Toeplitz operators and generated algebras on non-Hilbertian spaces

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Abstract

In this thesis we study Toeplitz operators on spaces of holomorphic and pluriharmonic functions. The main part of the thesis is concerned with such operators on the Fock spaces of holomorphic functions, F_t^p for $p \in [1, \infty]$.

We establish a notion of *Correspondence Theory* between symbols and Toeplitz operators, based on extended notions of convolutions as developed by Reinhard Werner [130], which gives rise to many important results on Toeplitz operators and the algebras they generate. Here, we find new proofs for old theorems, extending them to a larger range of values of p, and also provide entirely new results. We manage to include even the non-reflexive cases of $p = 1, \infty$ in our studies.

Based on the notions of *band-dominated* and *limit operators*, we establish a general criterion for an operator in the Toeplitz algebra over F_t^p to be Fredholm: Such an operator is Fredholm if and only if all of its limit operators are invertible.

As an example of a Toeplitz algebra over the Fock space, we study the Resolvent Algebra (in the sense of Detlev Buchholz and Hendrik Grundling [42]) in its Fock space representation.

Partially following the methods of Correspondence Theory as discussed in this thesis, we manage to extend a classical result on the boundedness of Toeplitz operators (the Berger-Coburn estimates) to the setting of p-Fock spaces.

Also based on results derived from the Correspondence Theory, we discuss several new characterizations of the full Toeplitz algebra on Fock spaces, at least in the reflexive range $p \in (1, \infty)$.

In the last part, we discuss several results on spectral theory and quantization estimates for Toeplitz operators acting on Bergman and Fock spaces of pluriharmonic functions.

Zusammenfassung

In dieser Arbeit werden Toeplitzoperatoren auf Räumen holomorpher und pluriharmonischer Funktionen studiert. Der Hauptteil dieser Arbeit befasst sich mit Operatoren auf den Fockräumen holomorpher Funktionen, F_t^p für $p \in [1, \infty]$.

Basierend auf gewissen verallgemeinerten Faltungen, wie sie von Reinhard Werner [130] eingeführt wurden, diskutieren wir eine Korrespondenztheorie von Symbolen und Toeplitzoperatoren. Mittels dieser Korrespondenztheorie lassen sich viele wichtige Aussagen zu Toeplitzoperatoren und den von ihnen erzeugten Algebren herleiten. Basierend darauf geben wir neue Beweise für bereits bekannte Sätze, deren Aussagen wir teilweise auf weitere Werte von p ausweiten können, finden aber auch einige komplett neue Ergebnisse. Insbesondere ist es mit diesen Methoden möglich, die nichtreflexiven Fälle $p = 1, \infty$ zu behandeln.

Mittels sogenannter banddominierter Operatoren und Grenzoperatoren geben wir ein Kriterium für die Fredholmeigenschaft beliebiger Operatoren aus der Toeplitzalgebra über F_t^p: Ein solcher Operator ist Fredholm genau dann, wenn alle seine

Grenzoperatoren invertierbar sind.

Weiterhin studieren wir die Resolventenalgebra (im Sinne von Detlev Buchholz und Hendrik Grundling [42]) in ihrer Fockraum-Darstellung, welche ein Beispiel für eine Toeplitzalgebra darstellt.

Wir erweitern die klassischen Ergebnisse von Charles Berger und Lewis Coburn zur Beschränktheit von Toeplitzoperatoren auf dem Fockraum auf den Fall beliebiger Werte von $p \in [1, \infty]$. Dafür benutzen wir teilweise Methoden der bereits oben erwähnten Korrespondenztheorie.

Ebenfalls basierend auf einigen der aus der Korrespondenztheorie hergeleiteten Ergebnisse diskutieren wir neue Charakterisierungen der Toeplitzalgebra auf dem Fockraum, zumindest für den reflexiven Bereich $p \in (1, \infty)$.

Zuletzt befassen wir uns mit einigen Ergebnissen aus der Spektraltheorie und Quantisierungsabschätzungen von Toeplitzoperatoren auf Bergman- und Fockräumen pluriharmonischer Funktionen.

Keywords: Toeplitz operators, Fock space, operator algebras

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Contents

Chapter 1

Introduction

Among the many different classes of linear operators, multiplication operators are certainly one of the best understood. Of an entirely different nature, but also relatively easy to understand, are orthogonal projections and restrictions. To an apprentice of operator theory, it may come as a surprise that the combination of such harmless objects may give rise to particularly difficult mathematical problems. Of course, we refer to Toeplitz operators: Let (X, μ) be a measure space, A a closed subspace of $L^p(X, \mu)$ and $P \in \mathcal{L}(L^p(X, \mu))$ a bounded projection onto A. Usually, A is chosen to be a space of holomorphic functions, but this is not necessary. For any suitable function f from X to $\mathbb{C}$, say measurable and essentially bounded, the Toeplitz operator T_f is defined as the compression of the operator of multiplication by f,

$$T_f = PM_f|_A : A \to A.$$

These operators are now an established subject of mathematical research. They originate from a work by Otto Toeplitz [123], where he considered operators on $\ell^2(\mathbb{N})$, the so-called Toeplitz matrices, which are equivalent to Toeplitz operators on the Hardy space $H^2(S^1)$. Since then, many different geometric settings have been considered in the studies of Toeplitz operators. Besides the classical situation of Hardy spaces on the circle [36, 105], some of the more studied geometric settings are Toeplitz operators on Kähler manifolds [8, 95] and on Bergman and Hardy spaces of (strictly) pseudoconvex domains [92, 126] and of bounded symmetric domains [21, 124]. In particular, the intersection of the previous two classes, the complex unit balls, have been studied intensively [125, 136].

The majority of this thesis deals with the study of Toeplitz operators on the Segal-Bargmann-Fock spaces F_t^p, i.e. on spaces of holomorphic functions on $\mathbb{C}^n$ which are p-integrable with respect to a Gaussian measure. For simplicity, we will only speak of Fock spaces, without wanting to downplay the roles of I. Segal and V. Bargmann in their initial explorations.

In principle, the study of Toeplitz operators can be distinguished into two different, but not disjoint subjects: One can either try to understand properties of a single Toeplitz operator or study algebras generated by a collection of these. Of course, it is most fruitful to combine the two approaches. As a motivation, let us mention a (by now classical) result from each direction.

Upon studying individual Toeplitz operators, one usually tries to derive properties of the operator T_f from its symbol. Here, "symbol" can refer to two different functions: The obvious one, which is usually named symbol (or *contravariant symbol* in the works of F. A. Berezin [22–26] and M. A. Šubin [27, 127]), is the function f used for defining T_f. The other one is the so-called *Berezin symbol* (or *covariant symbol* according to Berezin) of T_f, being denoted as $\mathcal{B}(f)$, is again a function on X and is constructed by relating the symbol f to the *reproducing kernel structure* of the underlying Bergman, Hardy or Fock space. A classical result (which holds true in many different geometric situations) is the following:

$$T_f \text{ is compact } \iff \mathcal{B}(f) \text{ vanishes at the boundary of } X.$$

A (by now well-known) result in the study of algebras generated by Toeplitz operators is the characterization of commutative Toeplitz algebras on Bergman spaces of the unit disk [125], which closely relates geometric structures with the algebras at hand.

While both results seemingly belong to only one of the two parts, i.e. either the study of individual operators or of entire algebras, each one benefits from the other subject: On the one hand, the compactness characterization in terms of the Berezin symbol carries over to every operator belonging to the *full Toeplitz algebra*, i.e. the Banach algebra generated by all Toeplitz operators with bounded symbols, provided that the underlying geometric space has plenty of symmetries [19, 82, 104, 122]. On the other hand, the characterization of commutative Toeplitz algebras (again in the presence of sufficiently many symmetries on the underlying geometry) hinges on a good understanding of individual operators and their invariances (along with invariances of their symbols) with respect to such symmetries. While some of those results could be considered classical by now, they still have a strong influence on current research: The works [1, 18, 64–66, 88, 90, 131, 138] are just a small selection of publications from the recent years related to those problems.

Toeplitz operators are also often studied as tools for quantization: The quantization map $f \mapsto T_f$, which in this setting is usually called *Berezin-Toeplitz quantization*, can be considered as a model for passing from the classical world (i.e. functions/symbols) to the quantum realm (i.e. operators on Hilbert spaces). Usually, a semiclassical parameter is introduced, which we will denote for various reasons by t or λ instead of the more physical $\hbar$. We then obtain a quantization scheme depending on this parameter, say $f \mapsto T_f^t$. Upon studying such quantization procedures, it is usually imposed that the quantized objects, T_f^t, behave as the classical objects, the symbols f, in the "classical limit" (which corresponds to letting $t \to 0$ in our notation). Here, the important difference between quantized and classical objects is commutativity: While functions clearly commute, $fg = gf$, this is in general not true for operators: $T_f^t T_g^t \neq T_g^t T_f^t$. Hence, when studying the quantization properties of the map $f \mapsto T_f^t$ one usually studies the behavior of the product $T_f^t T_g^t$ in the limit $t \to 0$. As it turns out, at least for suitable underlying geometries, Berezin-Toeplitz quantization provides a pleasant framework for quantization, as it satisfies many of the desired properties [2, 48, 57, 59–62]. In particular, it falls within the framework of Rieffel's *strict quantization* [114], asking that the asymptotics

$$\begin{aligned} \|T_f^t\| &\to \|f\|_\infty, \\ \|T_f^t T_g^t - T_{fg}^t\| &\to 0, \\ \left\| \frac{1}{t}[T_f^t, T_g^t] - iT_{\{f,g\}}^t \right\| &\to 0, \end{aligned}$$

as $t \to 0$ hold true for a suitable symbol space and a Poisson bracket $\{\cdot, \cdot\}$. It has been a focus of research in recent years to figure out the minimal assumptions on the symbols f, g such that the above asymptotics prevail [9, 12, 13, 17, 35, 81].

Let us now discuss the contents of this thesis. Since most of this work will be on operators on Fock spaces, we will give a thorough introduction to these spaces in Chapter 2, which consists mostly of well-known results, but also some new aspects appear there. Let us already mention the following important aspect of Fock space theory here: A particular group of automorphisms of $\mathbb{C}^n$ plays an important role in studying operators on Fock spaces, namely the shifts: $w \mapsto w - z$. They induce a group action α of $\mathbb{C}^n$ on the set of all (measurable, bounded) functions $f : \mathbb{C}^n \to \mathbb{C}$ by defining $\alpha_z(f)$ through

$$\alpha_z(f)(w) = f(w - z).$$

Similarly, the shifts define a group action on $\mathcal{L}(F_t^p)$: If W_z^t denotes the Weyl operators on F_t^p (i.e. they are a family of weighted shift operators on F_t^p with weight parameter $t > 0$), then define for $A \in \mathcal{L}(F_t^p)$

$$\alpha_z(A) := W_z^t A W_{-z}^t.$$

The point about the importance of these group actions is the following: If we denote the Toeplitz operator with symbol f on F_t^p by T_f^t, then we have the relation

$$\alpha_z(T_f^t) = T_{\alpha_z(f)}^t.$$

The importance of this relation for the study of Toeplitz operators on Fock spaces cannot be overestimated.

A similar structure of group actions exists in the setting of functions on $\mathbb{R}^{2n}$ and linear operators on $L^2(\mathbb{R}^n)$, being realized as *phase and space shifts.* This structure led R. Werner in [130] to the development of what he called *Quantum Harmonic Analysis.* Implementing Quantum Harmonic Analysis in the setting of the (non-Hilbertian) spaces F_t^p will be the content of the first part of this thesis. Motived by Werner's work, we will implement a notion of *convolution between functions and operators* on the Fock space, that is: We will define convolutions $f * B$, $A * g$, $A * B$ for certain functions $f, g : \mathbb{C}^n \to \mathbb{C}$ and bounded linear operators A, B on F_t^p. The study of these operations and their applications is the content of Chapter 3. The key features of these convolutions are that they naturally extend the convolution of functions on $\mathbb{C}^n$, $f * g$, and enjoy many of the natural properties of convolutions between functions. A thorough study of these convolutions will finally lead us to the *Correspondence Theorem*, which will ultimately establish a unique correspondence between closed, α-invariant subspaces of $\mathrm{BUC}(\mathbb{C}^n)$, the bounded uniformly continuous functions on $\mathbb{C}^n$, and closed, α-invariant subspaces of the full Toeplitz algebra on F_t^p. In particular, we will spend some effort to establish these results even in the non-reflexive cases $p = 1, \infty$. Having the Correspondence Theorem at hand, we will show that many important results on Toeplitz operators can be derived easily. Right now we only want to mention the compactness characterization cited above: It will turn out to be a simple consequence of the Correspondence Theorem and, which to the best of the author's knowledge has not been done before, will also be discussed in the non-reflexive setting.

Chapter 4 will be dedicated to applying the Correspondence Theorem to gain understanding of Toeplitz algebras. This can be motivated by the following surprising theorem, due to J. Xia: He proved in [131] that the full Toeplitz algebra on F_t^2 is the same as the operator norm closure of the space of all Toeplitz operators with bounded symbols, i.e. if we use for a subspace $\mathcal{D}_0$ of $L^\infty(\mathbb{C}^n)$ the notations

$$\mathcal{T}_{lin}^{2,t}(\mathcal{D}_0) := \overline{\{T_f^t \in \mathcal{L}(F_t^2);\ f \in \mathcal{D}_0\}},$$
$$\mathcal{T}_{*}^{2,t}(\mathcal{D}_0) := C^*(\{T_f^t \in \mathcal{L}(F_t^2);\ f \in \mathcal{D}_0\}),$$

he proved

$$\mathcal{T}_{*}^{2,t}(L^\infty(\mathbb{C}^n)) = \mathcal{T}_{lin}^{2,t}(L^\infty(\mathbb{C}^n)).$$

Thanks to the Correspondence Theorem, we will see that we can reduce the situation to symbols from $\mathrm{BUC}(\mathbb{C}^n)$, i.e.

$$\mathcal{T}_{*}^{2,t}(L^\infty(\mathbb{C}^n)) = \mathcal{T}_{*}^{2,t}(\mathrm{BUC}(\mathbb{C}^n)) = \mathcal{T}_{lin}^{2,t}(\mathrm{BUC}(\mathbb{C}^n)).$$

A similar result, which is well-known and closely related to the aforementioned compactness characterization, is

$$\mathcal{T}_{*}^{2,t}(C_0(\mathbb{C}^n)) = \mathcal{T}_{lin}^{2,t}(C_0(\mathbb{C}^n)).$$

These observations motivate the theme to which Chapter 4 is dedicated: Study (closed, α-invariant) subspaces $\mathcal{D}_0$ of $\mathrm{BUC}(\mathbb{C}^n)$ for which $\mathcal{T}_{lin}^{2,t}(\mathcal{D}_0)$ is a C^* algebra. Indeed, we will obtain the following characterization: If $\mathcal{D}_0 \subset \mathrm{BUC}(\mathbb{C}^n)$ is closed, α- and U-invariant (U being the action $Uf(z) = f(-z)$), then the following are equivalent:

$$\mathcal{D}_0 \text{ is a } C^* \text{ algebra} \iff \mathcal{T}_{lin}^{2,t}(\mathcal{D}_0) \text{ is a } C^* \text{ algebra for every } t > 0.$$

Here, t plays the role of a quantization constant in the construction of the Fock spaces F_t^p. Since the methods are partially based on the Correspondence Theorem, which is not restricted to the Hilbert space setting, our method of proof will work for any p. Further, we will obtain a similar result regarding α-invariant ideals. The presentations of Chapter 3 and 4 are based on, and partially improve, the author's results in [72].

Having already characterized compactness of Toeplitz operators, it is obvious to look for a characterization of the Fredholm property. If the symbol f behaves nicely at infinity (in terms of a certain oscillatory behaviour), conditions equivalent to this property in terms of the behavior of the Berezin symbol at infinity can be obtained. For a general L^∞ symbol or possibly an arbitrary operator from the Toeplitz algebra however the answer is not so simple. This is where we have to consider the *limit operators* of the operator we started with. Simply speaking, using the group action α_z we obtain the limit operators of $A \in \mathcal{L}(F_t^p)$ as all possible limit points of $\alpha_z(A)$ when z goes to infinity. If we take A from the Toeplitz algebra, then the limit operators indeed yield (spectral) information on A, i.e.

$$A \text{ is Fredholm} \iff \text{every limit operator of } A \text{ is invertible.}$$

Proving this result is the content of Chapter 5. We want to emphasize that the idea of studying limit operators for determining the Fredholm property is nothing new: In the setting of band-dominated operators on sequence spaces, these techniques are well-known [44, 96]. The approach we use for deriving our result has previously been established by R. Hagger [80] for achieving similar results for Toeplitz operators on Bergman spaces over the unit balls and has been worked out, in the Fock space setting, by the author together with R. Hagger in [73].

A classical object of study in theoretical physics is the *canonical commutator relations*. By this, we mean the following: If (X, σ) is a symplectic space, we want to study $\mathbb{R}$-linear maps ϕ from X into the space of self-adjoint operators on a Hilbert space $\mathcal{H}$ satisfying the relation

$$[\phi(f), \phi(g)] = i\sigma(f, g), \quad f, g \in X.$$

In general, the elements $\phi(f)$ turn out to be unbounded operators. Since algebraic expressions involving unbounded operators can be problematic, it is customary to pass to bounded operators generated by the $\phi(f)$ using the functional calculus. The most common approach is possibly passing to the unitary operators generated by the $\phi(f)$, i.e. one studies the C^* algebra generated by the elements $\exp(i\phi(f))$. The properties of such algebras, which are known as *CCR algebras* or *Weyl algebras*, are well-understood [40]. However, since there are certain drawbacks to using these algebras for physical considerations, it was proposed in [42] to instead use the C^* algebra generated by the resolvents of the $\phi(f)$. If we consider the standard symplectic space on $\mathbb{C}^n$, then this algebra of resolvents can actually be represented as a Toeplitz algebra, as we shall describe and study in Chapter 6. This chapter is based on ongoing work, which is done jointly with W. Bauer.

One of the most elementary properties of Toeplitz operators is that they are bounded whenever their symbol is bounded. The converse statement is in general wrong: There are many bounded Toeplitz operators with unbounded symbols. The understanding of this phenomenon is far from complete, i.e. it is an open problem to characterize boundedness of Toeplitz operators with unbounded symbol. The possibly most important contribution in that direction, at least on Fock spaces, was given by C. A. Berger and L. A. Coburn in [31], cf. also the recent works [50, 51]. There, they provided an upper and a lower bound for the operator norm of T_f^t in terms of the *heat transform* of the symbol f. More precisely, if $\widetilde{f}^{(s)}$ denotes the heat transform at time $s/4$, which is closely related to the Berezin transform in the setting of the Fock space, they proved estimates of the form

$$\begin{aligned} \|\widetilde{f}^{(s)}\|_\infty &\lesssim \|T_f^t\|_{F_t^2 \to F_t^2}, \quad && s \in (t/2, 2t), \\ \|T_f^t\|_{F_t^2 \to F_t^2} &\lesssim \|\widetilde{f}^{(s)}\|_\infty, && s \in (0, t/2), \end{aligned}$$

under certain technical assumptions on the symbol f. The content of Chapter 7 will be to establish analogous estimates for the cases $p \neq 2$. While, for the first estimate, this in principle boils down to a more technical version of the initial proof, the second estimate needs to be proven entirely different. The original proof of the second estimate depended on transforming the problem into a problem on pseudodifferential operators on $L^2(\mathbb{R}^n)$, using the Bargmann transform, and then applying the Calderón-Vaillancourt Theorem. These tools are not available in the non-Hilbertian setting. Instead, we use direct estimates for the integral kernel to establish an analogous estimate. This Chapter is partially based on joint work with W. Bauer [15].

The very short Chapter 8 will establish a handful of new characterizations of the full Toeplitz algebra on F_t^p. Building on results from Chapters 3 and 4 and on a theorem established by R. Hagger [79], we will present a few estimates which give further characterizations of this algebra.

Finally, let us return to the beginning of this introduction. There, we mentioned that the closed subspace of $L^p(X, \mu)$, on which we define the Toeplitz operators is *usually* defined as a space of holomorphic functions. In Chapter 9, we ignore this convention and deal with Toeplitz operators on spaces of pluriharmonic functions. The presentation is based on our paper [71]. Initially, our work started as a project on pluriharmonic Fock spaces. Yet, at some point it was clear that the study of Toeplitz operators on pluriharmonic function spaces is, to a large extent, independent of the actual underlying geometry and can be reduced to applying results from the holomorphic situation in combination with several algebraic tricks. In particular, the methods work in the same way on, say, Bergman spaces of bounded symmetric domains. Therefore, we included a very brief introduction to such Bergman spaces at the beginning of this chapter. Afterwards, we study two different questions. First, we investigate the essential spectrum for Toeplitz operators on pluriharmonic spaces (over $\mathbb{C}^n$ or bounded symmetric domains), at least for relatively nice symbols. Secondly, we investigate “quantization estimates” in the pluriharmonic world. These quantization estimates are supposed to show that the Toeplitz quantization $f \mapsto T_f$ behaves as in the “classical world” (i.e. taking operator products is the same as taking products of functions), at least in the limit of the quantization parameter. As was already noted in [62], Berezin-Toeplitz quantization on pluriharmonic function spaces is not perfect, since one of the important estimates fails. Hence, we end with applying those estimates, which actually work, to the spectral theory of Toeplitz operators on a non-standard Bergman space.

We end this thesis with three short appendices on topics which are relevant for the presentation.

Chapter 2

Fock spaces and their operators

2.1 Basic definitions and facts

On the Borel-σ-algebra of $\mathbb{C}^n$ we consider the family of Gaussian measures

$$d\mu_t(z) = \frac{1}{(\pi t)^n} e^{-\frac{|z|^2}{t}} dV(z)$$

for each $t > 0$. Here, V denotes the Lebesgue measure on $\mathbb{C}^n \cong \mathbb{R}^{2n}$. To simplify notation, we will usually write $dz,\ dw, \dots$ instead of $dV(z),\ dV(w), \dots$ when integrating with respect to the Lebesgue measure. It is well-known and a simple exercise in integration that the measures μ_t are actually probability measures.

For each p such that $1 \leq p < \infty$ and $t > 0$ we define the spaces L_t^p as the Lebesgue spaces

$$L_t^p := L^p(\mathbb{C}^n, \mu_{2t/p}),$$

i.e. the norm on L_t^p is given by

$$\|f\|_{L_t^p} = \left(\int_{\mathbb{C}^n} |f(w)|^p d\mu_{2t/p}(w) \right)^{1/p}.$$

We will always suppress n in the notion of L_t^p for simplicity. It seems odd that the variance of the Gaussian measure $\mu_{2t/p}$ depends on p. The necessity of this will be clear later. For the case $p = \infty$ we set

$$\|f\|_{L_t^\infty} := \operatorname*{ess\,sup}_{z \in \mathbb{C}^n} |f(z)| e^{-\frac{|z|^2}{2t}}$$

and

$$L_t^\infty := \{f : \mathbb{C}^n \to \mathbb{C};\ \|f\|_{L_t^\infty} < \infty\}.$$

All those spaces are clearly Banach spaces, F_t^2 being a Hilbert space with the inner product

$$\langle f, g \rangle_t := \int_{\mathbb{C}^n} f(w)\overline{g(w)} d\mu_t(w), \quad f,\ g \in L_t^2.$$

The primary spaces of interest to us will not be L^p_t, but the following subspaces, called Fock spaces:

$$F^p_t := L^p_t \cap \mathrm{Hol}(\mathbb{C}^n),$$

where

$$\mathrm{Hol}(\mathbb{C}^n) := \{f : \mathbb{C}^n \to \mathbb{C};\ f \text{ holomorphic}\}.$$

We will usually ignore the imprecision that L^p_t consists of equivalence classes of functions and not actual functions - this will not cause any problems if not mentioned.

Lemma 2.1.1. *Let $K \subset \mathbb{C}^n$ be a compact subset. For each $1 \leq p \leq \infty$ and $t > 0$ there exists a constant $C = C(K, t, p)$ such that for all $f \in F^p_t$ the following estimate holds true:*

$$\|f\|_{\infty,K} := \sup_{z\in K} |f(z)| \leq C\|f\|_{L^p_t}.$$

Proof. Let $f \in F^p_t$ for $1 < p < \infty$. Further, let q be the conjugate exponent to p, i.e. $1/p + 1/q = 1$. By Corollary A.1.4 and Hölder's inequality we have the following estimate for $K \subset \mathbb{C}^n$ compact, where C_K is independent of f:

$$\begin{aligned}\sup_{z\in K} |f(z)| &\leq C_K \int_{\mathbb{C}^n} |f(w) \cdot 1| e^{-\frac{|w|^2}{2t}}\, dw \\ &\leq C_K \left(\int_{\mathbb{C}^n} |f(w)|^p e^{-\frac{p|w|^2}{2t}}\, dw\right)^{1/p} \left(\int_{\mathbb{C}^n} e^{-\frac{q|w|^2}{2t}}\, dw)\right)^{1/q} \\ &= C_K' \|f\|_{L^p_t},\end{aligned}$$

i.e. for each compact $K \subset \mathbb{C}^n$ there is a constant C_K' such that

$$\|f\|_{\infty,K} := \sup_{z\in K} |f(z)| \leq C_K'\|f\|_{L^p_t}$$

for all $f \in F^p_t$. The analogous statement for F^1_t follows also immediately from Corollary A.1.4, the same statement for F^∞_t is immediate from the definition. □

The previous lemma has the following important consequence:

Lemma 2.1.2. *For each $1 \leq p \leq \infty$ and $t > 0$, F^p_t is a closed subspace of L^p_t.*

Proof. Let $(f_n)_{n\in\mathbb{N}} \subset F^p_t$ converge to $f \in L^p_t$. Possibly after passing to a subsequence, we may assume that $f_n(z) \to f(z)$ almost everywhere. By Lemma 2.1.1 f_n converges uniformly on all compact subsets to f (or, the be precise: to a representative of f). Therefore, f (or its particular representative) has to be holomorphic as well. □

The following subspace of F^∞_t will also be of relevance:

$$f^\infty_t := \{f \in F^\infty_t;\ f(z)e^{-\frac{|z|^2}{2t}} \to 0 \text{ as } |z| \to \infty\}.$$

Lemma 2.1.3. *f^∞_t is a closed subspace of F^∞_t.*

Proof. Let $(f_n)_{n\in\mathbb{N}} \subset f_t^\infty$ converge to $f \in F_t^\infty$. In particular, $f_n e^{-\frac{|\cdot|^2}{2t}} \in C_0(\mathbb{C}^n)$ and $fe^{-\frac{|\cdot|^2}{2t}} \in C_b(\mathbb{C}^n)$, where $C_b(\mathbb{C}^n)$ are the bounded continuous functions on $\mathbb{C}^n$ and $C_0(\mathbb{C}^n)$ is the ideal of continuous functions vanishing at infinity. Since $C_0(\mathbb{C}^n)$ is of course closed in $C_b(\mathbb{C}^n)$ with respect to the uniform topology, this yields $fe^{-\frac{|\cdot|^2}{2t}} \in C_0(\mathbb{C}^n)$, i.e. $f \in f_t^\infty$. □

Remark 2.1.4. f_t^∞ is a proper subspace of F_t^∞. As an example, consider $f(z) = e^{\frac{z_1^2+\cdots+z_n^2}{2t}}$, which is contained in F_t^∞ but not in f_t^∞.

We also have the following estimate, which is similar to Lemma 2.1.1, but with a precise constant:

Lemma 2.1.5. *Let $1 \leq p \leq \infty$ and $t > 0$. Then, for $f \in F_t^p$ we have*

$$|f(z)| \leq \|f\|_{F_t^p} e^{\frac{|z|^2}{2t}}.$$

Proof. We refer to the proof of [137, Theorem 2.7]. □

In contrast to usual L^p spaces, the Fock spaces are included in each other in certain ways:

Proposition 2.1.6. *Let $1 < p < p' < \infty$ and $0 < s < t$. Then, the following inclusions are well-defined and continuous:*

$$F_t^1 \hookrightarrow F_t^p \hookrightarrow F_t^{p'} \hookrightarrow f_t^\infty \hookrightarrow F_t^\infty \hookrightarrow F_s^1.$$

Proof. Let us start with the trivial cases: The continuity of the inclusion $f_t^\infty \hookrightarrow F_t^\infty$ follows from the definition.

Let $f \in F_t^\infty$. Then,

$$\begin{aligned}
\|f\|_{F_s^1} &= \int_{\mathbb{C}^n} |f(w)| d\mu_{2s}(w) \\
&= \frac{1}{(2\pi s)^n} \int_{\mathbb{C}^n} |f(w)| e^{-\frac{|w|^2}{2s}}\, dw \\
&= \frac{1}{(2\pi s)^n} \int_{\mathbb{C}^n} |f(w)| e^{-\frac{|w|^2}{2t}} e^{-\frac{|w|^2}{2}(\frac{1}{s}-\frac{1}{t})}\, dw \\
&\leq \frac{1}{(2\pi s)^n} \|f\|_{F_t^\infty} \int_{\mathbb{C}^n} e^{-\frac{|w|^2}{2}(\frac{1}{s}-\frac{1}{t})}\, dw \\
&= C\|f\|_{F_t^\infty}.
\end{aligned}$$

Now, let $1 \leq p < p' < \infty$ and $f \in F_t^p$. Then, following the computations in the proof of [137, Theorem 2.10] and using Lemma 2.1.5,

$$\begin{aligned}
\|f\|_{F_t^{p'}}^{p'} &= \left(\frac{p'}{2\pi t}\right)^n \int_{\mathbb{C}^n} |f(w)|^p |f(w)|^{p'-p} e^{-\frac{p'|w|^2}{2t}}\, dw \\
&\leq \left(\frac{p'}{2\pi t}\right)^n \|f\|_{F_t^p}^{p'-p} \int_{\mathbb{C}^n} e^{\frac{(p'-p)|w|^2}{2t}} |f(w)|^p e^{-\frac{p'|w|^2}{2t}}\, dw \\
&= \left(\frac{p'}{2\pi t}\right)^n \|f\|_{F_t^p}^{p'-p} \int_{\mathbb{C}^n} |f(w)|^p e^{-\frac{p|w|^2}{2t}}\, dw \\
&= \left(\frac{p'}{p}\right)^n \|f\|_{F_t^p}^{p'-p} \|f\|_{F_t^p}^p,
\end{aligned}$$

which gives $\|f\|_{F_t^{p'}} \leq \left(\frac{p'}{p}\right)^{n/p'} \|f\|_{F_t^p}$. Continuity of the inclusion $F_t^{p'} \hookrightarrow F_t^\infty$ follows immediately from Lemma 2.1.5. Finally, the inclusion $F_t^{p'} \subset f_t^\infty$ follows since polynomials are dense in $F_t^{p'}$, which we prove later on (Proposition 2.1.9). □

It is now the right moment to discuss certain elements and subclasses of the Fock spaces. By $\mathcal{P}[z_1, \ldots, z_n] \subset \mathrm{Hol}(\mathbb{C}^n)$ we denote the algebra of holomorphic polynomials in $z_1, \ldots, z_n$. Elementary estimates show that $\mathcal{P}[z_1, \ldots, z_n] \subset F_t^p$ and $\mathcal{P}[z_1, \ldots, z_n] \subset f_t^\infty$ for all $1 \leq p \leq \infty$ and $t > 0$. For a multi-index $\alpha \in \mathbb{N}_0^n$ let us consider the monomials

$$e_\alpha^t(z) = \sqrt{\frac{1}{\alpha! t^{|\alpha|}}} z^\alpha, \quad z = (z_1, \ldots, z_n) \in \mathbb{C}^n,$$

where we used standard multi-index notation. It is not difficult to prove that $\{e_\alpha;\ \alpha \in \mathbb{N}_0^n\}$ is an orthonormal set in F_t^2. Indeed, by the product structure of the functions and Fubini's Theorem it suffices to check the case $n = 1$. Then, for $j, k \in \mathbb{N}_0$ we have, using polar coordinates,

$$\begin{aligned}
\langle z^j, z^k \rangle_{L_t^2} &= \frac{1}{\pi t} \int_{\mathbb{C}} z^j \overline{z}^k e^{-\frac{|z|^2}{t}}\, dz \\
&= \frac{1}{\pi t} \int_0^\infty \int_0^{2\pi} r^{j+k+1} e^{-\frac{r^2}{t}} e^{i\theta(j-k)}\, dr\, d\theta.
\end{aligned}$$

For $j \neq k$, the angular integral evaluates to 0, while for $j = k$ we obtain

$$\begin{aligned}
\langle z^j, z^j \rangle_{L_t^2} &= \frac{2}{t} \int_0^\infty r^{2j+1} e^{-\frac{r^2}{t}}\, dr \\
&= j! t^j.
\end{aligned}$$

Each $f \in F_t^2$ can be expressed as a power series around 0:

$$f(z) = \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha z^\alpha = \sum_{\alpha \in \mathbb{N}_0^n} \tilde{a}_\alpha e_\alpha^t(z),$$

where a_α are appropriate coefficients in the power series and $\tilde{a}_\alpha = a_\alpha \sqrt{\alpha! t^{|\alpha|}}$. Since $\{e_\alpha;\ \alpha \in \mathbb{N}_0^n\}$ is an orthonormal set, as discussed above, the power series converges not only pointwise but also in F_t^2 by some standard Hilbert space argument (e.g. Bessel's inequality). In particular, we obtain:

Lemma 2.1.7. *$\{e_\alpha;\ \alpha \in \mathbb{N}_0^n\}$ is an orthonormal basis in F_t^2.*

Moreover, the holomorphic polynomials are also dense in F_t^p for $1 \leq p < \infty$ and in f_t^∞. Before proving this, let us recall the following fact from integration theory (cf. [115, Chapter 3, Exercise 17]:

Theorem 2.1.8 (Riesz-Radon Theorem)**.** *Let μ be a positive measure on the measurable space $(X, \mathcal{A})$ and $1 \leq p < \infty$. If $f_n, f \in L^p(X, \mu)$ are such that $f_n(x) \to f(x)$ a.e. and $\|f_n\|_{L^p} \to \|f\|_{L^p}$, then $f_n \to f$ in $L^p(X, \mu)$.*

Proposition 2.1.9. *$\mathcal{P}[z_1, \dots, z_n]$ is dense in F_t^p for $1 \leq p < \infty$ and also in f_t^∞ for all $t > 0$.*

Proof. We first prove the result for the case of F_t^p. Our proof follows a standard method, as it is used e.g. in [137, Proposition 2.9]. For $f \in F_t^p$ and $0 < r < 1$ set

$$f_r(z) := f(rz).$$

Obviously, $f_r(z) \to f(z)$ pointwise as $r \to 1$. Let us compute $\|f_r\|_{L_t^p}$:

$$\begin{aligned} \|f_r\|_{L_t^p}^p &= \left(\frac{p}{2t\pi}\right)^n \int_{\mathbb{C}^n} |f(rw)|^p e^{-\frac{p|w|^2}{2t}}\, dw \\ &= \left(\frac{p}{2t\pi}\right)^n r^{-2n} \int_{\mathbb{C}^n} |f(w)|^p e^{-\frac{p|w|^2}{2tr^2}}\, dw. \end{aligned}$$

The last expression converges, by the Dominated Convergence Theorem, to $\|f\|_{L_t^p}^p$ as $r \to 1$. Therefore, by Thereom 2.1.8, $f_r \to f$ in F_t^p.

Let now $0 < r < 1$ be fixed. It suffices to prove that f_r can be approximated by polynomials. Let $s \in (t, \frac{t}{r^2})$. Using Lemma 2.1.5 we see that

$$|f_r(z)| \leq \|f\|_{F_t^p} e^{-\frac{r^2|z|^2}{2t}}, \tag{2.1}$$

which yields $f_r \in F_s^2$. By Proposition 2.1.6 there is a constant $C > 0$ such that

$$\|f_r - g\|_{F_t^p} \leq C\|f_r - g\|_{F_s^2}$$

for each polynomial g. In particular, approximating f_r by its Taylor expansion in F_s^2, due to Lemma 2.1.7 we obtain an approximation by polynomials in F_t^p.

Let us now deal with the case f_t^∞. Once we can prove, with the above notation, that $f_r \to f$ in f_t^∞ for each $f \in f_t^\infty$, the second part of the proof, i.e. approximation by polynomials, can be carried out identically as in F_t^p.

