Skyrme and Faddeev models in the low-energy limit of 4d Yang–Mills–Higgs theories

Olaf Lechtenfeld [a,b,*], Alexander D. Popov [a]

[a] *Institut für Theoretische Physik, Leibniz Universität Hannover, Appelstraße 2, 30167 Hannover, Germany*
[b] *Riemann Center for Geometry and Physics, Leibniz Universität Hannover, Appelstraße 2, 30167 Hannover, Germany*

Abstract

Firstly, we consider $U(N_c)$ Yang–Mills gauge theory on $\mathbb{R}^{3,1}$ with $N_f > N_c$ flavours of scalar fields in the fundamental representation of $U(N_c)$. The moduli space of vacua is the Grassmannian manifold $Gr(N_c, N_f)$. It is shown that for strong gauge coupling this 4d Yang–Mills–Higgs theory reduces to the Faddeev sigma model on $\mathbb{R}^{3,1}$ with $Gr(N_c, N_f)$ as target. Its action contains the standard two-derivative sigma-model term as well as the four-derivative Skyrme-type term, which stabilizes solutions against scaling. Secondly, we consider a Yang–Mills–Higgs model with $N_f = 2N_c$ and a Higgs potential breaking the flavour group $U(N_f) = U(2N_c)$ to $U_+(N_c) \times U_-(N_c)$, realizing the simplest $A_2 \oplus A_2$-type quiver gauge theory. The vacuum moduli space of this model is the group manifold $U_h(N_c)$ which is the quotient of $U_+(N_c) \times U_-(N_c)$ by its diagonal subgroup. When the gauge coupling constant is large, this 4d Yang–Mills–Higgs model reduces to the Skyrme sigma model on $\mathbb{R}^{3,1}$ with $U_h(N_c)$ as target. Thus, both the Skyrme and the Faddeev model arise as effective field theories in the infrared of Yang–Mills–Higgs models.

© 2019 The Authors. Published by Elsevier B.V. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/). Funded by SCOAP³.

* Corresponding author.
  E-mail addresses: olaf.lechtenfeld@itp.uni-hannover.de (O. Lechtenfeld), alexander.popov@itp.uni-hannover.de (A.D. Popov).

1. Introduction and summary

In 1975, Faddeev introduced a (3+1)-dimensional SU(2)/U(1) coset sigma model that includes a term quartic in derivatives to stabilize classical solutions [1]. This model is similar to the Skyrme model [2], which features maps from $\mathbb{R}^{3,1}$ into SU(2). Despite their similarity, these models are quite different from one another. Topological solitons of the Skyrme model have a point-like core and are supposed to describe baryons and nuclei (see e.g. [3] for a review and [4–6] for some recent works). On the other hand, solitons in the Faddeev model take the form of stable knotted strings characterized by the Hopf charge (homotopy class of maps $S^3 \rightarrow S^2$). It is conjectured that Faddeev-model solitons describe glueballs (see e.g. [7–9] for reviews).

The standard Skyrme model [2] supposedly describes pions. Other mesons can be incorporated into an *extended* 4d Skyrme model, which is obtained from 5d Yang–Mills theory on an AdS-type manifold M^5 with boundary $\partial M^5 = \mathbb{R}^{3,1}$ as derived from D-brane configurations in string theory and the holographic approach [10] (see e.g. [11–13] for reviews). This extended Skyrme model also arises in the adiabatic limit of the 5d Yang–Mills system on $\mathbb{R}^{3,1} \times \mathcal{I}$, where $\mathcal{I}$ is a short interval [14].[1] Similarly, also an *extended* 4d Faddeev model can emerge in a low-energy limit of 5d maximally supersymmetric Yang–Mills theory with its five adjoint scalars [22]. In contrast to the extended Skyrme model, for the extended Faddeev model one needs to keep one of the five adjoint scalars and must modify the fifth dimension from $\mathcal{I}$ to the half-line $\mathbb{R}_+$. The boundary conditions required for the reduction to $\mathbb{R}^{3,1}$ are encoded in Nahm equations along the fifth dimension [23,24], which reduce to a "baby" Nahm equation on $\mathbb{R}_+$ for one adjoint scalar [22].

Quantum chromodynamics (QCD) as well as Yang–Mills theory are strongly coupled in the infrared limit, and hence the perturbative expansion for them breaks down. In the absence of a quantitative understanding of non-perturbative QCD, convenient alternatives at low energy are provided by effective models among which nonlinear sigma models play an important role, especially the Skyrme and Faddeev models. Both models are the standard two-derivative sigma models on $\mathbb{R}^{3,1}$ with a compact Lie group G and a coset space G/H as target spaces, respectively, completed with a four-derivative term which stabilizes classical solutions against scaling. In the Faddeev model, H is a closed subgroup of G such that G/H is a coadjoint orbit.

As we discussed above, both Skyrme and Faddeev models can be obtained as low-energy limit of 5d Yang–Mills–Higgs (YMH) theories on the classical level. On the other hand, in the strong-coupling or infrared limit, many YMH models on $\mathbb{R}^{d-1,1}$ with $d \geq 2$ reduce to standard two-derivative sigma models governing maps from $\mathbb{R}^{d-1,1}$ to a moduli space of Higgs vacua. In other words, YMH theories flow in the infrared to sigma models on the same space $\mathbb{R}^{d-1,1}$ (see e.g. [25] and references therein). For YMH models which are bosonic parts of supersymmetric QCD in $d=4$, these classical moduli spaces are non-trivial Kähler or hyper-Kähler manifolds [26, 27,25]. Here we will show that the four-derivative Skyrme term also naturally appears in these four-dimensional YMH models in the framework of the adiabatic approach.[2] To summarize, we demonstrate that both the Skyrme model and the Faddeev model occupy an infrared corner of 4d YMH models related with $\mathcal{N}=2$ supersymmetric QCD.

---

[1] The adiabatic approach was used in field theory for the first time by Manton [15]. For a review of this approach see [16,17]; brief discussions can be found e.g. in [18–21].

[2] Some steps in the derivation of Skyrme terms from YMH models were taken in [28,29], but for a different class of YMH models and without using the adiabatic method.

2. Yang–Mills–Higgs model

Notation. On Minkowski space $\mathbb{R}^{3,1} \ni x^\mu$ with the metric $(\eta_{\mu\nu}) = \mathrm{diag}(-1, 1, 1, 1)$, we consider $U(N_c)$ gauge theory with N_f flavours of scalar fields in the fundamental representation of $U(N_c)$, combined in an $N_c \times N_f$ matrix Φ. A gauge potential $\mathcal{A} = \mathcal{A}_\mu \, dx^\mu$ and the Yang–Mills field $\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ take values in the Lie algebra $u(N_c)$. Its components read $\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu]$, where $\partial_\mu := \partial/\partial x^\mu$ and $\mu, \nu = 0, 1, 2, 3$. For the generators $I_{\hat{\imath}}$ of the gauge group $U(N_c)$ we use the standard normalization $\mathrm{tr}(I_{\hat{\imath}} I_{\hat{\jmath}}) = -\frac{1}{2} \delta_{\hat{\imath}\hat{\jmath}}$.

