



Supersymmetric hyperbolic Calogero-Sutherland models by gauging

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Abstract

Novel $\mathcal{N}=2$ and $\mathcal{N}=4$ supersymmetric extensions of the Calogero-Sutherland hyperbolic systems are obtained by gauging the $U(n)$ isometry of matrix superfield models. The bosonic core of the $\mathcal{N}=2$ models is the standard A_{n-1} Calogero-Sutherland hyperbolic system, whereas the $\mathcal{N}=4$ model contains additional semi-dynamical spin variables and is an extension of the $U(2)$ spin Calogero-Sutherland hyperbolic system. We construct two different versions of the $\mathcal{N}=4$ model, with and without the interacting center-of-mass coordinate in the bosonic sector.

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1. Introduction

Multi-particle Calogero models [1] occupy a distinguished place among the integrable systems. Most studied are the rational Calogero (or Calogero-Moser) models respecting one-dimensional conformal invariance (see [2] for a review). The \mathcal{N} -extended supersymmetric versions of such systems are addressed in numerous works (see, for example, [3–10] and the review

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[11]). Beyond $\mathcal{N}=2$ however, some difficulties in a direct construction of minimal extensions of n -particle models (with $\mathcal{N}n$ fermionic variables) were encountered (see, e.g., [12–14]). A different approach was put forward in [15] (see also [16,11]), where $\mathcal{N}=1, 2, 4$ supersymmetric extensions of the rational Calogero model were derived by a gauging procedure [17,18] applied to matrix superfield systems. Recently, such an approach was applied for obtaining new superconformal Calogero-Moser systems with deformed supersymmetry and intrinsic mass parameter [19,20]. A characteristic feature of the gauging approach is the presence of extra fermionic fields (as compared to a minimal extension) and, in the $\mathcal{N}=4$ case, of bosonic semi-dynamical spinning variables. Within the Hamiltonian formalism, such a type of matrix system with an extended set of fermionic fields was further utilized in [21–23] for deriving general- \mathcal{N} supersymmetric rational Calogero models.

Rational Calogero systems constitute only one of the six types of multi-particle integrable systems classified in [24,25]. In addition to two rational Calogero systems (with and without oscillator potential), there also exist the Calogero-Sutherland trigonometric system with a potential proportional to the inverse square of the sine of the coordinate differences, the Calogero-Sutherland hyperbolic system with a potential proportional to the square of the hyperbolic sine of the coordinate differences, the Calogero elliptic system with a potential proportional to the Weierstraß function of the coordinate differences, as well as a wide class of Toda systems.

To date, supersymmetric generalizations of the Calogero-Sutherland systems [26] are understood rather badly. Their $\mathcal{N}=2$ supersymmetric generalizations were constructed at the component level [27–31]. For a relation to the representations of diverse superalgebras see [32,33]. A restricted set of $\mathcal{N}=4$ Calogero-Sutherland models with purely quantum potentials was obtained recently in [14] for BC_n , F_4 and G_2 root systems.

In this paper we apply the gauging procedure to superfield matrix systems to obtain the $\mathcal{N}n^2$ -type extensions of the hyperbolic A_{n-1} Calogero-Sutherland model, up to the $\mathcal{N}=4$ case. Such a system represents a particular limit of the elliptical Calogero system. The latter was used for the non-perturbative description of the vacuum sector of $\mathcal{N}=2$, $d=4$ supersymmetric gauge theories with matter in the Seiberg-Witten setting (see, e.g., [34,35]).

The plan of the paper is as follows. In Section 2 we show how to obtain the standard bosonic Calogero-Sutherland hyperbolic system through the matrix-gauging procedure. In Section 3 we construct a superfield formulation of the $\mathcal{N}=2$ supersymmetric Calogero-Sutherland hyperbolic model. This model is, in some sense, a direct $\mathcal{N}=2$ superfield generalization of the bosonic matrix model of Section 2. In contrast to the models of [28–31,27], this new $\mathcal{N}=2$ Calogero-Sutherland hyperbolic model involves $2n^2$ fermions, like the $\mathcal{N}=2$ rational matrix models of [15,11,17,18]. In Section 4 we consider the $\mathcal{N}=4$ superfield model in the $d=1$ harmonic superspace formulation. It essentially includes semi-dynamical $U(2)$ spin variables and provides an $\mathcal{N}=4$ supersymmetric generalization of the $U(2)$ -spin Calogero-Sutherland hyperbolic system. We construct two different versions of the $\mathcal{N}=4$ model. The first one is a direct analog of the $\mathcal{N}=4$ $U(2)$ rational Calogero model of [15], with an unremovable dependence on the center-of-mass coordinate. The second one secures a full decoupling of the center of mass in the bosonic sector.

2. Hyperbolic Calogero-Sutherland model by gauging

Here we consider a modification of the matrix formulation of the n -particle rational Calogero model with $U(n)$ gauge symmetry given in [15] (see also [36,37]). This slight modification will allow us to derive the hyperbolic Calogero-Sutherland model. Like in the case of rational mod-

els, the basic tool will be an Hermitian matrix $d=1$ field. The use of the Hermitian matrix model for describing the Calogero-Sutherland hyperbolic system was earlier discussed in [24,25].¹ As shown in [2], the analogous construction for the Calogero-Sutherland trigonometric system requires employing unitary matrices and so is more complicated.²

The matrix model we will deal with is underlaid by the positive definite Hermitian $n \times n$ -matrix field

$$X(t) := \|X_a^b(t)\|, \quad (X_a^b)^* = X_b^a, \quad \det X \neq 0,$$

$a, b = 1, \dots, n$, and the complex $U(n)$ -spinor field

$$Z(t) := \|Z_a(t)\|, \quad \bar{Z}^a = (Z_a)^*.$$

It also involves n^2 gauge fields

$$A(t) := \|A_a^b(t)\|, \quad (A_a^b)^* = A_b^a.$$

The gauge-invariant action has the following form

$$S = \int dt L, \quad L = \frac{1}{2} \text{Tr} \left(X^{-1} \nabla X X^{-1} \nabla X \right) + \frac{i}{2} \left(\bar{Z} \nabla Z - \nabla \bar{Z} Z \right) + c \text{Tr} A, \quad (2.1)$$

where the covariant derivatives are

$$\nabla X = \dot{X} + i[A, X], \quad \nabla Z = \dot{Z} + iAZ, \quad \nabla \bar{Z} = \dot{\bar{Z}} - i\bar{Z}A. \quad (2.2)$$

The last term (Fayet-Iliopoulos term) includes only $U(1)$ gauge field, c being a real constant.

