup \{H\})^H| + \ell - q$. Now

$$|\mathcal{A}'| - |(\mathcal{A}' \cup \{H\})^H| = d_\ell - 1$$

by condition (3) and hence

$$|\mathcal{A}| - |\mathcal{A}^H| = |\mathcal{A}'| + \ell - q + 1 - |(\mathcal{A}' \cup \{H\})^H| - \ell + q = d_\ell. \qquad \square$$

The second one is:

Lemma 24. *Let $\mathcal{A}$ be an MAT2-free arrangement. Then $\mathcal{A}$ has a free filtration, i.e. $\mathcal{A}$ is additionally free. In particular, the same is true, if $\mathcal{A}$ is MAT-free.*

Proof. Let $\mathcal{A}$ be MAT2-free. Then by definition there are $H_q, \ldots, H_\ell \in \mathcal{A}$ such that $\mathcal{A}' := \mathcal{A} \setminus \{H_q, \ldots, H_\ell\}$ is MAT2-free and conditions (1)–(3) are satisfied. Set $\mathcal{B} := \{H_q, \ldots, H_\ell\}$. By [AT19, Cor. 3.2] for all $\mathcal{C} \subseteq \mathcal{B}$ the arrangement $\mathcal{A}' \cup \mathcal{C}$ is free. Hence by induction $\mathcal{A}$ has a free filtration. □

An MAT2-free but not MAT-free arrangement

We now provide an example of an arrangement which is MAT2-free but not MAT-free.

Example 25. Let $\mathcal{A}$ be the arrangement defined by

$$\begin{aligned}\mathcal{A} &:= \{H_1, \ldots, H_{10}\} \\ &:= \{(1,0,0)^\perp, (0,1,0)^\perp, (0,0,1)^\perp, (1,1,0)^\perp, (1,2,0)^\perp, (0,1,1)^\perp, \\ &\qquad (1,3,0)^\perp, (1,1,1)^\perp, (2,3,0)^\perp, (1,3,1)^\perp\}.\end{aligned}$$

It is not hard to see that $\mathcal{A}$ is inductively free (actually supersolvable) with $\exp(\mathcal{A}) = (1, 4, 5)$.

Proposition 26. *The arrangement $\mathcal{A}$ from Example 25 is MAT2-free.*

Proof. Let $\mathcal{B}_1 = \{H_1, H_2, H_3\}$, $\mathcal{B}_2 = \{H_4\}$, $\mathcal{B}_3 = \{H_5, H_6\}$, $\mathcal{B}_4 = \{H_7, H_8\}$, $\mathcal{B}_5 = \{H_9, H_{10}\}$, and $\mathcal{A}_k = \cup_{i=1}^{k} \mathcal{B}_i$ for $1 \leqslant k \leqslant 5$. It is clear that $\mathcal{A}_1$ is MAT2-free. A simple linear algebra computation shows that the addition of $\mathcal{B}_{i+1}$ to $\mathcal{A}_i$ for $1 \leqslant i \leqslant 4$ satisfies Condition (1)–(3) of Definition 15. Hence $\mathcal{A} = \mathcal{A}_5$ is MAT2-free. □

Proposition 27. *The arrangement $\mathcal{A}$ from Example 25 is not MAT-free.*

Proof. Suppose $\mathcal{A}$ is MAT-free and $\pi = (\pi_1, \dots, \pi_5)$ is an MAT-partition. Since $\exp(\mathcal{A}) = (1, 4, 5)$ the last block π_5 has to be a singleton, i.e. $\pi_5 = \{H\}$. By Condition (3) of Lemma 19 we have $|\mathcal{A}^H| = 5$ and the only hyperplane with this property is $H_9 = (2, 3, 0)^\perp$. Similarly π_4 can only contain one of H_3, H_6, H_8, H_{10}. But looking at their intersections we see that all of the latter are contained in another hyperplane of $\mathcal{A}$, e.g. $H_3 \cap H_8 \subseteq H_4$. This contradicts Condition (2). Hence $\mathcal{A}$ is not MAT-free. □

As a direct consequence we get:

Proposition 28. *We have*

$$\mathfrak{MF} \subsetneq \mathfrak{MF}'.$$

Products of MAT-free and MAT2-free arrangements

As for freeness in general (Proposition 7), the product construction is compatible with the notion of MAT-freeness:

Theorem 29. *Let $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ be a product of two arrangements. Then $\mathcal{A} \in \mathfrak{MF}$ if and only if $\mathcal{A}_1 \in \mathfrak{MF}$ and $\mathcal{A}_2 \in \mathfrak{MF}$.*

