Some moduli spaces of curves and surfaces: topology and Kodaira dimension

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Zusammenfassung

Diese Doktorarbeit befasst sich mit der Untersuchung der Kohomologie und der Kodaira-Dimension einiger Modulräume. Im ersten Teil berechnen wir die Schnitt-Bettizahlen der GIT-Modelle von zwei Modulräumen. Sie parametrisieren jeweils nicht-hyperelliptische Petri-allgemeine Kurven vom Geschlecht vier und numerisch polarisierte Enriquesflächen vom Grad zwei. In beiden Fällen beruht die Strategie der kohomologischen Berechnung auf einer von Kirwan entwickelten allgemeinen Methode zur Berechnung der Kohomologie von GIT-Quotienten projektiver Varietäten. Dieses Verfahren basiert auf der äquivariant perfekten Stratifizierung der instabilen Punkte, die von Hesselink, Kempf, Kirwan und Ness untersucht wurde, und einer partiellen Auflösung von Singularitäten, die als Kirwan-Aufblasung bezeichnet wird. Im zweiten Teil der Doktorarbeit erforschen wir die Modulräume elliptischer K3-Flächen vor Picardzahl mindestens drei, das heißt $U \oplus \langle -2k \rangle$-polarisierter K3-Flächen. Wir beweisen, dass diese Modulräume für $k \geq 220$ von allgemeinem Typ sind. Der Beweis stützt sich auf den von Gritsenko, Hulek und Sankaran entwickelten Trick der Spitzenform von niedrigem Gewicht.

Schlüsselwörter: Modulräume, GIT, Schnitt-Kohomologie, Kurven vom Geschlecht vier, Enriquesflächen, Kodaira-Dimension, Modulformen, elliptische K3-Flächen.

Abstract

This thesis deals with the study of the cohomology and the Kodaira dimension of some moduli spaces. In the first part we compute the intersection Betti numbers of the GIT models of two moduli spaces. They parametrize non-hyperelliptic Petri-general curves of genus four and numerically polarized Enriques surfaces of degree two respectively. In both cases, the strategy of the cohomological calculation relies on a general method developed by Kirwan to compute the cohomology of GIT quotients of projective varieties. This procedure is based on the equivariantly perfect stratification of the unstable points studied by Hesselink, Kempf, Kirwan and Ness, and a partial resolution of singularities, called the Kirwan blow-up. In the second part of the thesis, we study the moduli spaces of elliptic K3 surfaces of Picard number at least three, i.e. $U \oplus \langle -2k \rangle$-polarized K3 surfaces. Such moduli spaces are proved to be of general type for $k \geq 220$. The proof relies on the low-weight cusp form trick developed by Gritsenko, Hulek and Sankaran.

Key words: moduli spaces, GIT, intersection cohomology, genus four curves, Enriques surfaces, Kodaira dimension, modular forms, elliptic K3 surfaces.

Contents

Introduction

> "And now, Harry, let us step out into the night and pursue that flighty temptress, adventure."
>
> Albus Dumbledore

One of the major goals of algebraic geometry is the classification of algebraic varieties. The theory of moduli spaces provides a powerful tool in this research field. Indeed, moduli spaces parametrize classes of geometric objects up to a certain notion of equivalence. Under suitable assumptions, moduli spaces inherit themselves a geometric structure which is interesting to study for its own features and in connection with the parametrized objects. This thesis focuses on two important geometric invariants, namely cohomology and Kodaira dimension.

In the first part of the thesis we discuss one of the fundamental methods to construct moduli spaces, the so-called Geometric Invariant Theory (GIT). It was developed by Mumford in the sixties and culminated in the seminal book [MFK94]. This theory provides a way to construct quotient spaces of algebraic schemes, and furthermore gives them a geometric structure. Hence, the problem of studying moduli spaces for various types of algebraic objects can be tackled by using the framework established by GIT. The rich geometry of moduli spaces can be then better understood in relation to the invariant families of objects that are parametrized. One of the interesting topological properties that are important to study is the cohomology. In this direction, a particularly relevant result was found by Kirwan, who successfully invented a procedure to compute (intersection) cohomology of GIT quotients. In turn, it can be applied to study the cohomology of many moduli spaces and their geometrically meaningful compactifications. In this thesis, we present two applications of Kirwan's method to the case of the moduli space of non-hyperelliptic genus four curves and to the case of the moduli space of degree two Enriques surfaces.

In the second part of the thesis, we examine another geometric property significant for the study of moduli spaces, that is the Kodaira dimension. It is a birational invariant which measures the dimension growth of the spaces of pluricanonical differential forms. Determining the Kodaira dimension of moduli spaces has revealed itself to be a highly non-trivial, though very challenging, task and has been fascinating many generations of algebraic geometers, who have been motivated to develop new and more sophisticated techniques. In this thesis, we focus on a class of geometric objects, whose importance has been rapidly growing in recent years, namely K3 surfaces. The seminal work [GHS07b] of Gritsenko, Hulek and Sankaran proved that the moduli space of polarized K3 surfaces of degree $2d$ is of general type, i.e. has maximal Kodaira dimension, for $d > 61$ and for other smaller values of d. It is then natural to address the general question about the Kodaira dimension of moduli spaces of lattice polarized K3 surfaces. We are interested in studying a particular class of such surfaces, namely elliptic K3 surfaces of Picard number at least 3, or equivalently $U \oplus \langle -2k \rangle$-polarized K3 surfaces. By using the methods of [GHS07b], we are able to prove that the moduli space of such K3 surfaces is of general type for $k \geq 220$ and for other smaller values of k.

In **Chapter 1**, we introduce the basic notions of Geometric Invariant Theory and present Kirwan's procedure to compute the intersection cohomology of GIT quotients, developed in [Kir84], [Kir85] and [Kir86]. This method essentially relies on the Hesselink-Kempf-Kirwan-Ness stratification, which is proved to be equivariantly perfect, and on a partial desingularization of the GIT quotient, known as Kirwan blow-up, obtained by blowing up the loci of strictly polystable points. Then one can compute the intersection cohomology of the GIT quotient from the Betti numbers of the Kirwan blow-up and the equivariant cohomology of the semistable locus.

Examples of application of Kirwan's method are the topological descriptions of the moduli space of points on the projective line [MFK94, §8], of K3 surfaces of degree two [KL89] and of hypersurfaces in $\mathbb{P}^n$ [Kir89], with explicit complete computations only in the case of plane curves up to degree six, cubic and quartic surfaces. More recently, the procedure has been applied to compactifications of the moduli space of cubic threefolds [CMGHL19].

In **Chapter 2**, we present the results of the author's article [For18] about the cohomology of the moduli space of non-hyperelliptic Petri-general curves of genus four. The canonical divisor of such curves induces an embedding into projective space, which realises the curves as complete intersection of a smooth quadric and a cubic surface. Hence, their moduli space has a natural compactification:

$$M := \mathbb{P}H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(3,3)) /\!/ \operatorname{Aut}(\mathbb{P}^1 \times \mathbb{P}^1),$$

as GIT quotient for the space of curves of bidegree $(3,3)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ under automorphism. The upshot of the article [For18] is the calculation of the intersection cohomology of M, via Kirwan's procedure outlined in Chapter 1, whose fundamental part is the definition of the Kirwan blow-up $\widetilde{M} \to M$. This leads to the first main theorem of the thesis (cf. Theorem 2.0.1).

Main Theorem 1. *For the moduli space M compactifying the space of non-hyperelliptic Petri-general curves of genus four, the intersection Betti numbers of M and the Betti numbers of the Kirwan blow-up $\widetilde{M}$ are as follows:*

i	0	2	4	6	8	10	12	14	16	18
$\dim IH^i(M, \mathbb{Q})$	1	1	2	2	3	3	2	2	1	1
$\dim H^i(\widetilde{M}, \mathbb{Q})$	1	4	7	11	14	14	11	7	4	1

while all the odd degree (intersection) Betti numbers vanish.

The variety M is a projective birational compactification for the moduli space of genus four curves M_4, which is the coarse moduli space associated to the moduli stack $\mathcal{M}_4$. The study of the birational models for M_g is the subject of the Hassett-Keel program (see [Has05]), which aims at giving a modular interpretation of the canonical model of the Deligne-Mumford compactification $\overline{M_g}$. The genus four case has attracted a lot of attention as a non-trivial instance of the aforementioned program. Specifically, Fedorchuk [Fed12] proved that M is the final non-trivial log canonical model for $\overline{M_4}$, namely

$$M \cong \overline{M_4}(\alpha) := \operatorname{Proj} \bigoplus_{n \geq 0} H^0(n(K_{\overline{M_4}} + \alpha\delta)), \quad \alpha \in \left(\frac{8}{17}, \frac{29}{60}\right] \cap \mathbb{Q},$$

where $\delta \subseteq \overline{M_4}$ is the boundary divisor. In [CMJL12] and [CMJL14], Casalaina-Martin, Jensen and Laza described the last steps of the Hassett-Keel program for log minimal models of $\overline{M_4}$, arising as VGIT quotients of the parameter space of $(2,3)$ complete intersections. On the other hand, Hassett, Hyeon and Lee (see [HH09], [HH13] and [HL14]) proved that the program starts with a divisorial contraction, followed by a flip and a small contraction, and gave a modular interpretation of the resulting spaces. In conclusion, most of the Hassett-Keel program for genus four is currently known. From our perspective, the salient point is the two ends of the program, namely the Deligne-Mumford compactification $\overline{M_4}$ and the GIT quotient for $(3,3)$ curves. One has a complete understanding of the rational cohomology of $\overline{M_4}$ due to Bergström-Tommasi in [BT07], and that of M_4 by Tommasi in [Tom05]. The purpose of the author's article [For18] is to calculate the cohomology at the other end of the Hassett-Keel program.

In **Chapter 3**, we report the results of the author's article [For20] about the cohomology of the moduli space of degree two Enriques surfaces. Horikawa in [Hor78a] was the first who constructed a projective model for such Enriques surfaces. The universal coverings of these Enriques surfaces are hyperelliptic quartic K3 surfaces obtained as double coverings of $\mathbb{P}^1 \times \mathbb{P}^1$ branched over a curve of bidegree $(4,4)$ invariant under a suitable involution $\iota : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ with four fixed points. This transformation together with the deck involution induces a fixed-point-free involution on the K3 surface, whose quotient is the desired Enriques surface. Therefore, by considering the isomorphism classes of such branch curves on $\mathbb{P}^1 \times \mathbb{P}^1$, we can construct the GIT quotient:

$$M^{GIT} := \mathbb{P}H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(4,4))^\iota /\!/ (\mathbb{C}^*)^2 \rtimes D_8,$$

where $\mathbb{P}H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(4,4))^\iota$ is the linear subsystem of $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(4,4)|$ of ι-invariant curves and $(\mathbb{C}^*)^2 \rtimes D_8$ is the subgroup of the automorphisms of $\mathbb{P}^1 \times \mathbb{P}^1$ that commute with ι. Then the space M^{GIT} can be proved to be a compactification of the moduli space of numerically polarized Enriques surfaces of degree two. The purpose of the article [For20] is to compute the intersection cohomology of M^{GIT}, by means of Kirwan's method described in Chapter 1, whose essential step amounts to constructing the Kirwan blow-up $M^K \to M^{GIT}$. The second main theorem of the thesis is hence stated in the following (cf. Theorem 3.0.1).

Main Theorem 2. *For the moduli space M^{GIT} compactifying the space of degree two Enriques surfaces, the intersection Betti numbers of M^{GIT} and the Betti numbers of the Kirwan blow-up M^K are as follows:*

i	0	2	4	6	8	10	12	14	16	18	20
$\dim IH^i(M^{GIT}, \mathbb{Q})$	1	1	2	2	3	3	3	2	2	1	1
$\dim H^i(M^K, \mathbb{Q})$	1	4	8	13	18	20	18	13	8	4	1

while all the odd degree (intersection) Betti numbers vanish.

The space M^{GIT} was extensively studied by Horikawa ([Hor78a] and [Hor78b]), Shah [Sha81] and Sterk ([Ste91] and [Ste95]). More precisely, Horikawa proved the Torelli theorem for Enriques surfaces and studied the period map (and its extension) from M^{GIT} to the period domain of Enriques surfaces. Shah classified all the projective degenerations of Horikawa's model of Enriques surfaces. Later Sterk built on these results by dealing with compactifications of the period space of Enriques surfaces which are of geometric interest. In particular, he gave a description of the boundary in the Baily-Borel compactification of the period space in [Ste91] and constructed a resolution of the period map via a new geometrically meaningful compactification, called the Shah compactification. This space can be obtained as a double weighted blow-up of M^{GIT} and its points include all the degenerations of Enriques surfaces classified in [Sha81]. Moreover, in [Ste95] the resolution of the period map was proved to factorize through a semi-toric compactification, obtained as normalized blow-up of the Baily-Borel compactification along the closure of the divisor describing periods of Enriques surfaces with a special quasi-polarization.

In **Chapter 4**, we present the results corresponding to the first part of the article [FM20] of the author in collaboration with G. Mezzedimi. As the first part of [FM20] appears in this thesis, the second part of it will appear in G. Mezzedimi's Ph.D. thesis and this subdivision corresponds to the respective contributions of the authors to the aforementioned article. Here we study the Kodaira dimension of the moduli spaces of elliptic K3 surfaces of Picard number at least three. A K3 surface is called elliptic if it admits a fibration onto the projective line in curves of genus one together with a section. The Néron-Severi group of such surfaces contains a lattice isomorphic to the hyperbolic plane U, generated by the classes of the fibre and the zero section, which coincides with the whole Néron-Severi group if the elliptic K3 surface is very general. Via the realisation of elliptic surfaces as Weierstrass fibrations, Miranda [Mir81] was able to construct the moduli space of elliptic K3 surfaces as GIT quotient. As a by-product, he furthermore showed that the moduli space is unirational, and in particular that it has minimal Kodaira dimension. Later, Lejarraga [Lej93] proved that this space is actually rational. We are interested in studying the divisors of the moduli space of elliptic K3 surfaces which parametrize the surfaces whose Néron-Severi groups contain primitively $U \oplus \langle -2k \rangle$, namely the moduli spaces $\mathcal{M}_{2k}$ of $U \oplus \langle -2k \rangle$-polarized K3 surfaces. From a geometric point of view, they are elliptic K3 surfaces admitting an extra class in their Néron-Severi group: if $k = 1$, it comes from a reducible fibre of the elliptic fibration, while if $k \geq 2$ it is represented by an extra section, intersecting the zero section in $k-2$ points. The third main theorem of the thesis deals with the Kodaira dimension of $\mathcal{M}_{2k}$ (cf. Theorem 4.0.1).

Main Theorem 3. *The moduli space $\mathcal{M}_{2k}$ of $U \oplus \langle -2k \rangle$-polarized K3 surfaces is of general type for $k \geq 220$, or*

$$k \geq 208,\ k \neq 211, 219,\ \text{or } k \in \{170, 185, 186, 188, 190, 194, 200, 202, 204, 206\}.$$

Moreover, the Kodaira dimension of $\mathcal{M}_{2k}$ is non-negative for $k \geq 176$, or

$$k \geq 164,\ k \neq 169, 171, 175 \text{ or } k \in \{140, 146, 150, 152, 154, 155, 158, 160, 162\}.$$

The Torelli theorem for K3 surfaces (see [PS72]) permits the moduli spaces $\mathcal{M}_{2k}$ to be realised as quotients of bounded hermitian symmetric domains $\Omega_{L_{2k}}$ of type IV and dimension 17 by the stable orthogonal groups $\widetilde{\mathrm{O}}^+(L_{2k})$, where the lattice L_{2k} is the orthogonal complement of $U \oplus \langle -2k \rangle$ in the K3-lattice $\Lambda_{K3} := 3U \oplus 2E_8(-1)$. Via this description, the low-weight cusp form trick, developed by Gritsenko, Hulek and Sankaran in [GHS07b], can be applied. This tool provides a sufficient condition for an orthogonal modular variety to be of general type. Indeed, it is enough to find a non-zero cusp form on $\Omega^\bullet_{L_{2k}}$ of weight strictly less than the dimension of $\Omega_{L_{2k}}$ vanishing along the ramification divisor of the projection $\Omega_{L_{2k}} \to \widetilde{\mathrm{O}}^+(L_{2k})\backslash\Omega_{L_{2k}}$. In our case, we construct the desired cusp form as quasi-pullback of the Borcherds form Φ_{12} associated to $L_{2,26}$, that is the unique even unimodular lattice of signature $(2,26)$. This allows us to reduce the proof of Main Theorem 3 to an arithmetic problem consisting of finding the values of k for which there exists a primitive embedding $L_{2k} \hookrightarrow L_{2,26}$ whose orthogonal complement contains a suitable number of roots.

Notation and conventions

We work over the field of complex numbers and all the cohomology and homology theories are taken with *rational* coefficients. The intersection cohomology will always be considered with respect to the middle perversity (see [KW06] for an excellent introduction). For any topological group G, we will denote by G^0 the connected component of the identity in G and by $\pi_0 G := G/G^0$ the finite group of connected components of G. The universal classifying bundle of G will be denoted by $EG \to BG$. If G acts on a topological space Y, its equivariant cohomology (see [AB83]) will be defined to be $H^*_G(Y) := H^*(Y \times_G EG)$. The Hilbert-Poincaré series is denoted by

$$P_t(Y) := \sum_{i\geq 0} t^i \dim H^i(Y),$$

and analogously for the intersection and equivariant cohomological theories. If F is a finite group acting on a vector space A, then A^F will indicate the subspace of elements in A fixed by F.

1 | Cohomology of GIT quotients

In this chapter, we review Kirwan's method to compute the intersection cohomology of GIT quotients, as explained in [Kir84], [Kir85] and [Kir86]. We start with a background section on Geometric Invariant Theory to introduce the basic definitions and notations. The first step consists of considering the Hesselink-Kempf-Kirwan-Ness stratification (see Theorem 1.2.1) of the unstable locus in order to compute the equivariant cohomology of the semistable locus (see Theorem 1.2.2). Since in general the cohomology of a GIT quotient does not coincide with the equivariant cohomology of the semistable locus, one needs to take a partial desingularization of the quotient, known as Kirwan blow-up, having only finite quotient singularities, obtained by successively blowing up the loci parametrizing strictly polystable points. Then Kirwan explains how to compute the Betti numbers of the Kirwan blow-up (see Theorem 1.4.1) taking into account the geometry of the centres of the blow-ups and the exceptional divisors. In the last step, we descend back to the GIT quotient and calculate its cohomology as an application of the Decomposition Theorem (see Theorem 1.5.1).

1.1 Background on Geometric Invariant Theory

A fundamental method for constructing moduli spaces is based on *Geometric Invariant Theory* (GIT). This theory was developed by Mumford in [MFK94] and provides a tool to construct the quotient of a projective variety X with respect to the action of a reductive group G and to give it the structure of an algebraic variety. In particular, given a linearisation on an ample line bundle L on X, we are able to construct a quotient which is also a projective variety, up to restricting to an open subset of X. The so-called *GIT quotient* is then defined as

$$X/\!/G := \operatorname{Proj}\left(\bigoplus_{n=0}^{\infty} H^0(X, L^{\otimes n})^G\right).$$

The open subset where the quotient map $X \dashrightarrow X/\!/G$ is defined is called the *semistable locus* X^{ss}, that is the set of points $x \in X$ for which there exists a non-zero G-invariant section $s \in H^0(X, L^{\otimes n})^G$ for some $n > 0$ with $s(x) \neq 0$. On the contrary, a point that is not semistable is said to be *unstable*. Moreover, we recall that a point $x \in X$ is called *polystable* if it is semistable and its orbit is closed in X^{ss}. Finally a point $x \in X$ is called *stable* if it is polystable and its stabiliser is finite: the stable locus is denoted by X^s and it is open in X^{ss}.

The GIT quotient is a *categorical quotient*, namely the quotient map $X^{ss} \to X/\!/G$ is constant on the orbits of G and it is universal with respect to G-invariant morphisms. In general, the GIT quotient need not to be an orbit space, since there might be several orbits which are identified under the quotient map: this issue is related to the existence of non-closed orbits in the semistable locus. Indeed, the GIT quotient can be regarded set-theoretically as the quotient of X^{ss} under the equivalence relation:

$$x \sim y \Leftrightarrow \overline{G \cdot x} \cap \overline{G \cdot y} \neq \varnothing.$$

The points of the GIT quotient are therefore in one-to-one correspondence with the orbits of the polystable points. However, the restriction $X^s \to X^s/G$ of the quotient map to the stable locus is a *geometric quotient*, i.e. a categorical quotient which induces a one-to-one correspondence between the orbits of X^s and the points of the image X^s/G of X^s inside $X/\!/G$. From this perspective, when the stable locus is non-empty, the GIT quotient $X/\!/G$ can be seen as a compactification of the orbit space X^s/G, which typically encodes the geometric aspects one wants to study: this has tremendous consequences in the field of moduli spaces.

Finally, we want to mention the fundamental criterion for determining the (semi)-stability, which will be extensively used in the following chapters. Suppose that a reductive group G acts on a projective variety $X \subseteq \mathbb{P}^n$ via a linear representation $\rho : G \to \mathrm{GL}(n+1, \mathbb{C})$. This can be achieved by taking a very ample G-linearised line bundle L on X. We denote by x^* a representative in $\mathbb{C}^{n+1}$ of a point $x \in X$. Consider a *1-parameter subgroup* (1-PS) of G, i.e. a non-constant regular homomorphism $\lambda : \mathbb{C}^* \to G$. In appropriate coordinates one can assume that λ acts diagonally as

$$\lambda(t) \cdot x^* = (t^{a_0} x_0^*, ..., t^{a_n} x_n^*),$$

where $a_0, ..., a_n \in \mathbb{Z}$. Let us set

$$\mu(x, \lambda) := -\min_i \{a_i : x_i^* \neq 0\}.$$

Intrinsically, $\mu(x, \lambda)$ is the unique integer μ such that $\lim_{t \to 0} t^\mu \lambda(t) \cdot x^*$ exists and is non-zero. Then the *Hilbert-Mumford Criterion* asserts that for a point $x \in X$:

$$x \in X^{ss} \Leftrightarrow \mu(x, \lambda) \geq 0 \text{ for all 1-PS's } \lambda \text{ of } G;$$
$$x \in X^{s} \Leftrightarrow \mu(x, \lambda) > 0 \text{ for all 1-PS's } \lambda \text{ of } G.$$

1.2 The Hesselink-Kempf-Kirwan-Ness stratification

In this section, we discuss the first step of Kirwan's method to compute the cohomology of GIT quotients. It consists of an equivariant stratification of the parameter space measuring the instability of every point under the group action (cf. Theorem 1.2.1). This stratification turns out to be equivariantly perfect, in the sense that the equivariant Betti numbers of all strata sum up to the equivariant cohomology of the whole parameter space (cf. Theorem 1.2.2). From the results in [Kir84], the first step in Kirwan's procedure is to consider the *Hesselink-Kempf-Kirwan-Ness (HKKN) stratification* of the parameter space, which, from a symplectic viewpoint, coincides with the Morse stratification for the norm-square of an associated moment map.

In general, let $X \subseteq \mathbb{P}^n$ be a complex projective manifold, acted on by a complex reductive group G, inducing a linearisation on the very ample line bundle $L = \mathcal{O}_{\mathbb{P}^n}(1)|_X$. We pick a maximal compact subgroup $K \subseteq G$, whose complexification gives G, and a maximal torus $T \subseteq G$, such that $T \cap K$ is a maximal compact torus of K. Before describing the stratification, we need also to fix an inner product together with the associated norm $\|.\|$ on the dual Lie algebra $\mathfrak{t}^\vee := \mathrm{Lie}(T \cap K)^\vee$, e.g. the Killing form, invariant under the adjoint action of K.

Theorem 1.2.1. *[Kir84] In the above setting, there exists a natural stratification of X*

$$X = \bigsqcup_{\beta \in \mathcal{B}} S_\beta$$

by G-invariant locally closed subvarieties S_β, indexed by a finite partially ordered set $\mathcal{B} \subseteq$ $\mathrm{Lie}(T \cap K)$ *such that the minimal stratum $S_0 = X^{ss}$ is the semistable locus of the action and the closure of S_β is contained in $\bigcup_{\gamma \geq \beta} S_\gamma$, where $\gamma \geq \beta$ if and only if $\gamma = \beta$ or $\|\gamma\| > \|\beta\|$.*

We briefly sketch the construction of the strata appearing in Theorem 1.2.1 (see [Kir84] for a detailed description). Let $\{\alpha_0, ..., \alpha_n\} \subseteq \mathfrak{t}^\vee$ be the weights of the representation (a.k.a. the linearisation) of G on $\mathbb{C}^{n+1}$ and identify $\mathfrak{t}^\vee$ with $\mathfrak{t}$ via the invariant inner product. After choosing a positive Weyl chamber $\mathfrak{t}_+$, an element $\beta \in \overline{\mathfrak{t}_+}$ in the closure of $\mathfrak{t}_+$ belongs to the indexing set $\mathcal{B}$ of the stratification if and only if β is the closest point to the origin of the convex hull of some non-empty subset of $\{\alpha_0, ..., \alpha_n\}$. We define Z_β to be the linear section of X

$$Z_\beta := \{(x_0 : ... : x_n) \in X : x_i = 0 \text{ if } \alpha_i.\beta \neq \|\beta\|^2\}.$$

The stratum indexed by β is then

$$S_\beta := G \cdot \bar{Y}_\beta \setminus \bigcup_{\|\gamma\| > \|\beta\|} G \cdot \bar{Y}_\gamma,$$

where

$$\bar{Y}_\beta := \{(x_0 : ... : x_n) \in X : x_i = 0 \text{ if } \alpha_i.\beta < \|\beta\|^2\}$$

is the closure of

$$Y_\beta := \{(x_0 : ... : x_n) \in X : x_i = 0 \text{ if } \alpha_i.\beta < \|\beta\|^2 \text{ and } \exists\ x_j \neq 0 \text{ s.t. } \alpha_j.\beta = \|\beta\|^2\}.$$

Since Z_β sits in projective space, for any point $(x_0 : ... : x_n) \in Z_\beta$ there exists some $x_j \neq 0$ with $\alpha_j.\beta = \|\beta\|^2$. Thus we have $Z_\beta \subseteq Y_\beta$ and in fact there is a retraction $Y_\beta \to Z_\beta$ that sends x_i to 0 if $\alpha_i.\beta > \|\beta\|^2$ (see [Kir84, 12.18] and [MFK94, p. 173]).

The heart of Kirwan's results in [Kir84] is the proof that the equivariant Betti numbers of the strata sum up to the cohomology of the whole space.

Theorem 1.2.2. *[Kir84, 8.12] The stratification $\{S_\beta\}_{\beta\in\mathcal{B}}$, constructed in Theorem 1.2.1, is G-equivariantly perfect, namely the following holds:*

$$P_t^G(X^{ss}) = P_t^G(X) - \sum_{0\neq\beta\in\mathcal{B}} t^{2\mathrm{codim}(S_\beta)} P_t^G(S_\beta).$$

The starting point in the proof of Theorem 1.2.2 is the Thom-Gysin sequence, which relates the cohomology of a manifold Y to that of a submanifold Z via the cohomology of the complement $Y \smallsetminus Z$, as follows:

$$\cdots \to H^{i-d}(Z,\mathbb{Q}) \to H^i(Y,\mathbb{Q}) \to H^i(Y \smallsetminus Z,\mathbb{Q}) \to H^{i+1-d}(Z,\mathbb{Q}) \to \cdots,$$

where d is the codimension of Z in Y. In terms of Hilbert-Poincaré polynomials, we obtain

$$t^d P_t(Z) - P_t(Y) + P_t(Y \smallsetminus Z) = (1+t)Q(t),$$

where $Q(t) \in \mathbb{Z}[t]$ has non-negative coefficients. If we apply a G-equivariant version of the Thom-Gysin sequence to the HKKN stratification of Theorem 1.2.1, we obtain the following identity:

$$P_t^G(X) = \sum_{\beta\in\mathcal{B}} t^{2\mathrm{codim}(S_\beta)} P_t^G(S_\beta) - (1+t)Q^G(t),$$

where the polynomial $Q^G(t) \in \mathbb{Z}[t]$ has non-negative coefficients. Hence the proof of Theorem 1.2.2 boils down to show that $Q^G(t) \equiv 0$. This is achieved by considering a degenerate Morse function $f : X \to \mathbb{R}$ [Kir84, Definition 2.9], given as the composition of the momentum map $\mu : X \to \mathrm{Lie}(K)^\vee$ with the quadratic form $\|\cdot\| : \mathrm{Lie}(K)^\vee \to \mathbb{R}$ associated to the Killing form. Then Kirwan reinterpreted the strata S_β with respect to the gradient flow to the critical set for f (see [Kir84, Theorem 12.26]). Finally Theorem 1.2.2 is established by using techniques from Morse theory and symplectic geometry (cf. [Kir84, Theorem 6.18]).

Remark 1.2.1. If we denote by $\mathrm{Stab}\beta \subseteq G$ the stabiliser of $\beta \in \mathfrak{t}$ under the adjoint action of G, the equivariant Hilbert-Poincaré series of each stratum is

$$P_t^G(S_\beta) = P_t^{\mathrm{Stab}\beta}(Z_\beta^{ss}),$$

where Z_β^{ss} is the set of semistable points of Z_β with respect to an appropriate linearisation of the action of $\mathrm{Stab}\beta$ on Z_β, which is described in [Kir84, 8.11]. An equivalent definition of Z_β^{ss} is given in [Kir84, §12]. For any $x = (x_0 : \ldots : x_n) \in X \subseteq \mathbb{P}^n$ let $C(x) \subseteq \mathfrak{t}$ be the convex hull of the set of weights α_i such that $x_i \neq 0$. We denote by $\beta(x)$ the closest point to the origin in $C(x)$. Then for $\beta \neq 0$ we have the following equality:

$$Z_\beta^{ss} = \{x \in Z_\beta : \beta(x) = \beta, \text{ and for all } g \in G,\ ||\beta(g \cdot x)|| \leq ||\beta||\}. \tag{1.1}$$

Remark 1.2.2. The codimension of each stratum $S_\beta \subseteq X$ is equal to (see [Kir89, 3.1])

$$\mathrm{codim}(S_\beta) := d(\beta) = n(\beta) - \dim G/P_\beta,$$

where $n(\beta)$ is the number of weights α_I such that $\beta \cdot \alpha_I < ||\beta||^2$, i.e. the number of weights lying in the half-plane containing the origin and defined by β. Moreover, $P_\beta \subseteq G$ is the subgroup of elements in G which preserve $\bar{Y}_\beta$, then P_β is a parabolic subgroup, whose Levi component is the stabiliser $\mathrm{Stab}\beta$ of $\beta \in \mathfrak{t}$ under the adjoint action of G.

1.3 The Kirwan blow-up

In this section we recall the general construction of the Kirwan blow-up of a GIT quotient which provides an orbifold resolution of singularities. It is achieved by stratifying the GIT boundary $X/\!/G \setminus X^s/G$ in terms of the connected components R of the stabilisers of the associated polystable orbits. Then, one proceeds by blowing up these strata according to the dimension of the corresponding R.

In general the equivariant cohomology $H_G^*(X^{ss})$ of the semistable locus does not coincide with the cohomology $H^*(X/\!/G)$ of the GIT quotient, unless in the case when all semistable points are actually stable. The solution is given by constructing a *partial resolution of singularities* $\widetilde{X}/\!/G \to X/\!/G$, known as *Kirwan blow-up* (see [Kir85]), for which the group G acts with finite isotropy groups on the semistable points $\widetilde{X}^{ss}$. We briefly describe how it is constructed.

We consider again the setting, as in Section 1.2, of a smooth projective manifold $X \subseteq \mathbb{P}^n$ acted on by a reductive group G. We also assume that the stable locus $X^s \neq \varnothing$ is non-empty. In order to produce the Kirwan blow-up, we need to study the GIT boundary $X/\!/G \setminus X^s/G$ and stratify it in terms of the isotropy groups of the associated semistable points. More precisely, let $\mathcal{R}$ be a set of representatives for the conjugacy classes of connected components of stabilisers of strictly polystable points, i.e. semistable points with closed orbits, but infinite stabilisers. Let r be the maximal dimension of the groups in $\mathcal{R}$, and let $\mathcal{R}(r) \subseteq \mathcal{R}$ be the set of representatives for conjugacy classes of subgroups of dimension r. For every $R \in \mathcal{R}(r)$, consider the fixed locus

$$Z_R^{ss} := \{x \in X^{ss} : R \text{ fixes } x\} \subseteq X^{ss}. \tag{1.2}$$

In [Kir85, §5] Kirwan showed that the subset

$$\bigcup_{R \in \mathcal{R}(r)} G \cdot Z_R^{ss} \subseteq X^{ss}$$

is a disjoint union of smooth G-invariant closed subvarieties in X^{ss}. Now let $\pi_1 : X_1 \to X^{ss}$ be the blow-up of X^{ss} along $\bigcup_{R \in \mathcal{R}(r)} G \cdot Z_R^{ss}$ and $E \subseteq X_1$ be the exceptional divisor. We recall [Kir85, Corollary 8.3], namely the result of blowing up X^{ss} along $\bigcup_{R \in \mathcal{R}(r)} G \cdot Z_R^{ss}$ is the same as successively blowing up X^{ss} along $G \cdot Z_R^{ss}$ for each $R \in \mathcal{R}(r)$.

Since the centre of the blow-up is invariant under G, there is an induced action of G on X_1, linearised by a suitable ample line bundle. If $L = \mathcal{O}_{\mathbb{P}^n}(1)|_X$ is the very ample line bundle on X linearised by G, then there exists $d \gg 0$ such that $L_1 := \pi_1^* L^{\otimes d} \otimes \mathcal{O}_{X_1}(-E)$ is very ample and admits a G-linearisation (see [Kir85, 3.11]). After making this choice, the set $\mathcal{R}_1$ of representatives for the conjugacy classes of connected components of isotropy groups of strictly polystable points in X_1 will be strictly contained in $\mathcal{R}$ (see [Kir85, 6.1]). For any $R' \in \mathcal{R}_1$, the locus $Z_{R',1}^{ss} \subseteq X_1^{ss}$ given as in (1.2) is is the strict transform of the locus $Z_{R'}^{ss} \subseteq X^{ss}$ defined by considering R' as an element of $\mathcal{R}$. Moreover, the maximum among the dimensions of the reductive subgroups in $\mathcal{R}_1$ is strictly less than r. Now we restrict to the new semistable locus $X_1^{ss} \subseteq X_1$, so that we are ready to perform the same process as above again. We notice that outside the exceptional divisor the effect of the blow-up is to destabilise exactly those strictly semistable points that have orbit closure meeting the centre of the blow-up.

By repeating the above process, we obtain a finite sequence of *modifications*:

$$\widetilde{X}^{ss} := X_r^{ss} \xrightarrow{\pi_r} \dots \xrightarrow{\pi_2} X_1^{ss} \xrightarrow{\pi_1} X^{ss}, \tag{1.3}$$

by iteratively restricting to the semistable locus and blowing up smooth invariant centres (cf. [Kir85, 6.3]). The morphism π_j is the blow-up along the locus determined by the subgroups in $\mathcal{R}$ of dimension $r - j + 1$, where we allow some of these blow-ups to be the identity if there are no relevant subgroups in a given dimension. We want to stress that if $R_1, R_2 \in \mathcal{R}$ have different dimension, it may happen that $G \cdot Z^{ss}_{R_1}$ is not disjoint from $G \cdot Z^{ss}_{R_2}$.

In the last step, $\widetilde{X}^{ss}$ is equipped with a G-linearised ample line bundle such that G acts with finite stabilisers. In conclusion, we have the diagram

$$\begin{array}{ccc} \widetilde{X}^{ss} & \longrightarrow & X^{ss} \\ \downarrow & & \downarrow \\ \widetilde{X}/\!/G & \longrightarrow & X/\!/G, \end{array}$$

where the *Kirwan blow-up* $\widetilde{X}/\!/G$, having at most finite quotient singularities, gives a partial desingularization of $X/\!/G$ that in general has worse singularities.

1.4 Cohomology of the Kirwan blow-up

In this section, we recall the general theory to compute the Betti numbers of the Kirwan blow-up $\widetilde{X}/\!/G \to X/\!/G$ of a GIT quotient. Since $\widetilde{X}/\!/G$ has only finite quotient singularities, its rational cohomology coincides with the equivariant cohomology of the semistable locus $\widetilde{X}^{ss}$, which in turn can be computed from the equivariant cohomology of X^{ss} corrected by an error term (see Theorem 1.4.1). This error term is divided into a main and an extra contribution: the former takes into account the geometry of the centres of the blow-ups and the latter the action of G on the exceptional divisors.

The effect of the desingularization on the equivariant Poincaré series is explained in [Kir85] (see also [CMGHL19, §4.1]). We consider again the setting, as in Section 1.3, of a nonsingular projective variety X together with a linear action of a reductive group G. Assume that R is a connected reductive subgroup with the property that the fixed point set $Z^{ss}_R \subseteq X^{ss}$ is non-empty, but that $Z^{ss}_{R'} = \varnothing$ for all subgroups $R' \subseteq G$ of higher dimension than R. For simplicity we further assume that Z^{ss}_R is connected, which will always be the case in our results.

Let $\pi : \hat{X} \to X^{ss}$ be the blow-up of X^{ss} along $G \cdot Z^{ss}_R$. Then the equivariant cohomology of $\hat{X}$ is related to that of the exceptional divisor E by

$$H^*_G(\hat{X}) = H^*_G(X^{ss}) \oplus H^*_G(E)/H^*_G(G \cdot Z^{ss}_R) \tag{1.4}$$

(see [GH78, §4.6], [Kir85, 7.2]). If $\mathcal{N}^R$ denotes the normal bundle to $G \cdot Z^{ss}_R$ in X^{ss}, then the equivariant cohomology of the exceptional divisor $E = \mathbb{P}\mathcal{N}^R$ can be computed via a degenerating spectral sequence (see [GH78, Prop. p. 606] and [Kir84, p. 67]), namely

$$H^*_G(E) = H^*_G(G \cdot Z^{ss}_R)(1 + \ldots + t^{2(\operatorname{rk}\mathcal{N}^R - 1)}). \tag{1.5}$$

In [Kir85, 5.10] Kirwan proved that $G \cdot Z_R^{ss}$ is algebraically isomorphic to $G \times_N Z_R^{ss}$, where $N \subseteq G$ is the normaliser of R, hence we can compute

$$\operatorname{rk}\mathcal{N}^R = \dim X - \dim G \cdot Z_R^{ss} = \dim X - (\dim G + \dim Z_R^{ss} - \dim N) \tag{1.6}$$

and

$$H_G^*(G \cdot Z_R^{ss}) = H_N^*(Z_R^{ss}). \tag{1.7}$$

Therefore from (1.4), (1.5) and (1.7), it follows that

$$P_t^G(\hat{X}) = P_t^G(X^{ss}) + P_t^N(Z_R^{ss})(t^2 + ... + t^{2(\operatorname{rk}\mathcal{N}^R - 1)}). \tag{1.8}$$

If we consider the HKKN stratification $\{S_{\hat{X},\hat{\beta}}\}_{\hat{\beta}\in\mathcal{B}_{\hat{X}}}$ associated to the induced action of G on $\hat{X}$ (see Theorem 1.2.1), we can apply Theorem 1.2.2 to deduce the equivariant Hilbert-Poincaré series of the semistable locus:

$$P_t^G(\hat{X}^{ss}) = P_t^G(\hat{X}) - \sum_{0\neq\hat{\beta}\in\mathcal{B}_{\hat{X}}} t^{2\operatorname{codim}(S_{\hat{X},\hat{\beta}})} P_t^G(S_{\hat{X},\hat{\beta}}). \tag{1.9}$$

If we denote by $\operatorname{Stab}\hat{\beta} \subseteq G$ the stabiliser of $\hat{\beta}$ under the adjoint action of G, it is proved in [Kir85, p. 72] that

$$S_{\hat{X},\hat{\beta}} = G \times_{N\cap\operatorname{Stab}\hat{\beta}} (Z_{\hat{X},\hat{\beta}}^{ss} \cap \pi^{-1}(Z_R^{ss})), \tag{1.10}$$

where $Z_{\hat{X},\hat{\beta}}^{ss} \subseteq \hat{X}$ is defined as in (1.1). Therefore from (1.8), (1.9) and (1.10) we obtain

$$\begin{aligned} P_t^G(\hat{X}^{ss}) = {} & P_t^G(X^{ss}) + P_t^N(Z_R^{ss})(t^2 + ... + t^{2(\operatorname{rk}\mathcal{N}^R-1)}) \\ & - \sum_{0\neq\hat{\beta}\in\mathcal{B}_{\hat{X}}} t^{2\operatorname{codim}(S_{\hat{X},\hat{\beta}})} P_t^{N\cap\operatorname{Stab}\hat{\beta}}(Z_{\hat{X},\hat{\beta}}^{ss} \cap \pi^{-1}(Z_R^{ss})). \end{aligned} \tag{1.11}$$

Now we want to relate the last term of (1.11) to the representation on the normal slice to the orbit. For this, let $x \in Z_R^{ss}$ be a general point and consider the normal vector space $\mathcal{N}_x^R$ to $G \cdot Z_R^{ss}$ in X^{ss} at this point. Since the action of R on X^{ss} keeps this point x fixed, there is a natural induced representation $\rho : R \to \operatorname{GL}(\mathcal{N}_x^R)$ of R on this vector space. Now let $\mathfrak{t}_R \subseteq \mathfrak{t}$ be the inclusion of real Lie algebras induced by the inclusion of the maximal compact torus of R into that of G and we use on $\mathfrak{t}_R$ the metric induced from that of $\mathfrak{t}$. We then consider the HKKN stratification $\{S_{\beta',\rho}\}_{\beta'\in\mathcal{B}(\rho)}$ associated to the action of R on the projective slice $\mathbb{P}\mathcal{N}_x^R$ (see Theorem 1.2.1). We recall that the indexing set $\mathcal{B}(\rho)$ is given by those points in $\mathfrak{t}_R$ that are the closest points to the origin for some convex hull of a nonempty set of weights of the representation ρ.

If we further assume that $\hat{X}$ is the parameter space $\widetilde{X}$ of the full Kirwan blow-up, it is proved in [Kir85, §7] that $\mathcal{B}_{\hat{X}}$ can be identified with a subset of $\mathcal{B}(\rho)$. Given

$\hat{\beta} \in \mathcal{B}_{\hat{X}}$, the Weyl group $W(G)$ orbit of $\hat{\beta}$ decomposes into a finite number of $W(R)$ orbits. There is a unique $\beta' \in \mathcal{B}(\rho)$ in each $W(R)$ orbit contained in the $W(G)$ orbit of $\hat{\beta}$. We thus denote by $w(\beta', R, G)$ the number of $\beta' \in \mathcal{B}(\rho)$ lying in the Weyl group orbit $W(G) \cdot \hat{\beta}$. For any $\hat{\beta} \in \mathcal{B}_{\hat{X}}$, we have by [Kir85, Lemma 7.9]

$$S_{\hat{X},\hat{\beta}} \cap \mathbb{P}\mathcal{N}_x^R = \bigcup_{\beta' \in \mathcal{B}(\rho) \cap W(G) \cdot \hat{\beta}} S_{\beta',\rho},$$

and by [Kir85, Lemma 7.11] we obtain that the codimension of $S_{\hat{X},\hat{\beta}} \subseteq \hat{X}$ is equal to the codimension $d(\mathbb{P}\mathcal{N}_x^R, \beta')$ of $S_{\beta',\rho} \subseteq \mathbb{P}\mathcal{N}_x^R$, which can be computed via Remark 1.2.2.