The action (2.1) is invariant with respect to the local $U(n)$ transformations, $g(\tau) \in U(n)$,

$$X \rightarrow gXg^\dagger, \quad Z \rightarrow gZ, \quad \bar{Z} \rightarrow \bar{Z}g^\dagger, \quad A \rightarrow gAg^\dagger + i\dot{g}g^\dagger. \quad (2.3)$$

Using the gauge transformations (2.3) and the standard representation $X = URU^\dagger$ for Hermitian matrix X , with U an unitary matrix and R a diagonal matrix, we can impose a (partial) gauge-fixing

$$X_a^b = 0, \quad a \neq b. \quad (2.4)$$

In the gauge (2.4) the matrix variable X takes the form

$$X_a^b = x_a \delta_a^b. \quad (2.5)$$

Then $[X, A]_a^b = (x_a - x_b)A_a^b$, and, therefore, $\text{Tr}[X, A] = 0$, $\text{Tr}(\dot{X}[X, A]) = 0$, $\text{Tr}([X, A][X, A]) = -\sum_{a,b} (x_a - x_b)^2 A_a^b A_b^a$. As a result, the Lagrangian (2.1) becomes

$$L = \frac{1}{2} \sum_{a,b} \left[\frac{\dot{x}_a \dot{x}_a}{(x_a)^2} + i(\bar{Z}^a \dot{Z}_a - \dot{\bar{Z}}^a Z_a) + \frac{(x_a - x_b)^2 A_a^b A_b^a}{x_a x_b} - 2\bar{Z}^a A_a^b Z_b + 2c A_a^a \right]. \quad (2.6)$$

The third term in (2.6) involves only non-diagonal elements of the matrix A , A_a^b with $a \neq b$. Therefore the Lagrangian (2.6) possesses a residual invariance under the gauge abelian $[U(1)]^n$ group with local parameters $\varphi_a(t)$:

¹ For an alternative derivation from free motion on the $GL(n, \mathbb{C})$ group manifold see [38].

² An extension of our approach to the trigonometric models will be considered elsewhere.

$$Z_a \rightarrow e^{i\varphi_a} Z_a, \quad \bar{Z}^a \rightarrow e^{-i\varphi_a} \bar{Z}^a, \quad A_a^a \rightarrow A_a^a - \dot{\varphi}_a \quad (\text{no sum over } a). \quad (2.7)$$

Using this residual gauge invariance, we can impose the further gauge condition

$$\bar{Z}^a = Z_a. \quad (2.8)$$

In this gauge, the second term in the Lagrangian (2.6) vanishes and the action takes the form

$$S = \frac{1}{2} \int dt \sum_{a,b} \left[\frac{\dot{x}_a \dot{x}_a}{(x_a)^2} + \frac{(x_a - x_b)^2 A_a^b A_b^a}{x_a x_b} - 2Z_a Z_b A_a^b + 2c A_a^a \right]. \quad (2.9)$$

The equation of motion for A_a^b amounts to the relations

$$A_a^b = \frac{x_a x_b}{(x_a - x_b)^2} Z_a Z_b \quad \text{for } a \neq b, \quad (2.10)$$

$$(Z_a)^2 = c \quad \forall a \quad (\text{no sum over } a). \quad (2.11)$$

The conditions (2.11) imply $c > 0$. Inserting (2.10) and (2.11) in the action (2.6), we obtain the final gauge-fixed action

$$S = \frac{1}{2} \int dt \left[\sum_a \frac{\dot{x}_a \dot{x}_a}{(x_a)^2} - \sum_{a \neq b} \frac{x_a x_b c^2}{(x_a - x_b)^2} \right]. \quad (2.12)$$

Introducing the variables q_a as

$$x_a = e^{q_a}, \quad (2.13)$$

we cast (2.1) in the form

$$S = \frac{1}{2} \int dt \left[\sum_a \dot{q}_a \dot{q}_a - \sum_{a \neq b} \frac{c^2}{4 \sinh^2 \frac{q_a - q_b}{2}} \right], \quad (2.14)$$

which is just the standard action of the hyperbolic Calogero-Sutherland system of the A_{n-1} -root type [26,24,25].

Note that imposing the gauge-invariant condition

$$\det X = 1 \quad (2.15)$$

in the action (2.1), we will end up with the hyperbolic Calogero-Sutherland system described by the action (2.14) without the center-of-mass coordinate, *i.e.* with $\sum_a q_a = 0$.

In the next sections we will consider supersymmetric generalizations of the bosonic construction presented here. We will limit our study to the most interesting $\mathcal{N}=2$ and $\mathcal{N}=4$ Calogero-Sutherland systems.

3. $\mathcal{N}=2$ supersymmetric extension

3.1. $\mathcal{N}=2$ superfield action and symmetries

We will use the following basic $\mathcal{N}=2$ superfields: the hermitian $n \times n$ -matrix superfield

$$\mathcal{X}_a^b(t, \theta, \bar{\theta}), \quad (\mathcal{X})^\dagger = \mathcal{X},$$

and chiral $U(n)$ -spinor superfield

$$\mathcal{Z}_a(t_L, \theta), \quad \bar{\mathcal{Z}}^a(t_R, \bar{\theta}) = (\mathcal{Z}_a)^\dagger, \quad t_{L,R} = t \pm i\theta\bar{\theta}$$

$a, b = 1, \dots, n$, satisfying

$$\bar{D}\mathcal{Z}_a = 0, \quad D\bar{\mathcal{Z}}^a = 0. \tag{3.1}$$

Here, the covariant spinor derivatives are defined as

$$D = \partial_\theta - i\bar{\theta}\partial_t, \quad \bar{D} = -\partial_{\bar{\theta}} + i\theta\partial_t, \quad \{D, \bar{D}\} = 2i\partial_t.$$

Gauge superfields in the $\mathcal{N}=2$ case are encompassed by $n \times n$ complex “bridge” matrix

$$b_a{}^b(t, \theta, \bar{\theta}), \quad \bar{b}_a{}^b := (b_b{}^a)^* \quad (\bar{b} := b^\dagger).$$

Let us consider the following gauge invariant action

$$S^{N=2} = \frac{1}{2} \int dt d^2\theta \left[\text{Tr} \left(\mathcal{X}^{-1} \bar{D}\mathcal{X} \mathcal{X}^{-1} D\mathcal{X} \right) - \bar{\mathcal{Z}} e^{2V} \mathcal{Z} + 2c \text{Tr} V \right]. \tag{3.2}$$

Here the covariant derivatives of the superfield \mathcal{X} are defined by

$$D\mathcal{X} = \mathcal{D}\mathcal{X} - i[A, \mathcal{X}], \quad \bar{D}\mathcal{X} = \bar{\mathcal{D}}\mathcal{X} - i[\bar{A}, \mathcal{X}] \tag{3.3}$$

where the gauge potentials are expressed through the bridges as

$$A = i e^{i\bar{b}} (D e^{-i\bar{b}}), \quad DA = i A A, \tag{3.4}$$

$$\bar{A} = i e^{ib} (\bar{D} e^{-ib}), \quad \bar{D}\bar{A} = i \bar{A} \bar{A}. \tag{3.5}$$

The gauge prepotential V is defined as

$$e^{2V} = e^{-i\bar{b}} e^{ib}. \tag{3.6}$$

The action (3.2) is invariant with respect to the two types of the local $U(n)$ transformations:

- τ -transformations with the hermitian $n \times n$ -matrix parameter

$$\tau_a{}^b(t, \theta, \bar{\theta}) \in u(n), \quad (\tau)^\dagger = \tau;$$

- λ -transformations with n^2 complex gauge parameters, $\lambda = \|\lambda_a{}^b\|$, $a, b = 1, \dots, n$, which are (anti)chiral superfields

$$\lambda(t_L, \theta) \in u(n), \quad \bar{\lambda}(t_R, \bar{\theta}) = (\lambda)^\dagger \in u(n).$$

The gauge transformations leaving the action (3.2) invariant are realized as:

$$e^{ib'} = e^{i\tau} e^{ib} e^{-i\lambda}, \quad e^{i\bar{b}'} = e^{i\tau} e^{i\bar{b}} e^{-i\bar{\lambda}}, \tag{3.7}$$

$$\mathcal{X}' = e^{i\tau} \mathcal{X} e^{-i\tau}, \quad \mathcal{Z}' = e^{i\lambda} \mathcal{Z}, \quad \bar{\mathcal{Z}}' = \bar{\mathcal{Z}} e^{-i\bar{\lambda}}. \tag{3.8}$$