Transformations of fields. The covariant derivative of the complex Higgs field Φ in the bi-fundamental representation of $U(N_c) \times U(N_f)$ with $N_f > N_c$ reads

$$D_\mu \Phi = \partial_\mu \Phi + \mathcal{A}_\mu \Phi \tag{2.1}$$

since the $U(N_f)$ flavour group acts on Φ only by global transformations

$$\Phi \mapsto \Phi \, g_f \,. \tag{2.2}$$

We denote by $\mathcal{G}$ the infinite-dimensional group $C^\infty(\mathbb{R}^{3,1}, U(N_c))$ of gauge transformations which are parametrized by $g_c(x) \in \mathcal{G}$ for $x \in \mathbb{R}^{3,1}$. Then $\mathcal{A}$ and Φ are transformed as

$$\mathcal{A} \mapsto {}^{g_c}\!\mathcal{A} = g_c \mathcal{A} g_c^{-1} + g_c d g_c^{-1} \qquad \text{and} \qquad \Phi \mapsto {}^{g_c}\!\Phi = g_c \Phi \,. \tag{2.3}$$

For the infinitesimal action of $\mathcal{G}$ we have

$$\mathcal{A} \mapsto \delta_{\epsilon_c} \mathcal{A} = d\epsilon_c - [\mathcal{A}, \epsilon_c] \qquad \text{and} \qquad \delta_{\epsilon_c} \Phi = \epsilon_c \Phi \qquad \text{with} \qquad g_c = \exp(\epsilon_c) \,. \tag{2.4}$$

Similarly, for the $U(N_f)$ flavour symmetry we have

$$\delta_{\epsilon_f} \mathcal{A} = 0 \qquad \text{and} \qquad \delta_{\epsilon_f} \Phi = \Phi \epsilon_f \,, \tag{2.5}$$

where $\epsilon_c \in \mathrm{Lie}\mathcal{G} = C^\infty(\mathbb{R}^{3,1}, u(N_c))$ and $\epsilon_f \in u(N_f)$.

Lagrangian. We consider the Yang–Mills–Higgs (YMH) action functional

$$S = - \int\limits_{\mathbb{R}^{3,1}} d^4x \left\{ \mathrm{tr}\left(\tfrac{1}{2e^2} \mathcal{F}_{\mu\nu}^\dagger \mathcal{F}^{\mu\nu} + D_\mu \Phi (D^\mu \Phi)^\dagger \right) + \tfrac{e^2}{4} V(\Phi) \right\} , \tag{2.6}$$

where † denotes Hermitian conjugation, e is the gauge coupling constant, and

$$V(\Phi) = \mathrm{tr}\left(M^2 \mathbf{1}_{N_c} - \Phi\Phi^\dagger \right)^2 \tag{2.7}$$

is the Higgs potential with a mass parameter M. The Lagrangian from (2.6) is related with the bosonic part of the Lagrangian for $\mathcal{N}=2$ supersymmetric QCD, and such Lagrangians are often considered in the literature (see e.g. [25,30] and references therein).

The energy density $\mathcal{H}$ of YMH configurations described by (2.6) is

$$\mathcal{H} = \mathrm{tr}\left(\tfrac{1}{e^2} \mathcal{F}_{0a}^\dagger \mathcal{F}_{0a} + D_0 \Phi (D_0 \Phi)^\dagger + \tfrac{1}{2e^2} \mathcal{F}_{ab}^\dagger \mathcal{F}_{ab} + D_a \Phi (D_a \Phi)^\dagger \right) + \tfrac{e^2}{4} V(\Phi) \,, \tag{2.8}$$

where $a, b = 1, 2, 3$. Here both $V(\Phi)$ and $\mathcal{H}$ are positive-semidefinite and gauge-invariant functions. They are also $U(N_f)$-invariant.

Vacua. A YMH vacuum configuration $(\hat{\mathcal{A}}, \hat{\mathcal{F}}, \hat{\Phi})$ is defined by the vanishing of the energy density (2.8). This is achieved by

$$\hat{\mathcal{F}}_{\mu\nu} = 0 , \quad \hat{D}_{\mu}\hat{\Phi} = 0 \quad \text{and} \quad V(\hat{\Phi}) = 0 , \tag{2.9}$$

where the last equation defines the Higgs vacuum manifold. Denote by $\widetilde{\mathcal{M}}$ the space of solutions of (2.9) with $\hat{A} = 0 = \hat{\mathcal{F}}$ and $\hat{\Phi} \in \mathrm{Mat}(N_c, N_f; \mathbb{C})$ (complex $N_c \times N_f$ matrices) such that

$$\hat{\Phi}\hat{\Phi}^{\dagger} = M^2 \mathbf{1}_{N_c} , \tag{2.10}$$

i.e. $\widetilde{\mathcal{M}}$ is the space of solutions to (2.10). The group $\mathrm{U}(N_c)$ acts freely on $\widetilde{\mathcal{M}}$ by left multiplication, $\hat{\Phi} \mapsto g_c\hat{\Phi}$. It is not difficult to show [31] that $\widetilde{\mathcal{M}}$ is fibred over the Grassmannian

$$\mathrm{Gr}(N_c, N_f) = [\mathrm{U}(N_c) \times \mathrm{U}(N_f - N_c)]\backslash \mathrm{U}(N_f) =: \mathcal{M} \tag{2.11}$$

with the projection

$$\pi : \quad \widetilde{\mathcal{M}} \overset{\mathrm{U}(N_c)}{\longrightarrow} \mathcal{M} \tag{2.12}$$

and the group $\mathrm{U}(N_c)$ as fibres.

It is important to distinguish between the Higgs field Φ depending on $x \in \mathbb{R}^{3,1}$ and the vacua $\hat{\Phi} \in \mathrm{Mat}(N_c, N_f; \mathbb{C})$, which solve (2.10). The moduli space of vacua $\mathcal{M}$ is the Grassmannian (2.11), any element of which can be obtained from a reference vacuum $\hat{\Phi}_0$. We choose $\hat{\Phi}_0 = (\mathbf{1}_{N_c} \ \mathbf{0}_{N_c \times (N_f - N_c)})$ so that the isotropy group of $\hat{\Phi}_0$ for the right $\mathrm{U}(N_f)$ action is

$$\mathrm{U}(N_c) \times \mathrm{U}(N_f - N_c) = \left\{ g_f \in \mathrm{U}(N_f) : \ \hat{\Phi}_0\, g_f = g_c\hat{\Phi}_0 \ \text{for some } g_c \in \mathrm{U}(N_c) \right\} . \tag{2.13}$$

It is obvious [31] that such g_f have the form $\mathrm{diag}(g_c, g_{f-c})$ with $g_{f-c} \in \mathrm{U}(N_f - N_c)$. In other words, the right action of the isotropy group $\mathrm{U}(N_c) \times \mathrm{U}(N_f - N_c)$ on $\hat{\Phi}_0$ is equivalent to the left action of the gauge group $\mathrm{U}(N_c)$, and we simply have

$$\widetilde{\mathcal{M}} = \mathrm{U}(N_f - N_c)\backslash \mathrm{U}(N_f) . \tag{2.14}$$

3. Moduli space of vacua

Geometry of $\mathrm{Gr}(N_c, N_f)$. The space $\widetilde{\mathcal{M}}$ in (2.12) parametrizes all vacua for the model (2.6), and the Grassmannian $\mathcal{M}$ in (2.11) and (2.12) parametrizes gauge inequivalent vacua, i.e. the vacuum moduli space. Both $\widetilde{\mathcal{M}}$ and $\mathcal{M}$ are homogeneous spaces with a right action of $\mathrm{U}(N_f)$. Note that right cosets can be changed to left cosets by interchanging $\hat{\Phi}$ with $\hat{\Phi}^{\dagger}$.