Proof. Assume $\mathcal{A}_i$ is an arrangement in the vector space V_i of dimension ℓ_i for $i = 1, 2$. We argue by induction on $|\mathcal{A}|$. If $|\mathcal{A}| = 0$, i.e. $\mathcal{A}_1 = \Phi_{\ell_1}$, and $\mathcal{A}_2 = \Phi_{\ell_2}$ then the statement is clear. Assume $\mathcal{A}_1$ is MAT-free with $\exp(\mathcal{A}_1) = (d_1^1, \dots, d_{\ell_1}^1)_\leqslant$ and $\mathcal{A}_2$ is MAT-free with $\exp(\mathcal{A}_1) = (d_1^2, \dots, d_{\ell_2}^2)_\leqslant$. Then without loss of generality $d := d_{\ell_1}^1 \geqslant d_{\ell_2}^2$. Let q_i be the multiplicity of the exponent d in $\exp(\mathcal{A}_i)$ for $i = 1, 2$ (note that $q_2 = 0$ if $d > d_{\ell_2}^2$). Then since $\mathcal{A}_i$ is MAT-free there are hyperplanes $\{H_1^i, \dots, H_{q_i}^i\} \subseteq \mathcal{A}_i$ such that $\mathcal{A}_i' := \mathcal{A}_i \setminus \{H_1^i, \dots, H_{q_i}^i\}$ is MAT-free, i.e. they satisfy Conditions (1)–(3) from Definition 13. Now by the induction hypothesis $\mathcal{A}' = \mathcal{A}_1' \times \mathcal{A}_2'$ is MAT-free and clearly $\{H_1^1 \oplus V_2, \dots, H_{q_1}^1 \oplus V_2\} \cup \{V_1 \oplus H_1^2, \dots, V_1 \oplus H_{q_2}^2\}$ satisfy Conditions (1)–(3). Hence $\mathcal{A}$ is MAT-free.

Conversely assume $\mathcal{A}$ is MAT-free with $\exp(\mathcal{A}) = (d_1, \dots, d_\ell)_\leqslant$. By Proposition 7 both factors $\mathcal{A}_1$ and $\mathcal{A}_2$ are free with $\exp(\mathcal{A}_i) = (d_1^i, \dots, d_{\ell_i}^i)_\leqslant$ and without loss of generality $d_\ell = d_{\ell_1}^1 \geqslant d_{\ell_2}^2$. Assume further that q_i is the multiplicity of d_ℓ in $\exp(\mathcal{A}_i)$ and q is the multiplicity of d_ℓ in $\exp(\mathcal{A})$, i.e. $q = q_1 + q_2$. There are hyperplanes $\{H_1, \dots, H_q\} \subset \mathcal{A}$ such that $\mathcal{A}' = \mathcal{A} \setminus \{H_1, \dots, H_q\}$ is MAT-free with $\exp(\mathcal{A}') = (d_1, \dots, d_{\ell-q}, d_{\ell-q+1} - 1, \dots, d_\ell - 1)_\leqslant$, and Conditions (1)–(3) are satisfied. We may further assume that $H_i = H_i^1 \oplus V_2$ for $1 \leqslant i \leqslant q_1$ and $H_j = V_1 \oplus H_{j-q_1}^2$ for $q_1 + 1 \leqslant j \leqslant q$. Let $\mathcal{A}_i' = \mathcal{A}_i \setminus \{H_1^i, \dots, H_{q_i}^i\}$ for $i = 1, 2$. Note that if $d_\ell > d_{\ell_2}^2$ we have $q_2 = 0$ and $\mathcal{A}_2' = \mathcal{A}_2$. But at least we have $\mathcal{A}_1' \subsetneq \mathcal{A}_1$. Then $\mathcal{A}' = \mathcal{A}_1' \times \mathcal{A}_2'$, $|\mathcal{A}'| < |\mathcal{A}|$ and by the induction hypothesis $\mathcal{A}_1'$ and $\mathcal{A}_2'$ are MAT-free and Conditions (1) and (2) are clearly satified for $\mathcal{A}_i'$ and $\{H_1^i, \dots, H_{q_i}^i\}$. But since

$$\begin{aligned} d_\ell - 1 &= |\mathcal{A}'| - |(\mathcal{A}' \cup \{H_i\})^{H_i}| \\ &= |\mathcal{A}_1'| + |\mathcal{A}_2'| - (|(\mathcal{A}_1 \cup \{H_i^1\})^{H_i^1}| + |\mathcal{A}_2'|) \\ &= |\mathcal{A}_1'| - |(\mathcal{A}_1 \cup \{H_i^1\})^{H_i^1}| \end{aligned}$$

for $1 \leqslant i \leqslant q_1$ and

$$\begin{aligned} d_\ell - 1 &= |\mathcal{A}'| - |(\mathcal{A}' \cup \{H_j\})^{H_j}| \\ &= |\mathcal{A}'_1| + |\mathcal{A}'_2| - (|(\mathcal{A}_1 \cup \{H^2_{j-q_1}\})^{H^2_{j-q_1}}| + |\mathcal{A}'_2|) \\ &= |\mathcal{A}'_1| - |(\mathcal{A}_1 \cup \{H^2_{j-q_1}\})^{H^2_{j-q_1}}| \end{aligned}$$

for $q_1 + 1 \leqslant j \leqslant q_2$, Condition (3) is also satisfied for $\mathcal{A}'_1$ and $\mathcal{A}'_2$. Hence both $\mathcal{A}_1$ and $\mathcal{A}_2$ are MAT-free. □

Altenatively, one can prove Theorem 29 by observing that MAT-Partitions for $\mathcal{A}_1$ and $\mathcal{A}_2$ are directly obtained from an MAT-Partition for $\mathcal{A}$: take the non-empty factors of each block in the same order, and vise versa: take the products of the blocks of partitions for $\mathcal{A}_1$ and $\mathcal{A}_2$.