The potentials are transformed by the τ -group

$$A' = e^{i\tau} A e^{-i\tau} + i e^{i\tau} (D e^{-i\tau}), \quad \bar{A}' = e^{i\tau} \bar{A} e^{-i\tau} + i e^{i\tau} (\bar{D} e^{-i\tau}) \tag{3.9}$$

whereas the prepotential V undergoes λ -transformations

$$e^{2V'} = e^{i\bar{\lambda}} e^{2V} e^{-i\lambda}. \tag{3.10}$$

Note that the second term in the action (3.2) can be rewritten as

$$\bar{Z} e^{2V} Z = \bar{\Phi} \Phi, \quad \Phi \equiv e^{ib} Z, \quad \bar{\Phi} \equiv \bar{Z} e^{-ib}, \quad (3.11)$$

where $\Phi, \bar{\Phi}$ are subject to the τ -transformations only,

$$\Phi' = e^{i\tau} \Phi, \quad \bar{\Phi}' = \Phi e^{-i\tau}. \quad (3.12)$$

3.2. Prepotential formulation and Wess-Zumino gauge

It will be convenient to pass to the new field variables on which only λ transformations act (“ λ frame”). One defines new Hermitian $n \times n$ matrix superfield

$$\mathcal{X} = e^{-ib} \mathcal{X} e^{ib}, \quad (3.13)$$

in terms of which the action (3.2) is rewritten as

$$S^{N=2} = \frac{1}{2} \int dt d^2\theta \left[\text{Tr} \left(\mathcal{X}^{-1} \bar{D} \mathcal{X} \mathcal{X}^{-1} D \mathcal{X} \right) - \bar{Z} e^{2V} Z + 2c \text{Tr} V \right], \quad (3.14)$$

where the covariant derivatives of the superfield \mathcal{X} are defined by

$$D \mathcal{X} = D \mathcal{X} + e^{-2V} (D e^{2V}) \mathcal{X}, \quad \bar{D} \mathcal{X} = \bar{D} \mathcal{X} - \mathcal{X} e^{2V} (\bar{D} e^{-2V}). \quad (3.15)$$

The superfields present in the action (3.14) undergo only chiral λ -transformations

$$\mathcal{X}' = e^{i\lambda} \mathcal{X} e^{-i\lambda}, \quad Z' = e^{i\lambda} Z, \quad \bar{Z}' = \bar{Z} e^{-i\lambda}, \quad e^{2V'} = e^{i\lambda} e^{2V} e^{-i\lambda}. \quad (3.16)$$

The component contents of the involved superfields are as follows

$$V = v + \theta \Phi - \bar{\theta} \bar{\Phi} + \theta \bar{\theta} A, \quad (3.17)$$

$$\mathcal{X} = X + \theta \Psi - \bar{\theta} \bar{\Psi} + \theta \bar{\theta} Y, \quad Z = Z + 2i\theta \Upsilon - i\theta \bar{\theta} \dot{Z}, \quad \bar{Z} = \bar{Z} + 2i\bar{\theta} \bar{\Upsilon} + i\theta \bar{\theta} \dot{\bar{Z}}, \quad (3.18)$$

where $\Psi_a^b, \bar{\Psi}_a^b = (\Psi_b^a)^* (\bar{\Psi} = \Psi^\dagger)$, $\Phi_a^b, \bar{\Phi}_a^b = (\Phi_b^a)^* (\bar{\Phi} = \Phi^\dagger)$ and $\Upsilon_a, \bar{\Upsilon}^a = (\Upsilon_a)^*$ are fermionic fields.

In order to pass to the component form of the action (3.14), we are as usual led to choose Wess-Zumino gauge

$$V(t, \theta, \bar{\theta}) = \theta \bar{\theta} A(t), \quad (3.19)$$

in which

$$e^{2V} = 1 + 2\theta \bar{\theta} A, \quad e^{-2V} D e^{2V} = 2DV = 2\bar{\theta} A, \quad e^{2V} \bar{D} e^{-2V} = -2\bar{D}V = -2\theta A$$

and

$$D \mathcal{X} = \Psi + \bar{\theta} (Y - i\dot{X} + 2AX) - \theta \bar{\theta} (2A\Psi - i\dot{\Psi}),$$

$$\bar{D} \mathcal{X} = \bar{\Psi} + \theta (Y + i\dot{X} + 2XA) - \theta \bar{\theta} (2\bar{\Psi}A + i\dot{\bar{\Psi}}).$$

After substituting these Wess-Zumino-gauge expressions into the action (3.14), integrating there over Grassmann coordinates ($\int d^2\theta (\theta\bar{\theta}) = 1$) and eliminating auxiliary component fields by their equations of motion,

$$Y = - \left(AX + XA + \frac{1}{2} \bar{\Psi} X^{-1} \Psi - \frac{1}{2} \Psi X^{-1} \bar{\Psi} \right), \quad \Upsilon = 0,$$

the superfield action (3.14) yields the on-shell component action

$$S^{N=2} = \int dt L^{N=2}, \tag{3.20}$$

$$L^{N=2} = \frac{1}{2} \text{Tr} \left(X^{-1} \nabla X X^{-1} \nabla X \right) + \frac{i}{2} \left(\bar{Z} \nabla Z - \nabla \bar{Z} Z \right) + c \text{Tr} A \\ + \frac{i}{2} \text{Tr} \left(X^{-1} \bar{\Psi} X^{-1} \nabla \Psi - X^{-1} \nabla \bar{\Psi} X^{-1} \Psi \right) \\ - \frac{1}{4} \text{Tr} \left(X^{-1} \bar{\Psi} X^{-1} \bar{\Psi} X^{-1} \Psi X^{-1} \Psi \right). \tag{3.21}$$

Here, ∇Z and $\nabla \bar{Z}$ are defined as in (2.2) and

$$\nabla \Psi = \dot{\Psi} + i [A, \Psi], \quad \nabla \bar{\Psi} = \dot{\bar{\Psi}} + i [A, \bar{\Psi}]. \tag{3.22}$$

We observe that the bosonic part of the model (3.20) (first line of (3.21)) is exactly the action of the hyperbolic Calogero system (2.1). Hence we have constructed the sought $\mathcal{N}=2$ supersymmetric extension of hyperbolic Calogero-Sutherland system.

It is interesting that, as opposed to the $\mathcal{N}=2$ supersymmetrization of the rational Calogero model [15], $\mathcal{N}=2$ superextension of the hyperbolic Calogero-Sutherland model (3.20) involves a term quartic in the fermionic fields (the last line in (3.21)).

The action (3.21) is invariant with respect to the local $U(n)$ transformations, $g(\tau) \in U(n)$, which act on the bosonic fields according to (2.3) and on the fermionic fields as

$$\Psi \rightarrow g \Psi g^\dagger, \quad \bar{\Psi} \rightarrow g \bar{\Psi} g^\dagger. \tag{3.23}$$

We can fix this residual gauge freedom by choosing the gauge (2.4), (2.5), in which the matrix X is diagonal. Then, by analogy with the action (2.6), the action obtained possesses the invariance under gauge Abelian $[U(1)]^n$ symmetry realized by the local transformations (2.7) and

$$\Psi_a^b \rightarrow e^{i\varphi_a} \bar{\Psi}_a^b e^{-i\varphi_b}, \quad \bar{\Psi}_a^b \rightarrow e^{i\varphi_a} \bar{\Psi}_a^b e^{-i\varphi_b}. \tag{3.24}$$

Finally, we can fix the gauge (2.8)

$$\bar{Z}^a = Z_a, \tag{3.25}$$

after which the variables Z_a can be eliminated by their equations of motion.