Now given any $\beta' \in \mathcal{B}(\rho)$, consider the unique element $\hat{\beta} \in \mathcal{B}_{\hat{X}}$ with β' in its $W(G)$-orbit. In general the subgroups $N \cap \mathrm{Stab}\hat{\beta}$ and $N \cap \mathrm{Stab}\beta'$ are different, but both of them have a well-defined action on Z_R^{ss}, as they are $W(G)$-conjugate subgroups of N. If we furthermore substitute $Z_{\hat{X},\hat{\beta}}^{ss} \subseteq \hat{X}$ with the isomorphic locus $Z_{\hat{X},\beta'}^{ss} \subseteq \hat{X}$ defined via (1.1), we obtain a well-defined action of $N \cap \mathrm{Stab}\beta'$ on $Z_{\hat{X},\hat{\beta}}^{ss} \cap \pi^{-1}(Z_R^{ss})$, such that

$$P_t^{N\cap \mathrm{Stab}\hat{\beta}}(Z_{\hat{X},\hat{\beta}}^{ss} \cap \pi^{-1}(Z_R^{ss})) = P_t^{N\cap \mathrm{Stab}\beta'}(Z_{\hat{X},\beta'}^{ss} \cap \pi^{-1}(Z_R^{ss})).$$

In summary, we have (cf. [Kir85, (7.15)] and [Kir89, (3.2), (3.4)]):

$$\begin{aligned} P_t^G(\hat{X}^{ss}) = P_t^G(X^{ss}) + P_t^N(Z_R^{ss})(t^2 + \dots + t^{2(\mathrm{rk}\mathcal{N}^R - 1)}) \\ - \sum_{0 \neq \beta' \in \mathcal{B}(\rho)} \frac{1}{w(\beta', R, G)} t^{2d(\mathbb{P}\mathcal{N}_x^R, \beta')} P_t^{N\cap \mathrm{Stab}\beta'}(Z_{\hat{X},\beta'}^{ss} \cap \pi^{-1}(Z_R^{ss})). \end{aligned}$$

Finally, we notice that for each $\beta' \in \mathcal{B}(\rho)$, there is an $(N \cap \mathrm{Stab}\beta')$-equivariant fibration

$$\pi : Z_{\beta',R}^{ss} := Z_{\hat{X},\beta'}^{ss} \cap \pi^{-1}(Z_R^{ss}) \to Z_R^{ss} \tag{1.12}$$

with all fibres isomorphic to $Z_{\beta',\rho}^{ss}$ as defined in (1.1) for the representation ρ of R on $\mathbb{P}\mathcal{N}_x^R$. Moreover, if $Z_{\beta',\rho}^{ss} = Z_{\beta',\rho}$, the spectral sequence of rational equivariant cohomology associated to the fibration $\pi : Z_{\beta',R}^{ss} \to Z_R^{ss}$ degenerates and hence (cf. [Kir85, 7.2] and [KL89, 4.1 (4)]):

$$P_t^{N\cap \mathrm{Stab}\beta'}(Z_{\beta',R}^{ss}) = P_t^{N\cap \mathrm{Stab}\beta'}(Z_R^{ss}) \cdot P_t(Z_{\beta',\rho}). \tag{1.13}$$

A repeated application of this argument leads to a formula to compute inductively the equivariant cohomology $H_G^*(\widetilde{X}^{ss})$ of the semistable locus $\widetilde{X}^{ss}$, whose GIT quotient gives the Kirwan blow-up. Since G acts on $\widetilde{X}^{ss}$ with finite stabilisers, its equivariant Hilbert-Poincaré polynomial coincides with that of the partial desingularization $\widetilde{X}/\!/G$. We summarise all the previous considerations under the following

Theorem 1.4.1. *([Kir85, 7.4], [Kir89, 3.4]) In the above setting, the cohomology of the Kirwan blow-up is given by*

$$P_t(\widetilde{X}/\!/G) = P_t^G(\widetilde{X}^{ss}) = P_t^G(X^{ss}) + \sum_{R\in\mathcal{R}} A_R(t),$$

where the error term $A_R(t)$ can be divided into main and extra terms, as follows:

$$\text{(Main term)} \qquad A_R(t) = P_t^{N(R)}(\hat{Z}_R^{ss})(t^2 + ... + t^{2(\mathrm{rk}\mathcal{N}^R - 1)})$$

$$\text{(Extra term)} \qquad - \sum_{0\neq\beta'\in\mathcal{B}_R(\rho)} \frac{1}{w(\beta', R, G)} t^{2d(\mathbb{P}\mathcal{N}_x^R, \beta')} P_t^{N(R)\cap \mathrm{Stab}\beta'}(\hat{Z}_{\beta',R}^{ss}),$$

where $N(R) \subseteq G$ denotes the normaliser of R, the subvariety $\hat{Z}_R^{ss}$ is the strict transform of Z_R^{ss} in the appropriate stage of the modification process (1.3) and $\hat{Z}_{\beta',R}^{ss}$ is defined analogously to (1.12) under the relevant blow-up step.

1.5 Intersection cohomology of the GIT quotient

In this section, we recall Kirwan's procedure to compare the cohomology of $\widetilde{X}/\!/G$ and the intersection cohomology of $X/\!/G$, as explained in [Kir86] (see also [CMGHL19, §5.1]). This is in turn an application of the *Decomposition Theorem* by Beĭlinson, Bernstein, Deligne and Gabber (cf. [BBD82]).

We start with the setting of Section 1.3, where a projective manifold X is acted on by a reductive group G and we have fixed a maximal dimensional connected component $R \in \mathcal{R}$ among the stabilisers of strictly polystable points of X in order to perform the blow-up $\pi : \hat{X} \to X^{ss}$ along the locus $G \cdot Z_R^{ss}$ as in (1.2). As the centre of the blow-up is G-invariant, the map π induces a blow-down morphism $\pi_G : \hat{X}/\!/G \to X/\!/G$ at the level of GIT quotients. For simplicity we further assume that Z_R^{ss} is connected, which will always be the case in our results.

Considering GIT quotients, the diagram in Figure 1.1 (cf. [Kir86, Diagram 1]) summarises the current situation. We recall that $\mathcal{N}^R$ denotes the normal bundle to $G \cdot Z_R^{ss}$ in X^{ss}, $E = \mathbb{P}\mathcal{N}^R$ is the exceptional divisor of the blow-up $\pi : \hat{X} \to X^{ss}$ and E^{ss} is the intersection of E with the semistable locus $\hat{X}^{ss}$. Let $\hat{\mathcal{N}}^R$ be the normal bundle to E^{ss} in $\hat{X}^{ss}$. The map $d\pi : \hat{\mathcal{N}}^R \to \mathcal{N}^R$ is induced by the differential of π. The G-actions extend naturally to all of the spaces in the diagram, so that one can consider the corresponding quotients in the sense of geometric invariant theory. The group $N \subseteq G$ is the normaliser of R in G and we can identify $G \cdot Z_R^{ss}/\!/G = Z_R/\!/N$ via the the isomorphism $G \cdot Z_R^{ss} = G \times_N Z_R^{ss}$ [Kir85, 5.10].

Now we want to use the Decomposition Theorem [BBD82] to compare the intersection cohomology groups $IH^*(\hat{X}/\!/G)$ and $IH^*(X/\!/G)$. An immediate application

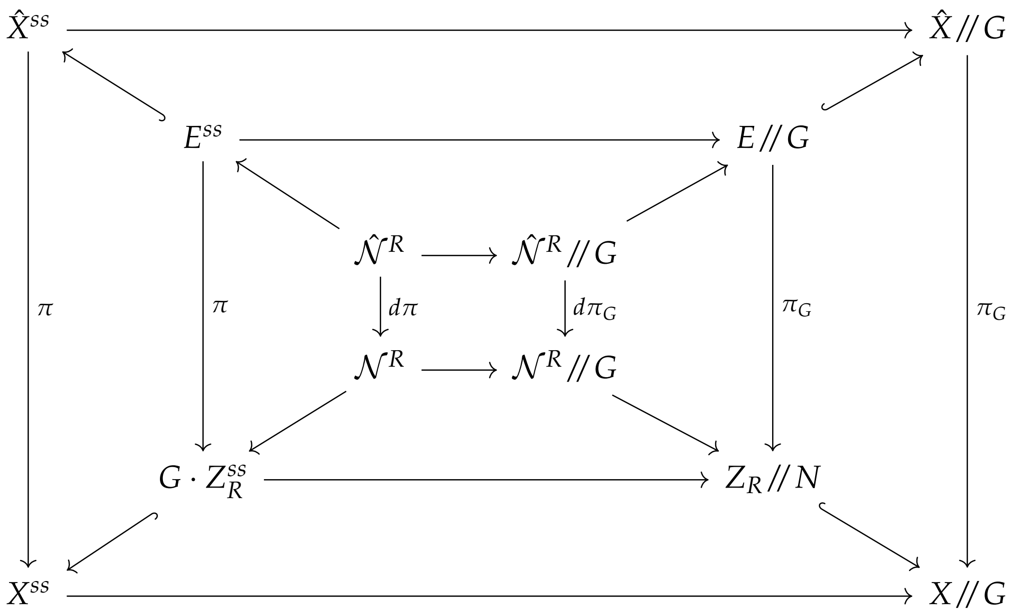


Figure 1.1

of the aforementioned theorem is the fact that $IH^*(X/\!/G)$ is a direct summand of $IH^*(\hat{X}/\!/G)$, as π_G is a birational morphism. Hence we can write

$$IP_t(X/\!/G) = IP_t(\hat{X}/\!/G) - B_R(t), \tag{1.14}$$

for some polynomial $B_R(t)$ with non-negative integral coefficients. Our goal is hence to determine this error polynomial.

We start by considering the following fibre product diagram:

$$\begin{array}{ccccc} E/\!/G & \hookrightarrow & \hat{U} & \hookrightarrow & \hat{X}/\!/G \\ \downarrow & & \downarrow & & \downarrow \pi_G \\ Z_R/\!/N & \hookrightarrow & U & \hookrightarrow & X/\!/G, \end{array}$$

where U is an open neighbourhood of $Z_R/\!/N$ in $X/\!/G$ and $\hat{U} := \pi_G^{-1}(U)$ is its inverse image in $\hat{X}/\!/G$, which is an open neighbourhood of $E/\!/G$ in $\hat{X}/\!/G$. Since π_G is a birational morphism that is an isomorphism on the complement of the closed subvariety $Z_R/\!/N$, by [Kir86, Lemma 2.8] we have

$$IP_t(X/\!/G) = IP_t(\hat{X}/\!/G) - IP_t(\hat{U}) + IP_t(U).$$

In [Kir86, Lemma 2.9] Kirwan shows that there is a homeomorphism between a neighbourhood of $Z_R/\!/N$ in $X/\!/G$ and a neighbourhood of $Z_R/\!/N$ in $\mathcal{N}^R/\!/G$, which fixes $Z_R/\!/N$ pointwise and lifts to a homeomorphism of neighbourhoods of $E/\!/G$ in $\hat{X}/\!/G$

and $\hat{\mathcal{N}}^R/\!/G$. Therefore we obtain [Kir86, Corollary 2.11]

$$IP_t(X/\!/G) = IP_t(\hat{X}/\!/G) - IP_t(\hat{\mathcal{N}}^R/\!/G) + IP_t(\mathcal{N}^R/\!/G).$$

By observing that there is an isomorphism $\mathcal{N}^R/\!/G \cong \mathcal{N}^R|_{Z_R^{ss}}/\!/N$ and $\hat{\mathcal{N}}^R/\!/G \cong \hat{\mathcal{N}}^R|_{\pi^{-1}(Z_R^{ss})}/\!/N$ (see [Kir86, p.493]) and that the intersection cohomology of the quotient by a finite group is the part of the intersection cohomology that is invariant under the action of the finite group (see [Kir86, Lemma 2.12]), we find that

$$\begin{aligned} IH^*(\mathcal{N}^R/\!/G) &\cong [IH^*(\mathcal{N}^R|_{Z_R^{ss}}/\!/N^0)]^{\pi_0 N}, \\ IH^*(\hat{\mathcal{N}}^R/\!/G) &\cong [IH^*(\hat{\mathcal{N}}^R|_{\pi^{-1}(Z_R^{ss})}/\!/N^0)]^{\pi_0 N}. \end{aligned} \tag{1.15}$$

The Leray spectral sequence of intersection cohomology associated to the morphisms $\mathcal{N}^R|_{Z_R^{ss}}/\!/N^0 \to Z_R/\!/N^0$ and $\hat{\mathcal{N}}^R|_{\pi^{-1}(Z_R^{ss})}/\!/N^0 \to Z_R/\!/N^0$ degenerates [Kir86, Proposition 2.13, p. 493], hence we can deduce

$$\begin{aligned} IH^*(\mathcal{N}^R|_{Z_R^{ss}}/\!/N^0) &\cong IH^*(\mathcal{N}_x^R/\!/R) \otimes H^*(Z_R/\!/N^0), \\ IH^*(\hat{\mathcal{N}}^R|_{\pi^{-1}(Z_R^{ss})}/\!/N^0) &\cong IH^*(\hat{\mathcal{N}}_x^R/\!/R) \otimes H^*(Z_R/\!/N^0), \end{aligned} \tag{1.16}$$

where $x \in Z_R^{ss}$ is a general point and the normal space $\mathcal{N}_x^R$ is acted on by the group R via the representation $\rho : R \to \mathrm{GL}(\mathcal{N}_x^R)$. Here by $\mathcal{N}_x^R/\!/R$ and $\hat{\mathcal{N}}_x^R/\!/R$ we mean the affine varieties associated to the corresponding invariant rings. Moreover, the intersection cohomology of $Z_R/\!/N^0$ equals its singular cohomology, as it has at worst finite quotient singularities. Combining (1.15) with (1.16) yields [Kir86, 2.20]

$$\begin{aligned} IH^*(\mathcal{N}^R/\!/G) &\cong [IH^*(\mathcal{N}_x^R/\!/R) \otimes H^*(Z_R/\!/N^0)]^{\pi_0 N}, \\ IH^*(\hat{\mathcal{N}}^R/\!/G) &\cong [IH^*(\hat{\mathcal{N}}_x^R/\!/R) \otimes H^*(Z_R/\!/N^0)]^{\pi_0 N}. \end{aligned}$$

Finally Kirwan proves that $IH^*(\hat{\mathcal{N}}_x^R/\!/R) \cong IH^*(\mathbb{P}\mathcal{N}_x^R/\!/R)$ [Kir86, Lemma 2.15], and that there is a natural surjection $IH^i(\mathbb{P}\mathcal{N}_x^R/\!/R) \to IH^i(\mathcal{N}_x^R/\!/R)$, whose kernel is isomorphic to $IH^{i-2}(\mathbb{P}\mathcal{N}_x^R/\!/R)$ if $i \leq \dim \mathbb{P}\mathcal{N}_x^R/\!/R$, and to $IH^i(\mathbb{P}\mathcal{N}_x^R/\!/R)$ otherwise [Kir86, Corollary 2.17]. In conclusion, we obtain [Kir86, Proposition 2.1]

$$\dim IH^i(X/\!/G) = \dim IH^i(\hat{X}/\!/G) - \sum_{p+q=i} \dim [H^p(Z_R/\!/N^0) \otimes IH^{\hat{q}}(\mathbb{P}\mathcal{N}_x^R/\!/R)]^{\pi_0 N},$$

where $\hat{q} = q - 2$ for $q \leq \dim \mathbb{P}\mathcal{N}_x^R/\!/R$, and $\hat{q} = q$ otherwise.

By repeating the above argument, we can find an iterative formula for the intersection cohomology of $X/\!/G$ from that of the Kirwan blow-up $\widetilde{X}/\!/G$. Since the partial desingularization $\widetilde{X}/\!/G$ has only finite quotient singularities, its intersection cohomology is isomorphic to its singular cohomology, which can be computed via Theorem 1.4.1. Eventually, we will be able to find the intersection Betti numbers of $X/\!/G$, by means of the following:

Theorem 1.5.1. *[Kir86, Theorem 3.1] In the above setting, the intersection Hilbert-Poincaré polynomial of the GIT quotient $X/\!/G$ is related to that of the Kirwan blow-up via the equality*

$$IP_t(X/\!/G) = P_t(\widetilde{X}/\!/G) - \sum_{R\in\mathcal{R}} B_R(t),$$

where the error term is given by

$$B_R(t) = \sum_{p+q=i} t^i \dim[H^p(\hat{Z}_R/\!/N(R)^0) \otimes IH^{\hat{q}_R}(\mathbb{P}\mathcal{N}_x^R/\!/R)]^{\pi_0 N(R)},$$

where the integer $\hat{q}_R = q-2$ for $q \leq \dim \mathbb{P}\mathcal{N}_x^R/\!/R$ and $\hat{q}_R = q$ otherwise. The subvariety $\hat{Z}_R$ is the strict transform of Z_R in the appropriate stage of the modification process (1.3), while $N(R) \subseteq G$ denotes the normaliser of R. The GIT quotient $\mathbb{P}\mathcal{N}_x^R/\!/R$ is constructed from the induced action of R on the normal slice $\mathcal{N}_x^R$ to the orbit $G \cdot Z_R^{ss}$ in X^{ss} at a general point $x \in Z_R^{ss}$.

Remark 1.5.1. If $\hat{Z}_R/\!/N(R)^0$ is simply connected, which will always be the case in the following chapters, then the action of $\pi_0 N(R)$ on the tensor product splits (see [Kir86, §2]), thus the error term for the subgroup R is

$$B_R(t) = \sum_{p+q=i} t^i \dim H^p(\hat{Z}_R/\!/N(R)) \cdot \dim[IH^{\hat{q}_R}(\mathbb{P}\mathcal{N}_x^R/\!/R)]^{\pi_0 N(R)}.$$

2 | Cohomology of the moduli space of non-hyperelliptic genus four curves

In this chapter, we present the results of the author's article [For18] about the cohomology of the moduli space of non-hyperelliptic Petri-general genus four curves. The canonical model of such curves is a complete intersection of a smooth quadric and a cubic surface in projective space. This moduli space hence carries a natural compactification:

$$M := \mathbb{P}H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(3,3)) /\!/ \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1),$$

as GIT quotient for the space of curves of bidegree $(3,3)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ under the automorphism group of $\mathbb{P}^1 \times \mathbb{P}^1$. We are interested in examining this space from a cohomological point of view. The strategy to compute the intersection Betti numbers of M relies on Kirwan's procedure, explained in Chapter 1, whose crucial step consists of the construction of the Kirwan blow-up $\widetilde{M} \to M$. Our result is summarised by the following:

Theorem 2.0.1. *The intersection Betti numbers of M and the Betti numbers of the Kirwan blow-up $\widetilde{M}$ are as follows:*

i	0	2	4	6	8	10	12	14	16	18
$\dim IH^i(M,\mathbb{Q})$	1	1	2	2	3	3	2	2	1	1
$\dim H^i(\widetilde{M},\mathbb{Q})$	1	4	7	11	14	14	11	7	4	1

while all the odd degree (intersection) Betti numbers vanish.

The structure of the chapter reflects the steps of Kirwan's machinery. In Section 2.1 we recall the construction of M as GIT quotient $X /\!/ G$ together with the geometric description of the semistable and stable loci. In Section 2.2, we calculate the equivariant Hilbert-Poincaré polynomial of the semistable locus X^{ss} in the parameter space of $(3,3)$ curves (see Proposition 2.2.1): this is done by computing the Hesselink-Kempf-Kirwan-Ness stratification of the unstable locus from Section 1.2. In Section 2.3, we explicitly construct the partial desingularization $\widetilde{M} \to M$, by blowing up three G-invariant loci in the GIT boundary of M, corresponding to strictly polystable curves (see Definition 2.3.1). These subspaces are given by triple conics in $\mathbb{P}^1 \times \mathbb{P}^1$, curves with two D_4 or two D_8 singularities, called *D-curves*, and curves with two singularities of type A_5, called *A-curves*. Section 2.4 is devoted to the computation of the rational Betti numbers of the Kirwan blow-up $\widetilde{M}$ (see Theorem 2.4.1). Here the correction terms arising from the modification process $\widetilde{M} \to M$ are calculated by following the results of Section 1.4. In the end, the intersection Betti numbers of M are computed in Section 2.5, as an application of Theorem 1.5.1 (see Theorem 2.5.1). In Section 2.6 we conclude with a geometric interpretation of some Betti numbers, via a description of the classes of curves in the GIT boundary which generate some cohomology groups.

2.1 GIT for $(3,3)$ curves in $\mathbb{P}^1 \times \mathbb{P}^1$

A smooth non-hyperelliptic curve of genus 4 is realised by the canonical embedding as a complete intersection of a quadric and a cubic surface in the projective space $\mathbb{P}^3$. If the quadric is smooth, the curve is said to be *Petri-general* and thus defines a point in the complete linear system

$$X := \mathbb{P}H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(3,3)) = \mathbb{P}(\mathrm{Sym}^3(\mathbb{C}^2)^\vee \otimes \mathrm{Sym}^3(\mathbb{C}^2)^\vee) \cong \mathbb{P}^{15}$$

of curves of bidegree $(3,3)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. Since every such curve admits a unique pair of g^1_3 systems, it follows that these curves are abstractly isomorphic as algebraic curves if and only if they lie in the same $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$-orbit.

We consider the reductive group $G := (\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})) \rtimes \mathbb{Z}/2\mathbb{Z}$, which is only isogenous to $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{P}\mathrm{O}(4,\mathbb{C})$, but has the advantage to define a linearisation of the hyperplane bundle of X. We will work with this linearisation throughout all the results. The action of G on X is induced by the natural action of $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$ on $\mathbb{P}^1 \times \mathbb{P}^1$ via change of coordinates and the $\mathbb{Z}/2\mathbb{Z}$-extension interchanges the rulings of $\mathbb{P}^1 \times \mathbb{P}^1$. Geometric Invariant Theory [MFK94] provides a good categorical projective quotient with respect to the linearisation $\mathcal{O}_X(1)$:

$$M := X /\!/ G,$$

whose cohomology we aim to compute. In particular, intersection cohomology satisfies Poincaré duality, allowing us to compute the Betti numbers up to dimension $9 = \dim M$. However, we prefer to carry out the computations in all dimensions for the sake of completeness, and to report also the results *mod* t^{10} for the sake of readability.

Now we want to present a description of the semistability condition for non-hyperelliptic Petri-general curves of genus 4. This is provided by the following:

Theorem 2.1.1. *[Fed12, 2.2] A curve C is unstable (i.e. non-semistable) for the action of* $(\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})) \rtimes \mathbb{Z}/2\mathbb{Z}$ *on* $\mathbb{P}H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(3,3))$ *if and only if one of the following holds:*

(i) C contains a double ruling;

(ii) C contains a ruling and the residual curve C' intersects this ruling in a unique point that is also a singular point of C'.

The GIT boundary $M \smallsetminus M^s$ consisting of strictly polystable points is described by the following:

Theorem 2.1.2. *[Fed12, §2.2] [CMJL14, 3.7] The strictly polystable curves for the action of* $(\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})) \rtimes \mathbb{Z}/2\mathbb{Z}$ *on* $\mathbb{P}H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(3,3))$ *fall into four categories:*

(i) Triple conics;

(ii) Unions of a smooth double conic and a conic that is nonsingular along the double conic. These form a 1*-dimensional family;*

(iii) Unions of three conics meeting in two D_4 singularities. These form a 2*-dimensional family;*

(iv) Unions of two lines of the same ruling, meeting the residual curve in two A_5 singularities.

2.2 The HKKN stratification for $(3,3)$ curves in $\mathbb{P}^1 \times \mathbb{P}^1$

In this section, we apply Theorem 1.2.2 to the case of $(3,3)$ curves in $\mathbb{P}^1 \times \mathbb{P}^1$ and prove the following:

Proposition 2.2.1. *The G-equivariant Hilbert-Poincaré series of the semistable locus is*

$$\begin{aligned} P_t^G(X^{ss}) &= \frac{1+t^2+t^4+t^6+2t^8+2t^{10}+t^{12}-t^{14}-t^{16}-t^{18}-t^{20}-t^{22}}{1-t^4} \\ &\equiv P_t^G(X) \equiv 1+t^2+2t^4+2t^6+4t^8 \bmod t^{10}. \end{aligned}$$

We need to start computing the equivariant Hilbert-Poincaré series $P_t^G(X)$. Since X is compact, its equivariant cohomology ring is the invariant part under the action of $\pi_0 G = \mathbb{Z}/2\mathbb{Z}$ of $H^*_{G^0}(X)$, which splits into the tensor product $H^*(BG^0) \otimes H^*(X)$ (see [Kir84, 8.12]). Then

$$\begin{aligned} H^*_G(X) &= H^*_{G^0}(X)^{\mathbb{Z}/2\mathbb{Z}} \\ &= (H^*(B(\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}))) \otimes H^*(\mathbb{P}^{15}))^{\mathbb{Z}/2\mathbb{Z}} \\ &= (\mathbb{Q}[c_1,c_2] \otimes \mathbb{Q}[h]/(h^{16}))^{\mathbb{Z}/2\mathbb{Z}}. \end{aligned}$$

In fact $H^*(B\mathrm{SL}(2,\mathbb{C})) \cong \mathbb{Q}[c]$, where c has degree 4, and $H^*(\mathbb{P}^n) = \mathbb{Q}[h]/(h^{n+1})$, with $\deg(h) = 2$. The extension $\mathbb{Z}/2\mathbb{Z}$ acts by interchanging c_1 and c_2, while it fixes the hyperplane class $h \in H^2(\mathbb{P}^{15})$. Therefore the ring of invariants is generated by $c_1 + c_2$, c_1c_2 and h:

$$H^*_G(X) = \mathbb{Q}[c_1 + c_2, c_1c_2] \otimes \mathbb{Q}[h]/(h^{16}).$$

Since $\deg(c_1 + c_2) = 4$ and $\deg(c_1c_2) = 8$, we have

$$P^G_t(X) = \frac{1 + t^2 + ... + t^{30}}{(1 - t^4)(1 - t^8)} \equiv 1 + t^2 + 2t^4 + 2t^6 + 4t^8 \bmod t^{11}. \tag{2.1}$$

According to Theorem 1.2.2, we need to subtract the contributions coming from the unstable strata. In our case, the indexing set $\mathcal{B}$ of the HKKN stratification can be visualised by means of Figure 2.1, called *Hilbert diagram*.

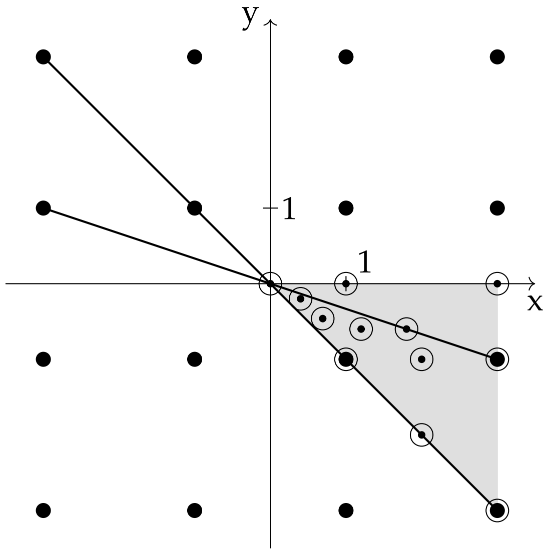


Figure 2.1: Hilbert diagram. The circled dots describe the indexing set $\mathcal{B}$. The two lines pass through the weights of strictly semistable points (see Proposition 2.3.1).

There are 16 black nodes in this square, and each of these nodes represents a monomial $x_0^i x_1^{3-i} y_0^j y_1^{3-i}$ in $H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(3,3))$, for $0 \leq i, j \leq 3$. This square is simply the diagram of weights $\alpha_I = \alpha_{(i,j)}$ of the representation of G on $H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(3,3))$ with respect to the standard maximal torus

$$T := (\mathrm{diag}(a, a^{-1}), \mathrm{diag}(b, b^{-1}), 1) \subseteq G.$$

Each of the nodes denotes a weight of this representation, namely

$$x_0^i x_1^{3-i} y_0^j y_1^{3-j} \leftrightarrow (3 - 2i, 3 - 2j), \text{ for } i, j = 0, ..., 3. \tag{2.2}$$

There is a non-degenerate inner product (the Killing form) defined in the Cartan subalgebra $\mathfrak{t} := \mathrm{Lie}(T \cap (\mathrm{SU}(2,\mathbb{C}) \times \mathrm{SU}(2,\mathbb{C})))$ in $\mathrm{Lie}(\mathrm{SU}(2,\mathbb{C}) \times \mathrm{SU}(2,\mathbb{C})) \otimes \mathbb{C} \cong \mathrm{Lie}(G)$. Using this inner product, we can identify the Lie algebra $\mathfrak{t}$ with its dual $\mathfrak{t}^\vee$, and the above square can be thought of as lying in $\mathfrak{t}$. The axes of the Hilbert diagram thus coincide with the Lie algebras of the two factors of the maximal compact torus.

The Weyl group $W(G) := N(T)/T \cong (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \ltimes \mathbb{Z}/2\mathbb{Z}$ coincides with the dihedral group D_8 of all symmetries of the square. It operates on the Hilbert diagram as follows: the first two involutions are reflections along the axes, while the third one is along the principal diagonal. It is easy to see that the grey region is the portion of the square which lies inside a fixed positive Weyl chamber $\mathfrak{t}_+$.

By definition, the indexing set $\mathcal{B}$ consists of vectors β such that β lies in the closure $\overline{\mathfrak{t}_+}$ of the positive Weyl chamber and is also the closest point to the origin of a convex hull spanned by a non-empty set of weights of the representation of G on $H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(3,3))$. In this situation, we may assume that such a convex hull is either a single weight or it is cut out by a line segment joining two weights, which will be denoted by $\langle\beta\rangle$ (see Figure 2.1).

All the contributions coming from the unstable strata are summarised in Table 2.1 and were computed looking at the Figure 2.1.

weights in $\langle\beta\rangle$	$n(\beta)$	Stabβ	$2d(\beta)$	$P_t^G(S_\beta)$
$(3,-3)$	15	$\langle T, \iota\rangle$	26	$(1-t^2)^{-1}(1-t^4)^{-1}$
$(3,-1),(1,-3)$	13	$\langle T, \iota\rangle$	22	$(1-t^2)^{-1}$
$(3,1),(1,-1),(-1,-3)$	10	$\langle T, \iota\rangle$	16	$\frac{1+t^2-t^6}{(1-t^2)(1-t^4)}$
$(1,-3),(3,1)$	12	T	20	$(1-t^2)^{-1}$
$(3,3),(1,-1)$	10	T	16	$(1-t^2)^{-1}$
$(1,1),(-1,-3)$	8	T	12	$(1-t^2)^{-1}$
$(3,-1)$	14	T	24	$(1-t^2)^{-2}$
$(1,-3),(3,3)$	11	T	18	$(1-t^2)^{-1}$
$(-1,-3),(3,3)$	9	T	14	$(1-t^2)^{-1}$
$(3,-3),(3,-1),(3,1),(3,3)$	12	$\mathbb{C}^* \times \mathrm{SL}(2,\mathbb{C})$	22	$(1-t^2)^{-1}$
$(1,-3),(1,-1),(1,1),(1,3)$	8	$\mathbb{C}^* \times \mathrm{SL}(2,\mathbb{C})$	14	$(1-t^2)^{-1}$

Table 2.1: Cohomology of the unstable strata.

The element

$$\iota := \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, -1 \right)$$

is a generator of $\mathbb{Z}/2\mathbb{Z}$ in the semidirect product $\langle T, \iota \rangle \cong (\mathbb{C}^*)^2 \rtimes \mathbb{Z}/2\mathbb{Z}$, with automorphism $(a, b) \leftrightarrow (b^{-1}, a^{-1})$, which is a double covering of the maximal torus T. For every $\beta \in \mathcal{B}$, the first column of Table 2.1 shows the weights contained in the segment $\langle \beta \rangle$ orthogonal to the vector $\beta \in \mathfrak{t}$ (see Figure 2.1): then via the correspondence (2.2) one can obtain an explicit geometric interpretation of the curve contained in each unstable stratum. The terms appearing in the second, third and fourth columns are determined easily from the Hilbert diagram. We recall that the value $n(\beta)$ is the number of weights α_I such that $\beta \cdot \alpha_I < ||\beta||^2$, i.e. the number of weights lying in the half-plane containing the origin and defined by β. The subgroup $\mathrm{Stab}\beta \subseteq G$ is the stabiliser of $\beta \in \mathfrak{t}$ under the adjoint action of G (cf. Remark 1.2.1) and the codimension $d(\beta)$ of each stratum $S_\beta \subseteq X$ can be computed via Remark 1.2.2. The computations in the last column follow from applying Theorem 1.2.2 to the action of $\mathrm{Stab}\beta$ on Z_β, in order to compute the equivariant cohomology of each unstable stratum $P_t^{\mathrm{Stab}\beta}(Z_\beta^{ss}) = P_t^G(S_\beta)$ (see Remark 1.2.1).

We shall discuss the cases of Table 2.1 below.

Lemma 2.2.1. *There are exactly two unstable strata indexed by β, as listed in Table 2.1, such that $Z_\beta \cong \mathbb{P}^0$, and their equivariant Hilbert-Poincaré series are $P_t^G(S_\beta) = (1-t^2)^{-1}(1-t^4)^{-1}$ if $\mathrm{Stab}\beta \cong \langle T, \iota \rangle$, and $P_t^G(S_\beta) = (1-t^2)^{-2}$ if $\mathrm{Stab}\beta \cong T$.*

Proof. The cases under consideration correspond to the first and seventh rows of Table 2.1, where the line orthogonal to β contains only the weight β itself, giving the point $Z_\beta \cong \mathbb{P}^0$. Hence by Remark 1.2.1, if $\mathrm{Stab}\beta \cong \langle T, \iota \rangle$, the equivariant cohomology of the corresponding stratum is

$$P_t^G(S_\beta) = P_t^{\langle T, \iota \rangle}(\mathbb{P}^0) = P_t(B((\mathbb{C}^*)^2 \rtimes \mathbb{Z}/2\mathbb{Z})) = \frac{1}{(1-t^2)(1-t^4)},$$

while, if $\mathrm{Stab}\beta \cong T$, it is

$$P_t^G(S_\beta) = P_t^T(\mathbb{P}^0) = P_t(B(\mathbb{C}^*)^2) = \frac{1}{(1-t^2)^2}.$$

□

Lemma 2.2.2. *There are exactly six unstable strata indexed by β, as listed in Table 2.1, such that $Z_\beta \cong \mathbb{P}^1$, and their equivariant Hilbert-Poincaré series is $P_t^G(S_\beta) = (1-t^2)^{-1}$.*

Proof. Looking at Figure 2.1, there are 6 unstable strata indexed by $\beta \in \mathcal{B}$ such that the segment $\langle \beta \rangle$ orthogonal to the vector β contains two weights that generate the line $Z_\beta \subseteq X$. As summarised in Table 2.1, in five of these cases the stabiliser Stabβ is isomorphic to the maximal torus T and hence by Remark 1.2.1:

$$P_t^G(S_\beta) = \frac{1+t^2}{(1-t^2)^2} - \frac{2t^2}{(1-t^2)^2} = \frac{1}{1-t^2}.$$

In the remaining case, corresponding to the second row of Table 2.1, the stabiliser is Stab$\beta \cong \langle T, \iota \rangle$ and the cohomology of the corresponding stratum is

$$P_t^G(S_\beta) = \frac{1+t^2}{(1-t^2)(1-t^4)} - \frac{t^2}{(1-t^2)^2} = \frac{1}{1-t^2}.$$

□

Lemma 2.2.3. *There is exactly one unstable stratum indexed by β, as listed in Table 2.1, such that $Z_\beta \cong \mathbb{P}^2$, and its equivariant Hilbert-Poincaré series is $P_t^G(S_\beta) = (1+t^2-t^6)(1-t^2)^{-1}(1-t^4)^{-1}$.*

Proof. The case under consideration corresponds to the third row of Table 2.1, where the segment orthogonal to β contains three weights spanning $Z_\beta \cong \mathbb{P}^2$. Hence, by Theorem 1.2.2, the equivariant cohomological series of the correspondent stratum is

$$P_t^G(S_\beta) = \frac{1+t^2+t^4}{(1-t^2)(1-t^4)} - \frac{t^4}{(1-t^2)^2} = \frac{1+t^2-t^6}{(1-t^2)(1-t^4)}.$$

□

Lemma 2.2.4. *There are exactly two unstable strata indexed by β, as listed in Table 2.1, such that $Z_\beta \cong \mathbb{P}^3$, and their equivariant Hilbert-Poincaré series is $P^G(S_\beta) = (1-t^2)^{-1}$.*

Proof. The cases under consideration correspond to the last two rows of Table 2.1, where the segment orthogonal to β contains four weights spanning a $\mathbb{P}^3$. The linear subspace Z_β is acted on by the group Stab$\beta = \mathbb{C}^* \times \mathrm{SL}(2,\mathbb{C})$. The first factor is central and acts trivially on Z_β, while the action of the second factor can be identified with the action on the space $\mathrm{Sym}^3\mathbb{P}^1 \cong \mathbb{P}^3$ of binary cubic forms by change of coordinates. This leads to

$$P^G(S_\beta) = P_t(B\mathbb{C}^*)P_t^{SL(2,\mathbb{C})}((\mathrm{Sym}^3\mathbb{P}^1)^{ss}) = P_t(B\mathbb{C}^*)P(M_{0,3}) = \frac{1}{1-t^2},$$

where $M_{0,3} \cong \mathbb{P}^0$ is the moduli space of curves of genus 0 with 3 marked points. □

We are finally ready to prove Proposition 2.2.1:

Proof of Proposition 2.2.1. According to Theorem 1.2.2, we need to subtract all the contributions of the unstable strata, appearing in Table 2.1, from the G-equivariant cohomology of X computed in (2.1). □

2.3 The Kirwan blow-up for $(3,3)$ curves in $\mathbb{P}^1 \times \mathbb{P}^1$

In this section we describe the construction of the Kirwan blow-up $\widetilde{M} \to M$ in our case (see Definition 2.3.1). It is obtained by blowing up three loci of strictly polystable points, geometrically described in Theorem 2.1.2 (see also Proposition 2.3.2).

By following Section 1.3, we need to find the indexing set $\mathcal{R}$ of the Kirwan blow-up and the corresponding spaces Z_R^{ss}, for all $R \in \mathcal{R}$. Namely, one must compute the connected components of the identity in the stabilisers among all the four families of polystable curves listed in Theorem 2.1.2. Compared to [Fed12, §2.2], we provide a more explicit, but equivalent, way to find the indexing set $\mathcal{R}$, which has also the advantage to compute Z_R and Z_R^{ss} in the coordinate system of X.

The goal is to find which non-trivial connected reductive subgroups $R \subseteq G$ fix at least one semistable point. Firstly, since R is connected, R must be contained in $G^0 = \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$. Secondly, since we are interested only in the conjugacy class of R, we may assume that its intersection $T_R := R \cap T$ with the maximal torus is a maximal torus of R and $R \cap (\mathrm{SU}(2,\mathbb{C}) \times \mathrm{SU}(2,\mathbb{C}))$ is a maximal compact subgroup. Since $0 \in \mathfrak{t}$ is not a weight, it follows that $T \cong (\mathbb{C}^*)^2$ fixes no semistable points. Therefore T_R is a subtorus of rank one.

The fixed point set Z_R^{ss} in X^{ss} consists of all semistable points whose representatives in $H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(3,3)) \cong \mathbb{C}^{16}$ are fixed by the linear action of R. Thus $\mathrm{Fix}(T_R, \mathbb{C}^{16})$ is spanned by those weight vectors which lie on a line through the centre of the Hilbert diagram and orthogonal to the Lie subalgebra $\mathrm{Lie}(T_R \cap (\mathrm{SU}(2,\mathbb{C}) \times \mathrm{SU}(2,\mathbb{C}))) \subseteq \mathfrak{t}$. Up to the action of a suitable element of the Weyl group $W(G)$, we can assume that the line passes through the chosen closed positive Weyl chamber $\bar{\mathfrak{t}}_+$. We have only two possibilities, see Figure 2.1.

Therefore we proved the following

Proposition 2.3.1. *If $R \in \mathcal{R}$ is a subgroup in the indexing set of Kirwan's partial resolution, let T_R denote the maximal torus of R and let Z_R^{ss} denote the fixed-point set of R in X^{ss}. Then, up to conjugation, there are two possibilities for T_R and Z_R^{ss}:*

(i) $T_R = T_1 := \{(\mathrm{diag}(t,t^{-1}), \mathrm{diag}(t,t^{-1}), 1) : t \in \mathbb{C}^*\}$ *and Z_R^{ss} is contained in the projective space* $\mathbb{P}\{a x_0^3 y_1^3 + b x_0^2 x_1 y_0 y_1^2 + c x_0 x_1^2 y_0^2 y_1 + d x_1^3 y_0^3\} \cong \mathbb{P}^3$ *spanned by the polynomials* $x_0^3 y_1^3$, $x_0^2 x_1 y_0 y_1^2$, $x_0 x_1^2 y_0^2 y_1$ *and* $x_1^3 y_0^3$*;*

(ii) $T_R = T_2 := \{(\mathrm{diag}(t,t^{-1}), \mathrm{diag}(t^3,t^{-3}), 1) : t \in \mathbb{C}^*\}$ *and Z_R^{ss} is contained in the projective space* $\mathbb{P}\{a x_0^3 y_0 y_1^2 + b x_1^3 y_0^2 y_1\} \cong \mathbb{P}^1$ *spanned by the polynomials* $x_0^3 y_0 y_1^2$ *and* $x_1^3 y_0^2 y_1$.

We start analysing the second case. We can easily see from the characterisation of semistable points (Theorem 2.1.2) that all the semistable curves are given by $y_0y_1(ax_0^3y_1 + bx_1^3y_0)$ with $a \neq 0$ and $b \neq 0$. Geometrically these curves contain two lines of the same ruling and the residual curve intersects them in 2 points, giving 2 singularities of type A_5. We will call these curves *A-curves*. Their singular points are $((0:1),(0:1))$ and $((1:0),(1:0))$ in $\mathbb{P}^1 \times \mathbb{P}^1$; see the Figure 2.2.

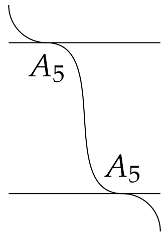


Figure 2.2: Curve with $2A_5$ singularities.

By rescaling the variables x_0 and x_1, all the semistable A-curves are equivalent to the curve C_{2A_5} defined by

$$C_{2A_5} := \{F_{C_{2A_5}} := y_0y_1(x_0^3y_1 + x_1^3y_0) = 0\}.$$

Through this geometric description, it is now easy to show that in this case actually $R = T_R$. We recall that R is the connected component of the identity in the stabiliser of the A-curves: up to conjugation, we can think just of C_{2A_5}. Yet every element of R, stabilising the point corresponding to C_{2A_5} in X, will induce an automorphism of C_{2A_5}, which a fortiori must preserve the singular locus. Therefore every element of R must fix $((0:1),(0:1))$ and $((1:0),(1:0))$ or interchange them. Hence

$$R \subseteq T \sqcup \left\{ \left(\begin{pmatrix} 0 & \alpha \\ -\alpha^{-1} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \beta \\ -\beta^{-1} & 0 \end{pmatrix}, 1 \right) : \alpha, \beta \in \mathbb{C}^* \right\} \subseteq G.$$

From the connectedness of R, it follows $R \subseteq T$, hence $R = T \cap R = T_R = T_2$.