Remark 30. Thanks to the preceding theorem, our classification of MAT-free irreducible reflection arrangements proved in the next 2 sections gives actually a classification of all MAT-free reflection arrangements: a reflection arrangement $\mathcal{A}(W)$ is MAT-free if and only if it has no irreducible factor isomorphic to one of the non-MAT-free irreducible reflection arrangements listed in Theorem 1.

In contrast to MAT-freeness, the weaker notion of MAT2-freeness is not compatible with products as the following example shows:

Example 31. Let $\mathcal{A}_1$ be the MAT2-free but not MAT-free arrangement of Example 25 with exponents $\exp(\mathcal{A}_1) = (1, 4, 5)$. Let $\zeta = \frac{1}{2}(-1 + i\sqrt{3})$ be a primitive cube root of unity, and let $\mathcal{A}_2$ be the arrangement defined by the following linear forms:

$$\begin{aligned} \mathcal{A}_2 &:= \{H^2_1, \ldots, H^2_{10}\} \\ &:= \{(1,0,0)^\perp, (0,1,0)^\perp, (0,0,1)^\perp, (1,-\zeta,0)^\perp, (1,0,-\zeta)^\perp \\ &\qquad (1,-\zeta^2,0)^\perp, (1,0,-\zeta^2)^\perp, (1,-1,0)^\perp, (1,0,-1)^\perp, (0,1,-\zeta)^\perp\}. \end{aligned}$$

A linear algebra computation shows that $\pi = (1,2,3|4,5|6,7|8,9|10)$ is an MAT-partition for $\mathcal{A}_2$. In particular $\mathcal{A}_2$ is MAT2-free with $\exp(\mathcal{A}_2) = (1,4,5)$.

Now by Proposition 7 the product $\mathcal{A} := \mathcal{A}_1 \times \mathcal{A}_2$ is free with $\exp(\mathcal{A}) = (1,1,4,4,5,5)$. Suppose $\mathcal{A}$ is MAT2-free. Then either there are hyperplanes $H_1 \in \mathcal{A}_1$ and $H_2 \in \mathcal{A}_2$ such that $\mathcal{A}' = \mathcal{A}'_1 \times \mathcal{A}'_2$ is MAT2-free with exponents $\exp(\mathcal{A}') = (1,1,4,4,4,4)$ where $\mathcal{A}'_i = \mathcal{A}_i \setminus \{H_i\}$. Or there are hyperplanes $H^1_1, H^1_2 \in \mathcal{A}_1$, $H^2_1, H^2_2 \in \mathcal{A}_2$ such that $\mathcal{A}' = \mathcal{A}'_1 \times \mathcal{A}'_2$ is MAT2-free with exponents $\exp(\mathcal{A}') = (1,1,3,3,4,4)$ where $\mathcal{A}'_i = \mathcal{A}_i \setminus \{H^i_1, H^i_2\}$.

In the first case $\mathcal{A}'$ is actually MAT-free by Remark 16. But then by Theorem 29 $\mathcal{A}'_2$ is MAT-free and $\mathcal{A}_2$ is MAT-free too which is a contradiction.

In the second case $H^1_1 \oplus V_2, H^1_2 \oplus V_2, V_1 \oplus H^2_1, V_1 \oplus H^2_2$ satisfy Condition (1)–(3) of Definition 15. But by Condition (3) we have

$$|\mathcal{A}'_1| - |(\mathcal{A}'_1 \cup \{H^1_1\})^{H^1_1}| = 4$$

and

$$|\mathcal{A}_1'| - |(\mathcal{A}_1' \cup \{H_2^1\})^{H_2^1}| = 3.$$

But an easy calculation shows that there are no two hyperplanes in $\mathcal{A}_1$ with this property and which also satisfy Condition (2)–(3). This is a contradiction and hence $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ is not MAT2-free.

4 MAT-free imprimitive reflection groups

Definition 32 ([OT92, §6.4])**.** Let $x_1, \ldots, x_\ell$ be a basis of V^*. Let $\zeta = \exp(\frac{2\pi i}{r})$ ($r \in \mathbb{N}$) be a primitive r-th root of unity. Define the linear forms $\alpha_{ij}(\zeta^k) \in V^*$ by

$$\alpha_{ij}(\zeta^k) = x_i - \zeta^k x_j$$

and the hyperplanes

$$H_{ij}(\zeta^k) = \ker(\alpha_{ij}(\zeta^k)).$$

for $1 \leqslant i, j \leqslant \ell$ and $1 \leqslant k \leqslant r$. Then the reflection arrangement of the imprimitive complex reflection group $G(r, 1, \ell)$ can be defined by:

$$\mathcal{A}(G(r,1,\ell)) = \{\ker(x_i) \mid 1 \leqslant i \leqslant \ell\} \dot\cup \{H_{ij}(\zeta^k) \mid 1 \leqslant i < j \leqslant \ell,\, 1 \leqslant k \leqslant r\}.$$