Fix $f \in f_t^\infty$ and let $0 < r < 1$. Then,

$$|f_r(z)|e^{-\frac{|z|^2}{2t}} \leq \|f\|_{f_t^\infty} e^{\frac{r^2|z|^2}{2t}} e^{-\frac{|z|^2}{2t}} \to 0 \quad \text{as } |z| \to \infty,$$

which yields $f_r \in f_t^\infty$. Let $\varepsilon > 0$. Then, there is some $R > 0$ such that for $|z| \geq R$ we have $|f(z)|e^{-\frac{|z|^2}{2t}} < \frac{\varepsilon}{2}$. Further, since $f(z)e^{-\frac{|z|^2}{2t}} \in C_0(\mathbb{C}^n)$, the functions $z \mapsto f(z)e^{-\frac{|z|^2}{2t}}$ and $z \mapsto e^{-\frac{|z|^2}{2t}}$ are uniformly continuous. In particular, there is $\delta > 0$ such that

$$|z - w| < \delta \Longrightarrow |f(z)e^{-\frac{|z|^2}{2t}} - f(w)e^{-\frac{|w|^2}{2t}}|, |e^{-\frac{|z|^2}{2t}} - e^{-\frac{|w|^2}{2t}}| < \varepsilon.$$

Fix $0 < r < 1$ such that $r > \max\left\{\frac{R}{R+1}, 1 - \frac{\delta}{R+1}, \sqrt{1 - \left(\frac{\delta}{R+1}\right)^2}\right\}$. Then, for $|z| > R+1$ we have $r|z| \geq R$ and therefore

$$\begin{aligned}
|f(z) - f(rz)|e^{-\frac{|z|^2}{2t}} &\leq |f(z)|e^{-\frac{|z|^2}{2t}} + |f(rz)|e^{-\frac{|z|^2}{2t}} \\
&\leq \frac{\varepsilon}{2} + |f(rz)|e^{-\frac{|rz|^2}{2t}} e^{-\frac{(1-r^2)|z|^2}{2t}} \\
&\leq \varepsilon.
\end{aligned}$$

For $|z| \leq R+1$ we have $|z - rz| = (1-r)|z| \leq (1-r)(R+1) < \delta$ and therefore obtain

$$\begin{aligned}
|f(z) - f(rz)|e^{-\frac{|z|^2}{2t}} &\leq |f(z)e^{-\frac{|z|^2}{2t}} - f(rz)e^{-\frac{|rz|^2}{2t}}| + |f(rz)e^{-\frac{|rz|^2}{2t}} - f(rz)e^{-\frac{|z|^2}{2t}}| \\
&\leq \varepsilon + |f(rz)|e^{-\frac{|rz|^2}{2t}}|1 - e^{-\frac{(1-r^2)|z|^2}{2t}}| \\
&\leq \varepsilon + \|f\|_{F_t^\infty}.
\end{aligned}$$

By assumption on r we have for z with $|z| \leq R+1$ that $|z|\sqrt{1-r^2} \leq \delta$ and therefore obtain

$$|f(z) - f(rz)|e^{-\frac{|z|^2}{2t}} \leq (1 + \|f\|_{F_t^\infty})\varepsilon.$$

In particular, $\|f - f_r\|_{F_t^\infty} \leq (1 + \|f\|_{F_t^\infty})\varepsilon$, which finishes the proof. □

The previous result has the following important consequence:

Corollary 2.1.10. *In each F_t^p, $1 \leq p < \infty$, and also in f_t^∞, $\{e_m^t;\ m \in \mathbb{N}_0^n\}$ is a Schauder basis. In particular, each of those spaces has the approximation property.*

By Lemma 2.1.1, the point evaluations

$$\delta_z : F_t^p \to \mathbb{C}, \quad f \mapsto f(z)$$

are bounded linear functionals for each $1 \leq p \leq \infty, t > 0$. In the Hilbert space case $p = 2$, the Riesz Representation Theorem now implies that for each $z \in \mathbb{C}^n$ there is a function $K_z^t \in F_t^2$ such that

$$f(z) = \langle f, K_z^t \rangle_t$$

for all $f \in F_t^2$. Let us set $K^t(w,z) := K_z^t(w)$. Observe that

$$K^t(w,z) = K_z^t(w) = \langle K_z^t, K_w^t \rangle_t = \overline{\langle K_w^t, K_z^t \rangle_t} = \overline{K^t(z,w)}$$

and

$$\|K_z^t\|_{L_2^t}^2 = \langle K_z^t, K_z^t \rangle_t = K_z^t(z) = K^t(z,z).$$

In particular, $K(w,z)$ is holomorphic in w and anti-holomorphic in z.

The function $K_z^t(w) = K^t(w,z)$ is called the *reproducing kernel function* of F_t^2. There is a general theory of *reproducing kernel Hilbert spaces* of independent interest, cf. [3, 106]. The following result holds in full generality in such spaces and is one of the keys to compute the reproducing kernel. We repeat the standard proof for completeness.

Lemma 2.1.11. *Let $\{g_j;\ j \in J\}$ an orthonormal basis for F_t^2, where J is a countable index set. Then, the reproducing kernel $K^t(w,z)$ can be computed as*

$$K^t(w,z) = \sum_{j \in J} \overline{g_j(z)} g_j(w).$$

Proof. Since $K_z^t \in F_t^2$, we can express it in the orthonormal basis:

$$\begin{aligned} K_z^t &= \sum_{j \in J} \langle K_z^t, g_j \rangle_t g_j \\ &= \sum_{j \in J} \overline{\langle g_j, K_z^t \rangle_t} g_j \\ &= \sum_{j \in J} \overline{g_j(z)} g_j. \end{aligned}$$

Since point evaluations in F_t^2 are continuous, the series does not only converge in F_t^2 but also pointwise:

$$K_z^t(w) = \sum_{j \in J} \overline{g_j(z)} g_j(w).$$

This concludes the proof. □

Using the above formula, let us compute $K^t(w,z)$. Using the Multinomial Theorem and standard multi-index notation, we have

$$\begin{aligned} K^t(w,z) &= \sum_{\alpha \in \mathbb{N}_0^n} e_\alpha^t(w) \overline{e_\alpha^t(z)} \\ &= \sum_{k=0}^\infty \sum_{|\alpha|=k} \frac{w_1^{\alpha_1} \overline{z_1}^{\alpha_1}}{\alpha_1! t^{\alpha_1}} \cdot \ldots \cdot \frac{w_n^{\alpha_n} \overline{z_n}^{\alpha_n}}{\alpha_n! t^{\alpha_n}} \\ &= \sum_{k=0}^\infty \sum_{|\alpha|=k} \binom{k}{\alpha} \frac{w^\alpha \overline{z}^\alpha}{k! t^k} \end{aligned}$$

$$= \sum_{k=0}^{\infty} \frac{(w \cdot \overline{z})^k}{k! t^k}$$
$$= e^{\frac{w \cdot \overline{z}}{t}}.$$

Here and in what follows, we will always use the convention

$$w \cdot \overline{z} = w_1 \overline{z_1} + \cdots + w_n \overline{z_n}.$$

By definition we have $K_z^t \in F_t^2$. Indeed, the reproducing kernel functions are contained in every Fock space:

Lemma 2.1.12. *It holds true that $K_z^t \in F_t^p$ and $K_z^t \in f_t^\infty$ for all $1 \leq p \leq \infty$, $t > 0$ with $\|K_z^t\|_{F_t^p} = e^{\frac{|z|^2}{2t}}$ for all such p.*

Proof. Using Fubini's Theorem it is straightforward to see that we may assume $n = 1$. For $1 \leq p < \infty$ we now compute the norm as follows:

$$\begin{aligned}
\|K_z^t\|_{F_t^p}^p &= \frac{p}{2\pi t} \int_{\mathbb{C}} \left| e^{\frac{w \cdot \overline{z}}{t}} \right|^p e^{-\frac{p|w|^2}{2t}} \, dw \\
&= \frac{p}{2\pi t} \int_{\mathbb{C}} e^{\frac{p \operatorname{Re}(w \cdot \overline{z})}{t}} e^{-\frac{p|w|^2}{2t}} \, dw \\
&= \frac{p}{2\pi t} \int_{\mathbb{C}} e^{\frac{p}{2t} w \cdot \overline{z}} e^{\frac{p}{2t} z \cdot \overline{w}} e^{-\frac{p|w|^2}{2t}} \, dw \\
&= \langle K_z^{2t/p}, K_z^{2t/p} \rangle_{2t/p} \\
&= e^{\frac{p|z|^2}{2t}}.
\end{aligned}$$

This gives $\|K_z^t\|_{F_t^p} = e^{\frac{|z|^2}{2t}}$.

For $p = \infty$ the identity

$$e^{-\frac{|w|^2}{2t} - \frac{|z|^2}{2t} + \frac{\operatorname{Re}(w \cdot \overline{z})}{t}} = e^{-\frac{|w-z|^2}{2t}},$$

which is readily verified, yields

$$\|K_z^t\|_{F_t^\infty} = \sup_{w \in \mathbb{C}} e^{\frac{\operatorname{Re}(w \cdot \overline{z})}{2t}} e^{-\frac{|w|^2}{2t}} = \sup_{w \in \mathbb{C}} e^{-\frac{|w-z|^2}{2t}} e^{\frac{|z|^2}{2t}} = e^{\frac{|z|^2}{2t}}.$$

Finally, since $\operatorname{Re}(w \cdot \overline{z}) - |w|^2 \to -\infty$ as $|w| \to \infty$, $K_z^t \in f_t^\infty$ follows as well. □

If X is a linear space and $A \subset X$ a subset, we will always denote by $\operatorname{Span} A$ the linear hull of A in X.

Lemma 2.1.13. $\operatorname{Span}\{K_z^t;\ z \in \mathbb{C}^n\}$ *is a dense subset of F_t^p, $1 \leq p < \infty$ and f_t^∞ for all $t > 0$.*

Proof. As a first observation we note that $\mathrm{Span}\{K_z^t;\ z \in \mathbb{C}^n\}$ is always the same set independently of t, as the weight parameter can be incorporated into the base point z.

Consider $p = 2$. In this case, we can equivalently show that $\mathrm{Span}\{K_z^t;\ z \in \mathbb{C}^n\}$ is dense in $(F_t^2)'$, i.e. we will prove that

$$\langle f, \phi \rangle_{F_t^2} = 0 \text{ for all } \phi \in \mathrm{Span}\{K_z^t;\ z \in \mathbb{C}^n\} \Longrightarrow f = 0.$$

But $\langle f, \phi \rangle_{F_t^2} = 0$ for all such ϕ means in particular $\langle f, K_z^t \rangle_{F_t^2} = f(z) = 0$ for all z, i.e. $f = 0$.

Let $1 \leq p < \infty$. By Lemma 2.1.6 for each $s > t$ there exists a $C > 0$ such that $F_s^2 \subset F_t^p$ and $\|f\|_{F_t^p} \leq C\|f\|_{F_s^2}$ for all $f \in F_s^2$. Now, if g is a holomorphic polynomial, then g lies in the closure of $\{\,K_z^t;\ z \in \mathbb{C}^n\}$ with respect to the F_s^2 norm by the above discussion, and by the above norm inequality g lies also into the closure with respect to the F_t^p norm. Since polynomials are dense, the same holds for arbitrary $f \in F_t^p$.

The same proof works for f_t^∞. □

2.2 Duality and interpolation of Fock spaces

The following result describes the duality of the Fock spaces under the dual pairing coming from F_t^2:

Proposition 2.2.1 ([91])**.** *Under the dual pairing $\langle \cdot, \cdot \rangle_t$ the following spaces are isomorphic:*

1) For $1 \leq p < \infty$: $(F_t^p)' \cong F_t^q$, where $1/p + 1/q = 1$;

2) $(f_t^\infty)' \cong F_t^1$;

3) F_t^1 is isomorphic to a subspace of $(F_t^\infty)'$

In particular, we have the following equivalences of norms (where $\langle \cdot, g \rangle_t$ denotes the linear functional induced by the dual pairing for fixed g):

1) For $1 \leq p < \infty$ and $g \in F_t^p$:

$$\|g\|_{F_t^q} \lesssim \|\langle \cdot, g \rangle_t\|_{(F_t^p)'} \lesssim \|g\|_{F_t^q}.$$

2) For $g \in F_t^1$:

$$\|g\|_{F_t^1} \lesssim \|\langle \cdot, g \rangle_t\|_{(f_t^\infty)'} \lesssim \|g\|_{F_t^1}.$$

Proof. Showing that the map $g \mapsto \langle \cdot, g \rangle_t$ is an isomorphism in all the cases stated above follow from standard methods, cf. [91, 137]. From this, the equivalence of norms follows automatically (i.e. by continuity of $g \mapsto \langle \cdot, g \rangle_t$ and its inverse). □

Remark 2.2.2. It is possible to give explicit constants for the norm estimates. For $p \in (1, \infty)$ and $g \in F_t^q$, Hölder's inequality yields

$$\|\langle \cdot, g\rangle_t\|_{(F_t^p)'} \leq \left(\frac{2}{p}\right)^{n/p} \left(\frac{2}{q}\right)^{n/q} \|g\|_{F_t^q}.$$

Further, it was proven in [76] that we have for the first inequality

$$\|g\|_{F_t^q} \leq \|\langle \cdot, g\rangle_t\|_{(F_t^p)'}.$$

For the case of $(f_t^\infty)'$, we obtain from Hölder's inequality

$$\|\langle \cdot, g\rangle_t\|_{(f_t^\infty)'} \leq \frac{1}{2^n}\|g\|_{F_t^1}.$$

Even though [76] only deals with the case $p \in (1, \infty)$, the same methods yield

$$\|g\|_{F_t^1} \leq \|\langle \cdot, g\rangle_t\|_{(f_t^\infty)'}.$$

One can show that F_t^1 is isomorphic to a strict subspace of $(F_t^\infty)'$ via the dual pairing. Therefore, we obtain the following:

Corollary 2.2.3. *The spaces* F_t^p*,* $1 < p < \infty$*, are reflexive, while* F_t^1*,* f_t^∞ *and* F_t^∞ *are not.*

We only cite the following important result on the interpolation behavior of Fock spaces and their ambient Lebesgue spaces under the Complex Interpolation Method:

Theorem 2.2.4 ([91, 137])**.** *For* $t > 0$ *the following holds true for* $0 \leq \theta < 1$*:*

$$(L_t^1, L_t^\infty)_{[\theta]} = L_t^{p_\theta},$$
$$(F_t^1, F_t^\infty)_{[\theta]} = F_t^{p_\theta}.$$

Here, $p_\theta = \frac{1}{1-\theta}$*.*

Remark 2.2.5. In [137], K. Zhu claims that the interpolation formulas in the above theorem are also valid for $\theta = 1$, which seems to be a typo. At least for the Fock spaces this is certainly not true: Part 3) of Theorem A.2.1, together with the inclusions in Proposition 2.1.6, yield that

$$(F_t^1, F_t^\infty)_{[1]} = f_t^\infty.$$

Therefore, Theorem A.2.1 also yields

$$(F_t^1, F_t^\infty)_{[\theta]} = (F_t^1, f_t^\infty)_{[\theta]} = F_t^{p_\theta}$$

for all $0 \leq \theta < 1$.

Let us just mention that the interpolation result for the ambient Lebesgue spaces is not particularly deep. Indeed, one can easily show that multiplication by $e^{-\frac{|\cdot|^2}{2t}}$ gives, up to a constant, an isometric isomorphism

$$M_{\exp\left(-\frac{|\cdot|^2}{2t}\right)} : L_t^p \to L^p(\mathbb{C}^n)$$

for all $1 \leq p \leq \infty$. In particular, the spaces L_t^p interpolate in the same way as the standard Lebesgue spaces $L^p(\mathbb{C}^n)$. After passing to the standard Lebesgue spaces, we can also easily identify the dual of the spaces L_t^p (even isometrically, up to a constant):

Lemma 2.2.6. *Let $1 \leq p < \infty$ and $q \in (1, \infty]$ such that $1/p + 1/q = 1$. Then, the dual of L_t^p can be identified with L_t^q and there is a constant $c_{t,n,p}$ such that*

$$\|g\|_{L_t^q} = c_{t,n,p} \|\langle \cdot, g\rangle_t\|_{(L_t^p)'}.$$

For completeness, we also note the following simple Fock space version of Littlewood's interpolation inequality with its standard proof:

Lemma 2.2.7. *For $f \in F_t^1$ and $1 < p < \infty$ we have*

$$\|f\|_{F_t^p} \leq p^{\frac{n}{p}} \|f\|_{F_t^1}^{1/p} \|f\|_{F_t^\infty}^{1-1/p}.$$

Proof. Follows easily from Hölder's inequality:

$$\begin{aligned}
\|f\|_{F_t^p}^p &= \left(\frac{p}{2\pi t}\right)^n \int_{\mathbb{C}^n} |f(w)e^{-\frac{|w|^2}{2t}}|^p \, dw \\
&= \left(\frac{p}{2\pi t}\right)^n \int_{\mathbb{C}^n} |f(w)e^{-\frac{|w|^2}{2t}}||f(w)e^{-\frac{|w|^2}{2t}}|^{p-1} \, dw \\
&\leq \left(\frac{p}{2\pi t}\right)^n \int_{\mathbb{C}^n} |f(w)|e^{-\frac{|w|^2}{2t}} \, dw \|fe^{-\frac{|\cdot|^2}{2t}}\|_\infty^{p-1} \\
&= p^n \|f\|_{F_t^1} \|f\|_{F_t^\infty}^{p-1}.
\end{aligned}$$

□

2.3 Toeplitz and Hankel operators

Recall that the Hilbert space F_t^2 is a closed subspace of L_t^2. Therefore, there is an orthogonal projection from L_t^2 onto F_t^2, which we denote by P_t. For a function $f \in L_t^2$ we have

$$\begin{aligned}
P_t f(z) &= \langle P_t f, K_z^t\rangle_t \\
&= \langle f, K_z^t\rangle_t \\
&= \int_{\mathbb{C}^n} f(w)e^{\frac{z\cdot\overline{w}}{t}} d\mu_t(w),
\end{aligned}$$

i.e. the projection acts as the integral operator which has the reproducing kernel as its integral kernel. It is desirable to have such a projection from L_t^p to F_t^p for each

$p \in [1, \infty]$, not only in the Hilbert space case. It turns out the integral operator given by the same integral kernel does the job, and this is really why we made the seemingly odd choice of $L_t^p = L^p(\mathbb{C}^n, \mu_{2t/p})$.

Proposition 2.3.1 ([91]). *For all $1 \le p \le \infty$ and $t > 0$ the linear operator given by*

$$P_t f(z) = \int_{\mathbb{C}^n} f(w) e^{\frac{z \cdot \overline{w}}{t}} d\mu_t(w)$$

is a bounded linear projection on L_t^p with range F_t^p. In particular, $P_t|_{F_t^p} = I$.

Proof. The inequalities

$$\|P_t f\|_{L_t^1} \le 2^n \|f\|_{L_t^1}$$
$$\|P_t f\|_{L_t^\infty} \le 2^n \|f\|_{L_t^\infty}$$

follow immediately from Fubini's Theorem and simple integral computations. By Theorem 2.2.4 we therefore obtain

$$\|P_t\|_{L_t^p \to L_t^p} \le 2^n.$$

We will now show that the range is contained in F_t^p, i.e. we need to show that $P_t f$ is holomorphic for $f \in L_t^p$. By Morera's Theorem, it suffices to prove that for each triangle $\Delta \subset \mathbb{C}$ and each $j = 1, \ldots, n$ we have

$$\int_{\partial\Delta} P_t f(z_1, \ldots, z_n) \, dz_j = 0.$$

Indeed, we have

$$\begin{aligned}
\int_{\partial\Delta} P_t f(z_1, \ldots, z_n) \, dz_j &= \frac{1}{(\pi t)^n} \int_{\partial\Delta} \int_{\mathbb{C}^n} f(w) e^{\frac{z \cdot \overline{w}}{t}} e^{-\frac{|w|^2}{t}} \, dw \, dz_j \\
&= \frac{1}{(\pi t)^n} \int_{\mathbb{C}^n} f(w) e^{-\frac{|w|^2}{t}} \int_{\partial\Delta} e^{\frac{z \cdot \overline{w}}{t}} \, dz_j \, dw \\
&= 0
\end{aligned}$$

using the holomorphicity of $z_j \mapsto e^{\frac{z \cdot \overline{w}}{t}}$. Here, for the case $1 < p < \infty$ the use of Fubini's Theorem is indeed justified by the following estimate, where q is the exponent conjugate to p and all but the jth entry of $z = (z_1, \ldots, z_n)$ are fixed:

$$\begin{aligned}
&\int_{\mathbb{C}^n} \int_{\partial\Delta} |f(w) e^{\frac{z \cdot \overline{w}}{t}}| e^{-\frac{|w|^2}{t}} \, dz_j \, dw \\
&\qquad \le \left(\frac{2\pi t}{p}\right)^{\frac{1}{p}} \|f\|_{L_t^p} \left(\int_{\mathbb{C}^n} \left(\int_{\partial\Delta} |e^{\frac{z \cdot \overline{w}}{t}}| \, dz_j \right)^q e^{-\frac{q|w|^2}{2t}} \, dw \right)^{\frac{1}{q}} \\
&\qquad \le \left(\frac{2\pi t}{p}\right)^{\frac{1}{p}} \|f\|_{L_t^p} \ell(\partial\Delta) \left(\int_{\mathbb{C}^n} e^{\frac{\sqrt{n} c q |w|}{t} - \frac{q|w|^2}{2t}} \, dw \right)^{\frac{1}{q}} < \infty.
\end{aligned}$$

We denote in these estimates by $\ell(\partial\Delta)$ the length of the boundary of the triangle and c is such that

$$\sup_{z_j \in \partial\Delta} |z| = \sup_{z_j \in \partial\Delta} \sqrt{|z_1|^2 + \cdots + |z_n|^2} \leq c.$$

Fubini's Theorem can be justified similarly in the cases $p = 1$ and $p = \infty$.

Finally, we need to prove that P_t acts as the identity on F_t^p. Recall that this is certainly true on F_t^2, and therefore on $\mathcal{P}[z_1, \ldots, z_n]$. By continuity of P_t and denseness of the polynomials in F_t^p by Proposition 2.1.9, the statement follows for $1 \leq p < \infty$. For $p = \infty$, observe for $f \in F_t^\infty$ and $g \in F_t^1$ we have

$$\langle g, P_t f \rangle_t = \langle P_t g, f \rangle_t = \langle g, f \rangle_t,$$

again by a direct application of Fubini's Theorem. Therefore, the statement follows also for F_t^∞. □

Remark 2.3.2. In the proof we have obtained the estimate $\|P_t\|_{L_t^p \to L_t^p} \leq 2^n$ for all p by interpolating between L_t^1 and L_t^∞. Since P_t is a (nontrivial) orthogonal projection on L_t^2, we certainly have $\|P_t\|_{L_t^2 \to L_t^2} = 1$. In particular, for $p \neq 1, \infty$ the norm estimate can be improved by interpolating between L_t^1 and L_t^2 or L_t^2 and L_t^∞.

We can now define Toeplitz and Hankel operators. For each $1 \leq p \leq \infty$ and a measurable function $f : \mathbb{C}^n \to \mathbb{C}$ we define the Toeplitz operator $T_f^t : D(T_f^t) \to F_t^p$ with domain

$$D(T_f^t) := \{g \in F_t^p; fg \in L_t^p\}$$

by

$$T_f^t(g) = P_t(fg).$$

Further, we define the Hankel operator $H_f^t : D(H_f^t) \to L_t^p$ with domain $D(H_f^t) := D(T_f^t)$ through

$$H_f^t(g) = (I - P_t)(fg).$$

Note that we ignore p in the notation for those operators.

When it comes to Banach space adjoints of linear operators, we will use two different notations: If $A : X \to Y$ is a linear operator between two Banach spaces, we will denote by A^* its Banach space adjoint with respect to a sesquilinear dual pairing (such as $\langle \cdot, \cdot \rangle_t$), while A' will denote the dual operator with respect to a bilinear dual pairing (such as the standard pairing between X and Y').

We list some properties:

Proposition 2.3.3. *For $f \in L^\infty(\mathbb{C}^n)$ the following facts hold true:*

1) The maps $f \mapsto T_f^t$ and $f \mapsto H_f^t$ are linear;

2) $\|T_f^t\| \leq \|P_t\|\|f\|_\infty$ *and* $\|H_f^t\| \leq (1+\|P_t\|)\|f\|_\infty$*;*

3) The restriction of $T_f^t : F_t^\infty \to F_t^\infty$ *to* f_t^∞ *has its range in* f_t^∞*, i.e.* $T_f^t|_{f_t^\infty} \in \mathcal{L}(f_t^\infty)$*;*

4) The adjoint of $T_f^t : f_t^\infty \to f_t^\infty$ *is* $(T_f^t)^* = T_{\overline{f}}^t : F_t^1 \to F_t^1$*;*

5) For $1 \leq p < \infty$ *the adjoint of* $T_f^t : F_t^p \to F_t^p$ *is* $(T_f^t)^* = T_{\overline{f}}^t : F_t^q \to F_t^q$*, where* q *is the exponent conjugate to* p*;*

6) For $1 \leq p < \infty$ *the adjoint of* $H_f^t : F_t^p \to L_t^p$ *is* $(H_f^t)^* = P_t M_{\overline{f}}(I - P_t) : L_t^q \to F_t^q$*;*

7) For $g \in L^\infty(\mathbb{C}^n)$ *the following identity holds true:*

$$T_f^t T_g^t - T_{fg}^t = -(H_{\overline{f}}^t)^* H_g^t.$$

Proof. All results except for point 3) follow from immediate computations and the duality relations in Proposition 2.2.1. 3) is a direct consequence of Remark 2.2.5. □

An important consequence of 3) from the previous proposition is the following:

Proposition 2.3.4. *Let* $t > 0$*,* $p \in [1, \infty]$ *and* $A \in \mathcal{L}(F_t^p)$*. Further, assume that* A *is contained in the Banach algebra generated by all Toeplitz operators with bounded symbols, i.e.* A *can be approximated by sums of products of Toeplitz operators. Then,* A *has a pre-adjoint.*

Proof. For the reflexive case, i.e. $p \in (1, \infty)$, this is trivial. For $p = 1$ we can argue as follows: For any Toeplitz operator $T_f^t \in \mathcal{L}(F_t^1)$ we have $(T_f^t)^* = T_{\overline{f}}^t \in \mathcal{L}(F_t^\infty)$, $(T_f^t)^*|_{f_t^\infty} = T_{\overline{f}}^t \in \mathcal{L}(f_t^\infty)$ and $\left((T_f^t)^*|_{f_t^\infty}\right)^* = T_f^t \in \mathcal{L}(F_t^1)$. The same scheme works for any operator from the Banach algebra generated by the Toeplitz operators. Analogous reasoning yields the result for $p = \infty$. □

Recall that K_z^t denotes the reproducing kernel functions $K_z^t(w) = e^{\frac{w \cdot \overline{z}}{t}}$. By Lemma 2.1.12 we have $\|K_z^t\|_{F_t^p} = e^{\frac{|z|^2}{2t}}$ for all p. We set

$$k_z^t(w) = \frac{K_z^t(w)}{\|K_z^t\|_{F_t^2}},$$

which is therefore of norm 1 in F_t^p for all p. For a linear operator A on F_t^p, possibly unbounded, such that $\mathrm{Span}\{K_z^t;\ z \in \mathbb{C}^n\} \subset D(A)$, we define its Berezin transform as a function on $\mathbb{C}^n$ by

$$\mathcal{B}(A)(z) := \widetilde{A}(z) := \langle A k_z^t, k_z^t \rangle_t.$$

If A is a bounded operator, then $\widetilde{A}$ is clearly a bounded function. For $f : \mathbb{C}^n \to \mathbb{C}$ such that $f K_z^t \in L_t^p$ for some p we can also define the Berezin transform of f by

$$\mathcal{B}_t(f)(z) := \widetilde{f}^{(t)}(z) := \widetilde{T_f^t}(z) = \langle P_t(f k_z^t), k_z^t \rangle_t = \langle f k_z^t, k_z^t \rangle_t.$$

As we will see later and is well known, properties of a linear operator are closely related to properties of its Berezin transform.

Obviously, the properties of a Toeplitz operator are closely related to properties of its symbol. We introduce several symbol classes, which we will use throughout this work.

By $C_b(\mathbb{C}^n)$ we denote the C^* algebra of bounded, continuous functions on $\mathbb{C}^n$ and $\mathrm{BUC}(\mathbb{C}^n)$ is its subalgebra of bounded and uniformly continuous functions. $C_0(\mathbb{C}^n)$ is the ideal of functions vanishing at infinity. $\mathrm{UC}(\mathbb{C}^n)$ denotes the uniformly continuous functions on $\mathbb{C}^n$ (not necessarily bounded). The following two symbol spaces have a long-known importance for the theory of Toeplitz and Hankel operators, cf. [30,32,135]. By $\mathrm{VO}_\partial(\mathbb{C}^n)$ we will denote the functions of *vanishing oscillation at infinity*, i.e.

$$\mathrm{VO}_\partial(\mathbb{C}^n) := \{f \in C_b(\mathbb{C}^n); \sup_{w:\ |z-w|<1} |f(z) - f(w)| \to 0, \quad |z| \to \infty\}.$$

For $f \in L^\infty(\mathbb{C}^n)$ we define the *mean oscillation* as

$$\mathrm{MO}_t(f)(z) := \mathcal{B}_t(|f|^2)(z) - |\mathcal{B}_t(f)(z)|^2$$

and denote by $\mathrm{VMO}^t_\partial(\mathbb{C}^n)$ the space of functions with *vanishing mean oscillation at infinity*:

$$\mathrm{VMO}^t_\partial(\mathbb{C}^n) := \{f \in L^\infty(\mathbb{C}^n); \ \mathrm{MO}_t(f)(z) \to 0, \quad |z| \to \infty\}.$$

Note that we always assume VMO^t_∂ functions to be bounded. It can be shown that the space $\mathrm{VMO}^t_\partial(\mathbb{C}^n)$ is actually indeed independent of the parameter $t > 0$ [84]. We therefore define

$$\mathrm{VMO}_\partial(\mathbb{C}^n) := \mathrm{VMO}^t_\partial(\mathbb{C}^n).$$

Here is an important property:

Theorem 2.3.5 ([10, 84, 107])**.** *Let $f \in \mathrm{VMO}_\partial(\mathbb{C}^n)$. Then H^t_f is compact for any $p \in (1, \infty)$.*

For $f \in L^1_{loc}(\mathbb{C}^n)$ and $E \subset \mathbb{C}^n$ bounded and measurable with $V(E) > 0$ we set

$$f_E = \frac{1}{V(E)} \int_E f(w)\ dw.$$

Recall that V is the Lebesgue measure on $\mathbb{C}^n$. For $z \in \mathbb{C}^n$ and $\rho > 0$ we now let

$$A_2(f, z, \rho) := \frac{1}{V(B(z,\rho))} \int_{B(z,\rho)} |f(w) - f_{B(z,\rho)}|^2\ dw$$

and set

$$\mathrm{BMO}^{2,\rho}(\mathbb{C}^n) := \{f \in L^1_{loc}(\mathbb{C}^n); \ \sup_{z\in\mathbb{C}^n} A_2(f, z, \rho) < \infty\}$$

$$\text{VMO}(\mathbb{C}^n) := \{f \in L^1_{loc}(\mathbb{C}^n);\ \lim_{\rho\to 0} A_2(f,z,\rho) = 0 \text{ uniformly on } \mathbb{C}^n\}.$$

Here, BMO stands for *bounded mean oscillation* and VMO stands for *vanishing mean oscillation* (in the interior). It is well-known that $\text{BMO}^{2,\rho}(\mathbb{C}^n)$ is independent of $\rho > 0$. We will use the abbreviations

$$\text{BMO}(\mathbb{C}^n) := \text{BMO}^{2,1}(\mathbb{C}^n)$$
$$\text{VMO}_b(\mathbb{C}^n) := \text{VMO}(\mathbb{C}^n) \cap L^\infty(\mathbb{C}^n).$$

We want to mention the following important results:

Theorem 2.3.6 ([12,13])**.** *For $f \in L^\infty(\mathbb{C}^n)$ the following holds true:*

$$\lim_{t\to 0} \|\widetilde{f}^{(t)}\|_\infty = \lim_{t\to 0} \|T_f^t\|_{F_t^2\to F_t^2} = \|f\|_\infty.$$

Furthermore:

1) If $f \in L^\infty(\mathbb{C}^n)$, then $\widetilde{f}^{(t)} \to f$ pointwise almost everywhere as $t \to 0$;

2) If $f \in C_b(\mathbb{C}^n)$, then $\widetilde{f}^{(t)} \to f$ pointwise as $t \to 0$;

3) If $f \in \text{UC}(\mathbb{C}^n)$, then $\widetilde{f}^{(t)} \to f$ uniformly.

Recall that Toeplitz operators with unbounded symbols in general turn out to be unbounded operators. Hence, if we take two unbounded functions $f, g : \mathbb{C}^n \to \mathbb{C}$, it is at least questionable if the operator product $T_f^t T_g^t$ is well-defined. If the symbols are at least uniformly continuous, then the product is indeed well-behaved: It turns out that there is a dense, self-adjoint subspace $\mathcal{D}_t$ of L_t^2 which is invariant under P_t and M_f for each $f \in \text{UC}(\mathbb{C}^n)$ [11]. Hence, for uniformly continuous symbols we obtain that $\mathcal{D}_t \cap F_t^2$ is invariant under T_f^t. Then, the product $T_f^t T_g^t$ is well-defined on $\mathcal{D}_t \cap F_t^2$ if at least one of the symbols is in $\text{UC}(\mathbb{C}^n)$ and the other one is either in $\text{UC}(\mathbb{C}^n)$ or $L^\infty(\mathbb{C}^n)$.

Theorem 2.3.7 ([13, Theorem 3.4, Theorem 4.9])**.** *For a symbol $f \in \text{UC}(\mathbb{C}^n)$ or $f \in \text{VMO}_b(\mathbb{C}^n)$ the following holds true:*

$$\lim_{t\to 0} \|H_f^t\|_{F_t^2\to L_t^2} = 0.$$

In light of 7) in 2.3.3 it follows that

$$\lim_{t\to 0} \|T_f^t T_g^t - T_{fg}^t\|_{F_t^2\to F_t^2} = 0$$

for each $g \in L^\infty(\mathbb{C}^n)$ or $g \in \text{UC}(\mathbb{C}^n)$.

Remark 2.3.8. The above two results, together with higher order results (cf. [81] and references therein), say that the map $f \mapsto T_f^t$ serves as a good model for strict quantization, i.e. in the classical limit $t \to 0$ it resembles the "classical world". We will come back to this later.

For $z \in \mathbb{C}^n$ we define the operator $W_z^t \in \mathcal{L}(F_t^p)$ by

$$W_z^t f(w) = k_z^t(w) f(w - z).$$

We will occasionally refer to the W_z^t as *Weyl operators*. An easy substitution shows that W_z^t is actually isometric on F_t^p for all $p \in [1, \infty]$. Further, since $W_z^t f \to W_{z_0}^t f$ pointwise as $z \to z_0$, W_z^t acts strongly continuously on F_t^p for $1 \leq p < \infty$ by Theorem 2.1.8. Since the shifts

$$\alpha_z(f)(w) := f(w - z)$$

act strongly continuously on $C_0(\mathbb{C}^n)$, it is easily seen that $z \mapsto W_z^t$ is strongly continuous on f_t^∞. We fix this for later reference:

Lemma 2.3.9. *Let $p \in [1, \infty)$ and $t > 0$. Then, $z \mapsto W_z^t$ is strongly continuous on F_t^p and on f_t^∞. Further, the operators satisfy $(W_z^t)^{-1} = W_{-z}^t$ and $(W_z^t)^* = W_{-z}^t$, where the adjoint of course acts on the space dual to F_t^p and f_t^∞, respectively.*

Remark 2.3.10. Indeed, $z \mapsto W_z^t$ is not strongly continuous on F_t^∞. As an example, consider again the function $f(z) = e^{\frac{z_1^2 + \dots + z_n^2}{2t}}$ from Remark 2.1.4. For simplicity, we only deal with the case $n = 1$. The higher dimensional examples can be worked out analogously. We have

$$\begin{aligned} |f(z) - W_w^t f(z)| e^{-\frac{|z|^2}{2t}} &= |e^{\frac{z^2 - |z|^2}{2t}} - e^{\frac{z \cdot \overline{w}}{t} - \frac{(z-w)^2}{2t} - \frac{|w|^2}{2t} - \frac{|z|^2}{2t}}| \\ &= |e^{\frac{z^2 - |z|^2}{2t}} - e^{\frac{i \operatorname{Im}(z \cdot \overline{w})}{t} - \frac{(z-w)^2}{2t} - \frac{|z-w|^2}{2t}}|. \end{aligned}$$

Letting now $w = ix$ for $x \in (0, \infty)$ and $z = \frac{\pi t}{2x}$ we get

$$|f(z) - W_w^t f(z)| e^{-\frac{|z|^2}{2t}} = |1 + e^{-\frac{\pi^2 t}{4x^2}}| \to 1, \quad x \to 0$$

and therefore $W_w^t f$ does not converge to f in F_t^∞ as $w \to 0$.