Now we analyse the first case. We can easily see via the Hilbert-Mumford numerical criterion [MFK94, 2.1] that all the semistable curves are given by

$$ax_0^3y_1^3 + bx_0^2x_1y_0y_1^2 + cx_0x_1^2y_0^2y_1 + dx_1^3y_0^3 = 0,$$

where (a,b) are not simultaneously zero and (c,d) are not simultaneously zero, i.e. $Z_{T_1}^{ss} = \mathbb{P}^3 \smallsetminus \{a = b = 0,\ c = d = 0\}$. Moreover we can write every curve

$$ax_0^3y_1^3 + bx_0^2x_1y_0y_1^2 + cx_0x_1^2y_0^2y_1 + dx_1^3y_0^3 = L_1L_2L_3,$$

$$L_i = \alpha_i x_0 y_1 + \beta_i x_1 y_0,\ (\alpha_i : \beta_i) \in \mathbb{P}^1,\ i = 1,2,3,$$

as the union of three conics in the class $(1,1)$, all meeting at points $((0:1),(0:1))$ and $((1:0),(1:0))$ in $\mathbb{P}^1 \times \mathbb{P}^1$. We find three cases depending on how many L_i's coincide.

(i) Assume that all the L_i coincide, namely the curve is a triple conic, which turns out to be equivalent to $3C$, defined by

$$3C := \{F_{3C} := (x_0y_1 - x_1y_0)^3 = 0\}.$$

This curve is nothing but a triple line $\mathbb{P}^1 \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ diagonally embedded. Thus the connected component of the identity in the stabiliser of $3C$ is $\mathbb{P}\mathrm{GL}(2,\mathbb{C})$ diagonally embedded, too. We get a non-splitting central extension of groups:

$$1 \to \mu_2 \times \mu_2 \to H \to \mathbb{P}\mathrm{GL}(2,\mathbb{C}) \to 1, \tag{2.3}$$

where $H := \{(A, \pm A) : A \in \mathrm{SL}(2,\mathbb{C})\}$ is the stabiliser of $3C$ in G^0, that is to say the preimage of $\mathbb{P}\mathrm{GL}(2,\mathbb{C})$ under the natural homomorphism $G^0 = \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}) \to \mathbb{P}\mathrm{GL}(2,\mathbb{C}) \times \mathbb{P}\mathrm{GL}(2,\mathbb{C})$. Here $\mu_2 \times \mu_2$ must be thought of as the subgroup $\{(\pm I, \pm I), (\pm I, \mp I)\} \subseteq H$. Therefore we find the indexing subgroup $R = \mathrm{SL}(2,\mathbb{C})$ diagonally embedded in G^0 and the associated spaces $Z_R = Z_R^{ss} = \{3C\}$ are one point.

(ii) Assume that two L_i coincide and the third one does not. The semistable curves of this type are unions of a smooth double conic and a conic that is nonsingular along the double conic. They intersect at the points $((0:1),(0:1))$ and $((1:0),(1:0))$, which consist of singularities of type D_8; see Figure 2.3.

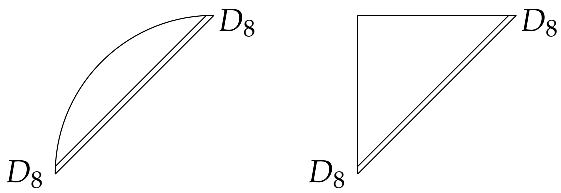


Figure 2.3: Curves with $2D_8$ singularities.

Now we can argue like in the case of C_{2A_5}, noticing that every element of R must preserve the D_8 singular points. Therefore $R \subseteq T$, so that $R = T_1$.

(iii) Assume all the L_i are distinct from each other. The semistable curves of this kind are unions of three conics meeting in two D_4 singularities. These singular points are again $((0:1),(0:1))$ and $((1:0),(1:0))$; see Figure 2.4.

Arguing once more as before, we find that $R = T_1$.

In conclusion, we proved the following:

Proposition 2.3.2. *The indexing set $\mathcal{R}$ of the Kirwan blow-up and the fixed loci Z_R^{ss}, for $(3,3)$ curves in $\mathbb{P}^1 \times \mathbb{P}^1$, can be described as follows:*

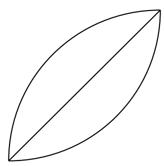


Figure 2.4: Curve with $2D_4$ singularities.

(i) $R_C := \mathrm{SL}(2,\mathbb{C})$, *diagonally embedded in* G^0, *and in this case* $Z_{R_C} = Z^{ss}_{R_C} = \{3C\}$ *is the triple conic.*

(ii) $R_D := \{(\mathrm{diag}(t,t^{-1}),\mathrm{diag}(t,t^{-1}),1) : t \in \mathbb{C}^*\} \cong \mathbb{C}^*$ *and in this case* $Z^{ss}_{R_D} = \mathbb{P}\{ax_0^3y_1^3 + bx_0^2x_1y_0y_1^2 + cx_0x_1^2y_0^2y_1 + dx_1^3y_0^3\} \smallsetminus \{a=b=0, c=d=0\}$ *is the set of D-curves.*

(iii) $R_A := \{(\mathrm{diag}(t,t^{-1}),\mathrm{diag}(t^3,t^{-3}),1) : t \in \mathbb{C}^*\} \cong \mathbb{C}^*$ *and in this case* $Z^{ss}_{R_A} = \mathbb{P}\{ax_0^3y_0y_1^2 + bx_1^3y_0^2y_1\} \smallsetminus \{a=0, b=0\}$ *is the set of A-curves.*

Moreover, the following holds:

$$R_D \subseteq R_C,\ R_A \cap R_C = \{(\pm I, \pm I, 1)\};$$
$$G \cdot Z^{ss}_{R_C} \subseteq G \cdot Z^{ss}_{R_D},\ G \cdot Z^{ss}_{R_A} \cap G \cdot Z^{ss}_{R_D} = \emptyset.$$

We recall that Kirwan's partial desingularization process consists of successively blowing up X^{ss} along the (strict transforms of the) loci $G \cdot Z^{ss}_R$ in order of $\dim R$, to obtain the space $\widetilde{X}^{ss}$, and then taking the induced GIT quotient $\widetilde{X}/\!/G$ with respect to a suitable linearisation. In our situation, we get the diagram:

$$\begin{array}{ccc}
\widetilde{X}^{ss} = (\mathrm{Bl}_{G\cdot Z^{ss}_{R_A}} X_2^{ss})^{ss} & \longrightarrow & \widetilde{M} \\
\downarrow & & \\
X_2^{ss} = (\mathrm{Bl}_{G\cdot Z^{ss}_{R_D,1}} X_1^{ss})^{ss} & & \\
\downarrow & & \\
X_1^{ss} = (\mathrm{Bl}_{G\cdot Z^{ss}_{R_C}} X^{ss})^{ss} & & \\
\downarrow & & \downarrow \\
X^{ss} & \longrightarrow & M.
\end{array}$$

The space $\widetilde{X}^{ss}$ is obtained by firstly blowing up the orbit of the triple conic $G \cdot Z^{ss}_{R_C}$, followed by the blow-up of $G \cdot Z^{ss}_{R_D,1}$, namely the strict transform of the locus of D-curves $G \cdot Z^{ss}_{R_D}$ under the first bow-up. In the end we need to blow up the orbit $G \cdot Z^{ss}_{R_A}$ of C_{2A_5}. We also observe that the third blow-up commutes with the other two, because the orbit of A-curves is disjoint from the locus of D-curves. Thus we find

Definition 2.3.1. The *Kirwan blow-up* $\widetilde{M} := \widetilde{X}/\!/G \to M$ is defined as the GIT quotient of the blown up variety $\widetilde{X}^{ss}$ constructed above.

Intrinsically at the level of moduli spaces, $\widetilde{M}$ is obtained by first blowing up the point $G \cdot Z^{ss}_{R_C}/\!/G$ corresponding to the orbit of triple conics, then the strict transform $\mathrm{Bl}_{G\cdot Z^{ss}_{R_C}/\!/G}(G \cdot Z^{ss}_{R_D}/\!/G)$ of the surface corresponding to the D-curves and eventually blowing up the point $G \cdot Z_{R_A}/\!/G$ of A-curves. Nevertheless, for computational reasons, we will prefer the first description.

2.4 Cohomology of the Kirwan blow-up for $(3,3)$ curves in $\mathbb{P}^1 \times \mathbb{P}^1$

This section is devoted to the proof of the following theorem, which is an application of Theorem 1.4.1 to the case of $(3,3)$ curves in $\mathbb{P}^1 \times \mathbb{P}^1$.

Theorem 2.4.1. *The Hilbert-Poincaré polynomial of the Kirwan blow-up $\widetilde{M}$ is*

$$P_t(\widetilde{M}) = 1 + 4t^2 + 7t^4 + 11t^6 + 14t^8 + 14t^{10} + 11t^{12} + 7t^{14} + 4t^{16} + t^{18}.$$

Due to the role they play in Theorem 1.4.1, we compute the normalisers of the reductive subgroups in $\mathcal{R}$.

Proposition 2.4.1. *The normalisers of the reductive subgroups in $\mathcal{R} = \{R_C, R_D, R_A\}$ are given as follows:*

(i) $N(R_C) = H \rtimes \mathbb{Z}/2\mathbb{Z} \subseteq G$, *where* $H = \{((A, \pm A), 1) : A \in \mathrm{SL}(2,\mathbb{C})\}$ *fits into the central extension (2.3):*

$$1 \to \mu_2 \times \mu_2 \to H \to \mathbb{P}\mathrm{GL}(2,\mathbb{C}) \to 1,$$

and the semidirect product structure descends from that of G.

(ii) $N(R_D) = S \rtimes \mathbb{Z}/2\mathbb{Z} \subseteq G$, *where S is the subgroup of some generalised permutation matrices, namely*

$$S = T \sqcup \left\{ \left(\begin{pmatrix} 0 & \alpha \\ -\alpha^{-1} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \beta \\ -\beta^{-1} & 0 \end{pmatrix}, 1 \right) : \alpha, \beta \in \mathbb{C}^* \right\} \subseteq G,$$

and the semidirect product structure descends from that of G.

(iii) $N(R_A) = S$*, as above.*

Proof. The proof of (ii) and (iii) is straightforward from the definition of normaliser.

In the case (i), we prove that the normaliser N' of R_C in $G^0 = \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$ is H, then the statement will follow from the symmetry of R_C. Since R_C has index two in H, it is normal in H, hence $H \subseteq N'$. For the converse, we need:

Claim. *For every* $n \in N'$*, there exists a* $g \in R_C$ *with* $gn \in T \cap N'$*, where* T *is the maximal torus.*

Proof of the Claim. Any element $n \in N'$ must conjugate the standard maximal torus $R_D \subseteq R_C$. Since all the maximal tori in R_C are conjugate under the action of R_C, it follows that there must exists a $g' \in R_C$ such that $\tilde{n} = g'n$ fixes the maximal torus R_D, that is to say $\tilde{n}$ belongs to the normaliser S of R_D in G^0. If $\tilde{n} := g'n \in T$, just take $g = g'$. Otherwise $\tilde{n} \in \sigma T$, where

$$\sigma := \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1 \right) \in R_C.$$

In this case, take $g = \sigma^{-1} g'$ and the claim is proved. □

By a straightforward matrix computation, we have that $T \cap N' \subseteq H$. Now we can prove that $N' \subseteq H$. Indeed, for every element $n \in N'$, there is a $g \in R_C$ with $gn \in T \cap N' \subseteq H$, by the *Claim*. Therefore $n \in g^{-1}H = H$. □

2.4.1 Main error terms

This subsection is devoted to computing the main error terms for all the three stages of the partial desingularization $\widetilde{M} \to M$, according to Theorem 1.4.1.

As we have seen, the first step in the Kirwan blow-up process is to blow up the locus corresponding to triple conics.

Proposition 2.4.2. *For the group* $R_C \cong \mathrm{SL}(2,\mathbb{C})$*, the main term of* $A_{R_C}(t)$ *is given by*

$$\begin{aligned} P_t^{N(R_C)}(Z_{R_C}^{ss})(t^2 + ... + t^{2(\mathrm{rk}\mathcal{N}^{R_C} - 1)}) &= \frac{t^2 + ... + t^{22}}{1 - t^4} \\ &\equiv t^2 + t^4 + 2t^6 + 2t^8 \mod t^{10}. \end{aligned}$$

Proof. We saw in Proposition 2.3.2 that $Z_{R_C}^{ss}$ consists of a single point, and we described the normaliser $N(R_C)$ in Proposition 2.4.1. Hence, we obtain

$$H^*_{N(R_C)}(Z_{R_C}^{ss}) = H^*(BN(R_C)) = H^*(B(H \rtimes \mathbb{Z}/2\mathbb{Z})) = H^*(BH)^{\mathbb{Z}/2\mathbb{Z}}.$$

We recall that H fits into the central extension (2.3):

$$1 \to \mu_2 \times \mu_2 \to H \to \mathbb{P}\mathrm{GL}(2, \mathbb{C}) \to 1,$$

hence $H^*(BH)^{\mathbb{Z}/2\mathbb{Z}} = H^*(B\mathbb{P}\mathrm{GL}(2,\mathbb{C}))^{\mathbb{Z}/2\mathbb{Z}}$, with the induced $\mathbb{Z}/2\mathbb{Z}$-action. From the description of R_C, we saw that this copy of $\mathbb{P}\mathrm{GL}(2,\mathbb{C})$ must be thought of as diagonally embedded in $\mathbb{P}\mathrm{GL}(2,\mathbb{C}) \times \mathbb{P}\mathrm{GL}(2,\mathbb{C})$, and $\mathbb{Z}/2\mathbb{Z}$ simply interchanges the two factors, acting trivially on the diagonal. This means that

$$H^*_{N(R_C)}(Z^{ss}_{R_C}) = H^*(B\mathbb{P}\mathrm{GL}(2,\mathbb{C}))^{\mathbb{Z}/2\mathbb{Z}} = H^*(B\mathbb{P}\mathrm{GL}(2,\mathbb{C}))$$

and $P_t^{N(R_C)}(Z^{ss}_{R_C}) = P_t(B\mathbb{P}\mathrm{GL}(2,\mathbb{C})) = (1-t^4)^{-1}$.

Finally, we can compute the rank of the normal bundle from (1.6):

$$\mathrm{rk}\mathcal{N}^{R_C} = \dim X - (\dim G + \dim Z^{ss}_{R_C} - \dim N(R_C)) = 12.$$

□

In the second step, we need to blow up the locus of D-curves.

Proposition 2.4.3. *For the group $R_D \cong \mathbb{C}^*$, the main term of $A_{R_D}(t)$ is given by*

$$\begin{aligned} P_t^{N(R_D)}(Z^{ss}_{R_D,1})(t^2 + \dots + t^{2(\mathrm{rk}\mathcal{N}^{R_D}-1)}) &= \frac{1+t^2}{1-t^2}(t^2 + \dots + t^{14}) \\ &\equiv t^2 + 3t^4 + 5t^6 + 7t^8 \mod t^{10}. \end{aligned}$$

Proof. For brevity, write $R = R_D$ and $N = N(R_D) = S \rtimes \mathbb{Z}/2\mathbb{Z}$ (see Proposition 2.3.2 and 2.4.1). Recall that $Z^{ss}_{R,1}$ is the strict transform of Z^{ss}_R in X^{ss}_1 under the first blow-up. We want to give an easier to handle geometric description of $Z^{ss}_{R,1}$.

We saw in Proposition 2.3.2 that

$$Z^{ss}_R = \mathbb{P}\{ax_0^3y_1^3 + bx_0^2x_1y_0y_1^2 + cx_0x_1^2y_0^2y_1 + dx_1^3y_0^3\} \setminus \{a=b=0,\ c=d=0\}.$$

The centre of the first blow-up is the orbit of the triple conic $3C$ which intersects Z^{ss}_R along the twisted cubic

$$G \cdot 3C \cap Z^{ss}_R = C^{ss} = \{(u^3 : 3u^2v : 3uv^2 : v^3) : (u:v) \in \mathbb{P}^1,\ u,v \neq 0\} \subseteq Z^{ss}_R,$$

corresponding to the union of three conics that are actually coincident. Therefore:

$$Z^{ss}_{R,1} = (\mathrm{Bl}_{C^{ss}} Z^{ss}_R)^{ss},$$

because we recall that, after taking the proper transform, one has to restrict only to the semistable points in $X_1 \to X$ for the induced action of G. We want to stress that the Kirwan blow-up is a blow-up, followed by a restriction to the semistable points. Nevertheless, by [Kir86, 1.9], $Z^{ss}_{R,1}$ is the set of semistable points for the natural action of N/R on $\mathrm{Bl}_{C^{ss}} Z^{ss}_R$. But every point of Z^{ss}_R is actually stable for the action of N/R and it will remain stable after the blow-up (see [Kir85, 3.2]). This means that every point of $\mathrm{Bl}_{C^{ss}} Z^{ss}_R$ is indeed stable and, a fortiori semistable for N/R, hence:

$$Z^{ss}_{R,1} = \mathrm{Bl}_{C^{ss}} Z^{ss}_R.$$

In conclusion, $Z^{ss}_{R,1}$ is the blow-up of $\mathbb{P}^3 \setminus \{a = b = 0, c = d = 0\}$ along the twisted cubic C^{ss} and we need to compute $P^N_t(Z^{ss}_{R,1}) = P^N_t(\mathrm{Bl}_{C^{ss}}(\mathbb{P}^3)^{ss})$. According to (1.4), the equivariant cohomology of the blow-up is related to the centre by the formula:

$$P^N_t(\mathrm{Bl}_{C^{ss}}(\mathbb{P}^3)^{ss}) = P^N_t((\mathbb{P}^3)^{ss}) + t^2 P^N_t(C^{ss}).$$

The action of N on C^{ss} is transitive and the stabiliser of $3C = (1 : -3 : 3 : -1)$ in N is $(H \cap S) \rtimes \mathbb{Z}/2\mathbb{Z}$, where

$$H \cap S = \left\{ \left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \begin{pmatrix} \pm\lambda & 0 \\ 0 & \pm\lambda^{-1} \end{pmatrix}, 1 \right) : \lambda \in \mathbb{C}^* \right\}$$
$$\sqcup \left\{ \left(\begin{pmatrix} 0 & \eta \\ -\eta^{-1} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \pm\eta \\ \mp\eta^{-1} & 0 \end{pmatrix}, 1 \right) : \eta \in \mathbb{C}^* \right\},$$

so $P^N_t(C^{ss}) = P_t(B(H \cap S))^{\mathbb{Z}/2\mathbb{Z}}$. The natural homomorphism $\mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C}) \to \mathbb{P}\mathrm{GL}(2, \mathbb{C}) \times \mathbb{P}\mathrm{GL}(2, \mathbb{C})$ induces a central extension

$$1 \to \mu_2 \times \mu_2 \to H \cap S \to K \to 1,$$

where $K \subseteq \mathbb{P}\mathrm{GL}(2, \mathbb{C}) \times \mathbb{P}\mathrm{GL}(2, \mathbb{C})$ is the image of $H \cap S$. Here K has a structure of semidirect product $\mathbb{C}^* \rtimes \mathrm{S}_2$, where S_2 acts on $\mathbb{C}^*$ by inversion. Hence:

$$\begin{aligned} H^*_N(C^{ss}) &= H^*(B(H \cap S))^{\mathbb{Z}/2\mathbb{Z}} \\ &= H^*(BK)^{\mathbb{Z}/2\mathbb{Z}} \\ &= (H^*(B\mathbb{C}^*)^{\mathrm{S}_2})^{\mathbb{Z}/2\mathbb{Z}} \\ &= (\mathbb{Q}[c]^{\mathrm{S}_2})^{\mathbb{Z}/2\mathbb{Z}} = \mathbb{Q}[c^2], \end{aligned}$$

because S_2 acts on $H^2(B\mathbb{C}^*) = \mathbb{Q}\langle c \rangle$ by $c \leftrightarrow -c$ and $\mathbb{Z}/2\mathbb{Z}$ acts trivially. This means that $P^N_t(C^{ss}) = (1 - t^4)^{-1}$.

Now we compute $P^N_t((\mathbb{P}^3)^{ss})$: we consider the action of N on $\mathbb{P}^3 \cong Z_R$ and the usual equivariantly perfect stratification (Theorem 1.2.1 and 1.2.2), giving

$$P^N_t((\mathbb{P}^3)^{ss}) = P^N_t(\mathbb{P}^3) - \sum_{0 \neq \beta \in \mathcal{B}} t^{2d(\beta)} P^{\mathrm{Stab}\beta}_t(Z^{ss}_\beta).$$

Firstly we compute $P_t^N(\mathbb{P}^3)$. Notice that N is disconnected, with connected component of the identity equal to $N^0 = T$, and $\pi_0 N = \mathbb{Z}/2\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Since $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ acts by linear transformation on $\mathbb{P}^3$, it acts trivially on cohomology $H^*(\mathbb{P}^3) = \mathbb{Q}[h]/(h^4)$ and hence:

$$\begin{aligned} H_N^*(\mathbb{P}^3) &= (H^*(\mathbb{P}^3) \otimes H^*(BT))^{\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}} \\ &= \mathbb{Q}[h]/(h^4) \otimes \mathbb{Q}[c_1, c_2]^{\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}}, \end{aligned}$$

where $\deg(c_1) = \deg(c_2) = 2$ and the action of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ on $H^2(BT) = \mathbb{Q}\langle c_1, c_2\rangle$ is represented by the matrices

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

By Molien's formula (see [Muk03, Theorem 1.10]), we find that $P_t^N(\mathbb{P}^3) = (1 + t^2 + t^4 + t^6)(1 - t^4)^{-2}$.

Secondly, since the action of T on $\mathbb{P}^3$ has weights

$$(3, -3),\ (1, -1),\ (-1, 1),\ (-3, 3),$$

which correspond to the weights on the antidiagonal of the Hilbert diagram (Figure 2.1) in the Lie algebra $\mathfrak{t}$, the indexing set of this stratification is

$$\mathcal{B} = \{(0, 0),\ (1, -1),\ (3, -3)\}.$$

The real codimension of the strata are $2d((1, -1)) = 4$ and $2d((3, -3)) = 6$, while for both indices $Z_\beta = Z_\beta^{ss} = \mathbb{P}^0$ and

$$\mathrm{Stab}\beta = \langle T, \iota\rangle \cong (\mathbb{C}^*)^2 \rtimes \mathbb{Z}/2\mathbb{Z},$$

as in Table 2.1, so that by Molien's formula $P_t^{\mathrm{Stab}\beta}(Z_\beta^{ss}) = P_t(B\mathrm{Stab}\beta) = (1 - t^2)^{-1}(1 - t^4)^{-1}$. In conclusion, putting everything together, we get

$$P_t^N(\mathrm{Bl}_{C^{ss}}(\mathbb{P}^3)^{ss}) = \frac{1 + t^2 + t^4 + t^6}{(1 - t^4)^2} - \frac{t^4 + t^6}{(1 - t^2)(1 - t^4)} + \frac{t^2}{1 - t^4} = \frac{1 + t^2}{1 - t^2}.$$

The result follows by computing the rank $\mathrm{rk}\mathcal{N}^R$, via the formula (1.6). □

In the last step we need to blow up the locus of A-curves. Recall that this locus remains unaltered after the first two resolutions.

Proposition 2.4.4. *For the group $R_A \cong \mathbb{C}^*$, the main term of $A_{R_A}(t)$ is given by*

$$\begin{aligned} P_t^{N(R_A)}(Z_{R_A}^{ss})(t^2 + \ldots + t^{2(\mathrm{rk}\mathcal{N}^{R_A} - 1)}) &= \frac{t^2 + \ldots + t^{18}}{1 - t^4} \\ &\equiv t^2 + t^4 + 2t^6 + 2t^8 \mod t^{10}. \end{aligned}$$

Proof. To compute $P_t^{N(R_A)}(Z_{R_A}^{ss})$, we use the equality [Kir86, 1.17]:

$$H_N^*(Z_{R_A}^{ss}) = (H^*(Z_{R_A} /\!/ N^0(R_A)) \otimes H^*(BR))^{\pi_0 N(R_A)}.$$

In our case the action of $N(R_A)^0 = T$ on $Z_{R_A}^{ss}$ is transitive, hence:

$$H^*(Z_{R_A}^{ss} /\!/ T) = H^*(\text{point}) = \mathbb{Q}.$$

Moreover $\pi_0 N(R_A) = \mathbb{Z}/2\mathbb{Z}$ acts on $R \cong \mathbb{C}^*$ by inversion, so that

$$\begin{aligned} H_{N(R_A)}^*(Z_{R_A}^{ss}) &= (H^*(Z_{R_A}^{ss} /\!/ T) \otimes H^*(B\mathbb{C}^*))^{\mathbb{Z}/2\mathbb{Z}} \\ &= (\mathbb{Q} \otimes \mathbb{Q}[c])^{\mathbb{Z}/2\mathbb{Z}} \\ &= \mathbb{Q}[c^2], \end{aligned}$$

where $\deg(c) = 2$ and the $\mathbb{Z}/2\mathbb{Z}$ operates on $\mathbb{Q}[c]$ by $c \leftrightarrow -c$. Hence $P_t^{N(R_A)}(Z_{R_A}^{ss}) = (1-t^4)^{-1}$. The result follows by computing the rank $\text{rk}\mathcal{N}^{R_A}$, via formula (1.6). □

2.4.2 Extra terms

To complete the computation of the contribution $A_R(t)$, we need to calculate the extra terms, as stated in Theorem 1.4.1. The crucial point is to analyse for each $R \in \mathcal{R}$ the representation $\rho : R \to \text{GL}(\mathcal{N}_x^R)$ on the normal slice to the orbit $G \cdot Z_R^{ss}$ at a generic point $x \in Z_R^{ss}$. Since here we are dealing only with a local geometry around x, we can restrict to consider the normal slice to the orbit $G^0 \cdot Z_R^{ss}$, which is the connected component of $G \cdot Z_R^{ss}$ at x.

We start reviewing a general approach to computing the tangent space to an orbit for the case of hypersurfaces in $\mathbb{P}^1 \times \mathbb{P}^1$ and then we apply it to our case of $(3,3)$ curves. If $F \in H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(d,d))$ is a bihomogeneous form of bidegree (d,d), it will define a hypersurface $V(F) \subseteq \mathbb{P}^1 \times \mathbb{P}^1$. We wish to describe the tangent space to the orbit $\text{GL}(2,\mathbb{C}) \times \text{GL}(2,\mathbb{C}) \cdot F$. We are actually interested in the normal space to the orbit

$$\text{SL}(2,\mathbb{C}) \times \text{SL}(2,\mathbb{C}) \cdot \{V(F)\} \subseteq \mathbb{P}H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(d,d)).$$

Since the normal space of any submanifold Y in a projective space $\mathbb{P}(W)$ can, via the Euler sequence, be identified with the normal space to its cone $C(Y) \subseteq W$, we can alternatively study the $\text{GL}(2,\mathbb{C}) \times \text{GL}(2,\mathbb{C})$-orbit of F in $H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(d,d))$, rather than the $\text{SL}(2,\mathbb{C}) \times \text{SL}(2,\mathbb{C})$-orbit of $V(F)$ in $\mathbb{P}H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(d,d))$.

The strategy to compute the tangent space to the $\text{GL}(2,\mathbb{C}) \times \text{GL}(2,\mathbb{C})$-orbit of F is to work with the Lie algebra $\mathfrak{gl}(2,\mathbb{C}) \times \mathfrak{gl}(2,\mathbb{C})$ and use the exponential map $\exp : \mathfrak{gl}(2,\mathbb{C}) \times \mathfrak{gl}(2,\mathbb{C}) \to \text{GL}(2,\mathbb{C}) \times \text{GL}(2,\mathbb{C})$. Given an element $e \in \mathfrak{gl}(2,\mathbb{C}) \times$

$\mathfrak{gl}(2,\mathbb{C})$, the derivative $\frac{d}{dt}(\exp(te)F)_{t=0}$ gives a vector in the tangent space to the orbit $\mathrm{GL}(2,\mathbb{C})\times\mathrm{GL}(2,\mathbb{C})\cdot F$. If we take a basis of $\mathfrak{gl}(2,\mathbb{C})\times\mathfrak{gl}(2,\mathbb{C})$, we then obtain generators for the tangent space to the orbit $\mathrm{GL}(2,\mathbb{C})\times\mathrm{GL}(2,\mathbb{C})\cdot F$. In practice, we choose the elementary matrices $e^1_{ij}=(\delta_{ij})_{i,j=1,2}$ as a basis of $\mathfrak{gl}(2,\mathbb{C})\times 0$ and $e^2_{ij}=(\delta_{ij})_{i,j=1,2}$ for $0\times\mathfrak{gl}(2,\mathbb{C})$, then we set:

$$(DF)^k_{ij} := \frac{d}{dt}(\exp(te^k_{ij})F)|_{t=0}, \quad 1\le i,j,k\le 2.$$

In conclusion, the tangent space to the orbit $\mathrm{GL}(2,\mathbb{C})\times\mathrm{GL}(2,\mathbb{C})\cdot F$ is spanned by the entries of the matrix $DF=((DF)^1_{ij}|(DF)^2_{ij})_{i,j=1,2}$.

Coming back to our situation, we carry this procedure out for the equations of strictly polystable hypersurfaces of $\mathbb{P}^1\times\mathbb{P}^1$ of bidegree $(3,3)$. Indeed the tangent space to the orbit $\mathrm{GL}(2,\mathbb{C})\times\mathrm{GL}(2,\mathbb{C})\cdot F$ is given by the span of the entries of the following matrices:

(i) For $F=ax_0^3y_1^3+bx_0^2x_1y_0y_1^2+cx_0x_1^2y_0^2y_1+dx_1^3y_0^3$, the matrix $DF=(DF^1|DF^2)$ is given by

(2.4)
$$DF^1=\begin{pmatrix} 3ax_0^3y_1^3+2bx_0^2x_1y_0y_1^2+cx_0x_1^2y_0^2y_1 & 3ax_0^2x_1y_1^3+2bx_0x_1^2y_0y_1^2+cx_1^3y_0^2y_1 \\ bx_0^3y_0y_1^2+2cx_0^2x_1y_0^2y_1+3dx_0x_1^2y_0^3 & bx_0^2x_1y_0y_1^2+2cx_0x_1^2y_0^2y_1+3dx_1^3y_0^3 \end{pmatrix},$$

(2.5)
$$DF^2=\begin{pmatrix} bx_0^2x_1y_0y_1^2+2cx_0x_1^2y_0^2y_1+3dx_1^3y_0^3 & bx_0^2x_1y_1^3+2cx_0x_1^2y_0y_1^2+3dx_1^3y_0^2y_1 \\ 3ax_0^3y_0y_1^2+2bx_0^2x_1y_0^2y_1+cx_0x_1^2y_0^3 & 3ax_0^3y_1^3+2bx_0^2x_1y_0y_1^2+cx_0x_1^2y_0^2y_1 \end{pmatrix}.$$

The set of linear relations satisfied by the entries of DF is

$$\begin{cases} (DF)^1_{11}=(DF)^2_{22}; \\ (DF)^1_{22}=(DF)^2_{11}. \end{cases} \tag{2.6}$$

(ii) For $F=F_{3C}$, which corresponds to $(a:b:c:d)=(1:-3:3:-1)$ in the above equations, the set of linear relations satisfied by the entries of DF_{3C} consists of the previous ones with the two further relations:

$$\begin{cases} (DF_{3C})^1_{12}+(DF_{3C})^2_{12}=0; \\ (DF_{3C})^1_{21}+(DF_{3C})^2_{21}=0. \end{cases} \tag{2.7}$$

(iii) For $F=F_{C_{2A_5}}=y_0y_1(x_0^3y_1+x_1^3y_0)$, the matrix $DF_{C_{2A_5}}$ is given by

(2.8)
$$DF_{C_{2A_5}}=\left(\begin{array}{cc|cc} 3x_0^3y_0y_1^2 & 3x_0^2x_1y_0y_1^2 & x_0^3y_0y_1^2+2x_1^3y_0^2y_1 & x_0^3y_1^3+2x_1^3y_0y_1^2 \\ 3x_0x_1^2y_0y_1^2 & 3x_1^3y_0^2y_1 & 2x_0^3y_0^2y_1+x_1^3y_0^3 & 2x_0^3y_0y_1^2+x_1^3y_0^2y_1 \end{array}\right).$$

The set of linear relations satisfied by the entries of $DF_{C_{2A_5}}$ is

$$\begin{cases} (DF)^1_{11} + (DF)^1_{22} = (DF)^2_{11} + (DF)^2_{22}; \\ (DF)^1_{22} - (DF)^1_{11} = 3(DF)^2_{11} - 3(DF)^2_{22}. \end{cases} \tag{2.9}$$

We compute the extra contribution coming from the blow-up of the triple conic.

Proposition 2.4.5. *For the group $R_C \cong \mathrm{SL}(2,\mathbb{C})$ the extra term of $A_{R_C}(t)$ is given by*

$$\sum_{0\neq\beta'\in\mathcal{B}(\rho)} \frac{1}{w(\beta',R_C,G)} t^{2d(\mathbb{P}\mathcal{N}^{R_C},\beta')} P_t^{N(R_C)\cap\mathrm{Stab}\beta'}(Z^{ss}_{\beta',R_C}) = \frac{t^{12}(1+t^2+t^4+t^6+t^8)}{1-t^2}$$

$$\equiv 0 \mod t^{10}.$$

In the following lemma we describe the weights of the representation $\rho : R_C \to \mathrm{GL}(\mathcal{N}_x^{R_C})$, where $x = 3C$.

Lemma 2.4.1. *For $R_C \cong \mathrm{SL}(2,\mathbb{C})$, $\dim \mathcal{N}_x^{R_C} = 12$, the weights of the representation ρ of R_C on $\mathcal{N}_x^{R_C}$ are as follows with the respective multiplicities:*

$$(\pm 6)\times 1, (\pm 4)\times 2, (\pm 2)\times 2, (0)\times 2.$$

Proof. The maximal torus $T_1 = \{(\mathrm{diag}(t,t^{-1}), \mathrm{diag}(t,t^{-1}), 1)\}$ in R_C acts on the coordinates $((x_0 : x_1), (y_0 : y_1))$ diagonally. Thus each monomial is an eigenspace for the action of T_1. Hence $H^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(3,3)) = \mathbb{C}^{16}$ decomposes as a sum of 1-dimensional representations of T_1 with the following multiplicities of weights:

$$(\pm 6)\times 1, (\pm 4)\times 2, (\pm 2)\times 3, (0)\times 4.$$

The tangent space to the orbit $G \cdot C_{3C}$ is generated by the entries of the matrices (2.4) and (2.5) at $3C$. Each polynomial spans an eigenspace for the action of T_1 with weight equal to

$$(\pm 2)\times 2, (0)\times 4.$$

Now the relations (2.6) are among the weight 0 generators, thus we may drop two of them in forming a basis of the tangent space. The two further relations (2.7) are among generators of weights 2 and -2, respectively, so we can drop one generator of weight 2 and -2. In total, the weights on the tangent space to the orbit are given by

$$(\pm 2)\times 1, (0)\times 2.$$

By subtracting the weights of the representation of the tangent space to the orbit from the weights of the representation of T_1 on $\mathbb{C}^{16}$, we obtain the weights of the action on the normal space. □

Proof of Proposition 2.4.5. From the description of the weights of ρ in the Lemma 2.4.1, we see that we can take $\mathcal{B}(\rho) = \{0, 2, 4, 6\}$. We can compute the codimension of the strata Z^{ss}_{β', R_C} by means of the formula (1.2.2):

$$d(\mathbb{P}\mathcal{N}_x^{R_C}, \beta') = n(\beta') - \dim(R_C / P_{\beta'}),$$

where $n(\beta')$ is the number of weights less than β' and $P_{\beta'}$ is the associated parabolic subgroup of dimension 2. After noticing that for every weight, $w(\beta', R_C, G) = 1$ and $N(R_C) \cap \mathrm{Stab}\beta' = \hat{T}_1 \rtimes \mathbb{Z}/2\mathbb{Z}$, where $\hat{T}_1 := \{(\mathrm{diag}(t, t^{-1}), \mathrm{diag}(\pm t, \pm t^{-1}), 1) : t \in \mathbb{C}^*\}$ is a double covering of T_1, the result follows. □

We compute the extra contribution coming from the blow-up of the D-curves.

Proposition 2.4.6. *For the group $R_D \cong \mathbb{C}^*$, the extra term of $A_{R_D}(t)$ is given by*

$$\sum_{0 \neq \beta' \in \mathcal{B}(\rho)} \frac{1}{w(\beta', R_D, G)} t^{2d(\mathbb{P}\mathcal{N}^{R_D}, \beta')} P_t^{N(R_D) \cap \mathrm{Stab}\beta'}(\hat{Z}^{ss}_{\beta', R_D}) =$$

$$= \frac{(1+t^2)^2}{1-t^2}(t^8 + t^{10} + t^{12} + t^{14}) \equiv t^8 \mod t^{10}.$$

This lemma describes the weights of the representation $\rho : R_D \to \mathrm{GL}(\mathcal{N}_x^{R_D})$. Here $x \in Z^{ss}_{R_D}$ is a general point: for our purposes it is enough to pick the point x not belonging to the locus of triple conics, but to fix an explicit point we set $x = V(F' := x_0^3 y_1^3 + x_1^3 y_0^3)$.

Lemma 2.4.2. *For $R_D \cong \mathbb{C}^*$, $\dim \mathcal{N}_x^{R_C} = 8$, the weights of the representation ρ of R_D on $\mathcal{N}_x^{R_D}$ are*

$$(\pm 6) \times 1, (\pm 4) \times 2, (\pm 2) \times 1.$$

Proof. The vector space $H^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(3,3)) = \mathbb{C}^{16}$ decomposes as a sum of 1-dimensional representations of R_D with the same multiplicities of weights as in the previous case:

$$(\pm 6) \times 1, (\pm 4) \times 2, (\pm 2) \times 3, (0) \times 4.$$

The tangent space to the orbit $\mathrm{GL}(2, \mathbb{C}) \times \mathrm{GL}(2, \mathbb{C}) \cdot F'$ is generated by the entries of the matrices (2.4) and (2.5), with $a, d = 1$ and $b, c = 0$. Each polynomial spans an eigenspace for the action of R_D with weights equal to

$$(\pm 2) \times 2, (0) \times 4.$$

Now the relations (2.6) are among the weight 0 generators, thus we may drop two of them in forming a basis of the tangent space. In total, the weights for R_D on the tangent space to the orbit $\mathrm{GL}(2, \mathbb{C}) \times \mathrm{GL}(2, \mathbb{C}) \cdot F'$ are given by

$$(\pm 2) \times 2, (0) \times 2.$$

However, we are interested in the normal space $\mathcal{N}_x^{R_D}$ to the orbit $G \cdot Z_{R_D}^{ss}$. We know that $Z_{R_D}^{ss}/\!/N$ is 2-dimensional, thus the tangent space $T_x(G \cdot Z_{R_D}^{ss})$, when lifted to $\mathbb{C}^{16}$, is the sum of $T_{F'}(\mathrm{GL}(2,\mathbb{C}) \times \mathrm{GL}(2,\mathbb{C}) \cdot F')$ together with two tangent vectors representing the direction along $Z_{R_D}^{ss}/\!/N$. These two further vectors can be thought of as coming from varying A and B around 0 in the equation (cf. [Fed12, p. 5658]):

$$F_{A,B} = x_0^3 y_1^3 + A x_0^2 x_1 y_0 y_1^2 + B x_0 x_1^2 y_0^2 y_1 + x_1^3 y_0^3.$$

The derivatives in these directions are $\frac{d}{dA} F_{A,B} = x_0^2 x_1 y_0 y_1^2$ and $\frac{d}{dB} F_{A,B} = x_0 x_1^2 y_0^2 y_1$, which, as expected, are of weight 0 and do not lie in the span of the weight-0 space of the orbit. Thus the lift to $\mathbb{C}^{16}$ of the tangent space to the orbit $G \cdot Z_{R_D}^{ss}$ is given by a space with weights

$$(\pm 2) \times 2, (0) \times 4.$$

By subtracting the weights of the representation of the tangent space to the orbit from the weights of the representation of R_D on $\mathbb{C}^{16}$, we obtain the weights of the action on the normal space. $\square$

Proof of Proposition 2.4.6 . From the description of the weights of ρ in the Lemma 2.4.2, we see that we can take $\mathcal{B}(\rho) = \{\pm 6, \pm 4, \pm 2, 0\}$. We can compute the codimension of the strata Z_{β', R_D}^{ss} via Remark 1.2.2:

$$d(\mathbb{P}\mathcal{N}_x^{R_D}, \beta') = n(\beta') - \dim(R_D / P_{\beta'}),$$

where $n(\beta')$ is the number of weights α such that $\alpha \cdot \beta' < ||\beta'||^2$ and $P_{\beta'}$ is the associated parabolic subgroup. Due to the symmetry, the coefficient for every weight is $w(\beta', R_D, G) = 2$ and, according to (1.13):

$$P_t^{N(R_D)\cap \mathrm{Stab}\beta'}(Z_{\beta',R_D}^{ss}) = P_t^{N(R_D)\cap \mathrm{Stab}\beta'}(Z_{R_D,1}^{ss}) P_t(Z_{\beta',\rho}),$$

because $Z_{\beta',\rho} = Z_{\beta',\rho}^{ss}$ is either $\mathbb{P}^0$ or $\mathbb{P}^1$. One can easily compute the stabiliser $\mathrm{Stab}\beta' = T \rtimes \mathbb{Z}/2\mathbb{Z} \subseteq N(R_D)$, where the semidirect product is induced from G. Arguing analogously to the main term of R_D (see Proposition 2.4.3), one finds that

$$P_t^{T \rtimes \mathbb{Z}/2\mathbb{Z}}(Z_{R_D,1}^{ss}) = \frac{(1+t^2)^2}{1-t^2},$$

completing the proof. $\square$

We compute the extra contribution coming from the blow-up of the A-curves.

Proposition 2.4.7. *For the group $R_A \cong \mathbb{C}^*$, the extra term of $A_{R_A}(t)$ is given by*

$$\sum_{0 \neq \beta' \in \mathcal{B}(\rho)} \frac{1}{w(\beta', R_A, G)} t^{2d(\mathbb{P}\mathcal{N}^{R_A}, \beta')} P_t^{N(R_A)\cap \mathrm{Stab}\beta'}(Z_{\beta',R_A}^{ss}) = \frac{t^{10}+t^{12}+t^{14}+t^{16}+t^{18}}{1-t^2}$$
$$\equiv 0 \mod t^{10}.$$

We need to describe the weights of the representation $\rho : R_A \to \mathrm{GL}(\mathcal{N}_x^{R_A})$, where $x = C_{2A_5}$.