Let $\mathfrak{m}$ be the tangent space to the Grassmannian $\mathcal{M}$ at the fixed point $\hat{\Phi}_0$. Then we have the splitting

$$u(N_f) = \mathfrak{m} \oplus u(N_c) \oplus u(N_f - N_c) , \tag{3.1}$$

and $\tilde{\mathfrak{m}} = \mathfrak{m} \oplus u(N_c)$ can be identified with the tangent space of $\widetilde{\mathcal{M}}$ at any given point. For $u(N_f)$ we choose a basis

$$\{I_i\} = \{I_{\bar{\imath}}, I_{\hat{\imath}}, I_{i'}\} \quad \text{with} \quad \begin{cases} \bar{\imath} &= 1, \dots, \dim \mathfrak{m} = 2N_c(N_f - N_c) , \\ \hat{\imath} &= \dim \mathfrak{m} + 1, \dots, \dim \mathfrak{m} + N_c^2 , \\ i' &= \dim \mathfrak{m} + N_c^2 + 1, \dots, \dim \mathfrak{m} + N_c^2 + (N_f - N_c)^2 , \end{cases} \tag{3.2}$$

so that $I_{\bar{\imath}}$, $I_{\hat{\imath}}$ and $I_{i'}$ form orthogonal bases for $\mathfrak{m}$, $u(N_c)$ and $u(N_f - N_c)$, respectively. One can associate to I_i vector fields V_i on $\mathrm{U}(N_f)$ and a basis $\{e^i\} = \{e^{\bar{\imath}}, e^{\hat{\imath}}, e^{i'}\}$ of one-forms which is dual to $\{V_i\}$, i.e. $V_i \lrcorner e^j = \delta_i^j$. These one-forms obey the Maurer-Cartan equations

$$
\begin{aligned}
\mathrm{d}e^{\bar{\imath}} &= -f^{\bar{\imath}}_{\hat{\jmath}\bar{k}}\, e^{\hat{\jmath}} \wedge e^{\bar{k}} - f^{\bar{\imath}}_{\hat{\jmath}'\bar{k}}\, e^{\hat{\jmath}'} \wedge e^{\bar{k}}\,, \\
\mathrm{d}e^{\hat{\imath}} &= -\tfrac{1}{2} f^{\hat{\imath}}_{\hat{\jmath}\bar{k}}\, e^{\hat{\jmath}} \wedge e^{\bar{k}} - \tfrac{1}{2} f^{\hat{\imath}}_{\hat{\jmath}\hat{k}}\, e^{\hat{\jmath}} \wedge e^{\hat{k}}\,, \\
\mathrm{d}e^{\hat{\imath}'} &= -\tfrac{1}{2} f^{\hat{\imath}'}_{\hat{\jmath}\bar{k}}\, e^{\hat{\jmath}} \wedge e^{\bar{k}} - \tfrac{1}{2} f^{\hat{\imath}'}_{\hat{\jmath}'\hat{k}'}\, e^{\hat{\jmath}'} \wedge e^{\hat{k}'}\,,
\end{aligned}
\tag{3.3}
$$

where we used the fact that $\mathrm{Gr}(N_c, N_f)$ is a symmetric space.

The Grassmannian $\mathcal{M} = \mathrm{Gr}(N_c, N_f)$ supports an orthonormal frame of one-forms $\{e^{\bar{\imath}}\}$ locally giving the $\mathrm{U}(N_f)$-invariant metric as

$$
\mathrm{d}s^2_{\mathcal{M}} = \delta_{\bar{\imath}\bar{\jmath}}\, e^{\bar{\imath}} e^{\bar{\jmath}} = \delta_{\bar{\imath}\bar{\jmath}}\, e^{\bar{\imath}}_{\alpha} e^{\bar{\jmath}}_{\beta}\, \mathrm{d}X^{\alpha} \mathrm{d}X^{\beta}
$$

$$
=: g_{\alpha\beta}\, \mathrm{d}X^{\alpha} \mathrm{d}X^{\beta}
$$

$$
\text{for} \quad \alpha, \beta = 1, \ldots, 2N_c(N_f - N_c),
\tag{3.4}
$$

where $\{X^{\alpha}\}$ is a set of real local coordinates of a point $X \in \mathrm{Gr}(N_c, N_f)$, and $\partial_{\alpha} = \partial/\partial X^{\alpha}$ will denote derivatives with respect to them.

Canonical connection. On the principal $\mathrm{U}(N_c)$-bundle (2.12) there exists a unique $\mathrm{U}(N_f)$-equivariant connection, the so-called canonical connection (see e.g. [32–35]),

$$
\mathcal{A}_{\mathrm{Gr}} = \mathcal{A}^{\mathrm{Gr}}_{\alpha}\, \mathrm{d}X^{\alpha} = e^{\hat{\imath}} I_{\hat{\imath}} = e^{\hat{\imath}}_{\alpha} I_{\hat{\imath}}\, \mathrm{d}X^{\alpha}
\tag{3.5}
$$

taking values in $u(N_c)$. It satisfies both Yang–Mills and generalized instanton equations on $\mathrm{Gr}(N_c, N_f)$ [33–35]. The curvature of the canonical connection (3.5) in the bundle (2.12) follows as

$$
\mathcal{F}_{\mathrm{Gr}} = \tfrac{1}{2} \mathcal{F}^{\mathrm{Gr}}_{\alpha\beta}\, \mathrm{d}X^{\alpha} \wedge \mathrm{d}X^{\beta} = -\tfrac{1}{2} f^{\hat{\imath}}_{\bar{\jmath}\bar{k}}\, I_{\hat{\imath}}\, e^{\bar{\jmath}} \wedge e^{\bar{k}} = -\tfrac{1}{2} f^{\hat{\imath}}_{\bar{\jmath}\bar{k}}\, I_{\hat{\imath}}\, e^{\bar{\jmath}}_{\alpha} e^{\bar{k}}_{\beta}\, \mathrm{d}X^{\alpha} \wedge \mathrm{d}X^{\beta}\,.
\tag{3.6}
$$

Variation of $\hat{\Phi}$. By letting X^{α} run over $\mathcal{M}$ we obtain a local section $\hat{\Phi}(X^{\alpha})$ of the bundle (2.12).