We would like to point out that the $\mathcal{N}=0$ and $\mathcal{N}=2$ gauge constructions above differ from those relevant to the rational (super)Calogero models merely by the choice of the initial structure of the kinetic term for the matrix field X and its superfield counterpart \mathcal{X} . While in the rational case they are free [15,11], in the hyperbolic case they are of sigma-model type. The $\mathcal{N}=1$ hyperbolic model shares this feature and so can be easily constructed following the same steps as in the $\mathcal{N}=1$ rational case (described in detail in [11]), with the properly modified kinetic term for the matrix Hermitian $\mathcal{N}=1$ superfield. The $\mathcal{N}=1$ component action can be recovered from the action (3.20), (3.21) by performing there the reduction $\Psi = \bar{\Psi}$. Note that the last quartic term in (3.21) vanishes upon such a reduction.

4. $\mathcal{N}=4$ supersymmetric extensions

4.1. Generalities

We shall construct the hyperbolic model with $\mathcal{N}=4$ supersymmetry by analogy with the previously considered $\mathcal{N}=0$ and $\mathcal{N}=2$ cases, as well as with the $\mathcal{N}=4$ superextension of the rational Calogero model. The relevant action for the basic matrix superfield proves to be a non-trivial generalization of that in the rational $\mathcal{N}=4$ model [15,39,40,11].

We shall gauge an action of n^2 superfields which form Hermitian $n \times n$ matrix \mathcal{X} and describe off-shell $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ multiplets. The superfield \mathcal{X} is subject to the kinematic constrains

$$D^i D_i \mathcal{X} = 0, \quad \bar{D}_i \bar{D}^i \mathcal{X} = 0, \quad [D^i, \bar{D}_i] \mathcal{X} = 0, \quad (4.1)$$

where

$$D^i = \frac{\partial}{\partial \theta_i} - i \bar{\theta}^i \partial_t, \quad \bar{D}_i = \frac{\partial}{\partial \bar{\theta}^i} - i \theta_i \partial_t.$$

These constraints are solved by

$$\mathcal{X}(t, \theta_i, \bar{\theta}^i) = X + \theta_i \Psi^i + \bar{\Psi}_i \bar{\theta}^i + i \theta^i \bar{\theta}^k N_{ik} - \frac{i}{2} (\theta)^2 \dot{\Psi}_i \bar{\theta}^i - \frac{i}{2} (\bar{\theta})^2 \theta_i \dot{\Psi}^i + \frac{1}{4} (\theta)^2 (\bar{\theta})^2 \ddot{X}, \quad (4.2)$$

where $(\theta)^2 := \theta_k \theta^k$, $(\bar{\theta})^2 := \bar{\theta}^k \bar{\theta}_k$ and $X^\dagger = X$, $(\Psi^i)^\dagger = \bar{\Psi}_i$, $(N^{ik})^\dagger = N_{ik} = N_{(ik)}$ or, using a more detailed notation,

$$(X_a^b)^* = X_b^a, \quad (\Psi_a^i)^* = \bar{\Psi}_i^a, \quad (N^{ik}_a{}^b)^* = N_{ikb}^a.$$

The Wess-Zumino term (a counterpart of \mathcal{Z} -term in $\mathcal{N}=2$ case) will be described by n constrained superfields forming $SU(n)$ spinor and describing n $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ multiplets. In the $d=1$ harmonic superspace (HSS) approach [41] they are represented by n commuting analytic superfields. Before going further, it is instructive to recall the necessary elements of the $d=1$ HSS formalism.

The $\mathcal{N}=4$, $d=1$ HSS (in the ‘‘central’’ basis) is parametrized by the coordinates

$$(t, \theta^\pm, \bar{\theta}^\pm, u_i^\pm), \quad \theta^\pm = \theta^i u_i^\pm, \quad \bar{\theta}^\pm = \bar{\theta}^i u_i^\pm, \quad u^{+i} u_i^- = 1.$$

Harmonic analytic superspace is formed by the coordinate set

$$(\zeta, u) = (t_A, \theta^+, \bar{\theta}^+, u_i^\pm), \quad t_A = t + i(\theta^+ \bar{\theta}^- + \theta^- \bar{\theta}^+).$$

The analytic basis in $\mathcal{N}=4$, $d=1$ HSS is given by the coordinate set

$$[(\zeta, u), \theta^-, \bar{\theta}^-].$$

The invariant integration measures are defined as

$$\mu_H = du dt d^4 \theta = \mu_A^{(-2)} (D^+ \bar{D}^+), \\ \mu_A^{(-2)} = du d\zeta^{(-2)} = du dt_A d\theta^+ d\bar{\theta}^+ = du dt_A (D^- \bar{D}^-).$$

The covariant derivatives in the analytic basis are given by the expressions:

$$D^+ = \frac{\partial}{\partial \theta^-}, \quad \bar{D}^+ = -\frac{\partial}{\partial \bar{\theta}^-}, \quad D^- = -\frac{\partial}{\partial \theta^+} - 2i \bar{\theta}^- \partial_{t_A}, \quad \bar{D}^- = \frac{\partial}{\partial \bar{\theta}^+} - 2i \theta^- \partial_{t_A}, \quad (4.3)$$

$$D^{\pm\pm} = \partial^{\pm\pm} + 2i\theta^\pm \bar{\theta}^\pm \partial_{t_A} + \theta^\pm \frac{\partial}{\partial \theta^\mp} + \bar{\theta}^\pm \frac{\partial}{\partial \bar{\theta}^\mp}. \tag{4.4}$$

Below we consider two $\mathcal{N}=4$ supersymmetric models which yield $\mathcal{N}=4$ generalizations of the hyperbolic Calogero-Sutherland system. In first case this $\mathcal{N}=4$ generalization contains additional interactions in the bosonic sector including the center of mass. In second case the $\mathcal{N}=4$ model produces only the $U(2)$ -spin hyperbolic Calogero-Sutherland system, with the center-of-mass coordinate fully decoupled.

4.2. System I

4.2.1. Action

Now we are prepared for describing the basic ingredients of our $\mathcal{N} = 4$ hyperbolic Calogero model.