Lemma 2.4.3. *For $R_A \cong \mathbb{C}^*$, $\dim \mathcal{N}_x^{R_C} = 10$, the weights of the representation ρ of R_A on $\mathcal{N}_x^{R_A}$ are*

$$(\pm 12) \times 1, (\pm 10) \times 1, (\pm 8) \times 1, (\pm 6) \times 1, (\pm 4) \times 1.$$

Proof. Recall that x is a general point of $Z^{ss}_{R_A}$, but since $G \cdot Z^{ss}_{R_A} = G \cdot C_{2A_5}$ we can take $x = C_{2A_5}$. Hence to describe $\mathcal{N}_x^{R_A}$, we must simply describe the normal space to the orbit $G \cdot C_{2A_5}$ at C_{2A_5}.

The vector space $H^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(3,3)) = \mathbb{C}^{16}$ decomposes as a sum of 1-dimensional representations of R_A with the following multiplicities of weights:

$$(\pm 12) \times 1, (\pm 10) \times 1, (\pm 8) \times 1, (\pm 6) \times 2, (\pm 4) \times 1, (\pm 2) \times 1, (0) \times 2.$$

The tangent space to the orbit $G \cdot C_{2A_5}$ is generated by the entries of the matrix (2.9). Each polynomial spans an eigenspace for the action of R_A with weight equal to

$$(\pm 6) \times 1, (\pm 2) \times 1, (0) \times 4.$$

Now the relations (2.9) are among the weight 0 generators, thus we may drop two of them in forming a basis of the tangent space. In total, the weights for R_A on the tangent space to the orbit are given by

$$(\pm 6) \times 1, (\pm 2) \times 1, (0) \times 2.$$

By subtracting the weights of the representation of the tangent space to the orbit from the weights of the representation of R_A on $\mathbb{C}^{16}$, we obtain the weights of the action on the normal space. □

Proof of Proposition 2.4.7 . From the description of the weights of ρ in Lemma 2.4.3, we see that we can take $\mathcal{B}(\rho) = \{\pm 12, \pm 10, \pm 8, \pm 6, \pm 4, 0\}$. We can calculate the codimension via Remark 1.2.2:

$$d(\mathbb{P}\mathcal{N}_x^{R_A}, \beta') = n(\beta') - \dim(R_A / P_{\beta'}),$$

where $n(\beta')$ is the number of weights α such that $\alpha \cdot \beta' < ||\beta'||^2$ and $P_{\beta'}$ is the associated parabolic subgroup, in this case equal to R_A since R_A is a torus. Moreover, we notice that for every non-zero weight, $w(\beta', R_A, G) = 2$ and $N(R_A) \cap \mathrm{Stab}\beta' = T$, so by (1.13) we find

$$P_t^{N(R_A) \cap \mathrm{Stab}\beta'}(Z^{ss}_{\beta', R_A}) = P_t^T(Z^{ss}_{R_A}) P_t(\mathbb{P}^0) = \frac{1}{1 - t^2},$$

because $Z_{\beta', \rho} = \mathbb{P}^0$ for all $\beta' \in \mathcal{B}(\rho)$. □

2.4.3 Cohomology of $\widetilde{M}$

We complete the proof of Theorem 2.4.1.

Proof of Theorem 2.4.1. Using Theorem 1.4.1, we need to put all the previous results together to find the Betti numbers of the Kirwan partial desingularization $\widetilde{M}$. For the sake of readability, we record only the polynomials modulo t^{10}, but one can double-check the result with the entire Hilbert-Poincaré series and observe that Poincaré duality effectively holds.

$$
\begin{aligned}
&P_t(\widetilde{M}) = P_t^G(\widetilde{X}^{ss}) \equiv \\
&\text{(Semistable locus)} && 1 + t^2 + 2t^4 + 2t^6 + 4t^8 \\
&\text{(Error term for triple conic)} && + t^2 + t^4 + 2t^6 + 2t^8 - 0 \\
&\text{(Error term for } D\text{-curves)} && + t^2 + 3t^4 + 5t^6 + 7t^8 - t^8 \\
&\text{(Error term for } A\text{-curves)} && + t^2 + t^4 + 2t^6 + 2t^8 - 0 \\
& && \equiv 1 + 4t^2 + 7t^4 + 11t^6 + 14t^8 \quad \mod t^{10}.
\end{aligned}
$$

□

2.5 Cohomology of blow-downs for $(3,3)$ curves in $\mathbb{P}^1 \times \mathbb{P}^1$

In this section, we compute the intersection cohomology of M descending from $\widetilde{M}$ and thus prove the following:

Theorem 2.5.1. *The intersection Hilbert-Poincaré polynomial of M is*

$$IP_t(M) = 1 + t^2 + 2t^4 + 2t^6 + 3t^8 + 3t^{10} + 2t^{12} + 2t^{14} + t^{16} + t^{18}.$$

We follow Kirwan's results described in Section 1.5 and study the variation of the intersection Betti numbers at the level of the parameter spaces X^{ss} and $\widetilde{X}^{ss}$, under each stage of the resolution, instead of applying the Decomposition Theorem directly to the blow-down map $\widetilde{M} \to M$ at the level of GIT quotients. In order to apply Theorem 1.5.1 to the moduli space of non-hyperelliptic Petri-general curves of genus 4, we will need to follow the steps backwards of the blow-down operations of A-curves, then D-curves, and eventually triple conics.

In the first step we need to blow down the locus of A-curves.

Proposition 2.5.1. *For the group $R_A \cong \mathbb{C}^*$, we have:*

(i) $Z_{R_A} /\!/ N(R_A)$ *is a point;*

(ii) $IP_t(\mathbb{P}\mathcal{N}_x^{R_A}/\!/R_A) = 1 + 2t^2 + 3t^4 + 4t^6 + 5t^8 + 4t^{10} + 3t^{12} + 2t^{14} + t^{16}$.

The term $B_{R_A}(t)$ is given by

$$\begin{aligned} B_{R_A}(t) &= t^2 + t^4 + 2t^6 + 2t^8 + 2t^{10} + 2t^{12} + t^{14} + t^{16} \\ &\equiv A_{R_A}(t) \mod t^{10}. \end{aligned}$$

Proof. For brevity we write $R = R_A$, $N = N(R_A)$ and $\mathbb{P}^9 \cong \mathbb{P}\mathcal{N}_x^{R_A}$. (i) follows from the fact that N acts transitively on Z_R^{ss}.

In Lemma 2.4.3 the weights of the representation $\rho : R \to \mathrm{GL}(\mathcal{N}_x^R)$ were computed. It follows that there are no strictly semistable points in $\mathbb{P}^9$, so that the GIT quotient $\mathbb{P}^9/\!/R$ is a projective toric variety of dimension 8 with at worst finite quotient singularities. Thus $IP_t(\mathbb{P}^9/\!/R) = P_t(\mathbb{P}^9/\!/R) = P_t^R((\mathbb{P}^9)^{ss})$ and using the usual R-equivariantly perfect stratification (see Theorem 1.2.1 and 1.2.2) we obtain

$$\begin{aligned} P_t^R((\mathbb{P}^9)^{ss}) &= P_t(\mathbb{P}^9)P_t(BR) - \sum_{0 \neq \beta' \in \mathcal{B}(\rho)} t^{2d(\beta')} P_t^R(S_{\beta'}) \\ &= \frac{1 + \ldots + t^{18}}{1 - t^2} - 2\frac{t^{10} + \ldots + t^{18}}{1 - t^2} \\ &= 1 + 2t^2 + 3t^4 + 4t^6 + 5t^8 + 4t^{10} + 3t^{12} + 2t^{14} + t^{16}. \end{aligned}$$

Now we need to know the dimensions $\dim IH^{\hat{q}}(\mathbb{P}^9/\!/R)^{\pi_0 N}$, where the action is induced by an action of $\pi_0 N$ on $\mathbb{P}^9/\!/R$. We have seen that $\pi_0 N \cong \mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{P}^9/\!/R$ via permutation of the coordinates $((x_0 : x_1), (y_0 : y_1)) \leftrightarrow ((x_1 : x_0), (y_1 : y_0))$. Thus the action on the cohomology of $\mathbb{P}^9$ is trivial, while $\mathbb{Z}/2\mathbb{Z}$ acts on the torus $\mathbb{C}^*$ via $\lambda \leftrightarrow \lambda^{-1}$, hence in cohomology $H^*(B\mathbb{C}^*) = \mathbb{Q}[c]$ by $c \leftrightarrow -c$, and on the strata interchanging the positive-indexed ones with the negative-indexed ones. Eventually,

$$\begin{aligned} IP_t(\mathbb{P}^9/\!/R)^{\pi_0 N} &= \frac{1 + \ldots + t^{18}}{1 - t^4} - \frac{t^{10} + \ldots + t^{18}}{1 - t^2} \\ &= 1 + t^2 + 2t^4 + 2t^6 + 3t^8 + 2t^{10} + 2t^{12} + t^{14} + t^{16}. \end{aligned}$$

Now the final statement easily follows from the definition of $B_R(t)$. □

In the second step, we need to blow down the locus of D-curves.

Proposition 2.5.2. *For the group $R_D \cong \mathbb{C}^*$ with the notation as in the proof of Proposition 2.4.3, we have:*

(i) $Z_{R_D,1}/\!/N(R_D)$ *is a simply connected surface and* $P_t(Z_{R_D,1}/\!/N(R_D)) = 1 + 2t^2 + t^4$;

(ii) $IP_t(\mathbb{P}\mathcal{N}_x^{R_D}/\!/R_D) = 1 + 2t^2 + 3t^4 + 4t^6 + 3t^8 + 2t^{10} + t^{12}$.

The term $B_{R_D}(t)$ is equal to

$$B_{R_D}(t) = t^2 + 3t^4 + 5t^6 + 7t^8 + 7t^{10} + 5t^{12} + 3t^{14} + t^{16}.$$

Proof. For brevity we write $R = R_D$, $N = N(R_D)$ and $\mathbb{P}^7 \cong \mathbb{P}\mathcal{N}_x^{R_D}$. The GIT quotient $Z_{R,1}/\!/N \cong Z_{R,1}/\!/(N/R)$ is a rational surface with finite quotient singularities, hence simply connected by [Kol93, Theorem 7.8]. Its cohomology can be computed by means of the equality [Kir86, 1.17]:

$$H^*_N(Z^{ss}_{R,1}) = (H^*(Z_{R,1}/\!/N^0) \otimes H^*(BR))^{\pi_0 N}.$$

The action of $\pi_0 N$ splits on the tensor product, because also $Z_{R,1}/\!/N^0$ is simply connected, giving

$$H^*_N(Z^{ss}_{R_D,1}) = H^*(Z^{ss}_{R_D,1}/\!/N) \otimes H^*(BR)^{\pi_0 N}.$$

Recall that $\pi_0 N = N/T = \mathbb{Z}/2\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$: the first factor acts on $R \cong \mathbb{C}^*$ by inversion, while the second one acts trivially. Therefore:

$$H^*(BR)^{\pi_0 N} = \mathbb{Q}[c]^{\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}} = \mathbb{Q}[c^2],\ \deg(c) = 2.$$

In the proof of Proposition 2.4.3, we have already computed $P^N_t(Z^{ss}_{R_D,1})$, thus:

$$P_t(Z^{ss}_{R_D,1}/\!/N) = \frac{1+t^2}{1-t^2}(1-t^4) = 1 + 2t^2 + t^4,$$

completing the proof of (i).

In Lemma 2.4.2 the weights of the representation $\rho : R \to \mathrm{GL}(\mathcal{N}_x^R)$ were computed. It follows that there are no strictly semistable points in $\mathbb{P}^7$, so that the GIT quotient $\mathbb{P}^7/\!/R$ is a projective variety of dimension 6 with at worst finite quotient singularities. Thus $IP_t(\mathbb{P}^7/\!/R) = P_t(\mathbb{P}^7/\!/R) = P^R_t((\mathbb{P}^7)^{ss})$ and using the usual R-equivariantly perfect stratification (see Theorem 1.2.1 and 1.2.2) we obtain

$$\begin{aligned} P^R_t((\mathbb{P}^7)^{ss}) &= P_t(\mathbb{P}^7)P_t(BR) - \sum_{0 \neq \beta' \in \mathcal{B}(\rho)} t^{2d(\beta')} P^R_t(S_{\beta'}) \\ &= \frac{1 + \ldots + t^{14}}{1-t^2} - 2\frac{t^8 + t^{10}(1+t^2) + t^{14}}{1-t^2} \\ &= 1 + 2t^2 + 3t^4 + 4t^6 + 3t^8 + 2t^{10} + t^{12}. \end{aligned}$$

Now we need to know the dimensions $\dim IH^{\hat{q}}(\mathbb{P}^7/\!/R)^{\pi_0 N}$. We have seen that $\pi_0 N \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{P}^7/\!/R$ as follows: the first $\mathbb{Z}/2\mathbb{Z}$ factor via permutation of the coordinates $((x_0 : x_1), (y_0 : y_1)) \leftrightarrow ((x_1 : x_0), (y_1 : y_0))$, while the second one by interchanging the rulings of $\mathbb{P}^1 \times \mathbb{P}^1$. Thus the action on the cohomology of $\mathbb{P}^7$ is trivial, while the first factor of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ acts on the torus $\mathbb{C}^*$ via $\lambda \leftrightarrow \lambda^{-1}$, hence in cohomology $H^*(B\mathbb{C}^*) = \mathbb{Q}[c]$ by $c \leftrightarrow -c$, and the second factor acts trivially. Moreover $\pi_0 N$ acts on the strata interchanging the positive-indexed ones with the negative-indexed ones:

$$\begin{aligned} IP_t(\mathbb{P}^7 /\!/ R)^{\pi_0 N} &= \frac{1 + ... + t^{14}}{1 - t^4} - \frac{t^8 + ... + t^{14}}{1 - t^2} \\ &= 1 + t^2 + 2t^4 + 2t^6 + 2t^8 + t^{10} + t^{12}. \end{aligned}$$

Now the final statement easily follows from the definition of $B_R(t)$. □

The last step is blowing down the triple conics.

Lemma 2.5.1. *The intersection cohomology of the GIT quotient* $\mathbb{P}\mathcal{N}_x^{R_C} /\!/ R_C$ *is*

$$IP_t(\mathbb{P}\mathcal{N}_x^{R_C} /\!/ R_C) = 1 + t^2 + 2t^4 + 2t^6 + 2t^8 + 2t^{10} + 2t^{12} + t^{14} + t^{16}.$$

Proof. For brevity we write $R = R_C \cong \mathrm{SL}(2, \mathbb{C})$ and $\mathbb{P}^{11} = \mathbb{P}\mathcal{N}_x^{R_C}$. From the weights of the slice representation (Lemma 2.4.1) and the usual R-equivariantly perfect stratification (see Theorem 1.2.1 and 1.2.2) one can compute the equivariant Poincaré series of the semistable locus:

$$P_t^R((\mathbb{P}^{11})^{ss}) = \frac{1 + ... + t^{22}}{1 - t^4} - \frac{t^{12}(1 + t^2) + t^{16}(1 + t^2) + t^{20}}{1 - t^2}.$$

The space $\mathbb{P}^{11} /\!/ R$ is not rationally smooth, thus $P_t^R((\mathbb{P}^{11})^{ss})$ is a priori neither $P_t(\mathbb{P}^{11} /\!/ R)$ nor $IP_t(\mathbb{P}^{11} /\!/ R)$. The remedy for this is first to blow up the orbit associated to the subgroup $T_1 := \{\mathrm{diag}(t, t^{-1}) : t \in \mathbb{C}^*\} \subseteq R$, which fixes strictly polystable points. Using the same procedure as before, we obtain a partial desingularization $\widetilde{\mathbb{P}^{11}} /\!/ R$, whose cohomology is related to the R-equivariant cohomology of $(\mathbb{P}^{11})^{ss}$ by the error term (see Theorem 1.4.1):

$$A_{T_1}(t) = \frac{1 + t^2}{1 - t^4}(t^2 + ... + t^{14}) - \frac{1 + t^2}{1 - t^2}(t^8 + t^{10}(1 + t^2) + t^{14}).$$

Hence the cohomology of the Kirwan blow-up is given by

$$\begin{aligned} P_t(\widetilde{\mathbb{P}^{11}} /\!/ R) &= P_t^R((\mathbb{P}^{11})^{ss}) + A_{T_1}(t) \\ &= 1 + 2t^2 + 4t^4 + 5t^6 + 6t^8 + 5t^{10} + 4t^{12} + 2t^{14} + t^{16}. \end{aligned}$$

Now by the blow-down procedure (see Theorem 1.5.1), we need to subtract the error term

$$B_{T_1}(t) = t^2 + 2t^4 + 3t^6 + 4t^8 + 3t^{10} + 2t^{12} + t^{14}.$$

Now the statement follows from $IP_t(\mathbb{P}^{11} /\!/ R) = P_t(\widetilde{\mathbb{P}^{11}} /\!/ R) - B_{T_1}(t)$. □

Proposition 2.5.3. *For the group $R_C \cong \mathrm{SL}(2,\mathbb{C})$, the error term $B_{R_C}(t)$ is given by*

$$\begin{aligned} B_{R_C}(t) &= t^2 + t^4 + 2t^6 + 2t^8 + 2t^{10} + 2t^{12} + t^{14} + t^{16} \\ &\equiv A_{R_C}(t) \mod t^{10}. \end{aligned}$$

Proof. The result easily follows from the definition of $B_{R_C}(t)$, after noticing that $Z_{R_C} /\!/ N(R_C)$ is a point and the group $\pi_0 N(R_C)$ acts trivially on $IH^*(\mathbb{P}\mathcal{N}_x^{R_C} /\!/ R)$ (cf. Proposition 2.4.2), which we computed in Lemma 2.5.1. □

2.5.1 Intersection cohomology of M

We complete the proof of Theorem 2.5.1.

Proof of Theorem 2.5.1. From Theorem 1.5.1, putting all the previous results together, we obtain that the intersection Hilbert-Poincaré polynomial of the moduli space of non-hyperelliptic Petri-general genus 4 curves $M = X/\!/G$ is

$$\begin{aligned} IP_t(M) &= P_t(\widetilde{M}) - \sum_{R\in\mathcal{R}} B_R(t) \\ &= P_t^G(X^{ss}) + \sum_{R\in\mathcal{R}} (A_R(t) - B_R(t)) \\ &\equiv 1 + t^2 + 2t^4 + 2t^6 + 4t^8 + 0 - t^8 + 0 \mod t^{10} \\ &\equiv 1 + t^2 + 2t^4 + 2t^6 + 3t^8 \mod t^{10}. \end{aligned}$$

□

Together with Theorem 2.4.1, this also completes the proof of the main Theorem 2.0.1.

Remark 2.5.1. From [Kir86, Remark 3.4] we can also deduce the ordinary Betti numbers of $X/\!/G$:

$$H^i(X/\!/G) = IH^i(X/\!/G) \text{ for } 12 \le i \le 18,$$

and

$$H^i(X^s/G) = IH^i(X/\!/G) \text{ for } 0 \le i \le 6,$$

where $X^s/G = X/\!/G \setminus \bigcup_{R\in\mathcal{R}} Z_R/\!/N(R)$ is the orbit space of GIT-stable curves.

2.6 Geometric interpretation

In conclusion, we give a geometric interpretation of some Betti numbers of the compactification M, by describing the classes of curves which generate the cohomology groups.

Let $U \subseteq M$ be the affine open subset corresponding to smooth non-hyperelliptic Petri-general curves of genus four. Tommasi [Tom05, Theorem 1.2] computed the rational cohomology of U, as geometric quotient of the complement of a discriminant, namely

$$H^i(U) = \begin{cases} 1 & i = 0, 5 \\ 0 & \text{otherwise.} \end{cases}$$

We now consider the Gysin long exact sequence (cf. [Ful98, §19.1 (6)]) associated to the inclusion $U \hookrightarrow M$:

$$\ldots \to H_{k+1}(U) \to H_k(M \smallsetminus U) \to H_k(M) \to H_k(U) \to \ldots$$

where H_* denotes the rational Borel-Moore homology theory (cf. [Ful98, Example 19.1.1]). As U has at most finite quotient singularities, by Poincaré duality $\dim H_{k+1}(U) = 1$ for $k = 12, 17$ and vanishes in all other degrees.

The dimensions of $H_k(M \smallsetminus U) \cong H_k(X^s/G \smallsetminus U)$, for $k \geq 12$, can be also computed from Remark 2.5.1 via the Gysin sequence related to the inclusion $U \hookrightarrow X^s/G$. Therefore, the geometry of the curves in $M \smallsetminus U$ suggests the following geometric interpretation of the Betti numbers:

- $H_{18}(M)$ is obviously generated by the fundamental class of M;
- $H_{16}(M)$ is generated by the fundamental class of $M \smallsetminus U$, i.e. the locus of singular curves;
- $H_{14}(M)$ is generated by the fundamental classes of the following subvarieties of $M \smallsetminus U$ (cfr. [Tom05, §3 and Table 1]): the closure of the locus of curves with at least two nodes and the closure of the locus of curves with a cusp;
- $H_{12}(M)$ is generated by the fundamental classes of the following subvarieties of $M \smallsetminus U$ (cfr. [Tom05, §3 and Table 1]): the closure of the locus of curves with at least three points in general position, the locus of reducible curves with a line as component and the closure of the locus of curves with at least a node and cusp. These three classes generate $H_{12}(M \smallsetminus U) \cong \mathbb{Q}^3$, but are linearly dependent in $H_{12}(M) \cong \mathbb{Q}^2$ and the space of relations con be identified with $H_{13}(U) \cong \mathbb{Q}$.

Similar (but dual) considerations can be applied to the Betti numbers of the stable quotient X^s/G. The geometric interpretation explained above hence confirms the results about $IH^i(M)$ for $i \leq 6$.

3 | Cohomology of the moduli space of degree two Enriques surfaces

This chapter deals with the results of the author's article [For20] about the cohomology of the moduli space of degree two Enriques surfaces. The projective model of degree 2 Enriques surfaces was first constructed by Horikawa in [Hor78a]. The K3 coverings of these Enriques surfaces are given as double coverings of $\mathbb{P}^1 \times \mathbb{P}^1$ branched over a curve of bidegree $(4,4)$ invariant under a suitable involution ι of $\mathbb{P}^1 \times \mathbb{P}^1$ with four fixed points. By looking at the isomorphism classes of such branch curves on $\mathbb{P}^1 \times \mathbb{P}^1$, we can construct the GIT quotient:

$$M^{GIT} := \mathbb{P}H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(4,4))^\iota // (\mathbb{C}^*)^2 \rtimes D_8,$$

where $\mathbb{P}H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(4,4))^\iota$ is the linear subsystem of $|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(4,4)|$ of ι-invariant curves and $(\mathbb{C}^*)^2 \rtimes D_8$ is the subgroup of the automorphisms of $\mathbb{P}^1 \times \mathbb{P}^1$ that commute with ι. Here D_8 denotes the dihedral group of symmetries of the square. The quotient M^{GIT} can thus be seen as a compactification of the moduli space of numerically polarized Enriques surfaces of degree 2 (see Theorem 3.2.1). The purpose of this chapter is to compute the intersection cohomology of M^{GIT}. The strategy of the proof relies on Kirwan's procedure, explained in Chapter 1, whose crucial step consists of the construction of the Kirwan blow-up $M^K \to M^{GIT}$. Our result is summarised by the following:

Theorem 3.0.1. *The intersection Betti numbers of M^{GIT} and the Betti numbers of the Kirwan blow-up M^K are as follows:*

i	0	2	4	6	8	10	12	14	16	18	20
$\dim IH^i(M^{GIT}, \mathbb{Q})$	1	1	2	2	3	3	3	2	2	1	1
$\dim H^i(M^K, \mathbb{Q})$	1	4	8	13	18	20	18	13	8	4	1

while all the odd degree (intersection) Betti numbers vanish.

The structure of the chapter reflects the steps of Kirwan's machinery. Section 3.1 is devoted to the description of Horikawa's model, which gives rise to Enriques surfaces with a non-special polarization of degree 2. In Section 3.2 we use this model to construct the moduli space M^{GIT} as GIT quotient $X//G$, which can be seen as a compactification of the moduli space of numerically polarized Enriques surfaces of degree 2 (cf. Theorem 3.2.1). Moreover, the geometric description of the semistable and stable loci are presented. In Section 3.3, we calculate the equivariant Hilbert-Poincaré polynomial of the semistable locus X^{ss} in the parameter space of $(4,4)$ ι-invariant curves (see Proposition 3.3.1): this is done by computing the Hesselink-Kempf-Kirwan-Ness stratification of the unstable locus from Section 1.2. In Section 3.4, we explicitly construct the partial desingularization $M^K \to M^{GIT}$, by blowing up three G-invariant loci in the GIT boundary of M^{GIT}, corresponding to strictly polystable curves (cf. Definition 3.4.1). Section 3.5 is devoted to the computation of the rational Betti numbers of the Kirwan blow-up M^K (see Theorem 2.4.1). Here the correction terms arising from the modification process $M^K \to M^{GIT}$ are calculated by following the results of Section 1.4. In the end, the intersection Betti numbers of M^{GIT} are computed in Section 3.6, as an application of Theorem 1.5.1 (see Theorem 3.6.1).

3.1 Horikawa's model

An Enriques surface is a smooth compact complex surface S such that $H^1(\mathcal{O}_S) = 0$ and its canonical bundle ω_S is not trivial, but $\omega_S^{\otimes 2} \cong \mathcal{O}_S$. The last condition implies the existence of an étale double covering $T \to S$ and by surface classification T is a K3 surface, that is T is simply connected and $H^0(T, \Omega_T^2)$ is spanned by a non-degenerate holomorphic 2-form. Moreover, every Enriques surface is algebraic, in particular $\mathrm{NS}(S) \cong H^2(S,\mathbb{Z})$. The canonical class is the only torsion element in the Néron-Severi group and there is a non-canonical splitting $H^2(S,\mathbb{Z}) = H^2(S,\mathbb{Z})_f \oplus \mathbb{Z}/2\mathbb{Z}$ where $H^2(S,\mathbb{Z})_f = H^2(S,\mathbb{Z})/\text{torsion}$ is a free module of rank 10. The intersection product endows this with a lattice structure and

$$H^2(S,\mathbb{Z})_f = \mathrm{Num}(S) \cong U \oplus E_8(-1),$$

where U denotes the hyperbolic plane and $E_8(-1)$ is the only negative definite, even, unimodular lattice of rank 8.

A polarized (resp. numerically polarized) Enriques surface is a pair (S, H), where S is an Enriques surface and $H \in \mathrm{NS}(S)$ (resp. $H \in \mathrm{Num}(S)$) is the (numerical) class of an ample line bundle. Moreover, a quasi-polarization is a nef and big line bundle, not necessarily ample. The degree of a (numerical) (quasi-)polarization is its self-intersection and it is always even by adjunction.

We will consider only quasi-polarizations of degree 2. By [CDL20, Remark 5.7.10], each numerical quasi-polarization of degree 2 can be represented as a sum of two isotropic classes $f + g$ in $U \oplus E_8(-1)$, with one of the following properties:

(i) Both f and g are nef: in this case $f + g$ is ample and is called *non-special* polarization;

(ii) The class $f - g$ represents an effective divisor R with $R^2 = -2$, $Rf = -1$, and hence $f = g + R$ is not nef: in this case $f + g$ is not ample and is called *special* quasi-polarization.

We now present a geometric construction of Enriques surfaces together with a non-special numerical polarization of degree 2, given by Horikawa in [Hor78a] (cf. also [BHPVdV04, V.23]). Let $\mathbb{P}^1 \times \mathbb{P}^1$ be acted on by the involution:

$$\iota : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$$

$$((x_0 : x_1), (y_0 : y_1)) \mapsto ((x_0 : -x_1), (y_0 : -y_1)).$$

The morphism ι has four isolated fixed points, namely

$$\Delta := \{((0:1),(0:1)),\ ((0:1),(1:0)),\ ((1:0),(1:0)),\ ((1:0),(0:1))\}.$$

Let B be a reduced curve on $\mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(4, 4)$ which is invariant under ι, does not pass through any point of Δ and has at worst simple singularities. The minimal resolution of the double covering of $\mathbb{P}^1 \times \mathbb{P}^1$ branched over B is a K3 surface $T \to \mathbb{P}^1 \times \mathbb{P}^1$. The pull-back of the $(1, 1)$-class on $\mathbb{P}^1 \times \mathbb{P}^1$ endows T with a polarization of degree 4, which splits as a sum of two genus one fibrations corresponding to the pull-backs of the two rulings on $\mathbb{P}^1 \times \mathbb{P}^1$. Moreover, the involution ι on $\mathbb{P}^1 \times \mathbb{P}^1$, composed with the deck transformation of the double covering, induces a fixed point free involution σ on T. Therefore the quotient $T \to S := T/\langle\sigma\rangle$ is an Enriques surface. As the degree 4 polarization on T is invariant under σ, it induces a polarization L of degree 2 on the Enriques surface S. This ample line bundle on S splits as a sum $L = E + F$ of two half pencils of elliptic curves with $E^2 = F^2 = 0$ and $EF = 1$, where E and F come from the two rulings of $\mathbb{P}^1 \times \mathbb{P}^1$. The linear system $|2L|$ maps S to a quartic del Pezzo surface $D \subseteq \mathbb{P}^4$ with four A_1 singularities, which coincides with the quotient $\mathbb{P}^1 \times \mathbb{P}^1/\langle\iota\rangle$. We notice that the image of the branch curve $B \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ under the quotient map is cut out on D by a quadric, hence S can be also viewed as a double covering of a 4-nodal quartic del Pezzo surface branched over a quadric section. Summarising we have the commutative diagram:

$$\begin{array}{ccc} T & \xrightarrow{2:1} & \mathbb{P}^1 \times \mathbb{P}^1 \\ \downarrow{\scriptstyle /\langle\sigma\rangle} & & \downarrow{\scriptstyle /\langle\iota\rangle} \\ S & \xrightarrow{|2L|} & D \end{array}$$

In [Hor78a] Horikawa proved that a general Enriques surface admits a non-special polarization of degree 2.

Theorem 3.1.1. *[Hor78a, Theorem 4.1] [BHPVdV04, Proposition VIII 18.1] Let S be a general Enriques surface. Then there exists a ι-invariant $(4,4)$-curve B on $\mathbb{P}^1 \times \mathbb{P}^1$ such that the universal covering T of S is the minimal resolution of the double covering of $\mathbb{P}^1 \times \mathbb{P}^1$ ramified over B. The curve B is reduced with at worst simple singularities and does not contain any fixed point of ι. The Enriques involution on T is induced by the involution ι on $\mathbb{P}^1 \times \mathbb{P}^1$ and the deck transformation of the double covering.*

To obtain a representation of all Enriques surfaces, one still needs to treat the special case (see [Hor78a, Theorem 4.2] and [BHPVdV04, Proposition VIII 18.2]). In a similar way as above, one can construct an Enriques surface from a quadric cone in $\mathbb{P}^3$ together with an involution. Indeed, the minimal resolution of the double covering of the cone branched over a curve cut out by a quartic polynomial is a K3 surface. The involution on the cone and the deck transformation of the double covering induce a fixed point free involution on the K3 surface, whose quotient is an Enriques surface. The hyperplane class of the cone induces a quasi-polarization L of degree 2 on the Enriques surface, which splits as a sum $L = 2E + R$, where E is a half pencil of elliptic curves and R is a (-2)-curve with $ER = 1$. Notice that this degree 2 line bundle is big and nef, but not ample, as it is orthogonal to the class of R coming from the resolution of the vertex of the cone.

By [Hor78a] every Enriques surface admits a special quasi-polarization or a non-special polarization of degree 2. In the following, we will consider only the non-special polarization, as the general Enriques surface can be endowed with it.

3.2 GIT for degree two Enriques surfaces

Via Horikawa's model, one can construct a GIT compactification of the moduli space of non-special Enriques surfaces of degree 2 by looking at the isomorphism classes of branch curves B on $\mathbb{P}^1 \times \mathbb{P}^1$. The ι-invariant polynomials of bidegree $(4,4)$ form a 13-dimensional vector space with a basis consisting of

$$x_0^i x_1^{4-i} y_0^j y_1^{4-j} \text{ for } i + j \equiv 0 \bmod 2,$$

which is explicitly

$$x_0^{2k} x_1^{4-2k} y_0^{2l} y_1^{4-2l},\ 0 \le k, l \le 2,$$
$$x_0^3 x_1 y_0^3 y_1,\ x_0 x_1^3 y_0^3 y_1,\ x_0^3 x_1 y_0 y_1^3,\ x_0 x_1^3 y_0 y_1^3.$$

We denote the corresponding linear system on $\mathbb{P}^1 \times \mathbb{P}^1$ by

$$X := \mathbb{P}H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(4,4))^\iota \cong \mathbb{P}^{12}.$$

Let G be the subgroup of the automorphism group of $\mathbb{P}^1 \times \mathbb{P}^1$, commuting with the involution ι or, equivalently, fixing the set Δ. The group has dimension 2 and has the structure of a semidirect product:

$$G = (\mathbb{C}^*)^2 \rtimes D_8,$$

where D_8 is the dihedral group of symmetries of the square. The group D_8 has a structure of a semidirect product:

$$D_8 = (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \ltimes \mathbb{Z}/2\mathbb{Z},$$

where the first two copies of $\mathbb{Z}/2\mathbb{Z}$ are generated by the reflections along the axes of the square, and the third copy of $\mathbb{Z}/2\mathbb{Z}$ corrsponds to the reflection along a diagonal of the square. In the structure of G the group D_8 acts on $(\mathbb{C}^*)^2$ as follows: the first two involutions act via inversion on every factor of the torus, while the third interchanges the two factors.

In [Sha81] Shah describes explicitly the group $G \subseteq \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ and its action on X in the following way. Let I_1 be the involution on $\mathbb{P}^1$ which keeps x_0 fixed and sends $x_1 \mapsto -x_1$, and let I_2 be the involution of $\mathbb{P}^1$ which keeps y_0 fixed and sends $y_1 \mapsto -y_1$. Let γ denote the automorphism of $\mathbb{P}^1 \times \mathbb{P}^1$ which interchanges the factors; let $\langle\gamma\rangle$ be the group generated by γ. For $i = 1, 2$ let G_i be the subgroup of $\mathbb{P}\mathrm{GL}(2, \mathbb{C})$ which commutes with I_i: G_i is the stabiliser of the set of fixed points of I_i. Then, G_1 and G_2 are isomorphic to the semidirect product $\mathbb{C}^* \rtimes \mathbb{Z}/2\mathbb{Z}$, where $\mathbb{C}^*$ acts via the transformations:

$$(x_0 : x_1) \mapsto (ax_0 : a^{-1}x_1), \quad a \in \mathbb{C}^*;$$
$$(y_0 : y_1) \mapsto (by_0 : b^{-1}y_1), \quad b \in \mathbb{C}^*.$$

The subgroup $\mathbb{Z}/2\mathbb{Z}$ is generated by the involution which interchanges x_0 and x_1 as an element of G_1, and interchanges y_0 and y_1 as an element of G_2. The group G is therefore isomorphic to

$$G \cong (G_1 \times G_2) \rtimes \langle\gamma\rangle.$$

We are now ready to construct the relevant moduli space of degree 2 Enriques surfaces. Geometric Invariant Theory [MFK94] provides a good categorical projective quotient with respect to the linearisation $\mathcal{O}_X(1)$:

$$M^{GIT} := X/\!/G,$$

which can be thought of as a compactification of the moduli space of non-special Enriques surfaces of degree 2. Via Horikawa's model of Section 3.1, one can equivalently construct the same quotient by considering the linear system of quadric sections $|\mathcal{O}_D(2)| \cong \mathbb{P}^{12}$ on a 4-nodal del Pezzo surface $D \subseteq \mathbb{P}^4$ modulo the action of the automorphism group of D, which is again isomorphic to G.

By a lattice theoretical result [CDL20, Corollary 1.5.4.], the non-special polarization of degree 2 constructed by Horikawa is the unique numerical polarization of degree 2 on an Enriques surface, up to an isometry of the Enriques lattice. Indeed, it is defined only up to numerical equivalence, since it is induced by an ample line bundle on the K3 covering.

Theorem 3.2.1. *[CDL20, Theorem 5.8.5] The GIT quotient M^{GIT} is a compactification of the moduli space of numerically polarized Enriques surfaces of degree 2 and it is rational.*

We aim at computing the intersection Betti numbers of M^{GIT}. We recall that intersection cohomology satisfies Poincaré duality, allowing us to compute the Betti numbers up to dimension $10 = \dim M^{GIT}$. Hence we will report the results *mod* t^{11} for the sake of readability. Nevertheless, we prefer to carry out the computations in all dimensions as a good way to double-check the calculations.

In order to find the intersection cohomology of M^{GIT}, we need to study the semistability conditions for the branch curves in X. In our case this description is provided by the following results of Shah [Sha81], which in turn come from the Hilbert-Mumford criterion [MFK94]. Here the four coordinate lines $x_0 = 0$, $x_1 = 0$, $y_0 = 0$ and $y_1 = 0$ in $\mathbb{P}^1 \times \mathbb{P}^1$ are called *edges*.

Theorem 3.2.2. *[Sha81, Proposition 5.1.] A curve in X is not semistable under the action of G if and only if either it has a point of multiplicity greater than 4 (which must necessarily be in Δ) or it has a quadruple point in Δ with an edge as a tangent of multiplicity greater than 3 at that point.*

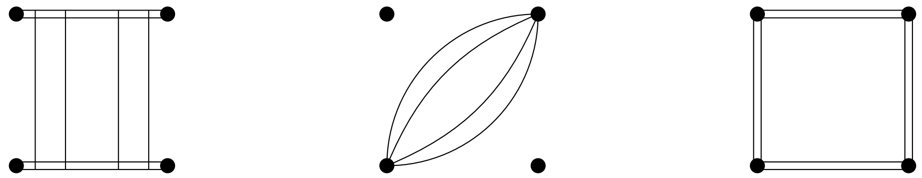
Theorem 3.2.3. *[Sha81, Proposition 5.2.(a)] A curve in X is strictly semistable, that is semistable, but not stable, under the action of G if and only if either it has an edge as a component with multiplicity 2 or it has a quadruple point in Δ.*

Theorem 3.2.4. *[Sha81, Proposition 5.2.(b)] The strictly polystable curves in X under the action of G fall into three categories:*

(i) Unions of two skew double edges and the components of the residual curve are mutually disjoint lines, none of which is an edge (see for example Figure 3.1(a));

(ii) Unions of four ι-invariant curves of bidegree $(1,1)$, each of which passes through two quadruple points in Δ. Moreover, these curves are not necessarily distinct and do not contain an edge as a component with multiplicity 2 (see for example Figure 3.1(b));

(iii) Union of all the edges with multiplicity 2 (see Figure 3.1(c)).



(a) (b) (c)

Figure 3.1: Strictly polystable curves

Remark 3.2.1. Each family of strictly polystable points described in Theorem 3.2.4 (i) and (ii) consists of two disjoint irreducible components in X, which are interchanged by the action of the Weyl group of G. Every connected component is an open subset of a linear subspace of X. Instead, the family of Theorem 3.2.4 (iii) consists of one point. We refer to Proposition 3.4.1 for a description of these loci with respect to the coordinates of X.

3.3 The HKKN stratification for degree 2 Enriques surfaces

In this section, we apply Theorem 1.2.2 to the case of degree 2 Enriques surfaces and prove the following:

Proposition 3.3.1. *The G-equivariant Hilbert-Poincaré series of the semistable locus X^{ss} is*

$$P_t^G(X^{ss}) = \\ = \frac{1+t^2+t^4+t^6+t^8+t^{10}+t^{12}-2t^{16}-3t^{18}-3t^{20}-2t^{22}+t^{26}+t^{28}+t^{30}+t^{32}}{(1-t^4)(1-t^8)}$$

$$\equiv P_t^G(X) \equiv 1+t^2+2t^4+2t^6+4t^8+4t^{10} \mod t^{11}.$$

We need to start computing the equivariant Hilbert-Poincaré series $P_t^G(X)$. Since X is compact, its equivariant cohomology ring is the invariant part under the action of $\pi_0 G = D_8$ of $H^*_{G^0}(X)$, which splits into the tensor product $H^*(BG^0) \otimes H^*(X)$ (see [Kir84, 8.12]). Then:

$$H^*_G(X) = (H^*(\mathbb{P}^{12}) \otimes H^*(B(\mathbb{C}^*)^2))^{D_8} \\ = (\mathbb{Q}[h]/(h^{13}) \otimes \mathbb{Q}[c_1, c_2])^{D_8}. \tag{3.1}$$

In fact $H^*(B(\mathbb{C})^2) \cong \mathbb{Q}[c_1, c_2]$, where c_1 and c_2 have degree 2, and $H^*(\mathbb{P}^n) = \mathbb{Q}[h]/(h^{n+1})$, with $\deg(h) = 2$. The group $D_8 = (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \ltimes \mathbb{Z}/2\mathbb{Z}$ acts on

$(\mathbb{C}^*)^2$ as follows: the first two involutions act via inversion $a \leftrightarrow a^{-1}$ on every factor of the torus, while the third interchanges the two factors. Moreover D_8 fixes the hyperplane class $h \in H^2(\mathbb{P}^{12})$, as it acts on $\mathbb{P}^{12}$ by change of coordinates. Therefore the ring of invariants is generated by $c_1^2 + c_2^2$, $c_1^2 c_2^2$ and h:

$$H^*_G(X) = \mathbb{Q}[c_1^2 + c_2^2, c_1^2 c_2^2] \otimes \mathbb{Q}[h]/(h^{13}).$$

Since $\deg(c_1^2 + c_2^2) = 4$ and $\deg(c_1^2 c_2^2) = 8$, we have

$$P_t^G(X) = \frac{1 + ... + t^{24}}{(1-t^4)(1-t^8)}. \tag{3.2}$$

According to Theorem 1.2.2, we need to subtract the contributions coming from the unstable strata. In our case, the indexing set $\mathcal{B}$ of the stratification can be visualised by means of the Hilbert diagram in Figure 3.2.

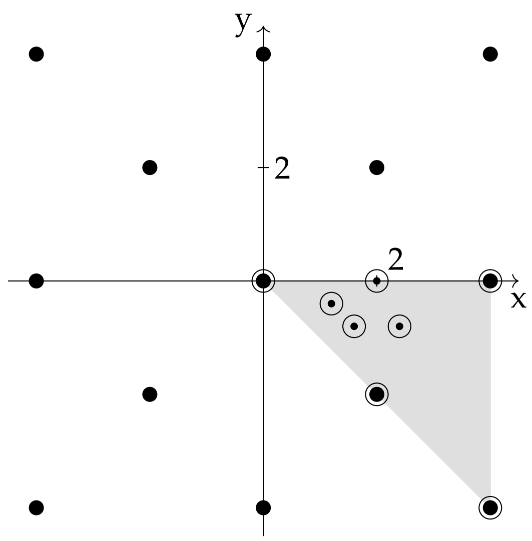


Figure 3.2: Hilbert diagram. The circled dots describe the indexing set $\mathcal{B}$.