The infinitesimal changes of this section are given by the covariant derivatives (cf. [18,19])

$$
\delta_{\alpha} \hat{\Phi} = \partial_{\alpha} \hat{\Phi} + \mathcal{A}^{\mathrm{Gr}}_{\alpha}\, \hat{\Phi}\,,
\tag{3.7}
$$

where $\mathcal{A}^{\mathrm{Gr}}_{\alpha}$ are the components of the connection (3.5) in the principal $\mathrm{U}(N_c)$ bundle (2.12).

4. Faddeev model in the infrared limit of 4d YMH

Dependence on x^{μ}. Now we return to Yang–Mills–Higgs theory on $\mathbb{R}^{3,1}$. In Section 3 we described the moduli space $\mathcal{M} = \mathrm{Gr}(N_c, N_f)$ of vacua for the YMH model (2.6)–(2.8). For small exitations around $\mathcal{M}$, in the strong gauge-coupling limit $e^2 \gg 1$, the Higgs field $\Phi(x)$ can be considered as a map

$$
\Phi: \quad \mathbb{R}^{3,1} \rightarrow \mathrm{Gr}(N_c, N_f)
\tag{4.1}
$$

since for $e^2 \gg 1$ it should be at a minimum of the Higgs potential (2.7). The moduli-space approximation then postulates that all fields depend on the spacetime coordinates $x = \{x^{\mu}\}$ only via coordinates $X^{\alpha} = X^{\alpha}(x)$ on $\mathcal{M}$ (see e.g. [15–21] and references therein). By substituting $\Phi(X^{\alpha}(x))$ and $\mathcal{A}(X^{\alpha}(x))$ into the initial action (2.6), we obtain an effective field theory describing small fluctuations around the vacuum moduli space $\mathcal{M}$.

Two-derivative part of effective action. Multiplying (3.7) by $\partial_\mu X^\alpha$, we obtain

$$\partial_\mu \Phi \;=\; (\partial_\mu X^\alpha)\, \partial_\alpha \Phi \;=\; (\partial_\mu X^\alpha) \delta_\alpha \Phi - \epsilon_\mu \Phi \quad \text{with} \quad \epsilon_\mu = (\partial_\mu X^\alpha) \mathcal{A}_\alpha^{\text{Gr}}, \tag{4.2}$$

where $\epsilon_\mu \in u(N_c)$ is the pull-back of $\mathcal{A}^{\text{Gr}}$ from $\text{Gr}(N_c, N_f)$ to $\mathbb{R}^{3,1}$. It immediately follows that

$$D_\mu \Phi \;=\; \partial_\mu \Phi + \mathcal{A}_\mu \Phi \;=\; (\partial_\mu X^\alpha) \delta_\alpha \Phi + (\mathcal{A}_\mu - \epsilon_\mu) \Phi\,. \tag{4.3}$$

We see that $D_\mu \Phi$ are tangent[3] to $C^\infty(\mathbb{R}^{3,1}, \text{Gr}(N_c, N_f))$ if

$$\mathcal{A}_\mu = \epsilon_\mu\,. \tag{4.4}$$

Substituting (4.3) with (4.4) into the (2.6), we obtain

$$S_{\text{kin}} \;=\; -\int_{\mathbb{R}^{3,1}} \mathrm{d}^4 x \; \eta^{\mu\nu} \operatorname{tr}\{D_\mu \Phi (D_\nu \Phi)^\dagger\} \;=\; -\frac{M^2}{2} \int_{\mathbb{R}^{3,1}} \mathrm{d}^4 x \; \eta^{\mu\nu}\, g_{\alpha\beta}\, \partial_\mu X^\alpha \partial_\nu X^\beta\,, \tag{4.5}$$

where

$$g_{\alpha\beta} \;=\; \tfrac{2}{M^2} \operatorname{tr}\{\delta_\alpha \Phi\, (\delta_\beta \Phi)^\dagger\} \;=\; \delta_{\bar i j}\, e_\alpha^{\bar i} e_\beta^{\bar j} \tag{4.6}$$

are the components of the metric (3.4) on $\text{Gr}(N_c, N_f)$ pulled back to $\mathbb{R}^{3,1}$, so $g_{\alpha\beta}(X^\gamma(x))$ now depend on x. We introduced the mass scale M from (2.7) into (4.6) to render $g_{\alpha\beta}$ dimensionless. Thus, this part of the action (2.6) reduces to the standard non-linear sigma model on $\mathbb{R}^{3,1}$ with the Grassmannian $\text{Gr}(N_c, N_f)$ as its target.

Four-derivative part of effective action. As discussed earlier, the potential term in (2.6) vanishes since $\Phi(x)$ takes values in the manifold $\mathcal{M} = \text{Gr}(N_c, N_f)$ of gauge-inequivalent vacua. For calculating the first term in (2.6), we use (4.4), and for the curvature of $\mathcal{A} = \mathcal{A}_\mu \mathrm{d}x^\mu$ we obtain

$$\mathcal{F} \;=\; \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A} \;=\; \tfrac{1}{2} \mathcal{F}_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \;=\; -\tfrac{1}{2} f_{\bar j \bar k}^{\hat i}\, I_{\hat i}\, e_\alpha^{\bar j} e_\beta^{\bar k}\, \partial_\mu X^\alpha \partial_\nu X^\beta \, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\,, \tag{4.7}$$

allowing one to extract the components $\mathcal{F}_{\mu\nu}$. Substituting (4.7) into (2.6) we arrive at

$$\begin{aligned}
S_{\text{Fad}} &\;=\; -\tfrac{1}{2e^2} \int_{\mathbb{R}^{3,1}} \mathrm{d}^4 x \; \operatorname{tr}(\mathcal{F}_{\mu\nu}^\dagger \mathcal{F}^{\mu\nu}) \\
&\;=\; -\tfrac{1}{4e^2} \int_{\mathbb{R}^{3,1}} \mathrm{d}^4 x \; \delta_{\hat i \hat j} f_{\bar l \bar k}^{\hat i} f_{\bar m \bar n}^{\hat j}\, e_\alpha^{\bar l} e_\beta^{\bar k} e_\gamma^{\bar m} e_\delta^{\bar n}\, \partial_\mu X^\alpha \partial_\nu X^\beta \partial^\mu X^\gamma \partial^\nu X^\delta\,,
\end{aligned} \tag{4.8}$$

where $\partial^\mu := \eta^{\mu\sigma} \partial_\sigma$. Thus, in the infrared limit the Yang–Mills–Higgs action (2.6) is reduced to the Faddeev action,

$$\begin{aligned}
S_{\text{eff}} \;=\; -\int_{\mathbb{R}^{3,1}} \mathrm{d}^4 x \; \Big\{ &\frac{M^2}{2} g_{\alpha\beta} \partial_\mu X^\alpha \partial^\mu X^\beta \\
&+ \frac{1}{4e^2} \delta_{\hat i \hat j} f_{\bar l \bar k}^{\hat i} f_{\bar m \bar n}^{\hat j}\, e_\alpha^{\bar l} e_\beta^{\bar k} e_\gamma^{\bar m} e_\delta^{\bar n}\, \partial_\mu X^\alpha \partial_\nu X^\beta \partial^\mu X^\gamma \partial^\nu X^\delta \Big\}
\end{aligned} \tag{4.9}$$

for scalar fields X^α with values in the Grassmannian $\text{Gr}(N_c, N_f)$.