The $\mathcal{N}=4$ supersymmetric and $U(n)$ gauge invariant action is defined as

$$S^{N=4} = S_{\mathcal{X}} + S_{WZ} + S_{FI}. \tag{4.5}$$

The first term in (4.5) is

$$S_{\mathcal{X}} = \frac{a}{2} \int \mu_H \text{Tr} \left(\ln \mathcal{X} \right), \tag{4.6}$$

where $\mathcal{X} = \|\mathcal{X}_a^b\|$ is subjected to the constraints

$$\mathcal{D}^{++} \mathcal{X} = 0, \tag{4.7}$$

$$\mathcal{D}^+ \mathcal{D}^- \mathcal{X} = 0, \quad \bar{\mathcal{D}}^+ \bar{\mathcal{D}}^- \mathcal{X} = 0, \quad (\mathcal{D}^+ \bar{\mathcal{D}}^- + \bar{\mathcal{D}}^+ \mathcal{D}^-) \mathcal{X} = 0, \tag{4.8}$$

which are none other than the gauge-covariant version of the original constraints (4.1). The constraint (4.7) contains the gauge-covariantized harmonic derivative

$$\mathcal{D}^{++} = D^{++} + i V^{++}, \tag{4.9}$$

where the gauge connection $V^{++}(\zeta, u)$ is an analytic unconstrained matrix superfield. The gauge connections of all other involved covariant derivatives are properly expressed through $V^{++}(\zeta, u)$. In the analytic basis (and the analytic gauge frame) the spinor derivatives \mathcal{D}^+ and $\bar{\mathcal{D}}^+$ remain “short”, $\mathcal{D}^+ = D^+$, $\bar{\mathcal{D}}^+ = \bar{D}^+$. Note that \mathcal{X} is in adjointed representation of $U(n)$ and, therefore,

$$\mathcal{D}^{++} \mathcal{X} = D^{++} \mathcal{X} + i [V^{++}, \mathcal{X}],$$

etc.

The last term in (4.5) is FI term

$$S_{FI} = -\frac{ic}{2} \int \mu_A^{(-2)} \text{Tr}(V^{++}). \tag{4.10}$$

The second term in (4.5) is Wess-Zumino term. It has the following explicit form

$$S_{WZ} = \frac{b}{2} \int \mu_A^{(-2)} \mathcal{V}_0 \tilde{\Phi}^+ \Phi^+. \tag{4.11}$$

Here, $\Phi^+ = (\Phi_a^+)$, $a = 1, \dots, n$, and its conjugate are commuting analytic superfields satisfying the constraints

$$\mathcal{D}^{++} \Phi^+ = 0, \quad \mathcal{D}^+ \Phi^+ = 0, \quad \bar{\mathcal{D}}^+ \Phi^+ = 0. \tag{4.12}$$

The real superfield \mathcal{V}_0 appearing in (4.11) is analytic,

$$\mathcal{V}_0 = \mathcal{V}_0(\zeta, u), \quad D^+ \mathcal{V}_0 = 0, \quad \bar{D}^+ \mathcal{V}_0 = 0, \quad (4.13)$$

and it is a prepotential solving the constraints for the singlet part of \mathcal{X} defined as:

$$\mathcal{X}_0 := \text{Tr}(\mathcal{X}). \quad (4.14)$$

For the superfield (4.14) the constraints (4.8) do not contain any connections and so take the form (4.1):

$$D^i D_i \mathcal{X}_0 = 0, \quad \bar{D}_i \bar{D}^i \mathcal{X}_0 = 0, \quad [D^i, \bar{D}_i] \mathcal{X}_0 = 0. \quad (4.15)$$

These constraints are solved in terms of \mathcal{V}_0 (4.13) through the integral transform [18]

$$\mathcal{X}_0(t, \theta_i, \bar{\theta}^i) = \int du \mathcal{V}_0(t_A, \theta^+, \bar{\theta}^+, u^\pm) \Big|_{\theta^\pm = \theta^i u_i^\pm, \bar{\theta}^\pm = \bar{\theta}^i u_i^\pm}. \quad (4.16)$$

In the actions (4.6), (4.11), a and b are some non-vanishing coupling constants.

The action (4.5) is invariant with respect to the local $U(n)$ transformations:

$$\mathcal{X}' = e^{i\lambda} \mathcal{X} e^{-i\lambda}, \quad (4.17)$$

$$\Phi^{+'} = e^{i\lambda} \Phi^+, \quad \tilde{\Phi}^{+'} = \tilde{\Phi}^+ e^{-i\lambda}, \quad (4.18)$$

$$V^{++'} = e^{i\lambda} V^{++} e^{-i\lambda} + i e^{i\lambda} (D^{++} e^{-i\lambda}), \quad (4.19)$$

$$\mathcal{X}'_0 = \mathcal{X}_0, \quad \mathcal{V}'_0 = \mathcal{V}_0, \quad (4.20)$$

where $\lambda_a{}^b(\zeta, u^\pm) \in u(n)$ is the ‘‘hermitian’’ $n \times n$ -matrix parameter, $\tilde{\lambda} = \lambda$. Just due to the choice of such an analytic gauge group, the derivatives \mathcal{D}^+ , $\bar{\mathcal{D}}^+$ in the equations (4.8) and (4.12) contain no gauge connections

$$\mathcal{D}^+ = D^+, \quad \bar{\mathcal{D}}^+ = \bar{D}^+, \quad (4.21)$$

the feature already mentioned earlier.

4.2.2. Wess-Zumino gauge

Using the analytic gauge freedom (4.19), we can choose Wess-Zumino gauge for V^{++} :

$$V^{++} = 2i \theta^+ \bar{\theta}^+ A(t_A), \quad \mathcal{D}^{++} = D^{++} - 2\theta^+ \bar{\theta}^+ A. \quad (4.22)$$

From the relation $[\mathcal{D}^{++}, \mathcal{D}^{--}] = D^0$ we find

$$\mathcal{D}^{--} = D^{--} - 2\theta^- \bar{\theta}^- A. \quad (4.23)$$

Then,

$$\mathcal{D}^- = [\mathcal{D}^{--}, D^+] = D^- + 2\bar{\theta}^- A, \quad \bar{\mathcal{D}}^- = [\mathcal{D}^{--}, \bar{D}^+] = \bar{D}^- + 2\theta^- A. \quad (4.24)$$

So the Wess-Zumino gauge form of the covariant derivatives is as follows

$$\mathcal{D}^+ = \frac{\partial}{\partial \theta^-}, \quad \bar{\mathcal{D}}^+ = -\frac{\partial}{\partial \bar{\theta}^-}, \quad \mathcal{D}^- = -\frac{\partial}{\partial \theta^+} - 2i \bar{\theta}^- \nabla_{t_A}, \quad \bar{\mathcal{D}}^- = \frac{\partial}{\partial \bar{\theta}^+} - 2i \theta^- \nabla_{t_A}, \quad (4.25)$$

$$\mathcal{D}^{\pm\pm} = \partial^{\pm\pm} + 2i \theta^\pm \bar{\theta}^\pm \nabla_{t_A} + \theta^\pm \frac{\partial}{\partial \theta^\mp} + \bar{\theta}^\pm \frac{\partial}{\partial \bar{\theta}^\mp}, \quad (4.26)$$

where

$$\nabla_{t_A} = \partial_{t_A} + i A. \tag{4.27}$$

The solution of the constraint (4.7), (4.8) in Wess-Zumino gauge (4.22) reads

$$\begin{aligned} \mathcal{X} = & X + i\theta^-\bar{\theta}^-N^{++} + i\theta^+\bar{\theta}^+N^{--} - i(\theta^-\bar{\theta}^+ + \theta^+\bar{\theta}^-)N + \theta^-\bar{\theta}^-\theta^+\bar{\theta}^+D \\ & + \theta^-\Psi^+ + \bar{\theta}^-\bar{\Psi}^+ - \theta^+\Psi^- - \bar{\theta}^+\bar{\Psi}^- + \theta^-\bar{\theta}^-(\theta^+\Omega^+ + \bar{\theta}^+\bar{\Omega}^+), \end{aligned} \tag{4.28}$$

where,

$$N^{\pm\pm} = N^{ik}u_i^\pm u_k^\pm, \quad N = \nabla_{t_A}X + N^{ik}u_i^+u_k^-, \quad D = 2\nabla_{t_A}\nabla_{t_A}X + 2\nabla_{t_A}N^{ik}u_i^+u_k^-, \tag{4.29}$$

$$\Psi^\pm = \Psi^i u_i^\pm, \quad \bar{\Psi}^\pm = \bar{\Psi}^i u_i^\pm, \quad \Omega^+ = 2i\nabla_{t_A}\Psi^i u_i^+, \quad \bar{\Omega}^+ = 2i\nabla_{t_A}\bar{\Psi}^i u_i^+ \tag{4.30}$$

and

$$X(t_A), \quad N^{ik} = N^{(ik)}(t_A), \quad \Psi^i(t_A), \quad \bar{\Psi}^i(t_A) \tag{4.31}$$

are usual $d=1$ fields having no harmonic dependence.