There are 13 black nodes in this square, and each of these nodes represents a monomial

$$x_0^i x_1^{4-i} y_0^j y_1^{4-j} \text{ for } i + j \equiv 0 \text{ mod } 2$$

in $H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(4,4))^\iota$. This square is simply the diagram of weights $\alpha_I = \alpha_{(i,j)}$ of the representation of G on $H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(4,4))^\iota$ with respect to the standard maximal torus $T := (\mathrm{diag}(a, a^{-1}), \mathrm{diag}(b, b^{-1}), 1)$ in G. Each of the nodes denotes a weight of this representation, namely

$$x_0^i x_1^{4-i} y_0^j y_1^{4-j} \leftrightarrow (4-2i, 4-2j), \text{ for } i + j \equiv 0 \text{ mod } 2. \tag{3.3}$$

There is a non-degenerate inner product (the Killing form) defined on the Lie algebra $\mathfrak{t} := \mathrm{Lie}(T) \cong \mathrm{Lie}(G)$. Using this inner product, we can identify the Lie algebra $\mathfrak{t}$

with its dual $\mathfrak{t}^\vee$, and the above square can be thought of as lying in $\mathfrak{t}$. The axes of the Hilbert diagram thus coincide with the Lie algebras of the two factors of the maximal compact torus.

The Weyl group $W(G) := N(T)/T \cong (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \ltimes \mathbb{Z}/2\mathbb{Z}$ coincides with the dihedral group D_8 of all symmetries of the square. It operates on the Hilbert diagram as follows: the first two involutions are reflections along the axes, while the third one is along the principal diagonal. It is easy to see that the grey region is the portion of the square which lies inside a fixed positive Weyl chamber $\mathfrak{t}_+$.

By definition, the indexing set $\mathcal{B}$ consists of vectors β such that β lies in the closure $\overline{\mathfrak{t}_+}$ of the positive Weyl chamber and is also the closest point to the origin of a convex hull spanned by a non-empty set of weights of the representation of G on $H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(4,4))^\iota$. In this situation, we may assume that such a convex hull is either a single weight or it is cut out by a line segment joining two weights, which will be denoted by $\langle\beta\rangle$ (see Figure 3.2).

All the contributions coming from the unstable strata are summarised in Table 3.1 and can be deduced by analysing Figure 3.2.

weights in $\langle\beta\rangle$	$n(\beta)$	Stabβ	$2d(\beta)$	$P_t^G(S_\beta)$
$(4,-4)$	12	$(\mathbb{C}^*)^2 \rtimes \mathbb{Z}/2\mathbb{Z}$	24	$(1-t^2)^{-1}(1-t^4)^{-1}$
$(4,0),(2,-2),(0,-4)$	9	$(\mathbb{C}^*)^2 \rtimes \mathbb{Z}/2\mathbb{Z}$	18	$\frac{1+t^2-t^6}{(1-t^2)(1-t^4)}$
$(4,4),(2,-2)$	9	$(\mathbb{C}^*)^2$	18	$(1-t^2)^{-1}$
$(2,2),(0,-4)$	7	$(\mathbb{C}^*)^2$	14	$(1-t^2)^{-1}$
$(4,4),(0,-4)$	8	$(\mathbb{C}^*)^2$	16	$(1-t^2)^{-1}$
$(2,2),(2,-2)$	8	$\mathbb{C}^* \times G_2$	16	$(1-t^2)^{-1}$
$(4,4),(4,0),(4,-4)$	10	$\mathbb{C}^* \times G_2$	20	$\frac{1+t^2-t^6}{(1-t^2)(1-t^4)}$

Table 3.1: Cohomology of the unstable strata.

For every $\beta \in \mathcal{B}$, the first column of Table 3.1 shows the weights contained in the segment $\langle\beta\rangle$ orthogonal to the vector $\beta \in \mathfrak{t}$ (see Figure 3.2): then via the correspondence (3.3) one can obtain an explicit geometric interpretation of the curve contained in each unstable stratum. The terms appearing in the second, third and fourth columns are determined easily from the Hilbert diagram. We recall that the value $n(\beta)$ is the number of weights α_I such that $\beta \cdot \alpha_I < ||\beta||^2$, i.e. the number of weights lying in the half-plane containing the origin and defined by β. The subgroup Stab$\beta \subseteq G$ is the stabiliser of $\beta \in \mathfrak{t}$ under the adjoint action of G (cf. Remark 1.2.1) and the codimension $d(\beta)$ of each stratum $S_\beta \subseteq X$ can be computed via Remark

1.2.2: in our case, the parabolic subgroup $P_\beta \cong T$ has always dimension 2, hence $\dim(G/P_\beta) = 0$. Here $(\mathbb{C}^*)^2 \rtimes \mathbb{Z}/2\mathbb{Z}$ is a double covering of the maximal torus of G, determined by the extension $(a,b) \leftrightarrow (b^{-1}, a^{-1})$. The computations in the last column follow from applying Theorem 1.2.2 to the action of Stabβ on Z_β, in order to compute the equivariant cohomology of each unstable stratum $P_t^{\mathrm{Stab}\beta}(Z_\beta^{ss}) = P_t^G(S_\beta)$ (see Remark 1.2.1).

We shall discuss all the cases of Table 3.1 below.

Lemma 3.3.1. *There is exactly one unstable stratum indexed by β, as listed in Table 3.1, such that $Z_\beta \cong \mathbb{P}^0$, and its equivariant Hilbert-Poincaré series is $P_t^G(S_\beta) = (1-t^2)^{-1}(1-t^4)^{-1}$.*

Proof. The case under consideration corresponds to the first row of Table 3.1, where the line orthogonal to β contains only the weight β itself, giving the point $Z_\beta \cong \mathbb{P}^0$. Hence by Remark 1.2.1 the equivariant cohomology of the corresponding stratum is

$$P_t^G(S_\beta) = P_t^{(\mathbb{C}^*)^2 \rtimes \mathbb{Z}/2\mathbb{Z}}(\mathbb{P}^0) = \frac{1}{(1-t^2)(1-t^4)}.$$

□

Lemma 3.3.2. *There are exactly four unstable strata indexed by β, as listed in Table 3.1, such that $Z_\beta \cong \mathbb{P}^1$, and their equivariant Hilbert-Poincaré series is $P_t^G(S_\beta) = (1-t^2)^{-1}$.*

Proof. Looking at Figure 3.2, there are four unstable strata indexed by $\beta \in \mathcal{B}$ such that the segment $\langle \beta \rangle$ orthogonal to the vector β contains two weights that generate the line $Z_\beta \subseteq X$. As summarised in Table 3.1, in three of these cases the stabiliser Stabβ is isomorphic to the maximal torus $(\mathbb{C}^*)^2$ and hence by Remark 1.2.1:

$$P_t^G(S_\beta) = \frac{1+t^2}{(1-t^2)^2} - \frac{2t^2}{(1-t^2)^2} = \frac{1}{1-t^2}.$$

In the remaining case, corresponding to the sixth row of Table 3.1, the stabiliser is Stab$\beta \cong \mathbb{C}^* \times G_2$ and the cohomology of the corresponding stratum is

$$P_t^G(S_\beta) = \frac{1+t^2}{(1-t^2)(1-t^4)} - \frac{t^2}{(1-t^2)^2} = \frac{1}{1-t^2}.$$

□

Lemma 3.3.3. *There are exactly two unstable strata indexed by β, as listed in Table 3.1, such that $Z_\beta \cong \mathbb{P}^2$, and its equivariant Hilbert-Poincaré series is $P_t^G(S_\beta) = (1+t^2-t^6)(1-t^2)^{-1}(1-t^4)^{-1}$.*

Proof. The cases under consideration correspond to the second and last row of Table 3.1, where the segment orthogonal to β contains three weights spanning $Z_\beta \cong \mathbb{P}^2$. By Theorem 1.2.2 the equivariant cohomological series of the correspondent stratum is

$$P_t^G(S_\beta) = \frac{1+t^2+t^4}{(1-t^2)(1-t^4)} - \frac{t^4}{(1-t^2)^2} = \frac{1+t^2-t^6}{(1-t^2)(1-t^4)}.$$

□

We are finally ready to prove Proposition 3.3.1:

Proof of Proposition 3.3.1. According to Theorem 1.2.2, we need to subtract all the contributions of the unstable strata, appearing in Table 3.1, from the G-equivariant cohomology of X computed in (3.2). □

3.4 The Kirwan blow-up for degree 2 Enriques surfaces

In this section we describe the construction of the Kirwan blow-up $M^K \to M^{GIT}$ in the case of degree 2 Enriques surfaces (see Definition 3.4.1). It is obtained by blowing up three loci of strictly polystable points, geometrically described in Theorem 3.2.4 (see also Proposition 3.4.1).

By following Section 1.3, we need to find the indexing set $\mathcal{R}$ of the Kirwan blow-up and the corresponding spaces Z_R^{ss}, for all $R \in \mathcal{R}$. Namely, one must compute the conjugacy classes of the connected components of the identity in the stabilisers among all three families of polystable curves listed in Theorem 3.2.4. Compared to [Sha81, Proposition 5.2], we provide a more explicit, but equivalent, way to find the indexing set $\mathcal{R}$, which has also the advantage that one can compute Z_R and Z_R^{ss} in the coordinate system of X.

The goal is to find which non-trivial connected reductive subgroups $R \subseteq G$ fix at least one semistable point. Firstly, since R is connected, R must be contained in $G^0 = (\mathbb{C}^*)^2$, therefore it is a subtorus of rank 1 or 2. Secondly, since we are interested only in the conjugacy class of R, we may assume that its intersection $R \cap (S^1)^2$ with the maximal compact torus is a maximal compact subgroup. The fixed point set Z_R^{ss} in X^{ss} consists of all semistable points whose representatives in $H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(4,4))^\iota \cong \mathbb{C}^{13}$ are fixed by the linear action of R.

If R is a torus of rank 2, then it coincides with the whole $(\mathbb{C}^*)^2$ and clearly $Z_R = \{x_0^2x_1^2y_0^2y_1^2\}$. Instead, if R has rank 1, Z_R is spanned by those weight vectors which lie on a line through the centre of the Hilbert diagram (Figure 3.2) and are orthogonal to the Lie subalgebra $\mathrm{Lie}(R) \subseteq \mathfrak{t}$. Up to the action of a suitable element of the Weyl group $W(G)$, we can assume that the line passes through the chosen closed positive

Weyl chamber $\bar{\mathfrak{t}}_+$. We have only two possibilities (see Figure 3.2), namely the x-axis and the bisector of the II and III quadrants. These considerations lead to the following:

Proposition 3.4.1. *The indexing set of the Kirwan blow-up and the fixed loci Z_R^{ss} for ι-invariant $(4,4)$ curves in $\mathbb{P}^1 \times \mathbb{P}^1$ can be described as follows:*

(i) $R_0 = G^0 = (\mathbb{C}^*)^2$ *and in this case:*

$$Z_{R_0} = Z_{R_0}^{ss} = G \cdot Z_{R_0}^{ss} = \{x_0^2 x_1^2 y_0^2 y_1^2\};$$

(ii) $R_1 = \{(t,t) \in G^0 : t \in \mathbb{C}^*\} \cong \mathbb{C}^*$ *and in this case:*

$$\begin{aligned} Z_{R_1} = \mathbb{P}\{Ax_0^4 y_1^4 + Bx_0^3 x_1 y_0 y_1^3 + Cx_0^2 x_1^2 y_0^2 y_1^2 + Dx_0 x_1^3 y_0^3 y_1 + Ex_1^4 y_0^4\} = \\ = \mathbb{P}(\mathbb{C}[x_0 y_1, x_1 y_0]_4) \cong \mathbb{P}^4, \end{aligned}$$

$$Z_{R_1}^{ss} = \mathbb{P}^4 \setminus \{A = B = C = 0,\ C = D = E = 0\},$$

$$\begin{aligned} G \cdot Z_{R_1}^{ss} = Z_{R_1}^{ss} \cup \mathbb{P}\{A' x_0^4 y_0^4 + B' x_0^3 x_1 y_0^3 y_1 + C' x_0^2 x_1^2 y_0^2 y_1^2 + D' x_0 x_1^3 y_0 y_1^3 + E' x_1^4 y_1^4\} \\ \setminus \{A' = B' = C' = 0,\ C' = D' = E' = 0\}; \end{aligned}$$

(iii) $R_2 = \{(1,t) \in G^0 : t \in \mathbb{C}^*\} \cong \mathbb{C}^*$ *and in this case:*

$$Z_{R_2} = \mathbb{P}\{ax_0^4 y_0^2 y_1^2 + bx_0^2 x_1^2 y_0^2 y_1^2 + cx_1^4 y_0^2 y_1^2\} = \mathbb{P}(y_0^2 y_1^2 \cdot \mathbb{C}[x_0^2, x_1^2]_2) \cong \mathbb{P}^2$$

$$Z_{R_2}^{ss} = \mathbb{P}^2 \setminus \{a = b = 0,\ b = c = 0\},$$

$$G \cdot Z_{R_2}^{ss} = Z_{R_2}^{ss} \cup \mathbb{P}\{a' x_0^2 x_1^2 y_0^4 + b' x_0^2 x_1^2 y_0^2 y_1^2 + c' x_0^2 x_1^2 y_1^4\} \setminus \{a' = b' = 0,\ b' = c' = 0\}.$$

Moreover, the following holds:

$$G \cdot Z_{R_1}^{ss} \cap G \cdot Z_{R_2}^{ss} = Z_{R_0}.$$

We recall that Kirwan's partial desingularization process consists of successively blowing up X^{ss} along the (strict transforms of the) loci $G \cdot Z_R^{ss}$ in order of $\dim R$, to obtain the space $\widetilde{X}^{ss}$, and then taking the induced GIT quotient $\widetilde{X} /\!/ G$ with respect to a suitable linearisation. In our situation, we get the following diagram:

$$\begin{array}{ccc} \widetilde{X}^{ss} = (\mathrm{Bl}_{G \cdot Z_{R_2,1}^{ss}} X_2^{ss})^{ss} & \longrightarrow & M^K \\ \downarrow & & \downarrow \\ X_2^{ss} = (\mathrm{Bl}_{G \cdot Z_{R_1,1}^{ss}} X_1^{ss})^{ss} & & \\ \downarrow & & \\ X_1^{ss} = (\mathrm{Bl}_{Z_{R_0}} X^{ss})^{ss} & & \\ \downarrow & & \\ X^{ss} & \longrightarrow & M^{GIT}. \end{array}$$

The space $\widetilde{X}^{ss}$ is obtained by blowing up firstly the point $Z^{ss}_{R_0}$, followed by the blow-up of $G \cdot Z^{ss}_{R_1,1}$, namely the strict transform of the locus $G \cdot Z^{ss}_{R_1}$ under the first blow-up. In the end we need to blow up the strict transform $G \cdot Z^{ss}_{R_2,1}$ of the orbit $G \cdot Z^{ss}_{R_2}$. We also observe that the third blow-up commutes with the second one, because the strict transforms

$$G \cdot Z^{ss}_{R_1,1} \cap G \cdot Z^{ss}_{R_2,1} = \varnothing$$

are disjoint. Thus we find

Definition 3.4.1. The *Kirwan blow-up* $M^K := \widetilde{X}/\!/G \to M^{GIT}$ is defined as the GIT quotient of the blown up variety $\widetilde{X}^{ss}$ constructed above.

Intrinsically at the level of moduli spaces, M^K is obtained by first blowing up the point $G \cdot Z_{R_0}/\!/G$ corresponding to the union of the four double edges (cf. Theorem 3.2.4 (iii)). Then one needs to blow up the strict transform $\mathrm{Bl}_{G\cdot Z_{R_0}/\!/G}(G \cdot Z_{R_1}/\!/G)$ of the threefold parametrizing the unions of four conics (cf. Theorem 3.2.4 (ii)). Eventually the blow-up of the curve $\mathrm{Bl}_{G\cdot Z_{R_0}/\!/G}(G \cdot Z_{R_2}/\!/G)$ corresponding to the union of two skew double edges and two skew lines (cf. Theorem 3.2.4 (i)) completes the construction of M^K. Nevertheless, for computational reasons, we will prefer the description at the level of parameter spaces.

3.5 Cohomology of the Kirwan blow-up for degree 2 Enriques surfaces

This section is devoted to the proof of the following theorem, which is an application of Theorem 1.4.1 to the case of degree 2 Enriques surfaces.

Theorem 3.5.1. *The Hilbert-Poincaré polynomial of the Kirwan blow-up M^K is*

$$P_t(M^K) = 1 + 4t^2 + 8t^4 + 13t^6 + 18t^8 + 20t^{10} + 18t^{12} + 13t^{14} + 8t^{16} + 4t^{18} + t^{20}.$$

Due to the role they play in Theorem 1.4.1, we compute the normalisers of the reductive subgroups in $\mathcal{R}$.

Proposition 3.5.1. *The normalisers of the reductive subgroups in $\mathcal{R} = \{R_0, R_1, R_2\}$ (see Proposition 3.4.1) are given as follows:*

(i) $N(R_0) = G$*;*

(ii) $N(R_1) = (\mathbb{C}^*)^2 \rtimes (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$ *with action of the first* $\mathbb{Z}/2\mathbb{Z}$ *by* $(a,b) \leftrightarrow (b,a)$ *and the second* $\mathbb{Z}/2\mathbb{Z}$ *by* $(a,b) \leftrightarrow (b^{-1},a^{-1})$*;*

(iii) $N(R_2) = G_1 \times G_2$*, where* $G_1 \cong G_2 \cong \mathbb{C}^* \rtimes \mathbb{Z}/2\mathbb{Z}$ *with* $\mathbb{Z}/2\mathbb{Z}$ *acting via* $\lambda \leftrightarrow \lambda^{-1}$*.*

Proof. The result follows from the group structure of $G = (\mathbb{C}^*)^2 \rtimes D_8$. □

3.5.1 Main error terms

This subsection is devoted to computing the main error terms for all the three stages of the partial desingularization $M^K \to M^{GIT}$, according to Theorem 1.4.1.

As we have seen, the first step in Kirwan's process is to blow up the point representing the union of all the edges with multiplicity 2.

Proposition 3.5.2. *For the group $R_0 \cong (\mathbb{C}^*)^2$, the main term is*

$$P_t^{N(R_0)}(Z_{R_0}^{ss})(t^2 + ... + t^{2(\mathrm{rk}\mathcal{N}^{R_0}-1)}) = \frac{t^2 + ... + t^{22}}{(1-t^4)(1-t^8)}$$
$$\equiv t^2 + t^4 + 2t^6 + 2t^8 + 4t^{10} \mod t^{11}.$$

Proof. We saw in Proposition 3.4.1 that $Z_{R_0}^{ss}$ consists of a single point, and in Proposition 3.5.1 that the normaliser $N(R_0) = G$, therefore:

$$H^*_{N(R_0)}(Z_{R_0}^{ss}) = H^*(BN(R_0)) = H^*(BG).$$

We already showed in (3.2) that $H^*(BG) = \mathbb{Q}[c_1^2c_2^2,\ c_1^2 + c_2^2]$ with $\deg c_1 = \deg c_2 = 2$, so:

$$P_t^{N(R_0)}(Z_{R_0}^{ss}) = (1-t^4)^{-1}(1-t^8)^{-1}.$$

In (1.6) we explained how to compute the rank of the normal bundle:

$$\mathrm{rk}\mathcal{N}^{R_0} = \dim X - (\dim G + \dim Z_{R_0}^{ss} - \dim N(R_0)) = 12 - (2 + 0 - 2) = 12.$$

□

In the second step, we need to blow up the locus corresponding to the subgroup $R_1 \in \mathcal{R}$.

Proposition 3.5.3. *For the group $R_1 \cong \mathbb{C}^*$, the main term is*

$$P_t^{N(R_1)}(Z_{R_1,1}^{ss})(t^2 + ... + t^{2(\mathrm{rk}\mathcal{N}^{R_1}-1)}) = \frac{1+t^2+t^4}{1-t^2}(t^2 + ... + t^{14})$$
$$\equiv t^2 + 3t^4 + 6t^6 + 9t^8 + 12t^{10} \mod t^{11}.$$

Proof. For brevity, write $R = R_1$ and $N = N(R_1) = (\mathbb{C}^*)^2 \rtimes (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$ (see Proposition 3.4.1 and Proposition 3.5.1). Recall that $Z_{R,1}^{ss}$ is the strict transform of Z_R^{ss} in X_1^{ss} under the first blow-up.

We saw in Proposition 3.4.1 that $Z_R \cong \mathbb{P}^4$ and

$$Z_R^{ss} = \mathbb{P}\{Ax_0^4y_1^4 + Bx_0^3x_1y_0y_1^3 + Cx_0^2x_1^2y_0^2y_1^2 + Dx_0x_1^3y_0^3y_1 + Ex_1^4y_0^4\} \smallsetminus \{A = B = C = 0,\ C = D = E = 0\}. \tag{3.4}$$

In this system of coordinates, the centre of the first blow-up consists of the point $p = (0 : 0 : 1 : 0 : 0)$. Therefore:

$$Z_{R,1}^{ss} = (\mathrm{Bl}_p Z_R^{ss})^{ss}, \tag{3.5}$$

because we recall that, after taking the proper transform, one has to restrict to the semistable points in $X_2 \to X_1$ for the induced action of G. We want to stress that the Kirwan blow-up is a blow-up, followed by a restriction to the semistable locus.

To compute $P_t^N(Z_{R,1}^{ss})$ we notice that we can use Theorem 1.4.1. Indeed, the restriction of the first blow-up to Z_R^{ss} coincides with the unique step of Kirwan's procedure applied to the action of N on Z_R. Hence by Theorem 1.4.1, we obtain

$$P_t^N(Z_{R,1}^{ss}) = P_t^N(Z_R^{ss}) + P_t^N(\{p\})(t^2 + t^4 + t^6) - \sum_{0 \neq \beta' \in \mathcal{B}'} \frac{1}{w(\beta', R_0, N)} t^{2d(\mathbb{P}\mathcal{N}_p, \beta')} P_t^{N \cap \mathrm{Stab}\beta'}(Z_{\beta', R_0}^{ss}),$$

where $\mathcal{B}'$ is the indexing set of the HKKN stratification induced on the exceptional divisor $\mathbb{P}\mathcal{N}_p \cong \mathbb{P}^3$. We now clarify how to calculate all the contributions appearing in the equality above.

Firstly, we choose to compute $P_t^N(\{p\})$. The equivariant cohomology of a point is

$$H_N^*(\{p\}) = H^*(BN) = H^*(B(\mathbb{C}^*)^2)^{\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}} = \mathbb{Q}[c_1, c_2]^{\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}},$$

where c_1 and c_2 are the generating classes of the cohomology of $B(\mathbb{C}^*)^2$ and have both degree 2, while the action of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is described in Proposition 3.5.1(ii). By Molien's formula (see [Muk03, Theorem 1.10]) we obtain $P_t^N(\{p\}) = (1 - t^4)^{-2}$.

Secondly, we compute $P_t^N(Z_R^{ss})$. We can once again apply Theorem 1.2.2 and Remark 1.2.1, namely we consider the HKKN equivatiantly perfect stratification induced by the action of N on Z_R and we find

$$P_t^N(Z_R^{ss}) = P_t^N(Z_R) - \sum_{0 \neq \beta \in \mathcal{B}} t^{2\mathrm{codim}(S_\beta)} P_t^{\mathrm{Stab}\beta}(Z_\beta^{ss}). \tag{3.6}$$

The indexing set of the previous stratification is $\mathcal{B} = \{(0,0), (2,-2), (4,-4)\}$ and the data can be summarised as follows:

$\mathcal{B} \smallsetminus \{(0,0)\}$	$Z_\beta^{ss} \subseteq Z_R$	$\mathrm{Stab}\beta$	$\mathrm{codim}(S_\beta)$
$(2,-2)$	$(0 : 1 : 0 : 0 : 0)$	$(\mathbb{C}^*)^2 \rtimes \mathbb{Z}/2\mathbb{Z}$	3
$(4,-4)$	$(1 : 0 : 0 : 0 : 0)$	$(\mathbb{C}^*)^2 \rtimes \mathbb{Z}/2\mathbb{Z}$	4

The extension $(\mathbb{C}^*)^2 \rtimes \mathbb{Z}/2\mathbb{Z}$ is given by the involution $(a,b) \leftrightarrow (b^{-1}, a^{-1})$. Recalling that $P_t^N(Z_R) = P_t(\mathbb{P}^4)P_t(BN)$, we obtain

$$P_t^N(Z_R^{ss}) = \frac{1+t^2+t^4+t^6+t^8}{(1-t^4)^2} - \frac{t^6+t^8}{(1-t^2)(1-t^4)}.$$

Finally, we need to consider the contribution coming from the stratification of the exceptional divisor $\mathbb{P}\mathcal{N}_p$. The indexing set of this stratification is $\mathcal{B}' = \{(0,0), \pm(2,-2), \pm(4,-4)\}$ and the data we need to compute are summarised as follows:

$\mathcal{B}' \smallsetminus \{(0,0)\}$	$w(\beta', R_0, N)$	$N \cap \mathrm{Stab}\beta'$	$d(\mathbb{P}\mathcal{N}_p, \beta')$	$P_t^{N\cap \mathrm{Stab}\beta'}(Z^{ss}_{\beta',R_0})$
$\pm(2,-2)$	2	$(\mathbb{C}^*)^2 \rtimes \mathbb{Z}/2\mathbb{Z}$	2	$(1-t^2)^{-1}(1-t^4)^{-1}$
$\pm(4,-4)$	2	$(\mathbb{C}^*)^2 \rtimes \mathbb{Z}/2\mathbb{Z}$	3	$(1-t^2)^{-1}(1-t^4)^{-1}$

The extension $(\mathbb{C}^*)^2 \rtimes \mathbb{Z}/2\mathbb{Z}$ is given by the involution $(a,b) \leftrightarrow (b^{-1}, a^{-1})$. By (1.13), the equivariant Hilbert-Poincaré polynomial of each stratum is

$$\begin{aligned} P_t^{N\cap \mathrm{Stab}\beta'}(Z^{ss}_{\beta',R_0}) &= P_t^{N\cap \mathrm{Stab}\beta'}(\{p\})P_t(\mathbb{P}^0) \\ &= P_t(B((\mathbb{C}^*)^2 \rtimes \mathbb{Z}/2\mathbb{Z})) \\ &= (1-t^2)^{-1}(1-t^4)^{-1}. \end{aligned}$$

Combining the three steps of calculations above leads to

$$\begin{aligned} P_t^N(Z^{ss}_{R,1}) = \frac{1+t^2+t^4+t^6+t^8}{(1-t^4)^2} - \frac{t^6+t^8}{(1-t^2)(1-t^4)} + \\ + \frac{t^2+t^4+t^6}{(1-t^4)^2} - \frac{t^4+t^6}{(1-t^2)(1-t^4)} = \frac{1+t^2+t^4}{1-t^2}. \end{aligned}$$

To complete the proof of the Proposition 3.5.3 we need to compute the rank of the normal bundle:

$$\mathrm{rk}\mathcal{N}^R = \dim X - (\dim G + \dim Z_R^{ss} - \dim N) = 12 - (2+4-2) = 8.$$

□

In the last step we need to blow up the locus corresponding to $R_2 \in \mathcal{R}$. Recall that this locus remains unaltered under the second blow-up.

Proposition 3.5.4. *For the group $R_2 \cong \mathbb{C}^*$, the main term is*

$$\begin{aligned} P_t^{N(R_2)}(Z^{ss}_{R_2,1})(t^2 + ... + t^{2(\mathrm{rk}\mathcal{N}^{R_2}-1)}) &= \frac{1}{1-t^2}(t^2+...+t^{18}) \\ &\equiv t^2 + 2t^4 + 3t^6 + 4t^8 + 5t^{10} \mod t^{11}. \end{aligned}$$

Proof. For brevity, write $R = R_2$ and $N = N(R_2) = G_1 \times G_2$ (see Proposition 3.4.1 and Proposition 3.5.1). We recall that $Z^{ss}_{R,1}$ is the strict transform of Z^{ss}_R in X^{ss}_2 under the second blow-up.

Proposition 3.4.1 describes $Z_R \cong \mathbb{P}^2$ and

$$Z^{ss}_R = \mathbb{P}\{Ax_0^4y_0^2y_1^2 + Bx_0^2x_1^2y_0^2y_1^2 + Cx_1^4y_0^2y_1^2\} \setminus \{(0:0:1),(1:0:0)\}. \tag{3.7}$$

In this coordinate system, the centre of the first blow-up corresponds to the point $p = (0:1:0)$. Hence we have

$$Z^{ss}_{R,1} = (\mathrm{Bl}_p Z^{ss}_R)^{ss}. \tag{3.8}$$

Indeed, after taking the proper transform of Z^{ss}_R, one has to restrict to the semistable points in $X_2 \to X_1$ for the induced action of G. We recall that the Kirwan blow-up is a blow-up operation, followed by a restriction to the semistable locus.

In order to calculate $P^N_t(Z^{ss}_{R,1})$ we can apply Theorem 1.4.1. Indeed, the restriction of the second blow-up to Z^{ss}_R coincides with the unique step of Kirwan's procedure for the action of N on Z_R. Hence by Theorem 1.4.1, we obtain

$$P^N_t(Z^{ss}_{R,1}) = P^N_t(Z^{ss}_R) + P^N_t(\{p\})t^2 - \sum_{0\neq\beta'\in\mathcal{B}'} \frac{1}{w(\beta',R_0,N)} t^{2d(\mathbb{P}\mathcal{N}_p,\beta')} P_t^{N\cap\mathrm{Stab}\beta'}(Z^{ss}_{\beta',R_0}),$$

where $\mathcal{B}'$ is the indexing set of the HKKN stratification induced on the exceptional divisor $\mathbb{P}\mathcal{N}_p \cong \mathbb{P}^1$. We now explain how to compute all the contributions appearing in the equality above.

Firstly, we choose to compute $P^N_t(\{p\})$. The equivariant cohomology of a point is

$$H^*_N(\{p\}) = H^*(BN) = H^*(BG_1) \otimes H^*(BG_2) = \mathbb{Q}[c_1]^{\mathbb{Z}/2\mathbb{Z}} \otimes \mathbb{Q}[d_1]^{\mathbb{Z}/2\mathbb{Z}},$$

where c_1 and d_1 are the generating classes of the cohomology of $B\mathbb{C}^*$ and have both degree 2. In both cases the action of $\mathbb{Z}/2\mathbb{Z}$ interchanges the cohomology class with its opposite. By Molien's formula (see [Muk03, Theorem 1.10]) we get $P^N_t(\{p\}) = (1-t^4)^{-2}$.

Secondly, we calculate $P^N_t(Z^{ss}_R)$. We can once again apply Theorem 1.2.2 and Remark 1.2.1, namely we consider the HKKN equivatiantly perfect stratification induced by the action of N on Z_R and we get

$$P^N_t(Z^{ss}_R) = P^N_t(Z_R) - \sum_{0\neq\beta\in\mathcal{B}} t^{2\mathrm{codim}(S_\beta)} P_t^{\mathrm{Stab}\beta}(Z^{ss}_\beta). \tag{3.9}$$

The indexing set of the HKKN stratification is $\mathcal{B} = \{(0,0),(4,0)\}$ and the contributions can be summarised as follows:

$\mathcal{B} \smallsetminus \{(0,0)\}$	$Z^{ss}_\beta \subseteq Z_R$	$\mathrm{Stab}\beta$	$\mathrm{codim}(S_\beta)$
$(4,0)$	$(1:0:0)$	$\mathbb{C}^* \times G_2$	2

Recalling that $P_t^N(Z_R) = P_t(\mathbb{P}^4)P_t(BN)$ and $P_t^{\mathbb{C}^*\times G_2}(\mathbb{P}^0) = P_t(B(\mathbb{C}^* \times G_2)) = (1-t^2)^{-1}(1-t^4)^{-1}$, we obtain

$$P_t^N(Z_R^{ss}) = \frac{1+t^2+t^4}{(1-t^4)^2} - \frac{t^4}{(1-t^2)(1-t^4)}.$$

Finally, we need to take into consideration the contribution coming from the stratification of the exceptional divisor $\mathbb{P}\mathcal{N}_p$. The indexing set of this HKKN stratification is $\mathcal{B}' = \{(0,0), \pm(4,0)\}$ and the data we need to calculate are summarised as follows:

$\mathcal{B}' \smallsetminus \{(0,0)\}$	$w(\beta', R_0, N)$	$N \cap \mathrm{Stab}\beta'$	$d(\mathbb{P}\mathcal{N}_p, \beta')$	$P_t^{N\cap\mathrm{Stab}\beta'}(Z^{ss}_{\beta',R_0})$
$\pm(4,0)$	2	$\mathbb{C}^* \times G_2$	1	$(1-t^2)^{-1}(1-t^4)^{-1}$

By (1.13), the equivariant Hilbert-Poincaré polynomial of each stratum is

$$P_t^{N\cap\mathrm{Stab}\beta'}(Z^{ss}_{\beta',R_0}) = P_t^{N\cap\mathrm{Stab}\beta'}(\{p\})P_t(\mathbb{P}^0) = P_t(B(\mathbb{C}^* \times G_2)) = (1-t^2)^{-1}(1-t^4)^{-1}.$$

Putting together the three steps of calculations above leads to

$$P_t^N(Z^{ss}_{R,1}) = \frac{1+t^2+t^4}{(1-t^4)^2} - \frac{t^4}{(1-t^2)(1-t^4)} + \frac{t^2}{(1-t^4)^2} - \frac{t^2}{(1-t^2)(1-t^4)} = \frac{1}{1-t^2}.$$

In order to complete the proof of the Proposition 3.5.4, we need to compute the rank of the normal bundle:

$$\mathrm{rk}\mathcal{N}^R = \dim X - (\dim G + \dim Z_R^{ss} - \dim N) = 12 - (2+2-2) = 10.$$

□

3.5.2 Extra terms

To complete the computation of the contributions $A_R(t)$, we need to calculate the extra terms, as stated in Theorem 1.4.1. The crucial point is to analyse for each $R \in \mathcal{R}$ the representation $\rho : R \to \mathrm{GL}(\mathcal{N}_x^R)$ on the normal slice to the orbit $G \cdot Z_R^{ss}$ at a generic point $x \in Z_R^{ss}$. Since here we are dealing only with the local geometry around x, we can restrict to consider the normal slice to the orbit $G^0 \cdot Z_R^{ss}$, which is the connected component of $G \cdot Z_R^{ss}$ at x.

In order to compute the extra contribution coming from the blow-up of the point corresponding to R_0 (see Proposition 3.5.5), we need to describe the weights of the representation $\rho : R_0 \to \mathrm{GL}(\mathcal{N}_x^{R_0})$, where $x = Z^{ss}_{R_0}$.

Lemma 3.5.1. *For $R = R_0$, $\dim \mathcal{N}_x^{R_0} = 12$, the weights of the representation ρ of R_0 on $\mathcal{N}_x^{R_0}$ are described by the diagram in Figure 3.3.*

Proof. Each monomial in $H^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(4,4))^\iota$ is an eigenspace for the action of $R_0 = G^0$. Hence $H^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(4,4))^\iota$ decomposes as a direct sum of 1-dimensional representations of R_0 with multiplicities one, as described by the Hilbert diagram in Figure 3.2. The tangent space to the orbit $G \cdot Z^{ss}_{R_0}$ at $x = Z^{ss}_{R_0}$ is 0-dimensional and the group R_0 acts on it with weight 0. Therefore the weights of the representation on the normal slice $\mathcal{N}_x^{R_0}$ are all the ones in $H^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(4,4))^\iota$ except for the origin: they are pictured as larger dots in the Hilbert diagram of Figure 3.3. □

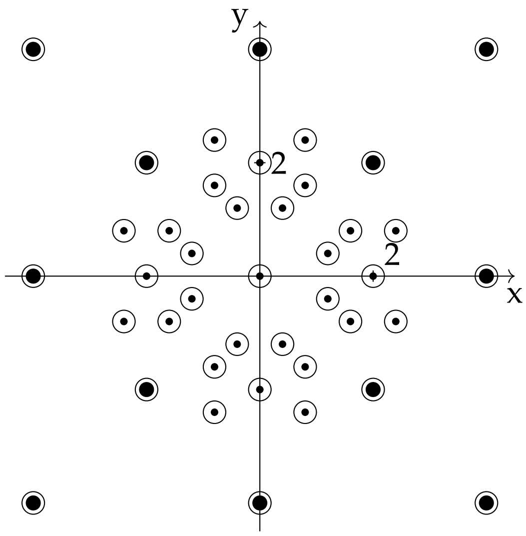


Figure 3.3: Hilbert diagram of the weights in the exceptional divisor $\mathbb{P}\mathcal{N}_x^{R_0}$ and the indexing set of its HKKN stratification.

We can now compute the extra error corresponding to the subgroup R_0.

Proposition 3.5.5. *For the group $R_0 \cong (\mathbb{C}^*)^2$ the extra term of $A_{R_0}(t)$ is given by*

$$\sum_{0\neq\beta'\in\mathcal{B}(\rho)} \frac{1}{w(\beta', R_0, G)} t^{2d(\mathbb{P}\mathcal{N}^{R_0},\beta')} P_t^{N(R_0)\cap\mathrm{Stab}\beta'}(Z^{ss}_{\beta',R_0}) = \frac{t^{12}(1+2t^2+t^4-t^{12})}{(1-t^2)(1-t^4)}$$
$$\equiv 0 \mod t^{11}.$$

Proof. Let $\{\alpha_1, ..., \alpha_{12}\} \subseteq \mathrm{Lie}_{\mathbb{R}}(R_0) \cong \mathbb{R}^2$ be the weights of the representation $\rho : R_0 \to \mathrm{GL}(\mathcal{N}_x^{R_0})$ at $x = Z^{ss}_{R_0}$ as computed in Lemma 3.5.1. After choosing a positive Weyl chamber for the adjoint action of $R_0 \cong (\mathbb{C}^*)^2$, which coincides with the whole of $\mathrm{Lie}_{\mathbb{R}}(R_0)$ in this case, we recall that an element $\beta' \in \mathrm{Lie}_{\mathbb{R}}(R_0)$ belongs to the indexing set $\mathcal{B}(\rho)$ of the stratification if and only if β' is the closest point to the origin of the convex hull of some non-empty subset of $\{\alpha_1, ..., \alpha_{12}\}$.

Table 3.2 displays all the data required to compute the extra term for R_0. We notice that they clearly coincide with the information in Table 3.1 except for the codimension of the strata which is decreased by two, as the weight zero is missing. The value $w(\beta', R_0, G)$ can be easily deduced from the diagram in Figure 3.3, while the equivariant Hilbert-Poincaré series $P_t^{N(R_0)\cap \mathrm{Stab}\beta'}(Z^{ss}_{\beta',\rho})$ can be computed as in Lemma 3.3.1, Lemma 3.3.2 and Lemma 3.3.3. □

weights in $\langle\beta'\rangle$	$w(\beta')$	Stabβ'	$2d(\beta')$	$P_t^{N(R_0)\cap \mathrm{Stab}\beta'}(Z^{ss}_{\beta',\rho})$
$(4,-4)$	4	$(\mathbb{C}^*)^2 \rtimes \mathbb{Z}/2\mathbb{Z}$	22	$(1-t^2)^{-1}(1-t^4)^{-1}$
$(4,0),(2,-2),(0,-4)$	4	$(\mathbb{C}^*)^2 \rtimes \mathbb{Z}/2\mathbb{Z}$	16	$\frac{1+t^2-t^6}{(1-t^2)(1-t^4)}$
$(4,4),(2,-2)$	8	$(\mathbb{C}^*)^2$	16	$(1-t^2)^{-1}$
$(2,2),(0,-4)$	8	$(\mathbb{C}^*)^2$	12	$(1-t^2)^{-1}$
$(4,4),(0,-4)$	8	$(\mathbb{C}^*)^2$	14	$(1-t^2)^{-1}$
$(2,2),(2,-2)$	4	$\mathbb{C}^* \times G_2$	14	$(1-t^2)^{-1}$
$(4,4),(4,0),(4,-4)$	4	$\mathbb{C}^* \times G_2$	18	$\frac{1+t^2-t^6}{(1-t^2)(1-t^4)}$

Table 3.2: Cohomology of the unstable strata in the exceptional divisor. The value $w(\beta')$ stands for $w(\beta', R_0, G)$, while the term $2d(\beta')$ indicates $2d(\mathbb{P}\mathcal{N}^{R_0}, \beta')$.

The next lemma describes the weights of the representation $\rho : R_1 \to \mathrm{GL}(\mathcal{N}_x^{R_1})$, where $x \in Z^{ss}_{R_1}$ is a general point: for our purposes it is enough to pick the point x different from $Z^{ss}_{R_0}$.

Lemma 3.5.2. *For $R = R_1$, $\dim \mathcal{N}_x^{R_1} = 8$, the weights of the representation ρ of R_1 on $\mathcal{N}_x^{R_1}$ are as follows with the respective multiplicities:*

$$(\pm 8) \times 1,\ (\pm 4) \times 3.$$

Proof. The torus R_1 acts on the coordinates $((x_0 : x_1), (y_0 : y_1))$ of $\mathbb{P}^1 \times \mathbb{P}^1$ diagonally. Thus each monomial in $H^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(4,4))^\iota$ is an eigenspace for the action of R_1. Hence $H^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(4,4))^\iota = \mathbb{C}^{13}$ decomposes as a sum of 1-dimensional representations of R_1 with the following multiplicities of weights:

$$(\pm 8) \times 1,\ (\pm 4) \times 3,\ (0) \times 5.$$

The orbit $G^0 \cdot Z^{ss}_{R_1}$ is an open part of a linear subspace, since it coincides with $Z^{ss}_{R_1}$. Therefore the tangent space at every point $x \in G^0 \cdot Z^{ss}_{R_2}$ can be identified, via the Euler sequence, with the corresponding vector subspace

$$\langle x_0^4 y_1^4,\ x_0^3 x_1 y_0 y_1^3,\ x_0^2 x_1^2 y_0^2 y_1^2,\ x_0 x_1^3 y_0^3 y_1,\ x_1^4 y_0^4 \rangle \subseteq H^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(4,4))^\iota.$$

Each monomial spans an eigenspace for the action of R_1 with weight zero, because R_1 is contained in the stabiliser of every point $x \in G^0 \cdot Z^{ss}_{R_2}$.

By subtracting the weights $(0) \times 5$ of the representation of the tangent space to the orbit from the weights of the representation of R_1 on $H^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(4,4))^\iota$, we obtain the weights of the action on the normal space. $\square$

Now we can calculate the extra error coming from the blow-up of the locus $G \cdot Z^{ss}_{R_1,1}$.

Proposition 3.5.6. *For the group $R_1 \cong \mathbb{C}^*$ the extra term of $A_{R_1}(t)$ is given by*

$$\sum_{0\neq\beta'\in\mathcal{B}(\rho)} \frac{1}{w(\beta', R_1, G)} t^{2d(\mathbb{P}\mathcal{N}^{R_1},\beta')} P_t^{N(R_1)\cap \mathrm{Stab}\beta'}(\hat{Z}^{ss}_{\beta',R_1}) =$$
$$= \frac{1+2t^2+2t^4+t^6}{1-t^2}(t^8+t^{10}+t^{12}+t^{14}) \equiv t^8+4t^{10} \mod t^{11}.$$

Proof. For brevity, we write $R = R_1$ and $N = N(R_1)$. By Lemma 3.5.2 we can take $\mathcal{B}(\rho) = \{\pm 8, \pm 4, 0\}$ as indexing set of the stratification on the projective normal slice $\mathbb{P}\mathcal{N}^R_x$ at a point $x \in G \cdot Z^{ss}_R$. We can compute the codimension of the strata $Z^{ss}_{\beta',R}$ via Remark 1.2.2:

$$d(\mathbb{P}\mathcal{N}^R_x, \beta') = n(\beta') - \dim(R/P_{\beta'}),$$

where $n(\beta')$ is the number of weights α such that $\alpha \cdot \beta' < ||\beta'||^2$ and $P_{\beta'}$ is the associated parabolic subgroup. We have $d(\mathbb{P}\mathcal{N}^R_x, \pm 4) = 4$ and $d(\mathbb{P}\mathcal{N}^R_x, \pm 8) = 7$. Due to the symmetry, the coefficient for every weight is $w(\beta', R, G) = 2$ and the stabiliser is $\mathrm{Stab}\beta' = N \cap \mathrm{Stab}\beta' = (\mathbb{C}^*)^2 \rtimes \mathbb{Z}/2\mathbb{Z}$. The extension $(\mathbb{C}^*)^2 \rtimes \mathbb{Z}/2\mathbb{Z}$ is given by the involution $(a,b) \leftrightarrow (b,a)$.