Proposition 33. *Let $\mathcal{A} = \mathcal{A}(G(r,1,\ell))$. Let*

$$\pi_{11} := \{\ker(x_i) \mid 1 \leqslant i \leqslant \ell\},$$

and

$$\pi_{ij} := \{H_{(i-1)k}(\zeta^j) \mid i \leqslant k \leqslant \ell\},$$

for $2 \leqslant i \leqslant \ell$, $1 \leqslant j \leqslant r$. Then

$$\begin{aligned} \pi &= (\pi_{ij})_{\substack{1 \leqslant i \leqslant \ell, \\ 1 \leqslant j \leqslant m_i}}, \ m_i = \begin{cases} 1 & \text{for } i = 1 \\ r & \text{for } 2 \leqslant i \leqslant \ell \end{cases} \\ &= (\pi_{11} | \pi_{21} | \cdots | \pi_{2r} | \cdots | \pi_{\ell r}) \end{aligned}$$

is an MAT-partition of $\mathcal{A}$. In particular $\mathcal{A} \in \mathfrak{MF}$ with exponents

$$\exp(\mathcal{A}) = (1, r+1, 2r+1, \ldots, (l-1)r+1).$$

Proof. We verify Conditions (1)–(3) from Lemma 19 in turn.

Let

$$\mathcal{A}_{ij} := (\bigcup_{\substack{1 \leqslant a \leqslant i-1, \\ 1 \leqslant b \leqslant m_a}} \pi_{ab}) \cup (\bigcup_{1 \leqslant b \leqslant j} \pi_{ib})$$

and

$$\mathcal{A}_{ij}' := (\bigcup_{\substack{1 \leqslant a \leqslant i-1, \\ 1 \leqslant b \leqslant m_a}} \pi_{ab}) \cup (\bigcup_{1 \leqslant b \leqslant j-1} \pi_{ib}).$$

For π_{11} we clearly have $|\pi_{11}| = \mathrm{rk}(\pi_{11}) = \ell$. Similarly for $2 \leqslant i \leqslant \ell$, $1 \leqslant j \leqslant r$ we have $|\pi_{ij}| = \mathrm{rk}(\pi_{ij}) = \ell - i + 1$ since all the defining linear forms $\alpha_{(i-1)k}(\zeta^j)$ $(i \leqslant k \leqslant \ell)$ for the hyperplanes in π_{ij} are linearly independent. Thus Condition (1) holds.

Furthermore, the forms $\{\alpha_{ac}(\zeta^b)\}\dot\cup\{\alpha_{(i-1)k}(\zeta^j) \mid i \leqslant k \leqslant \ell\}$ are linearly independent for all $1 \leqslant a \leqslant i-1$, $1 \leqslant b \leqslant j-1$, and $a+1 \leqslant c \leqslant \ell$, i.e. $\cap_{H\in\pi_{ij}} H =: X_{ij} \nsubseteq H$ for all $H \in \mathcal{A}'_{ij}$. Hence Condition (2) is also satisfied.

To verify Condition (3) let $H = H_{(i-1)k}(\zeta^j) \in \pi_{ij}$ for a fixed $1 \leqslant k \leqslant r$. We show

$$|\mathcal{A}'_{ij}| - (j + (i-2)r) = |(\mathcal{A}'_{ij})^H|.$$

Let $H'_a := H_{(i-1)k}(\zeta^a) \in \mathcal{A}'_{ij}$, $1 \leqslant a \leqslant j-1$. Then

$$\mathcal{B} := (\mathcal{A}'_{ij})_{H\cap H'_a} = \{\ker(x_{i-1}), \ker(x_k)\}\dot\cup\{H'_b \mid 1 \leqslant b \leqslant j-1\},$$

and $\mathrm{rk}(\mathcal{B}) = 2$. So all $H' \in \mathcal{B}$ give the same intersection with H and $|\mathcal{B}| = j+1$. For $H' = H_{a(i-1)}(\zeta^b) \in \mathcal{A}'_{ij}$ with $a \leqslant i-2$, and $1 \leqslant b \leqslant r$ we have

$$\mathcal{C} := (\mathcal{A}'_{ij})_{H\cap H'} = \{H', H_{ak}(\zeta^(j+b))\},$$

$|\mathcal{C}| = 2$ and there are exactly $(i-2)r$ such H'. All other $H'' \in \mathcal{A}'_{ij}$ intersect H simply. Hence

$$\begin{aligned}|(\mathcal{A}'_{ij})^H)| &= |\mathcal{A}'_{ij}| - (|\mathcal{B}|-1) - (i-2)r(|\mathcal{C}|-1) \\ &= |\mathcal{A}'_{ij}| - j - (i-2)r,\end{aligned}$$

or $|\mathcal{A}'_{ij}| - |(\mathcal{A}'_{ij})^H)| = \sum_{a=1}^{i-1} m_i + (j-1)$. This finishes the proof. □

Proposition 34. *Let $\mathcal{A} = \mathcal{A}(G(r,r,\ell))$ $(r,\ell \geqslant 3)$. Then $\mathcal{A}$ is not MAT2-free. In particular $\mathcal{A}$ is not MAT-free.*

Proof. By [OT92, Prop. 6.85] the arrangement $\mathcal{A}$ is free with $\exp(\mathcal{A}) = (d_1,\dots,d_\ell) = (1, r+1, 2r+1, \dots, (\ell-2)r+1, (\ell-1)(r-1))$. In particular we have $(\ell-1)(r-1) = d_\ell$ and $|\mathcal{A}| = \frac{\ell(\ell-1)}{2}r$. But for all $H \in \mathcal{A}$ by [OT92, Prop. 6.82, 6.85] we have $|\mathcal{A}^H| = \frac{(\ell-1)(\ell-2)}{2}r+1$. Hence $|\mathcal{A}| - |\mathcal{A}^H| = (\ell-1)r - 1 \neq d_\ell$ and by Lemma 23 the arrangement $\mathcal{A}$ is not MAT2-free. □