---

[3] This is a key requirement of the adiabatic approach. It is necessary for the description of *small* fluctuations around the initial moduli space when the dynamical fields are collective coordinates (see e.g. [18,19,36]).

5. $A_2 \oplus A_2$-quiver gauge theory

Fields. It is possible to obtain not only the Faddeev model but also the standard Skyrme model from Yang–Mills–Higgs theory in four dimensions. To achieve this, we should consider a 4d YMH model with a group manifold, say U(N), as the moduli space $\mathcal{M}$ of vacua. The simplest way to do this is to specialize the model (2.6) to $N_f = 2N_c =: 2N$ but with a potential different from (2.7). We parametrize

$$\Phi =: (\phi_-, \phi_+) \quad \text{with} \quad \phi_\pm \in \mathrm{Mat}(N, N; \mathbb{C}) . \tag{5.1}$$

Thus, we have a $u(N)$-valued gauge field $\mathcal{F}$, an $N \times 2N$ complex Higgs field $\Phi = (\phi_-, \phi_+)$, the group of gauge transformations $\mathcal{G} = C^\infty(\mathbb{R}^{3,1}, \ \mathrm{U}(N))$ and transformations (2.2)–(2.5) for $N_f = 2N_c = 2N$.

Action. We consider the Yang–Mills–Higgs (YMH) action functional

$$S = - \int_{\mathbb{R}^{3,1}} \mathrm{d}^4x \ \left\{ \mathrm{tr}\left(\tfrac{1}{2e^2} \mathcal{F}_{\mu\nu}^\dagger \mathcal{F}^{\mu\nu} + D_\mu \Phi (D^\mu \Phi)^\dagger \right) + \tfrac{e^2}{4} V(\Phi) \right\} , \tag{5.2}$$

and the two-term potential

$$V(\Phi) = \mathrm{tr}\left(m^2 \mathbf{1}_N - \phi_- \phi_-^\dagger\right)^2 + \mathrm{tr}\left(m^2 \mathbf{1}_N - \phi_+ \phi_+^\dagger\right)^2 \tag{5.3}$$

with a mass parameter m. This action can be obtained from $A_2^+ \oplus A_2^-$ quiver gauge theory (see e.g. [37,38] and references therein) corresponding to a direct sum of quivers

$$A_2^\pm : \quad \mathbb{C}^N \ \xrightarrow{\phi_\pm} \ \mathbb{C}^N , \tag{5.4}$$

where four copies of $\mathbb{C}^N$ at four vertices carry the fundamental U(N) representation, and the arrows $\phi_\pm$ denote maps between them.

The form (5.3) of the Higgs potential breaks the flavour group U($2N$) to the subgroup $G = \mathrm{U}_-(N) \times \mathrm{U}_+(N)$. Let $\{I_i\}$ be a basis of the Lie algebra $\mathfrak{g} = \mathrm{Lie}\,G = u_-(N) \oplus u_+(N)$ realized as $2N \times 2N$ block-diagonal matrices with the normalization $\mathrm{tr}(I_i I_j) = -\tfrac{1}{2}\delta_{ij}$ for $i = 1, \ldots, 2N^2$. The covariant derivative in (5.2) reads

$$D_\mu \Phi = (D_\mu \phi_-, \ D_\mu \phi_+) \quad \text{with} \quad D_\mu \phi_\pm = \partial_\mu \phi_\pm + \mathcal{A}_\mu \phi_\pm , \tag{5.5}$$

with a $u(N)$-valued gauge potential $\mathcal{A} = \mathcal{A}_\mu \mathrm{d}x^\mu$.

Vacua. The energy density of YMH configurations described by the action (5.2) has the form (2.8) with $V(\Phi)$ given by (5.3). The vacuum configurations are defined by (2.9), which implies

$$\hat{\phi}_- \hat{\phi}_-^\dagger = m^2 \mathbf{1}_N \qquad \text{and} \qquad \hat{\phi}_+ \hat{\phi}_+^\dagger = m^2 \mathbf{1}_N . \tag{5.6}$$

Equations (5.6) are solved by some

$$(\hat{\phi}_-, \hat{\phi}_+) \in \mathrm{U}_-(N) \times \mathrm{U}_+(N) = \widetilde{\mathcal{M}} \tag{5.7}$$

subject to global gauge transformations

$$(\hat{\phi}_-, \hat{\phi}_+) \mapsto (h\hat{\phi}_-, h\hat{\phi}_+) \quad \text{for} \quad h \in \mathrm{U}(N) . \tag{5.8}$$

The group U(N) acts freely on the vacuum manifold $\widetilde{\mathcal{M}}$ by left multiplication, and one can define the projection

$$\pi : \quad \widetilde{\mathcal{M}} \overset{\mathrm{U}(N)}{\longrightarrow} \mathcal{M} \qquad \text{via} \qquad (\hat{\phi}_-, \hat{\phi}_+) \longmapsto (\mathbf{1}_N, \hat{\phi}) \qquad \text{with} \quad \hat{\phi} = \hat{\phi}_-^{-1} \hat{\phi}_+ . \tag{5.9}$$

Hence, the moduli space of vacua

$$\mathcal{M} = \mathrm{U}(N) \backslash [\mathrm{U}_-(N) \times \mathrm{U}_+(N)] \tag{5.10}$$

is diffeomorphic to the group manifold $\mathrm{U}(N)$, any element of which can be obtained from a reference vacuum $\hat{\Phi}_0$. We choose $\frac{1}{m} \hat{\Phi}_0 = (\mathbf{1}_N, \mathbf{1}_N)$ so that the isotropy group for the right G action is

$$\mathrm{U}(N) = \left\{ g \in G : \hat{\Phi}_0 \, g = h \, \hat{\Phi}_0 \quad \text{for some } h \in \mathrm{U}(N) \right\} . \tag{5.11}$$

It is obvious that $g = \mathrm{diag}(h, h)$, i.e. the isotropy group is

$$\mathrm{diag}(G) \cong \mathrm{U}_{\mathrm{diag}}(N) = \mathrm{U}(N) =: H , \tag{5.12}$$

and the global gauge transformations form the stability subgroup in a realization of the group manifold $\mathrm{U}(N)$ as the coset space $H \backslash G$ in (5.10). This is also seen from the fact that $(h\hat{\phi}_-)^{-1}(h\hat{\phi}_+) = \hat{\phi}_-^{-1}\hat{\phi}_+$, i.e. $\hat{\phi}$ is inert under the action of H. From (5.9) it follows the decomposition

$$\mathfrak{g} = u_-(N) \oplus u_+(N) = \mathfrak{m} \oplus \mathfrak{h} = \mathfrak{m} \oplus u(N)_{\mathrm{diag}} \quad \text{with} \quad \mathfrak{h} = \left\{ (\eta, \eta) \mid \eta \in u(N) \right\} . \tag{5.13}$$