We will later be able to prove the following result, which we already state now:

Proposition 2.3.11. *Let $t > 0$. Then, we have*

$$f_t^\infty = \{f \in F_t^\infty;\ z \mapsto W_z^t f \text{ is continuous in } F_t^\infty\}.$$

The operators W_z^t can also be considered as operators on L_t^p, acting by the same formula. They are also isometric and satisfy $(W_z^t)^{-1} = W_{-z}^t$ and $(W_z^t)^* = W_{-z}^t$ (with respect to the standard dual pairing induced by L_t^2), at least for $p \in [1, \infty)$.

It is important to note that the W_z^t (acting on F_t^p) are actually Toeplitz operators. Letting

$$g_z^t(w) := e^{\frac{|z|^2}{2t} + \frac{2i \operatorname{Im}(w \cdot \overline{z})}{t}},$$

one can show that $W_z^t = T_{g_z^t}^t$. Further, the map $z \mapsto W_z^t$ is a projective representation of $\mathbb{C}^n$ on F_t^p, i.e. for $z, w \in \mathbb{C}^n$ we have the following identity:

$$W_z^t W_w^t = e^{-\frac{i \operatorname{Im}(z \cdot \overline{w})}{t}} W_{z+w}^t.$$

In particular, if we define for $z \in \mathbb{C}^n$ and $A \in \mathcal{L}(F_t^p)$ or $A \in \mathcal{L}(f_t^\infty)$

$$\alpha_z(A) := W_z^t A W_{-z}^t,$$

one obtains

$$\alpha_z(\alpha_w(A)) = \alpha_{z+w}(A). \tag{2.2}$$

The formally identical equality holds true obviously for functions:

$$\alpha_z(\alpha_w(f)) = \alpha_{z+w}(f).$$

We will also consider the operator U, which acts as

$$Uf(z) = f(-z).$$

This operator acts on all (measurable) functions. It leaves the Fock spaces invariant and is isometric on them. We have the following relation between the Weyl operators and U:

$$UW_z^t U = W_{-z}^t. \tag{2.3}$$

Here are some important properties of the Berezin transform.

Lemma 2.3.12. *Let $p \in [1, \infty)$ and $t > 0$. Then, the Berezin transform is a bounded linear map*

$$\mathcal{L}(F_t^p) \to C_b(\mathbb{C}^n).$$

Further, the Berezin transform is injective.

The same statements are true for the Berezin transform on $\mathcal{L}(f_t^\infty)$.

Proof. Boundedness and linearity are obvious. Let us show that $\widetilde{A}$ is continuous for $A \in \mathcal{L}(F_t^p)$. This follows from the following computations:

$$\begin{aligned}|\widetilde{A}(z) - \widetilde{A}(w)| &\leq |\langle A(k_z^t - k_w^t), k_z^t\rangle_t| + |\langle Ak_w^t, k_z^t - k_w^t\rangle_t| \\ &\leq \|A\|_{op}\|k_z^t - k_w^t\|_{F_t^p}\|k_z^t\|_{F_q^t} + \|A\|_{op}\|k_w^t\|_{F_t^p}\|k_z^t - k_w^t\|_{F_t^q} \\ &\leq \|A\|_{op}\|k_z^t - k_w^t\|_{F_t^p} + \|A\|_{op}\|k_z^t - k_w^t\|_{F_t^q}.\end{aligned}$$

Now observe that $k_z^t = W_z^t(1)$, therefore $\|k_z^t - k_w^t\|_{F_t^p} \to 0$ as $z \to w$, since $z \mapsto W_z^t$ is strongly continuous. The same holds for $\|k_z^t - k_w^t\|_{F_t^q}$. In the case $p = 1$, observe that strong continuity of W_z^t on $f_t^\infty \subset (F_t^1)'$ suffices. Analogous arguments yield that $\widetilde{A}$ is continuous for $A \in \mathcal{L}(f_t^\infty)$.

We prove the injectivity: Let $A \in \mathcal{L}(F_t^p)$ such that $\widetilde{A} = 0$. Let us consider the function $\widetilde{A}(z, w) = \langle Ak_z^t, k_w^t\rangle_t$. Standard arguments (e.g. using Morera's Theorem) show that this function is holomorphic in w and anti-holomorphic in z. Then, [69, Proposition 1.69] and $\widetilde{A}(z, z) = \widetilde{A}(z) \equiv 0$ shows that $\widetilde{A}(z, w) = 0$ for all z, w. Therefore, we have

$$\widetilde{A}(z, w) = \langle Ak_z^t, k_w^t\rangle_t = e^{-\frac{|w|^2}{2t}} Ak_z^t(w) = 0$$

for all z, w, i.e. $Ak_z^t = 0$ for all z. Since $\mathrm{Span}\{k_z^t;\ z \in \mathbb{C}^n\}$ is dense in F_t^p, this shows $A = 0$. The same reasoning works for $A \in \mathcal{L}(f_t^\infty)$. □

Remark that injectivity of the Berezin transform simply does not hold true in the case $p = \infty$. This is shown by the following example, which was inspired by [96, Example 1.26(b)].

Example 2.3.13. Recall that for any $f \in F_t^\infty$, $z \mapsto f(z)e^{-\frac{|z|^2}{2t}}$ is bounded and continuous on $\mathbb{C}^n$. In particular, the function continuously extends to $\beta\mathbb{C}^n$, the Stone-Čech compactification of $\mathbb{C}^n$. Fix $x \in \beta\mathbb{C}^n \setminus \mathbb{C}^n$ and set

$$\nu_x(f) = f(x)e^{-\frac{|x|^2}{2t}},$$

interpreted in the sense of the continuous extension. Then,

$$f \mapsto \nu_x(f)$$

is a bounded linear functional on F_t^∞ which vanishes on f_t^∞. Fix $g \in F_t^\infty$ and define $A \in \mathcal{L}(F_t^\infty)$ by

$$A(f) = \nu_x(f)g.$$

Then, A vanishes on f_t^∞. In particular,

$$\widetilde{A}(z) = \langle Ak_z^t, k_z^t \rangle_t = \nu_x(k_z^t)\langle g, k_z^t \rangle_t = 0$$

for any $z \in \mathbb{C}^n$.

The following result is well-known, at least for the case $1 < p < \infty$ (cf. [73, Proposition 7] and also [10, Theorem 3.1(b)] for the case $p = 2$). We could not find the result in the literature for the cases $p = 1, \infty$, hence we provide a proof.

Proposition 2.3.14. *Let $f \in L^\infty(\mathbb{C}^n)$ have compact support. Then, $M_f P_t, P_t M_f : L_t^p \to L_t^p$ are compact for all $1 \leq p \leq \infty$.*

The proof is based on the following criterion, which in turn is a nice application of the Riesz-Kolmogorov Theorem.

Theorem 2.3.15 ([67, Corollary 5.1])**.** *Let $h : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ be a measurable function such that $h(z, \cdot) \in L^1(\mathbb{C}^n)$ for almost all $z \in \mathbb{C}^n$. Assume there is a constant $M > 0$ such that for almost all $z \in \mathbb{C}^n$ we have*

$$\int_{\mathbb{C}^n} |h(z, w)| \, dw < M.$$

Denote by T the integral operator on $L^1(\mathbb{C}^n)$ defined by

$$Tg(w) = \int_{\mathbb{C}^n} h(z, w)g(z) \, dz.$$

Then, the following are equivalent:

1) T is compact;

2) For every $\varepsilon > 0$ there exist $\delta > 0$ and $R > 0$ such that for almost all $z \in \mathbb{C}^n$ and every $v \in \mathbb{C}^n$ with $|v| < \delta$:

$$\int_{\mathbb{C}^n \setminus B(0,R)} |h(z,w)| \, dw < \varepsilon, \qquad \int_{\mathbb{C}^n} |h(z,w+v) - h(z,w)| \, dw < \varepsilon.$$

Proof of Proposition 2.3.14. As we have already mentioned before, the space L_t^p is isometrically equivalent to $L^p(\mathbb{C}^n)$ via a constant multiple of the multiplication operator $M_{\exp\left(-\frac{|\cdot|^2}{2t}\right)} : L_t^p \to L^p(\mathbb{C}^n)$.

We will use the previous theorem to prove compactness of the operators over L_t^1. Compactness over L_t^∞ then follows by duality. The other cases will follow from applying interpolation. Note that it is in general an open problem if the (complex) interpolation of compact operators is again compact, cf. [52] for a recent survey on that problem. In our case, compactness of the interpolated operators is indeed verified by a classical theorem of Krasnosel'skiĭ [94]. As already stated, for the cases $1 < p < \infty$ compactness can also be directly verified estimating the dual norm of the kernel, as was done in [73, Proposition 7].

Recall that the operator P_t is an integral operator, hence $P_t M_f$ is also an integral operator:

$$P_t M_f g(w) = \int_{\mathbb{C}^n} f(z) g(z) e^{\frac{w\cdot\bar{z}}{t}} \, d\mu_t(z).$$

Analogously, $M_f P_t$ is given by

$$M_f P_t g(w) = \int_{\mathbb{C}^n} f(w) g(z) e^{\frac{w\cdot\bar{z}}{t}} \, d\mu_t(z).$$

Adjoining by the operator $M_{\exp\left(\frac{|\cdot|^2}{2t}\right)}$, we obtain the following integral operators on $L^1(\mathbb{C}^n)$:

$$M_{\exp\left(-\frac{|\cdot|^2}{2t}\right)} P_t M_f M_{\exp\left(\frac{|\cdot|^2}{2t}\right)} g(w) = \frac{1}{(\pi t)^n} \int_{\mathbb{C}^n} f(z) g(z) e^{\frac{w\cdot\bar{z}}{t} - \frac{|w|^2}{2t} - \frac{|z|^2}{2t}} \, dz,$$
$$M_{\exp\left(-\frac{|\cdot|^2}{2t}\right)} M_f P_t M_{\exp\left(\frac{|\cdot|^2}{2t}\right)} g(w) = \frac{1}{(\pi t)^n} \int_{\mathbb{C}^n} f(w) g(z) e^{\frac{w\cdot\bar{z}}{t} - \frac{|w|^2}{2t} - \frac{|z|^2}{2t}} \, dz.$$

First, using the notation from Theorem 2.3.15 we set $h(z,w) = f(z) e^{\frac{z\cdot\overline{w}}{t} - \frac{|w|^2}{2t} - \frac{|z|^2}{2t}}$. Hence, the operator T on $L^1(\mathbb{C}^n)$ is

$$T = (\pi t)^n M_{\exp\left(-\frac{|\cdot|^2}{2t}\right)} P_t M_f M_{\exp\left(\frac{|\cdot|^2}{2t}\right)}.$$

Observe the trivial estimate $\int_{\mathbb{C}^n} |h(z,w)| \, dw \leq \|f\|_\infty (2\pi t)^n$, which places us within the framework of the theorem. Let $\varepsilon > 0$. We choose $R > 0$ such that

$$\operatorname{supp}(f) \subset B(0, R/2), \tag{2.4}$$

$$\|f\|_\infty (4\pi t)^n e^{-\frac{R^2}{16t}} < \varepsilon, \tag{2.5}$$

$$\|f\|_\infty \int_{\mathbb{C}^n \setminus B(0,\frac{R}{2})} e^{-\frac{|w|^2}{2t}}\, dw < \frac{\varepsilon}{4}. \tag{2.6}$$

Then, for $|z| \geq \frac{R}{2}$ we trivially have

$$\int_{\mathbb{C}^n \setminus B(0,R)} h(z,w)\, dw = 0.$$

For $|z| < \frac{R}{2}$ we obtain $|z-w| > \frac{R}{2}$ for every $w \in \mathbb{C}^n \setminus B(0,R)$ and therefore

$$\begin{aligned}\int_{\mathbb{C}^n \setminus B(0,R)} |h(z,w)|\, dw &\leq \|f\|_\infty \int_{\mathbb{C}^n \setminus B(0,R)} e^{-\frac{|z-w|^2}{2t}}\, dw \\ &\leq \|f\|_\infty e^{-\frac{R^2}{16t}} \int_{\mathbb{C}^n \setminus B(0,R)} e^{-\frac{|z-w|^2}{4t}}\, dw \\ &\leq \|f\|_\infty (4\pi t)^n e^{-\frac{R^2}{16t}} < \varepsilon.\end{aligned}$$

Let us consider the second estimate. We have

$$\int_{\mathbb{C}^n} |h(z,w+v) - h(z,w)|\, dw = |f(z)| \int_{\mathbb{C}^n} |e^{\frac{(w+v)\cdot\overline{z}}{t} - \frac{|w+v|^2}{2t} - \frac{|z|^2}{2t}} - e^{\frac{w\cdot\overline{z}}{t} - \frac{|w|^2}{2t} - \frac{|z|^2}{2t}}|\, dw.$$

For $|z| > \frac{R}{2}$ this trivially evaluates to 0 by (2.4). Hence, assume $|z| \leq \frac{R}{2}$. Let $\delta > 0$ be such that

$$B(z-v, R/2) \subset B(z,R), \quad |v| < \delta.$$

Then, by (2.6) we obtain

$$\begin{aligned}&\int_{\mathbb{C}^n \setminus B(z,R)} |h(z,w+v) - h(z,w)|\, dw \\ &\quad\leq \|f\|_\infty \left(\int_{\mathbb{C}^n \setminus B(z-v,\frac{R}{2})} e^{-\frac{|w-(z-v)|^2}{2t}}\, dw + \int_{\mathbb{C}^n \setminus B(z,\frac{R}{2})} e^{-\frac{|w-z|^2}{2t}}\, dw \right) \\ &\quad\leq \frac{\varepsilon}{2}.\end{aligned}$$

On $(z,w) \in \overline{B(0,\frac{R}{2})} \times \overline{B(0,2R)}$ the function

$$(z,w) \mapsto e^{\frac{w\cdot\overline{z}}{t} - \frac{|w|^2}{2t} - \frac{|z|^2}{2t}}$$

is uniformly continuous, hence we can choose δ small enough such that

$$\|f\|_\infty \left| e^{\frac{(w+v)\cdot\overline{z}}{t} - \frac{|w+v|^2}{2t} - \frac{|z|^2}{2t}} - e^{\frac{w\cdot\overline{z}}{t} - \frac{|w|^2}{2t} - \frac{|z|^2}{2t}} \right| < \frac{\varepsilon/2}{V(B(0,R))}$$

for $|z| \leq \frac{R}{2}$, $w \in B(z,R)$ and $|v| < \delta$. This gives

$$\int_{B(z,r)} |h(z,w+v) - h(z,w)| \; dw \leq \frac{\varepsilon}{2}.$$

Combining both estimates we obtain

$$\int_{\mathbb{C}^n} |h(z,w+v) - h(z,w)| \; dw < \varepsilon$$

with the above choice of δ. This proves compactness of $P_t M_f$ on L^1_t. The compactness of $M_f P_t$ on L^1_t follows from similar arguments. We just give a very brief sketch. The kernel is now $h(z,w) = f(w)e^{\frac{w\cdot\bar{z}}{t} - \frac{|w|^2}{2t} - \frac{|z|^2}{2t}}$. If we fix $R > 0$ such that $\operatorname{supp}(f) \subset B(0,R)$, then

$$\int_{\mathbb{C}^n \setminus B(0,R)} |h(z,w)| \; dw = 0.$$

For the second estimate, since f has compact support we observe that

$$\begin{aligned} &\int_{\mathbb{C}^n} |h(z,w+v) - h(z,w)| \; dw \\ &\quad \leq \int_{B(0,2R)} |f(w+v) - f(w)| e^{-\frac{|w-z|}{2t}} \; dw \\ &\quad + \|f\|_\infty \int_{B(0,2R)} |e^{\frac{(w+v)\cdot\bar{z}}{t} - \frac{|w+v|^2}{2t} - \frac{|z|^2}{2t}} - e^{\frac{w\cdot\bar{z}}{t} - \frac{|w|^2}{2t} - \frac{|z|^2}{2t}}| \; dw. \end{aligned}$$

For $|v| < \delta$ and δ small enough, the first integral is now less than $\varepsilon/2$ by uniform continuity of f, the second can be estimated as in the computations for the first operator. □

2.4 Remarks

The study of the spaces F^2_t goes back to the important works by Valentine Bargmann [5–7] and Irving Segal [117–119] in connection with their study of what is nowadays called the *Bargmann transform*, due to which the spaces F^p_t are also often called *Segal-Bargmann spaces*. The name *Fock space* for F^p_t arises from the following fact: If we let $F = \bigoplus_{k=0}^\infty (\mathbb{C}^n)^{\circ k}$ be the *symmetric* (or *bosonic*) *Fock space*, where $\circ k$ denotes the k-fold symmetric tensor product, then this symmetric Fock space can be canonically identified with F^2_t (over $\mathbb{C}^n$) by identifying the symmetric elementary tensor

$$e_{j_1} \circ \cdots \circ e_{j_k},$$

where the e_j are the standard basis elements of $\mathbb{C}^n$, with the orthonormal basis element e^t_α, which is (up to the normalizing constant) the monomial

$$\prod_{l=1}^k z_{j_l}.$$

Using this identification, the Fock spaces F_t^2 serve as a natural setting for studying the annihilation and creation operators (which actually turn out to be unbounded Toeplitz operators).

It is not entirely clear to the author where the first occurrence of the non-Hilbertian Fock spaces F_t^p was. Certainly the most important work on these spaces was [91], where many of the properties presented here were first discussed. In particular, the study of duality and interpolation behaviour of Fock spaces was first done in that paper.

The history of mathematical studies of Toeplitz operators is now well over 100 years old. The notion of a *Toeplitz operator* originates from Otto Toeplitz' work [123] on infinite Toeplitz matrices, or equivalently, on Toeplitz operators on the Hardy space $H^2(\mathbb{D})$.

A comprehensive discussion on the history of Toeplitz operators could probably fill a treatise on its own, therefore we defer from this. Let us only mention that the systematic study of Toeplitz operators on Fock spaces seemingly started with the papers of Berger and Coburn [28–30], even though certain aspect have already been studied by Berezin [22]. Among all the follow-up works published on Toeplitz operators on Fock spaces, let us only mention the textbook [137] by K. Zhu, which so far is the only systematic collection of results concerning operator theory, in particular of Toeplitz operators, on Fock spaces.

Essentially all results presented in this chapter are well-known, most can be found in [91, 137]. The possibly only new results presented here are Proposition 2.3.11 and 2.3.14. While the latter is certainly not surprising, we could not locate it anywhere in the literature. For $p \in (1, \infty)$, a proof was given by Raffael Hagger and the author in [73], the full result for $p \in [1, \infty]$ seems to have been written down here for the first time. Proposition 2.3.11, the proof of which will be given in a later chapter, seems to be entirely new and adds an interesting aspect to the studies of Fock spaces.

Chapter 3

Correspondence Theory

In this chapter, we closely follow the presentation of the author's own work [72]. Many statements and proofs are taken verbatim from there. We want to emphasize that the initial work [72] and therefore also this presentation was inspired by R. Werner's approach to Quantum Harmonic Analysis [130], a topic which recently inspired further research in Harmonic Analysis [102, 103]. Observe that, in contrast to [72], we will spend some extra effort to include the non-reflexive case f_t^∞ into the discussion.

3.1 The convolution formalism

Until stated otherwise, the underlying space will always be F_t^p for $p \in (1, \infty)$ or f_t^∞. If we do not clarify in which of these two cases we are working, it will not make any difference. We exclude F_t^1 from the discussions, since the Weyl operators W_z^t in general are not well-behaved on the dual space $(F_t^1)' \cong F_t^\infty$ (in a sense which we will specify below). To simplify notation, we will use the following abbreviations:

$$\begin{aligned}
\mathcal{N} &= \mathcal{N}(F_t^p) \text{ for } p \in (1, \infty) \text{ or } \mathcal{N}(f_t^\infty), \\
\mathcal{K} &= \mathcal{K}(F_t^p) \text{ for } p \in (1, \infty) \text{ or } \mathcal{K}(f_t^\infty), \\
\mathcal{L} &= \mathcal{L}(F_t^p) \text{ for } p \in (1, \infty) \text{ or } \mathcal{L}(f_t^\infty), \\
\mathcal{S}^{p_0} &= \mathcal{S}^{p_0}(F_t^p) \text{ for } p \in (1, \infty) \text{ or } \mathcal{S}^{p_0}(f_t^\infty).
\end{aligned}$$

Of course, in all occurrences of $\mathcal{N}, \mathcal{K}, \mathcal{L}$ or $\mathcal{S}^{p_0}$, the underlying space F_t^p or f_t^∞ is always assumed to be the same. Here, $\mathcal{S}^{p_0}$ denotes the interpolated space between $\mathcal{N}$ and $\mathcal{L}$, cf. Appendix A.3.

The first goal will be to understand the action of α_z on certain operators and functions. The necessary facts are summarized in the following lemma:

Lemma 3.1.1. *1) Let $p_0 \in [1, \infty)$. Then, α_z acts strongly continuously on $L^{p_0}(\mathbb{C}^n)$ and on $\mathcal{S}^{p_0}$;*

2) α_z acts strongly continuously on $C_0(\mathbb{C}^n)$ and on $\mathcal{K}$;

3) α_z *acts weak* continuously on* $L^\infty(\mathbb{C}^n)$ *and on* $\mathcal{L}$*. The latter means that for every* $A \in \mathcal{L}$ *and* $N \in \mathcal{N}$ *we have* $\mathrm{Tr}(\alpha_z(A)N) \to \mathrm{Tr}(AN)$ *as* $z \to 0$*.*

Proof. 1) Strong continuity on $L^{p_0}(\mathbb{C}^n)$ follows immediately from Theorem 2.1.8. Let $A = y \otimes x$ be a rank one operator, where $x \in F_t^p$ and $y \in (F_t^p)' \cong F_t^q$. Then, one easily verifies that $\alpha_z(A) = (W_z^t y) \otimes (W_z^t x)$. In particular, for $z \to 0$,

$$\begin{aligned}\|\alpha_z(A) - A\|_{\mathcal{N}} &\leq \|(W_z^t y - y) \otimes x\|_{\mathcal{N}} + \|(W_z^t y) \otimes (x - W_z^t x)\|_{\mathcal{N}} \\ &\leq \|W_z^t y - y\|_{(F_t^p)'}\|x\|_{F_t^p} + \|W_z^t y\|_{(F_t^p)'}\|x - W_z^t x\|_{F_t^p} \\ &\lesssim \|W_z^t y - y\|_{F_t^q}\|x\|_{F_t^p} + \|y\|_{F_t^q}\|x - W_z^t x\|_{F_t^p} \\ &\to 0, \quad z \to 0,\end{aligned}$$

where we used the strong continuity of $z \mapsto W_z^t$ on F_t^p and F_t^q. This estimate now carries over to all finite rank operators. Since $\|\cdot\|_{\mathcal{S}^{p_0}} \leq \|\cdot\|_{\mathcal{N}}$, the estimate carries over to $\mathcal{S}^{p_0}$ as well. Observe that

$$\|\alpha_z(A)\|_{\mathcal{N}} = \|A\|_{\mathcal{N}} \text{ and } \|\alpha_z(B)\|_{op} = \|B\|_{op}$$

for all $A \in \mathcal{N}(F_t^p)$, $B \in \mathcal{L}(F_t^p)$. By exactness of the Complex Interpolation Method, we obtain

$$\|\alpha_z(A)\|_{\mathcal{S}^{p_0}} \leq \|A\|_{\mathcal{S}^{p_0}}$$

for all $A \in \mathcal{S}^{p_0}$. Using this and the fact that finite rank operators are dense in $\mathcal{S}^{p_0}$, it is now standard to show that

$$\|\alpha_z(A) - A\|_{\mathcal{S}^{p_0}} \to 0, \quad z \to 0.$$

Finally, by Equation (2.2) it suffices to show continuity at 0, so we are done. Note that the same proof works over f_t^∞.

2) The strong continuity of the shifts on $C_0(\mathbb{C}^n)$ is standard. On $\mathcal{K}$, the same argument as for the Schatten classes works.

3) Weak* continuity follows from continuity on $L^1(\mathbb{C}^n)$ and $\mathcal{N}(F_t^p)$. Note that $\mathcal{L}(f_t^\infty)$ is strictly contained in $(\mathcal{N}(f_t^\infty))'$ under the trace duality $\varphi_A(N) = \mathrm{Tr}(AN)$, which does not cause any problems. For $A \in \mathcal{L}$ and $N \in \mathcal{N}$ we have

$$\begin{aligned}\mathrm{Tr}(\alpha_z(A)N) &= \mathrm{Tr}(A\alpha_{-z}(N)) \\ &\to \mathrm{Tr}(AN),\end{aligned}$$

since W_z^t is strongly continuous on $\mathcal{N}$. □

We will need to consider those subspaces of $L^\infty(\mathbb{C}^n)$ and of $\mathcal{L}$ on which the action of α_z is "well-behaved". Let

$$\mathcal{C}_0 := \{f \in L^\infty(\mathbb{C}^n);\ \|\alpha_z(f) - f\|_\infty \to 0,\ z \to 0\},$$

$$\mathcal{C}_1 := \{A \in \mathcal{L};\ \|\alpha_z(A) - A\|_{op} \to 0,\ z \to 0\}.$$

$\mathcal{C}_0$ is clearly a C^* algebra, and so is $\mathcal{C}_1$ for $p = 2$ (being a Banach algebra in general). Note that the defining conditions are equivalent, respectively for $\mathcal{C}_0$ and $\mathcal{C}_1$, to

$$\|\alpha_z(f) - \alpha_w(f)\|_\infty \to 0, \quad z \to w$$
$$\|\alpha_z(A) - \alpha_w(A)\|_{op} \to 0, \quad z \to w$$

for all $w \in \mathbb{C}^n$, since α_{z-w} acts isometrically and thanks to Equation (2.2). It is a matter of standard computations to show that

$$\mathcal{C}_0 = \mathrm{BUC}(\mathbb{C}^n). \tag{3.1}$$

Recall that the convolution of $f, g \in L^1(\mathbb{C}^n)$ is defined as

$$f * g(z) = \int_{\mathbb{C}^n} f(w)g(z-w)\ dw.$$

We will define two additional notions of convolution. For $f \in L^1(\mathbb{C}^n)$ and $A \in \mathcal{N}$, set

$$f * A := \int_{\mathbb{C}^n} f(z)\alpha_z(A)\ dz,$$

which is defined as a Bochner integral in $\mathcal{N}$. Further, for $A, B \in \mathcal{N}$ set

$$A * B(z) := \mathrm{Tr}(A\alpha_z(UBU)),$$

which is a function from $\mathbb{C}^n$ to $\mathbb{C}$. Those convolutions have the following properties:

Lemma 3.1.2. *1) For $f, g \in L^1(\mathbb{C}^n)$ we have*

$$f * g \in L^1(\mathbb{C}^n),$$
$$\|f * g\|_{L^1} \le \|f\|_{L^1}\|g\|_{L^1},$$
$$\int_{\mathbb{C}^n} f * g(z)\ dz = \int_{\mathbb{C}^n} f(z)\ dz\ \int_{\mathbb{C}^n} g(z)\ dz.$$

2) For $f \in L^1(\mathbb{C}^n)$ and $A \in \mathcal{N}$ we have

$$f * A \in \mathcal{N},$$
$$\|f * A\|_{\mathcal{N}} \le \|f\|_{L^1}\|A\|_{\mathcal{N}},$$
$$\mathrm{Tr}(f * A) = \int_{\mathbb{C}^n} f(z)\ dz\, \mathrm{Tr}(A).$$

3) For $A, B \in \mathcal{N}$ we have

$$A * B \in L^1(\mathbb{C}^n) \cap C_b(\mathbb{C}^n),$$
$$\|A * B\|_{L^1} \le (\pi t)^n \|A\|_{\mathcal{N}}\|B\|_{\mathcal{N}},$$
$$\int_{\mathbb{C}^n} A * B(z)\ dz = (\pi t)^n\, \mathrm{Tr}(A)\, \mathrm{Tr}(B).$$

Proof. 1) This is well-known and follows from the Dominated Convergence Theorem and Fubini's Theorem.

2) The membership of $f * A$ in $\mathcal{N}$ is trivial, since the operator is defined as a Bochner integral in that space. Further, since W_z^t is isometric,

$$\begin{aligned}\|f * A\|_{\mathcal{N}} &\leq \int_{\mathbb{C}^n} |f(z)|\ \|W_z^t A W_{-z}^t\|_{\mathcal{N}}\ dz \\ &= \int_{\mathbb{C}^n} |f(z)|\ \|A\|_{\mathcal{N}}\ dz,\end{aligned}$$

which proves the second claim. Finally, since the trace map is continuous on the nuclear operators, we can exchange the order of applying the trace and evaluating the Bochner integral and obtain

$$\begin{aligned}\operatorname{Tr}(f * A) &= \operatorname{Tr}\left(\int_{\mathbb{C}^n} f(z) W_z^t A W_{-z}^t\ dz\right) \\ &= \int_{\mathbb{C}^n} f(z) \operatorname{Tr}(W_z^t A W_{-z}^t)\ dz \\ &= \int_{\mathbb{C}^n} f(z) \operatorname{Tr}(A)\ dz.\end{aligned}$$

3) The continuity of $A * B$ is immediate, since $z \mapsto \alpha_z(UBU)$ is continuous in the ideal of nuclear operators. Further, boundedness follows from

$$|\operatorname{Tr}(A\alpha_z(UBU))| \leq \|A\|_{\mathcal{N}} \|\alpha_z(UBU)\|_{op} = \|\mathcal{A}\|_{\mathcal{N}} \|B\|_{op}.$$

We copy the rest of the proof almost verbatim from [72, Lemma 2.3] with only minor changes. For simplicity, we state the remaining proof only for the case $A, B \in \mathcal{N}(F_t^p)$, the case $\mathcal{N}(f_t^\infty)$ is identical.

Assume that A and B are both rank one operators. Hence,

$$A = y_1 \otimes x_1, \quad B = y_2 \otimes x_2$$

with $y_j \in (F_t^p)' \cong F_t^q,\ x_j \in F_t^p$. Then,

$$\begin{aligned}A\alpha_z(UBU) &= (y_1 \otimes x_1)\alpha_z((Uy_2) \otimes (Ux_2)) \\ &= (y_1 \otimes x_1)((W_z^t U y_2) \otimes (W_z^t U x_2)).\end{aligned}$$

This is again a rank one operator and one readily checks

$$A\alpha_z(UBU) = ((W_z^t U y_2) \otimes x_1)\langle W_z^t U x_2, y_1\rangle_t.$$

Furthermore,

$$\begin{aligned}\operatorname{Tr}(A\alpha_z(UBU)) &= \langle x_1, W_z^t U y_2\rangle_t \langle W_z^t U x_2, y_1\rangle_t \\ &= \int_{\mathbb{C}^n} [W_{-z}^t x_1](w)\overline{[Uy_2](w)} d\mu_t(w) \int_{\mathbb{C}^n} [W_z^t U x_2](v)\overline{y_1(v)} d\mu_t(v) \\ &= \int_{\mathbb{C}^n} x_1(w+z) k_{-z}^t(w)\overline{y_2(-w)} d\mu_t(w) \int_{\mathbb{C}^n} x_2(z-v) k_z^t(v)\overline{y_1(v)} d\mu_t(v).\end{aligned}$$

Assume for the moment that x_j and y_j are polynomials (which are dense in F_t^p and F_t^q, respectively). We can then apply Fubini's Theorem and obtain:

$$\begin{aligned}&\int_{\mathbb{C}^n} \operatorname{Tr}(A\alpha_z(UBU))\, dz \\ &\quad = \int_{\mathbb{C}^n} \overline{y_2(-w)} \int_{\mathbb{C}^n} \overline{y_1(v)} \int_{\mathbb{C}^n} x_1(w+z) x_2(z-v) k_{-z}^t(w) k_z^t(v)\, dz d\mu_t(w) d\mu_t(v).\end{aligned}$$

Since x_1, x_2 are polynomials in $z_1, \ldots, z_n$, they (and their product) are in F_t^2 as well and it holds

$$\begin{aligned}&\int_{\mathbb{C}^n} x_1(w+z) x_2(z-v) k_{-z}^t(w) k_z^t(v) dz \\ &\qquad = (\pi t)^n \int_{\mathbb{C}^n} x_1(w+z) x_2(z-v) e^{\frac{(v-w)\cdot \overline{z}}{t}} d\mu_t(z) \\ &\qquad = (\pi t)^n \langle x_1(w+\cdot) x_2(\cdot - v), K_{v-w}^t\rangle_t \\ &\qquad = (\pi t)^n x_1(v) x_2(-w).\end{aligned}$$

We therefore get

$$\begin{aligned}\int_{\mathbb{C}^n} \operatorname{Tr}(A\alpha_z(UBU)) dz &= (\pi t)^n \int_{\mathbb{C}^n} x_1(v)\overline{y_1(v)} d\mu_t(v) \int_{\mathbb{C}^n} x_2(-w)\overline{y_2(-w)} d\mu_t(w) \\ &= (\pi t)^n \operatorname{Tr}(y_1 \otimes x_1) \operatorname{Tr}(y_2 \otimes x_2).\end{aligned}$$

Setting $x = x_1 = y_2$, $y = x_2 = y_1$ (which is possible, since we still assume that they are polynomials) we have

$$\int_{\mathbb{C}^n} |\langle y, W_z^t U x\rangle_t|^2\, dz = (\pi t)^n |\operatorname{Tr}(y \otimes x)|^2$$

and hence it holds $\langle y, W_z^t U x\rangle_t \in L^2(\mathbb{C}^n)$ as a function of z with

$$\|\langle y, W_z^t U x\rangle_t\|_{L^2} \leq (\pi t)^{n/2} |\operatorname{Tr}(y \otimes x)| \leq (\pi t)^{n/2} \|y\|_{F_t^q} \|x\|_{F_t^p}.$$

Therefore,

$$\operatorname{Tr}(A\alpha_z(UBU)) = \langle W_z^t U y_2, x_1\rangle_t \langle y_1, W_z^t U x_2\rangle_t \in L^1(\mathbb{C}^n)$$

(understood as a function of z) and the Cauchy-Schwarz inequality yields the estimate

$$\|\operatorname{Tr}(A\alpha_z(UBU))\|_{L^1} \le (\pi t)^n \|y_1\|_{F_t^q}\|y_2\|_{F_t^q}\|x_1\|_{F_t^p}\|x_2\|_{F_t^p}.$$

Now, let $x_j \in F_t^p$, $y_j \in F_t^q$ be arbitrary. Let $(x_j^m)_m$, $(y_j^m)_m$ be sequences of polynomials converging to x_j and y_j in F_t^p and F_t^q, respectively. Then,

$$\begin{aligned}\operatorname{Tr}(A\alpha_z(UBU)) &= \langle W_z^t U y_2, x_1\rangle_t \langle y_1, W_z^t U x_2\rangle_t \\ &= \lim_{m\to\infty} \langle W_z^t U y_2^m, x_1^m\rangle_t \langle y_1^m, W_z^t U x_2^m\rangle_t.\end{aligned}$$

By Fatou's Lemma we get

$$\|\operatorname{Tr}(A\alpha_z(UBU))\|_{L^1} \le (\pi t)^n \|y_1\|_{F_t^q}\|y_2\|_{F_t^q}\|x_1\|_{F_t^p}\|x_2\|_{F_t^p}.$$

Recall the inequality of norms from Remark 2.2.2, $\|y\|_{F_t^q} \le \|y\|_{(F_t^p)'}$. This yields

$$\begin{aligned}\|\operatorname{Tr}(A\alpha_z(UBU))\|_{L^1} &\le (\pi t)^n \|y_1\|_{(F_t^p)'}\|y_2\|_{(F_t^p)'}\|x_1\|_{F_t^p}\|x_2\|_{F_t^p} \\ &= (\pi t)^n \|y_1 \otimes x_1\|_{\mathcal{N}} \|y_2 \otimes x_2\|_{\mathcal{N}},\end{aligned}$$

which proves the result for arbitrary rank one operators. Having this estimate, it is easy to derive

$$\int_{\mathbb{C}^n} A * B(z)\, dz = (\pi t)^n \operatorname{Tr}(A)\operatorname{Tr}(B)$$

for operators of finite rank. Finally, it is standard to generalize the results for arbitrary nuclear operators. □

Remark 3.1.3. Upon applying the equation $\int_{\mathbb{C}^n} A * C(z)\, dz = (\pi t)^n \operatorname{Tr}(A)\operatorname{Tr}(C)$ to the operator $C = UBU$, one obtains

$$\int_{\mathbb{C}^n} \operatorname{Tr}(AW_z^t B W_{-z}^t)\, dz = (\pi t)^n \operatorname{Tr}(A)\operatorname{Tr}(B), \tag{3.2}$$

since U is formally self-adjoint (i.e. the adjoint of U under the standard dual pairing is $U^* = U$) and therefore $\operatorname{Tr}(UBU) = \operatorname{Tr}(B)$.