Theorem 35. *Let $\mathcal{A} = \mathcal{A}(W)$ be the reflection arrangement of the imprimitive complex reflection group $W = G(r,e,\ell)$ $(r,\ell \geqslant 3)$. Then $\mathcal{A}$ is MAT-free if and only if it is MAT2-free if and only if $e \neq r$.*

Proof. Since $\mathcal{A} = \mathcal{A}(G(r,1,\ell))$ if and only if $r \neq e$, this is Proposition 33 and Proposition 34. □

5 MAT-free exceptional complex reflection groups

To prove the MAT-freeness of one of the following reflection arrangements, we explicitly give a realization by linear forms.

First note that if W is an exceptional Weyl group, or a group of rank $\leqslant 2$, then by Example 17 $\mathcal{A}(W)$ is MAT-free.

Proposition 36. *Let $\mathcal{A}$ be the reflection arrangement of the reflection group H_3 (G_{23}). Then $\mathcal{A}$ is MAT-free. In particular $\mathcal{A}$ is MAT2-free.*

Proof. Let $\tau = \frac{1+\sqrt{5}}{2}$ be the golden ratio and $\tau' = 1/\tau$ its reciprocal. The arrangement $\mathcal{A}$ can be defined by the following linear forms:

$$\begin{aligned}\mathcal{A} &= \{H_1, \dots, H_{15}\} \\ &= \{(1,0,0)^\perp, (0,1,0)^\perp, (0,0,1)^\perp, (1,\tau,\tau')^\perp, (\tau',1,\tau)^\perp, (\tau,\tau',1)^\perp, \\ &\quad (1,-\tau,\tau')^\perp, (\tau',1,-\tau)^\perp, (-\tau,\tau',1)^\perp, (1,\tau,-\tau')^\perp, (-\tau',1,\tau)^\perp, \\ &\quad (\tau,-\tau',1)^\perp, (1,-\tau,-\tau')^\perp, (-\tau',1,-\tau)^\perp, (-\tau,-\tau',1)^\perp\}.\end{aligned}$$

With this linear ordering of the hyperplanes the partition

$$\pi = (13, 14, 15|10, 12|5, 6|4, 11|8, 9|7|3|2|1)$$

satisfies Conditions (1)–(3) of Lemma 19 as one can verify by an easy linear algebra computation. Hence π is an MAT-partition and $\mathcal{A}$ is MAT-free. ☐

Proposition 37. *Let $\mathcal{A}$ be the reflection arrangement of the complex reflection group G_{24}. Then $\mathcal{A}$ is not MAT2-free. In particular $\mathcal{A}$ is not MAT-free.*

Proof. The arrangement $\mathcal{A}$ is free with $\exp(\mathcal{A}) = (1, 9, 11)$ and $|\mathcal{A}| - |\mathcal{A}^H| = 13$ for all $H \in \mathcal{A}$ by [OT92, Tab. C.5]. Hence by Lemma 23 $\mathcal{A}$ is not MAT2-free. ☐

Proposition 38. *Let $\mathcal{A}$ be the reflection arrangement of the complex reflection group G_{25}. Then $\mathcal{A}$ is MAT-free. In particular $\mathcal{A}$ is MAT2-free.*

Proof. Let $\zeta = \frac{1}{2}(-1 + i\sqrt{3})$ be a primitive cube root of unity. The reflecting hyperplanes of $\mathcal{A}$ can be defined by the following linear forms (cf. [LT09, Ch. 8, 5.3]):

$$\begin{aligned}\mathcal{A} &= \{H_1, \dots, H_{12}\} \\ &= \{(1,0,0)^\perp, (0,1,0)^\perp, (0,0,1)^\perp, (1,1,1)^\perp, (1,1,\zeta)^\perp, (1,1,\zeta^2)^\perp, \\ &\quad (1,\zeta,1)^\perp, (1,\zeta,\zeta)^\perp, (1,\zeta,\zeta^2)^\perp, (1,\zeta^2,1)^\perp, (1,\zeta^2,\zeta)^\perp, (1,\zeta^2,\zeta^2)^\perp\}.\end{aligned}$$

With this linear ordering of the hyperplanes the partition

$$\pi = (7, 4, 3|8, 5|9, 6|2, 1|10|11|12)$$

satisfies the three conditions of Lemma 19 as one can easily verify by a linear algebra computation. Hence π is an MAT-partition and $\mathcal{A}$ is MAT-free. ☐

Proposition 39. *Let $\mathcal{A}$ be the reflection arrangement of the complex reflection group G_{26}. Then $\mathcal{A}$ is MAT-free. In particular $\mathcal{A}$ is MAT2-free.*