Geometry of $H \backslash G$. The geometry of a group manifold considered as a homogeneous space has some characteristic features (see e.g. [31,39,40]) which we briefly describe here. In the split (5.13), $\mathfrak{m}$ is not necessarily orthogonal to $\mathfrak{h}$ with respect to the Cartan–Killing form. In fact, there are three natural reductive decompositions of $\mathfrak{g}$ with the following versions of $\mathfrak{m}$:

$$\mathfrak{m}_0 = \left\{ (-\theta, \theta) \right\} , \qquad \mathfrak{m}_- = \left\{ (-\theta, 0) \right\} , \qquad \mathfrak{m}_+ = \left\{ (0, \theta) \right\} , \qquad \text{with} \quad \theta \in u(N) . \tag{5.14}$$

The first case yields $H \backslash G$ as a symmetric space with $\mathfrak{m}_0$ orthogonal to $\mathfrak{h}$. With the choice $\mathfrak{m}_+$ or $\mathfrak{m}_-$ the coset (5.10) becomes a nonsymmetric homogeneous manifold. Obviously, $\mathfrak{m} \cong u(N)$ in all three cases. The choices of $\mathfrak{m}_0$, $\mathfrak{m}_-$ and $\mathfrak{m}_+$ correspond to the gauges $\hat{\phi}_- = \hat{\phi}_+^\dagger$, $\hat{\phi}_+ = m\mathbf{1}_N$ and $\hat{\phi}_- = m\mathbf{1}_N$, respectively, which determine different coset representatives, i.e. sections of the bundle (5.9) with $\widetilde{\mathcal{M}} = G$ and $\mathcal{M} = H \backslash G$.

We split the basis of $\mathfrak{g}$ according to the decomposition (5.13),

$$\{I_i\} = \{I_{\bar{\imath}}, I_{\hat{\imath}}\} \quad \text{with} \quad \begin{cases} \bar{\imath} = 1, \ldots, N^2 & \text{for} \quad \mathfrak{m} , \\ \hat{\imath} = N^2+1, \ldots, 2N^2 & \text{for} \quad \mathfrak{h} . \end{cases} \tag{5.15}$$

We have an orthonormal frame of one-forms $\{e^{\bar{\imath}}\}$ on $H \backslash G$, the metric (3.4) with $\alpha, \beta = 1, \ldots, N^2$ and the canonical connection $\mathcal{A}^{\mathrm{can}} = e^{\hat{\imath}} I_{\hat{\imath}} = e^{\hat{\imath}}_\alpha I_{\hat{\imath}} \, dX^\alpha$ for all three cases $\mathfrak{m}_0$, $\mathfrak{m}_-$ and $\mathfrak{m}_+$. However, the Maurer–Cartan equations depend on the case:

$$\mathfrak{m}_0 : \quad de^{\bar{\imath}} = -f^{\bar{\imath}}_{\hat{\jmath}\bar{k}} e^{\hat{\jmath}} \wedge e^{\bar{k}} \qquad \text{and} \qquad de^{\hat{\imath}} = -\tfrac{1}{2} f^{\hat{\imath}}_{\bar{\jmath}\bar{k}} e^{\bar{\jmath}} \wedge e^{\bar{k}} - \tfrac{1}{2} f^{\hat{\imath}}_{\hat{\jmath}\hat{k}} e^{\hat{\jmath}} \wedge e^{\hat{k}} ,$$

$$\mathfrak{m}_- : \quad de^{\bar{\imath}} = -f^{\bar{\imath}}_{\hat{\jmath}\bar{k}} e^{\hat{\jmath}} \wedge e^{\bar{k}} + \tfrac{1}{2} f^{\bar{\imath}}_{\bar{\jmath}\bar{k}} e^{\bar{\jmath}} \wedge e^{\bar{k}} \qquad \text{and} \qquad de^{\hat{\imath}} = -\tfrac{1}{2} f^{\hat{\imath}}_{\hat{\jmath}\hat{k}} e^{\hat{\jmath}} \wedge e^{\hat{k}} , \tag{5.16}$$

$$\mathfrak{m}_+ : \quad de^{\bar{\imath}} = -f^{\bar{\imath}}_{\hat{\jmath}\bar{k}} e^{\hat{\jmath}} \wedge e^{\bar{k}} - \tfrac{1}{2} f^{\bar{\imath}}_{\bar{\jmath}\bar{k}} e^{\bar{\jmath}} \wedge e^{\bar{k}} \qquad \text{and} \qquad de^{\hat{\imath}} = -\tfrac{1}{2} f^{\hat{\imath}}_{\hat{\jmath}\hat{k}} e^{\hat{\jmath}} \wedge e^{\hat{k}} .$$

Furthermore, on the group manifold (5.10) one can introduce a family of connections

$$\mathcal{A}^{\varkappa}_{\mathrm{U}(N)} = \varkappa \, e^{\hat{\imath}} I_{\hat{\imath}} = \varkappa \, e^{\hat{\imath}}_{\alpha} I_{\hat{\imath}} \mathrm{d} X^{\alpha} =: \mathcal{A}^{\varkappa}_{\alpha} \, \mathrm{d} X^{\alpha} \quad \text{with} \quad \varkappa \in \mathbb{R} \tag{5.17}$$

with curvature

$$\mathcal{F}^{\varkappa}_{\mathrm{U}(N)} = \tfrac{1}{2} \varkappa (\varkappa - 1) f^{\hat{\imath}}_{\hat{\jmath}\hat{k}} I_{\hat{\imath}} \, e^{\hat{\jmath}} \wedge e^{\hat{k}} - \tfrac{1}{2} \varkappa \, f^{\hat{\imath}}_{\hat{\jmath}\bar{k}} I_{\hat{\imath}} \, e^{\hat{\jmath}} \wedge e^{\bar{k}} . \tag{5.18}$$

For the cases $\mathfrak{m}_{\pm}$ the last term in (5.18) vanishes. The connection (5.17) is the unique G-equivariant family of connections on the bundle (5.9) [31,39].

Variation of $\hat{\Phi}$. In the following we adopt the gauge $\frac{1}{m} \hat{\phi}_{-} = \mathbf{1}_{N}$ fixing $\mathfrak{m} = \mathfrak{m}_{+}$, so $f^{\hat{\imath}}_{\hat{\jmath}\bar{k}} = 0$ in (5.18). Then, abbreviating $\hat{\phi}_{+} \equiv \hat{\phi}$,

$$\mathcal{A}^{\varkappa}_{\mathrm{U}(N)} = \varkappa \, \hat{\phi} \, (\partial_{\alpha} \hat{\phi}^{-1}) \, \mathrm{d} X^{\alpha} \quad \Rightarrow \quad \mathcal{A}^{\varkappa}_{\alpha} = \varkappa \, \hat{\phi} \, \partial_{\alpha} \hat{\phi}^{-1} , \tag{5.19}$$

and letting X^{α} run over $\mathcal{M} \cong \mathrm{U}(N)$ we obtain a local section $\hat{\Phi}(X^{\alpha}) = m(\mathbf{1}_{N}, \hat{\phi}(X^{\alpha}))$ of the bundle (5.9). Infinitesimal changes of this section are given by the covariant derivatives (cf. [18, 19])

$$\delta_{\alpha} \hat{\Phi} = \partial_{\alpha} \hat{\Phi} + \mathcal{A}^{\varkappa}_{\alpha} \hat{\Phi} = m \big(\mathcal{A}^{\varkappa}_{\alpha}, \, \partial_{\alpha} \hat{\phi} + \mathcal{A}^{\varkappa}_{\alpha} \hat{\phi} \big) . \tag{5.20}$$