Lemma 3.1.4. *The following relations hold for $f, g \in L^1(\mathbb{C}^n)$ and $A, B, C \in \mathcal{N}$:*

$$\begin{aligned} f * (g * A) &= (f * g) * A, \\ f * (A * B) &= (f * A) * B, \\ (A * B) * C &= (B * C) * A, \\ A * B &= B * A. \end{aligned}$$

Proof. In this proof we will frequently use the following fact: Every $T \in \mathcal{L}$ induces a bounded linear functional on $\mathcal{N}$ via

$$N \mapsto \operatorname{Tr}(NT).$$

Since $k_z^t \otimes k_z^t \in \mathcal{N}$, we have for $T_1, T_2 \in \mathcal{L}$:

$$\begin{aligned} &\operatorname{Tr}(NT_1) = \operatorname{Tr}(NT_2) \text{ for all } N \in \mathcal{N} \\ \implies &\operatorname{Tr}((k_z^t \otimes k_z^t)T_1) = \operatorname{Tr}((k_z^t \otimes k_z^t)T_2) \text{ for all } z \in \mathbb{C}^n. \end{aligned}$$

But

$$\operatorname{Tr}((k_z^t \otimes k_z^t)T) = \operatorname{Tr}((k_z^t \otimes (T^* k_z^t)) = \langle k_z^t, T^* k_z^t \rangle_t = \widetilde{T}(z),$$

i.e. if two operators T_1, T_2 induce the same linear functional on $\mathcal{N}$, they have to be the same operators, since the Berezin transform is injective.

Let $N \in \mathcal{N}$. Then:

$$\begin{aligned} \operatorname{Tr}(N(f * (g * A))) &= \operatorname{Tr}\left(N \int_{\mathbb{C}^n} f(z) W_z^t \int_{\mathbb{C}^n} g(w) W_w^t A W_{-w}^t \, dw \; W_{-z}^t dz \right) \\ &= \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} f(z) g(w) \operatorname{Tr}(N W_{z+w}^t A W_{-(z+w)}^t) \, dw \, dz \\ &= \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} f(z) g(u-z) \operatorname{Tr}(N W_u^t A W_{-u}^t) \, du \, dz. \end{aligned}$$

Since $u \mapsto \operatorname{Tr}(N W_u^t A W_{-u}^t)$ is bounded we may apply Fubini's Theorem and obtain

$$\begin{aligned} \operatorname{Tr}(N(f * (g * A))) &= \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} f(z) g(u-z) \, dz \operatorname{Tr}(N W_u^t A W_{-u}^t) \, du \\ &= \operatorname{Tr}(N((f * g) * A)). \end{aligned}$$

Since $N \in \mathcal{N}$ was arbitrary, this proves $(f * g) * A = f * (g * A)$.

The second identity follows easily:

$$\begin{aligned} (f * A) * B(z) &= \operatorname{Tr}\left(\int_{\mathbb{C}^n} f(w) W_w^t A W_{-w}^t \, dz W_z^t U B U W_{-z}^t \right) \\ &= \int_{\mathbb{C}^n} f(w) \operatorname{Tr}(W_w^t A W_{-w}^t W_z^t U B U W_{-z}^t) \, dz \\ &= \int_{\mathbb{C}^n} f(w) \operatorname{Tr}(A W_{-w}^t W_z^t U B U W_{-z}^t W_w^t) \, dz \\ &= \int_{\mathbb{C}^n} f(w) \operatorname{Tr}(A W_{z-w}^t U B U W_{w-z}^t) \, dz. \end{aligned}$$

The fourth identity follows immediately from the definition and Equation 2.3. We reproduce the proof of the third identity from [103, Proposition 4.4], which discusses essentially the same convolution formalism in the Hilbert space case of the Schrödinger representation. Let $N \in \mathcal{N}$. Then,

$$\begin{aligned}
&\mathrm{Tr}(N((B*C)*A)) \\
&\qquad = \mathrm{Tr}(N((C*B)*A) \\
&\qquad = \mathrm{Tr}\left(N\int_{\mathbb{C}^n} \mathrm{Tr}(CW_z^tUBUW_{-z}^t)W_z^tAW_{-z}^t\ dz\right) \\
&\qquad = \int_{\mathbb{C}^n} \mathrm{Tr}(W_z^tAW_{-z}^tN)\,\mathrm{Tr}(CW_z^tUBUW_{-z}^t)\ dz.
\end{aligned}$$

Applying Equation (3.2) we get

$$\begin{aligned}
&= \frac{1}{(\pi t)^n}\int_{\mathbb{C}^n}\int_{\mathbb{C}^n} \mathrm{Tr}(W_z^tAW_{-z}^tNW_w^tCW_z^tUBUW_{-z}^tW_{-w}^t)\ dw\ dz \\
&= \frac{1}{(\pi t)^n}\int_{\mathbb{C}^n}\int_{\mathbb{C}^n} \mathrm{Tr}(W_z^tAW_{-z}^tNW_w^tCW_{-w}^tW_w^tW_z^tUBUW_{-z}^tW_{-w}^t)\ dw\ dz.
\end{aligned}$$

We now apply Equation (2.2):

$$\begin{aligned}
&= \frac{1}{(\pi t)^n}\int_{\mathbb{C}^n}\int_{\mathbb{C}^n} \mathrm{Tr}(W_z^tAW_{-z}^tNW_w^tCW_{-w}^tW_z^tW_w^tUBUW_{-w}^tW_{-z}^t)\ dw\ dz \\
&= \frac{1}{(\pi t)^n}\int_{\mathbb{C}^n}\int_{\mathbb{C}^n} \mathrm{Tr}(NW_w^tCW_{-w}^tW_z^tW_w^tUBUW_{-w}^tW_{-z}^tW_z^tAW_{-z}^t)\ dw\ dz \\
&= \frac{1}{(\pi t)^n}\int_{\mathbb{C}^n}\int_{\mathbb{C}^n} \mathrm{Tr}(NW_w^tCW_{-w}^tW_z^tW_w^tUBUW_{-w}^tAW_{-z}^t)\ dw\ dz.
\end{aligned}$$

Using Lemma 3.1.2 one can show that Fubini's Theorem applies here, which gives

$$\begin{aligned}
&= \frac{1}{(\pi t)^n}\int_{\mathbb{C}^n}\int_{\mathbb{C}^n} \mathrm{Tr}(NW_w^tCW_{-w}^tW_z^tW_w^tUBUW_{-w}^tAW_{-z}^t)\ dz\ dw \\
&= \int_{\mathbb{C}^n} \mathrm{Tr}(NW_w^tCW_{-w}^t)\,\mathrm{Tr}(W_w^tUBUW_{-w}^tA)\ dw \\
&= \mathrm{Tr}(N((A*B)*C)),
\end{aligned}$$

once again having used Equation (3.2). □

Lemma 3.1.5. *Let $f, g \in L^1(\mathbb{C}^n)$ and $A_1, A_2 \in \mathcal{N}$. Then, we have for all $z \in \mathbb{C}^n$:*

$$\begin{aligned}
\alpha_z(f*g) &= \alpha_z(f)*g = f*\alpha_z(g), \\
\alpha_z(f*A_1) &= \alpha_z(f)*A_1 = f*\alpha_z(A_1), \\
\alpha_z(A_1*A_2) &= \alpha_z(A_1)*A_2 = A_1*\alpha_z(A_2).
\end{aligned}$$

Proof. For the convolution of two functions the identity follows readily. For $f \in L^1(\mathbb{C}^n)$ and A_1 nuclear we get

$$\begin{aligned}\alpha_z(f * A_1) &= W_z^t \int_{\mathbb{C}^n} f(w) W_w^t A_1 W_{-w}^t \; dw \; W_{-z}^t \\ &= \int_{\mathbb{C}^n} f(w) W_z^t W_w^t A_1 W_{-w}^t W_{-z}^t \; dw \\ &= \int_{\mathbb{C}^n} f(w) W_w^t W_z^t A_1 W_{-z}^t W_{-w}^t \; dw \\ &= f * (\alpha_z(A_1)).\end{aligned}$$

The equality with $\alpha_z(f) * A_1$ follows now by a formal substitution. This can be made rigorous in the following way: On pairing with an arbitrary $N \in \mathcal{N}$, we obtain

$$\begin{aligned}\operatorname{Tr}(N\alpha_z(f * A_1)) &= \int_{\mathbb{C}^n} f(w) \operatorname{Tr}(N W_{w+z}^t A_1 W_{-(w+z)}^t) \; dw \\ &= \int_{\mathbb{C}^n} f(w-z) \operatorname{Tr}(N W_w^t A_1 W_{-w}^t) \; dw \\ &= \operatorname{Tr}\left(N(\alpha_z(f) * A_1)\right).\end{aligned}$$

The identities for the convolution of two nuclear operators follow easily from properties of the trace and the Equations (2.2) and (2.3). □

Recall that the convolution between $f \in L^1(\mathbb{C}^n)$ and $g \in L^\infty(\mathbb{C}^n)$ is also well-defined, producing a function in $L^\infty(\mathbb{C}^n)$. Analogously, we will now generalize the convolution to an operation between elements from $\mathcal{N}$ and $\mathcal{L}$, $L^1(\mathbb{C}^n)$ and $\mathcal{L}$, $L^\infty(\mathbb{C}^n)$ and $\mathcal{N}$. For a unified notation, we will denote the trace duality pairing by

$$\langle A, B\rangle_{tr} = \operatorname{Tr}(AB), \quad A \in \mathcal{N}, \; B \in \mathcal{L}.$$

Analogously, we will write

$$\langle f, g\rangle_{tr} = \int_{\mathbb{C}^n} f(z) g(z) \; dz,$$

which is well-defined for $g \in L^1(\mathbb{C}^n)$ and $f \in L^1(\mathbb{C}^n)$ or $f \in L^\infty(\mathbb{C}^n)$. Let us first note the following important identities:

Lemma 3.1.6. *Let $f \in L^1(\mathbb{C}^n)$ and $A_1, A_2 \in \mathcal{N}$. Then, we have*

$$\begin{aligned}\langle f * A_1, B\rangle_{tr} &= \langle f, B * (UA_1U)\rangle_{tr}, \quad B \in \mathcal{N}, \\ \langle f * A_2, B\rangle_{tr} &= \langle A_2, (Uf) * B\rangle_{tr}, \quad B \in \mathcal{N}, \\ \langle A_1 * A_2, g\rangle_{tr} &= \langle A_1, g * (UA_2U)\rangle_{tr}, \quad g \in L^1(\mathbb{C}^n).\end{aligned}$$

Proof. The first identity follows immediately from the properties of the Bochner integral and the definitions:

$$\begin{aligned}
\langle f * A_1, B\rangle_{tr} &= \operatorname{Tr}\left(\int_{\mathbb{C}^n} f(z)W_z^t A_1 W_{-z}^t \,dzB\right) \\
&= \int_{\mathbb{C}^n} f(z)\operatorname{Tr}(W_z^t A_1 W_{-z}^t B)\,dz \\
&= \langle f, B * (UA_1U)\rangle_{tr}.
\end{aligned}$$

The second identity follows, using the substitution $z \mapsto -z$:

$$\begin{aligned}
\langle f * A_2, B\rangle_{tr} &= \int_{\mathbb{C}^n} f(z)\operatorname{Tr}(W_z^t A_2 W_{-z}^t B)\,dz \\
&= \int_{\mathbb{C}^n} f(-z)\operatorname{Tr}(W_{-z}^t A_2 W_z^t B)\,dz \\
&= \int_{\mathbb{C}^n} f(-z)\operatorname{Tr}(A_2 W_z^t B W_{-z}^t)\,dz \\
&= \operatorname{Tr}\left(A_2 \int_{\mathbb{C}^n} Uf(z)W_z^t B W_{-z}^t\,dz\right).
\end{aligned}$$

The third identity follows equally easily:

$$\begin{aligned}
\langle A_1 * A_2, g\rangle_{tr} &= \int_{\mathbb{C}^n} g(z)\operatorname{Tr}(A_1 W_z^t UA_2 U W_{-z}^t)\,dz \\
&= \operatorname{Tr}\left(A_1 \int_{\mathbb{C}^n} g(z)W_z^t UA_2UW_{-z}^t\,dz\right) \\
&= \langle A_1,\ g * (UA_2U)\rangle_{tr}.
\end{aligned}$$

□

We can now set up the following definition:

Definition 3.1.7. Let $f \in L^1(\mathbb{C}^n)$, $g \in L^\infty(\mathbb{C}^n)$, $A \in \mathcal{N}$ and $B \in \mathcal{L}$. Then, $f*B \in \mathcal{N}'$, $A*g \in \mathcal{N}'$ and $A*B \in (L^1(\mathbb{C}^n))' = L^\infty(\mathbb{C}^n)$ are defined through the following relations:

$$\begin{aligned}
f * B(N) &= \langle B, Uf * N\rangle_{tr}, \quad N \in \mathcal{N}, \\
A * g(N) &= \langle g, N * (UAU)\rangle_{tr}, \quad N \in \mathcal{N}, \\
\langle A * B, h\rangle_{tr} &= \langle B, h * (UAU)\rangle_{tr}, \quad h \in L^1(\mathbb{C}^n).
\end{aligned}$$

Note that Lemma 3.1.6 states that the convolutions are well-defined. In the reflexive case of F_t^p ($p \in (1,\infty)$), one can isometrically identify $\mathcal{L}(F_t^p) \cong (\mathcal{N}(F_t^p))'$ and therefore we can identify the linear functionals $f * B$ and $A * g$ with elements from $\mathcal{L}(F_t^p)$. In the case of f_t^∞, which has the approximation property, we still have the isometric identifications

$$\mathcal{N}(f_t^\infty) \cong (f_t^\infty)'\hat{\otimes}_\pi f_t^\infty,$$

where, $\hat{\otimes}_\pi$ denotes the projective tensor product (see e.g. [116] for an introduction to tensor products of Banach spaces). Here, we identify each element

$$\sum_{j=1}^{\infty} x_j \otimes y_j \in (f_t^\infty)' \hat{\otimes}_\pi f_t^\infty$$

with the nuclear operator

$$f_t^\infty \ni f \mapsto \sum_{j=1}^{\infty} x_j(f) y_i.$$

Further, we can isometrically identify

$$((f_t^\infty)' \hat{\otimes}_\pi f_t^\infty)' \cong \mathcal{L}((f_t^\infty)')$$

by associating with each $T \in \mathcal{L}((f_t^\infty)')$ the functional

$$\psi_T : ((f_t^\infty)' \hat{\otimes}_\pi f_t^\infty) \to \mathbb{C}, \quad \psi_T \left(\sum_{j=1}^{\infty} x_j \otimes y_j \right) = \sum_{j=1}^{\infty} (T x_j)(y_j).$$

Each element of $\mathcal{N}(f_t^\infty)'$ arises in this form. It is important to note that each $A \in \mathcal{L}(f_t^\infty)$ induces a linear functional on $\mathcal{N}(f_t^\infty)$ via

$$\varphi_A : \mathcal{N}(f_t^\infty) \to \mathbb{C}, \quad \varphi_A(N) = \operatorname{Tr}(NA).$$

If $N = \sum_{j=1}^{\infty} x_j \otimes y_j$, then

$$\begin{aligned}
\varphi_A(N) &= \operatorname{Tr} \left(\sum_{j=1}^{\infty} x_j \otimes (A y_j) \right) \\
&= \operatorname{Tr} \left(\sum_{j=1}^{\infty} (A' y_j) \otimes x_j \right) \\
&= \sum_{j=1}^{\infty} (A' y_j)(x_j) \\
&= \psi_{A'}(N).
\end{aligned}$$

Since $\|A\|_{op} = \|A'\|_{\mathcal{L}((f_t^\infty)')} = \|\psi_{A'}\|$, we obtain:

Lemma 3.1.8. *Let $g \in L^\infty(\mathbb{C}^n)$ and $A \in \mathcal{N}$. If $A * g \in \mathcal{N}'$ is induced by some $B \in \mathcal{L}$, i.e. $A * g = \varphi_B$, then we have*

$$\|B\|_{op} = \|A * g\|_{\mathcal{N}'}$$

and the following estimate holds true:

$$\|B\|_{op} \le (\pi t)^n \|g\|_\infty \|A\|_{\mathcal{N}}.$$

Proof. The equality of the norms follows from the above discussion. Further, we have for $N \in \mathcal{N}$:

$$\begin{aligned} |A * g(N)| &= |\langle g, N * (UAU)\rangle| \\ &\leq \|g\|_\infty \|N * (UAU)\|_{L^1} \\ &\leq (\pi t)^n \|g\|_\infty \|N\|_{\mathcal{N}} \|A\|_{\mathcal{N}}, \end{aligned}$$

which gives the norm estimate. □

We will show in the following that for the case of f_t^∞, at least in the cases most interesting to us, the functionals $A * g$ on $\mathcal{N}$ are also induced by bounded linear operators. In case the functionals are actually induced by operators, we will never distinguish between the functional and the corresponding operator.

Lemma 3.1.9. *Let $A \in \mathcal{N}$, $B \in \mathcal{L}$. Then, we have*

$$A * B(z) = \operatorname{Tr}(AW_z^t UBUW_{-z}^t)$$

and the following estimate holds true:

$$\|A * B\|_\infty \leq \|A\|_{\mathcal{N}} \|B\|_{op}.$$

Proof. For $h \in L^1(\mathbb{C}^n)$ it is

$$\begin{aligned} \langle \operatorname{Tr}(AW_{(\cdot)}^t UBUW_{(-\cdot)}^t), h\rangle_{tr} &= \int_{\mathbb{C}^n} h(z) \operatorname{Tr}(AW_z^t UBUW_{-z}^t)\, dz \\ &= \int_{\mathbb{C}^n} h(z) \operatorname{Tr}(BW_z^t UAUW_{-z}^t)\, dz \\ &= \operatorname{Tr}\left(B \int_{\mathbb{C}^n} h(z) W_z^t UAUW_{-z}^t\, dz\right) \\ &= \langle B, h * (UAU)\rangle_{tr}, \end{aligned}$$

i.e. the function satisfies the defining relation. The norm estimate is now immediate. □

Lemma 3.1.10. *For $f \in L^1(\mathbb{C}^n)$ and $A \in \mathcal{C}_1$ we have*

$$f * A = \int_{\mathbb{C}^n} f(z) W_z^t A W_{-z}^t\, dz \in \mathcal{C}_1$$

and

$$\|f * A\|_{op} \leq \|f\|_{L^1} \|A\|_{op}.$$

Proof. First observe that the above integral exists as a Bochner integral in $\mathcal{C}_1$. Now,

$$\begin{aligned}\left\langle \int_{\mathbb{C}^n} f(z) W_z^t A W_{-z}^t \, dz, N \right\rangle_{tr} &= \int_{\mathbb{C}^n} f(z) \operatorname{Tr}(A W_{-z}^t N W_z^t) \, dz \\ &= \int_{\mathbb{C}^n} f(-z) \operatorname{Tr}(A W_z^t N W_{-z}^t) \, dz \\ &= \operatorname{Tr}\left(A(Uf * N)\right) \\ &= \langle A, Uf * N \rangle_{tr},\end{aligned}$$

i.e. the integral satisfies the defining relation. The norm estimate follows directly from basic properties of the Bochner integral. □

Remark 3.1.11. Since $z \mapsto W_z^t$ acts weak* continuously on $\mathcal{L}$ (Lemma 3.1.1), the integral

$$\int_{\mathbb{C}^n} f(z) W_z^t A W_{-z}^t \, dz$$

exists as a weak* integral in $\mathcal{L}$ and satisfies the defining relation for $f * A$. Thus $f * A \in \mathcal{L}$ for all $f \in L^1(\mathbb{C}^n)$, $A \in \mathcal{L}$. Since we will not need this, we do not go into the details here.

It will be a crucial point of the following discussions that $g * B$ is also a linear operator for all $g \in L^\infty(\mathbb{C}^n)$, at least for one particular choice of $B \in \mathcal{N}$. But before showing this, let us state the following important observations:

Lemma 3.1.12. *For $f \in L^1(\mathbb{C}^n)$, $g \in L^\infty(\mathbb{C}^n)$, $A \in \mathcal{N}$ and $B \in \mathcal{L}$ we have*

$$\begin{aligned}\alpha_z(f * B) &= \alpha_z(f) * B = f * \alpha_z(B), \\ \alpha_z(A * g) &= \alpha_z(A) * g = A * \alpha_z(g), \\ \alpha_z(A * B) &= \alpha_z(A) * B = A * \alpha_z(B).\end{aligned}$$

Proof. For the cases $f*B$ and $A*g$ note the following: While the statement of the lemma is also true in larger generality (considering the action of α_z on $\mathcal{N}(f_t^\infty)' \cong \mathcal{L}((f_t^\infty)')$ as the action on $\mathcal{L}(F_t^1)$ via $(f_t^\infty)' \cong F_t^1$), we will discuss the relations only for the case that the convolutions are induced by linear operators from $\mathcal{L}$.

In this case, the identities follow easily from the defining relations of the convolutions, which now read as

$$\begin{aligned}\langle f * B, N \rangle_{tr} &= \langle B, Uf * N \rangle_{tr}, \\ \langle A * g, N \rangle_{tr} &= \langle g, N * (UAU) \rangle_{tr}, \\ \langle A * B, h \rangle_{tr} &= \langle B, h * (UAU) \rangle_{tr},\end{aligned}$$

and an application of Lemma 3.1.5. We discuss this for the identity $\alpha_z(A*g) = \alpha_z(A)*g$ as an example. Note that α_z leaves $\mathcal{L}$ and $\mathcal{N}$ invariant, so $\alpha_z(A * g) \in \mathcal{N}'$ is induced by an operator from $\mathcal{L}$ if and only if $A * g$ is. Then:

$$
\begin{aligned}
\langle \alpha_z(A * g), N\rangle_{tr} &= \mathrm{Tr}(W_z^t (A * g) W_{-z}^t N) \\
&= \mathrm{Tr}((A * g) W_{-z}^t N W_z^t) \\
&= \langle A * g, W_{-z}^t N W_z^t \rangle_{tr} \\
&= \langle g, (W_{-z}^t N W_z^t) * (UAU)\rangle_{tr} \\
&= \int_{\mathbb{C}^n} g(w)\, \mathrm{Tr}(W_{-z}^t N W_z^t W_w^t A W_{-w}^t)\, dw \\
&= \int_{\mathbb{C}^n} g(w)\, \mathrm{Tr}(N W_z^t W_w^t A W_{-w}^t W_{-z}^t)\, dw \\
&= \int_{\mathbb{C}^n} g(w)\, \mathrm{Tr}(N W_w^t W_z^t A W_{-z}^t W_{-w}^t)\, dw \\
&= \langle g, N * (U\alpha_z(A)U)\rangle_{tr} \\
&= \langle \alpha_z(A) * g, N\rangle_{tr}.
\end{aligned}
$$

Similar computations prove the other identities. □

We now obtain the following important consequence:

Proposition 3.1.13. *1) Let $A \in \mathcal{N}$, $B \in \mathcal{L}$. Then, $A * B \in \mathrm{BUC}(\mathbb{C}^n)$.*

2) Let $A \in \mathcal{N}$ and $g \in L^\infty(\mathbb{C}^n)$ such that $A * g \in \mathcal{L}$. Then, $A * g \in \mathcal{C}_1$.

3) Let $f \in L^1(\mathbb{C}^n)$ and $B \in \mathcal{L}$ such that $f * B \in \mathcal{L}$. Then, $f * B \in \mathcal{C}_1$.

Proof. 1) This follows from the previous results, since as $z \to 0$,

$$
\begin{aligned}
\|A * B - \alpha_z(A * B)\|_\infty &= \|(A - \alpha_z(A)) * B\|_\infty \\
&\leq \|A - \alpha_z(A)\|_{\mathcal{N}} \|B\|_{op} \\
&\to 0, \quad z \to 0,
\end{aligned}
$$

having used the strong continuity of α_z on $\mathcal{N}$.

2) The reasoning here is similar:

$$
\begin{aligned}
\|A * g - \alpha_z(A * g)\|_{op} &= \|(A - \alpha_z(A)) * g\|_{op} \\
&\lesssim \|A - \alpha_z(A)\|_{\mathcal{N}} \|g\|_\infty \\
&\to 0, \quad z \to 0.
\end{aligned}
$$

3) Follows analogously to 2). □

We deduce several associativity relations from the properties of the convolutions on $\mathcal{N}$ and $L^1(\mathbb{C}^n)$. The one important to us is the following:

Lemma 3.1.14. *Let $A, B \in \mathcal{N}$ and $C \in \mathcal{L}$. Then, $A * (B * C) = (A * B) * C$.*

Proof. It is not difficult to verify that $(UAU) * (UBU) = U(A * B)$. In particular, for $N \in \mathcal{N}$ we obtain

$$(N * (UAU)) * (UBU) = N * ((UAU) * (UBU)) = N * (U(A * B)).$$

Therefore,

$$\begin{aligned}(A * (B * C))(N) &= \langle B * C, N * (UAU)\rangle_{tr} \\ &= \langle C, (N * (UAU)) * (UBU)\rangle_{tr} \\ &= \langle C, N * (U(A * B))\rangle_{tr} \\ &= ((A * B) * C)(N),\end{aligned}$$

as required. □

Let us also note the following fact, which will we use later:

Lemma 3.1.15. *Let $p_0 \in [1, \infty)$. For $A \in \mathcal{N}$, $B \in \mathcal{S}^{p_0}$ the convolution $A * B$ is well-defined and satisfies*

$$\|A * B\|_{L^{p_0}} \leq C\|A\|_{\mathcal{N}}\|B\|_{\mathcal{S}^{p_0}}$$

for some constant $C > 0$ depending on n, t, p and p_0.

Proof. This follows immediately by applying the Complex Interpolation Method to the maps

$$\begin{aligned}B &\mapsto A * B,\ \mathcal{N} \to L^1(\mathbb{C}^n) \\ B &\mapsto A * B,\ \mathcal{L} \to L^\infty(\mathbb{C}^n)\end{aligned}$$

which satisfy estimates of the form

$$\begin{aligned}\|A * B\|_{L^1} &\lesssim \|A\|_{\mathcal{N}}\|B\|_{\mathcal{N}} \\ \|A * B\|_\infty &\lesssim \|A\|_{\mathcal{N}}\|B\|_{op}.\end{aligned}$$

□

3.2 Connections with Toeplitz operators and the Berezin transform

Let us denote by $P_{\mathbb{C}}$ the operator

$$P_{\mathbb{C}} = 1 \otimes 1 \in \mathcal{N},$$

i.e.

$$P_{\mathbb{C}}(f) = f(0)$$

and its normalized version $\mathcal{R}_t = \frac{1}{(\pi t)^n} P_{\mathbb{C}}$.

Proposition 3.2.1. *Let $A \in \mathcal{L}$. Then, we have*

$$P_{\mathbb{C}} * A(z) = \widetilde{A}(z).$$

Proof. One easily verifies

$$AW_z^t U P_{\mathbb{C}} U W_{-z}^t = AW_z^t U(1 \otimes 1)UW_{-z}^t = k_z^t \otimes (Ak_z^t)$$

and therefore

$$P_{\mathbb{C}} * A(z) = \mathrm{Tr}(AW_z^t U P_{\mathbb{C}} U W_{-z}^t) = \mathrm{Tr}(k_z^t \otimes (Ak_z^t)) = \langle Ak_z^t, k_z^t \rangle_t = \widetilde{A}(z). \qquad \square$$

Let us note the following: Since $P_{\mathbb{C}} * A \in L^1(\mathbb{C}^n)$ for $A \in \mathcal{N}$ we obtain:

Lemma 3.2.2. *Let $A \in \mathcal{N}$. Then, $\widetilde{A} \in L^1(\mathbb{C}^n)$.*

Proposition 3.2.3. *For $f \in L^1(\mathbb{C}^n)$ we have $f * \mathcal{R}_t = T_f^t$. In particular, $T_f^t \in \mathcal{N}$ and $\|T_f^t\|_{\mathcal{N}} \lesssim \|f\|_{L^1}$.*

Proof. Recall that $f * \mathcal{R}_t$ is defined through the Bochner integral

$$f * \mathcal{R}_t = \int_{\mathbb{C}^n} f(z) W_z^t \mathcal{R}_t W_{-z}^t \, dz.$$

Let us evaluate this integral at $g \in F_t^p$ for $p \in (1, \infty)$ or $g \in f_t^\infty$:

$$\begin{aligned} f * \mathcal{R}_t(g) &= \frac{1}{(\pi t)^n} \int_{\mathbb{C}^n} f(z)(k_z^t \otimes k_z^t)(g) \, dz \\ &= \frac{1}{(\pi t)^n} \int_{\mathbb{C}^n} f(z) \langle g, K_z^t \rangle_t K_z^t e^{-\frac{|z|^2}{t}} \, dz \\ &= \frac{1}{(\pi t)^n} \int_{\mathbb{C}^n} f(z) g(z) K_z^t e^{-\frac{|z|^2}{t}} \, dz \\ &= T_f^t g. \end{aligned}$$

The norm estimates now follow from the corresponding estimates for the convolutions. $\square$

Proposition 3.2.4. *Let $f \in L^\infty(\mathbb{C}^n)$. Then, we have $\mathcal{R}_t * f = T_f^t \in \mathcal{L}$.*

Proof. Recall that we have to prove that T_f^t satisfies the relation

$$\langle T_f^t, N \rangle_{tr} = \langle f, N * (U\mathcal{R}_t U) \rangle_{tr}$$

for each $N \in \mathcal{N}$. More specifically, since $\mathrm{Span}\{k_z^t;\ z \in \mathbb{C}^n\}$ is dense in F_t^p and $(F_t^p)' \cong F_t^q$ (respectively, in f_t^∞ and $(f_t^\infty)' \cong F_t^1$), it suffices to prove this for $N = k_z^t \otimes k_w^t$, $z, w \in \mathbb{C}^n$. In this case, we have

$$\begin{aligned}\langle T_f^t, N\rangle_{tr} &= \mathrm{Tr}((k_z^t \otimes k_w^t)T_f^t)\\ &= \mathrm{Tr}((T_f^t)^* k_z^t) \otimes k_w^t)\\ &= \langle k_w^t, (T_f^t)^* k_z^t\rangle_t\\ &= \langle T_f^t k_w^t, k_z^t\rangle_t\\ &= \widetilde{f}^{(t)}(w,z).\end{aligned}$$

On the other hand, one can easily verify that

$$(k_z^t \otimes k_w^t) W_u^t (1 \otimes 1) W_{-u}^t = (k_z^t \otimes k_w^t)(k_u^t \otimes k_u^t) = \langle k_u^t, k_z^t\rangle_t (k_u^t \otimes k_w^t)$$

and therefore

$$\begin{aligned}\langle f, N * (U\mathcal{R}_t U)\rangle_{tr} &= \frac{1}{(\pi t)^n}\int_{\mathbb{C}^n} f(u)\,\mathrm{Tr}((k_z^t \otimes k_w^t) W_u^t (1\otimes 1) W_{-u}^t)\ dz\\ &= \frac{1}{(\pi t)^n}\int_{\mathbb{C}^n} f(u)\langle k_u^t, k_z^t\rangle_t \langle k_w^t, k_u^t\rangle_t\ du\\ &= \int_{\mathbb{C}^n} f(u) e^{\frac{z\cdot\overline{u}}{t} - \frac{|z|^2}{2t}} e^{\frac{u\cdot\overline{w}}{t} - \frac{|w|^2}{2t}}\, d\mu_t(u)\\ &= \langle f k_w^t, k_z^t\rangle_t\\ &= \widetilde{f}^{(t)}(w,z).\end{aligned}$$

□

We have the following important consequence:

Lemma 3.2.5. *Let $f \in L^\infty(\mathbb{C}^n)$. Then, $T_f^t \in \mathcal{C}_1$.*

Proof. This follows from the previous proposition and Proposition 3.1.13. □

So far we have obtained continuous linear maps

$$\begin{aligned}L^1(\mathbb{C}^n) \to \mathcal{N}, &\quad f \mapsto T_f^t,\\ L^\infty(\mathbb{C}^n) \to \mathcal{L}, &\quad f \mapsto T_f^t,\\ \mathcal{N} \to L^1(\mathbb{C}^n), &\quad A \mapsto \widetilde{A},\\ \mathcal{L} \to L^\infty(\mathbb{C}^n), &\quad A \mapsto \widetilde{A}.\end{aligned}$$

Applying the Complex Interpolation Method gives continuous linear maps

$$\begin{aligned}L^{p_0}(\mathbb{C}^n) \to \mathcal{S}^{p_0}, &\quad f \mapsto T_f^t,\\ \mathcal{S}^{p_0} \mapsto L^{p_0}(\mathbb{C}^n), &\quad A \mapsto \widetilde{A}.\end{aligned}$$

Here, we use the notation $\mathcal{S}^{p_0} = \mathcal{S}^{p_0}(F_t^p)$ or $\mathcal{S}^{p_0} = \mathcal{S}^{p_0}(f_t^\infty)$ for the ideal obtained from complex interpolation between $\mathcal{N}$ and $\mathcal{L}$ for $1 \leq p_0 < \infty$, cf. Appendix A.3. In particular, this proves that there is some constant $c > 0$ such that the estimates

$\|T_f^t\|_{\mathcal{S}^{p_0}} \leq c\|f\|_{L^{p_0}}$, $\|\widetilde{A}\|_{L^{p_0}} \leq c\|A\|_{\mathcal{S}^{p_0}}$ hold true. Since the finite rank operators are dense in $\mathcal{S}^{p_0}$, it is not difficult to see that α_z acts strongly continuous on $\mathcal{S}^{p_0}$. In particular, we can define for $f \in L^1(\mathbb{C}^n)$ the convolution with $A \in \mathcal{S}^{p_0}$ as

$$f * A = \int_{\mathbb{C}^n} f(z)\alpha_z(A)\ dz,$$

which converges as a Bochner integral (or alternatively, obtaining the same estimate, interpolate the convolution). One of the few important (to us) facts about this convolution is the obvious estimate

$$\|f * A\|_{\mathcal{S}^{p_0}} \leq \|f\|_{L^1}\|A\|_{\mathcal{S}^{p_0}}.$$

Further, for $A \in \mathcal{N}$, $B \in \mathcal{S}^{p_0}$ we can still set

$$A * B(z) = \operatorname{Tr}(AW_z^t UBUW_{-z}^t).$$

In the following, we will denote for $s > 0$ by f_s the Gaussian functions

$$f_s(z) = \frac{1}{(\pi s)^n} e^{-\frac{|z|^2}{s}}.$$

An easy computation shows that $\mathcal{R}_t * P_{\mathbb{C}} = f_t$. Using Lemma 3.1.14 and Propositions 3.2.1 and 3.2.4 we obtain the following important identity, which is well known in the Hilbert space setting [30, Theorem 6].

Lemma 3.2.6. *For $A \in \mathcal{L}$ the following holds true:*

$$f_t * A = T_{\widetilde{A}}^t.$$

3.3 Correspondence Theory

Recall that convolution by f_s is an approximate identity in $\mathrm{BUC}(\mathbb{C}^n)$, i.e. for each $g \in \mathrm{BUC}(\mathbb{C}^n)$ we have $f_s * g \to g$ uniformly as $s \to 0$. The following fact is of key importance:

Lemma 3.3.1. *Convolution by f_s is an approximate identity of $\mathcal{S}^{p_0}$, $1 \leq p_0 < \infty$, and also of $\mathcal{C}_1$.*

Proof. Recall that α_z acts strongly continuously on $\mathcal{S}^{p_0}$ and on $\mathcal{C}_1$. Let us denote by $\|\cdot\|$ the norm of either $\mathcal{S}^{p_0}$ or $\mathcal{C}_1$ for the moment. Then, by the reverse triangle inequality, $z \mapsto \|\alpha_z(A) - A\|$ is uniformly continuous. Therefore,

$$\begin{aligned}\|f_s * A - A\| &\leq \int_{\mathbb{C}^n} f_s(z)\|\alpha_z(A) - A\|\ dz \\ &= f_s * \|\alpha_{(-\cdot)}(A) - A\|(0) \\ &\to \|\alpha_{(-0)}(A) - A\| = 0\end{aligned}$$

as $s \to 0$, which is what we wanted to prove. ∎

Let us recall the following important result due to Norbert Wiener. Here, $\hat{f}$ denotes the Fourier transform of the function f.