Proof. Let $\zeta = \frac{1}{2}(-1+i\sqrt{3})$ be a primitive cube root of unity. The reflection arrangement $\mathcal{A}$ is the union of the reflecting hyperplanes of $\mathcal{A}(G_{25})$ and $\mathcal{A}(G(3,3,3))$ (cf. [LT09, Ch. 8, 5.5]). In particular the hyperplanes contained in $\mathcal{A}$ can be defined by the following linear forms:

$$\begin{aligned}\mathcal{A} &= \{H_1,\ldots,H_{21}\}\\ &= \{(1,0,0)^\perp,(0,1,0)^\perp,(0,0,1)^\perp,(1,1,1)^\perp,(1,1,\zeta)^\perp,(1,1,\zeta^2)^\perp,\\ &\quad (1,\zeta,1)^\perp,(1,\zeta,\zeta)^\perp,(1,\zeta,\zeta^2)^\perp,(1,\zeta^2,1)^\perp,(1,\zeta^2,\zeta)^\perp,(1,\zeta^2,\zeta^2)^\perp,\\ &\quad (1,-\zeta,0)^\perp,(1,-\zeta^2,0)^\perp,(1,-1,0)^\perp,(1,0,-\zeta)^\perp,(1,0,-\zeta^2)^\perp,\\ &\quad (1,0,-1)^\perp,(0,1,-\zeta)^\perp,(0,1,-\zeta^2)^\perp,(0,1,-1)^\perp\}.\end{aligned}$$

With this linear ordering of the hyperplanes the partition

$$\pi = (12,19,20|16,18|13,15|17,21|10,14|6,11|8,9|7|5|4|3|2|1)$$

satisfies the three conditions of Lemma 19 as one can verify by a standard linear algebra computation. Hence π is an MAT-partition and $\mathcal{A}$ is MAT-free. □

Proposition 40. *Let $\mathcal{A}$ be the reflection arrangement of the complex reflection group G_{27}. Then $\mathcal{A}$ is not MAT2-free. In particular $\mathcal{A}$ is not MAT-free.*

Proof. The arrangement $\mathcal{A}$ is free with $\exp(\mathcal{A}) = (1,19,25)$ and $|\mathcal{A}| - |\mathcal{A}^H| = 29$ for all $H \in \mathcal{A}$ by [OT92, Tab. C.8]. Hence by Lemma 23 $\mathcal{A}$ is not MAT2-free. □

Proposition 41. *Let $\mathcal{A}$ be the reflection arrangement of the reflection group H_4 (G_{30}). Then $\mathcal{A}$ is MAT-free. In particular $\mathcal{A}$ is MAT2-free.*

Proof. Let $\tau = \frac{1+\sqrt{5}}{2}$ be the golden ratio and $\tau' = 1/\tau$ its reciprocal. The arrangement $\mathcal{A}$ can be defined by the following linear forms:

$$\begin{aligned}\mathcal{A} &= \{H_1,\ldots,H_{60}\}\\ &= \{(1,0,0,0)^\perp,(0,1,0,0)^\perp,(0,0,1,0)^\perp,(0,0,0,1)^\perp,(1,\tau,\tau',0)^\perp,\\ &\quad (1,0,\tau,\tau')^\perp,(1,\tau',0,\tau)^\perp,(\tau,1,0,\tau')^\perp,(\tau',1,\tau,0)^\perp,(0,1,\tau',\tau)^\perp,\\ &\quad (\tau,\tau',1,0)^\perp,(0,\tau,1,\tau')^\perp,(\tau',0,1,\tau)^\perp,(\tau,0,\tau',1)^\perp,(\tau',\tau,0,1)^\perp,\\ &\quad (0,\tau',\tau,1)^\perp,(-1,\tau,\tau',0)^\perp,(1,-\tau,\tau',0)^\perp,(1,\tau,-\tau',0)^\perp,(-1,0,\tau,\tau')^\perp,\\ &\quad (1,0,-\tau,\tau')^\perp,(1,0,\tau,-\tau')^\perp,(-1,\tau',0,\tau)^\perp,(1,-\tau',0,\tau)^\perp,(1,\tau',0,-\tau)^\perp,\\ &\quad (-\tau,1,0,\tau')^\perp,(\tau,-1,0,\tau')^\perp,(\tau,1,0,-\tau')^\perp,(-\tau',1,\tau,0)^\perp,(\tau',-1,\tau,0)^\perp,\\ &\quad (\tau',1,-\tau,0)^\perp,(0,-1,\tau',\tau)^\perp,(0,1,-\tau',\tau)^\perp,(0,1,\tau',-\tau)^\perp,(-\tau,\tau',1,0)^\perp,\\ &\quad (\tau,-\tau',1,0)^\perp,(\tau,\tau',-1,0)^\perp,(0,-\tau,1,\tau')^\perp,(0,\tau,-1,\tau')^\perp,(0,\tau,1,-\tau')^\perp,\end{aligned}$$

$$(-\tau', 0, 1, \tau)^{\perp}, (\tau', 0, -1, \tau)^{\perp}, (\tau', 0, 1, -\tau)^{\perp}, (-\tau, 0, \tau', 1)^{\perp}, (\tau, 0, -\tau', 1)^{\perp},$$
$$(\tau, 0, \tau', -1)^{\perp}, (-\tau', \tau, 0, 1)^{\perp}, (\tau', -\tau, 0, 1)^{\perp}, (\tau', \tau, 0, -1)^{\perp}, (0, -\tau', \tau, 1)^{\perp},$$
$$(0, \tau', -\tau, 1)^{\perp}, (0, \tau', \tau, -1)^{\perp}, (1, 1, 1, 1)^{\perp}, (-1, 1, 1, 1)^{\perp}, (1, -1, 1, 1)^{\perp},$$
$$(1, 1, -1, 1)^{\perp}, (1, 1, 1, -1)^{\perp}, (-1, -1, 1, 1)^{\perp}, (-1, 1, -1, 1)^{\perp}, (-1, 1, 1, -1)^{\perp}\}.$$