6. Skyrme model in the infrared limit of 4d YMH theory

The derivation of the Skyrme model as an effective theory for the 4d YMH model (5.2) is similar to the derivation of the Faddeev model from the YMH action (2.6). The main difference is that now the vacuum moduli space $\mathcal{M} = H \backslash G = \mathrm{U}(N)$ is a group manifold, whose geometry was described in Section 5. According to the philosophy of the adiabatic method, we assume that the gauge potential $\mathcal{A} = \mathcal{A}_{\mu} \mathrm{d} x^{\mu}$ and the Higgs field Φ depend on the $\mathbb{R}^{3,1}$ coordinates x only via real coordinates $X^{\alpha} = X^{\alpha}(x)$ on $\mathrm{U}(N)$, and we substitute $\mathcal{A}(X^{\alpha}(x))$ and $\Phi(X^{\alpha}(x))$ into the action (5.2) by using results of Section 5.

Kinetic term. Multiplying (5.20) by $\partial_{\mu} X^{\alpha}$, we obtain

$$D_{\mu} \Phi = \partial_{\mu} X^{\alpha} \, \delta_{\alpha} \Phi + (A_{\mu} - \epsilon_{\mu}) \Phi , \tag{6.1}$$

where $\epsilon_{\mu} = (\partial_{\mu} X^{\alpha}) \mathcal{A}^{\varkappa}_{\alpha} \in u(N)$ is the pull-back of $\mathcal{A}^{\varkappa}_{\mathrm{U}(N)}$ from $\mathrm{U}(N)$ to $\mathbb{R}^{3,1}$. To render $(D_{\mu} \Phi) \Phi^{\dagger}$ tangent to $C^{\infty}(\mathbb{R}^{3,1}, \mathrm{U}(N))$, we choose

$$A_{\mu} = \epsilon_{\mu} = \varkappa \, (\partial_{\mu} X^{\alpha}) \, \phi \, \partial_{\alpha} \phi^{-1} = \varkappa \, \phi \, \partial_{\mu} \phi^{-1} , \tag{6.2}$$

where ϕ is a $\mathrm{U}(N)$-valued function. Notice that (5.19) and (5.20) imply

$$\delta_{\alpha} \Phi = -m \big(\varkappa \, (\partial_{\alpha} \phi) \, \phi^{\dagger}, (\varkappa - 1) \, \partial_{\alpha} \phi \big) \quad \Rightarrow \quad D_{\mu} \Phi = -m \big(\varkappa \, (\partial_{\mu} \phi) \, \phi^{\dagger}, (\varkappa - 1) \, \partial_{\mu} \phi \big) . \tag{6.3}$$

Substituting (6.1)–(6.3) into (5.2), we obtain

$$S_{\mathrm{kin}} = -\int_{\mathbb{R}^{3,1}} \mathrm{d}^4 x \, \eta^{\mu\nu} \, \mathrm{tr} \big\{ D_{\mu} \Phi (D_{\nu} \Phi)^{\dagger} \big\} = \tfrac{1}{4} f^2_{\pi} \int_{\mathbb{R}^{3,1}} \mathrm{d}^4 x \, \eta^{\mu\nu} \, \mathrm{tr}(R_{\mu} R_{\nu}) \tag{6.4}$$

with

$$R_\mu := \phi\,\partial_\mu\phi^{-1} \qquad \text{and} \qquad \tfrac{1}{4}\,f_\pi^2 = \left(\varkappa^2 + (\varkappa-1)^2\right)m^2\,, \tag{6.5}$$

where f_π may be interpreted as the pion decay constant. Thus, this part of the action (5.2) reduces to the standard non-linear sigma model on $\mathbb{R}^{3,1}$ with a U(N) target space.

Skyrme term. For calculating the $\mathcal{F}^2$-terms in (5.2) we employ (6.2) and find

$$\mathcal{F} = d\mathcal{A} + \mathcal{A}\wedge\mathcal{A} = \varkappa(\varkappa-1)\,\phi\,d\phi^{-1}\wedge\phi\,d\phi^{-1} = \tfrac{1}{2}\,\varkappa(\varkappa-1)\,[R_\mu, R_\nu]\,dx^\mu\wedge dx^\nu \tag{6.6}$$

since $\mathcal{A}_\mu = \varkappa\,\phi\,\partial_\mu\phi^{-1} = \varkappa R_\mu$ after the pull-back to $\mathbb{R}^{3,1}$. Substituting (6.6) into (5.2), we obtain

$$S_{\text{Sky}} = -\frac{1}{2e^2}\int_{\mathbb{R}^{3,1}} d^4x\;\text{tr}\left(\mathcal{F}_{\mu\nu}^\dagger\mathcal{F}^{\mu\nu}\right) = \frac{1}{32\zeta^2}\int_{\mathbb{R}^{3,1}} d^4x\;\eta^{\mu\lambda}\eta^{\nu\sigma}\,\text{tr}\left([R_\mu, R_\nu][R_\lambda, R_\sigma]\right)\,, \tag{6.7}$$

where

$$\frac{1}{32\zeta^2} = \frac{\varkappa^2(\varkappa-1)^2}{8e^2}\,, \tag{6.8}$$

and ζ is the dimensionless Skyrme parameter. Hence, in the infrared limit the Yang–Mills–Higgs action (5.2) is reduced to the action of the Skyrme model,

$$S_{\text{eff}} = \int_{\mathbb{R}^{3,1}} d^4x\left\{\frac{f_\pi^2}{4}\,\eta^{\mu\nu}\,\text{tr}\left(R_\mu R_\nu\right) + \frac{1}{32\zeta^2}\,\eta^{\mu\lambda}\eta^{\nu\sigma}\,\text{tr}\left([R_\mu, R_\nu][R_\lambda, R_\sigma]\right)\right\}\,. \tag{6.9}$$

Thus, both Skyrme and Faddeev models appear as effective field theories in the infrared of Yang–Mills–Higgs models.