Theorem 3.3.2 (Wiener's Approximation Theorem). *Let $f \in L^1(\mathbb{R}^n)$. Then, the following holds true:*

$$\hat{f}(\xi) \neq 0 \text{ for all } \xi \in \mathbb{R}^n \iff \operatorname{Span}\{f(\cdot - x);\ x \in \mathbb{R}^n\} \text{ is dense in } L^1(\mathbb{R}^n).$$

A proof of this well-known theorem can be found in [70, Corollary 4.70]. Since the Fourier transform of a Gaussian is again a Gaussian, which vanishes nowhere, we obtain the following: For each $N \in \mathbb{N}$ there are constants $M_N \in \mathbb{N}$, $c_j^N \in \mathbb{C}$ and $z_j^N \in \mathbb{C}^n$ $(j = 1, \dots, M_N)$ such that

$$\left\| f_{\frac{1}{N}} - \sum_{j=1}^{M_N} c_j^N \alpha_{z_j^N}(f_1) \right\|_{L^1} < \frac{1}{N}.$$

Let us fix these constants. It is a matter of a simple substitution to show that the same constants satisfy

$$\left\| f_{\frac{t}{N}} - \sum_{j=1}^{M_N} c_j^N \alpha_{\sqrt{t}z_j^N}(f_t) \right\|_{L^1} < \frac{1}{N}$$

for all $t > 0$.

Theorem 3.3.3. *Let $A \in \mathcal{S}^{p_0}$ for $1 \leq p_0 < \infty$ or $A \in \mathcal{C}_1$. Then:*

$$\left\| A - \sum_{j=1}^{M_N} c_j^N \alpha_{\sqrt{t}z_j^N}(T^t_{\widetilde{A}}) \right\| \to 0, \quad N \to \infty,$$

where $\|\cdot\|$ is the norm of the space from which A is taken.

Proof. Recall that $T^t_{\widetilde{A}} = f_t * A$ by Lemma 3.2.6. By Lemmas 3.1.12 and 3.3.1 we receive

$$\begin{aligned}
\left\| A - \sum_{j=1}^{M_N} c_j^N \alpha_{\sqrt{t}z_j^N}(T^t_{\widetilde{A}}) \right\| &\leq \left\| A - f_{\frac{t}{N}} * A \right\| + \left\| f_{\frac{t}{N}} * A - \sum_{j=1}^{M_N} c_j^N \alpha_{\sqrt{t}z_j^N}(T^t_{\widetilde{A}}) \right\| \\
&= \left\| A - f_{\frac{t}{N}} * A \right\| + \left\| f_{\frac{t}{N}} * A - \sum_{j=1}^{M_N} c_j^N \alpha_{\sqrt{t}z_j^N}(f_t) * A \right\| \\
&\leq \left\| A - f_{\frac{t}{N}} * A \right\| + \left\| f_{\frac{t}{N}} - \sum_{j=1}^{M_N} c_j^N \alpha_{\sqrt{t}z_j^N}(f_t) \right\|_{L^1} \|A\| \\
&\leq \left\| A - f_{\frac{t}{N}} * A \right\| + \frac{1}{N}\|A\| \\
&\to 0
\end{aligned}$$

as $N \to \infty$. □

Let us introduce some notation. For a subspace $\mathcal{D}_0 \subset L^\infty(\mathbb{C}^n)$ we denote for $1 \le p < \infty$ and $t > 0$:

$$\begin{aligned}
\mathcal{T}_{lin}^{p,t}(\mathcal{D}_0) &:= \overline{\{T_f^t \in \mathcal{L}(F_t^p);\ f \in \mathcal{D}_0\}},\\
\mathcal{T}^{p,t}(\mathcal{D}_0) &:= \overline{\mathrm{Alg}}\{T_f^t \in \mathcal{L}(F_t^p);\ f \in \mathcal{D}_0\},\\
\mathcal{T}_*^{2,t}(\mathcal{D}_0) &:= C^*(\{T_f^t \in \mathcal{L}(F_t^2);\ f \in \mathcal{D}_0\}),
\end{aligned}$$

i.e. $\mathcal{T}_{lin}^{p,t}(\mathcal{D}_0)$ is the operator norm closure of the Toeplitz operators with symbol in $\mathcal{D}_0$, $\mathcal{T}^{p,t}(\mathcal{D}_0)$ is the Banach algebra and $\mathcal{T}_*^{2,t}(\mathcal{D}_0)$ the C^* algebra generated by them. Analogously,

$$\begin{aligned}
\mathcal{T}_{lin}^{\infty,t}(\mathcal{D}_0) &:= \overline{\{T_f^t \in \mathcal{L}(f_t^\infty);\ f \in \mathcal{D}_0\}},\\
\mathcal{T}^{\infty,t}(\mathcal{D}_0) &:= \overline{\mathrm{Alg}}\{T_f^t \in \mathcal{L}(f_t^\infty);\ f \in \mathcal{D}_0\}.
\end{aligned}$$

For a lack of a better notation, we will write

$$\begin{aligned}
\mathcal{T}_{lin}^{\infty+,t}(\mathcal{D}_0) &:= \overline{\{T_f^t \in \mathcal{L}(F_t^\infty);\ f \in \mathcal{D}_0\}},\\
\mathcal{T}^{\infty+,t}(\mathcal{D}_0) &:= \overline{\mathrm{Alg}}\{T_f^t \in \mathcal{L}(F_t^\infty);\ f \in \mathcal{D}_0\}.
\end{aligned}$$

Further, we will often abbreviate $\mathcal{T}^{p,t} = \mathcal{T}^{p,t}(L^\infty(\mathbb{C}^n))$ $(1 \le p \le \infty)$ and $\mathcal{T}^{\infty+,t} = \mathcal{T}^{\infty+,t}(L^\infty(\mathbb{C}^n))$ for the full Toeplitz algebra.

Observe the following consequence of Theorem 3.3.3, parts of which are well-known (i.e. $\mathcal{T}^{2,t} = \mathcal{T}_{lin}^{2,t}(\mathrm{BUC}(\mathbb{C}^n))$ is obtained from the Fock space version of [131, Theorem 1.5] combined with [19, Theorem 3.7] and $\mathcal{N} = \overline{\{T_f^t;\ f \in L^1(\mathbb{C}^n)\}}$ for $p = 2$ is [30, Theorem 9]).

Corollary 3.3.4. *Let $1 < p \le \infty$ and $t > 0$. Then:*

1) $\mathcal{C}_1 = \mathcal{T}^{p,t} = \mathcal{T}_{lin}^{p,t}(\mathrm{BUC}(\mathbb{C}^n))$;

2) $\mathcal{S}^{p_0} = \overline{\{T_f^t;\ f \in L^{p_0}(\mathbb{C}^n)\}}$ for each $1 \le p_0 < \infty$.

In the second part of the Corollary, the closure is taken with respect to the $\mathcal{S}^{p_0}$ norm. A construction analogous to the one above works for functions. If $f \in L^{p_0}(\mathbb{C}^n)$ or $f \in \mathrm{BUC}(\mathbb{C}^n)$, then we have

$$\begin{aligned}
\left\| f - \sum_{j=1}^{M_N} c_j^N \alpha_{\sqrt{t}z_j^N}(\widetilde{f}^{(t)}) \right\| &\le \left\| f - f_{\frac{t}{N}} * f \right\| + \left\| f_{\frac{t}{N}} * f - \sum_{j=1}^{M_N} c_j^N \alpha_{\sqrt{t}z_j^N}(f_t) * f \right\|\\
&\le \left\| f - f_{\frac{t}{N}} * f \right\| + \left\| f_{\frac{t}{N}} - \sum_{j=1}^{M_N} c_j^N \alpha_{\sqrt{t}z_j^N}(f_t) \right\|_{L^1} \|f\|\\
&\to 0
\end{aligned}$$

as $N \to \infty$. We therefore obtain:

Proposition 3.3.5. *Let $f \in L^{p_0}(\mathbb{C}^n)$ or $f \in \mathrm{BUC}(\mathbb{C}^n)$. Then, we have*

$$\left\| f - \sum_{j=1}^{M_N} c_j^N \alpha_{\sqrt{t}z_j^N}(\widetilde{f}^{(t)}) \right\| \to 0$$

as $N \to \infty$. Here, $\|\cdot\|$ is the norm coming from the space $L^{p_0}(\mathbb{C}^n)$ or $\mathrm{BUC}(\mathbb{C}^n)$. In particular, we have

$$L^{p_0}(\mathbb{C}^n) = \overline{\{\widetilde{A};\ A \in \mathcal{S}^{p_0}\}},$$
$$\mathrm{BUC}(\mathbb{C}^n) = \overline{\{\widetilde{A};\ A \in \mathcal{C}_1\}},$$

for all $1 \le p_0 < \infty$.

It is noteworthy that the last two equalities in the previous result are independent of p and t.

The cases F_t^1 and F_t^∞

We will now try to carry over the results we obtained to operators on F_t^1 and F_t^∞. In both cases, we can still define the space $\mathcal{C}_1$ analogously, i.e. as the space of operators A from $\mathcal{L}(F_t^1)$ or $\mathcal{L}(F_t^\infty)$ on which the shifts $z \mapsto \alpha_z(A)$ act strongly continuously. Further, for $A \in \mathcal{C}_1$ and $f \in L^1(\mathbb{C}^n)$, the convolution

$$f * A := \int_{\mathbb{C}^n} f(z) W_z^t A W_{-z}^t \, dz$$

is still well-defined as a Bochner integral in $\mathcal{C}_1$ and satisfies the estimate $\|f * A\|_{op} \le \|f\|_{L^1}\|A\|_{op}$. If one wants to imitate the preceding convolution approach, one encounters the problem that the compact operators (and even the nuclear operators) are not entirely contained in $\mathcal{C}_1$. As an example, consider the function $f(z) = e^{\frac{z_1^2 + \cdots + z_n^2}{2t}}$ from Remark 2.1.4. Since $z \mapsto W_z^t(f)$ is not continuous in F_t^∞, one can show that $z \mapsto \alpha_z(f \otimes 1)$ is not continuous in $\mathcal{N}(F_t^1)$. Analogously, $z \mapsto \alpha_z(1 \otimes f)$ is not continuous in $\mathcal{N}(F_t^\infty)$. While we will not talk about the $\mathcal{S}^{p_0}$ part of the theory on F_t^1 or F_t^∞, the $\mathcal{C}_1$ part can be rescued. Proving that $T_f^t \in \mathcal{C}_1$ for $f \in L^\infty(\mathbb{C}^n)$ cannot be done by imitating the proofs from above. Instead, this follows from duality: For $f \in L^\infty(\mathbb{C}^n)$ we know that $z \mapsto \alpha_z(T_{\overline{f}}^t)$ is continuous in $\mathcal{L}(f_t^\infty)$. Since the Banach space adjoint of $T_{\overline{f}}^t$ is $T_f^t \in \mathcal{L}(F_t^1)$, and the norms of F_t^1 and $(f_t^\infty)'$ are equivalent, we obtain by duality that $T_f^t \in \mathcal{C}_1$ over F_t^1. Using duality again, we obtain that $T_{\overline{f}}^t$ is in $\mathcal{C}_1$ over F_t^∞. The equality $f_t * A = T_{\widetilde{A}}^t$ for $A \in \mathcal{C}_1$ is still valid over F_t^1 and F_t^∞: While the proof using convolutions does not work anymore, both sides are still well-defined. Comparing their Berezin transforms shows that they indeed define the same operator. Using the same proof as for the cases above, we obtain:

Theorem 3.3.6. *For $A \in \mathcal{C}_1$ over F_t^1 or F_t^∞ we have for $N \to \infty$:*

$$\sum_{j=1}^{M_N} c_j^N \alpha_{\sqrt{t}z_j^N}(T_{\widetilde{A}}^t) \to A.$$

Consequently, the equalities

$$\mathcal{C}_1 = \mathcal{T}^{1,t} = \mathcal{T}_{lin}^{1,t}(\mathrm{BUC}(\mathbb{C}^n)),$$
$$\mathcal{C}_1 = \mathcal{T}^{\infty+,t} = \mathcal{T}_{lin}^{\infty+,t}(\mathrm{BUC}(\mathbb{C}^n))$$

also holds true.

Note the following: Since every Toeplitz operator on F_t^∞ leaves f_t^∞ invariant, we obtain that every operator in $\mathcal{C}_1$ over F_t^∞ leaves f_t^∞ invariant by the above theorem, which is not at all obvious.

To simplify notation, we will in the following refer to the case F_t^∞ by $p = \infty+$. To clarify this, here are two examples: If a statement is supposed to hold true for $1 < p \leq \infty$, then it holds true over F_t^p, $1 < p < \infty$, and over f_t^∞ as well. The validity of a statement over the range $1 < p \leq \infty+$ refers to all the spaces F_t^p, $1 < p < \infty$, f_t^∞ and F_t^∞. Analogously we will encounter the range $1 \leq p \leq \infty+$.

We can now also conclude a proof from Chapter 2:

Proof of Proposition 2.3.11. Recall that the equality which we want to prove is the following:

$$f_t^\infty = \{f \in F_t^\infty;\ z \mapsto W_z^t f \text{ is continuous in } F_t^\infty\}.$$

The inclusion "$\subseteq$" was already noted in Lemma 2.3.9. Hence, let $f \in F_t^\infty$ be such that $z \mapsto W_z^t f$ is continuous in F_t^∞-norm. Thus, for the rank one operator $1 \otimes f$ from $\mathcal{L}(F_t^\infty)$ we obtain

$$\begin{aligned}
\|\alpha_z(1 \otimes f) - (1 \otimes f)\|_{\mathcal{N}(F_t^\infty)} &= \|(k_z^t \otimes W_z^t(f)) - (1 \otimes f)\|_{\mathcal{N}(F_t^\infty)} \\
&\leq \|(k_z^t - 1) \otimes f\|_{\mathcal{N}(F_t^\infty)} + \|1 \otimes (f - W_z^t(f))\|_{\mathcal{N}(F_t^\infty)} \\
&\lesssim \|k_z^t - 1\|_{F_t^1} \|f\|_{F_t^\infty} + \|1\|_{F_t^1} \|f - W_z^t(f)\|_{F_t^\infty} \\
&\to 0
\end{aligned}$$

as $z \to 0$. In particular, $1 \otimes f \in \mathcal{C}_1 = \mathcal{T}^{\infty+,t}$. Therefore, by Proposition 2.3.3.3), $1 \otimes f$ leaves f_t^∞ invariant. Hence, $(1 \otimes f)(1) = f \in f_t^\infty$. □

The main correspondence result

The following result is now the essential part of Werner's Correspondence Theorem [130]. The result holds true for all $1 \leq p \leq \infty+$.

Theorem 3.3.7. *1) Let $\mathcal{D}_0 \subset \mathrm{BUC}(\mathbb{C}^n)$ be a closed and α-invariant subspace. Then, there is a unique closed and α-invariant subspace $\mathcal{D}_1 \subset \mathcal{C}_1$ such that the following holds true: For each $A \in \mathcal{C}_1$ we have*

$$A \in \mathcal{D}_1 \Longleftrightarrow \widetilde{A} \in \mathcal{D}_0.$$

2) Let $\mathcal{D}_1 \subset \mathcal{C}_1$ be a closed and α-invariant subspace. Then, there is a unique closed and α-invariant subspace $\mathcal{D}_0 \subset \mathrm{BUC}(\mathbb{C}^n)$ such that the following holds true: For each $f \in \mathrm{BUC}(\mathbb{C}^n)$ we have

$$f \in \mathcal{D}_0 \Longleftrightarrow T_f^t \in \mathcal{D}_1.$$

This correspondence of spaces is symmetric, i.e. if $\mathcal{D}_1$ is the unique closed and α-invariant subspace associated to $\mathcal{D}_0$, then $\mathcal{D}_0$ is the unique closed and α-invariant subspace associated to $\mathcal{D}_1$ and vice versa. Finally, this correspondence, which we will write by $\mathcal{D}_0 \longleftrightarrow \mathcal{D}_1$, is given by

$$\overline{\{\widetilde{A};\ A \in \mathcal{D}_1\}} = \mathcal{D}_0 \longleftrightarrow \mathcal{D}_1 = \mathcal{T}_{lin}^{p,t}(\mathcal{D}_0).$$

Proof. First observe that, granted the existence of such corresponding spaces, they need to be $\mathcal{T}_{lin}^{p,t}(\mathcal{D}_0)$ and $\overline{\{\widetilde{A};\ A \in \mathcal{D}_1\}}$. Indeed, let $\mathcal{D}_0$ be as assumed in the theorem and $\mathcal{D}_1$ the space corresponding to it. Since $\mathcal{D}_0$ is α-invariant and closed, we obtain $f_t * f = \widetilde{f}^{(t)} = \widetilde{T_f^t} \in \mathcal{D}_0$ for every $f \in \mathcal{D}_0$, and therefore we receive $\mathcal{T}_{lin}^{p,t}(\mathcal{D}_0) \subset \mathcal{D}_1$. Since $\mathcal{D}_1$ is α-invariant and closed, it is invariant under convolutions by $L^1(\mathbb{C}^n)$. In particular, $f_t * A = T_{\widetilde{A}}^t \in \mathcal{D}_1$ for $A \in \mathcal{D}_1$. Therefore, any $A \in \mathcal{D}_1$ can be approximated by elements in $\mathcal{T}_{lin}^{p,t}(\mathcal{D}_0)$ by Theorems 3.3.3 and 3.3.6. This means that $\mathcal{T}_{lin}^{p,t}(\mathcal{D}_0)$ is dense in $\mathcal{D}_1$. Since both spaces are closed, we obtain $\mathcal{T}_{lin}^{p,t}(\mathcal{D}_0) = \mathcal{D}_1$.

On the other hand, if we start with $\mathcal{D}_1$ and consider its associated space $\mathcal{D}_0$, analogous reasoning yields $\mathcal{D}_0 = \overline{\{\widetilde{A};\ A \in \mathcal{D}_1\}}$. This settles the uniqueness part. Now, it remains to show that these spaces actually satisfy the claimed properties.

Let us start with the first part, i.e. $\mathcal{D}_0 \subset \mathrm{BUC}(\mathbb{C}^n)$ is given. Let $A \in \mathcal{C}_1$. Assume $A \in \mathcal{T}_{lin}^{p,t}(\mathcal{D}_0)$. Then, we can approximate A in operator norm by Toeplitz operators $T_{g_k}^t$ with $g_k \in \mathcal{D}_0$. Since $\widetilde{T_{g_k}^t} = \widetilde{g_k}^{(t)} = f_t * g_k \in \mathcal{D}_0$ for all k, and $\widetilde{T_{g_k}^t} \to \widetilde{A}$ uniformly as $n \to \infty$, we obtain $\widetilde{A} \in \mathcal{D}_0$. On the other hand, if $\widetilde{A} \in \mathcal{D}_0$, then Theorem 3.3.3 or Theorem 3.3.6 prove that $A \in \mathcal{T}_{lin}^{p,t}(\mathcal{D}_0)$.

Now, assume $\mathcal{D}_1 \subset \mathcal{C}_1$ is given. Let $f \in \mathrm{BUC}(\mathbb{C}^n)$. If we assume $f \in \overline{\{\widetilde{A};\ A \in \mathcal{D}_1\}}$, let $A_k \in \mathcal{D}_1$ such that $\widetilde{A_k} \to f$ uniformly. Then, $T_{\widetilde{A_k}}^t \to T_f^t$ in operator norm. But $T_{\widetilde{A_k}}^t = f_t * A_k \in \mathcal{D}_1$, therefore $T_f^t \in \mathcal{D}_1$. Conversely, if we assume that $T_f^t \in \mathcal{D}_1$, then clearly $f \in \overline{\{\widetilde{A};\ A \in \mathcal{D}_1\}}$. □

In what follows, we will always denote by $\mathcal{D}_0,\ \mathcal{D}_1$ such a pair of corresponding spaces. For later reference, let us fix the following easy observations, which follow immediately from the Correspondence Theorem.

Lemma 3.3.8. *Let $\mathcal{D}_0 \longleftrightarrow \mathcal{D}_1$. Then the following statements hold true:*

1) $\mathcal{D}_0$ is U-invariant, i.e. $Uf \in \mathcal{D}_0$ for all $f \in \mathcal{D}_0$, if and only if $\mathcal{D}_1$ is U-invariant, i.e. $UAU \in \mathcal{D}_1$ for all $A \in \mathcal{D}_1$;

2) $1 \in \mathcal{D}_0$ if and only if $1 \in \mathcal{D}_1$. Here, 1 denotes the respective unit elements in the Banach algebras $\mathrm{BUC}(\mathbb{C}^n)$ and $\mathcal{C}_1$;

3) If $\mathcal{E}_0, \mathcal{E}_1$ is another pair of corresponding spaces in the sense of Theorem 3.3.7, then

$$\mathcal{D}_0 \subset \mathcal{E}_0 \Longleftrightarrow \mathcal{D}_1 \subset \mathcal{E}_1;$$

4) For $p = 2$: $\mathcal{D}_0$ is self-adjoint (i.e. $f^* \in \mathcal{D}_0$ for $f \in \mathcal{D}_0$) if and only if $\mathcal{D}_1$ is self-adjoint ($A^* \in \mathcal{D}_1$ for $A \in \mathcal{D}_1$).

We already know one example of corresponding spaces: By Corollary 3.3.4 and Theorem 3.3.6 we obtain

$$\mathrm{BUC}(\mathbb{C}^n) \longleftrightarrow \mathcal{C}_1 = \mathcal{T}^{p,t}$$

for all p and t. Here is another example:

Theorem 3.3.9. *For all p, t we have*

$$C_0(\mathbb{C}^n) \longleftrightarrow \mathcal{C}_1 \cap \mathcal{K}.$$

Proof. Let us first consider the case $1 \leq p \leq \infty$. If $f \in C_c(\mathbb{C}^n)$, then $f \in L^1(\mathbb{C}^n)$ and therefore $T_f^t \in \mathcal{N}$. In particular, $T_f^t \in \mathcal{K}$. Approximating an arbitrary $f \in C_0(\mathbb{C}^n)$ by compactly supported functions yields $T_f^t \in \mathcal{K}$ for $f \in C_0(\mathbb{C}^n)$. Since the adjoint of a compact operator is again compact, we obtain $T_f^t \in \mathcal{K}$ for all $1 \leq p \leq \infty+$ if $f \in C_0(\mathbb{C}^n)$.

Let $\mathcal{D}_0$ be the space corresponding to $\mathcal{K} \cap \mathcal{C}_1$. Then, we have by the above

$$C_0(\mathbb{C}^n) \longleftrightarrow \mathcal{T}_{lin}^{p,t}(C_0(\mathbb{C}^n)) \subset \mathcal{K} \cap \mathcal{C}_1 \longleftrightarrow \mathcal{D}_0$$

and therefore by Lemma 3.3.8 $C_0(\mathbb{C}^n) \subset \mathcal{D}_0$. In the cases $1 < p \leq \infty$, the normalized reproducing kernels k_z^t converge weakly to 0 as $|z| \to \infty$ (this follows from the inclusions in Proposition 2.1.6 and the dualities in Proposition 2.2.1). Therefore, for $A \in \mathcal{K} \cap \mathcal{C}_1$ we obtain in these cases (since every compact operator over a Banach space is completely continuous) that $\|Ak_z^t\| \to 0$ as $|z| \to \infty$ and hence

$$|\widetilde{A}(z)| = |\langle Ak_z^t, k_z^t \rangle_t| \lesssim \|Ak_z^t\| \to 0, \quad z \to \infty,$$

i.e. $\widetilde{A} \in C_0(\mathbb{C}^n)$. Over F_t^∞, $A \in \mathcal{K} \cap \mathcal{C}_1$ leaves f_t^∞ invariant and therefore the restriction $A|_{f_t^\infty}$ is in $\mathcal{K} \cap \mathcal{C}_1$ over f_t^∞ and has the same Berezin transform, hence $\widetilde{A} \in C_0(\mathbb{C}^n)$ in this case. For the last case, $A \in \mathcal{K} \cap \mathcal{C}_1$ over F_t^1, the dual A^* is in $\mathcal{K} \cap \mathcal{C}_1$ over F_t^∞ and for the Berezin transform of A^* is the complex conjugate of $\widetilde{A}$, which is contained in $C_0(\mathbb{C}^n)$.

This discussion shows that $\widetilde{A} \in C_0(\mathbb{C}^n)$ for $A \in \mathcal{K} \cap \mathcal{C}_1$ in all cases. Therefore, $\mathcal{D}_0 \subset C_0(\mathbb{C}^n)$. Hence, we have proven $\mathcal{D}_0 = C_0(\mathbb{C}^n)$, which shows the correspondence. □

The following corollary is the well-known characterization of compact operators over the Fock spaces, cf. [19, Theorem 1.1] for the first proof in the case $1 < p < \infty$. To the best of the author's knowledge, the limit cases f_t^∞, F_t^1 and F_t^∞ have not been dealt with before.

Corollary 3.3.10. *1) For $1 < p \leq \infty$ the following holds true for any $A \in \mathcal{L}$:*

$$A \text{ is compact } \Leftrightarrow A \in \mathcal{T}^{p,t} \text{ and } \widetilde{A} \in C_0(\mathbb{C}^n).$$

2) For $p = 1, \infty+$ the following holds true for $A \in \mathcal{C}_1$:

$$A \text{ is compact } \Leftrightarrow \widetilde{A} \in C_0(\mathbb{C}^n).$$

Proof. Follows immediately from the correspondence $C_0(\mathbb{C}^n) \longleftrightarrow \mathcal{K} \cap \mathcal{C}_1$. Observe that we have $\mathcal{K} \subset \mathcal{C}_1$ by Lemma 3.1.1 for the cases $1 < p \leq \infty$. ☐

Observe that we necessarily get a weaker statement for the cases of F_t^1 and F_t^∞. If we consider for $n = 1$ the rank one operator $A = e^{\frac{(\cdot)^2}{2t}} \otimes 1 \in \mathcal{L}(F_t^1)$, which is certainly compact, then A is not contained in $\mathcal{C}_1$. Indeed, its adjoint, considered as an element of $\mathcal{L}(F_t^\infty)$, is $A^* = 1 \otimes e^{\frac{(\cdot)^2}{2t}}$. Since $e^{\frac{(\cdot)^2}{2t}} \in F_t^\infty \setminus f_t^\infty$, A^* clearly does not leave f_t^∞ invariant, hence cannot be in $\mathcal{T}^{\infty+,t}$.

Over F_t^∞ it is seemingly difficult to verify whether an operator belongs to $\mathcal{C}_1$. Therefore, we present a different characterization of compactness in this space:

Corollary 3.3.11. *Let $A \in \mathcal{L}(F_t^\infty)$ be such that it leaves f_t^∞ invariant. Then, the following holds true:*

$$A \text{ is compact } \Leftrightarrow A \in \mathcal{T}^{\infty+,t} \text{ and } \widetilde{A} \in C_0(\mathbb{C}^n).$$

Proof. By identifying $(f_t^\infty)''$ with F_t^∞, we obtain $A = (A|_{f_t^\infty})^{**}$. Now the result follows easily from the compactness characterization over f_t^∞. ☐

Here is another important example of correspondences:

Lemma 3.3.12. *The following correspondence holds true:*

$$\mathrm{VO}_\partial(\mathbb{C}^n) \longleftrightarrow \mathrm{esscom}(\mathcal{C}_1, \mathcal{C}_1),$$

i.e. $\mathrm{esscom}(\mathcal{C}_1, \mathcal{C}_1) = \mathcal{T}_{lin}^{p,t}(\mathrm{VO}_\partial(\mathbb{C}^n))$.

Recall that $\mathrm{VO}_\partial(\mathbb{C}^n)$ denotes the functions of vanishing oscillation at infinity. It is not difficult to see that this is a closed and α-invariant subspace of $\mathrm{BUC}(\mathbb{C}^n)$ (and even a unital C^* subalgebra, as will become important later). Further, $\mathrm{esscom}(\mathcal{C}_1, \mathcal{C}_1)$ denotes the essential commutant of $\mathcal{C}_1$ in $\mathcal{C}_1$, i.e.

$$\mathrm{esscom}(\mathcal{C}_1, \mathcal{C}_1) := \{A \in \mathcal{C}_1;\ [A, B] \in \mathcal{K} \text{ for all } B \in \mathcal{C}_1\}.$$

We will later see that this agrees with the full essential commutant $\mathrm{esscom}(\mathcal{C}_1, \mathcal{L})$, i.e. with

$$\mathrm{esscom}(\mathcal{C}_1, \mathcal{L}) := \{A \in \mathcal{L};\ [A, B] \in \mathcal{K} \text{ for all } B \in \mathcal{C}_1\},$$

at least in the cases $1 < p \leq \infty$.

Proof of Lemma 3.3.12. It is elementary to verify that $\mathrm{VO}_\partial(\mathbb{C}^n)$ and $\mathrm{esscom}(\mathcal{C}_1, \mathcal{C}_1)$ are α-invariant and closed. Since Toeplitz operators with $\mathrm{BUC}(\mathbb{C}^n)$ symbols are dense in $\mathcal{C}_1$, we have

$$A \in \mathrm{esscom}(\mathcal{C}_1, \mathcal{C}_1) \Leftrightarrow [A, T_g^t] \in \mathcal{K} \quad \text{for all } g \in \mathrm{BUC}(\mathbb{C}^n).$$

Hence, we need to show that

$$\mathcal{T}_{lin}^{p,t}(\mathrm{VO}_\partial(\mathbb{C}^n)) = \{A \in \mathcal{C}_1;\ [A, T_g^t] \in \mathcal{K} \text{ for all } g \in \mathrm{BUC}(\mathbb{C}^n)\}.$$

We know that $\mathrm{esscom}(\mathcal{C}_1, \mathcal{C}_1) = \mathcal{T}_{lin}^{p,t}(\mathcal{D}_0)$ for some $\mathcal{D}_0 \subset \mathrm{BUC}(\mathbb{C}^n)$ α-invariant and closed, i.e.

$$\mathcal{D}_0 = \{f \in \mathrm{BUC}(\mathbb{C}^n);\ [T_f^t, T_g^t] \in \mathcal{K} \text{ for all } g \in \mathrm{BUC}(\mathbb{C}^n)\}.$$

Now observe two things: First, for $f \in \mathrm{BUC}(\mathbb{C}^n)$ we have $[T_f^t, T_g^t] \in \mathcal{K}$ for all $g \in \mathrm{BUC}(\mathbb{C}^n)$ if and only if $([T_f^t, T_g^t])^\sim \in C_0(\mathbb{C}^n)$ for all $g \in \mathrm{BUC}(\mathbb{C}^n)$ by Corollary 3.3.10. This condition is indeed independent of p, since the Berezin transform of the formally p-independent integral operator $[T_f^t, T_g^t]$ (i.e. its integral expression does not depend on p) does not depend on p. On the other hand, [13, Proposition 3.6] shows $\mathrm{VO}_\partial(\mathbb{C}^n) \subset \mathcal{D}_0$ for $p = 2$, hence for all p.

Finally, if $f \in \mathcal{D}_0$, we can conclude as in [30, Proof of Theorem D]: Since the operators W_z^t are Toeplitz operators with bounded symbols, the assumption implies that for any $z \in \mathbb{C}^n$ we have $[T_f^t, W_z^t] \in \mathcal{K}$ and therefore

$$W_z^t T_f^t W_{-z}^t - T_f^t = \alpha_z(T_f^t) - T_f^t \in \mathcal{K}$$

for all $z \in \mathbb{C}^n$. This gives

$$T_{\widetilde{f}^{(t)}}^t - T_f^t = \int_{\mathbb{C}^n} f_t(z)[\alpha_z(T_t^t) - T_f^t]\ dz \in \mathcal{K}, \tag{3.3}$$

which implies by Corollary 3.3.10

$$(\widetilde{f}^{(t)} - f)^{\sim(t)} \in C_0(\mathbb{C}^n).$$

Since $\widetilde{f}^{(t)} - f \in \mathrm{BUC}(\mathbb{C}^n)$, Proposition 3.3.5 yields $\widetilde{f}^{(t)} - f \in C_0(\mathbb{C}^n)$. Finally, this yields $f \in \mathrm{VO}_\partial(\mathbb{C}^n)$ by [30, Corollary to Theorem 5]. Therefore, $\mathcal{D}_0 \subset \mathrm{VO}_\partial(\mathbb{C}^n)$. □

Note that the upshot of Lemma 3.3.12 is not the correspondence itself, which is as expected, but rather the fact that it can be deduced for any p from the well-studied Hilbert space case. We will encounter this reasoning again later. We also have the following consequence of Equation (3.3), which is again well-known for the Hilbert space case:

Corollary 3.3.13. *The following holds true for any $1 \leq p \leq \infty+$:*

$$\mathrm{esscom}(\mathcal{C}_1, \mathcal{C}_1) = \mathcal{T}_{lin}^{p,t}(\mathrm{VO}_\partial(\mathbb{C}^n)) = \{T_f^t;\ f \in \mathrm{VO}_\partial(\mathbb{C}^n)\} + (\mathcal{K} \cap \mathcal{C}_1).$$

We will later see more examples of correspondences. Let us for the moment continue with the general theory.

Note that $C_0(\mathbb{C}^n)$ is an ideal in $\mathrm{BUC}(\mathbb{C}^n)$ and similarly $\mathcal{T}_{lin}^{p,t}(C_0(\mathbb{C}^n)) = \mathcal{K} \cap \mathcal{C}_1$ is an ideal in $\mathcal{T}_{lin}^{p,t}(\mathrm{BUC}(\mathbb{C}^n)) = \mathcal{C}_1$. We will later see that this is not just a coincidence. For the moment, we will focus on the quotients of those spaces. Recall that for $1 < p \leq \infty$ we have even $\mathcal{T}_{lin}^{p,t}(C_0(\mathbb{C}^n)) = \mathcal{K}$, which is now an ideal in both $\mathcal{T}^{p,t}$ and $\mathcal{L}$. Therefore, we can consider the Coburn algebra $\mathcal{T}^{p,t}/\mathcal{K}$ as a closed subalgebra of the Calkin algebra $\mathcal{L}/\mathcal{K}$. Recall that the natural norm on the Calkin algebra is:

$$\|A + \mathcal{K}\| = \|A\|_{ess} := \inf_{K \in \mathcal{K}} \|A + K\|_{op}.$$

Further, since the group action α_z leaves $\mathcal{K}$ invariant, it descends to a group action in $\mathcal{L}/\mathcal{K}$: $\alpha_z(A + \mathcal{K}) = (\alpha_z(A) + \mathcal{K})$. In particular, once $z \mapsto \alpha_z(A + \mathcal{K})$ is continuous for fixed $A + \mathcal{K} \in \mathcal{L}/\mathcal{K}$, we can define its convolution by $f \in L^1(\mathbb{C}^n)$ as the Bochner integral

$$f * (A + \mathcal{K}) := \int_{\mathbb{C}^n} f(z)\alpha_z(A + \mathcal{K})\, dz.$$

Theorem 3.3.14. *For $1 < p \leq \infty$ the following equalities hold true:*

$$\begin{aligned}\mathcal{T}^{p,t}/\mathcal{K} &= \{A + \mathcal{K} \in \mathcal{L}/\mathcal{K};\ z \mapsto \alpha_z(A + \mathcal{K}) \text{ is norm continuous}\}\\ &= \{A + \mathcal{K} \in \mathcal{L}/\mathcal{K};\ f_s * (A + \mathcal{K}) \to (A + \mathcal{K}) \text{ in norm as } s \to 0\}.\end{aligned}$$

Further, for $A + \mathcal{K} \in \mathcal{T}^{p,t}/\mathcal{K}$, the same approximation scheme as in Theorem 3.3.3 works:

$$\left\| (A + \mathcal{K}) - \left(\sum_{j=1}^{M_N} c_j^N \alpha_{\sqrt{t} z_j^N}(T_{\tilde{A}}^t) + \mathcal{K} \right) \right\| \to 0 \tag{3.4}$$

as $N \to \infty$, where the coefficients are as earlier.