The constraints (4.12) in Wess-Zumino gauge (4.22) are solved by

$$\Phi^+ = Z^+ + \theta^+\varphi + \bar{\theta}^+\phi - 2i\theta^+\bar{\theta}^+Z^-, \quad \tilde{\Phi}^+ = \tilde{Z}^+ + \theta^+\bar{\phi} - \bar{\theta}^+\bar{\varphi} - 2i\theta^+\bar{\theta}^+\tilde{Z}^-, \tag{4.32}$$

with

$$Z^+ = Z^i u_i^+, \quad \tilde{Z}^+ = \bar{Z}_i u^{+i}, \quad \bar{Z}_i = (Z^i)^*, \tag{4.33}$$

$$Z^- = \nabla_{t_A}Z^i u_i^-, \quad \tilde{Z}^- = \nabla_{t_A}\bar{Z}_i u^{-i}, \tag{4.34}$$

and

$$Z^i(t_A), \quad \varphi(t_A), \quad \phi(t_A) \tag{4.35}$$

being $d=1$ fields.

Using (4.16), we can also determine the real analytic superfield \mathcal{V}_0 . First, from the expression (4.28) we find the singlet part $\mathcal{X}_0 = \text{Tr}(\mathcal{X})$ (in the central basis):

$$\begin{aligned} \mathcal{X}_0(t, \theta_i, \bar{\theta}^i) = & X_0 + i\theta^i\bar{\theta}^k N_{0ik} + \frac{1}{4}(\theta)^2(\bar{\theta})^2\check{X}_0 - \theta^i\Psi_{0i} - \bar{\theta}^i\bar{\Psi}_{0i} + \frac{i}{2}(\theta)^2\bar{\theta}^i\dot{\Psi}_{0i} \\ & + \frac{i}{2}(\bar{\theta})^2\theta^i\dot{\bar{\Psi}}_{0i}, \end{aligned} \tag{4.36}$$

where

$$X_0 := \text{Tr}(X), \quad N_0^{ik} := \text{Tr}(N^{ik}), \quad \Psi_0^i := \text{Tr}(\Psi^i), \quad \bar{\Psi}_0^i := \text{Tr}(\bar{\Psi}^i). \tag{4.37}$$

Then, from (4.16) we restore \mathcal{V}_0 as³

³ Note that $\int du u^{+i}u_k^- = -\int du u^{-i}u_k^+ = \frac{1}{2}\delta_k^i$, $\int du u^{+(i_1 u^{+i_2})}u_{(k_1}^-u_{k_2)}^- = \int du u^{-(i_1 u^{-i_2})}u_{(k_1}^+u_{k_2)}^+ = -2\int du u^{+(i_1 u^{-i_2})}u_{(k_1}^+u_{k_2)}^- = \frac{1}{3}\delta_{(k_1}^{(i_1}\delta_{k_2)}^{i_2)}$ and $(\theta)^2 = -2\theta^+\theta^-$, $(\bar{\theta})^2 = 2\bar{\theta}^+\bar{\theta}^-$.

$$\mathcal{V}_0(\zeta, u) = X_0(t_A) + 3i\theta^+\bar{\theta}^+ N_0^{ik}(t_A) u_i^- u_k^- - 2\theta^+ \Psi_0^i(t_A) u_i^- - 2\bar{\theta}^+ \bar{\Psi}_0^i(t_A) u_i^-. \quad (4.38)$$

While finding this expression, we made use of the abelian gauge freedom of the representation (4.16), $\mathcal{V}_0' = \mathcal{V}_0 + D^{++}\Lambda^{--}$, $\Lambda^{--} = \Lambda^{--}(\zeta, u)$.

Inserting the expressions (4.28), (4.32), (4.38) in the action (4.5), using the identity

$$\partial \text{Tr}(\ln \mathcal{X}) \equiv \partial \ln(\det \mathcal{X}) = \text{Tr}(\mathcal{X}^{-1} \partial \mathcal{X}) \quad (4.39)$$

and, finally, integrating over Grassmann and harmonic coordinates, we obtain

$$S^{N=4} = S_{\mathcal{X}} + S_{WZ} + S_{FI}, \quad (4.40)$$

with

$$\begin{aligned} S_{\mathcal{X}} = & \frac{a}{2} \text{Tr} \int dt \left(X^{-1} \nabla X X^{-1} \nabla X + i X^{-1} \nabla \bar{\Psi}_k X^{-1} \Psi^k - i X^{-1} \bar{\Psi}_k X^{-1} \nabla \Psi^k \right. \\ & + \frac{1}{2} X^{-1} N^{ik} X^{-1} N_{ik} - i [X^{-1} \Psi^i, X^{-1} \bar{\Psi}^k] X^{-1} N_{ik} \\ & - \frac{1}{3} [X^{-1} \Psi^i, X^{-1} \bar{\Psi}^k] [X^{-1} \Psi_{(i}, X^{-1} \bar{\Psi}_{k)}] - \frac{1}{3} X^{-1} \Psi^i X^{-1} \Psi^k X^{-1} \bar{\Psi}_{(i} X^{-1} \bar{\Psi}_{k)} \\ & \left. + \frac{1}{2} X^{-1} \Psi^i X^{-1} \Psi_i X^{-1} \bar{\Psi}^k X^{-1} \bar{\Psi}_k \right), \end{aligned} \quad (4.41)$$

$$S_{FI} = c \text{Tr} \int dt A, \quad (4.42)$$

$$\begin{aligned} S_{WZ} = & \frac{ib}{2} \int dt X_0 \left(\nabla \bar{Z}_k Z^k - \bar{Z}_k \nabla Z^k \right) - \frac{ib}{2} \int dt N_0^{ik} \bar{Z}_i Z_k \\ & + \frac{b}{2} \int dt \left[\Psi_0^k (\bar{Z}_k \phi + \bar{\varphi} Z_k) + \bar{\Psi}_0^k (\bar{\phi} Z_k - \bar{Z}_k \varphi) - X_0 (\bar{\phi} \phi + \bar{\varphi} \varphi) \right]. \end{aligned} \quad (4.43)$$

In these expressions,

$$\begin{aligned} \nabla X &= \dot{X} + i[A, X], & \nabla \Psi^k &= \dot{\Psi}^k + i[A, \Psi^k], & \nabla \bar{\Psi}_k &= \dot{\bar{\Psi}}_k + i[A, \bar{\Psi}_k], \\ \nabla Z^k &= \dot{Z}^k + iAZ^k, & \nabla \bar{Z}_k &= \dot{\bar{Z}}_k - i\bar{Z}_k A. \end{aligned}$$