By (1.13), we obtain for every $\beta' \in \mathcal{B}(\rho) \setminus \{0\}$

$$P_t^{N\cap\mathrm{Stab}\beta'}(Z^{ss}_{\beta',R}) = P_t^{N\cap\mathrm{Stab}\beta'}(Z^{ss}_{R,1})P_t(Z_{\beta',\rho}),$$

because

$$Z_{\beta',\rho} = Z^{ss}_{\beta',\rho} = \begin{cases} \mathbb{P}^2 & \beta' = \pm 4 \\ \mathbb{P}^0 & \beta' = \pm 8. \end{cases}$$

Therefore, we just need to compute $P_t^{(\mathbb{C}^*)^2\rtimes\mathbb{Z}/2\mathbb{Z}}(Z^{ss}_{R,1})$ in a way similar to Proposition 3.5.3. Recall that by (3.4) and (3.5) $Z^{ss}_{R,1}$ is isomorphic to the semistable locus in the blow-up of $Z^{ss}_R \subseteq Z^{ss}_R \cong \mathbb{P}^4$ at $p = (0:0:1:0:0)$. By Theorem 1.4.1, the action of $(\mathbb{C}^*)^2 \rtimes \mathbb{Z}/2\mathbb{Z}$ on Z_R leads to

$$\text{(3.10)} \quad P_t^{(\mathbb{C}^*)^2\rtimes\mathbb{Z}/2\mathbb{Z}}(Z^{ss}_{R,1}) = P_t^{(\mathbb{C}^*)^2\rtimes\mathbb{Z}/2\mathbb{Z}}(Z^{ss}_R) + P_t^{(\mathbb{C}^*)^2\rtimes\mathbb{Z}/2\mathbb{Z}}(\{p\})(t^2+t^4+t^6)$$
$$- \sum_{0\neq\beta'\in\mathcal{B}'} \frac{1}{w(\beta', R_0, (\mathbb{C}^*)^2\rtimes\mathbb{Z}/2\mathbb{Z})} t^{2d(\mathbb{P}\mathcal{N}_p,\beta')} P_t^{(\mathbb{C}^*)^2\rtimes\mathbb{Z}/2\mathbb{Z}\cap\mathrm{Stab}\beta'}(Z^{ss}_{\beta',R_0}),$$

where $\mathcal{B}'$ is the indexing set of the HKKN stratification induced on the exceptional divisor $\mathbb{P}\mathcal{N}_p \cong \mathbb{P}^3$. We now clarify how to calculate all the contributions appearing in the equality above.

Firstly, we choose to compute $P_t^{(\mathbb{C}^*)^2 \rtimes \mathbb{Z}/2\mathbb{Z}}(\{p\})$. The equivariant cohomology of a point is

$$\begin{aligned} H^*_{(\mathbb{C}^*)^2 \rtimes \mathbb{Z}/2\mathbb{Z}}(\{p\}) &= H^*(B((\mathbb{C}^*)^2 \rtimes \mathbb{Z}/2\mathbb{Z})) \\ &= H^*(B\mathbb{C}^*)^{\mathbb{Z}/2\mathbb{Z}} \\ &= \mathbb{Q}[c_1, c_2]^{\mathbb{Z}/2\mathbb{Z}} = \mathbb{Q}[c_1 + c_2, c_1 c_2], \end{aligned}$$

where c_1 and c_2 are the generating classes of the cohomology of $B(\mathbb{C}^*)^2$ and have both degree 2. The action of $\mathbb{Z}/2\mathbb{Z}$ interchanges the two classes, so we obtain $P_t^{(\mathbb{C}^*)^2 \rtimes \mathbb{Z}/2\mathbb{Z}}(\{p\}) = (1-t^2)^{-1}(1-t^4)^{-1}$.

Secondly, we compute $P_t^{(\mathbb{C}^*)^2 \rtimes \mathbb{Z}/2\mathbb{Z}}(Z_R^{ss})$. We can once again apply Theorem 1.2.2 and Remark 1.2.1, namely we consider the HKKN equivatiantly perfect stratification induced by the action of $(\mathbb{C}^*)^2 \rtimes \mathbb{Z}/2\mathbb{Z}$ on Z_R and we find

$$P_t^{(\mathbb{C}^*)^2 \rtimes \mathbb{Z}/2\mathbb{Z}}(Z_R^{ss}) = P_t^{(\mathbb{C}^*)^2 \rtimes \mathbb{Z}/2\mathbb{Z}}(Z_R) - \sum_{0 \neq \beta \in \mathcal{B}} t^{2\mathrm{codim}(S_\beta)} P_t^{\mathrm{Stab}\beta}(Z_\beta^{ss}). \tag{3.11}$$

The indexing set of the previous stratification is $\mathcal{B} = \{(0,0), (2,-2), (4,-4)\}$ and the data can be summarised as follows:

$\mathcal{B}' \smallsetminus \{(0,0)\}$	$Z_\beta^{ss} \subseteq Z_R$	$\mathrm{Stab}\beta$	$\mathrm{codim}(S_\beta)$
$(2,-2)$	$(0:1:0:0:0)$	$(\mathbb{C}^*)^2$	3
$(4,-4)$	$(1:0:0:0:0)$	$(\mathbb{C}^*)^2$	4

Recalling that $P_t^{(\mathbb{C}^*)^2 \rtimes \mathbb{Z}/2\mathbb{Z}}(Z_R) = P_t(\mathbb{P}^4) P_t(B((\mathbb{C}^*)^2 \rtimes \mathbb{Z}/2\mathbb{Z}))$ and $P_t^{(\mathbb{C}^*)^2}(\mathbb{P}^0) = (1-t^2)^{-2}$, we obtain

$$P_t^{(\mathbb{C}^*)^2 \rtimes \mathbb{Z}/2\mathbb{Z}}(Z_R^{ss}) = \frac{1+t^2+t^4+t^6+t^8}{(1-t^2)(1-t^4)} - \frac{t^6+t^8}{(1-t^2)^2} = \frac{1+t^2+t^4-t^8-t^{10}}{(1-t^2)(1-t^4)}.$$

Finally, we need to consider the contribution coming from the stratification of the exceptional divisor $\mathbb{P}\mathcal{N}_p$. The indexing set of this stratification is

$$\mathcal{B}' = \{(0,0), \pm(2,-2), \pm(4,-4)\}$$

and the data we need to compute are summarised as follows, where $w(\beta')$ stands for $w(\beta', R_0, (\mathbb{C}^*)^2 \rtimes \mathbb{Z}/2\mathbb{Z})$ and $d(\beta')$ indicates $d(\mathbb{P}\mathcal{N}_p, \beta')$:

$\mathcal{B}' \smallsetminus \{(0,0)\}$	$w(\beta')$	$(\mathbb{C}^*)^2 \rtimes \mathbb{Z}/2\mathbb{Z} \cap \mathrm{Stab}\beta'$	$d(\beta')$	$P_t^{(\mathbb{C}^*)^2 \rtimes \mathbb{Z}/2\mathbb{Z} \cap \mathrm{Stab}\beta'}(Z_{\beta',R_0}^{ss})$
$\pm(2,-2)$	2	$(\mathbb{C}^*)^2$	2	$(1-t^2)^{-2}$
$\pm(4,-4)$	2	$(\mathbb{C}^*)^2$	3	$(1-t^2)^{-2}$

By (1.13), the equivariant Hilbert-Poincaré polynomial of each stratum is

$$\begin{aligned} P_t^{(\mathbb{C}^*)^2 \rtimes \mathbb{Z}/2\mathbb{Z} \cap \mathrm{Stab}\beta'}(Z^{ss}_{\beta',R_0}) &= P_t^{(\mathbb{C}^*)^2 \rtimes \mathbb{Z}/2\mathbb{Z} \cap \mathrm{Stab}\beta'}(\{p\}) P_t(\mathbb{P}^0) \\ &= P_t(B(\mathbb{C}^*)^2) = (1-t^2)^{-2}. \end{aligned}$$

Combining the three steps of calculations above leads to the result of (3.10):

$$\begin{aligned} P_t^{(\mathbb{C}^*)^2 \rtimes \mathbb{Z}/2\mathbb{Z}}(Z^{ss}_{R,1}) &= \frac{1+t^2+t^4-t^8-t^{10}}{(1-t^2)(1-t^4)} + \frac{t^2+t^4+t^6}{(1-t^2)(1-t^4)} - \frac{t^4+t^6}{(1-t^2)^2} \\ &= \frac{1+2t^2+2t^4+t^6}{1-t^2}. \end{aligned}$$

□

Finally, we need to describe the weights of the representation $\rho : R_2 \to \mathrm{GL}(\mathcal{N}_x^{R_2})$, where $x \in Z^{ss}_{R_2}$ is a point different from $Z^{ss}_{R_0}$.

Lemma 3.5.3. *For $R = R_2$, $\dim \mathcal{N}_x^{R_2} = 10$, the weights of the representation ρ of R_2 on $\mathcal{N}_x^{R_2}$ are as follows with the respective multiplicities:*

$$(\pm 4) \times 3,\ (\pm 2) \times 2.$$

Proof. The torus R_2 acts on the coordinates $((x_0 : x_1), (y_0 : y_1))$ of $\mathbb{P}^1 \times \mathbb{P}^1$ diagonally. Thus each monomial in $H^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(4,4))^\iota$ is an eigenspace for the action of R_2. Hence $H^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(4,4))^\iota = \mathbb{C}^{13}$ decomposes as a sum of 1-dimensional representations of R_2 with the following multiplicities of weights:

$$(\pm 4) \times 3,\ (\pm 2) \times 2,\ (0) \times 3.$$

The orbit $G^0 \cdot Z^{ss}_{R_2}$ is an open part of a linear subspace, since it clearly coincides with $Z^{ss}_{R_2}$. Therefore the tangent space at every point $x \in G^0 \cdot Z^{ss}_{R_2}$ can be identified, via the Euler sequence, with the corresponding vector subspace

$$\langle x_0^4 y_0^2 y_1^2,\ x_0^2 x_1^2 y_0^2 y_1^2,\ x_1^4 y_0^2 y_1^2 \rangle \subseteq H^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(4,4))^\iota.$$

Each monomial spans an eigenspace for the action of R_2 with weight zero, because R_2 is contained in the stabiliser of every point $x \in G^0 \cdot Z^{ss}_{R_2}$.

By subtracting the weights $(0) \times 3$ of the representation of the tangent space to the orbit from the weights of the representation of R_2 on $H^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(4,4))^\iota$, we obtain the weights of the action on the normal space. □

The next proposition deals with the extra contribution coming from the blow-up of $G \cdot Z^{ss}_{R_2,1}$.

Proposition 3.5.7. *For the group $R_2 \cong \mathbb{C}^*$ the extra term of $A_{R_2}(t)$ is given by*

$$\sum_{0\neq\beta'\in\mathcal{B}(\rho)} \frac{1}{w(\beta',R_2,G)} t^{2d(\mathbb{P}\mathcal{N}^{R_2},\beta')} P_t^{N(R_2)\cap\mathrm{Stab}\beta'}(\hat{Z}^{ss}_{\beta',R_2}) =$$

$$= \frac{1+t^2}{1-t^2}(t^{10}+t^{12}+t^{14}+t^{16}+t^{18}) \equiv t^{10} \mod t^{11}.$$

Proof. For brevity, we write $R = R_2$ and $N = N(R_2)$. By Lemma 3.5.3 we can take $\mathcal{B}(\rho) = \{\pm 4, \pm 2, 0\}$ as indexing set of the stratification on the projective normal slice $\mathbb{P}\mathcal{N}^R_x$ at a point $x \in G \cdot Z^{ss}_R$. We can compute the codimension of the strata $Z^{ss}_{\beta',R}$ via Remark 1.2.2:

$$d(\mathbb{P}\mathcal{N}^R_x, \beta') = n(\beta') - \dim(R/P_{\beta'}),$$

where $n(\beta')$ is the number of weights α such that $\alpha \cdot \beta' < ||\beta'||^2$ and $P_{\beta'}$ is the associated parabolic subgroup. We have $d(\mathbb{P}\mathcal{N}^R_x, \pm 2) = 5$ and $d(\mathbb{P}\mathcal{N}^R_x, \pm 4) = 7$. Due to the symmetry, the coefficient for every weight is $w(\beta', R, G) = 2$ and the stabiliser is $\mathrm{Stab}\beta' = N \cap \mathrm{Stab}\beta' = G_1 \times \mathbb{C}^*$.

By (1.13), we obtain for every $\beta' \in \mathcal{B}(\rho) \setminus \{0\}$

$$P_t^{N\cap\mathrm{Stab}\beta'}(Z^{ss}_{\beta',R}) = P_t^{N\cap\mathrm{Stab}\beta'}(Z^{ss}_{R,1}) P_t(Z_{\beta',\rho}),$$

because

$$Z_{\beta',\rho} = Z^{ss}_{\beta',\rho} = \begin{cases} \mathbb{P}^1 & \beta' = \pm 2 \\ \mathbb{P}^2 & \beta' = \pm 4. \end{cases}$$

Therefore, we just need to compute $P_t^{G_1\times\mathbb{C}^*}(Z^{ss}_{R,1})$ in a way similar to Proposition 3.5.4. Recall that by (3.7) and (3.8) $Z^{ss}_{R,1}$ is isomorphic to the semistable locus in the blow-up of $Z^{ss}_R \subseteq Z^{ss}_R \cong \mathbb{P}^2$ at $p = (0:1:0)$. By Theorem 1.4.1, the action of $G_1 \times \mathbb{C}^*$ on Z_R leads to

$$P_t^{G_1\times\mathbb{C}^*}(Z^{ss}_{R,1}) = P_t^{G_1\times\mathbb{C}^*}(Z^{ss}_R) + P_t^{G_1\times\mathbb{C}^*}(\{p\})t^2 - \sum_{0\neq\beta'\in\mathcal{B}'} \frac{1}{w(\beta',R_0,G_1\times\mathbb{C}^*)} t^{2d(\mathbb{P}\mathcal{N}_p,\beta')} P_t^{G_1\times\mathbb{C}^*\cap\mathrm{Stab}\beta'}(Z^{ss}_{\beta',R_0}), \tag{3.12}$$

where $\mathcal{B}'$ is the indexing set of the HKKN stratification induced on the exceptional divisor $\mathbb{P}\mathcal{N}_p \cong \mathbb{P}^1$. We now clarify how to calculate all the contributions appearing in the equality above.

Firstly, we choose to compute $P_t^{G_1\times\mathbb{C}^*}(\{p\})$. The equivariant cohomology of a point is

$$H^*_{G_1\times\mathbb{C}^*}(\{p\}) = H^*(B(G_1\times\mathbb{C}^*)) = H^*(BG_1)\otimes H^*(B\mathbb{C}^*) = \mathbb{Q}[c_1]^{\mathbb{Z}/2\mathbb{Z}} \otimes \mathbb{Q}[d_1],$$

where c_1 and d_1 are the generating classes of the cohomology of $B\mathbb{C}^*$ and have both degree 2. The action of $\mathbb{Z}/2\mathbb{Z}$ interchanges the cohomology class with its opposite. By Molien's formula (see [Muk03, Theorem 1.10]) we obtain $P_t^{G_1\times\mathbb{C}^*}(\{p\}) = (1-t^2)^{-1}(1-t^4)^{-1}$.

Secondly, we compute $P_t^{G_1\times\mathbb{C}^*}(Z_R^{ss})$. We can once again apply Theorem 1.2.2 and Remark 1.2.1, namely we consider the HKKN equivatiantly perfect stratification induced by the action of $G_1\times\mathbb{C}^*$ on Z_R and we find

$$P_t^{G_1\times\mathbb{C}^*}(Z_R^{ss}) = P_t^{G_1\times\mathbb{C}^*}(Z_R) - \sum_{0\neq\beta\in\mathcal{B}} t^{2\mathrm{codim}(S_\beta)} P_t^{\mathrm{Stab}\beta}(Z_\beta^{ss}). \tag{3.13}$$

The indexing set of the previous stratification is $\mathcal{B} = \{(0,0),(4,0)\}$ and the data can be summarised as follows:

$\mathcal{B}\smallsetminus\{(0,0)\}$	$Z_\beta^{ss}\subseteq Z_R$	$\mathrm{Stab}\beta$	$\mathrm{codim}(S_\beta)$
$(4,0)$	$(1:0:0)$	$(\mathbb{C}^*)^2$	2

Recalling that $P_t^{G_1\times\mathbb{C}^*}(Z_R) = P_t(\mathbb{P}^2)P_t(B(G_1\times\mathbb{C}^*))$ and $P_t^{(\mathbb{C}^*)^2}(\mathbb{P}^0) = (1-t^2)^{-2}$, we obtain

$$P_t^{G_1\times\mathbb{C}^*}(Z_R^{ss}) = \frac{1+t^2+t^4}{(1-t^2)(1-t^4)} - \frac{t^4}{(1-t^2)^2} = \frac{1+t^2-t^6}{(1-t^2)(1-t^4)}.$$

In the end, we need to consider the contribution coming from the stratification of the exceptional divisor $\mathbb{P}\mathcal{N}_p$. The indexing set of this stratification is $\mathcal{B}' = \{(0,0),\ \pm(4,0)\}$ and the data we need to compute are summarised as follows, where $w(\beta')$ stands for $w(\beta',R_0,G_1\times\mathbb{C}^*)$:

$\mathcal{B}'\smallsetminus\{(0,0)\}$	$w(\beta')$	$G_1\times\mathbb{C}^*\cap\mathrm{Stab}\beta'$	$d(\mathbb{P}\mathcal{N}_p,\beta')$	$P_t^{G_1\times\mathbb{C}^*\cap\mathrm{Stab}\beta'}(Z_{\beta',R_0}^{ss})$
$\pm(4,0)$	2	$(\mathbb{C}^*)^2$	1	$(1-t^2)^{-2}$

By (1.13), the equivariant Hilbert-Poincaré polynomial of each stratum is

$$P_t^{G_1\times\mathbb{C}^*\cap\mathrm{Stab}\beta'}(Z_{\beta',R_0}^{ss}) = P_t^{G_1\times\mathbb{C}^*\cap\mathrm{Stab}\beta'}(\{p\})P_t(\mathbb{P}^0) = P_t(B(\mathbb{C}^*)^2) = (1-t^2)^{-2}.$$

Combining the three steps of calculations above leads to the result of (3.12):

$$P_t^{G_1\times\mathbb{C}^*}(Z_{R,1}^{ss}) = \frac{1+t^2-t^6}{(1-t^2)(1-t^4)} + \frac{t^2}{(1-t^2)(1-t^4)} - \frac{t^2}{(1-t^2)^2} = \frac{1+t^2}{1-t^2}.$$

□

3.5.3 Cohomology of M^K

We complete the proof of Theorem 3.5.1.

Proof of Theorem 3.5.1. From Theorem 1.4.1, we need to put all the previous results together to find the Betti numbers of the Kirwan partial desingularization M^K. For the sake of readability, we report only the polynomials modulo t^{10}, but one can double-check the result with the entire Hilbert-Poincaré series and observe that Poincaré duality effectively holds.

$$
\begin{aligned}
& P_t(M^K) = P_t^G(\widetilde{X}^{ss}) \equiv \\
\text{(Semistable locus)} \quad & 1 + t^2 + 2t^4 + 2t^6 + 4t^8 + 4t^{10} \\
\text{(Error term for } R_0\text{)} \quad & + t^2 + t^4 + 2t^6 + 2t^8 + 4t^{10} - 0 \\
\text{(Error term for } R_1\text{)} \quad & + t^2 + 3t^4 + 6t^6 + 9t^8 + 12t^{10} - (t^8 + 4t^{10}) \\
\text{(Error term for } R_2\text{)} \quad & + t^2 + 2t^4 + 3t^6 + 4t^8 + 5t^{10} - t^{10} \\
& \equiv 1 + 4t^2 + 8t^4 + 13t^6 + 18t^8 + 20t^{10} \mod t^{11}.
\end{aligned}
$$

□

3.6 Cohomology of blow-downs for degree 2 Enriques surfaces

In this section, we compute the intersection cohomology of M^{GIT} descending from M^K, and thus prove the following:

Theorem 3.6.1. *The intersection Hilbert-Poincaré polynomial of M^{GIT} is*

$$IP_t(M^{GIT}) = 1 + t^2 + 2t^4 + 2t^6 + 3t^8 + 3t^{10} + 3t^{12} + 2t^{14} + 2t^{16} + t^{18} + t^{20}.$$

We follow Kirwan's results described in Section 1.5 and study the variation of the intersection Betti numbers at the level of the parameter spaces X^{ss} and $\widetilde{X}^{ss}$, under each stage of the modification process. In order to apply Theorem 1.5.1 to the moduli space of non-special degree 2 Enriques surfaces, we will need to follow backwards the steps of the blow-down operations. The first contribution to consider is thus the one coming from the blow-up of the strictly polystable points fixed by R_2.

Proposition 3.6.1. *For the group $R_2 \cong \mathbb{C}^*$, we have:*

(i) $Z_{R_2,1} /\!/ N(R_2)$ *is isomorphic to* $\mathbb{P}^1$*;*

(ii) $IP_t(\mathbb{P}\mathcal{N}_x^{R_2}//R_2) = 1 + 2t^2 + 3t^4 + 4t^6 + 5t^8 + 4t^{10} + 3t^{12} + 2t^{14} + t^{16}$.

The term $B_{R_2}(t)$ *is equal to*

$$B_{R_2}(t) = t^2 + 2t^4 + 3t^6 + 4t^8 + 4t^{10} + 4t^{12} + 3t^{14} + 2t^{16} + t^{18}.$$

Proof. The GIT quotient $Z_{R_2,1}//N(R_2)$ is a normal unirational curve, hence isomorphic to the projective line.

In Lemma 3.5.3 the weights of the representation $\rho : R_2 \to \mathrm{GL}(\mathcal{N}_x^{R_2})$ were computed. Since there are no strictly semistable points, the GIT quotient $\mathbb{P}\mathcal{N}_x^{R_2}//R_2 = \mathbb{P}^9//R_2$ is a projective variety of dimension 8 with at worst finite quotient singularities. Therefore $IP_t(\mathbb{P}^9//R_2) = P_t(\mathbb{P}^9//R_2) = P_t^{R_2}((\mathbb{P}^9)^{ss})$ and using the usual R_2-equivariantly perfect stratification (see Theorems 1.2.1 and 1.2.2) we obtain

$$\begin{aligned} P_t^{R_2}((\mathbb{P}^9)^{ss}) &= P_t(\mathbb{P}^9)P_t(BR_2) - \sum_{0\neq\beta'\in\mathcal{B}(\rho)} t^{2d(\beta')}P_t^{R_2}(S_{\beta'}) \\ &= \frac{1+...+t^{18}}{1-t^2} - 2\frac{t^{10}(1+t^2)+t^{14}(1+t^2+t^4)}{1-t^2}. \end{aligned}$$

Now we need to know the dimension of $IH^{\hat{q}}(\mathbb{P}^9//R_2)^{\pi_0 N(R_2)}$. The action of $\pi_0 N(R_2) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ on the cohomology of $\mathbb{P}^9$ is trivial, while its action on R_2 is as follows: the first factor acts trivially and the second one acts by inversion. Moreover, $\pi_0 N(R_2)$ acts on the strata interchanging the positive-indexed ones with the negative-indexed ones:

$$\begin{aligned} IP_t(\mathbb{P}^9//R_2)^{\pi_0 N(R_2)} &= \frac{1+...+t^{18}}{1-t^4} - \frac{t^{10}+...+t^{18}}{1-t^2} \\ &= 1 + t^2 + 2t^4 + 2t^6 + 3t^8 + 2t^{10} + 2t^{12} + t^{14} + t^{16}. \end{aligned} \tag{3.14}$$

Now the final statement easily follows from the definition of $B_{R_2}(t)$ in Theorem 1.5.1 and Remark 1.5.1. □

Next we need to compute the error term given by the blow-down of the strictly polystable locus fixed by R_1.

Proposition 3.6.2. *For the group* $R_1 \cong \mathbb{C}^*$*, we have:*

(i) $Z_{R_1,1}//N(R_1)$ *is a simply connected threefold and* $P_t(Z_{R_1,1}//N(R_1)) = 1 + 2t^2 + 2t^4 + t^6$*;*

(ii) $IP_t(\mathbb{P}\mathcal{N}_x^{R_1}//R_1) = 1 + 2t^2 + 3t^4 + 4t^6 + 3t^8 + 2t^{10} + t^{12}$.

The term $B_{R_1}(t)$ *is equal to*

$$B_{R_1}(t) = t^2 + 3t^4 + 6t^6 + 9t^8 + 10t^{10} + 9t^{12} + 6t^{14} + 3t^{16} + t^{18}.$$

Proof. For brevity we write $R = R_1$, $N = N(R_1)$ and $\mathbb{P}^7 \cong \mathbb{P}\mathcal{N}_x^{R_1}$. The GIT quotient $Z_{R,1}/\!/N$ is a unirational threefold with finite quotient singularities, hence simply connected by [Kol93, Theorem 7.8.1]. Its cohomology can be computed by means of the equality [Kir86, 1.17]:

$$H^*_N(Z^{ss}_{R,1}) = (H^*(Z_{R,1}/\!/N^0) \otimes H^*(BR))^{\pi_0 N}.$$

Using Remark 1.5.1, the action of $\pi_0 N$ splits on the tensor product, because also $Z_{R,1}/\!/N^0$ is simply connected, giving

$$H^*_N(Z^{ss}_{R,1}) = H^*(Z_{R,1}/\!/N) \otimes H^*(BR)^{\pi_0 N}.$$

Recall that $\pi_0 N = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$: the first factor acts on $R \cong \mathbb{C}^*$ trivially, while the second one acts by inversion. Therefore:

$$H^*(BR)^{\pi_0 N} = \mathbb{Q}[c]^{\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}} = \mathbb{Q}[c^2], \ \deg(c) = 2.$$

In the proof of Proposition 3.5.3, we have already computed $P_t^N(Z^{ss}_{R,1})$, thus:

$$P_t(Z_{R,1}/\!/N) = \frac{1+t^2+t^4}{1-t^2}(1-t^4) = 1 + 2t^2 + 2t^4 + t^6,$$

completing the proof of (i).

In Lemma 3.5.2 the weights of the representation $\rho : R \to \mathrm{GL}(\mathcal{N}_x^R)$ were computed. Since there are no strictly semistable points, the GIT quotient $\mathbb{P}^7/\!/R$ is a projective variety of dimension 6 with at worst finite quotient singularities. Therefore $IP_t(\mathbb{P}^7/\!/R) = P_t(\mathbb{P}^7/\!/R) = P_t^R((\mathbb{P}^7)^{ss})$ and using the usual R-equivariantly perfect stratification (see Theorems 1.2.1 and 1.2.2) we obtain

$$\begin{aligned} P_t^R((\mathbb{P}^7)^{ss}) &= P_t(\mathbb{P}^7)P_t(BR) - \sum_{0 \neq \beta' \in \mathcal{B}(\rho)} t^{2d(\beta')} P_t^R(S_{\beta'}) \\ &= \frac{1 + \ldots + t^{14}}{1-t^2} - 2\frac{t^8(1+t^2+t^4)+t^{14}}{1-t^2} \\ &= 1 + 2t^2 + 3t^4 + 4t^6 + 3t^8 + 2t^{10} + t^{12}. \end{aligned}$$

Now we need to know the dimensions $\dim IH^{\hat{q}}(\mathbb{P}^7/\!/R)^{\pi_0 N}$. The action of $\pi_0 N \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ on the cohomology of $\mathbb{P}^7$ is trivial, while its action on R was explained above. Moreover, $\pi_0 N$ acts on the strata interchanging the positive-indexed ones with the negative-indexed ones:

$$\begin{aligned} IP_t(\mathbb{P}^7/\!/R)^{\pi_0 N} &= \frac{1 + \ldots + t^{14}}{1-t^4} - \frac{t^8 + \ldots + t^{14}}{1-t^2} \\ &= 1 + t^2 + 2t^4 + 2t^6 + 2t^8 + t^{10} + t^{12}. \end{aligned} \tag{3.15}$$

Now the final statement easily follows from the definition of $B_R(t)$ in Theorem 1.5.1. □

The last step consists of blowing down the point $Z_{R_0}^{ss}$.

Proposition 3.6.3. *For the group $R_0 \cong (\mathbb{C}^*)^2$, we have*

$$B_{R_0}(t) = \sum_{2\leq q\leq 18} t^q \dim IH^{\hat{q}_{R_0}}(\mathbb{P}\mathcal{N}_x^{R_0}/\!/R_0)^{\pi_0 N(R_0)}$$
$$= t^2 + t^4 + 2t^6 + 2t^8 + 3t^{10} + 2t^{12} + 2t^{14} + t^{16} + t^{18}.$$

Proof. For brevity we write $R = R_0$, $N = N(R_0)$ and $\mathbb{P}^{11} \cong \mathbb{P}\mathcal{N}_x^{R_0}$. Clearly $Z_R/\!/N$ is a point, thus we have to compute only the invariant intersection cohomology of the GIT quotient $\mathbb{P}^{11}/\!/R$. By looking at the weights of the representation of R on $\mathbb{P}^{11}$ from Lemma 3.5.1, we find that this action gives rise to strictly polystable points, hence we need to perform the entire Kirwan procedure again in this case. We also need to take care of the invariants with respect to the action of the finite group $\pi_0 N \cong D_8$ at every step.

The first step is to consider the R-equivarintly perfect stratification of $\mathbb{P}^{11}$, as explained in Theorems 1.2.1 and 1.2.2. This stratification was already considered in Proposition 3.5.5, leading to

$$P_t^R((\mathbb{P}^{11})^{ss})^{\pi_0 N} = \frac{1+\ldots+t^{22}}{(1-t^4)(1-t^8)} - \frac{t^{12}(1+2t^2+t^4-t^{12})}{(1-t^2)(1-t^4)}. \tag{3.16}$$

The first term in the above expression comes from the R-equivariant cohomology of $\mathbb{P}^{11}$ and can be computed as in (3.2), while the second one is the sum of the contributions from the unstable strata from Proposition 3.5.5. The group $\pi_0 N$ acts trivially on the R-equivariant cohomology of $\mathbb{P}^{11}$ as in 3.1, while it identifies the unstable strata in the same orbit under the action of the Weyl group of G (cf. Lemma 3.5.1).

The second step of Kirwan's method amounts to blowing up the strictly semistable loci in $(\mathbb{P}^{11})^{ss}$, which are indexed by $\mathcal{R}^0 = \{R_1, R_2, R_3, R_4\}$, where R_1 and R_2 are defined as in Proposition 3.4.1, while

$$R_3 = \{(t, t^{-1}) \in R : t \in \mathbb{C}^*\} \cong \mathbb{C}^* \text{ and } R_4 = \{(t, 1) \in R : t \in \mathbb{C}^*\} \cong \mathbb{C}^*.$$

The fixed loci of these subgroups are permuted by the action of $\pi_0 N$. Indeed, $Z_{R_1}^0$ is isomorphic to $Z_{R_3}^0$ and they are interchanged by the reflection $\langle\sigma\rangle \subseteq \pi_0 N$ along the x-axis (cf. Figure 3.3). Moreover, $Z_{R_2}^0$ is isomorphic to $Z_{R_4}^0$ and they are interchanged by the reflection $\langle\tau\rangle \subseteq \pi_0 N$ along the diagonal (cf. Figure 3.3). In the following we give the description of the fixed loci $Z_{R_i}^0$ for $i = 1, \ldots, 4$ and the weights of the action of R from Lemma 3.5.1:

(i) $Z_{R_1}^0 \cong \mathbb{P}^3$ and $(Z_{R_1}^0)^{ss} = \mathbb{P}^3 \smallsetminus \{z_0 = z_1 = 0,\ z_2 = z_3 = 0\}$ because
$$(a, b) \cdot (z_0 : z_1 : z_2 : z_3) = (a^{-4}b^4 z_0 : a^{-2}b^2 z_1 : a^2 b^{-2} z_2 : a^4 b^{-4} z_3),$$
for $(a, b) \in R$ and $(z_0 : z_1 : z_2 : z_3) \in Z_{R_1}^0$. The same holds for $Z_{R_3}^0$.

(ii) $Z^0_{R_2} \cong \mathbb{P}^1$ and $(Z^0_{R_2})^{ss} = \mathbb{P}^1 \setminus \{(0:1),\ (1:0)\}$ because

$$(a,b)\cdot(z_0:z_1) = (a^{-4}z_0 : a^4 z_1),$$

for $(a,b) \in R$ and $(z_0:z_1) \in Z^0_{R_2}$. The same holds for $Z^0_{R_4}$.

To construct the Kirwan blow-up $\widetilde{\mathbb{P}^{11}}/\!/R$, we need to blow up the orbit loci $\bigcup_{i=1}^4 R \cdot (Z^0_{R_i})^{ss}$. Notice that the order of the resolutions is irrelevant, since the centres of the blow-ups are disjoint.

$$\begin{array}{ccc} (\widetilde{\mathbb{P}^{11}})^{ss} = (\mathrm{Bl}_{\bigcup_{i=1}^4 R\cdot(Z^0_{R_i})^{ss}}(\mathbb{P}^{11})^{ss})^{ss} & \longrightarrow & (\mathbb{P}^{11})^{ss} \\ \downarrow & & \downarrow \\ \widetilde{\mathbb{P}^{11}}/\!/R & \longrightarrow & \mathbb{P}^{11}/\!/R. \end{array}$$

Following Theorem 1.4.1, we can now compute the cohomology of $\widetilde{\mathbb{P}^{11}}/\!/R$.

Claim. *The $\pi_0 N$-equivariant cohomology of the Kirwan blow-up $\widetilde{\mathbb{P}^{11}}/\!/R$ is*

$$P_t(\widetilde{\mathbb{P}^{11}}/\!/R)^{\pi_0 N} = 1 + 3t^2 + 5t^4 + 8t^6 + 10t^8 + 10t^{10} + 8t^{12} + 5t^{14} + 3t^{16} + t^{18}.$$

Proof of Claim. By Theorem 1.4.1, we have

$$\begin{aligned} P_t(\widetilde{\mathbb{P}^{11}}/\!/R)^{\pi_0 N} &= P_t^R((\widetilde{\mathbb{P}^{11}})^{ss})^{\pi_0 N} = P_t^R((\mathbb{P}^{11})^{ss})^{\pi_0 N} + \left(\sum_{i=1}^4 A^0_{R_i}(t)\right)^{\pi_0 N} \\ &= P_t^R((\mathbb{P}^{11})^{ss})^{\pi_0 N} + (A^0_{R_1}(t))^{\pi_0 N/\langle\sigma\rangle} + (A^0_{R_2}(t))^{\pi_0 N/\langle\tau\rangle}. \end{aligned} \tag{3.17}$$

The last equality follows from the fact that the fixed loci and consequently the exceptional divisors are permuted by $\pi_0 N$, as explained above. Hence we need to calculate the two contributions $(A^0_{R_1}(t))^{\pi_0 N/\langle\sigma\rangle}$ and $(A^0_{R_2}(t))^{\pi_0 N/\langle\tau\rangle}$ coming from the blow-ups. We distinguish the two cases.

(i) The main term of $(A^0_{R_1}(t))^{\pi_0 N/\langle\sigma\rangle}$ is

$$\left(\frac{1+t^2+t^4+t^6}{(1-t^4)^2} - \frac{t^4+t^6}{(1-t^2)(1-t^4)}\right)(t^2+\ldots+t^{14}),$$

where $P_t^R((Z^0_{R_1})^{ss})^{\pi_0 N/\langle\sigma\rangle} = P_t^{(\mathbb{C}^*)^2 \rtimes (\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z})}((Z^0_{R_1})^{ss})$ has been computed using Theorem 1.2.2 and it is completely analogous to the calculation of (3.6) in Proposition 3.5.3, while $\mathrm{rk}\mathcal{N}^{R_1} = 8$ in this case.

The extra term of $(A^0_{R_1}(t))^{\pi_0 N/\langle\sigma\rangle}$ is

$$\left(\frac{1+t^2+t^4+t^6}{(1-t^2)(1-t^4)} - \frac{t^4+t^6}{(1-t^2)^2}\right)(t^8+\ldots+t^{14}),$$

where $P_t^R((Z^0_{R_1})^{ss})^{\pi_0 N/\langle\sigma\rangle} = P_t^{(\mathbb{C}^*)^2\rtimes\mathbb{Z}/2\mathbb{Z}}((Z^0_{R_1})^{ss})$ has been computed using Theorem 1.2.2 and it is totally similar to the calculation of (3.11) in Proposition 3.5.6.

(ii) The main term of $(A^0_{R_2}(t))^{\pi_0 N/\langle\tau\rangle}$ is

$$\left(\frac{1+t^2}{(1-t^4)^2}-\frac{t^2}{(1-t^2)(1-t^4)}\right)(t^2+\ldots+t^{18}),$$

where $P_t^R((Z^0_{R_2})^{ss})^{\pi_0 N/\langle\tau\rangle} = P_t^{G_1\times G_2}((Z^0_{R_2})^{ss})$ has been computed using Theorem 1.2.2 and it is completely analogous to the calculation of (3.9) in Proposition 3.5.4, while $\mathrm{rk}\mathcal{N}^{R_2}=10$ in this case.

The extra term of $(A^0_{R_2}(t))^{\pi_0 N/\langle\tau\rangle}$ is

$$\left(\frac{1+t^2}{(1-t^2)(1-t^4)}-\frac{t^2}{(1-t^2)^2}\right)(t^{10}+\ldots+t^{18}),$$

where $P_t^R((Z^0_{R_2})^{ss})^{\pi_0 N/\langle\tau\rangle} = P_t^{G_1\times\mathbb{C}^*}((Z^0_{R_2})^{ss})$ has been computed using Theorem 1.2.2 and it is totally similar to the calculation of (3.13) in Proposition 3.5.7.

By summing and subtracting appropriately the previous terms according to Theorem 1.4.1, the result follows. □

The third step of Kirwan's procedure consists of computing the intersection cohomology of $\mathbb{P}^{11}/\!/R$ descending from $\widetilde{\mathbb{P}^{11}}/\!/R$ following Theorem 1.5.1. Since we need only the invariant part of $IP_t(\mathbb{P}^{11}/\!/R)$ under $\pi_0 N$, we argue as in (3.17) and find

$$\begin{aligned} IP_t(\mathbb{P}^{11}/\!/R)^{\pi_0 N} &= P_t(\widetilde{\mathbb{P}^{11}}/\!/R)^{\pi_0 N} - \left(\sum_{i=1}^{4} B^0_{R_i}(t)\right)^{\pi_0 N} \\ &= P_t(\widetilde{\mathbb{P}^{11}}/\!/R)^{\pi_0 N} - (B^0_{R_1}(t))^{\pi_0 N/\langle\sigma\rangle} - (B^0_{R_2}(t))^{\pi_0 N/\langle\tau\rangle}, \end{aligned}$$

where $B^0_{R_1}(t)$ and $B^0_{R_2}(t)$ are defined in Theorem 1.5.1. We now calculate the invariant part of these two contributions:

(i) $Z^0_{Z_1}/\!/R$ is a simply connected surface by [Kol93, Theorem 7.8.1], because it is unirational and has only finite quotient singularities. Hence we can compute $(B^0_{R_1}(t))^{\pi_0 N/\langle\sigma\rangle}$ by using Remark 1.5.1. The cohomology $P_t(Z^0_{R_1}/\!/R)^{\pi_0 N/\langle\sigma\rangle}$ can be calculated by means of the equality [Kir86, 1.17] in a totally analogous way to Proposition 3.6.2 (i):

$$P_t(Z^0_{R_1}/\!/R)^{\pi_0 N/\langle\sigma\rangle} = \frac{P_t^R((Z^0_{R_1})^{ss})^{\pi_0 N/\langle\sigma\rangle}}{P_t(BR_1)^{\pi_0 N/\langle\sigma\rangle}} = \frac{1+t^2+t^4}{1-t^4}(1-t^4) = 1+t^2+t^4.$$

Since the normal bundle of $R \cdot (Z^0_{R_1})^{ss}$ coincides with the one considered in Proposition 3.6.2, we obtain from (3.15) that

$$IP_t(\mathbb{P}\mathcal{N}_x^{R_1} /\!/ R_1)^{\pi_0 N/\langle\sigma\rangle} = 1 + t^2 + 2t^4 + 2t^6 + 2t^8 + t^{10} + t^{12}.$$

By Theorem 1.5.1 and Remark 1.5.1 we find

$$(B^0_{R_1}(t))^{\pi_0 N/\langle\sigma\rangle} = t^2 + 2t^4 + 4t^6 + 5t^8 + 5t^{10} + 4t^{12} + 2t^{14} + t^{16}.$$

(ii) $Z^0_{Z_1} /\!/ R$ is a point. Since the normal bundle of $R \cdot (Z^0_{R_2})^{ss}$ coincides with the one considered in Proposition 3.6.1, we obtain from (3.14) that

$$IP_t(\mathbb{P}\mathcal{N}_x^{R_2} /\!/ R_2)^{\pi_0 N/\langle\sigma\rangle} = 1 + t^2 + 2t^4 + 2t^6 + 3t^8 + 2t^{10} + 2t^{12} + t^{14} + t^{16}.$$

By Theorem 1.5.1 and Remark 1.5.1 we find

$$(B^0_{R_2}(t))^{\pi_0 N/\langle\tau\rangle} = t^2 + t^4 + 2t^6 + 2t^8 + 2t^{10} + 2t^{12} + t^{14} + t^{16}.$$

By (3.6), the blow-down operations give

$$IP_t(\mathbb{P}^{11} /\!/ R)^{\pi_0 N} = 1 + t^2 + 2t^4 + 2t^6 + 3t^8 + 3t^{10} + 2t^{12} + 2t^{14} + t^{16} + t^{18}.$$

Now the result follows the definition of $B_{R_0}(t)$. □

3.6.1 Intersection cohomology of M^{GIT}

We complete the proof of Theorem 3.6.1.

Proof of Theorem 3.6.1. From Theorem 1.5.1 putting all the previous results together, we obtain that the intersection Hilbert-Poincaré polynomial of the moduli space of non-special degree 2 Enriques surfaces $M^{GIT} = X /\!/ G$ is

$$\begin{aligned} IP_t(M^{GIT}) &= P_t(M^K) - \sum_{R \in \mathcal{R}} B_R(t) \\ &= P_t^G(X^{ss}) + \sum_{R \in \mathcal{R}} (A_R(t) - B_R(t)) \\ &\equiv 1 + t^2 + 2t^4 + 2t^6 + 4t^8 + 4t^{10} + (t^{10} - t^8 - 2t^{10} + 0) \mod t^{10} \\ &\equiv 1 + t^2 + 2t^4 + 2t^6 + 3t^8 + 3t^{10} \mod t^{10}. \end{aligned}$$

□

Together with Theorem 3.5.1, this also completes the proof of the main Theorem 3.0.1.