Proof. For $(A + \mathcal{K}) \in \mathcal{T}^{p,t}/\mathcal{K}$ the continuity of $z \mapsto \alpha_z(A + \mathcal{K})$ follows immediately from the fact that $\mathcal{T}^{p,t} = \mathcal{C}_1$ and the trivial estimate

$$\|A + \mathcal{K}\| \leq \|A\|_{op}.$$

If $z \mapsto \alpha_z(A + \mathcal{K})$ is continuous, we obtain as earlier:

$$\|(A + \mathcal{K}) - f_t * (A + \mathcal{K})\| \leq \int_{\mathbb{C}^n} f_t(z) \|(A + \mathcal{K}) - \alpha_z(A + \mathcal{K})\|\, dz \to 0, \quad t \to 0.$$

If $f_t * (A + \mathcal{K}) \to (A + \mathcal{K})$ as $t \to 0$, we can prove the approximation scheme in Equation (3.4) as earlier using Wiener's Approximation Theorem, hence we get $A + \mathcal{K} \in \mathcal{T}^{p,t}/\mathcal{K}$. □

While we cannot naturally consider $\mathcal{T}^{p,t}/(\mathcal{K} \cap \mathcal{C}_1)$ as a subalgebra of the Calkin algebra $\mathcal{L}/\mathcal{K}$ in the cases $p = 1, \infty+$, α_z still descends to a strongly continuous group action in the quotient $\mathcal{T}^{p,t}/(\mathcal{K} \cap \mathcal{C}_1)$. Therefore, one can analogously prove that the approximation from Equation (3.4) works equally well in these cases.

We just notice as a byproduct of the approximation method (3.4) that we also get a Correspondence Theorem in the Coburn algebra, which is proven analogously to the first Correspondence Theorem:

Corollary 3.3.15. *For $1 \leq p \leq \infty+$ there is a 1:1 correspondence between closed, α-invariant subspaces $\mathcal{D}_0/C_0(\mathbb{C}^n)$ of $\mathrm{BUC}(\mathbb{C}^n)/C_0(\mathbb{C}^n)$ and closed, α-invariant subspaces $\mathcal{D}_1/(\mathcal{K} \cap \mathcal{C}_1)$ of $\mathcal{T}^{p,t}/(\mathcal{K} \cap \mathcal{C}_1)$:*

$$\begin{aligned}\overline{\{\widetilde{A} + C_0(\mathbb{C}^n);\ A + (\mathcal{K} \cap \mathcal{C}_1) \in \mathcal{D}_1/(\mathcal{K} \cap \mathcal{C}_1)\}} &= \mathcal{D}_0/C_0(\mathbb{C}^n) \\ &\longleftrightarrow \mathcal{D}_1/(\mathcal{K} \cap \mathcal{C}_1) = \mathcal{T}^{p,t}_{lin}(\mathcal{D}_0)/(\mathcal{K} \cap \mathcal{C}_1).\end{aligned}$$

The usual correspondence statements hold true: For $f + C_0(\mathbb{C}^n) \in \mathrm{BUC}(\mathbb{C}^n)/C_0(\mathbb{C}^n)$ and $A + (\mathcal{K} \cap \mathcal{C}_1) \in \mathcal{T}^{p,t}/(\mathcal{K} \cap \mathcal{C}_1)$ we have

$$\begin{aligned} f + C_0(\mathbb{C}^n) \in \mathcal{D}_0/C_0(\mathbb{C}^n) &\Longleftrightarrow T_f^t + (\mathcal{K} \cap \mathcal{C}_1) \in \mathcal{D}_1/(\mathcal{K} \cap \mathcal{C}_1), \\ A + (\mathcal{K} \cap \mathcal{C}_1) \in \mathcal{D}_1/(\mathcal{K} \cap \mathcal{C}_1) &\Longleftrightarrow \widetilde{A} + C_0(\mathbb{C}^n) \in \mathcal{D}_0/C_0(\mathbb{C}^n).\end{aligned}$$

Another corollary to Theorem 3.3.14 is the following:

Corollary 3.3.16. *For $1 < p \leq \infty$ we have*

$$\mathrm{esscom}(\mathcal{C}_1, \mathcal{C}_1) = \mathrm{esscom}(\mathcal{C}_1, \mathcal{L}) = \mathcal{T}^{p,t}(\mathrm{VO}_\partial(\mathbb{C}^n)).$$

Here,

$$\mathrm{esscom}(\mathcal{C}_1, \mathcal{L}) := \{A \in \mathcal{L};\ [A, B] \in \mathcal{K} \text{ for all } B \in \mathcal{C}_1\}.$$

Proof. The inclusion

$$\mathrm{esscom}(\mathcal{C}_1, \mathcal{C}_1) \subset \mathrm{esscom}(\mathcal{C}_1, \mathcal{L})$$

is obvious. Assume $A \in \mathcal{L}$ is such that $[A, B] \in \mathcal{K}$ for all $B \in \mathcal{C}_1$. This of course implies $A - \alpha_z(A) \in \mathcal{K}$ for all $z \in \mathbb{C}^n$. Therefore, $(A + \mathcal{K}) - (\alpha_z(A) + \mathcal{K}) = 0$, i.e $A + \mathcal{K} \in \mathcal{T}^{p,t}/\mathcal{K}$ by Theorem 3.3.14. In particular, there are $B \in \mathcal{T}^{p,t}$ and $K \in \mathcal{K}$ such that $A = B + K$, which proves $A \in \mathrm{esscom}(\mathcal{C}_1, \mathcal{C}_1)$. Thus

$$\mathrm{esscom}(\mathcal{C}_1, \mathcal{C}_1) \supset \mathrm{esscom}(\mathcal{C}_1, \mathcal{L})$$

as stated. □

Here is one more corollary of the theorem:

Corollary 3.3.17. *For $1 < p \leq \infty$, $\mathcal{T}^{p,t}/\mathcal{K}$ is closed under inversion in $\mathcal{L}/\mathcal{K}$, i.e. if $A \in \mathcal{T}^{p,t}$ is such that there exists $B \in \mathcal{L}$ with*

$$AB = I + K_1, \quad BA = I + K_2$$

for some $K_1, K_2 \in \mathcal{K}$, then $B \in \mathcal{T}^{p,t}$. In particular, if $A \in \mathcal{T}^{p,t}$ is invertible, then $A^{-1} \in \mathcal{T}^{p,t}$.

Proof. The statement follows from the characterization

$$\mathcal{T}^{p,t}/\mathcal{K} = \{A + K \in \mathcal{L}/\mathcal{K};\ \|(A+\mathcal{K}) - \alpha_z(A+\mathcal{K})\| \to 0,\ z \to 0\},$$

the fact that $(W_z^t)^{-1} = W_{-z}^t$ and a standard Neumann series argument, i.e. if $(A+\mathcal{K})$ is invertible in $\mathcal{L}/\mathcal{K}$, then

$$\begin{aligned}\|(A+\mathcal{K})^{-1} - (\alpha_z(A+\mathcal{K}))^{-1}\| &= \|(A+\mathcal{K})^{-1} - \alpha_z((A+\mathcal{K})^{-1})\| \\ &\leq \frac{\|(A+\mathcal{K})^{-1}\|\ \|(A+\mathcal{K}) - \alpha_z(A+\mathcal{K})\|}{1 - \|(A+\mathcal{K})^{-1}\|\|(A+\mathcal{K}) - \alpha_z(A+\mathcal{K})\|}\end{aligned}$$

for $|z|$ sufficiently small. □

The most important part of the previous corollary can be salvaged for the cases of F_t^1 and F_t^∞.

Corollary 3.3.18. *For $p = 1, \infty+$ we have the following: If $A \in \mathcal{T}^{p,t}$ is invertible, then $A^{-1} \in \mathcal{T}^{p,t}$.*

Proof. This is in principle a consequence of Corollary 3.3.4. In fact, the result follows from the same Neumann series argument as in the proof before, but now in $\mathcal{T}^{p,t}$ instead of the quotient. □

Corollary 3.3.17 implies that the Fredholm property of operators from the Toeplitz algebra depends only on invertibility in $\mathcal{T}^{p,t}/\mathcal{K}$. Even though $\mathcal{K}$ is not entirely contained in $\mathcal{T}^{p,t}$ for $p = 1,\ \infty+$, this particular statement carries over:

Lemma 3.3.19. *Let $1 \leq p \leq \infty+$ and $t > 0$. Then, $A \in \mathcal{T}^{p,t}$ is Fredholm if and only if there are operators $B \in \mathcal{T}^{p,t}$ and $K_1, K_2 \in \mathcal{K} \cap \mathcal{C}_1$ such that*

$$AB = I + K_1, \quad BA = I + K_2.$$

This means that $A \in \mathcal{T}^{p,t}$ is Fredholm if and only if $A + (\mathcal{K} \cap \mathcal{C}_1)$ is invertible in $\mathcal{T}^{p,t}/(\mathcal{K} \cap \mathcal{C}_1)$.

Proof. We only need to discuss the cases $p = 1, \infty+$. Further, the nontrivial part of the proof is showing that Fredholmness of A implies the existence of such operators B, K_1, K_2. Recall that every operator from $\mathcal{T}^{1,t}$ is the adjoint of some operator in $\mathcal{T}^{\infty,t}$ and every operator from $\mathcal{T}^{\infty+,t}$ is the adjoint of some operator from $\mathcal{T}^{1,t}$. Let $A \in \mathcal{T}^{1,t}$ be Fredholm. Then, there is some $A^0 \in \mathcal{T}^{\infty,t}$ such that $(A^0)^* = A$. Since an operator is Fredholm if and only if its Banach space adjoint is Fredholm, A^0 needs to be Fredholm. By Corollary 3.3.17, there are $B^0 \in \mathcal{T}^{\infty,t}$ and $K_1^0, K_2^0 \in \mathcal{K}(f_t^\infty)$ such that

$$A^0B^0 = I + K_1^0, \quad B^0A^0 = I + K_2^0.$$

Passing to the adjoints, we obtain $B = (B^0)^*,\ K_j = (K_j^0)^*$ with $B \in \mathcal{T}^{1,t},\ K_j \in \mathcal{K}(F_t^1) \cap \mathcal{C}_1$ such that

$$AB = I + K_1, \quad BA = I + K_2.$$

For $A \in \mathcal{T}^{\infty+,t}$ Fredholm the corresponding operators are similarly coming from the pre-adjoint operators. □

3.4 Remarks

As already mentioned, the core ideas of *Quantum Harmonic Analysis* originate from Reinhard Werner's paper [130], cf. also [102, 103] for a more detailed exposition and recent results on the subject. Based on this, the author established an analogous notion of Quantum Harmonic Analysis on the reflexive Fock spaces F_t^p ($p \in (1, \infty)$) in [72] and applied it to certain problems of Toeplitz operators and Toeplitz algebras. This chapter, together with the next one, is based on that paper. Compared to [72], we took several changes into account. First and foremost, we were able to include the non-reflexive endpoint cases in the present discussion. Further, the discussion of correspondences in the quotients with its implications was not contained in [72].

Chapter 4

Invariant C^* algebras

In this chapter, we will investigate how the notions of Correspondence Theory interact with algebras of Toeplitz operators. An important tool for doing this will be the so-called limit operators. Before we can introduce and investigate these objects, we need to return to the symbol spaces.

4.1 Invariant C^* subalgebras of $\mathrm{BUC}(\mathbb{C}^n)$ and their maximal ideal spaces

For the whole of this section, let $\mathcal{A}$ be a unital C^* subalgebra of $\mathrm{BUC}(\mathbb{C}^n)$ which is invariant with respect to the actions α and U, i.e. if $f \in \mathcal{A}$, then $Uf \in \mathcal{A}$ and $\alpha_z(f) \in \mathcal{A}$ for all $z \in \mathbb{C}^n$. We will denote by $\mathcal{M}(\mathcal{A})$ the maximal ideal space of $\mathcal{A}$, i.e. the space of all nontrivial multiplicative linear functionals. Since we assume $\mathcal{A}$ to be unital, $\mathcal{M}(\mathcal{A})$ is a compact Hausdorff space when endowed with the weak* topology. Further, we can identify each point $z \in \mathbb{C}^n$ with $\delta_z \in \mathcal{M}(\mathcal{A})$, the functional of point evaluation, and the set of all these functionals is always dense in $\mathcal{M}(\mathcal{A})$. In general, the assignment $z \mapsto \delta_z$ is not injective. It is injective if and only if $\mathcal{A}$ separates the points of $\mathbb{C}^n$, i.e. if for each pair of points $z, w \in \mathbb{C}^n$ there is a function $f \in \mathcal{A}$ such that $f(z) \neq f(w)$. Even if this is not the case, α-invariance gives some structure among the points which are not separated:

Lemma 4.1.1. *The set*

$$per(\mathcal{A}) := \{z \in \mathbb{C}^n;\ f(z) = f(0) \text{ for every } f \in \mathcal{A}\}$$

forms a closed subgroup of $\mathbb{C}^n$.

The proof of this lemma is obvious. We also have the following equivalences.

Lemma 4.1.2. *The following are equivalent:*

1) $per(\mathcal{A}) = \{0\}$*;*

2) $\mathcal{A}$ *separates the points of* $\mathbb{C}^n$*;*

3) $\mathcal{M}(\mathcal{A})$ *is a compactification of* $\mathbb{C}^n$*.*

The closed subgroups of $\mathbb{C}^n$ are readily characterized:

Proposition 4.1.3 ([38, Chapter VII])**.** *Let* G *be a closed subgroup of* $\mathbb{C}^n \cong \mathbb{R}^{2n}$*. Then, there exists a real vector space basis* $(a_j)_{j=1}^{2n}$ *of* $\mathbb{C}^n$ *and integers* $0 \leq p \leq r \leq 2n$ *such that*

$$G = \left\{ \sum_{j=1}^{p} t_j a_j + \sum_{j=p+1}^{r} n_j a_j;\ t_j \in \mathbb{R},\ n_j \in \mathbb{Z} \right\}.$$

If we now decompose $per(\mathcal{A})$ as above,

$$per(\mathcal{A}) = \operatorname{Span}\{a_j;\ j = 1, \ldots, p\} \oplus \operatorname{lat}\{a_j;\ j = p+1, \ldots, r\}$$

where lat denotes the lattice with integer coefficients generated by the vectors, then we obtain:

Lemma 4.1.4. *Let* $f \in \mathcal{A}$*. Then,* $f(x) = f(0)$ *for all* $x \in \operatorname{Span}\{a_j;\ j = 1, \ldots, p\}$ *and* f *is periodic with respect to* $\operatorname{lat}\{a_j;\ j = p+1, \ldots, r\}$*.*

Each function in $\mathcal{A}$ descends to a function in $\mathbb{R}^{2n}/per(\mathcal{A})$, which is isomorphic to $\mathbb{R}^{n-r} \times \mathbb{T}^{r-p}$ [38]. Let us denote by π the map

$$\pi : \mathbb{R}^{2n} \to \mathbb{R}^{n-r} \times \mathbb{T}^{r-p}.$$

Then, for each $f \in \mathcal{A}$ the pushforward of f under the quotient map π is a uniformly continuous function on $\mathbb{R}^{n-r} \times \mathbb{T}^{r-p}$. In particular, π induces a *-isomorphism φ from $\mathcal{A}$ onto a C^* subalgebra of BUC($\mathbb{R}^{n-r} \times \mathbb{T}^{r-p}$). It is now easy to check that $\varphi(A)$ separates the points of $\mathbb{R}^{n-r} \times \mathbb{T}^{r-p}$. In particular:

Lemma 4.1.5. $\mathcal{M}(\mathcal{A})$ *is a compactification of* $\mathbb{R}^{n-r} \times \mathbb{T}^{r-p}$*.*

While this fact is not important for the following discussions, it gives the right picture one should have in mind when thinking about $\mathcal{M}(\mathcal{A})$: While it might fail to give an actual compactification of $\mathbb{C}^n$, as the natural map $\mathbb{C}^n \to \mathcal{M}(\mathcal{A})$ might fail to be injective, it always gives rise to a compactification of an essentially unique quotient of $\mathbb{C}^n$ by a subgroup.

More importantly, using the α- and U-invariance of $\mathcal{A}$, we can continue the actions of α and U to $\mathcal{M}(\mathcal{A})$: For each $x \in \mathcal{M}(\mathcal{A})$ and $z \in \mathbb{C}^n$ define $\alpha_z(x)$ and Ux by

$$\alpha_z(x)(f) := x(\alpha_z(f)),\ Ux(f) := x(Uf).$$

It is readily verified that these are indeed multiplicative linear functionals. For $w \in \mathbb{C}^n$ we have $\alpha_z(\delta_w) = \delta_{w-z}$ and $U\delta_w = \delta_{-w}$, i.e. the actions reproduce how they should act on $\mathbb{C}^n$. Further, since $\mathbb{C}^n$ is invariant under these actions, it is not difficult to see that $\mathcal{M}(\mathcal{A}) \setminus \mathbb{C}^n$ is also invariant under them.

For $x \in \mathcal{M}(\mathcal{A})$ and $f \in \mathcal{A}$ let us now define the function $f_x : \mathbb{C}^n \to \mathbb{C}$ via

$$f_x(w) := \alpha_w(x)(Uf) = x(\alpha_w(Uf)).$$

In an abuse of notation, we will write for $z \in \mathbb{C}^n$

$$f_z = f_{\delta_z}.$$

This ambiguity does not cause any trouble. Indeed, one immediately sees that $f_z = \alpha_z(f)$.

Let $x \in \mathcal{M}(\mathcal{A})$ and $(z_\gamma)_{\gamma\in\Gamma} \subset \mathbb{C}^n$ be a net such that $\lim_{\gamma\in\Gamma} z_\gamma = x$ in $\mathcal{M}(\mathcal{A})$. Then, since we always consider $\mathcal{M}(\mathcal{A})$ with the weak* topology, we obtain

$$f_{z_\gamma}(w) = \delta_{z_\gamma}(\alpha_w(Uf)) \overset{\gamma\in\Gamma}{\to} x(\alpha_w(Uf)) = f_x(w),$$

i.e. f_x is the pointwise limit of the net of functions f_{z_γ}. We can say even more.

Lemma 4.1.6. *For each $f \in \mathcal{A}$ and $x \in \mathcal{M}(\mathcal{A})$, f_x is bounded and uniformly continuous. Furthermore, the map*

$$\mathcal{M}(\mathcal{A}) \ni x \mapsto f_x$$

is continuous with respect to the compact-open topology.

Proof. Boundedness of f_x follows from the uniform boundedness of the functions f_{z_γ}. For the uniform continuity, let $\varepsilon > 0$ and $\delta > 0$ such that $|w_1 - w_2| < \delta$ implies $|f(w_1) - f(w_2)| < \varepsilon$. Then, observe that

$$\begin{aligned} |f_x(w_1) - f_x(w_2)| &= \lim_{\gamma\in\Gamma} |f_{z_\gamma}(w_1) - f_{z_\gamma}(w_2)| \\ &= \lim_{\gamma\in\Gamma} |f(w_1 - z_\gamma) - f(w_2 - z_\gamma)| \le \varepsilon, \end{aligned}$$

as $|(w_1 - z_\gamma) - (w_2 - z_\gamma)| < \delta$. Now, observe that $(f_{z_\gamma})_\gamma$ is a uniformly equicontinuous net of bounded functions which converge pointwise to f_x. The Arzelà-Ascoli Theorem therefore implies that $(f_{z_\gamma})_\gamma$ converges uniformly on compact subsets of $\mathbb{C}^n$ to f_x. Finally, since f_x is independent of the precise net $(z_\gamma)_{\gamma\in\Gamma}$ (as long as it converges to x), $x \mapsto f_x$ is continuous from $\mathcal{M}(\mathcal{A})$ to $\mathrm{BUC}(\mathbb{C}^n)$ with respect to the compact-open topology, cf. [37, Theorem 1, page 81] and [19, Lemma 5.2]. □

Let us consider some examples:

Examples 4.1.7. 1) For $\mathcal{A} = C_0(\mathbb{C}^n) \oplus \mathbb{C}1$, where 1 denotes the function being constantly one, $\mathcal{M}(\mathcal{A})$ is just the one point compactification $\alpha\mathbb{C}^n = \mathbb{C}^n \cup \{\infty\}$ of $\mathbb{C}^n$. For $f = g + \lambda \in \mathcal{A}$ with $g \in C_0(\mathbb{C}^n)$ and $\lambda \in \mathbb{C}$, $f_\infty = \lambda$.

2) Let $\mathcal{A} = \mathrm{VO}_\partial(\mathbb{C}^n)$. Then for every $f \in \mathcal{A}$ and $x \in \mathcal{M}(\mathcal{A}) \setminus \mathbb{C}^n$, f_x is constant. To see this, it suffices to prove that f_x is constant on $B(z,1)$ for every $z \in \mathbb{C}^n$: Indeed, if $(z_\gamma)_{\gamma\in\Gamma} \subset \mathbb{C}^n$ converges to x, then we necessarily have $|z_\gamma| \to \infty$. Therefore, for any $w \in B(z,1)$:

$$\begin{aligned} |f_x(z) - f_x(w)| &= \lim_{\gamma\in\Gamma} |f_{z_\gamma}(z) - f_{z_\gamma}(w)| \\ &= \lim_{\gamma\in\Gamma} |f(z - z_\gamma) - f(w - z_\gamma)| \\ &= 0, \end{aligned}$$

as $|z - z_\gamma| \to \infty$ and $|(z - z_\gamma) - (w - z_\gamma)| < 1$. Let us note at this point that every $x \in \mathcal{M}(\mathrm{VO}_\partial(\mathbb{C}^n))$ can already be obtained as a limit of nets in $\mathbb{Z}^{2n} \subset \mathbb{C}^n$. The class $\mathrm{VO}_\partial(\mathbb{C}^n)$ equals

$$\{f \in C_b(\mathbb{C}^n); \sup_{w:\ |z-w|<R} |f(z) - f(w)| \to 0,\ |z| \to \infty\}$$

for any $R > 0$. In particular, let us choose $R = \sqrt{2n}$. Let $(z_\gamma)_{\gamma\in\Gamma} \subset \mathbb{C}^n$ be a net converging to $x \in \mathcal{M}(\mathcal{A})$. By the choice of R, for each γ there is some $\widetilde{z}_\gamma \in \mathbb{Z}^{2n}$ such that $|z_\gamma - \widetilde{z}_\gamma| < R$. We obtain for every $f \in \mathrm{VO}_\partial(\mathbb{C}^n)$:

$$x(f) = \lim_{\gamma\in\Gamma} f(z_\gamma) = \lim_{\gamma\in\Gamma} [f(\widetilde{z}_\gamma) - (f(\widetilde{z}_\gamma) - f(z_\gamma))] = \lim_{\gamma\in\Gamma} f(\widetilde{z}_\gamma),$$

since $f(\widetilde{z}_\gamma) - f(z_\gamma) \xrightarrow{\gamma\in\Gamma} 0$.

3) If $\mathcal{A} = \mathrm{BUC}(\mathbb{C}^n)$, then the corresponding compactification of $\mathcal{M}(\mathcal{A})$ of $\mathbb{C}^n$ is sometimes called the Samuel compactification of $\mathbb{C}^n$.

4) Consider AP, the set of almost periodic functions, i.e.

$$\mathrm{AP} := \mathrm{AP}(\mathbb{C}^n) := \overline{\mathrm{Span}}\{z \mapsto e^{i\,\mathrm{Im}(z\cdot\overline{w})};\ w \in \mathbb{C}^n\}.$$

It is not difficult to see that this is indeed an α- and U-invariant C^* subalgebra of $\mathrm{BUC}(\mathbb{C}^n)$ which separates the points of $\mathbb{C}^n$. As is well-known,

$$\mathrm{AP} = \{f \in \mathrm{BUC}(\mathbb{C}^n);\ \{f_z;\ z \in \mathbb{C}^n\} \text{ is totally bounded in the uniform metric}\}.$$

The corresponding compactification $\mathcal{M}(\mathrm{AP})$ is the Bohr compactification of $\mathbb{C}^n$. Since the orbit of $f \in \mathrm{AP}$ under the group action is totally bounded, it is not difficult to prove that $f_{z_\gamma} \to f_x$ uniformly for $z_\gamma \to x \in \mathcal{M}(\mathrm{AP})$. This property also uniquely determines AP: If $f \in \mathrm{BUC}(\mathbb{C}^n)$, then the continuity of $\mathcal{M}(\mathrm{BUC}(\mathbb{C}^n)) \ni x \mapsto f_x$ with respect to the uniform topology implies $f \in \mathrm{AP}$. For proofs of these statements see [70, Theorem 4.79].

Let us briefly discuss α- and U-invariant ideals in $\mathcal{A}$. There is a 1:1 correspondence between closed ideals $\mathcal{I}$ in $\mathcal{A} \cong C(\mathcal{M}(\mathcal{A}))$ and closed subsets I of $\mathcal{M}(\mathcal{A})$ via

$$\mathcal{I} = \{f \in C(\mathcal{M}(\mathcal{A}));\ f(x) = 0 \text{ for all } x \in I\},$$
$$I = \{x \in \mathcal{M}(\mathcal{A});\ f(x) = 0 \text{ for all } f \in \mathcal{I}\}.$$

We denote $\mathcal{I}_I$ for the ideal corresponding to the closed set $I \subset \mathcal{M}(\mathcal{A})$.

Lemma 4.1.8. *If $\mathcal{I}_I$ is α-invariant and $I \cap \mathbb{C}^n \neq \emptyset$, then $\mathcal{I}_I = \{0\}$.*

Proof. Assume $z \in I \cap \mathbb{C}^n$. Let $f \in \mathcal{I}_I$. In particular, $f(z) = 0$. Since $\mathcal{I}_I$ is α-invariant, $\alpha_w(f) \in \mathcal{I}_I$ for all $w \in \mathbb{C}^n$, i.e. $\alpha_w(f)(z) = f(z - w) = 0$ for all $z \in \mathbb{C}^n$. Therefore, f vanishes on all of $\mathbb{C}^n$, i.e. $f = 0$. □

This result shows that nontrivial invariant ideals "live" in the corona $\mathcal{M}(\mathcal{A}) \setminus \mathbb{C}^n$, i.e. $\mathcal{I}_I$ can be a non-trivial α-invariant ideal only if $I \subset \mathcal{M}(\mathcal{A}) \setminus \mathbb{C}^n$.

Lemma 4.1.9. *Let $I \subset \mathcal{M}(\mathcal{A}) \setminus \mathbb{C}^n$. Then, $\mathcal{I}_I$ is α-invariant if and only if I is α-invariant if and only if UI is α-invariant.*

Proof. The proof follows immediately from the definitions. Assume $\mathcal{I}_I$ is α-invariant. Then, for $x \in I$ we need to show $\alpha_z(x) \in I$ for every $z \in \mathbb{C}^n$. For each $f \in \mathcal{I}_I$ we have

$$f(\alpha_z(x)) = \alpha_z(f)(x) = 0,$$

since $\alpha_z(f) \in \mathcal{I}_I$ by translation invariance. Hence, $\alpha_z(x) \in I$. On the other hand, α-invariance of I similarly implies α-invariance of $\mathcal{I}_I$. Finally, equivalence of I being α-invariant to UI being α-invariance follows immediately from the formula

$$U(\alpha_z(x)) = \alpha_{-z}(U(x)), \quad x \in \mathcal{M}(\mathcal{A})$$

the verification of which is again just a matter of writing down the definitions. □

Recall that *maximal* ideals in $C(\mathcal{M}(\mathcal{A})) \cong \mathcal{A}$ correspond to single points in $\mathcal{M}(\mathcal{A})$. A consequence of the previous lemma is the fact that maximal α-invariant ideals in $C(\mathcal{M}(\mathcal{A})) \cong \mathcal{A}$ are those ideals $\mathcal{I}_I$ with $I \subset \mathcal{M}(\mathcal{A}) \setminus \mathbb{C}^n$ which are minimal with the property that I is closed and α-invariant. Such ideals always exist by an easy application of Zorn's Lemma, but it is not always possible to explicitly describe them.

Example 4.1.10. We consider again the example of $\mathcal{A} = \mathrm{VO}_\partial(\mathbb{C}^n)$. From the definition of $\mathrm{VO}_\partial(\mathbb{C}^n)$ it is easy to see that for each $x \in \mathcal{M}(\mathcal{A}) \setminus \mathbb{C}^n$ and each $z \in \mathbb{C}^n$, $f_x = f_{\alpha_z(x)}$ for all $f \in \mathcal{A}$. Therefore, the α-orbits in $\mathcal{M}(\mathcal{A}) \setminus \mathbb{C}^n$ consist of exactly one point in this case. Since $\mathcal{M}(\mathrm{VO}_\partial(\mathbb{C}^n))$ is Hausdorff, the singletons $\{x\}$ are always closed and therefore give rise to maximal α-invariant ideals.

4.2 Limit operators

Let us return to Toeplitz operators. As in [19], where only the case of $\mathcal{A} = \mathrm{BUC}(\mathbb{C}^n)$, $1 < p < \infty$ was considered, we have the following result:

Proposition 4.2.1. *Let $\mathcal{A}$ be an α- and U-invariant C^* subalgebra of* $\mathrm{BUC}(\mathbb{C}^n)$. *Then, for any $f \in \mathcal{A}$ and for any $1 \leq p < \infty$ the map*

$$\mathcal{M}(\mathcal{A}) \ni x \mapsto T_{f_x}^t$$

is continuous with respect to the strong operator topology on F_t^p and also on f_t^∞.

Proof. Recall that $x \mapsto f_x$ is continuous with respect to the compact-open topology by Lemma 4.1.6. Therefore, it suffices to prove the following: If $(g_\gamma)_{\gamma \in \Gamma}$ is a uniformly bounded net of uniformly continuous functions, converging to $g \in \mathrm{BUC}(\mathbb{C}^n)$ in the compact-open topology, then $T_{g_\gamma}^t \to T_g^t$ in strong operator topology. Since the g_γ are uniformly bounded (and therefore also the corresponding Toeplitz operators), it suffices to prove strong convergence on a dense subset. Since holomorphic polynomials are dense in the Fock space F_t^p for $1 \leq p < \infty$ and in f_t^p, we need to prove strong convergence only on such polynomials. Let $h \in \mathcal{P}[z_1, \dots, z_n]$. For $p = 1$ we have

$$\|T_{g_\gamma - g}^t h\|_{F_t^1} \leq \frac{1}{2^n(\pi t)^{2n}} \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} |g_\gamma(w) - g(w)||h(w)|e^{-\frac{|v-w|^2}{2t} - \frac{|w|^2}{2t}} \, dw \, dv.$$

Since h is a polynomial, we can find a constant $C > 0$ such that $|h(w)| \leq Ce^{\frac{|w|^2}{4t}}$ for all $w \in \mathbb{C}^n$. Then, applying Fubini's Theorem:

$$\begin{aligned} \|T_{g_\gamma - g}^t h\|_{F_t^1} &\leq C \frac{1}{2^n(\pi t)^{2n}} \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} |g_\gamma(w) - g(w)| \, e^{-\frac{|v-w|^2}{2t} - \frac{|w|^2}{4t}} \, dw \, dv \\ &= C \frac{1}{(\pi t)^n} \int_{\mathbb{C}^n} |g_\gamma(w) - g(w)| e^{-\frac{|w|^2}{4t}} \, dw. \end{aligned}$$

One may now be tempted to derive convergence to 0 by the Dominated Convergence Theorem. Yet, the Dominated Convergence Theorem is a purely sequential statement, i.e. it does not hold true for nets in general. Instead, one concludes the convergence by cutting off the area of integration to a sufficiently large compact subset of $\mathbb{C}^n$ and using uniform convergence of the net there.

For $p = \infty$, the argument works similarly:

$$\begin{aligned} \|T_{g_\gamma - g}^t h\|_{F_t^\infty} &\leq \frac{1}{(\pi t)^n} \sup_{v \in \mathbb{C}^n} \int_{\mathbb{C}^n} |g_\gamma(w) - g(w)| \, |h(w)| e^{-\frac{|v-w|^2}{2t} - \frac{|w|^2}{2t}} \, dw \\ &\leq \frac{C}{(\pi t)^n} \sup_{v \in \mathbb{C}^n} \int_{\mathbb{C}^n} |g_\gamma(w) - g(w)| e^{-\frac{|v-w|^2}{2t} - \frac{|w|^2}{4t}} \, dw \\ &\leq \frac{C}{(\pi t)^n} \int_{\mathbb{C}^n} |g_\gamma(w) - g(w)| e^{-\frac{|w|^2}{4t}} \, dw. \end{aligned}$$

Now, apply again uniform convergence on compact subsets. For $1 < p < \infty$ and h a polynomial, the result follows from Littlewood's inequality (Lemma 2.2.7). ☐

Recall that $T^t_{f_z} = T^t_{\alpha_z(f)} = \alpha_z(T^t_f)$ for $z \in \mathbb{C}^n$. A simple density argument shows the following:

Corollary 4.2.2. *Let* $1 \leq p \leq \infty$ *and let* $\mathcal{A}$ *be a unital* α*- and* U*-invariant* C^* *subalgebra of* $\mathrm{BUC}(\mathbb{C}^n)$*. Then, for every* $A \in \mathcal{T}^{p,t}_{lin}(\mathcal{A})$ *and every net* $(z_\gamma)_{\gamma\in\Gamma}$ *converging to* $x \in \mathcal{M}(\mathcal{A})$ *we have*

$$\alpha_{z_\gamma}(A) = W^t_{z_\gamma} A W^t_{-z_\gamma} \overset{\gamma\in\Gamma}{\longrightarrow} B \tag{4.1}$$

in strong operator topology for a unique operator $B \in \mathcal{T}^{p,t}$*. This means that the map*

$$\mathbb{C}^n \ni z \mapsto \alpha_z(A)$$

extends to a continuous map from $\mathcal{M}(\mathcal{A})$ *to* $\mathcal{T}^{p,t}$ *with respect to the strong operator topology.*

Definition 4.2.3. Let $1 \leq p \leq \infty$, $\mathcal{A}$ a unital α- and U-invariant C^* subalgebra of $\mathrm{BUC}(\mathbb{C}^n)$ and $A \in \mathcal{T}^{p,t}_{lin}(\mathcal{A})$. For $x \in \mathcal{M}(\mathcal{A})$ we denote by $A_x \in \mathcal{T}^{p,t}$ the unique operator determined by Equation (4.1). If $x \in \mathcal{M}(\mathcal{A}) \setminus \mathbb{C}^n$ we say that A_x is a *limit operator* of A.

Observe the following simple, but very important fact:

Lemma 4.2.4. *Under the assumptions of Corollary 4.2.2, we have*

$$\big(\widetilde{A}\big)_x(w) = (A_x)^\sim(w)$$

for every $w \in \mathbb{C}^n$*.*

Proof. The statement follows immediately from the definitions:

$$\begin{aligned}(A_x)^\sim(w) &= \langle \lim_{z_\gamma\to x} W^t_{z_\gamma} A W^t_{-z_\gamma} k^t_w, k^t_w\rangle_t \\ &= \lim_{z_\gamma\to x} \langle A k^t_{w-z_\gamma}, k^t_{w-z_\gamma}\rangle_t \\ &= \big(\widetilde{A}\big)_x(w).\end{aligned}$$

□

Let us note the following consequence of Corollary 3.3.10 and the previous Lemma:

Lemma 4.2.5. *Let* $A \in \mathcal{T}^{p,t}$*. Then,* A *is compact if and only if* $A_x = 0$ *for all* $x \in \mathcal{M}(\mathrm{BUC}(\mathbb{C}^n)) \setminus \mathbb{C}^n$*.*

Proof. If A is compact, then $\widetilde{A} \in C_0(\mathbb{C}^n)$, i.e. $\lim_{z_\gamma\to x} \alpha_{z_\gamma}(\widetilde{A})(w) = (A_x)^\sim(w) = 0$ for all $w \in \mathbb{C}^n$. Since the Berezin transform is injective, this implies $A_x = 0$ for all $x \in \mathcal{M}(\mathrm{BUC}(\mathbb{C}^n)) \setminus \mathbb{C}^n$. On the other hand, if $A_x = 0$ for all such x, then $x(\widetilde{A}) = (U\widetilde{A})_x(0) = (UA_xU)^\sim(0) = 0$, i.e. $\widetilde{A}(x) = 0$ for all $x \in \mathcal{M}(\mathrm{BUC}(\mathbb{C}^n)) \setminus \mathbb{C}^n$, i.e. $\widetilde{A}$ vanishes at infinity. □

4.3 Invariant Toeplitz algebras

As we have already seen earlier, we have the two equalities

$$\mathcal{T}_{lin}^{p,t}(C_0(\mathbb{C}^n)) = \mathcal{T}^{p,t}(C_0(\mathbb{C}^n)),$$
$$\mathcal{T}_{lin}^{p,t}(\mathrm{BUC}(\mathbb{C}^n)) = \mathcal{T}^{p,t}(\mathrm{BUC}(\mathbb{C}^n)).$$

It seems interesting to investigate the following question: Determine those α-invariant and closed subspaces $\mathcal{D}_0$ of $\mathrm{BUC}(\mathbb{C}^n)$ for which the following holds true:

$$\mathcal{T}_{lin}^{p,t}(\mathcal{D}_0) = \mathcal{T}^{p,t}(\mathcal{D}_0).$$

Note the following: Given any subspace $\mathcal{D}_0 \subset \mathrm{BUC}(\mathbb{C}^n)$, α-invariant and closed, $\mathcal{T}^{p,t}(\mathcal{D}_0)$ is always α-invariant and, by definition, closed. By Theorem 3.3.7 there always exists $\mathcal{D}_0' \subset \mathrm{BUC}(\mathbb{C}^n)$ such that

$$\mathcal{T}^{p,t}(\mathcal{D}_0) = \mathcal{T}_{lin}^{p,t}(\mathcal{D}_0').$$

Therefore, the question is not about writing a Toeplitz algebra as the closure of some set of Toeplitz operators, but about determining whether it is just the closure of the set of its generators.