Acknowledgements

References

[1] L. Faddeev, Quantization of solitons, IAS preprint, Print-75-QS70, Princeton, 1975.
[2] T.H.R. Skyrme, A unified field theory of mesons and baryons, Nucl. Phys. 31 (1962) 556.
[3] I. Zahed, G.E. Brown, The Skyrme model, Phys. Rep. 142 (1986) 1.
[4] R.A. Battye, N.S. Manton, P.M. Sutcliffe, S.W. Wood, Light nuclei of even mass number in the Skyrme model, Phys. Rev. C 80 (2009) 034323, arXiv:0905.0099 [nucl-th].
[5] M. Gillard, D. Harland, E. Kirk, B. Maybee, M. Speight, A point particle model of lightly bound skyrmions, Nucl. Phys. B 917 (2017) 286, arXiv:1612.05481 [hep-th].
[6] C. Adam, M. Haberichter, T. Romanczukiewicz, A. Wereszczynski, Roper resonances and quasi-normal modes of Skyrmions, J. High Energy Phys. 03 (2018) 023, arXiv:1710.00837 [hep-th].
[7] L.D. Faddeev, Knots as possible excitations of the quantum Yang–Mills fields, in: M.-L. Ge, C.H. Oh, K.K. Phua (Eds.), Proc. Conf. in Honor of C.N. Yang's 85th Birthday, Statistical Physics, High Energy, Condensed Matter and Mathematical Physics, World Scientific, Singapore, 2008, p. 18, arXiv:0805.1624 [hep-th].
[8] P. Sutcliffe, Knots in the Skyrme–Faddeev model, Proc. R. Soc. Lond. A 463 (2007) 3001, arXiv:0705.1468 [hep-th].

[9] D. Harland, Topological energy bounds for the Skyrme and Faddeev models with massive pions, Phys. Lett. B 728 (2014) 518, arXiv:1311.2403 [hep-th].

[10] T. Sakai, S. Sugimoto, Low energy hadron physics in holographic QCD, Prog. Theor. Phys. 113 (2005) 843, arXiv: hep-th/0412141.

[11] J. Erdmenger, N. Evans, I. Kirsch, E. Threlfall, Mesons in gauge/gravity duals – a review, Eur. Phys. J. A 35 (2008) 81, arXiv:0711.4467 [hep-th].

[12] V. Kaplunovsky, D. Melnikov, J. Sonnenschein, Baryonic popcorn, J. High Energy Phys. 11 (2012) 047, arXiv: 1201.1331 [hep-th].

[13] P. Sutcliffe, Holographic skyrmions, Mod. Phys. Lett. B 29 (2015) 1540051.

[14] T.A. Ivanova, O. Lechtenfeld, A.D. Popov, Skyrme model from 6d $\mathcal{N} = (2, 0)$ theory, Phys. Lett. B 783 (2018) 222, arXiv:1805.07241 [hep-th].

[15] N.S. Manton, A remark on the scattering of BPS monopoles, Phys. Lett. B 110 (1982) 54.

[16] N.S. Manton, P. Sutcliffe, Topological Solitons, Cambridge University Press, Cambridge, 2004.

[17] E.J. Weinberg, P. Yi, Magnetic monopole dynamics, supersymmetry, and duality, Phys. Rep. 438 (2007) 65, arXiv: hep-th/0609055.

[18] J.A. Harvey, A. Strominger, String theory and the Donaldson polynomial, Commun. Math. Phys. 151 (1993) 221, arXiv:hep-th/9108020.

[19] J.A. Harvey, G.W. Moore, A. Strominger, Reducing S-duality to T-duality, Phys. Rev. D 52 (1995) 7161, arXiv: hep-th/9501022.

[20] A. Deser, O. Lechtenfeld, A.D. Popov, Sigma-model limit of Yang–Mills instantons in higher dimensions, Nucl. Phys. B 894 (2015) 361, arXiv:1412.4258 [hep-th].

[21] T.A. Ivanova, O. Lechtenfeld, A.D. Popov, Non-Abelian sigma models from Yang–Mills theory compactified on a circle, Phys. Lett. B 781 (2018) 322, arXiv:1803.07322 [hep-th].

[22] O. Lechtenfeld, A.D. Popov, Skyrme-Faddeev model from 5d super-Yang–Mills, Phys. Lett. B 786 (2018) 39, arXiv:1807.11803 [hep-th].

[23] D. Gaiotto, G.W. Moore, Y. Tachikawa, On 6d $\mathcal{N} = (2, 0)$ theory compactified on a Riemann surface with finite area, PTEP 2013 (2013) 013B03, arXiv:1110.2657 [hep-th].

[24] B. Assel, S. Schäfer-Nameki, J.M. Wong, M5-branes on $S^2 \times M_4$: Nahm's equations and 4d topological sigma-models, J. High Energy Phys. 09 (2016) 120, arXiv:1604.03606 [hep-th].

[25] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, N. Sakai, Solitons in the Higgs phase: the moduli matrix approach, J. Phys. A 39 (2006) R315, arXiv:hep-th/0602170.

[26] N. Seiberg, Exact results on the space of vacua of four-dimensional SUSY gauge theories, Phys. Rev. D 49 (1994) 6857, arXiv:hep-th/9402044.

[27] C. Beasley, E. Witten, New instanton effects in supersymmetric QCD, J. High Energy Phys. 01 (2005) 056, arXiv: hep-th/0409149.

[28] E. Babaev, Non-Meissner electrodynamics and knotted solitons in two-component superconductors, Phys. Rev. B 79 (2009) 104506, arXiv:0809.4468 [cond-mat.supr-con].

[29] A. Mohamadnejad, Skyrme-Faddeev Lagrangian from reformulated Georgi-Glashow model, Europhys. Lett. 123 (2018) 61001, arXiv:1609.05605 [hep-th].

[30] D. Tong, Quantum vortex strings: a review, Ann. Phys. 324 (2009) 30, arXiv:0809.5060 [hep-th].

[31] S. Kobayashi, K. Nomizu, Foundations of Differential Geometry, vol. 2, Interscience Publishers, New York, 1969.

[32] S. Kobayashi, K. Nomizu, Foundations of Differential Geometry, vol. 1, Interscience Publishers, New York, 1963.

[33] J.P. Harnad, J. Tafel, S. Shnider, Canonical connections on Riemannian symmetric spaces and solutions to the Einstein-Yang–Mills equations, J. Math. Phys. 21 (1980) 2236.

[34] D. Harland, T.A. Ivanova, O. Lechtenfeld, A.D. Popov, Yang-Mills flows on nearly Kähler manifolds and G_2-instantons, Commun. Math. Phys. 300 (2010) 185, arXiv:0909.2730 [hep-th].

[35] D. Harland, C. Nölle, Instantons and Killing spinors, J. High Energy Phys. 03 (2012) 082, arXiv:1109.3552 [hep-th].

[36] K. Uhlenbeck, Moduli spaces and adiabatic limits, Not. Am. Math. Soc. 42 (1995) 41.

[37] A.D. Popov, R.J. Szabo, Quiver gauge theory of non-Abelian vortices and noncommutative instantons in higher dimensions, J. Math. Phys. 47 (2006) 012306, arXiv:hep-th/0504025.

[38] O. Lechtenfeld, A.D. Popov, R.J. Szabo, Rank two quiver gauge theory, graded connections and noncommutative vortices, J. High Energy Phys. 09 (2006) 054, arXiv:hep-th/0603232.

[39] Y.A. Kubyshin, V.O. Malyshenko, D. Marin Ricoy, Invariant connections with torsion on group manifolds and their application in Kaluza–Klein theories, J. Math. Phys. 35 (1994) 310, arXiv:gr-qc/9304047.

[40] D. Harland, The Skyrme model and chiral perturbation theory, arXiv:1612.07259 [hep-th].