With this linear ordering of the hyperplanes the partition

$$\begin{aligned}\pi = (\, & 31, 43, 48, 54|29, 38, 51|23, 34, 58|18, 20, 25|17, 59, 60 \\ & |21, 47, 52|39, 41, 44|26, 32, 49|30, 35, 40|2, 3, 42|33, 46, 50 \\ & |4, 37|27, 57|19, 24|55, 56|10, 22|12, 45|16, 28|15, 36 \\ & |53|14|13|11|9|8|7|6|5|1)\end{aligned}$$

satisfies Conditions (1)–(3) of Lemma 19 as one can verify with a linear algebra computation. Hence π is an MAT-partition and $\mathcal{A}$ is MAT-free. In particular $\mathcal{A}$ is MAT2-free. □

We recall the following result about free filtration subarrangements of $\mathcal{A}(G_{31})$:

Proposition 42 ([Müc17, Pro. 3.8])**.** *Let $\mathcal{A} := \mathcal{A}(G_{31})$ be the reflection arrangement of the finite complex reflection group G_{31}. Let $\tilde{\mathcal{A}}$ be a minimal (w.r.t. the number of hyperplanes) free filtration subarrangement. Then $\tilde{\mathcal{A}} \cong \mathcal{A}(G_{29})$.*

Corollary 43. *Let $\mathcal{A}$ be the reflection arrangement of one of the complex reflection groups G_{29} or G_{31}. Then $\mathcal{A}$ has no free filtration.*

Proposition 44. *Let $\mathcal{A}$ be the reflection arrangement of one of the complex reflection groups G_{29} or G_{31}. Then $\mathcal{A}$ is not MAT2-free. In particular $\mathcal{A}$ is not MAT-free.*

Proof. By Corollary 43 both arrangements have no free filtration and hence are not MAT2-free by Lemma 24. □

Proposition 45. *Let $\mathcal{A}$ be the reflection arrangement of the complex reflection group G_{32}. Then $\mathcal{A}$ is not MAT-free and also not MAT2-free.*

Proof. Up to symmetry of the intersection lattice there are exactly 9 different choices of a basis, where a basis is a subarrangement $\mathcal{B} \subseteq \mathcal{A}$ with $|\mathcal{B}| = r(\mathcal{B}) = r(\mathcal{A}) = 4$. Suppose that $\mathcal{A}$ is MAT-free. Then the first block in an MAT-partition for $\mathcal{A}$ has to be one of these bases. But a computer calculation shows that non of these bases may be extended to an MAT-partition for $\mathcal{A}$. Hence $\mathcal{A}$ is not MAT-free. A similar but more cumbersome calculation shows that $\mathcal{A}$ is also not MAT2-free. □

Proposition 46. *Let $\mathcal{A}$ be the reflection arrangement of one of the complex reflection group G_{33} or G_{34}. Then $\mathcal{A}$ is not MAT2-free. In particular $\mathcal{A}$ is not MAT-free.*

Proof. First, let $\mathcal{A} = \mathcal{A}(G_{33})$. Then $\exp(\mathcal{A}) = (1, 7, 9, 13, 15)$ by [OT92, Tab. C.14]. But $|\mathcal{A}| - |\mathcal{A}^H| = 17$ for all $H \in \mathcal{A}$ also by [OT92, Tab. C.14]. So $\mathcal{A}$ is not MAT2-free by Lemma 23.

Similarly $\mathcal{A} = \mathcal{A}(G_{34})$ is free with $\exp(\mathcal{A}) = (1, 13, 19, 25, 31, 37)$ by [OT92, Tab. C.17] and $|\mathcal{A}| - |\mathcal{A}^H| = 41$ for all $H \in \mathcal{A}$. Hence $\mathcal{A}$ is not MAT2-free by Lemma 23. □

Comparing with Theorem 10 finishes the proofs of Theorem 1 and Theorem 2.

6 Further remarks on MAT-freeness

In their very recent note [HR19] Hoge and Röhrle confirmed a conjecture by Abe [Abe18a] by providing two examples $\mathcal{B}$, $\mathcal{D}$ of arrangements, related to the exceptional reflection arrangement $\mathcal{A}(E_7)$, which are additionally free but not divisionally free and in particular also not inductively free. The arrangements have exponents $\exp(\mathcal{B}) = (1, 5, 5, 5, 5, 5, 5)$ and $\exp(\mathcal{D}) = (1, 5, 5, 5, 5)$. Since both arrangements have only 2 different exponents by Remark 16 they are MAT-free if and only if they are MAT2-free. Now a computer calculation shows that both arrangements are not MAT-free and hence also not MAT2-free. In particular they provide no negative answer to Question 3 and Question 4.