Observe the following: If $\mathcal{D}_0 \subset \mathrm{BUC}(\mathbb{C}^n)$ is closed and α-invariant and so is $\mathcal{D}_0'$ with

$$\mathcal{T}^{p,t}(\mathcal{D}_0) = \mathcal{T}_{lin}^{p,t}(\mathcal{D}_0'),$$

then we know that $\mathcal{T}_{lin}^{p,t}(\mathcal{D}_0')$ is a Banach algebra. In particular, we obtain

$$\mathcal{T}_{lin}^{p,t}(\mathcal{D}_0') = \mathcal{T}^{p,t}(\mathcal{D}_0')$$

in this situation and therefore

$$\mathcal{T}^{p,t}(\mathcal{D}_0) = \mathcal{T}^{p,t}(\mathcal{D}_0').$$

While the relation

$$\mathcal{T}_{lin}^{p,t}(\mathcal{D}_0) = \mathcal{T}_{lin}^{p,t}(\mathcal{D}_0')$$

would clearly imply $\mathcal{D}_0 = \mathcal{D}_0'$ by the Correspondence Theorem, it is not clear if

$$\mathcal{T}^{p,t}(\mathcal{D}_0) = \mathcal{T}^{p,t}(\mathcal{D}_0')$$

implies any relations between $\mathcal{D}_0$ and $\mathcal{D}_0'$ (e.g. it could very well be that this implies $\mathrm{Alg}(\mathcal{D}_0) = \mathrm{Alg}(\mathcal{D}_0')$). We will later obtain a weak statement relating $\mathcal{D}_0$ and $\mathcal{D}_0'$ if $\mathcal{T}^{p,t}(\mathcal{D}_0) = \mathcal{T}_{lin}^{p,t}(\mathcal{D}_0')$.

Let us note the following simple but important observations:

Lemma 4.3.1. *Let $\mathcal{D}_0 \subset \mathrm{BUC}(\mathbb{C}^n)$ be an α-invariant and closed subspace. Then, the following are equivalent:*

1) $\mathcal{T}^{p,t}_{lin}(\mathcal{D}_0) = \mathcal{T}^{p,t}(\mathcal{D}_0)$*;*

2) $(T^t_{f_1} T^t_{f_2} \dots T^t_{f_k})^\sim \in \mathcal{D}_0$ *for all* $f_1, \dots, f_k \in \mathcal{D}_0$*.*

Furthermore, if we assume 1) and $\mathcal{I}_0 \subset \mathcal{D}_0$ is a closed and α-invariant subspace, we have the following equivalences of statements:

$$1l^*) \iff 2l^*), \quad 1r^*) \iff 2r^*), \quad 1^*) \iff 2^*)$$

The above statements are the following:

1l)* $\mathcal{T}^{p,t}_{lin}(\mathcal{I}_0)$ *is a left ideal in* $\mathcal{T}^{p,t}_{lin}(\mathcal{D}_0)$*;*

2l)* $\mathcal{T}^{p,t}_{lin}(\mathcal{I}_0) = \mathcal{T}^{p,t}(\mathcal{I}_0)$ *and* $(T^t_f T^t_g)^\sim \in \mathcal{I}_0$ *for* $f \in \mathcal{D}_0$, $g \in \mathcal{I}_0$*;*

1r)* $\mathcal{T}^{p,t}_{lin}(\mathcal{I}_0)$ *is a right ideal in* $\mathcal{T}^{p,t}_{lin}(\mathcal{D}_0)$*;*

2r)* $\mathcal{T}^{p,t}_{lin}(\mathcal{I}_0) = \mathcal{T}^{p,t}(\mathcal{I}_0)$ *and* $(T^t_g T^t_f)^\sim \in \mathcal{I}_0$ *for* $f \in \mathcal{D}_0$, $g \in \mathcal{I}_0$*;*

1)* $\mathcal{T}^{p,t}_{lin}(\mathcal{I}_0)$ *is a two-sided ideal in* $\mathcal{T}^{p,t}_{lin}(\mathcal{D}_0)$*;*

2)* $\mathcal{T}^{p,t}_{lin}(\mathcal{I}_0) = \mathcal{T}^{p,t}(\mathcal{I}_0)$ *and* $(T^t_f T^t_g)^\sim, (T^t_g T^t_f)^\sim \in \mathcal{I}_0$ *for* $f \in \mathcal{D}_0$, $g \in \mathcal{I}_0$*.*

Proof. All the statements follow similarly from the Correspondence Theorem 3.3.7. We prove 1) ⇔ 2) as an example.

Of course, 1) is equivalent to $T^t_{f_1} \dots T^t_{f_k} \in \mathcal{T}^{p,t}_{lin}(\mathcal{D}_0)$ for all $f_1, \dots, f_k \in \mathcal{D}_0$. By the Correspondence Theorem 3.3.7, this is equivalent to 2). □

While the condition 2), $2l^*), 2r^*)$ and $2^*)$ in the above lemma are in general not easy to verify, they have the following important consequence: The expression $(T^t_{f_1} T^t_{f_2} \dots T^t_{f_k})^\sim$, being some integral transform of $f_1, \dots, f_k$, does not depend on p, since neither the integral operators $T^t_{f_1}, \dots, T^t_{f_k}$ nor the Berezin transform depend on p. Hence, we obtain:

Lemma 4.3.2. *All the statements in Lemma 4.3.1 are independent of p, i.e. if they hold true for one $1 \leq p \leq \infty+$, then they hold true for all such p.*

We have the following positive result about the main question of this section:

Theorem 4.3.3. *Let $\mathcal{A} \subset \mathrm{BUC}(\mathbb{C}^n)$ be an α- and U-invariant C^*-subalgebra of $\mathrm{BUC}(\mathbb{C}^n)$. Further, let $\mathcal{I} \subset \mathcal{A}$ be an α-invariant ideal of $\mathcal{A}$. Then, the following hold true for all $1 \leq p \leq \infty+$ and all $t > 0$:*

1) $\mathcal{T}^{p,t}_{lin}(\mathcal{A}) = \mathcal{T}^{p,t}(\mathcal{A})$*. In particular,* $\mathcal{T}^{2,t}_{lin}(\mathcal{A})$ *is a* C^* *algebra;*

2) $\mathcal{T}^{p,t}_{lin}(\mathcal{I})$ *is a two-sided ideal in* $\mathcal{T}^{p,t}_{lin}(\mathcal{A})$*.*

Note that we do not need U-invariance of the ideal $\mathcal{I}$ in the above theorem. For proving the result, we will use the following well-known fact:

Lemma 4.3.4. *Let X be a Banach space and let $S \subset \mathcal{L}(X)$ be bounded with respect to the uniform topology. Then, the multiplication*

$$S \times \mathcal{L}(X) \to \mathcal{L}(X), \quad (A, B) \mapsto AB$$

is continuous with respect to the strong operator topology.

Proof of Theorem 4.3.3. Step 1: Assume $\mathcal{A}$ is unital. Let $A, B \in \mathcal{T}_{lin}^{p,t}(\mathcal{A})$. Since A_x is norm-bounded by $\|A\|$ and B_x is norm-bounded by $\|B\|$ for every $x \in \mathcal{M}(\mathcal{A})$, Lemma 4.3.4 implies that

$$z \mapsto \alpha_z(AB) = \alpha_z(A)\alpha_z(B),$$
$$z \mapsto \alpha_z(BA) = \alpha_z(B)\alpha_z(A),$$

both extend to continuous functions from $\mathcal{M}(\mathcal{A})$ to $\mathcal{T}^{p,t}$ with respect to the strong operator topology. For the Berezin transforms, this means that

$$U(\widetilde{AB})(z) = \widetilde{AB}(-z) = \langle \alpha_z(AB)1, 1\rangle_t,$$
$$U(\widetilde{BA})(z) = \widetilde{BA}(-z) = \langle \alpha_z(BA)1, 1\rangle_t$$

extend to continuous functions from $\mathcal{M}(\mathcal{A})$ to $\mathbb{C}$, i.e. $U(\widetilde{AB})$, $U(\widetilde{BA}) \in \mathcal{A}$. Since $\mathcal{A}$ is U-invariant, we also get $\widetilde{AB}$, $\widetilde{BA} \in \mathcal{A}$. The Correspondence Theorem 3.3.7 now yields AB, $BA \in \mathcal{T}_{lin}^{p,t}(\mathcal{A})$. Hence, $\mathcal{T}_{lin}^{p,t}(\mathcal{A})$ is closed under taking products, i.e. a Banach algebra. This yields $\mathcal{T}_{lin}^{p,t}(\mathcal{A}) = \mathcal{T}^{p,t}(\mathcal{A})$.

Step 2: Let $\mathcal{A}$ be unital and $\mathcal{I}$ a nontrivial, closed and α-invariant ideal in $\mathcal{A}$. Then $\mathcal{I} = \mathcal{I}_I$ for some $I \subset \mathcal{M}(\mathcal{A}) \setminus \mathbb{C}^n$ which is α-invariant by Lemma 4.1.9. Let $f \in \mathcal{I}$. It follows from the definitions that

$$f_x(w) = x(\alpha_w(Uf)) = 0$$

for every $x \in UI, w \in \mathbb{C}^n$. In particular, this implies $A_x = 0$ for every $A \in \mathcal{T}_{lin}^{p,t}(\mathcal{I})$ and $x \in UI$. If we now fix $A \in \mathcal{T}_{lin}^{p,t}(\mathcal{I})$ and $B \in \mathcal{T}_{lin}^{p,t}(\mathcal{A})$, then we obtain

$$(AB)_x = A_x B_x = 0 = B_x A_x = (BA)_x$$

for every $x \in UI$. Therefore,

$$U(\widetilde{AB})(x) = \widetilde{AB}(Ux) = 0 = \widetilde{BA}(Ux) = U(\widetilde{BA})(x)$$

for every $x \in UI$, hence

$$\widetilde{AB}(y) = 0 = \widetilde{BA}(y)$$

for every $y \in I$. This yields $\widetilde{AB}, \widetilde{BA} \in \mathcal{I}_I$ and thus, by the Correspondence Theorem, $AB, BA \in \mathcal{T}_{lin}^{p,t}(\mathcal{I})$.

Step 3: The general case. If $\mathcal{A}$ is not unital, then $\mathcal{A}$ is an ideal in $\mathcal{A} \oplus \mathbb{C}1 \subset \mathrm{BUC}(\mathbb{C}^n)$. Hence, Step 2 implies $\mathcal{T}_{lin}^{p,t}(\mathcal{A}) = \mathcal{T}^{p,t}(\mathcal{A})$. If further $\mathcal{I}$ is an α-invariant ideal in $\mathcal{A}$, then it is also an ideal in $\mathcal{A} \oplus \mathbb{C}1$, hence $\mathcal{T}_{lin}^{p,t}(\mathcal{I})$ is an ideal in $\mathcal{T}_{lin}^{p,t}(\mathcal{A} \oplus \mathbb{C}1)$ and therefore also in $\mathcal{T}_{lin}^{p,t}(\mathcal{A})$. □

The above theorem provides a sufficient condition for $\mathcal{T}_{lin}^{p,t}(\mathcal{D}_0) = \mathcal{T}^{p,t}(\mathcal{D}_0)$. There is also a necessary condition:

Theorem 4.3.5. *Let $\mathcal{D}_0 \subset \mathrm{BUC}(\mathbb{C}^n)$ be closed and α-invariant. Assume that for all $t > 0$ we have*

$$\mathcal{T}_{lin}^{p,t}(\mathcal{D}_0) = \mathcal{T}^{p,t}(\mathcal{D}_0).$$

Then, $\mathcal{D}_0$ is a Banach algebra. If $\mathcal{I}_0 \subset \mathcal{D}_0$ is closed and α-invariant such that $\mathcal{T}_{lin}^{p,t}(\mathcal{I}_0)$ is a left or right ideal in $\mathcal{T}_{lin}^{p,t}(\mathcal{D}_0)$ for all $t > 0$, then $\mathcal{I}_0$ is an ideal in $\mathcal{D}_0$.

Proof. By Lemma 4.3.2 we may assume $p = 2$. We only prove that $\mathcal{D}_0$ is a Banach algebra. The other statements follow from the same proof with the obvious changes. Let $f, g \in \mathcal{D}_0$. By assumption, $T_f^t T_g^t \in \mathcal{T}_{lin}^{p,t}(\mathcal{D}_0)$ and therefore $\widetilde{T_f^t T_g^t} \in \mathcal{D}_0$ for all $t > 0$. Recall the following facts:

1) $\|T_f^t T_g^t - T_{fg}^t\| \to 0$ as $t \to 0$ (Theoren 2.3.7),

2) $\|\widetilde{fg}^{(t)} - fg\|_\infty \to 0$ as $t \to 0$ (Theorem 2.3.6).

1) immediately implies $\|\widetilde{T_f^t T_g^t} - \widetilde{fg}^{(t)}\|_\infty \to 0$. Hence:

$$\|\widetilde{T_f^t T_g^t} - fg\|_\infty \leq \|\widetilde{T_f^t T_g^t} - \widetilde{T_{fg}^t}\|_\infty + \|\widetilde{fg}^{(t)} - fg\|_\infty \to 0 \text{ as } t \to 0.$$

Since $\mathcal{D}_0$ is closed, this yields $fg \in \mathcal{D}_0$. □

Combining all the above results, we obtain the following:

Theorem 4.3.6. *Let $\mathcal{D}_0 \subset \mathrm{BUC}(\mathbb{C}^n)$ be closed, α- and U-invariant and self-adjoint. Then, the following are equivalent:*

1) $\mathcal{D}_0$ is a C^* algebra;

2) $\mathcal{T}_{lin}^{p,t}(\mathcal{D}_0) = \mathcal{T}^{p,t}(\mathcal{D}_0)$ for all $1 \leq p \leq \infty+$ and $t > 0$;

3) $\mathcal{T}_{lin}^{2,t}(\mathcal{D}_0)$ is a C^* algebra for every $t > 0$.

If we assume that $\mathcal{D}_0$ satisfies the above conditions and $\mathcal{I}_0 \subset \mathcal{D}_0$ is a closed and α-invariant subspace, then the following are equivalent:

1) $\mathcal{I}_0$ is an ideal in $\mathcal{D}_0$;*

2) $\mathcal{T}_{lin}^{p,t}(\mathcal{I}_0)$ is a left ideal in $\mathcal{T}_{lin}^{p,t}(\mathcal{D}_0)$ for every $1 \leq p \leq \infty+$ and $t > 0$;*

3)* $\mathcal{T}_{lin}^{2,t}(\mathcal{I}_0)$ *is a left ideal in* $\mathcal{T}_{lin}^{2,t}(\mathcal{D}_0)$ *for every* $t > 0$*;*

4)* $\mathcal{T}_{lin}^{p,t}(\mathcal{I}_0)$ *is a right ideal in* $\mathcal{T}_{lin}^{p,t}(\mathcal{D}_0)$ *for every* $1 \leq p \leq \infty+$ *and* $t > 0$*;*

5)* $\mathcal{T}_{lin}^{2,t}(\mathcal{I}_0)$ *is a right ideal in* $\mathcal{T}_{lin}^{2,t}(\mathcal{D}_0)$ *for every* $t > 0$*;*

6)* $\mathcal{T}_{lin}^{p,t}(\mathcal{I}_0)$ *is a two-sided ideal in* $\mathcal{T}_{lin}^{p,t}(\mathcal{D}_0)$ *for every* $1 \leq p \leq \infty+$ *and* $t > 0$*;*

7)* $\mathcal{T}_{lin}^{2,t}(\mathcal{I}_0)$ *is a two-sided ideal in* $\mathcal{T}_{lin}^{2,t}(\mathcal{D}_0)$ *for every* $t > 0$*.*

Let us mention a few additional examples of correspondences between subspaces of $\mathrm{BUC}(\mathbb{C}^n)$ and $\mathcal{T}^{p,t}(\mathbb{C}^n)$.

Examples 4.3.7. 1) As we have already discussed in Example 4.1.7 4), the almost periodic functions

$$\mathrm{AP}(\mathbb{C}^n) = \overline{\mathrm{span}}\{w \mapsto e^{i\,\mathrm{Im}(w\cdot\overline{z})};\ z \in \mathbb{C}^n\}$$

are characterizied by the property that $\mathcal{M}(\mathrm{BUC}(\mathbb{C}^n)) \ni x \mapsto f_x$ is uniformly continuous. It follows now easily by the Correspondence Theorem that the corresponding Toeplitz algebra is

$$\mathcal{T}_{lin}^{p,t}(\mathrm{AP}(\mathbb{C}^n)) = \{A \in \mathcal{T}^{p,t};\ \mathcal{M}(\mathrm{BUC}(\mathbb{C}^n)) \ni x \mapsto A_x \text{ is } \|\cdot\|_{op}\text{-cont}\}.$$

As we have already mentioned earlier, the Weyl operator W_z^t is just the Toeplitz operator $T_{g_z^t}^t$ with symbol

$$g_z^t(w) = e^{\frac{|z|^2}{2t} + \frac{2i\,\mathrm{Im}(w\cdot\overline{z})}{t}}.$$

In particular,

$$\mathcal{T}_{lin}^{p,t}(\mathrm{AP}(\mathbb{C}^n)) = \mathcal{T}^{p,t}(\mathrm{AP}(\mathbb{C}^n)) = \overline{Alg}\{W_z^t;\ z \in \mathbb{C}^n\}.$$

The generators satisfy

$$W_z^t W_w^t = e^{-\frac{i\,\mathrm{Im}(z\cdot\overline{w})}{t}} W_{z+w}^t,$$

i.e. $\mathcal{T}_{lin}^{2,t}(\mathrm{AP}(\mathbb{C}^n))$ is just the CCR algebra for the symplectic space $(\mathbb{C}^n, \sigma_t)$, where $\sigma_t(z,w) = \frac{\mathrm{Im}(z\cdot\overline{w})}{t}$. This has already been studied in [28, 49].

2) Let us denote by $\mathrm{AA}(\mathbb{C}^n)$ the class of functions

$$\mathrm{AA}(\mathbb{C}^n) := \{f \in \mathrm{BUC}(\mathbb{C}^n);\ \forall x \in \mathcal{M}(\mathrm{BUC}(\mathbb{C}^n)):\ (f_x)_{Ux} = f\}.$$

Here, AA stands for *almost automorphism.* Almost automorphisms are a well-studied class of continuous functions on topological groups [34, Chapter 7]. While an almost automorphism is not necessarily uniformly continuous, we consider only those almost automorphisms from $\mathrm{BUC}(\mathbb{C}^n)$. $\mathrm{AA}(\mathbb{C}^n)$ strictly contains $\mathrm{AP}(\mathbb{C}^n)$. In some sense, $\mathrm{AA}(\mathbb{C}^n)$ is the opposite of $C_0(\mathbb{C}^n)$: For $f \in C_0(\mathbb{C}^n)$ we have $f_x \equiv 0$ for every $x \in \mathcal{M}(\mathrm{BUC}(\mathbb{C}^n))$, i.e. upon passing to the limit functions, all information is lost. On the other hand, for almost automorphisms every limit function contains all the information about the initial function (since the initial function can be recovered by shifting backwards). Indeed, $\mathrm{AA}(\mathbb{C}^n)$ is a unital α- and U-invariant C^* subalgebra of $\mathrm{BUC}(\mathbb{C}^n)$. It is not difficult to see that the corresponding Toeplitz algebra is

$$\mathcal{T}_{lin}^{p,t}(\mathrm{AA}(\mathbb{C}^n)) = \{A \in \mathcal{T}^{p,t};\ \forall x \in \mathcal{M}(\mathrm{BUC}(\mathbb{C}^n)) :\ (A_x)_{Ux} = A\}.$$

In Chapter 5 we will see that (for $p \in (1, \infty)$) $A \in \mathcal{T}^{p,t}$ is Fredholm if and only if A_x is invertible for every $x \in \mathcal{M}(\mathrm{BUC}(\mathbb{C}^n)) \setminus \mathbb{C}^n$. For $A \in \mathcal{T}_{lin}^{p,t}(\mathrm{AA}(\mathbb{C}^n))$ we therefore obtain

$$A \text{ is Fredholm } \Longleftrightarrow A_x \text{ is invertible } \Longrightarrow (A_x)_{Ux} = A \text{ is invertible.}$$

Therefore, in $\mathcal{T}_{lin}^{p,t}(\mathrm{AA}(\mathbb{C}^n))$ Fredholm operators are automatically invertible.

3) Let $G \subset \mathbb{R}^{2n}$ be a closed subgroup of $\mathbb{R}^{2n} \cong \mathbb{C}^n$. Set

$$\mathrm{BUC}_G := \{f \in \mathrm{BUC}(\mathbb{C}^n);\ \alpha_w(f) = f \text{ for all } w \in G\}.$$

Then, BUC_G is an α- and U-invariant C^* subalgebra of $\mathrm{BUC}(\mathbb{C}^n)$ and therefore we have $\mathcal{T}_{lin}^{p,t}(\mathrm{BUC}_G) = \mathcal{T}^{p,t}(\mathrm{BUC}_G)$. In the notation from above we have $G = per(\mathrm{BUC}_G)$. It is not difficult to see that

$$\mathcal{T}_{lin}^{p,t}(\mathrm{BUC}_G) = \{A \in \mathcal{T}^{p,t};\ \alpha_w(A) = A \text{ for all } w \in G\}.$$

If $z \in \mathbb{R}^{2n} \cong \mathbb{C}^n$ is such that $z \perp G$ and we let

$$C_{0,G,z} := \{f \in \mathrm{BUC}_G;\ \alpha_w(f)(\lambda z) \to 0 \text{ as } \mathbb{R} \ni \lambda \to \infty \text{ for every } w \in \mathrm{span}(G)\},$$

then $C_{0,G,z}$ is an α-invariant (but not necessarily U-invariant) ideal of BUC_G, hence $\mathcal{T}_{lin}^{p,t}(C_{0,G,z})$ is an ideal of $\mathcal{T}_{lin}^{p,t}(\mathrm{BUC}_G)$. In particular, one can show that

$$\mathcal{T}_{lin}^{p,t}(C_{0,G,z}) = \{A \in \mathcal{T}_{lin}^{p,t}(\mathrm{BUC}_G);\ \alpha_{\lambda z}(A) \to 0 \text{ as } \mathbb{R} \ni \lambda \to -\infty\}.$$

If we let G be a Lagrangian subspace $\mathcal{L}$ of $\mathbb{C}^n$, then we obtain in particular that

$$\mathcal{T}_{lin}^{p,t}(\mathrm{BUC}_{\mathcal{L}}) = \{A \in \mathcal{T}^{p,t};\ A \text{ is } \mathcal{L}\text{-invariant}\},$$

generalizing a result from [66] to arbitrary p.

Let us come back to the problem that this section started with:

Proposition 4.3.8. *Let $\mathcal{D}_0, \mathcal{D}_0' \subset \mathrm{BUC}(\mathbb{C}^n)$ be α- and U-invariant closed subspaces.*

1) If we have

$$\mathcal{T}_*^{2,t}(\mathcal{D}_0) = \mathcal{T}_{lin}^{2,t}(\mathcal{D}_0')$$

for some $t > 0$, then $\mathcal{D}_0 \subseteq \mathcal{D}_0' \subseteq C^*(\mathcal{D}_0)$ (where $C^*(\mathcal{D}_0)$ is the C^* algebra generated by $\mathcal{D}_0$).

2) If we have

$$\mathcal{T}_*^{2,t}(\mathcal{D}_0) = \mathcal{T}_{lin}^{2,t}(\mathcal{D}_0')$$

for all $t > 0$, then $\mathcal{D}_0' = C^*(\mathcal{D}_0)$.

Proof. 1) The result is obtained from Lemma 3.3.8 3) and the following diagram of inclusions, which is a consequence of Theorem 4.3.3:

$$\begin{array}{ccccc} \mathcal{T}_{lin}^{2,t}(\mathcal{D}_0) & \subseteq & \mathcal{T}_*^{2,t}(\mathcal{D}_0) & = & \mathcal{T}_{lin}^{2,t}(\mathcal{D}_0') \\ & & \cap\!\!| & & \\ & & \mathcal{T}_*^{2,t}(C^*(\mathcal{D}_0)) & = & \mathcal{T}_{lin}^{2,t}(C^*(\mathcal{D}_0)) \end{array}$$

2) Since we have for any $t > 0$ that

$$\mathcal{T}_*^{2,t}(\mathcal{D}_0) = \mathcal{T}_{lin}^{2,t}(\mathcal{D}_0'),$$

we know that $\mathcal{T}_{lin}^{2,t}(\mathcal{D}_0')$ is a C^* algebra for every $t > 0$, i.e.

$$\mathcal{T}_{lin}^{2,t}(\mathcal{D}_0') = \mathcal{T}_*^{2,t}(\mathcal{D}_0')$$

for every $t > 0$. Hence, Theorem 4.3.6 shows that $\mathcal{D}_0'$ is itself a C^* algebra. From 1) we know that $\mathcal{D}_0 \subseteq \mathcal{D}_0' \subseteq C^*(\mathcal{D}_0)$. In particular, $\mathcal{D}_0' = C^*(\mathcal{D}_0)$. □

Note that the significant assumption in 2) of the previous theorem is that the space corresponding to $\mathcal{T}_*^{2,t}(\mathcal{D}_0)$ is independent of $t > 0$. Working with this assumption is in practice not easy, as one might see in the example considered in Chapter 6.

4.4 Remarks

As already mentioned, there are several remaining open questions in connection with the results presented here. From the technical point of view, the following question is obvious: Is the assumption on $\mathcal{A}$ being U-invariant and/or self-adjoint in Theorem 4.3.3 necessary? While both assumptions are natural in the setting of the proof presented, one naively would expect the result to be true without these assumptions.

As we have noted earlier, it is an open problem to deduce a relation on $\mathcal{D}_0, \mathcal{D}_0'$ if they satisfy

$$\mathcal{T}^{p,t}(\mathcal{D}_0) = \mathcal{T}^{p,t}(\mathcal{D}_0')$$

or

$$\mathcal{T}_*^{2,t}(\mathcal{D}_0) = \mathcal{T}_*^{2,t}(\mathcal{D}_0').$$

One could naively expect that this implies something as

$$\mathrm{Alg}(\mathcal{D}_0) = \mathrm{Alg}(\mathcal{D}_0')$$

and

$$C^*(\mathcal{D}_0) = C^*(\mathcal{D}_1),$$

respectively. Note that this does not follow automatically from Proposition 4.3.8, even if the relations are satisfied for all $t > 0$: For each $t > 0$ there is certainly some $\mathcal{D}_0''$ such that

$$\mathcal{T}_*^{2,t}(\mathcal{D}_0) = \mathcal{T}_*^{2,t}(\mathcal{D}_0') = \mathcal{T}_{lin}^{2,t}(\mathcal{D}_0'')$$

but $\mathcal{D}_0''$ will a priori depend on the parameter t. Hence, the proposition does not apply here.

Since the statement of Theorem 4.3.3 is quite powerful, it could be interesting to search for more applications of that theorem for studying operator algebras on Fock spaces.

$\mathcal{D}_0 \subset \mathrm{BUC}(\mathbb{C}^n)$	$\mathcal{T}_{lin}^{p,t}(\mathcal{D}_0)$	Reference
α- and U-invariant C^* subalgebra	α- and U-invariant subalgebra of $\mathcal{T}^{p,t}$	Theorem 4.3.3
$\mathrm{BUC}(\mathbb{C}^n)$	$\mathcal{T}^{p,t} = \mathcal{C}_1$	Theorem 3.3.4
VO_∂	$\mathrm{esscomm}(\mathcal{T}^{p,t}, \mathcal{L})$	Lemma 3.3.12
$\mathrm{AP}(\mathbb{C}^n)$	$\{A \in \mathcal{T}^{p,t};\ \mathcal{M}(\mathrm{BUC}(\mathbb{C}^n)) \ni x \mapsto A_x \text{ is } \lVert \cdot \rVert_{op}\text{-cont.}\}$	Example 4.3.7 1)
$\mathrm{AA}(\mathbb{C}^n)$	$\{A \in \mathcal{T}^{p,t}(\mathbb{C}^n);\ \forall x \in \mathcal{M}(\mathrm{BUC}(\mathbb{C}^n)) : (A_x)_{Ux} = A\}$	Example 4.3.7 2)
BUC_G	$\{A \in \mathcal{T}^{p,t};\ \alpha_w(A) = A,\ w \in G\}$	Example 4.1.7 3)
$\mathrm{BUC}_{\mathcal{L}}$, $\mathcal{L}$ Lagrangian subspace	$\mathcal{L}$-invariant operators of $\mathcal{T}^{p,t}$	Example 4.1.7 3)
α-invariant ideal in $\mathcal{A}$	α-invariant ideal in $\mathcal{T}_{lin}^{p,t}(\mathcal{A})$	Theorem 4.3.3
$C_0(\mathbb{C}^n)$	$\mathcal{K}$	Theorem 3.3.9
$C_{0,G,z}$	$A \in \mathcal{T}_{lin}^{p,t}(\mathrm{BUC}_G)$ vanishing in direction of z	Example 4.1.7 3)

Table 4.1: Some corresponding spaces for $1 < p \leq \infty$

Chapter 5

Fredholm theory of Toeplitz operators on Fock spaces

The results presented in this chapter are taken from the joint paper with R. Hagger [73], which we also closely follow. The goal is the following: In Corollary 3.3.10, we deduced the well-known compactness characterization for linear operators on the Fock spaces, which characterizes the compactness in terms of the behaviour of the Berezin transform at infinity and membership in the Toeplitz algebra. Analogously one may ask whether one can characterize the Fredholm property for operators in the Toeplitz algebra in terms of some quantities at infinity. Such results are well-known for band-dominated operators on sequence spaces (cf. [96]) and have already been established in the case of Bergman spaces over unit balls by R. Hagger [80]. In general, the methods presented here are similar to those used in Hagger's paper on the unit ball case. In contrast to the paper [73], we will also try to obtain information on the Fredholm property for $p = 1, \infty, \infty+$ from the methods we present. Further, we will try to distinguish between left- and right-invertibility modulo compact operators. We will need the following definitions:

Definition 5.0.1. Let X be a Banach space and $A \in \mathcal{L}(X)$. We say that

1) A is *upper semi-Fredholm* if $\dim(\ker(A)) < \infty$ and $\operatorname{ran}(A)$ is closed;

2) A is *lower semi-Fredholm* if $\operatorname{codim}(\operatorname{ran}(A)) < \infty$;

3) A is *left Atkinson* if there exist $B \in \mathcal{L}(X)$, $K \in \mathcal{K}(X)$ such that $BA = I + K$;

4) A is *right Atkinson* if there exist $B \in \mathcal{L}(X)$, $K \in \mathcal{K}(X)$ such that $AB = I + K$.

We will denote the set of upper semi-Fredholm, lower semi-Fredholm, left Atkinson and right Atkinson operators on X by $\Phi_-(X)$, $\Phi_+(X)$, $\Phi_l(X)$ and $\Phi_r(X)$, respectively. Further, we will denote the class of Fredholm operators by $\Phi(X) = \Phi_-(X) \cap \Phi_+(X)$. Here are some important properties:

Proposition 5.0.2. *Let X be a Banach space and $A \in \mathcal{L}(X)$.*

1) $\Phi(X) = \Phi_l(X) \cap \Phi_r(X)$*;*

2) $A \in \Phi(X)$ *if and only if* $A' \in \Phi(X')$*;*

3) $A \in \Phi_l(X)$ *if and only if* $A \in \Phi_+(X)$ *and* $\ker(A)$ *is complemented in* X*;*

4) $A \in \Phi_r(X)$ *if and only if* $A \in \Phi_-(X)$ *and* $\operatorname{ran}(A)$ *is complemented in* X*;*

5) $A \in \Phi_l(X)$ *implies* $A' \in \Phi_r(X')$*;*

6) $A \in \Phi_r(X)$ *implies* $A' \in \Phi_l(X')$*;*

7) If X is reflexive: $A \in \Phi_r(X) \Leftrightarrow A' \in \Phi_l(X')$ *and* $A \in \Phi_l(X) \Leftrightarrow A' \in \Phi_r(X')$*.*

Proof. 1) is the well-known Atkinson Theorem. 2) is a consequence of Banach's closed range theorem [134, Theorem VII.5]. 3) and 4) are Theorems 4.3.2 and 4.3.3 in [43]. 5) and 6) follow immediately from the definitions. If X is reflexive, then 7) follows from 5) and 6), since $A \cong (A')'$. □

Let us also state the following elementary fact, which follows from simple estimates or, alternatively, is an immediate consequence of Jensen's inequality:

Lemma 5.0.3. *Let $p \in [1, \infty)$ and $x_1, \dots, x_k \geq 0$. Then,*

$$\left(\sum_{j=1}^{k} x_j \right)^p \leq k^p \sum_{j=1}^{k} x_j^p.$$

Here is an important fact, which we will frequently use:

Proposition 5.0.4. *Assume $1 \leq p < \infty$. Let $(U_j)_{j \in \mathbb{N}}$ be a sequence of measurable subsets of $\mathbb{C}^n$ such that every $z \in \mathbb{C}^n$ belongs to at most N of the sets U_j for some $N \in \mathbb{N}$ independent of z. Further, let $(f_j)_{j \in \mathbb{N}}$ be a sequence of measurable functions $f_j : \mathbb{C}^n \to \mathbb{C}$ such that* $\operatorname{supp}(f_j) \subseteq U_j$ *and $|f_j(z)| \leq 1$ for all $z \in \mathbb{C}^n$. Then, for every $g \in L_t^p$ we have*

$$\sum_{j=1}^{\infty} \int_{\mathbb{C}^n} |f_j(z) g(z)|^p d\mu_{2t/p}(z) \leq N \|g\|_{L_t^p}^p,$$

i.e.

$$\sum_{j=1}^{\infty} \|M_{f_j} g\|_{L_t^p}^p \leq N \|g\|_{L_t^p}^p.$$

Proof. The estimate is derived as follows:

$$\sum_{j=1}^{\infty} \int_{\mathbb{C}^n} |f_j(z) g(z)|^p d\mu_{2t/p}(z) \leq \sum_{j=1}^{\infty} \int_{U_j} |g(z)|^p d\mu_{2t/p}(z)$$

$$
\begin{aligned}
&= \lim_{M\to\infty} \int_{\mathbb{C}^n} \sum_{j=1}^{M} \chi_{U_j}(z)|g(z)|^p \, d\mu_{2t/p}(z) \\
&\leq N \int_{\mathbb{C}^n} |g(z)|^p \, d\mu_{2t/p}(z) \\
&= N\|g\|_{L_t^p}^p,
\end{aligned}
$$

as required. □

Here is another fact in the same spirit. Note that the analogous result on sequence spaces is originally due to I. B. Simonenko [121].

Lemma 5.0.5. *Let $1 \leq p < \infty$. For every $j \in \mathbb{N}$ let $a_j,\ b_j : \mathbb{C}^n \to [0,1]$ be measurable functions and assume that each $z \in \mathbb{C}^n$ belongs to at most N of the sets $\operatorname{supp}(a_j)$ and at most M of the sets $\operatorname{supp}(b_j)$. If $(A_j)_{j\in\mathbb{N}} \subset \mathcal{L}(L_t^p)$ such that there is a constant $C > 0$ with $\|A_j\| \leq C$ for all $j \in \mathbb{N}$, then the series*

$$\sum_{j=1}^{\infty} M_{a_j} A_j M_{b_j}$$

converges in the strong operator topology and $\|\sum_{j=1}^{\infty} M_{a_j} A_j M_{b_j}\| \leq NMC$.