Remark 3.6.1. From [Kir86, Remark 3.4] we can also deduce the ordinary Betti numbers of $X/\!/G$:

$$H^i(M^{GIT}) = IH^i(M^{GIT}) \text{ for } 13 \leq i \leq 20,$$

and

$$H^i(X^s/G) = IH^i(M^{GIT}) \text{ for } 0 \leq i \leq 7,$$

where $X^s/G = M^{GIT} \smallsetminus \bigcup_{R \in \mathcal{R}} Z_R/\!/N(R)$ is the orbit space of GIT-stable curves.

4 | The Kodaira dimension of some moduli spaces of elliptic K3 surfaces

This chapter deals with the first part of the joint work [FM20] of the author with G. Mezzedimi about the Kodaira dimension of some moduli spaces of elliptic K3 surfaces. A K3 surface X is said to be elliptic if it admits a fibration $X \to \mathbb{P}^1$ in curves of genus one together with a section. The classes of the fibre and the zero section in the Néron-Severi group generate a lattice isomorphic to the hyperbolic plane U, and they span the whole Néron-Severi group if the elliptic K3 surface is very general. The geometry of elliptic surfaces can be studied via their realisation as Weierstrass fibrations. By using this description, Miranda [Mir81] constructed the moduli space of elliptic K3 surfaces and showed its unirationality as a by-product. Later, Lejarraga [Lej93] proved that this space is actually rational. We want to study the divisors of the moduli space of elliptic K3 surfaces which parametrize the surfaces whose Néron-Severi groups have Picard number at least three, which means that they contain primitively $U \oplus \langle -2k \rangle$. These are the moduli spaces $\mathcal{M}_{2k}$ of $U \oplus \langle -2k \rangle$-polarized K3 surfaces. Geometrically we are considering elliptic K3 surfaces admitting an extra class in the Néron-Severi group: if $k = 1$, it comes from a reducible fibre of the elliptic fibration, while if $k \geq 2$ it is represented by an extra section, intersecting the zero section in $k - 2$ points with multiplicity. Our result is summarised by the following:

Theorem 4.0.1. *The moduli space $\mathcal{M}_{2k}$ is of general type for $k \geq 220$, or*

$$k \geq 208,\ k \neq 211, 219,\ \textit{or}\ k \in \{170, 185, 186, 188, 190, 194, 200, 202, 204, 206\}.$$

Moreover, the Kodaira dimension of $\mathcal{M}_{2k}$ is non-negative for $k \geq 176$, or

$$k \geq 164,\ k \neq 169, 171, 175\ \textit{or}\ k \in \{140, 146, 150, 152, 154, 155, 158, 160, 162\}.$$

The chapter is organised as follows. In Section 4.1 we review the general construction for the moduli spaces of lattice polarized K3 surfaces as orthogonal modular varieties. We give a description of the moduli spaces $\mathcal{M}_{2k}$, as quotients of bounded hermitian symmetric domains $\Omega_{L_{2k}}$ of type IV and dimension 17 by the stable orthogonal groups $\widetilde{\mathrm{O}}^+(L_{2k})$, where the lattice L_{2k} is the orthogonal complement of $U \oplus \langle -2k \rangle$ in the K3-lattice $\Lambda_{K3} := 3U \oplus 2E_8(-1)$. In Section 4.2 we describe the method used in proving Theorem 4.0.1, namely the low-weight cusp form trick (Theorem 4.2.1). This tool provides a sufficient condition for an orthogonal modular variety to be of general type. Namely, one has to find a non-zero cusp form on $\Omega^\bullet_{L_{2k}}$ of weight strictly less than 17 vanishing along the ramification divisor of the projection $\Omega_{L_{2k}} \to \widetilde{\mathrm{O}}^+(L_{2k})\backslash\Omega_{L_{2k}}$. The desired form is constructed as a quasi pull-back of the Borcherds form Φ_{12} (see Theorem 4.2.3) associated to the lattice $L_{2,26}$, i.e. the unique (up to isometry) even unimodular lattice of signature $(2, 26)$. If the number $N(L_{2k})$ of effective roots in the orthogonal complement $L_{2k}^\perp$ of $L_{2k} \hookrightarrow L_{2,26}$ under a suitable primitive embedding is positive, the quasi pull-back is a cusp-form of weight $12 + N(L_{2k})$. Section 4.3 is devoted to the proof of Proposition 4.3.1. For this, we study some special reflections in the stable orthogonal group $\widetilde{\mathrm{O}}^+(L_{2k})$. This is then used to impose the vanishing of the quasi pull-back $\Phi|_{L_{2k}}$ of the Borcherds form along the ramification divisor of the quotient map $\Omega_{L_{2k}} \to \mathcal{M}_{2k}$. In Section 4.4 we tackle Problem 4.4.1 of finding the values of k for which there exists a suitable primitive embedding $L_{2k} \hookrightarrow L_{2,26}$, whose orthogonal complement contains at least 2 and at most 8 roots. First, we prove that for any $k \geq 4900$ such an embedding exists. Then, we perform an exhaustive computer analysis to find explicit embeddings for the remaining values of k. It relies on the geometry of K3 surfaces with Néron-Severi group isometric to $U \oplus E_8(-1)$. In Section 4.5 we state without proof the result of the second part of the article [FM20] about the unirationality of $\mathcal{M}_{2k}$ for small values of k. It was further improved in the article [FHM20] of the author in collaboration with M. Hoff and G. Mezzedimi.

4.1 Moduli spaces of lattice polarized K3 surfaces

In this section we review the construction of the moduli spaces of lattice polarized K3 surfaces. The main reference to this subject is [Dol96].

First we recall some basic notions of lattice theory. Let L be an integral lattice of signature $(2, n)$. Let Ω_L be one of the two connected components of

$$\{[w] \in \mathbb{P}(L \otimes \mathbb{C}) \mid (w, w) = 0,\ (w, \bar{w}) > 0\}.$$

It is a hermitian symmetric domain of type IV and dimension n. We denote by $\mathrm{O}^+(L)$ the index two subgroup of the orthogonal group $\mathrm{O}(L)$ preserving Ω_L. If $\Gamma \subseteq \mathrm{O}^+(L)$ is a subgroup of finite index we denote by $\mathcal{F}_L(\Gamma)$ the quotient $\Gamma\backslash\Omega_L$. By a result of Baily and Borel [BB66], $\mathcal{F}_L(\Gamma)$ is a quasi-projective variety of dimension n, whose boundary consists of a finite union of 0-cusps and 1-cusps.

For every non-degenerate integral lattice L we denote by $L^\vee := \mathrm{Hom}(L, \mathbb{Z})$ its dual lattice. If L is even, the finite group $A_L := L^\vee/L$, called *discriminant group*, is endowed with a quadratic form q_L with values in $\mathbb{Q}/2\mathbb{Z}$, induced by the quadratic form on L. We define

$$\widetilde{\mathrm{O}}(L) := \ker(\mathrm{O}(L) \to \mathrm{O}(A_L))$$

and

$$\widetilde{\mathrm{O}}^+(L) := \widetilde{\mathrm{O}}(L) \cap \mathrm{O}^+(L).$$

Now we introduce the geometric object whose moduli space we want to study. We recall that a compact smooth complex surface X is a *K3 surface* if X is simply connected and $H^0(X, \Omega_X^2)$ is spanned by a non-degenerate holomorphic 2-form ω_X. The cohomology group $H^2(X, \mathbb{Z})$ is naturally endowed with a unimodular intersection pairing, making it isomorphic to the K3-lattice:

$$\Lambda_{K3} := 3U \oplus 2E_8(-1),$$

where U is the hyperbolic plane and $E_8(-1)$ is the unique (up to isometry) even unimodular negative definite lattice of rank 8. In particular the signature of $H^2(X, \mathbb{Z})$ is $(3, 19)$.

Fix an integral even lattice M of signature $(1, t)$ with $t \geq 0$, which can be embedded primitively into the K3-lattice. The cone $\{x \in M \otimes \mathbb{R} : (x, x) > 0\}$ has two connected components: we fix one and denote it by $\mathcal{C}_M$. Let

$$\Delta_M := \{d \in M : (d, d) = -2\}.$$

We fix a subset $\Delta_M^+ \subseteq \Delta_M$ such that:

(i) $\Delta_M = \Delta_M^+ \sqcup \Delta_M^-$, where $\Delta_M^- = \{-d : d \in \Delta_M^+\}$;

(ii) if $d_1, \ldots, d_k \in \Delta_M^+$ and $d = \sum n_i d_i$ with $n_i \geq 0$ then $d \in \Delta_M^+$.

The choice of a subset Δ_M^+ as above defines the subset:

$$\mathcal{C}_M^+ := \{h \in \mathcal{C}_M \cap M : (h, d) > 0 \text{ for all } d \in \Delta_M^+\}.$$

An *M-polarized K3 surface* is then a pair (X, j) where X is a K3 surface and $j : M \hookrightarrow \mathrm{NS}(X)$ is a primitive embedding. An isomorphism between M-polarized K3 surfaces is an isomorphism between the surfaces that commutes with the primitive embeddings. We say that (X, j) is a *pseudo-ample* (resp. *ample*) M-polarized K3 surface, if $j(\mathcal{C}_M^+) \subseteq \mathrm{NS}(X)$ contains a big and nef (resp. ample) class. The classical case of polarized K3 surfaces is the case where $t = 0$ and $M = \langle 2d \rangle$. For further purposes, we denote by

$$N := j(M)^{\perp}_{\Lambda_{K3}}$$

the orthogonal complement of M in Λ_{K3}. It is an integral even lattice of signature $(2, 19 - t)$.

By the Torelli theorem [PS72] (see also [Dol96, Corollary 3.2]), the moduli spaces of pseudo-ample M-polarized K3 surfaces can be identified with the quotient of a classical hermitian symmetric domain of type IV and dimension $19 - t$ by an arithmetic group. More precisely, the 2-form ω_X of an M-polarized K3 surface X determines a point in the *period domain*:

$$\Omega_N := \{[w] \in \mathbb{P}(N \otimes \mathbb{C}) \mid (w,w) = 0,\ (w,\bar{w}) > 0\}^+,$$

modulo the action of the group (cf. [Dol96, Proposition 3.3]):

$$\widetilde{\mathrm{O}}^+(N) = \{g \in \mathrm{O}^+(\Lambda_{K3}) \mid g|_M = \mathrm{id}\}.$$

Theorem 4.1.1. *[Dol96, §3] The variety $\mathcal{F}_N(\widetilde{\mathrm{O}}^+(N))$ is isomorphic to the coarse moduli space of pseudo-ample M-polarized K3 surfaces.*

In the following, we will study the moduli spaces of M-polarized K3 surfaces with $M = U \oplus \langle -2k \rangle$, i.e. elliptic K3 surfaces of Picard rank at least 3. The very general such K3 surface has Néron-Severi group isomorphic to $U \oplus \langle -2k \rangle$. Since the embedding $U \oplus \langle -2k \rangle \hookrightarrow \Lambda_{K3}$ is unique up to isometry by [Nik79a, Theorem 1.14.4], we get the isomorphism:

$$L_{2k} := U \oplus 2E_8(-1) \oplus \langle 2k \rangle \cong (U \oplus \langle -2k \rangle)^{\perp}_{\Lambda_{K3}}.$$

As we discussed above, the quotient variety

$$\mathcal{M}_{2k} := \mathcal{F}_{L_{2k}}(\widetilde{\mathrm{O}}^+(L_{2k}))$$

is the moduli space of $U \oplus \langle -2k \rangle$-polarized K3 surfaces. Notice that all these surfaces are elliptic, since their Picard lattices contain a copy of the hyperbolic plane U.

4.2 Low-weight cusp form trick

This section deals with a tool for computing the Kodaira dimension of modular orthogonal varieties, namely the low-weight cusp form trick, developed by Gritsenko, Hulek and Sankaran [GHS07b]. An early version of these methods was also used by Kōndo in [Kon93] and [Kon99].

We start by recalling that, if Y is a connected smooth projective variety of dimension n, the *Kodaira dimension* $\kappa(Y)$ of Y is defined by means of the transcendence degree of the ring of pluricanonical sections, namely

$$\kappa(Y) := \mathrm{tr.deg}\left(\bigoplus_{k=0}^{+\infty} H^0(Y, kK_Y)\right) - 1,$$

or $\kappa(Y) := -\infty$, if $H^0(Y, kK_Y) = 0$ for all $k > 0$. Thus $h^0(Y, kK_Y) \sim k^{\kappa(Y)}$ for k sufficiently divisible. The possible values of $\kappa(Y)$ are $-\infty, 0, 1, \ldots, n = \dim(Y)$, and Y is said to be *of general type* if $\kappa(Y) = \dim(Y)$. The Kodaira dimension is a birational invariant, so it makes sense to extend the definition to arbitrary irreducible quasi-projective varieties X by putting $\kappa(X) = \kappa(\tilde{X})$ for $\tilde{X}$ a desingularization of a compactification of X.

In the case of modular orthogonal varieties, differential forms on $\mathcal{F}_L(\Gamma)$ may be interpreted as modular forms for Γ. Therefore, arithmetic information about Γ may be used to obtain geometric information about $\mathcal{F}_L(\Gamma)$, such as its Kodaira dimension. In order to describe this relation, we need to introduce a little theory of modular forms on orthogonal groups.

Let L be an integral even lattice of signature $(2, n)$. A *modular form* of weight $k \in \mathbb{Z}$ and character $\chi : \Gamma \to \mathbb{C}^*$ for a finite index subgroup $\Gamma \subseteq \mathrm{O}^+(L)$ is a holomorphic function $F : \Omega_L^\bullet \to \mathbb{C}$ on the affine cone $\Omega_L^\bullet$ over Ω_L such that

$$F(tZ) = t^{-k}F(Z) \ \forall t \in \mathbb{C}^*, \quad \text{and} \quad F(gZ) = \chi(g)F(Z) \ \forall g \in \Gamma.$$

A modular form is a *cusp form* if it vanishes at every cusp. We denote the vector spaces of modular forms and cusp forms of weight k and character χ for Γ by $M_k(\Gamma, \chi)$ and $S_k(\Gamma, \chi)$ respectively.

The connection between modular forms for Γ and differential forms on $\mathcal{F}_L(\Gamma)$ relies on the following observation. One may choose a complex volume form dZ on Ω_L such that, if F is a modular form of weight mn and character $\det^m$ for Γ, then $F(dZ)^m$ is a Γ-invariant section of mK_{Ω_L}. It will then "descend" to a pluricanonical section on $\mathcal{F}_L(\Gamma)$: here one must be very careful and pay attention at all the obstructions that can be encountered in this process. The study of these obstructions led Gritsenko, Hulek and Sankaran to develop a powerful tool for computing the Kodaira dimension of modular orthogonal varieties, which plays a crucial role in the application of modular forms to moduli problems. This result goes under the name of *low-weight cusp form trick* and is stated in the following theorem, whose second part follows from a result of Freitag [Fre83, Hilfssatz 2.1, Kap. III].

Theorem 4.2.1. *[GHS07b, Theorem 1.1] Let L be an integral lattice of signature $(2, n)$ with $n \geq 9$, and let $\Gamma \subseteq \mathrm{O}^+(L)$ be a subgroup of finite index. The modular variety $\mathcal{F}_L(\Gamma)$ is of general type if there exists a nonzero cusp form $F \in S_k(\Gamma, \chi)$ of weight $k < n$ and character χ that vanishes along the ramification divisor of the projection $\pi : \Omega_L \to \mathcal{F}_L(\Gamma)$ and vanishes with order at least 1 at infinity.*

If $S_n(\Gamma, \det) \neq 0$, then the Kodaira dimension of $\mathcal{F}_L(\Gamma)$ is non-negative.

Remark 4.2.1. In [Ma21] the author shows the necessity for an additional hypothesis in Theorem 4.2.1 concerning the so-called irregular cusps (cf. [Ma21, Theorem 1.2]). However, this does not affect our case as explained in [Ma21, Example 4.10].

In order to apply Theorem 4.2.1, we need a description of the ramification divisor of orthogonal projections and a method to construct cusp forms on homogeneous domains of type IV. These will be the topics of the following subsections.

4.2.1 Ramification divisor

Now we want to describe the ramification divisor of orthogonal projections, which turns out to be the union of rational quadratic divisors associated to reflective vectors.

For any $v \in L \otimes \mathbb{Q}$ such that $v^2 < 0$ we define the *rational quadratic divisor* to be

$$\Omega_v(L) := \{[Z] \in \Omega_L \mid (Z, v) = 0\} \cong \Omega_{v_L^\perp},$$

where the orthogonal complement $v_L^\perp$ of v in L is an even integral lattice of signature $(2, n-1)$. The reflection with respect to the hyperplane defined by a non-isotropic vector $r \in L$ is given by

$$\sigma_r : l \longmapsto l - 2\frac{(l, r)}{r^2} r.$$

If $r \in L$ is primitive and σ_r fixes the integral structure of L, i.e. $\sigma_r \in \mathrm{O}(L)$, then we say that r is a *reflective vector*. We notice that r is always reflective if $r^2 = \pm 2$, and we call it *root* in this case. If $v \in L^\vee$ and $v^2 < 0$, the divisor $\Omega_v(L)$ is called a *reflective divisor* if $\sigma_v \in \mathrm{O}(L)$. The following theorem describes the ramification divisors of orthogonal projections in terms of reflective divisors.

Theorem 4.2.2. *[GHS07b, Corollary 2.13] For $n \geq 6$, the ramification divisor of the projection $\pi_\Gamma : \Omega_L \to \mathcal{F}_L(\Gamma)$ is the union of the reflective divisors with respect to $\Gamma \subseteq \mathrm{O}^+(L)$:*

$$\mathrm{Rdiv}(\pi_\Gamma) = \bigcup_{\substack{\mathbb{Z}r \subseteq L \\ \sigma_r \in \Gamma \cup -\Gamma}} \Omega_r(L).$$

4.2.2 Quasi pull-back

From Theorem 4.2.1, we have seen that we need a supply of modular forms for $\Gamma \subseteq \mathrm{O}^+(L)$ in order to prove that $\mathcal{F}_L(\Gamma)$ is of general type. These modular forms are provided by quasi pull-backs of modular forms with respect to some higher rank orthogonal group.

Let $L_{2,26}$ denote the unique (up to isometry) even unimodular lattice of signature $(2, 26)$, namely

$$L_{2,26} := 2U \oplus 3E_8(-1).$$

Borcherds proved [Bor95] that $M_{12}(\mathrm{O}^+(L_{2,26}), \det)$ is a 1-dimensional complex vector space spanned by a modular form Φ_{12}, called the *Borcherds form*. The zeroes of Φ_{12} lie on rational quadratic divisors defined by (-2)-vectors in $L_{2,26}$, i.e. $\Phi_{12}(Z) = 0$ if and only if there exists $r \in L_{2,26}$ with $r^2 = -2$ such that $(Z, r) = 0$. Moreover the multiplicity of the rational quadratic divisor of zeroes of Φ_{12} is one.

Given a primitive embedding of lattices $L \hookrightarrow L_{2,26}$, with L of signature $(2, n)$, we define

$$R_{L_{2,26}}(L) := \{r \in L_{2,26} \mid r^2 = -2,\ (r, L) = 0\}.$$

To construct a modular form for some subgroup of $\mathrm{O}^+(L)$, one might try to pull back Φ_{12} along the closed immersion $\Omega_L^\bullet \hookrightarrow \Omega_{L_{2,26}}^\bullet$. However, for any $r \in R_{L_{2,26}}(L)$ one has $\Omega_L^\bullet \subseteq \Omega_{r^\perp}^\bullet$ and hence Φ_{12} vanishes identically on $\Omega_L^\bullet$. The method of the *quasi pull-back*, first developed by Gritsenko, Hulek and Sankaran [GHS07b], deals with this issue by dividing out by appropriate linear factors.

Theorem 4.2.3. *[GHS15, Theorem 8.3] Let $L \hookrightarrow L_{2,26}$ be a primitive non-degenerate sublattice of signature $(2, n)$, $n \geq 3$, and let $\Omega_L \hookrightarrow \Omega_{L_{2,26}}$ be the corresponding embedding of the homogeneous domains. The set of (-2)-roots $R_{L_{2,26}}(L)$ in the orthogonal complement of L is finite. We put $N(L) := |R_{L_{2,26}}(L)|/2$. Then the function*

$$\Phi|_L(Z) := \left.\frac{\Phi_{12}(Z)}{\prod_{r \in R_{L_{2,26}}(L)/\pm 1}(Z, r)}\right|_{\Omega_L} \in M_{12+N(L)}(\widetilde{\mathrm{O}}^+(L), \det)$$

is non-zero, where in the product over r we have taken a finite system of representatives in $R_{L_{2,26}}(L)/\pm 1$. The modular form $\Phi|_L$ vanishes only on rational quadratic divisors of type $\Omega_v(L)$ where $v \in L^\vee$ is the orthogonal projection with respect to $L^\perp$ of a (-2)-root $r \in L_{2,26}$ on $L^\vee$.

Moreover, if $N(L) > 0$, then $\Phi|_L$ is a cusp form.

We want to apply the low-weight cusp form trick and Theorem 4.2.3 to the orthogonal variety $\mathcal{M}_{2k}$ isomorphic to the moduli space of $U \oplus \langle -2k \rangle$-polarized K3 surfaces. In our situation, we need to find a suitable primitive embedding of $L_{2k} \hookrightarrow L_{2,26}$, such that the quasi pull-back $\Phi|_{L_{2k}}$ is a cusp form of weight (strictly) less than 17 which vanishes along the ramification divisor of the projection

$$\pi : \Omega_{L_{2k}} \to \mathcal{M}_{2k} = \widetilde{\mathrm{O}}^+(L_{2k}) \backslash \Omega_{L_{2k}}.$$

Remark 4.2.2. By [GHS09, Theorem 1.7] the abelianization of $\widetilde{\mathrm{O}}^+(L_{2k})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. This is because L_{2k} is isometric to $2U \oplus E_8(-1) \oplus \langle -2k \rangle^\perp_{E_8(-1)}$, since the embedding $U \oplus \langle -2k \rangle \hookrightarrow \Lambda_{K3}$ is unique up to isometry (cf. [Nik79a, Theorem 1.14.4]). As a consequence, the Albanese varieties of the moduli spaces $\mathcal{M}_{2k}$ are all trivial (cf. [Kon88, Theorem 2.5]). Moreover, [GHS09, Corollary 1.8] implies that the unique non-trivial character of $\widetilde{\mathrm{O}}^+(L_{2k})$ is det.

4.3 Special reflections

In this section we prove the following result about the ramification divisor of the quasi pull-back $\Phi|_{L_{2k}}$ of the Borcherds form, depending on the embedding $L_{2k} \hookrightarrow L_{2,26}$.

Proposition 4.3.1. *The quasi pull-back $\Phi|_{L_{2k}}$ defined in Theorem 4.2.3 vanishes along the ramification divisor of*

$$\pi : \Omega_{L_{2k}} \to \mathcal{M}_{2k} = \widetilde{\mathrm{O}}^+(L_{2k})\backslash\Omega_{L_{2k}}$$

for any primitive embedding $L_{2k} \hookrightarrow L_{2,26}$ such that $(L_{2k})^{\perp}_{L_{2,26}}$ does not contain a copy of $E_8(-1)$.

For any $l \in L$ we define its *divisibility* $\mathrm{div}(l)$ to be the unique $m > 0$ such that $(l, L) = m\mathbb{Z}$ or, equivalently, the unique $m > 0$ such that $l/m \in L^\vee$ is primitive. Since $\mathrm{div}(r) > 0$ is the smallest intersection number of r with any other vector, $\mathrm{div}(r)$ divides r^2. Moreover, if r is reflective, the number $2\frac{(l,r)}{r^2}$ must be an integer, so r^2 divides $2(l, r)$ for all $l \in L$, i.e. $r^2 \mid 2\mathrm{div}(r)$. Summing up,

$$\mathrm{div}(r) \mid r^2 \mid 2\mathrm{div}(r).$$

The following proposition is similar to [GHS07b, Corollary 3.4] and identifies the reflective vectors with respect to $\pm\widetilde{\mathrm{O}}(L_{2k})$.

Proposition 4.3.2. *Let $r \in L_{2k}$ be a reflective vector. Then σ_r induces $\pm\mathrm{id}$ in $A_{L_{2k}}$, i.e. $\pm\sigma_r \in \widetilde{\mathrm{O}}(L_{2k})$, if and only if $r^2 = \pm 2$ or $r^2 = \pm 2k$ and $\mathrm{div}(r) \in \{k, 2k\}$.*

Proof. By [GHS07b, Proposition 3.1] the reflection $\sigma_r \in \widetilde{\mathrm{O}}(L_{2k})$ if and only if $r^2 = \pm 2$. Moreover, if $-\sigma_r \in \widetilde{\mathrm{O}}(L_{2k})$, then $r^2 = \pm 2k$ and $\mathrm{div}(r) \in \{k, 2k\}$ by [GHS07b, Proposition 3.2 (i)], as the exponent of the discriminant group $A_{L_{2k}} \cong \mathbb{Z}/2k\mathbb{Z}$ is $2k$. On the contrary, if $r^2 = \pm 2k$ and $\mathrm{div}(r) = 2k$, then $-\sigma_r \in \widetilde{\mathrm{O}}(L_{2k})$ by [GHS07b, Proposition 3.2 (iii)]. Finally, if $r^2 = \pm 2k$ and $\mathrm{div}(r) = k$, the vector r can be written as $r = km + xs$, where $m \in U \oplus 2E_8(-1)$, s is a generator of $\langle 2k \rangle$ and $x^2 = \pm 1 - km^2/2$. We find

$$\sigma_r\left(\frac{s}{2k}\right) = -\frac{s}{2k} \mp xm \pm \frac{m^2}{2}s \equiv -\frac{s}{2k} \bmod L_{2k}.$$

□

Now $\sigma_r \in \mathrm{O}^+(L \otimes \mathbb{R})$ if and only if $r^2 < 0$ (see [GHS07a]). Recall that an integral lattice is called 2-*elementary* if every element of its discriminant group has order 2.

Proposition 4.3.3. *Let $r \in L_{2k}$ be primitive with $r^2 = -2k$ and $\mathrm{div}(r) \in \{k, 2k\}$. Then $L_r := r^{\perp}_{L_{2k}}$ is a 2-elementary lattice of signature $(2, 16)$ and determinant 4.*

Proof. The determinant of L_r can be computed from the well-known formula (see for instance [GHS07b, Equation 20]):

$$\det(L_r) = \frac{\det(L_{2k}) \cdot r^2}{\operatorname{div}(r)^2} \in \{1, 4\}.$$

Since L_{2k} has signature $(2, 17)$ and $r^2 < 0$, we have that L_r has signature $(2, 16)$. Therefore $\det(L_r)$ cannot be 1, because there are no unimodular lattices with signature $(2, 16)$ (see [Nik79a, Theorem 0.2.1]). This shows that $\operatorname{div}(r) = k$. Therefore the reflection σ_r acts as $-\mathrm{id}$ on the discriminant group $A_{L_{2k}}$ (see [GHS07b, Corollary 3.4]). Now we can extend $-\sigma_r \in \widetilde{\mathrm{O}}(L_{2k})$ to an element $\bar{\sigma}_r \in \mathrm{O}(\Lambda_{K3})$ by defining $\bar{\sigma}_r|_{U \oplus \langle -2k \rangle} = \mathrm{id}$ on the orthogonal complement of $L_{2k} \hookrightarrow \Lambda_{K3}$. Put $S_r := (L_r)^{\perp}_{\Lambda_{K3}}$. It is easy to see that

$$\bar{\sigma}_r|_{L_r} = -\mathrm{id} \quad \text{and} \quad \bar{\sigma}_r|_{S_r} = \mathrm{id}.$$

Then L_r is 2-elementary by [Nik79a, Corollary I.5.2]. □

Proposition 4.3.4. *Given any embedding $L_{2k} \hookrightarrow L_{2,26}$, let $r \in L_{2k}$ be a primitive reflective vector with $r^2 = -2k$, and consider $L_r = r^{\perp}_{L_{2k}}$ as above. Under the chosen embedding, the orthogonal complement $(L_r)^{\perp}_{L_{2,26}}$ is isomorphic to either $D_{10}(-1)$ or $E_8(-1) \oplus 2A_1(-1)$.*

Proof. Since $L_{2,26}$ is unimodular, the discriminant groups of L_r and $(L_r)^{\perp}_{L_{2,26}}$ are isometric up to a sign. Proposition 4.3.3 thus implies that $(L_r)^{\perp}_{L_{2,26}}$ is a 2-elementary, negative definite lattice of rank 10 and determinant 4. By [Nik79a, Proposition 1.8.1], any 2-elementary discriminant form is isometric to a direct sum of finitely many quadratic forms, each of which is isometric to one of four quadratic forms, namely the discriminant forms of the 2-elementary lattices A_1, $A_1(-1)$, $U(2)$, D_4. Since $(L_r)^{\perp}_{L_{2,26}}$ has signature $-2 \pmod 8$ and determinant 4, it is immediate to see that its discriminant form must be isometric to the discriminant form of $2A_1(-1)$. Now we notice that the lattice $E_8(-1) \oplus 2A_1(-1)$ is a 2-elementary, negative definite lattice of rank 10 with the desired discriminant form. Finally, it is enough to compute the genus of $E_8(-1) \oplus 2A_1(-1)$. A quick check with the software `Magma` yields that the whole genus consists of $E_8(-1) \oplus 2A_1(-1)$ and $D_{10}(-1)$. Alternatively, we can use the Siegel mass formula [CS88] and check that the mass of the quadratic form f associated to the lattice $E_8(-1) \oplus 2A_1(-1)$ is

$$m(f) = \frac{5}{2^8 \cdot 4! \cdot 1814400} = \frac{1}{2229534720}.$$

We now notice that $D_{10}(-1)$ is in the genus of $E_8(-1) \oplus 2A_1(-1)$, as they have the same signature and their discriminant groups are isometric. Since the following equality holds:

$$\frac{1}{|\mathrm{O}(D_{10}(-1))|}+\frac{1}{|\mathrm{O}(E_8(-1)\oplus 2A_1(-1))|}=$$
$$=\frac{1}{3715891200}+\frac{1}{5573836800}=\frac{1}{2229534720}=m(f),$$

we can deduce that $\{D_{10}(-1), E_8(-1)\oplus 2A_1(-1)\}$ is the whole genus of $E_8(-1)\oplus 2A_1(-1)$. □

Now we are ready to prove Proposition 4.3.1.

Proof of Proposition 4.3.1. In order to prove that $\Phi|_{L_{2k}}$ vanishes along the ramification divisor of the projection π, we have to show that $\Phi|_{L_{2k}}$ vanishes on the $(-2k)$-divisors $\Omega_r(L_{2k})$ given by reflective vectors $r \in L_{2k}$ of norm $-2k$ (see Theorem 4.2.2), because $\Phi|_{L_{2k}}$ already vanishes on the (-2)-divisors by Theorem 4.2.3. Hence let r be a $(-2k)$-reflective vector. By Proposition 4.3.4, $(L_r)^{\perp}_{L_{2,26}}$ is a root lattice with at least 180 roots ($E_8(-1)\oplus 2A_1(-1)$ has 244 roots and $D_{10}(-1)$ has 180). On the other hand, since by assumption the orthogonal complement of L_{2k} in $L_{2,26}$ does not contain a copy of $E_8(-1)$, the root lattice generated by $R_{L_{2,26}}(L_{2k})$ has rank at most 9 and does not contain a copy of $E_8(-1)$. By going through finitely many possibilities for such root lattice, we obtain $|R_{L_{2,26}}(L_{2k})| \leq |\{\text{roots of } D_9\}| = 144$ (just recall that A_n has $n(n+1)$ roots, D_n has $2n(n-1)$ roots, E_6, E_7 have 72 and 126 roots respectively). Consequently, $\Phi|_{L_{2k}}$ vanishes along the $(-2k)$-divisor $\Omega_r(L_{2k})$ given by r with order $\geq (180-144)/2 > 0$, as claimed. □

4.4 Lattice engineering

In this section we reduce the proof of Theorem 4.0.1 to a problem of lattice engineering (see Problem 4.4.1) and we find a lower bound for the values of k such that $\mathcal{M}_{2k}$ is of general type (see Proposition 4.4.2). Then we deal with the (finitely many) remaining values of k by means of the geometry of K3 surfaces with Néron-Severi lattice isometric to $U \oplus E_8(-1)$.

Let $L_{2k} \hookrightarrow L_{2,26}$ be a primitive embedding. Since the embedding $U \oplus 2E_8(-1) \hookrightarrow L_{2,26}$ is unique up to isometry by [Nik79a, Theorem 1.14.4], we can assume that every summand of $U \oplus 2E_8(-1)$ is mapped identically onto the corresponding summand of $L_{2,26}$. Therefore, any choice of a primitive vector $l \in U \oplus E_8(-1)$ of norm $l^2 = 2k$ gives a primitive embedding

$$L_{2k} = U \oplus 2E_8(-1) \oplus \langle 2k \rangle \hookrightarrow L_{2,26}.$$

By Theorem 4.2.1 and 4.2.3 together with the previous discussion, we have transformed our original question of determining the Kodaira dimension of $\mathcal{M}_{2k}$ to the following:

Problem 4.4.1. For which $2k > 0$ does there exist a primitive vector $l \in U \oplus E_8(-1)$ with norm $l^2 = 2k$ such that l is orthogonal to at least 2 and at most 8 roots?

We want to find a lower bound for the values $2k$ answering Problem 4.4.1 positively (see Proposition 4.4.2). Since $U \oplus E_8(-1)$ contains infinitely many roots, we want to start by reducing to the more manageable case of $E_8(-1)$, whose number of roots is finite. For simplicity, we define

$$R(l) := \{r \in U \oplus E_8(-1) \ : \ r^2 = -2, (r,l) = 0\} = R_{L_{2,26}}(L_{2k}).$$

The following lemma is a slight generalisation of [TV19, Lemma 4.1,4.3].

Lemma 4.4.1. *[Pet19, Lemma 3.3 and 3.4] Let $l = \alpha e + \beta f + v$, where $U = \langle e, f \rangle$ such that $e^2 = f^2 = 0$ and $ef = 1$, $v \in E_8(-1)$ and $\alpha, \beta \in \mathbb{Z}$, with norm $l^2 = 2k > 0$. Let $r = \alpha' e + \beta' f + v'$ be a vector of $R(l)$, where $v' \in E_8(-1)$ and $\alpha', \beta' \in \mathbb{Z}$. If $\alpha \neq \beta$, $\alpha, \beta > \sqrt{k}$ and $\alpha\beta < \frac{5}{4}k$, then $\alpha' = \beta' = 0$.*

In other words, if $l = \alpha e + \beta f + v \in U \oplus E_8(-1)$ is a vector of norm $2k$ satisfying the assumptions of Lemma 4.4.1, then the roots of $U \oplus E_8(-1)$ orthogonal to l are roots of $E_8(-1)$. Therefore the set $R(l)$ coincides with the set of roots in $v^{\perp}_{E_8(-1)}$. The following lemma, inspired by [GHS07b, Theorem 7.1], controls the number of roots of $E_8(-1)$ orthogonal to v.

Lemma 4.4.2. *There exists $v \in E_8$ with $v^2 = 2n$ and such that $v^{\perp}_{E_8}$ contains at least 2 and at most 8 roots if the following inequality holds:*

$$2N_{E_7}(2n) > 28N_{E_6}(2n) + 63N_{D_6}(2n),$$

where $N_L(2n)$ denotes the number of representations of $2n$ by the positive definite lattice L, for $L = E_7, E_6, D_6$.

Proof. We follow closely [GHS07b, Theorem 7.1]. Let $a \in E_8$ be a root. Its orthogonal complement $E_7^{(a)} := a^{\perp}_{E_8}$ is isometric to E_7. The set of 240 roots in E_8 consists of the 126 roots in $E_7^{(a)}$ and 114 other roots, forming the subset X_{114}. Assume that every $v \in E_7^{(a)}$ with $v^2 = 2n$ is orthogonal to at least 10 roots in E_8, including $\pm a$. By [GHS07b, Lemma 7.2] we know that every such v is contained in the union

$$\bigcup_{i=1}^{28} (A_2^{(i)})^{\perp}_{E_8} \sqcup \bigcup_{j=1}^{63} (A_1^{(j)})^{\perp}_{E_7^{(a)}}, \tag{4.1}$$

where $A_2^{(i)}$ (resp. $A_1^{(j)}$) are root systems of type A_2 (resp. A_1) contained in X_{114} (resp. $E_7^{(a)}$). Denote by $n(v)$ the number of components in the union (4.1) containing v. Since $(A_2^{(i)})^{\perp}_{E_8} \cong E_6$ and $(A_1^{(j)})^{\perp}_{E_7^{(a)}} \cong D_6$, we have counted the vector v exactly $n(v)$ times in the sum

$$28N_{E_6}(2n) + 63N_{D_6}(2n).$$

We distinguish three cases.

(i) If $v \cdot c \neq 0$ for every $c \in X_{114} \setminus \{\pm a\}$, then v is orthogonal to at least 4 copies of A_1 in $E_7^{(a)}$, so $n(v) \geq 4$;

(ii) If v is orthogonal to only one $A_2^{(i)}$ (6 roots), then v is orthogonal to at least 2 copies of A_1 in $E_7^{(a)}$, so $n(v) \geq 3$;

(iii) If v is orthogonal to at least two $A_2^{(i)}$, then $n(v) \geq 2$.

In conclusion $n(v) \geq 2$ for every $v \in E_7^{(a)}$. Therefore, under our assumption that every $v \in E_7^{(a)}$ with $v^2 = 2n$ is orthogonal to at least 10 roots, we have shown that any such v is contained in at least 2 sets of the union (4.1), i.e.

$$2N_{E_7}(2n) \leq 28N_{E_6}(2n) + 63N_{D_6}(2n).$$

□

Proposition 4.4.1. *Let $n \geq 952$. Then there exists $v \in E_8(-1)$ with $v^2 = -2n$ such that $v^{\perp}_{E_8(-1)}$ contains at least 2 and at most 8 roots.*

Proof. [GHS07b, Equations (31), (33) and (34)] give the following estimates:

$$N_{E_7}(2n) > 123.8\, n^{5/2}, \qquad N_{E_6}(2n) < 103.69\, n^2, \qquad N_{D_6}(2n) < 75.13\, n^2.$$

By Lemma 4.4.2, we immediately obtain the claim. □

We are now ready to give a sufficient condition for answering Problem 4.4.1 positively:

Proposition 4.4.2. *Let $k \geq 4900$. Then there exists a primitive $l \in U \oplus E_8(-1)$ with $l^2 = 2k$ and $2 \leq |R(l)| \leq 8$.*

Proof. Pick $k > 0$ and consider $l = \alpha e + \beta f + v$, where $l^2 = 2k$, $v^2 = -2n$, so that $\alpha\beta = n + k$. Suppose that there exist α and β satisfying the hypotheses of Lemma 4.4.1 such that $n = \alpha\beta - k \geq 952$. Then Proposition 4.4.1 implies that we can find a $v \in E_8(-1)$ with $v^2 = -2n$ such that $v^{\perp}_{E_8(-1)}$ contains at least 2 and at most 8 roots. Moreover Lemma 4.4.1 ensures that the roots of $U \oplus E_8(-1)$ orthogonal to

$l = \alpha e + \beta f + v$ are contained in $E_8(-1)$, so that $l^{\perp}_{U \oplus E_8(-1)}$ also contains at least 2 and at most 8 roots. Therefore the existence of such α, β is sufficient for the existence of $l \in U \oplus E_8(-1)$ with $2 \leq |R(l)| \leq 8$.

Now let $k \geq 4900 = 70^2$, and consider

$$\alpha = \lceil \sqrt{k} + 6 \rceil, \qquad \beta = \alpha + 1.$$

Clearly $\alpha \neq \beta$, $\gcd(\alpha, \beta) = 1$ and $\alpha, \beta > \sqrt{k}$. Moreover,

$$\frac{5}{4}k - \alpha\beta \geq \frac{5}{4}k - (\sqrt{k}+7)(\sqrt{k}+8) = \frac{1}{4}k - 15\sqrt{k} - 56 > 0,$$

and

$$n = \alpha\beta - k \geq (\sqrt{k}+6)(\sqrt{k}+7) - k = 13\sqrt{k} + 42 \geq 952,$$

completing the proof. □

Now we want to tackle Problem 4.4.1 for the remaining values of k, i.e $k < 4900$. For this purpose, we make use of the geometry of K3 surfaces with Néron-Severi lattice isometric to $U \oplus E_8(-1)$. We start by recalling the main properties of such surfaces.

Let X be a K3 surface with $\mathrm{NS}(X) = U \oplus E_8(-1)$. Then X has finite automorphism group by [Nik79b, Theorem 0.2.2] (see also [Kon89]), and consequently a finite number of irreducible (-2)-curves by [Ste85, Proposition 2.5]. More precisely, if $|E|$ denotes the unique elliptic fibration on X, then the irreducible (-2)-curves on X are the 9 curves $C_2, \ldots, C_{10}$ contained in the unique reducible fibre of $|E|$, plus the unique section of E, which we will denote by C_1. The dual graph of such (-2)-curves is depicted in Figure 4.1, where $C_1, \ldots, C_7, C_9, C_{10}$ are the curves in the upper line, and C_8 is such that $C_7 C_8 = 1$.