Several computer experiments suggest that similar to the poset obtained from the positive roots of a Weyl group giving rise to an MAT-partition (cf. Example 17) MAT-free arrangements might in general satisfy a certain poset structure:

Problem 47. Can MAT-freeness be characterized by the existence of a partial order on the hyperplanes, generalizing the classical partial order on the positive roots of a Weyl group?

Recall that by Example 22 the restriction $\mathcal{A}^H$ is in general not MAT-free (MAT2-free) if the arrangement $\mathcal{A}$ is MAT-free (MAT2-free). But regarding localizations there is the following:

Problem 48. Is $\mathcal{A}_X$ MAT-free (MAT2-free) for all $X \in L(\mathcal{A})$ provided $\mathcal{A}$ is MAT-free (MAT2-free)?

Last but not least, related to the previous problem, our investigated examples suggest the following:

Problem 49. Suppose $\mathcal{A}'$ and $\mathcal{A} = \mathcal{A}' \cup \{H\}$ are free arrangements such that $\exp(\mathcal{A}') = (d_1, \dots, d_\ell)_\leqslant$ and $\exp(\mathcal{A}) = (d_1, \dots, d_{\ell-1}, d_\ell+1)_\leqslant$. Let $X \in L(\mathcal{A})$ with $X \subseteq H$. By [OT92, Thm. 4.37] both localizations $\mathcal{A}'_X$ and $\mathcal{A}_X$ are free. If $\exp(\mathcal{A}'_X) = (c_1, \dots, c_r)_\leqslant$ is it true that $\exp(\mathcal{A}) = (c_1, \dots, c_{r-1}, c_r + 1)_\leqslant$, i.e. if we only increase the highest exponent is the same true for all localizations?

Note that the answer is yes if we only look at localizations of rank $\leqslant 2$. Our proceeding investigation of Problem 47 suggests that this should be true at least for MAT-free arrangements. Furthermore, a positive answer to Problem 49 would imply (with a bit more work) a positive answer to Problem 48.

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References

[ABC+16] T. Abe, M. Barakat, M. Cuntz, T. Hoge, and H. Terao, *The freeness of ideal subarrangements of Weyl arrangements*, J. Eur. Math. Soc. (JEMS) **18** (2016), no. 6, 1339–1348.

[Abe16] T. Abe, *Divisionally free arrangements of hyperplanes*, Invent. Math. **204** (2016), no. 1, 317–346.

[Abe18a] T. Abe, *Addition-deletion theorem for free hyperplane arrangements and combinatorics*, arXiv:1811.03780v1, 2018

[Abe18b] T. Abe, *Deletion theorem and combinatorics of hyperplane arrangements*, Mathematische Annalen (2018).

[AT16] T. Abe and H. Terao, *Free filtrations of affine Weyl arrangements and the ideal-Shi arrangements*, J. Algebraic Combin. **43** (2016), no. 1, 33–44.

[AT19] T. Abe and H. Terao, *Multiple addition, deletion and restriction theorems for hyperplane arrangements*, Proc. Amer. Math. Soc. **147** (2019), no. 11, 4835–4845.

[BC12] M. Barakat and M. Cuntz, *Coxeter and crystallographic arrangements are inductively free*, Adv. Math. **229** (2012), no. 1, 691–709.

[CH15] M. Cuntz and T. Hoge, *Free but not recursively free arrangements*, Proc. Amer. Math. Soc. **143** (2015), no. 1, 35–40.

[CRS19] M. Cuntz, G. Röhrle, and A. Schauenburg, *Arrangements of ideal type are inductively free*, Internat. J. Algebra Comput. **29** (2019), no. 5, 761–773.

[Grü09] B. Grünbaum, *A catalogue of simplicial arrangements in the real projective plane*, Ars Math. Contemp. **2** (2009), no. 1, 1–25.

[HR15] T. Hoge and G. Röhrle, *On inductively free reflection arrangements*, J. Reine Angew. Math **701** (2015), 205–220.

[HR19] T. Hoge and G. Röhrle, *Some remarks on free arrangements*, arXiv:1903.01438v1, 2019

[LT09] G. I. Lehrer and D. E. Taylor, *Unitary Reflection Groups*, Australian Mathematical Society Lecture Series, Cambridge University Press, 2009.

[Müc17] P. Mücksch, *Recursively free reflection arrangements*, J. Algebra **474** (2017), 24–48.

[OT92] P. Orlik and H. Terao, *Arrangements of hyperplanes*, Grundlehren der mathematischen Wissenschaften, Springer, 1992.

[Sai80] K. Saito, *Theory of logarithmic differential forms and logarithmic vector fields*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **27** (1980), no. 2, 265–291.

[ST54] G. C. Shephard and J. A. Todd, *Finite unitary reflection groups*, Canadian Journal of Mathematics **6** (1954), 274–304.

[Ter80a] H. Terao, *Arrangements of hyperplanes and their freeness I*, J. Fac. Sci. Univ. Tokyo **27** (1980), 293–320.

[Ter80b] H. Terao, *Free arrangements of hyperplanes and unitary reflection groups*, Proc. Japan Acad. Ser. A Math. Sci. **56** (1980), no. 8, 389–392.

[Zie90] G. M. Ziegler, *Matroid representations and free arrangements*, Trans. Amer. Math. Soc. **320** (1990), no. 2, 525–541.