The equations of motion for the fields N^{ik} , ϕ , $\bar{\phi}$, φ , $\bar{\varphi}$ are algebraic and can be used to eliminate these fields:

$$N^{ik} = iX[X^{-1}\Psi^{(i}, X^{-1}\bar{\Psi}^{k)}] + \frac{ib}{a} (\bar{Z}^{(i} Z^{k)}) X^2, \quad (4.44)$$

$$\phi = -\frac{1}{X_0} \bar{\Psi}_0^k Z_k, \quad \bar{\phi} = \frac{1}{X_0} \Psi_0^k \bar{Z}_k, \quad \varphi = -\frac{1}{X_0} \Psi_0^k Z_k, \quad \bar{\varphi} = -\frac{1}{X_0} \bar{\Psi}_0^k \bar{Z}_k. \quad (4.45)$$

Inserting these expressions in (4.40), we obtain

$$\begin{aligned} S^{N=4} = & \frac{1}{2} \int dt \left[\text{Tr} \left(aX^{-1} \nabla X X^{-1} \nabla X + 2cA \right) + ib X_0 \left(\nabla \bar{Z}_k Z^k - \bar{Z}_k \nabla Z^k \right) \right. \\ & \left. + \frac{b^2}{2a} (\bar{Z}^{(i} Z^{k)}) (\bar{Z}_i Z_k) \text{Tr} \left(X^2 \right) \right] \\ & + \frac{a}{2} \text{Tr} \int dt \left(i X^{-1} \nabla \bar{\Psi}_k X^{-1} \Psi^k - i X^{-1} \bar{\Psi}_k X^{-1} \nabla \Psi^k \right. \\ & \left. - \frac{1}{6} [X^{-1} \Psi^i, X^{-1} \bar{\Psi}^k] [X^{-1} \Psi_{(i}, X^{-1} \bar{\Psi}_{k)}] \right) \end{aligned}$$

$$\begin{aligned}
 & -\frac{1}{3} X^{-1} \Psi^i X^{-1} \Psi^k X^{-1} \bar{\Psi}_{(i} X^{-1} \bar{\Psi}_{k)} \\
 & + \frac{1}{2} X^{-1} \Psi^i X^{-1} \Psi_i X^{-1} \bar{\Psi}^k X^{-1} \bar{\Psi}_k \\
 & + \frac{b}{2} \int dt (\bar{Z}_{(i} Z_{k)}) \left(\text{Tr} \left(X [X^{-1} \Psi^i, X^{-1} \bar{\Psi}^k] \right) - \frac{2}{X_0} \Psi_0^i \bar{\Psi}_0^k \right).
 \end{aligned} \tag{4.46}$$

4.2.3. Bosonic limit

Let us consider the bosonic limit of the action (4.46), i.e., the first two lines in it. For simplicity, we put $a = -b = 1$.

We also introduce the new fields

$$Z_a^i = (X_0)^{1/2} Z_a^i, \tag{4.47}$$

and, in what follows, omit the primes on the newly defined fields. The bosonic part of the action (4.46) becomes

$$\begin{aligned}
 S_{bose}^{N=4} = \frac{1}{2} \int dt \left[\text{Tr} \left(X^{-1} \nabla X X^{-1} \nabla X + 2c A \right) + i \left(\bar{Z}_k \nabla Z^k - \nabla \bar{Z}_k Z^k \right) \right. \\
 \left. + \frac{(\bar{Z}^{(i} Z^{k)}) (\bar{Z}_i Z_k) \text{Tr} (X^2)}{2(X_0)^2} \right].
 \end{aligned} \tag{4.48}$$

The action (4.48) respects the residual invariance (4.17)–(4.20) of Wess-Zumino gauge (4.22):

$$X' = e^{i\lambda} X e^{-i\lambda}, \tag{4.49}$$

$$Z_a^i{}' = e^{i\lambda} Z_a^i, \quad \bar{Z}_k^a{}' = \bar{Z}_k^a e^{-i\lambda}, \tag{4.50}$$

$$A' = e^{i\lambda} A e^{-i\lambda} + i e^{i\lambda} (\partial_t e^{-i\lambda}), \tag{4.51}$$

$$X_0' = X_0, \tag{4.52}$$

where $\lambda_a{}^b(t) \in u(n)$ are the gauge parameters depending only on t . Now we can impose the gauge-fixing conditions (2.4)

$$X_a{}^b = 0, \quad a \neq b. \tag{4.53}$$

As a result, the action (4.48) acquires the form (the sums over repeating indices $i, k = 1, 2$ are always assumed)

$$\begin{aligned}
 S_{bose}^{N=4} = \frac{1}{2} \int dt \sum_{a,b} \left[\frac{\dot{x}_a \dot{x}_a}{(x_a)^2} + i (\bar{Z}_k^a \dot{Z}_a^k - \dot{\bar{Z}}_k^a Z_a^k) + \frac{(x_a - x_b)^2 A_a{}^b A_b{}^a}{x_a x_b} \right. \\
 \left. - 2 \bar{Z}_k^a A_a{}^b Z_b^k + 2c A_a{}^a + \frac{\bar{Z}^{a(i} Z_a^k) \bar{Z}_{(i} Z_{k)b} \text{Tr} (X^2)}{2(X_0)^2} \right].
 \end{aligned} \tag{4.54}$$

Note that $X_0 = \sum_a x_a$.

The equations of motion for $A_a{}^b$ read

$$A_a{}^b = \frac{x_a x_b}{(x_a - x_b)^2} Z_a^i \bar{Z}_i^b \quad \text{for } a \neq b, \tag{4.55}$$

$$\bar{Z}_i^a Z_a^i = c \quad \forall a \quad (\text{no sum over the index } a). \tag{4.56}$$

Substituting this in (4.54), we obtain

$$S_{bose}^{N=4} = \frac{1}{2} \int dt \left\{ \sum_a \left[\frac{\dot{x}_a \dot{x}_a}{(x_a)^2} + i(\bar{Z}_k^a \dot{Z}_a^k - \dot{\bar{Z}}_k^a Z_a^k) \right] - \sum_{a \neq b} \frac{x_a x_b \bar{Z}_i^a Z_b^i \bar{Z}_k^b Z_a^k}{(x_a - x_b)^2} + \sum_{a,b} \frac{\bar{Z}^{a(i} Z_a^{k)} \bar{Z}_{(i} Z_{k)b} \text{Tr}(X^2)}{2(X_0)^2} \right\}. \tag{4.57}$$

The $d=1$ fields Z_a^k are subjected by the constraints (4.56) and carry the residual $[U(1)]^n$ gauge symmetry with local parameters $\varphi_a(t)$:

$$Z_a^k \rightarrow e^{i\varphi_a} Z_a^k, \quad \bar{Z}_k^a \rightarrow e^{-i\varphi_a} \bar{Z}_k^a. \tag{4.58}$$

Introducing the variables q_a as in (2.13),

$$x_a = e^{q_a}, \tag{4.59}$$

we observe that the action (4.57) takes the form

$$S_{bose}^{N=4} = \frac{1}{2} \int dt \left\{ \sum_a \left[\dot{q}_a \dot{q}_a + i(\bar{Z}_k^a \dot{Z}_a^k - \dot{\bar{Z}}_k^a Z_a^k) \right] - \sum_{a \neq b} \frac{(S_a)_i{}^k (S_b)_k{}^i}{4 \sinh^2 \frac{q_a - q_b}{2}} + \sum_{a,b} \frac{(S_a)^{(ik)} (S_b)_{(ik)} \text{Tr}(X^2)}{2(X_0)^2} \right\}, \tag{4.60}$$

where

$$\text{Tr}(X^2) = \sum_c e^{2q_c}, \quad X_0 = \sum_c e^{q_c} \tag{4.61}$$

and

$$(S_a)_i{}^k := \bar{Z}_i^a Z_a^k. \tag{4.62}$$

The quantities (4.62) generate n copies of $U(2)$ algebra. As a result, the above action, up to the last term, describes the hyperbolic $U(2)$ -spin Calogero-Sutherland system (see, e.g., [42,2]).