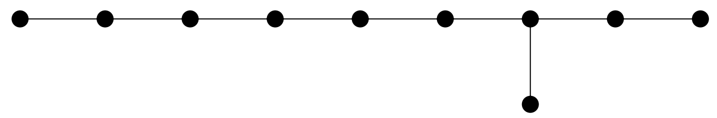


Figure 4.1: Dual graph of (-2)-curves

Now let $D \in \mathrm{NS}(X) = U \oplus E_8(-1)$ be a primitive divisor of X of norm $2k > 0$ with $2 \leq |R(D)| \leq 8$. In other words, $D^{\perp}$ contains at least 1 and at most 4 effective (-2)-divisors. The divisor D will play the role of the primitive vector $l \in U \oplus E_8(-1)$ from Problem 4.4.1. Up to the action of the Weyl group $W \subseteq \mathrm{O}(U \oplus E_8(-1))$, we can assume that D is nef (see [Huy16, Corollary 8.2.11]), since the isometries of $U \oplus E_8(-1)$ do not change the number of orthogonal roots. The nef cone of X is rational polyhedral (see [Huy16, Corollary 8.4.7]), and is the dual cone of the cone spanned by the (-2)-curves. A basis of the nef cone can be computed by means of the software

Magma. It turns out that such basis $\{D_1, \dots, D_{10}\}$ is the dual basis of $\{C_1, \dots, C_{10}\}$, i.e. $D_iC_j = \delta_{ij}$ for all $1 \le i, j \le 10$. For instance, $D_1 = E$ defines the only elliptic fibration on X, so $D_1^2 = 0$ and $D_1D_i > 0$ for $i \ge 2$, and $D_i^2 > 0$ for $i \ge 2$. Hence any nef divisor D is a linear combination of $D_1, \dots, D_{10}$ with non-negative coefficients:

$$D = \sum_{i=1}^{10} d_i D_i.$$

By construction, the (-2)-curve C_j is orthogonal to D if and only if $d_j = 0$. This implies that the root part of $D^\perp$ is a root lattice R generated by the (-2)-curves $\{C_j \mid d_j = 0\}$. Since R contains at most 4 effective roots, it is isomorphic to one of the following root lattices:

$$A_1(-1),\ 2A_1(-1),\ 3A_1(-1),\ 4A_1(-1),\ A_2(-1),\ A_2(-1) \oplus A_1(-1). \tag{4.2}$$

Now choose one of the the finitely many sub-diagrams $J \subseteq \{1, \dots, 10\}$ of the dual graph in Figure 4.1 giving rise to a root lattice $\langle C_j \mid j \in J \rangle$ isometric to one of the lattices in (4.2). The nef divisors D orthogonal precisely to $\{C_j \mid j \in J\}$ are all those of the form:

$$D = \sum_{i \notin J} d_i D_i,$$

for some $d_i > 0$. Since we are only interested in divisors of norm $2k < 2 \cdot 4900$, we can use the inequality

$$D^2 \ge \sum_{i \notin J} d_i^2 D_i^2 + 2 \sum_{1 \ne i \notin J} d_1 d_i (D_1 D_i),$$

to bound the d_i for $i \notin J$. More precisely, we have that

$$d_i^2 \le \frac{2 \cdot 4900}{D_i^2}\ \forall i \ge 2, \quad \text{and} \quad d_1 \le \frac{4900}{\sum_{1 \ne i \notin J} D_1 D_i}.$$

By varying the coefficients d_i's in these ranges for every choice of the root sublattice $\langle C_j \mid j \in J \rangle$, we obtain all primitive vectors $D \in U \oplus E_8(-1)$ with $D^2 \le 2 \cdot 4900$ and $2 \le |R(D)| \le 8$ up to the action of $\mathrm{O}(U \oplus E_8(-1))$. Therefore this search is completely exhaustive.

A similar list can be obtained if we allow D to have up to 10 orthogonal roots. All the previous discussion works analogously, with the only difference that the root part of $D^\perp$ can also be isometric to $5A_1(-1)$ or $A_2(-1) \oplus 2A_1(-1)$ besides the lattices listed in (4.2).

We use the software Magma to implement the search described above and give the source code in Appendix A. We get the following result:

Proposition 4.4.3. *A primitive vector $l \in U \oplus E_8(-1)$ with $l^2 = 2k < 2 \cdot 4900$ and $2 \leq |R(l)| \leq 8$ exists if and only if*

$$k \geq 208,\ k \neq 211, 219 \text{ or } k \in \{170, 185, 186, 188, 190, 194, 200, 202, 204, 206\}. \tag{4.3}$$

Moreover, a similar vector l with $2 \leq |R(l)| \leq 10$ exists if and only if

$$k \geq 164,\ k \neq 169, 171, 175 \text{ or } k \in \{140, 146, 150, 152, 154, 155, 158, 160, 162\}. \tag{4.4}$$

4.4.1 The Kodaira dimension of $\mathcal{M}_{2k}$

We are now ready to prove Theorem 4.0.1.

Proof of Theorem 4.0.1. Proposition 4.4.2 combined with Proposition 4.4.3 ensures that there exists a primitive $l \in U \oplus E_8(-1)$ with norm $l^2 = 2k$ and $2 \leq |R(l)| \leq 8$ if $k \geq 4900$ or k belongs to the list (4.3), in particular for any $k \geq 220$. Such an $l \in U \oplus E_8(-1)$ determines an embedding $L_{2k} \hookrightarrow L_{2,26}$ with the property

$$1 \leq N(L_{2k}) \leq 4,$$

where $N(L_{2k})$ is the number of effective roots in the orthogonal complement $(L_{2k})^{\perp}_{L_{2,26}}$. Hence Theorem 4.2.3 provides a non-zero cusp form $\Phi|_{L_{2k}}$ of weight $12 + N(L_{2k}) \leq 12 + 4 < 17 = \dim(\mathcal{M}_{2k})$. Moreover, this cusp form vanishes along the ramification divisor of $\pi : \Omega_{L_{2k}} \to \mathcal{M}_{2k}$ in view of Proposition 4.3.1, since $l^{\perp}$ does not contain $E_8(-1)$, otherwise l would be orthogonal to at least 240 roots. Then the low-weight cusp form trick (Theorem 4.2.1) ensures that $\mathcal{M}_{2k}$ is of general type.

An analogous argument shows that $\mathcal{M}_{2k}$ has non-negative Kodaira dimension if k belongs to the list (4.4) in Proposition 4.4.3, in particular for any $k \geq 176$. $\square$

4.5 Unirationality of $\mathcal{M}_{2k}$ for small k

For the sake of completeness, we state the result of the second part of [FM20] and of [FHM20], where it is proved that $\mathcal{M}_{2k}$ is unirational for small values of k.

Theorem 4.5.1. *The moduli space $\mathcal{M}_{2k}$ is unirational for $k \leq 50$, $k \notin \{11, 35, 42, 48\}$ and for the following values of k:*

$$\{52, 53, 54, 59, 60, 61, 62, 64, 68, 69, 73, 79, 81, 94, 97\}.$$

The strategy of the proof consists of constructing explicit projective models of $U \oplus \langle -2k \rangle$-polarized K3 surfaces, giving rise to a unirational parameter space $\mathcal{P}_{2k} \dashrightarrow \mathcal{M}_{2k}$. Some of the projective models we considered are Weierstrass fibrations, double coverings of $\mathbb{P}^2$ branched over a sextic curve, and double coverings of $\mathbb{P}^1 \times \mathbb{P}^1$

branched over a $(4,4)$-curve. Moreover, we studied all $U \oplus \langle -2k \rangle$-polarized K3 surfaces that can be realised as complete intersections of degree 4 in $\mathbb{P}^3$, 6 in $\mathbb{P}^4$ and 8 in $\mathbb{P}^5$, and contain two smooth rational or elliptic curves of suitable degrees and intersection number. The search for these projective models amounts to looking for lattice isomorphisms between $U \oplus \langle -2k \rangle$ and the Néron-Severi lattice of the general K3 surface described above. Then, we need to construct the parameter space $\mathcal{P}_{2k}$, which takes into account the projective embedding of the K3 surfaces together with the relevant curves. We also prove that $\mathcal{P}_{2k}$ is unirational: it is typically a projective bundle, and more generally an iterated Grassmannian, over a unirational variety. Finally, we prove that the map $\mathcal{P}_{2k} \dashrightarrow \mathcal{M}_{2k}$, sending a projective model of a K3 surface to its isomorphism class, is dominant, which leads to the unirationality of $\mathcal{M}_{2k}$ itself.

A | Appendix

We report the source code in `Magma` used to prove Proposition 4.4.3.

```
%Intersection matrix U+E_8(-1)
Int:=Matrix([
    [ -2, 1, 0, 0, 0, 0, 0, 0, 0, 0 ],
    [ 1, -2, 1, 0, 0, 0, 0, 0, 0, 0 ],
    [ 0, 1, -2, 1, 0, 0, 0, 0, 0, 0 ],
    [ 0, 0, 1, -2, 1, 0, 0, 0, 0, 0 ],
    [ 0, 0, 0, 1, -2, 1, 0, 0, 0, 0 ],
    [ 0, 0, 0, 0, 1, -2, 1, 0, 0, 0 ],
    [ 0, 0, 0, 0, 0, 1, -2, 1, 1, 0 ],
    [ 0, 0, 0, 0, 0, 0, 1, -2, 0, 0 ],
    [ 0, 0, 0, 0, 0, 0, 1, 0, -2, 1 ],
    [ 0, 0, 0, 0, 0, 0, 0, 0, 1, -2 ]
]);

%Euclidean lattice in ZZ^{10}
Euc:=LatticeWithGram(IdentityMatrix(Integers(),10));

%Generators of Nef cone of U+E_8(-1)
D:=[];
D[1]:=Vector([ 0, 1, 2, 3, 4, 5, 6, 3, 4, 2 ]);
D[2]:=Vector([ 1, 2, 4, 6, 8, 10, 12, 6, 8, 4]);
D[10]:=Vector([ 2, 4, 6, 8, 10, 12, 14, 7, 9, 4]);
D[3]:=Vector([ 2, 4, 6, 9, 12, 15, 18, 9, 12, 6]);
D[8]:=Vector([ 3, 6, 9, 12, 15, 18, 21, 10, 14, 7]);
D[4]:=Vector([ 3, 6, 9, 12, 16, 20, 24, 12, 16, 8]);
D[9]:=Vector([ 4, 8, 12, 16, 20, 24, 28, 14, 18, 9]);
D[5]:=Vector([ 4, 8, 12, 16, 20, 25, 30, 15, 20, 10]);
D[6]:=Vector([ 5, 10, 15, 20, 25, 30, 36, 18, 24, 12]);
D[7]:=Vector([ 6, 12, 18, 24, 30, 36, 42, 21, 28, 14]);

%Intersection pairing in U+E_8(-1)
function Product(v,w)
v1:=CoordinatesToElement(Euc,(v*Int));
v2:=CoordinatesToElement(Euc,w);
return (v1,v2);
end function;

%Initialize final lists
FinalListVectors:=[];
FinalListNorms:=[];

%List of all subgraphs of roots of U+E_8(-1) of type 4A_1(-1)
ListJ:=[];
for j1 in [1..10] do
```

```
J2:={ j2 : j2 in [j1..10] | Int[j1,j2] eq 0};
for j2 in J2 do
J3:={ j3 : j3 in [j2..10] | (Int[j1,j3] eq 0) and (Int[j2,j3] eq 0)};
for j3 in J3 do
J4:={ j4 : j4 in [j3..10] | (Int[j1,j4] eq 0) and (Int[j2,j4] eq 0) and (Int[j3,j4] eq
    0)};
for j4 in J4 do
Append(~ListJ,[j1,j2,j3,j4]);
end for;
end for;
end for;
end for;

%Maximum coefficients for a nef divisor if k<=4900
MAX:=[0,14,11,10,8,8,7,7,6,6];

%Compute all the values of k<=4900 s.t. there is a nef divisor of norm -2k and
    orthogonal root part 4A_1(-1)
for J in ListJ do

if (1 in J) then
Now:=[i : i in [1..10] | not (i in J)];
MAXNow:=MAX[Now];
List:=[d1*D[Now[1]]+d2*D[Now[2]]+d3*D[Now[3]]+d4*D[Now[4]]+d5*D[Now[5]]+d6*D[Now[6]] :
    d1 in [1..MAXNow[1]],d2 in [1..MAXNow[2]],d3 in [1..MAXNow[3]],d4 in [1..MAXNow[4]],
    d5 in [1..MAXNow[5]],d6 in [1..MAXNow[6]] | Product(d1*D[Now[1]]+d2*D[Now[2]]+d3*D[
    Now[3]]+d4*D[Now[4]]+d5*D[Now[5]]+d6*D[Now[6]],d1*D[Now[1]]+d2*D[Now[2]]+d3*D[Now
    [3]]+d4*D[Now[4]]+d5*D[Now[5]]+d6*D[Now[6]]) le 9800];
end if;

if not (1 in J) then
Now:=[i : i in [1..10] | not (i in J)];
MAXNow:=MAX[Now];
List:=[d1*D[Now[1]]+d2*D[Now[2]]+d3*D[Now[3]]+d4*D[Now[4]]+d5*D[Now[5]]+d6*D[Now[6]] :
    d1 in [1..45],d2 in [1..MAXNow[2]],d3 in [1..MAXNow[3]],d4 in [1..MAXNow[4]],d5 in
    [1..MAXNow[5]],d6 in [1..MAXNow[6]] | Product(d1*D[Now[1]]+d2*D[Now[2]]+d3*D[Now
    [3]]+d4*D[Now[4]]+d5*D[Now[5]]+d6*D[Now[6]],d1*D[Now[1]]+d2*D[Now[2]]+d3*D[Now[3]]+
    d4*D[Now[4]]+d5*D[Now[5]]+d6*D[Now[6]]) le 9800];
end if;

%Update final list
for D in List do
k:=Integers()!(Product(D,D)/2);
if not (k in FinalListNorms) then
Append(~FinalListVectors,D);
Append(~FinalListNorms,k);
end if;

end for;
end for;

%After running this code the maximum k not appearing in the final list is 235

%List of all subgraphs of roots of U+E_8(-1) of type 3A_1(-1)
ListJ:=[];
for j1 in [1..10] do
J2:={ j2 : j2 in [j1..10] | Int[j1,j2] eq 0};
for j2 in J2 do
```

```
J3:={ j3 : j3 in [j2..10] | (Int[j1,j3] eq 0) and (Int[j2,j3] eq 0)};
for j3 in J3 do
Append(~ListJ,[j1,j2,j3]);
end for;
end for;
end for;

%Maximum coefficients for a nef divisor if k<=235
MAX:=[0,10,7,6,4,4,3,3,2,2];

%Compute all the values of k<=235 s.t. there is a nef divisor of norm -2k and orthogonal
     root part 3A_1(-1)
for J in ListJ do

if (1 in J) then
Now:=[i : i in [1..10] | not (i in J)];
MAXNow:=MAX[Now];
List:=[d1*D[Now[1]]+d2*D[Now[2]]+d3*D[Now[3]]+d4*D[Now[4]]+d5*D[Now[5]]+d6*D[Now[6]]+d7*
    D[Now[7]] : d1 in [1..MAXNow[1]],d2 in [1..MAXNow[2]],d3 in [1..MAXNow[3]],d4 in
    [1..MAXNow[4]],d5 in [1..MAXNow[5]],d6 in [1..MAXNow[6]],d7 in [1..MAXNow[7]] |
    Product(d1*D[Now[1]]+d2*D[Now[2]]+d3*D[Now[3]]+d4*D[Now[4]]+d5*D[Now[5]]+d6*D[Now
    [6]]+d7*D[Now[7]],d1*D[Now[1]]+d2*D[Now[2]]+d3*D[Now[3]]+d4*D[Now[4]]+d5*D[Now[5]]+
    d6*D[Now[6]]+d7*D[Now[7]]) le 470];
end if;

if not (1 in J) then
Now:=[i : i in [1..10] | not (i in J)];
MAXNow:=MAX[Now];
List:=[d1*D[Now[1]]+d2*D[Now[2]]+d3*D[Now[3]]+d4*D[Now[4]]+d5*D[Now[5]]+d6*D[Now[6]]+d7*
    D[Now[7]] : d1 in [1..15],d2 in [1..MAXNow[2]],d3 in [1..MAXNow[3]],d4 in [1..MAXNow
    [4]],d5 in [1..MAXNow[5]],d6 in [1..MAXNow[6]],d7 in [1..MAXNow[7]] | Product(d1*D[
    Now[1]]+d2*D[Now[2]]+d3*D[Now[3]]+d4*D[Now[4]]+d5*D[Now[5]]+d6*D[Now[6]]+d7*D[Now
    [7]],d1*D[Now[1]]+d2*D[Now[2]]+d3*D[Now[3]]+d4*D[Now[4]]+d5*D[Now[5]]+d6*D[Now[6]]+
    d7*D[Now[7]]) le 470];
end if;

%Update final list
for D in List do
k:=Integers()!(Product(D,D)/2);
if not (k in FinalListNorms) then
Append(~FinalListVectors,D);
Append(~FinalListNorms,k);
end if;

end for;
end for;

%List of all subgraphs of roots of U+E_8(-1) of type 2A_1(-1)
ListJ:=[];
for j1 in [1..10] do
J2:={ j2 : j2 in [j1..10] | Int[j1,j2] eq 0};
for j2 in J2 do
Append(~ListJ,[j1,j2]);
end for;
end for;

%Compute all the values of k<=235 s.t. there is a nef divisor of norm -2k and orthogonal
     root part 2A_1(-1)
```

```
for J in ListJ do

if (1 in J) then
Now:=[i : i in [1..10] | not (i in J)];
MAXNow:=MAX[Now];
List:=[d1*D[Now[1]]+d2*D[Now[2]]+d3*D[Now[3]]+d4*D[Now[4]]+d5*D[Now[5]]+d6*D[Now[6]]+d7*
    D[Now[7]]+d8*D[Now[8]] : d1 in [1..MAXNow[1]],d2 in [1..MAXNow[2]],d3 in [1..MAXNow
    [3]],d4 in [1..MAXNow[4]],d5 in [1..MAXNow[5]],d6 in [1..MAXNow[6]],d7 in [1..MAXNow
    [7]],d8 in [1..MAXNow[8]] | Product(d1*D[Now[1]]+d2*D[Now[2]]+d3*D[Now[3]]+d4*D[Now
    [4]]+d5*D[Now[5]]+d6*D[Now[6]]+d7*D[Now[7]]+d8*D[Now[8]],d1*D[Now[1]]+d2*D[Now[2]]+
    d3*D[Now[3]]+d4*D[Now[4]]+d5*D[Now[5]]+d6*D[Now[6]]+d7*D[Now[7]]+d8*D[Now[8]]) le
    470];
end if;

if not (1 in J) then
Now:=[i : i in [1..10] | not (i in J)];
MAXNow:=MAX[Now];
List:=[d1*D[Now[1]]+d2*D[Now[2]]+d3*D[Now[3]]+d4*D[Now[4]]+d5*D[Now[5]]+d6*D[Now[6]]+d7*
    D[Now[7]]+d8*D[Now[8]] : d1 in [1..15],d2 in [1..MAXNow[2]],d3 in [1..MAXNow[3]],d4
    in [1..MAXNow[4]],d5 in [1..MAXNow[5]],d6 in [1..MAXNow[6]],d7 in [1..MAXNow[7]],d8
    in [1..MAXNow[8]] | Product(d1*D[Now[1]]+d2*D[Now[2]]+d3*D[Now[3]]+d4*D[Now[4]]+d5*D
    [Now[5]]+d6*D[Now[6]]+d7*D[Now[7]]+d8*D[Now[8]],d1*D[Now[1]]+d2*D[Now[2]]+d3*D[Now
    [3]]+d4*D[Now[4]]+d5*D[Now[5]]+d6*D[Now[6]]+d7*D[Now[7]]+d8*D[Now[8]]) le 470];
end if;

%Update final list
for D in List do
k:=Integers()!(Product(D,D)/2);
if not (k in FinalListNorms) then
Append(~FinalListVectors,D);
Append(~FinalListNorms,k);
end if;

end for;
end for;

%List of all subgraphs of roots of U+E_8(-1) of type A_1(-1)
ListJ:=[];
for j1 in [1..10] do
Append(~ListJ,[j1]);
end for;

%Compute all the values of k<=235 s.t. there is a nef divisor of norm -2k and orthogonal
     root part A_1(-1)
for J in ListJ do

if (1 in J) then
Now:=[i : i in [1..10] | not (i in J)];
MAXNow:=MAX[Now];
List:=[d1*D[Now[1]]+d2*D[Now[2]]+d3*D[Now[3]]+d4*D[Now[4]]+d5*D[Now[5]]+d6*D[Now[6]]+d7*
    D[Now[7]]+d8*D[Now[8]] : d1 in [1..MAXNow[1]],d2 in [1..MAXNow[2]],d3 in [1..MAXNow
    [3]],d4 in [1..MAXNow[4]],d5 in [1..MAXNow[5]],d6 in [1..MAXNow[6]],d7 in [1..MAXNow
    [7]],d8 in [1..MAXNow[8]] | Product(d1*D[Now[1]]+d2*D[Now[2]]+d3*D[Now[3]]+d4*D[Now
    [4]]+d5*D[Now[5]]+d6*D[Now[6]]+d7*D[Now[7]]+d8*D[Now[8]],d1*D[Now[1]]+d2*D[Now[2]]+
    d3*D[Now[3]]+d4*D[Now[4]]+d5*D[Now[5]]+d6*D[Now[6]]+d7*D[Now[7]]+d8*D[Now[8]]) le
    470];
end if;
```

```
if not (1 in J) then
Now:=[i : i in [1..10] | not (i in J)];
MAXNow:=MAX[Now];
List:=[d1*D[Now[1]]+d2*D[Now[2]]+d3*D[Now[3]]+d4*D[Now[4]]+d5*D[Now[5]]+d6*D[Now[6]]+d7*
    D[Now[7]]+d8*D[Now[8]] : d1 in [1..15],d2 in [1..MAXNow[2]],d3 in [1..MAXNow[3]],d4
    in [1..MAXNow[4]],d5 in [1..MAXNow[5]],d6 in [1..MAXNow[6]],d7 in [1..MAXNow[7]],d8
    in [1..MAXNow[8]] | Product(d1*D[Now[1]]+d2*D[Now[2]]+d3*D[Now[3]]+d4*D[Now[4]]+d5*D
    [Now[5]]+d6*D[Now[6]]+d7*D[Now[7]]+d8*D[Now[8]],d1*D[Now[1]]+d2*D[Now[2]]+d3*D[Now
    [3]]+d4*D[Now[4]]+d5*D[Now[5]]+d6*D[Now[6]]+d7*D[Now[7]]+d8*D[Now[8]]) le 470];
end if;

%Update final list
for D in List do
k:=Integers()!(Product(D,D)/2);
if not (k in FinalListNorms) then
Append(~FinalListVectors,D);
Append(~FinalListNorms,k);
end if;

end for;
end for;

%List of all subgraphs of roots of U+E_8(-1) of type A_1(-1)+A_2(-1)
ListJ:=[];
for j1 in [1..10] do
J2:={ j2 : j2 in [j1..10] | Int[j1,j2] eq 1};
for j2 in J2 do
J3:={ j3 : j3 in [1..10] | (Int[j1,j3] eq 0) and (Int[j2,j3] eq 0)};
for j3 in J3 do
Append(~ListJ,[j1,j2,j3]);
end for;
end for;
end for;

%Compute all the values of k<=235 s.t. there is a nef divisor of norm -2k and orthogonal
     root part A_1(-1)+A_2(-1)
for J in ListJ do

if (1 in J) then
Now:=[i : i in [1..10] | not (i in J)];
MAXNow:=MAX[Now];
List:=[d1*D[Now[1]]+d2*D[Now[2]]+d3*D[Now[3]]+d4*D[Now[4]]+d5*D[Now[5]]+d6*D[Now[6]]+d7*
    D[Now[7]] : d1 in [1..MAXNow[1]],d2 in [1..MAXNow[2]],d3 in [1..MAXNow[3]],d4 in
    [1..MAXNow[4]],d5 in [1..MAXNow[5]],d6 in [1..MAXNow[6]],d7 in [1..MAXNow[7]] |
    Product(d1*D[Now[1]]+d2*D[Now[2]]+d3*D[Now[3]]+d4*D[Now[4]]+d5*D[Now[5]]+d6*D[Now
    [6]]+d7*D[Now[7]],d1*D[Now[1]]+d2*D[Now[2]]+d3*D[Now[3]]+d4*D[Now[4]]+d5*D[Now[5]]+
    d6*D[Now[6]]+d7*D[Now[7]]) le 470];
end if;

if not (1 in J) then
Now:=[i : i in [1..10] | not (i in J)];
MAXNow:=MAX[Now];
List:=[d1*D[Now[1]]+d2*D[Now[2]]+d3*D[Now[3]]+d4*D[Now[4]]+d5*D[Now[5]]+d6*D[Now[6]]+d7*
    D[Now[7]] : d1 in [1..15],d2 in [1..MAXNow[2]],d3 in [1..MAXNow[3]],d4 in [1..MAXNow
    [4]],d5 in [1..MAXNow[5]],d6 in [1..MAXNow[6]],d7 in [1..MAXNow[7]] | Product(d1*D[
    Now[1]]+d2*D[Now[2]]+d3*D[Now[3]]+d4*D[Now[4]]+d5*D[Now[5]]+d6*D[Now[6]]+d7*D[Now
    [7]],d1*D[Now[1]]+d2*D[Now[2]]+d3*D[Now[3]]+d4*D[Now[4]]+d5*D[Now[5]]+d6*D[Now[6]]+
    d7*D[Now[7]]) le 470];
```

```
end if;

%Update final list
for D in List do
k:=Integers()!(Product(D,D)/2);
if not (k in FinalListNorms) then
Append(~FinalListVectors,D);
Append(~FinalListNorms,k);
end if;

end for;
end for;

%List of all subgraphs of roots of U+E_8(-1) of type A_2(-1)
ListJ:=[];
for j1 in [1..10] do
J2:={ j2 : j2 in [j1..10] | Int[j1,j2] eq 1};
for j2 in J2 do
Append(~ListJ,[j1,j2]);
end for;
end for;

%Compute all the values of k<=235 s.t. there is a nef divisor of norm -2k and orthogonal
     root part A_2(-1)
for J in ListJ do

if (1 in J) then
Now:=[i : i in [1..10] | not (i in J)];
MAXNow:=MAX[Now];
List:=[d1*D[Now[1]]+d2*D[Now[2]]+d3*D[Now[3]]+d4*D[Now[4]]+d5*D[Now[5]]+d6*D[Now[6]]+d7*
    D[Now[7]]+d8*D[Now[8]] : d1 in [1..MAXNow[1]],d2 in [1..MAXNow[2]],d3 in [1..MAXNow
    [3]],d4 in [1..MAXNow[4]],d5 in [1..MAXNow[5]],d6 in [1..MAXNow[6]],d7 in [1..MAXNow
    [7]],d8 in [1..MAXNow[8]] | Product(d1*D[Now[1]]+d2*D[Now[2]]+d3*D[Now[3]]+d4*D[Now
    [4]]+d5*D[Now[5]]+d6*D[Now[6]]+d7*D[Now[7]]+d8*D[Now[8]],d1*D[Now[1]]+d2*D[Now[2]]+
    d3*D[Now[3]]+d4*D[Now[4]]+d5*D[Now[5]]+d6*D[Now[6]]+d7*D[Now[7]]+d8*D[Now[8]]) le
    470];
end if;

if not (1 in J) then
Now:=[i : i in [1..10] | not (i in J)];
MAXNow:=MAX[Now];
List:=[d1*D[Now[1]]+d2*D[Now[2]]+d3*D[Now[3]]+d4*D[Now[4]]+d5*D[Now[5]]+d6*D[Now[6]]+d7*
    D[Now[7]]+d8*D[Now[8]] : d1 in [1..15],d2 in [1..MAXNow[2]],d3 in [1..MAXNow[3]],d4
    in [1..MAXNow[4]],d5 in [1..MAXNow[5]],d6 in [1..MAXNow[6]],d7 in [1..MAXNow[7]],d8
    in [1..MAXNow[8]] | Product(d1*D[Now[1]]+d2*D[Now[2]]+d3*D[Now[3]]+d4*D[Now[4]]+d5*D
    [Now[5]]+d6*D[Now[6]]+d7*D[Now[7]]+d8*D[Now[8]],d1*D[Now[1]]+d2*D[Now[2]]+d3*D[Now
    [3]]+d4*D[Now[4]]+d5*D[Now[5]]+d6*D[Now[6]]+d7*D[Now[7]]+d8*D[Now[8]]) le 470];
end if;

%Update final list
for D in List do
k:=Integers()!(Product(D,D)/2);
if not (k in FinalListNorms) then
Append(~FinalListVectors,D);
Append(~FinalListNorms,k);
end if;

end for;
```

```
end for;

%The values of k not appearing in FinalListNorms are precisely those of Proposition
    4.4.3 for general type

%We repeat the same process for non-negative Kodaira dimension
FinalListNorms:=[];
FinalListVectors2:=[];

%List of all subgraphs of roots of U+E_8(-1) of type 5A_1(-1)
ListJ:=[];
for j1 in [1..10] do
J2:={ j2 : j2 in [j1..10] | Int[j1,j2] eq 0};
for j2 in J2 do
J3:={ j3 : j3 in [j2..10] | (Int[j1,j3] eq 0) and (Int[j2,j3] eq 0)};
for j3 in J3 do
J4:={ j4 : j4 in [j3..10] | (Int[j1,j4] eq 0) and (Int[j2,j4] eq 0) and (Int[j3,j4] eq
    0)};
for j4 in J4 do
J5:={ j5 : j5 in [j4..10] | (Int[j1,j5] eq 0) and (Int[j2,j5] eq 0) and (Int[j3,j5] eq
    0) and (Int[j4,j5] eq 0)};
for j5 in J5 do
Append(~ListJ,[j1,j2,j3,j4,j5]);
end for;
end for;
end for;
end for;
end for;

%Compute all the values of k<=235 s.t. there is a nef divisor of norm -2k and orthogonal
     root part 5A_1(-1)
for J in ListJ do

if (1 in J) then
Now:=[i : i in [1..10] | not (i in J)];
MAXNow:=MAX[Now];
List:=[d1*D[Now[1]]+d2*D[Now[2]]+d3*D[Now[3]]+d4*D[Now[4]]+d5*D[Now[5]] : d1 in [1..
    MAXNow[1]],d2 in [1..MAXNow[2]],d3 in [1..MAXNow[3]],d4 in [1..MAXNow[4]],d5 in [1..
    MAXNow[5]] | Product(d1*D[Now[1]]+d2*D[Now[2]]+d3*D[Now[3]]+d4*D[Now[4]]+d5*D[Now
    [5]],d1*D[Now[1]]+d2*D[Now[2]]+d3*D[Now[3]]+d4*D[Now[4]]+d5*D[Now[5]]) le 440];
end if;

if not (1 in J) then
Now:=[i : i in [1..10] | not (i in J)];
MAXNow:=MAX[Now];
List:=[d1*D[Now[1]]+d2*D[Now[2]]+d3*D[Now[3]]+d4*D[Now[4]]+d5*D[Now[5]] : d1 in [1..25],
    d2 in [1..MAXNow[2]],d3 in [1..MAXNow[3]],d4 in [1..MAXNow[4]],d5 in [1..MAXNow[5]]
    | Product(d1*D[Now[1]]+d2*D[Now[2]]+d3*D[Now[3]]+d4*D[Now[4]]+d5*D[Now[5]],d1*D[Now
    [1]]+d2*D[Now[2]]+d3*D[Now[3]]+d4*D[Now[4]]+d5*D[Now[5]]) le 440];
end if;

%Update final list
for D in List do
k:=Integers()!(Product(D,D)/2);
if not (k in FinalListNorms) then
Append(~FinalListVectors2,D);
Append(~FinalListNorms,k);
end if;
```

```
end for;
end for;

%List of all subgraphs of roots of U+E_8(-1) of type 2A_1(-1)+A_2(-1)
ListJ:=[];
for j1 in [1..10] do
J2:={ j2 : j2 in [j1..10] | Int[j1,j2] eq 1};
for j2 in J2 do
J3:={ j3 : j3 in [1..10] | (Int[j1,j3] eq 0) and (Int[j2,j3] eq 0)};
for j3 in J3 do
J4:={ j4 : j4 in [j3..10] | (Int[j1,j4] eq 0) and (Int[j2,j4] eq 0) and (Int[j3,j4] eq
    0)};
for j4 in J4 do
Append(~ListJ,[j1,j2,j3,j4]);
end for;
end for;
end for;
end for;

%Compute all the values of k<=235 s.t. there is a nef divisor of norm -2k and orthogonal
     root part 2A_1(-1)+A_2(-1)
for J in ListJ do

if (1 in J) then
Now:=[i : i in [1..10] | not (i in J)];
MAXNow:=MAX[Now];
List:=[d1*D[Now[1]]+d2*D[Now[2]]+d3*D[Now[3]]+d4*D[Now[4]]+d5*D[Now[5]]+d6*D[Now[6]] :
    d1 in [1..MAXNow[1]],d2 in [1..MAXNow[2]],d3 in [1..MAXNow[3]],d4 in [1..MAXNow[4]],
    d5 in [1..MAXNow[5]],d6 in [1..MAXNow[6]] | Product(d1*D[Now[1]]+d2*D[Now[2]]+d3*D[
    Now[3]]+d4*D[Now[4]]+d5*D[Now[5]]+d6*D[Now[6]],d1*D[Now[1]]+d2*D[Now[2]]+d3*D[Now
    [3]]+d4*D[Now[4]]+d5*D[Now[5]]+d6*D[Now[6]]) le 440];
end if;

if not (1 in J) then
Now:=[i : i in [1..10] | not (i in J)];
MAXNow:=MAX[Now];
List:=[d1*D[Now[1]]+d2*D[Now[2]]+d3*D[Now[3]]+d4*D[Now[4]]+d5*D[Now[5]]+d6*D[Now[6]] :
    d1 in [1..20],d2 in [1..MAXNow[2]],d3 in [1..MAXNow[3]],d4 in [1..MAXNow[4]],d5 in
    [1..MAXNow[5]],d6 in [1..MAXNow[6]] | Product(d1*D[Now[1]]+d2*D[Now[2]]+d3*D[Now
    [3]]+d4*D[Now[4]]+d5*D[Now[5]]+d6*D[Now[6]],d1*D[Now[1]]+d2*D[Now[2]]+d3*D[Now[3]]+
    d4*D[Now[4]]+d5*D[Now[5]]+d6*D[Now[6]]) le 440];
end if;

%Update final list
for D in List do
k:=Integers()!(Product(D,D)/2);
if not (k in FinalListNorms) then
Append(~FinalListVectors2,D);
Append(~FinalListNorms,k);
end if;

end for;
end for;

%The values of k not appearing in FinalListNorms are precisely those of Proposition
    4.4.3 for non-negative Kodaira dimension
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Bibliography

[AB83] M. F. Atiyah and R. Bott. The Yang-Mills equations over Riemann surfaces. *Philos. Trans. Roy. Soc. London Ser. A*, 308(1505):523–615, 1983.

[BB66] W. L. Baily and A. Borel. Compactification of arithmetic quotients of bounded symmetric domains. *Ann. Math. (2)*, 84:442–528, 1966.

[BBD82] A. A. Beĭlinson, J. Bernstein, and P. Deligne. Faisceaux pervers. In *Analysis and topology on singular spaces, I (Luminy, 1981)*, volume 100 of *Astérisque*, pages 5–171. Soc. Math. France, Paris, 1982.

[BHPVdV04] W. Barth, K. Hulek, C. Peters, and A. Van de Ven. *Compact complex surfaces*, volume 4 of *Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics*. Springer-Verlag, Berlin, second edition, 2004.

[Bor95] R. E. Borcherds. Automorphic forms on $O_{s+2,2}(\mathbb{R})$ and infinite products. *Invent. Math.*, 120(1):161–213, 1995.

[BT07] J. Bergström and O. Tommasi. The rational cohomology of $\overline{M_4}$. *Math. Ann.*, 338(1):207–239, 2007.

[CDL20] F. Cossec, I. Dolgachev, and C. Liedtke. Enriques surfaces I, 2020. Available at the webpage http://www.math.lsa.umich.edu/ idolga/Enriques1.May2020.pdf.

[CMGHL19] S. Casalaina-Martin, S. Grushevsky, K. Hulek, and R. Laza. Cohomology of the moduli space of cubic threefolds and its smooth models, 2019. To appear in Mem. Amer. Math. Soc., Available at arXiv:1904.08728.

[CMJL12] S. Casalaina-Martin, D. Jensen, and R. Laza. The geometry of the ball quotient model of the moduli space of genus four curves. In *Compact moduli spaces and vector bundles*, volume 564 of *Contemp. Math.*, pages 107–136. Amer. Math. Soc., Providence, RI, 2012.

[CMJL14] S. Casalaina-Martin, D. Jensen, and R. Laza. Log canonical models and variation of GIT for genus 4 canonical curves. *J. Algebraic Geom.*, 23(4):727–764, 2014.

[CS88] J. H. Conway and N. J. A. Sloane. Low-dimensional lattices. IV: The mass formula. *Proc. R. Soc. Lond., Ser. A*, 419(1857):259–286, 1988.

[Dol96] I. V. Dolgachev. Mirror symmetry for lattice polarized $K3$ surfaces. *J. Math. Sci., New York*, 81(3):2599–2630, 1996.

[Fed12] M. Fedorchuk. The final log canonical model of the moduli space of stable curves of genus 4. *Int. Math. Res. Not.*, 2012(24):5650–5672, 2012.

[FHM20] M. Fortuna, M. Hoff, and G. Mezzedimi. Unirational moduli spaces of some elliptic K3 surfaces, 2020. Available at arXiv:2008.12077.

[FM20] M. Fortuna and G. Mezzedimi. The Kodaira dimension of some moduli spaces of elliptic K3 surfaces, 2020. Available at arXiv:2003.10957.

[For18] M. Fortuna. Cohomology of the moduli space of non-hyperelliptic genus four curves, 2018. To appear in Ann. Inst. Fourier, Available at arXiv:1812.03817.

[For20] M. Fortuna. Cohomology of the moduli space of degree two Enriques surfaces, 2020. Available at arXiv:2008.06934.

[Fre83] E. Freitag. *Siegelsche Modulfunktionen*, volume 254. Springer, Berlin, 1983.

[Ful98] W. Fulton. *Intersection theory*, volume 2 of *Results in Mathematics and Related Areas.* Springer-Verlag, Berlin, second edition, 1998.

[GH78] P. Griffiths and J. Harris. *Principles of algebraic geometry*. Wiley-Interscience [John Wiley & Sons], New York, 1978. Pure and Applied Mathematics.

[GHS07a] V. Gritsenko, K. Hulek, and G. K. Sankaran. The Hirzebruch-Mumford volume for the orthogonal group and applications. *Doc. Math.*, 12:215–241, 2007.

[GHS07b] V. A. Gritsenko, K. Hulek, and G. K. Sankaran. The Kodaira dimension of the moduli of $K3$ surfaces. *Invent. Math.*, 169(3):519–567, 2007.

[GHS09] V. Gritsenko, K. Hulek, and G. K. Sankaran. Abelianisation of orthogonal groups and the fundamental group of modular varieties. *J. Algebra*, 322(2):463–478, 2009.

[GHS15] V. Gritsenko, K. Hulek, and G. K. Sankaran. Moduli of $K3$ surfaces and irreducible symplectic manifolds. In *Handbook of moduli. Volume I*, pages 459–526. Somerville, MA: International Press; Beijing: Higher Education Press, 2015.

[Has05] B. Hassett. Classical and minimal models of the moduli space of curves of genus two. In *Geometric methods in algebra and number theory*, pages 169–192. Basel: Birkhäuser, 2005.

[HH09] B. Hassett and D. Hyeon. Log canonical models for the moduli space of curves: the first divisorial contraction. *Trans. Am. Math. Soc.*, 361(8):4471–4489, 2009.

[HH13] B. Hassett and D. Hyeon. Log minimal model program for the moduli space of stable curves: the first flip. *Ann. Math. (2)*, 177(3):911–968, 2013.

[HL14] D. Hyeon and Y. Lee. A birational contraction of genus 2 tails in the moduli space of genus 4 curves I. *Int. Math. Res. Not.*, 2014(13):3735–3757, 2014.

[Hor78a] E. Horikawa. On the periods of Enriques surfaces. I. *Math. Ann.*, 234(1):73–88, 1978.

[Hor78b] E. Horikawa. On the periods of Enriques surfaces. II. *Math. Ann.*, 235(3):217–246, 1978.

[Huy16] D. Huybrechts. *Lectures on K3 surfaces.*, volume 158. Cambridge: Cambridge University Press, 2016.

[Kir84] F. Kirwan. Cohomology of quotients in symplectic and algebraic geometry. Mathematical Notes, 31. Princeton, New Jersey: Princeton University Press. III, 210 p. $ 17.50 (1984)., 1984.

[Kir85] F. Kirwan. Partial desingularisations of quotients of nonsingular varieties and their betti numbers. *Ann. of Math. (2)*, 122(1):41–85, 1985.

[Kir86] F. Kirwan. Rational intersection cohomology of quotient varieties. *Invent. Math.*, 86(3):471–505, 1986.

[Kir89] F. Kirwan. Moduli spaces of degree d hypersurfaces in $\mathbf{P}_n$. *Duke Math. J.*, 58(1):39–78, 1989.

[KL89] F. Kirwan and R. Lee. The cohomology of moduli spaces of $K3$ surfaces of degree 2. I. *Topology*, 28(4):495–516, 1989.

[Kol93] J. Kollár. Shafarevich maps and plurigenera of algebraic varieties. *Invent. Math.*, 113(1):177–215, 1993.

[Kon88] S. Kondō. On the Albanese variety of the moduli space of polarized K3 surfaces. *Invent. Math.*, 91(3):587–593, 1988.

[Kon89] S. Kondō. Algebraic K3 surfaces with finite automorphism groups. *Nagoya Math. J.*, 116:1–15, 1989.

[Kon93] S. Kondō. On the Kodaira dimension of the moduli space of K3 surfaces. *Compos. Math.*, 89(3):251–299, 1993.

[Kon99] S. Kondō. On the Kodaira dimension of the moduli space of $K3$ surfaces. II. *Compos. Math.*, 116(2):111–117, 1999.

[KW06] F. Kirwan and J. Woolf. *An introduction to intersection homology theory*. Chapman & Hall/CRC, Boca Raton, FL, second edition, 2006.

[Lej93] P. Lejarraga. The moduli of Weierstrass fibrations over $\mathbb{P}^1$: Rationality. *Rocky Mt. J. Math.*, 23(2):649–650, 1993.

[Ma21] S. Ma. Irregular cusps of orthogonal modular varieties. *ArXiv Preprint: arXiv:2101.02950*, 2021.

[MFK94] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*, volume 34 of *Results in Mathematics and Related Areas (2)*. Springer-Verlag, Berlin, third edition, 1994.

[Mir81] R. Miranda. The moduli of Weierstrass fibrations over $\mathbb{P}^1$. *Math. Ann.*, 255:379–394, 1981.

[Muk03] S. Mukai. *An introduction to invariants and moduli. Transl. by W. M. Oxbury*, volume 81. Cambridge: Cambridge University Press, 2003.

[Nik79a] V. V. Nikulin. Integral symmetric bilinear forms and some geometric applications. *Izv. Akad. Nauk SSSR, Ser. Mat.*, 43:111–177, 1979.

[Nik79b] V. V. Nikulin. On factor groups of the automorphism groups of hyperbolic forms modulo subgroups generated by 2-reflections. *Sov. Math., Dokl.*, 20:1156–1158, 1979.

[Pet19] J. Petok. Kodaira dimension of moduli of special $K3^{[2]}$-fourfolds of degree 2. *ArXiv Preprint: arXiv:1911.03435*, 2019.

[PS72] I. I. Pyatetskij-Shapiro and I. R. Shafarevich. A Torelli theorem for algebraic surfaces of type K3. *Math. USSR, Izv.*, 5:547–588, 1972.

[Sha81] J. Shah. Projective degenerations of Enriques' surfaces. *Math. Ann.*, 256(4):475–495, 1981.

[Ste85] H. Sterk. Finiteness results for algebraic K3 surfaces. *Math. Z.*, 189(4):507–513, 1985.

[Ste91] H. Sterk. Compactifications of the period space of Enriques surfaces. I. *Math. Z.*, 207(1):1–36, 1991.

[Ste95] H. Sterk. Compactifications of the period space of Enriques surfaces. II. *Math. Z.*, 220(3):427–444, 1995.

[Tom05] O. Tommasi. Rational cohomology of the moduli space of genus 4 curves. *Compos. Math.*, 141(2):359–384, 2005.

[TV19] S. Tanimoto and A. Várilly-Alvarado. Kodaira dimension of moduli of special cubic fourfolds. *J. Reine Angew. Math.*, 752:265–300, 2019.

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