4.3. System II

In the total action (4.5) the Wess-Zumino term (4.11) involves the interaction of the trace part of the matrix $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ multiplet with spin $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ multiplets. Just this Wess-Zumino term was employed in [15] to construct the $\mathcal{N}=4$ rational Calogero system. This specific form of the Wess-Zumino term was chosen exclusively in order to ensure the $\mathcal{N}=4$ superconformal invariance of the full action.

However, the $\mathcal{N}=4$ hyperbolic model we are considering here lacks the superconformal invariance already in the kinetic term (4.6). So, there are no reasons to require such an invariance for the Wess-Zumino term as well. Keeping this in mind, we can alternatively consider the system with the action

$$\tilde{S}^{N=4} = S_{\mathcal{X}} + S'_{WZ} + S_{FI}, \tag{4.63}$$

with the simplest Wess-Zumino term

$$S'_{WZ} = \frac{b}{2} \int \mu_A^{(-2)} \tilde{\Phi}^+ \Phi^+, \tag{4.64}$$

and the same S_X and S_{FI} as in (4.6) and (4.10).

In the Wess-Zumino gauge (4.22) the action (4.64) acquires the following component form (cf. (4.43))

$$S'_{WZ} = \frac{b}{2} \int dt \left[i \left(\nabla \bar{Z}_k Z^k - \bar{Z}_k \nabla Z^k \right) - (\bar{\phi} \phi + \bar{\varphi} \varphi) \right], \tag{4.65}$$

implying the very simple on-shell expressions for the auxiliary fields:

$$N^{ik} = i X [X^{-1} \Psi^i, X^{-1} \bar{\Psi}^k], \quad \phi = \bar{\phi} = \varphi = \bar{\varphi} = 0, \tag{4.66}$$

in contrast to (4.44), (4.45). As a result, the component on-shell form of the action (4.63) is given by

$$\begin{aligned} \tilde{S}^{N=4} = & \frac{1}{2} \int dt \left[\text{Tr} \left(a X^{-1} \nabla X X^{-1} \nabla X + 2c A \right) + i b \left(\nabla \bar{Z}_k Z^k - \bar{Z}_k \nabla Z^k \right) \right] \\ & + \frac{a}{2} \text{Tr} \int dt \left(i X^{-1} \nabla \bar{\Psi}_k X^{-1} \Psi^k - i X^{-1} \bar{\Psi}_k X^{-1} \nabla \Psi^k \right. \\ & \quad - \frac{1}{6} [X^{-1} \Psi^i, X^{-1} \bar{\Psi}^k] [X^{-1} \Psi_{(i}, X^{-1} \bar{\Psi}_{k)}] \\ & \quad - \frac{1}{3} X^{-1} \Psi^i X^{-1} \Psi^k X^{-1} \bar{\Psi}_{(i} X^{-1} \bar{\Psi}_{k)} \\ & \quad \left. + \frac{1}{2} X^{-1} \Psi^i X^{-1} \Psi_i X^{-1} \bar{\Psi}^k X^{-1} \bar{\Psi}_k \right). \end{aligned} \tag{4.67}$$

The bosonic limit of this action (the first line in (4.67)) has already the correct form for the variables Z_a^i , so the rescaling (4.47) is unnecessary in this case. After gauge-fixing (4.53) and eliminating the auxiliary fields by the equations (4.55) at $a = -b = 1$ we obtain that the bosonic limit of (4.67) is described by the action

$$\tilde{S}_{bose}^{N=4} = \frac{1}{2} \int dt \left\{ \sum_a \left[\dot{q}_a \dot{q}_a + i (\bar{Z}_k^a \dot{Z}_a^k - \dot{\bar{Z}}_k^a Z_a^k) \right] - \sum_{a \neq b} \frac{(S_a)_i{}^k (S_b)_k{}^i}{4 \sinh^2 \frac{q_a - q_b}{2}} \right\}. \tag{4.68}$$

Recall that the last term in the previous bosonic action (4.57) involves $\text{Tr}(X^2)$ and X_0 . The latter coordinate (the center-of-mass coordinate) decouples only for the trivial cases $n = 1, 2$. In contrast, the action (4.63) for any n yields in the bosonic sector the pure hyperbolic $U(2)$ -spin Calogero-Sutherland system (4.68) without any additional interaction. The center-of-mass coordinate is fully detached and described by the free action in this sector.

Finally, note that in the one-particle case $n=1$ in the system (4.63) there comes about a separation of the dynamic X -sector from the semi-dynamic Φ sector. The Lagrangian of the dynamic sector describes a free system, not possessing any potential (see, e.g., [43]). Non-trivial cases start with the two-particle system $n=2$. In this case the relative-motion sector (no-center-of-mass sector) of the system (4.63) with the coordinate $y = (q_1 - q_2)/2$ describes an $\mathcal{N}=4$ generalization of a particular (hyperbolic) Pöschl-Teller system [44] with additional bosonic spin variables Z_a^k . As opposed to the $\mathcal{N}=4$ case, in the $\mathcal{N}=0$ and $\mathcal{N}=2$ cases (Sections 2 and 3) the relevant two-particle models describe Pöschl-Teller systems without any extra spin variables.⁴

⁴ For supersymmetric many-particle generalizations of the Pöschl-Teller system see, e.g., [45–49].

5. Conclusions

In this paper we have derived new models of multi-particle supersymmetric mechanics, which are $\mathcal{N}=2$ and $\mathcal{N}=4$ supersymmetric generalizations of the Calogero-Sutherland hyperbolic system of A_{n-1} root type. The input in our construction was superfield matrix systems with $U(n)$ gauge symmetry. Because of this, the resulting n -particle systems feature $\mathcal{N}n^2$ real physical fermions, like the rational $\mathcal{N}=2$ and $\mathcal{N}=4$ Calogero systems constructed in [15,16,11] by using a similar gauging approach. In the $\mathcal{N}=4$ case, the system involves additional semi-dynamical bosonic spin variables and so describes a $\mathcal{N}=4$ supersymmetric generalization of the $U(2)$ -spin Calogero-Sutherland hyperbolic system.

In subsequent works we aim at the explicit form of the supersymmetry generators in the models constructed here and plan carry out their quantization. An interesting question is as to whether the classical and quantum integrability of the bosonic Calogero-Sutherland model is preserved upon the supersymmetrization considered. Another option is the generalization of the $\mathcal{N}=4$ model to deformed $SU(2|1)$ supersymmetry along the lines of [19,20]. The intrinsic mass parameter (deformation parameter) in this case will amount to an additional oscillator potential in the corresponding actions.

Finally, the remaining integrable many-particle Calogero-type systems from the list of [24,25] wait to be supersymmetrized.

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