Proof. Let $f \in L_t^p$. Then,

$$
\begin{aligned}
\left\| \sum_{j=m}^{\infty} M_{a_j} A_j M_{b_j} f \right\|_{L_t^p}^p &= \int_{\mathbb{C}^n} \left| \sum_{j=m}^{\infty} (M_{a_j} A_j M_{b_j} f)(z) \right|^p \, d\mu_{2t/p}(z) \\
&\leq \int_{\mathbb{C}^n} \left(\sum_{j=m}^{\infty} |(M_{a_j} A_j M_{b_j} f)(z)| \right)^p \, d\mu_{2t/p}(z).
\end{aligned}
$$

By assumption, the sum in the integral is pointwise a finite sum with at most N terms. Using Lemma 5.0.3 and the Monotone Convergence Theorem, we conclude

$$
\begin{aligned}
&\leq \int_{\mathbb{C}^n} N^p \sum_{j=m}^{\infty} |(M_{a_j} A_j M_{b_j} f(z)|^p \, d\mu_{2t/p}(z) \\
&= N^p \sum_{j=m}^{\infty} \int_{\mathbb{C}^n} |(M_{a_j} A_j M_{b_j} f)(z)|^p \, d\mu_{2t/p}(z) \\
&= N^p \sum_{j=m}^{\infty} \|M_{a_j} A_j M_{b_j} f\|_{L_t^p}^p \\
&\leq N^p C^p \sum_{j=m}^{\infty} \|M_{b_j} f\|_{L_t^p}^p,
\end{aligned}
$$

where we used $\|a\|_\infty \leq 1$ in the last step. By Proposition 5.0.4 we obtain

$$\sum_{j=m}^{\infty} \|M_{b_j} f\|_{L_t^p}^p \to 0, \quad m \to \infty.$$

This shows convergence of the sum in the strong operator topology. The norm estimate follows from the same estimates as above (for $m = 1$) and a final application of Proposition 5.0.4. □

5.1 Band-dominated operators

Definition 5.1.1. 1) $A \in \mathcal{L}(L_t^p)$ is called a *band operator* if there is some $\omega > 0$ such that for any pair of functions $f, g \in L^\infty(\mathbb{C}^n)$, $\operatorname{dist}(\operatorname{supp}(f), \operatorname{supp}(g)) > \omega$ implies $M_f A M_g = 0$. We will denote the infimum of all such ω by $\omega(A)$ and call it the *band-width* (or *propagation*) of A.

2) $A \in \mathcal{L}(L_t^p)$ is said to be *band-dominated* if it can be approximated by band operators in operator norm. The set of all band-dominated operators on L_t^p will be denoted by BDO_t^p.

The first goal will be to derive certain characterizations of band-dominated operators. Let us denote by $|z|_\infty$ the supremum norm of $z \in \mathbb{C}^n \cong \mathbb{R}^{2n}$, i.e.

$$|z|_\infty = \max\{|\operatorname{Re}(z_j)|, \ |\operatorname{Im}(z_j)|; \ j = 1, \ldots, n\}.$$

For a subset $B \subseteq \mathbb{C}^n$ and $z \in \mathbb{C}^n$ we let

$$d_\infty(z, B) := \inf\{|z - w|_\infty; \ w \in B\}.$$

Let us consider the following collection of subsets of $\mathbb{C}^n$:

$$\zeta := \{[-3, 3)^{2n} + \sigma \subset \mathbb{R}^{2n}; \ \sigma \in 6\mathbb{Z}^{2n}\}.$$

Since ζ is clearly a countable set, we can fix an enumeration $\zeta = \{B_j\}_{j=1}^\infty$ such that $0 \in B_1$. Further, we will consider the sets

$$\Omega_k(B_j) := \{z \in \mathbb{C}^n; \ \operatorname{dist}_\infty(z, B_j) \leq k\}$$

for $k = 1, 2, 3$ and all $j \in \mathbb{N}$.

Lemma 5.1.2 ([19, Lemma 3.1])**.** *The sets B_j satisfy the following properties:*

1) $B_j \cap B_k = \emptyset$ for $j \neq k$;

2) Every $z \in \mathbb{C}^n$ belongs to at most 2^{2n} of the sets $\Omega_1(B_j)$ and to at most 4^{2n} of the sets $\Omega_3(B_j)$;

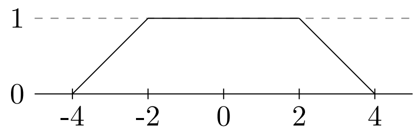

Figure 5.1: The function ϕ

3) $\mathrm{diam}(B_j) = 6\sqrt{2n}$*, where* diam *denotes the Euclidean diameter of the set.*

Let us denote by $\phi : \mathbb{R} \to [0,1]$ the function from Figure 5.1, i.e.

$$\phi(x) = \begin{cases} 0, & x \leq -4 \\ \frac{1}{2}x + 2; & -4 < x \leq -2 \\ 1, & -2 < x \leq 2 \\ -\frac{1}{2}x + 2; & 2 < x \leq 4 \\ 0, & x > 4 \end{cases}$$

Further, define the function $\varphi : \mathbb{C}^n \cong \mathbb{R}^{2n} \to [0,1]$ by

$$\varphi(x_1, \dots, x_{2n}) = \phi(x_1) \cdot \dots \cdot \phi(x_{2n}).$$

Let $(\sigma_j)_{j\in\mathbb{N}}$ be the enumeration of $6\mathbb{Z}^{2n}$ coinciding with the enumeration of ζ, i.e. $\sigma_j \in B_j$ for all $j \in \mathbb{N}$. Then, we consider the functions $\varphi_j : \mathbb{C}^n \cong \mathbb{R}^{2n} \to [0,1]$ given by

$$\varphi_j(x) = \varphi(x - \sigma_j).$$

Since $\{\phi(\cdot - 6k)\}_{k\in\mathbb{Z}}$ is a partition of unity of $\mathbb{R}$, where each element is Lipschitz with Lipschitz constant $1/2$ we obtain by induction:

1) $\mathrm{supp}(\varphi_j) = \Omega_1(B_j)$ for every $j \in \mathbb{N}$,

2) $\sum_{j=1}^{\infty} \varphi_j(z) = 1$ for every $z \in \mathbb{C}^n$,

3) φ_j is Lipschitz continuous with Lipschitz constant $\frac{1}{2} \cdot 2n$ for every $j \in \mathbb{N}$. In particular, the sequence $(\varphi_j)_{j\in\mathbb{N}}$ is uniformly equicontinuous.

Imitating the above construction, it is not a problem to find a uniformly equicontinuous sequence of functions $(\psi_j)_{j\in\mathbb{N}}$ such that the functions $\psi_j : \mathbb{C}^n \to [0,1]$ satisfy

1) $\psi_j(z) = 1$ for every $z \in \Omega_2(B_j)$;

2) $\mathrm{supp}(\psi_j) = \Omega_3(B_j)$.

Given $s > 0$ and $j \in \mathbb{N}$ we define $\varphi_{j,s}(z) = \varphi_j(sz)$ and $\psi_{j,s}(z) = \psi_j(sz)$. Here is the announced characterization of band-dominated operators:

Proposition 5.1.3. *Let $p \in [1,\infty)$, $t > 0$ and $A \in \mathcal{L}(L_t^p)$. Then, the following are equivalent:*

1) A is band-dominated,

2) $\lim\limits_{s\to 0} \sup\limits_{\|f\|=1} \sum\limits_{j=1}^{\infty} \|M_{\varphi_{j,s}} A M_{1-\psi_{j,s}} f\|_{L_t^p}^p = 0,$

3) $\lim\limits_{s\to 0} \left\| \sum\limits_{j=1}^{\infty} M_{\varphi_{j,s}} A M_{1-\psi_{j,s}} \right\| = 0$*, where the sum converges in strong operator topology.*

Proof. 1) ⇒ 2): Let $\varepsilon > 0$ and $B \in \mathcal{L}(L_t^p)$ a band operator such that $\|A - B\| < \varepsilon$. Observe that

$$\operatorname{supp}(\varphi_{j,s}) = \left[-\frac{4}{s}, \frac{4}{s}\right]^{2n} - \frac{\sigma_j}{s}, \quad \operatorname{supp}(1-\psi_{j,s}) = \mathbb{C}^n \setminus \left(\left[-\frac{5}{s}, \frac{5}{s}\right]^{2n} - \frac{\sigma_j}{s}\right). \tag{5.1}$$

In particular,

$$\begin{aligned} \operatorname{dist}(\operatorname{supp}(\varphi_{j,s}), \operatorname{supp}(1-\psi_{j,s})) &= \operatorname{dist}(\operatorname{supp}(\varphi_{1,s}), \operatorname{supp}(1-\psi_{1,s})) \\ &\geq \frac{1}{s} \to \infty, \quad s \to 0. \end{aligned}$$

Therefore, we may fix $s > 0$ small enough such that

$$\operatorname{dist}(\operatorname{supp}(\varphi_{j,s}), \operatorname{supp}(1-\psi_{j,s})) > \omega(A)$$

for all $j \in \mathbb{N}$. Hence, $M_{\varphi_{j,s}} B M_{1-\psi_{j,s}} = 0$ for all j. We therefore obtain for $f \in L_t^p$

$$\begin{aligned} \sum_{j=1}^{\infty} \|M_{\varphi_{j,s}} A M_{1-\psi_{j,s}} f\|_{L_t^p}^p &= \sum_{j=1}^{\infty} \|M_{\varphi_{j,s}} (A-B) M_{1-\psi_{j,s}} f\|_{L_t^p}^p \\ &\leq 2^p \sum_{j=1}^{\infty} (\|M_{\varphi_{j,s}} (A-B) f\|_{L_t^p}^p \\ &\quad + \|M_{\varphi_{j,s}} (A-B) M_{\psi_{j,s}} f\|_{L_t^p}^p) \\ &\leq 2^p 2^{2n} (\|(A-B) f\|_{L_t^p}^p + \|(A-B) M_{\psi_{j,s}} f\|_{L_t^p}^p), \end{aligned}$$

where we first applied Lemma 5.0.3 for the case $k = 2$ and afterwards Proposition 5.0.4. This yields

$$\sum_{j=1}^{\infty} \|M_{\varphi_{j,s}} A M_{1-\psi_{j,s}} f\|_{L_t^p}^p \leq 2^{p+1} 2^{2n} \varepsilon^p \|f\|_{L_t^p}^p.$$

Since $\varepsilon > 0$ was arbitrary, we therefore have shown

$$\limsup_{s\to 0} \sup_{\|f\|_{L_t^p}=1} \sum_{j=1}^{\infty} \|M_{\varphi_{j,s}} A M_{1-\psi_{j,s}} f\|_{L_t^p}^p = 0.$$

2) ⇒ 3): Note that the series in *3)* converges in the strong operator topology by Lemma 5.0.5. Using Lemma 5.0.3, Lemma 5.1.2 and the Monotone Convergence Theorem, we obtain for $f \in L_t^p$:

$$\begin{aligned}\left\|\sum_{j=1}^{\infty} M_{\varphi_{j,t}} A M_{1-\psi_{j,t}} f\right\|_{L_t^p}^p &= \int_{\mathbb{C}^n} \left|\sum_{j=1}^{\infty} (M_{\varphi_{j,t}} A M_{1-\psi_{j,t}} f)(z)\right|^p \, d\mu_{2t/p}(z)\\ &\le (2^{2n})^p \int_{\mathbb{C}^n} \sum_{j=1}^{\infty} |(M_{\varphi_{j,t}} A M_{1-\psi_{j,t}} f)(z)|^p \, d\mu_{2t/p}(z)\\ &= (2^{2n})^p \sum_{j=1}^{\infty} \|M_{\varphi_{j,t}} A M_{1-\psi_{j,t}} f\|_{L_t^p}^p.\end{aligned}$$

Taking now the supremum over all $f \in L_t^p$ with $\|f\|_{L_t^p} = 1$ yields *3)*.

3) ⇒ 1): For $m \in \mathbb{N}$ we have

$$\begin{aligned}\operatorname{supp}(\varphi_{j,\frac{1}{m}}) &= [-4m, 4m]^{2n} - m\sigma_j,\\ \operatorname{supp}(\psi_{j,\frac{1}{m}}) &= [-5m, 5m]^{2n} - m\sigma_j.\end{aligned}$$

Let $f, g \in L^\infty(\mathbb{C}^n)$ such that

$$\operatorname{dist}(\operatorname{supp}(f), \operatorname{supp}(g)) > \operatorname{diam}(\operatorname{supp}(\psi_{j,\frac{1}{m}})).$$

Then, only one of $\operatorname{supp}(f)$, $\operatorname{supp}(g)$ can have nontrivial intersection with the supports of $\varphi_{j,\frac{1}{m}}$ and $\psi_{j,\frac{1}{m}}$, i.e.

$$M_f M_{\varphi_{j,\frac{1}{m}}} A M_{\psi_{j,\frac{1}{m}}} M_g = 0.$$

Since this holds independently of j, we obtain that $A_m := \sum_{j=1}^{\infty} M_{\varphi_{j,\frac{1}{m}}} A M_{\psi_{j,\frac{1}{m}}}$ is a band operator. Recall that $(\varphi_{j,t})_{j\in\mathbb{N}}$ is a partition of unity, i.e. $\sum_{j=1}^{\infty} \varphi_{j,t} \equiv 1$ for all $t > 0$. Hence,

$$\begin{aligned}\|A - A_m\| &= \left\|\sum_{j=1}^{\infty} M_{\varphi_{j,\frac{1}{m}}} A - \sum_{j=1}^{\infty} M_{\varphi_{j,\frac{1}{m}}} A M_{\psi_{j,\frac{1}{m}}}\right\|\\ &= \left\|\sum_{j=1}^{\infty} M_{\varphi_{j,\frac{1}{m}}} A M_{1-\psi_{j,\frac{1}{m}}}\right\|\\ &\to 0\end{aligned}$$

as $m \to \infty$. □

There is another important characterization of band-dominated operators:

Proposition 5.1.4. *Let* $1 \leq p < \infty$ *and* $A \in \mathcal{L}(L_t^p)$. *Then, the following two statements are equivalent:*

1) $A \in \mathrm{BDO}_t^p$;

2) $\lim\limits_{s\to 0} \sup\limits_{\|f\|=1} \sum\limits_{j=1}^{\infty} \|[A, M_{\varphi_{j,s}}]f\|_{L_t^p}^p = 0.$

In this case, $\limsup_{s\to 0} \|[A, M_{\varphi_{j,s}}]\| = 0$.

The proof of this proposition will be divided in two steps. Note that we did not prove this statement in [73], as its proof is essentially identical to the unit ball case. For completeness, we give the proof here, which follows the discussion in [80] closely. The following lemma is analogous to [80, Lemma 12], from where we also took our proof almost literally.

Lemma 5.1.5. *Let* $\omega > 0$ *and for every* $j \in \mathbb{N}$, $s \in (0, \infty)$ *let* $a_{j,s} : \mathbb{C}^n \to [0, 1]$ *be measurable. If*

$$\lim_{s\to 0} \inf_{j\in\mathbb{N}} \operatorname{dist}(a_{j,s}^{-1}(U), a_{j,s}^{-1}(V)) = \infty$$

for all sets $U, V \subset [0, 1]$ *with* $\operatorname{dist}(U, V) > 0$*, then for every* $\varepsilon > 0$ *there exists some* $s_0 > 0$ *such that for all* $s \in (0, s_0)$ *and all band operators* A *of band-width at most* ω *the estimate*

$$\sup_{j\in\mathbb{N}} \|[A, M_{a_{j,s}}]\| \leq 3\|A\|\varepsilon$$

holds true.

Proof. Let $A \in \mathcal{L}(L_t^p)$ be a band operator such that $\omega(A) < \omega$. Let $\varepsilon > 0$ and set $m = \lceil \frac{1}{\varepsilon} \rceil$. For every $k = 1, \ldots, m$ we set

$$U_{k,s}^j := a_{j,s}^{-1}([k\varepsilon, 1]) \quad \text{and} \quad V_{k,s}^j := a_{j,s}^{-1}\left(\left[\left(k - \frac{1}{2}\right)\varepsilon, 1\right]\right).$$

Moreover, define

$$a_{j,s}^U := \varepsilon \sum_{k=1}^{m} \chi_{U_{k,s}^j} \quad \text{and} \quad a_{j,s}^V := \varepsilon \sum_{k=1}^{m} \chi_{V_{k,s}^j}.$$

Obviously, for every $z \in \mathbb{C}^n$ and $s > 0$ we have either $a_{j,s}(z) < \varepsilon$ or $a_{j,s}(z) \in [l\varepsilon, (l+1)\varepsilon)$ for some $l \in \mathbb{N}$. Hence, $|a_{j,s}(z) - a_{j,s}^U(z)| < \varepsilon$ for every $z \in \mathbb{C}^n$, i.e. $\sup_{j\in\mathbb{N}} \|a_{j,s} - a_{j,s}^U\|_\infty \leq \varepsilon$. In the same way one sees that $\sup_{j\in\mathbb{N}} \|a_{j,s} - a_{j,s}^V\|_\infty \leq \varepsilon$. In particular,

$$\sup_{j\in\mathbb{N}} \|M_{a_{j,s}} - M_{a_{j,s}^U}\| \leq \varepsilon \quad \text{and} \quad \sup_{j\in\mathbb{N}} \|M_{a_{j,s}} - M_{a_{j,s}^V}\| \leq \varepsilon.$$

Therefore,

$$\begin{aligned}
&\sup_{j\in\mathbb{N}}\|AM_{a_{j,s}} - M_{a_{j,s}}A\| \\
&\quad\le \sup_{j\in\mathbb{N}}\Big(\|A(M_{a_{j,s}} - M_{a^V_{j,s}})\| + \|AM_{a^V_{j,s}} - M_{a^U_{j,s}}A\| + \|(M_{a^U_{j,s}} - M_{a_{j,s}})A\|\Big) \\
&\quad\le 2\|A\|\varepsilon + \sup_{j\in\mathbb{N}}\|AM_{a^V_{j,s}} - M_{a^U_{j,s}}A\| \\
&\quad= 2\|A\|\varepsilon + \varepsilon\sup_{j\in\mathbb{N}}\left\|\sum_{k=1}^{m}(AM_{\chi_{V^j_{k,s}}} - M_{\chi_{U^j_{k,s}}}A)\right\| \\
&\quad\le 2\|A\|\varepsilon + \varepsilon\sup_{j\in\mathbb{N}}\left\|\sum_{k=1}^{m}M_{\chi_{(U^j_{k,s})^c}}AM_{\chi_{V^j_{k,s}}}\right\| + \varepsilon\sup_{j\in\mathbb{N}}\left\|\sum_{k=1}^{m}M_{\chi_{U^j_{k,s}}}AM_{\chi_{(V^j_{k,s})^c}}\right\|.
\end{aligned}$$

Recall that $U^j_{k,s} = a^{-1}_{j,s}([k\varepsilon, 1])$ and $(V^j_{k,s})^c = a^{-1}_{j,s}\left(\left[0, \left(k - \frac{1}{2}\right)\varepsilon\right)\right)$. Hence, by assumption we have for every k:

$$\inf_{j\in\mathbb{N}} \operatorname{dist}(U^j_{k,s}, (V^j_{k,s})^c) \to \infty, \quad s \to 0.$$

We may choose $s_0 > 0$ such that for every $s \in (0, s_0)$ and every $k = 1, \ldots, m$ we have $\inf_{j\in\mathbb{N}} \operatorname{dist}(U^j_{k,s}, (V^j_{k,s})^c) > \omega$. Recalling that ω was assumed to be larger than the band width of A, we obtain $M_{\chi_{U^j_{k,s}}}AM_{\chi_{(V^j_{k,s})^c}} = 0$ for all k. In particular, for such s we obtain the estimate

$$\sup_{j\in\mathbb{N}}\|AM_{a_{j,s}} - M_{a_{j,s}}A\| \le 2\|A\|\varepsilon + \varepsilon\sup_{j\in\mathbb{N}}\left\|\sum_{k=1}^{m}M_{\chi_{(U^j_{k,s})^c}}AM_{\chi_{V^j_{k,s}}}\right\|.$$

Let us set $U^j_{0,s} := \mathbb{C}^n$ and $V^j_{m+1,s} := \emptyset$. Then, by a similar argument as above and possibly making s_0 even smaller, we obtain

$$\begin{aligned}
&\sup_{j\in\mathbb{N}}\left\|\sum_{k=1}^{m}M_{\chi_{(U^j_{k,s})^c}}AM_{\chi_{V^j_{k,s}}}\right\| \\
&\quad\le \sup_{j\in\mathbb{N}}\left\|\sum_{k=1}^{m}M_{\chi_{(U^j_{k,s})^c}}AM_{\chi_{V^j_{k+1,s}}}\right\| \\
&\quad+ \sup_{j\in\mathbb{N}}\left\|\sum_{k=1}^{m}M_{\chi_{(U^j_{k-1,s})^c}}AM_{\chi_{V^j_{k,s}\setminus V^j_{k+1,s}}}\right\| \\
&\quad+ \sup_{j\in\mathbb{N}}\left\|\sum_{k=1}^{m}M_{\chi_{U^j_{k-1,s}\setminus U^j_{k,s}}}AM_{\chi_{V^j_{k,s}\setminus V^j_{k+1,s}}}\right\| \\
&\quad\le \sup_{j\in\mathbb{N}}\left\|\sum_{k=1}^{m}M_{\chi_{U^j_{k-1,s}\setminus U^j_{k,s}}}AM_{\chi_{V^j_{k,s}\setminus V^j_{k+1,s}}}\right\|.
\end{aligned}$$

Using the fact that the sets $U_{k-1,s}^j \setminus U_{k,s}^j$ are pairwise disjoint, we can compute for every $f \in L_t^p$ and $j \in \mathbb{N}$ arbitrary:

$$\begin{aligned}\left\| \sum_{k=1}^m M_{\chi_{U_{k-1,s}^j \setminus U_{k,s}^j}} A M_{\chi_{V_{k,s}^j \setminus V_{k+1,s}^j}} f \right\|_{L_t^p}^p &= \sum_{k=1}^m \left\| M_{\chi_{U_{k-1,s}^j \setminus U_{k,s}^j}} A M_{\chi_{V_{k,s}^j \setminus V_{k+1,s}^j}} f \right\|_{L_t^p}^p \\ &\leq \|A\|^p \sum_{k=1}^m \left\| M_{\chi_{V_{k,s}^j \setminus V_{k+1,s}^j}} f \right\|_{L_t^p}^p \\ &= \|A\|^p \left\| M_{\chi_{V_{1,s}^j}} f \right\|_{L_t^p}^p \\ &\leq \|A\|^p \|f\|_{L_t^p}^p .\end{aligned}$$

This establishes the desired estimate

$$\sup_{j \in \mathbb{N}} \|(AM_{a_{j,s}} - M_{a_{j,s}} A)\| \leq 3\|A\|\varepsilon$$

for $s \in (0, s_0)$. □

We now present the proof of the characterization of band-dominated operators in Proposition 5.1.4. Again, we did not present this proof in [73], since it can be concluded identically to the case of Bergman spaces over the unit ball. We reproduce the proof from [80, Proposition 11].

Proof of Proposition 5.1.4. Let $A \in \mathrm{BDO}_t^p$ and $\varepsilon > 0$. Then, we can pick a band operator A_m such that $\|A - A_m\| < \varepsilon$.

We claim that the functions $\varphi_{j,s}$ satisfy the assumptions of Lemma 5.1.5: Recall that the functions φ_j are by construction Lipschitz with Lipschitz constant n. For $U, V \subset [0,1]$ satisfying $\mathrm{dist}(U, V) > 0$, let $w_{j,s} \in \varphi_{j,s}^{-1}(U), z_{j,s} \in \varphi_{j,s}^{-1}(V)$. Then,

$$n|w_{j,s} - z_{j,s}| \geq \frac{1}{s}|\varphi_{j,s}(w_{j,s}) - \varphi_{j,s}(z_{j,s})| \geq \frac{1}{s}\,\mathrm{dist}(U, V) \to \infty, \quad s \to 0$$

independently of the choice of the points $w_{j,s}$ and $z_{j,s}$. Therefore

$$\lim_{s\to 0} \inf_{j \in \mathbb{N}} \mathrm{dist}(\varphi_{j,s}^{-1}(U), \varphi_{j,s}^{-1}(V)) = \infty.$$

By Lemma 5.1.5 there exists $s_0 > 0$ such that for all $s \in (0, s_0)$:

$$\begin{aligned}\sup_{j\in\mathbb{N}} \|[A, M_{\varphi_{j,s}}]\| &\leq \sup_{j\in\mathbb{N}} \|[A_m, M_{\varphi_{j,s}}]\| + \sup_{j\in\mathbb{N}} \|[A - A_m, M_{\varphi_{j,s}}]\| \\ &\leq 3\|A_m\|\varepsilon + 2\varepsilon \\ &\leq 3(\|A\| + \varepsilon)\varepsilon + 2\varepsilon.\end{aligned}$$

This of course implies

$$\lim_{s\to 0} \sup_{j\in\mathbb{N}} \|[A, M_{\varphi_{j,s}}]\| = 0.$$

Using Lemma 5.0.3 and the properties of the functions $\varphi_{j,s}$ and $\psi_{j,s}$ we estimate

$$\begin{aligned}
&\sup_{\|f\|=1} \sum_{j=1}^{\infty} \|[A, M_{\varphi_{j,s}}]f\|_{L_t^p}^p \\
&\quad \leq 2^p \sup_{\|f\|=1} \sum_{j=1}^{\infty} (\|[A, M_{\varphi_{j,s}}]M_{\psi_{j,s}}f\|_{L_t^p}^p + \|[A, M_{\varphi_{j,s}}]M_{1-\psi_{j,s}}f\|_{L_t^p}^p) \\
&\quad \leq 2^p \sup_{\|f\|=1} \sum_{j=1}^{\infty} (\|[A, M_{\varphi_{j,s}}]\|^p \|M_{\psi_{j,s}}f\|_{L_t^p}^p + \|M_{\varphi_{j,s}}AM_{1-\psi_{j,s}}f\|_{L_t^p}^p) \\
&\quad \leq 2^p 4^{2n} \sup_{j \in \mathbb{N}} \|[A, M_{\varphi_{j,s}}]\|^p + 2^p \sup_{\|f\|=1} \sum_{j=1}^{\infty} \|M_{\varphi_{j,s}}AM_{1-\psi_{j,s}}f\|_{L_t^p}^p.
\end{aligned}$$

Now, Proposition 5.1.3 yields

$$\lim_{s \to 0} \sup_{\|f\|=1} \sum_{j=1}^{\infty} \|[A, M_{\varphi_{j,s}}]f\|_{L_t^p}^p = 0.$$

Conversely, assume $\sup_{\|f\|=1} \sum_{j=1}^{\infty} \|[A, M_{\varphi_{j,s}}]f\|_{L_t^p}^p \to 0$ as $s \to 0$. Since

$$\sup_{j \in \mathbb{N}} \|[A, M_{\varphi_{j,s}}]\|^p \leq \sup_{j \in \mathbb{N}} \sup_{\|f\|=1} \|[A, M_{\varphi_{j,s}}]f\|_{L_t^p}^p \leq \sup_{\|f\|=1} \sum_{j=1}^{\infty} \|[A, M_{\varphi_{j,s}}]f\|_{L_t^p}^p,$$

this clearly implies $\sup_{j \in \mathbb{N}} \|[A, M_{\varphi_{j,s}}]\| \to 0$ as $s \to 0$. Therefore, by similar estimates as we had in the other direction of the proof,

$$\begin{aligned}
&\sup_{\|f\|=1} \sum_{j=1}^{\infty} \|M_{\varphi_{j,s}}AM_{1-\psi_{j,s}}f\|_{L_t^p}^p \\
&\quad \leq 2^p \sup_{\|f\|=1} \sum_{j=1}^{\infty} (\|[M_{\varphi_{j,s}}, A]f\|_{L_t^p}^p + \|[M_{\varphi_{j,s}}, A]M_{\psi_{j,s}}f\|_{L_t^p}^p) \\
&\quad \leq 2^p \sup_{\|f\|=1} \sum_{j=1}^{\infty} (\|[M_{\varphi_{j,s}}, A]f\|_{L_t^p}^p + \|[M_{\varphi_{j,s}}, A]\|^p \|M_{\psi_{j,s}}f\|_{L_t^p}^p) \\
&\quad \leq 2^p \sup_{\|f\|=1} \sum_{j=1}^{\infty} \|[M_{\varphi_{j,s}}, A]f\|_{L_t^p}^p + 2^p 4^{2n} \sup_{j \in \mathbb{N}} \|[M_{\varphi_{j,s}}, A]\|^p,
\end{aligned}$$

which converges to 0 as $s \to 0$ by assumption. $A \in \mathrm{BDO}_t^p$ follows now by Proposition 5.1.3. □

Here are some properties of BDO_t^p:

Proposition 5.1.6. *Let $1 \leq p < \infty$ and $t > 0$. Then,*

1) $M_f \in \mathrm{BDO}_t^p$ for all $f \in L^\infty(\mathbb{C}^n)$;

2) BDO_t^p is a closed subalgebra of $\mathcal{L}(L_t^p)$;

3) If $A \in \mathrm{BDO}_t^p$ is Fredholm and B a Fredholm regularizer of A, then $B \in \mathrm{BDO}_t^p$, and in particular, BDO_t^p is inverse closed;

4) $\mathcal{K}(L_t^p)$ is a closed and two-sided ideal in BDO_t^p;

5) If $A \in \mathrm{BDO}_t^p$ then $A^* \in \mathrm{BDO}_t^q$, where $\frac{1}{p} + \frac{1}{q} = 1$. In particular, BDO_t^2 is a C^* algebra.

Proof. 1) Clearly, multiplication operators are band operators.

2) Let A, B be band operators and $f, g \in L^\infty(\mathbb{C}^n)$ such that

$$\operatorname{dist}(\operatorname{supp}(f), \operatorname{supp}(g)) \geq 4(\omega(A) + \omega(B)).$$

Define

$$S := \{z \in \mathbb{C}^n;\ \operatorname{dist}(z, \operatorname{supp}(f)) \leq 2(\omega(A) + \omega(B))\}.$$

Then,

$$\begin{aligned}\operatorname{dist}(\operatorname{supp}(f), S^c) &> \omega(A), \\ \operatorname{dist}(\operatorname{supp}(g), S) &> \omega(B),\end{aligned}$$

and therefore

$$\begin{aligned} M_f A B M_g &= M_f A (M_{1-\chi_S} + M_{\chi_S}) B M_g \\ &= M_f A M_{1-\chi_S} B + M_f A M_{\chi_S} B M_g \\ &= 0, \end{aligned}$$

i.e. the band operators form an algebra. Therefore, also their closure is an algebra.

3) and 4): Since those facts will not be needed in this work, we refer to the identical proofs in [80, Proposition 13].

5) Follows immediately, since the adjoint of a band operator is again a band operator. □

Proposition 5.1.7. *Let $1 \leq p \leq \infty$ and $t > 0$. Then, $P_t \in \mathrm{BDO}_t^p$.*

By including the limit cases $p = 1, \infty$, we can actually give a proof which is somewhat simpler compared to the proof we gave in [73].

Proof. We will prove the statement using characterization 3) of Proposition 5.1.3. Let us first deal with the case $p = 1$. Recall that we obtain by Equation (5.1) that

$$\operatorname{dist}(\operatorname{supp}(\varphi_{j,s}), \operatorname{supp}(1-\psi_{j,s})) \geq \frac{1}{s}$$

for every $j \in \mathbb{N}$, $s > 0$. Let $f \in L_t^1$. Then, since $\varphi_{j,s}(z) \in [0,1]$ and $1 - \psi_{j,s}(w) \in [0,1]$ for all $z, w \in \mathbb{C}^n$, we obtain

$$\begin{aligned}
&\left\| \sum_{j=1}^{\infty} M_{\varphi_{j,s}} P_t M_{1-\psi_{j,s}} f \right\|_{L_t^1} \\
&\leq \frac{1}{2^n} \left(\frac{1}{\pi t}\right)^{2n} \int_{\mathbb{C}^n} \left| \sum_{j=1}^{\infty} \varphi_{j,s}(z) \int_{\mathbb{C}^n} (1-\psi_{j,s})(w) f(w) e^{\frac{z\cdot\overline{w}}{t} - \frac{|w|^2}{t}} \, dw \right| e^{-\frac{|z|^2}{2t}} \, dz \\
&\leq \frac{1}{2^n} \left(\frac{1}{\pi t}\right)^{2n} \sum_{j=1}^{\infty} \int_{\operatorname{supp}(\varphi_{j,s})} \varphi_{j,s}(z) \int_{\operatorname{supp}(1-\psi_{j,s})} |f(w)| e^{-\frac{|z-w|^2}{2t}} e^{-\frac{|w|^2}{2t}} \, dw \, dz \\
&\leq \frac{1}{2^n} \left(\frac{1}{\pi t}\right)^{2n} e^{-\frac{1}{4ts^2}} \sum_{j=1}^{\infty} \int_{\operatorname{supp}(\varphi_{j,s})} \varphi_{j,s}(z) \int_{\operatorname{supp}(1-\psi_{j,s})} |f(w)| e^{-\frac{|w|^2}{2t}} e^{-\frac{|z-w|^2}{4t}} \, dw \, dz \\
&\leq \frac{1}{2^n} \left(\frac{1}{\pi t}\right)^{2n} e^{-\frac{1}{4ts^2}} \sum_{j=1}^{\infty} \int_{\mathbb{C}^n} |f(w)| e^{-\frac{|w|^2}{2t}} \int_{\mathbb{C}^n} \varphi_{j,s}(z) e^{-\frac{|z-w|^2}{4t}} \, dz \, dw \\
&= \frac{1}{2^n} \left(\frac{1}{\pi t}\right)^{2n} e^{-\frac{1}{4ts^2}} \int_{\mathbb{C}^n} |f(w)| e^{-\frac{|w|^2}{2t}} \int_{\mathbb{C}^n} e^{-\frac{|z-w|^2}{4t}} \, dz \, dw \\
&= 4^n e^{-\frac{1}{4ts^2}} \|f\|_{L_t^1}.
\end{aligned}$$

Therefore,

$$\left\| \sum_{j=1}^{\infty} M_{\varphi_{j,s}} P_t M_{1-\psi_{j,t}} \right\|_{L_t^1 \to L_t^1} \leq 4^n e^{-\frac{1}{4ts^2}} \to 0, \quad s \to 0,$$

i.e. $P_t \in \mathrm{BDO}_t^1$. Since the adjoint of $P_t \in \mathcal{L}(L_t^1)$ is $P_t \in \mathcal{L}(L_t^\infty)$, we obtain $P_t \in \mathrm{BDO}_t^\infty$ from Proposition 5.1.6. Finally, using $1 - \psi_{j,s}(w) \in [0,1]$ for all $w \in \mathbb{C}^n$, we note that for $f \in L_t^\infty$

$$\begin{aligned}
&\left| \sum_{j=1}^{\infty} \varphi_{j,s}(z) P_t((1-\psi_{j,s}) f)(z) \right| e^{-\frac{|z|^2}{2t}} \\
&\leq \left(\frac{1}{\pi t}\right)^n \sum_{j=1}^{\infty} \varphi_{j,s}(z) \int_{\mathbb{C}^n} (1-\psi_{j,s}(w)) |f(w)| \, |e^{\frac{z\cdot\overline{w}}{t}}| e^{-\frac{|w|^2}{2t}} \, dw \, e^{-\frac{|z|^2}{2t}} \\
&\leq \left(\frac{1}{\pi t}\right)^n \sum_{j=1}^{\infty} \varphi_{j,s}(z) \|f\|_{L_t^\infty} \int_{\mathbb{C}^n} e^{-\frac{|z-w|^2}{2t}} \, dw
\end{aligned}$$

$$= 2^n \|f\|_{L_t^\infty} \sum_{j=1}^{\infty} \varphi_{j,s}(z)$$
$$= 2^n \|f\|_{L_t^\infty}$$

and therefore, for every $s > 0$,

$$\left\| \sum_{j=1}^{\infty} M_{\varphi_{j,s}} P_t M_{1-\psi_{j,s}} \right\|_{L_t^\infty \to L_t^\infty} \leq 2^n.$$

Interpolating between the limit cases $p = 1$ and $p = \infty$, we get

$$\left\| \sum_{j=1}^{\infty} M_{\varphi_{j,s}} P_t M_{1-\psi_{j,s}} \right\|_{L_t^p \to L_t^p} \to 0, \quad s \to 0$$

for every $p \in (1, \infty)$. Hence, $P_t \in \mathrm{BDO}_t^p$ for every $p \in [1, \infty]$. □

Given an operator $A \in \mathcal{L}(F_t^p)$, we will denote by $\widehat{A} \in \mathcal{L}(L_t^p)$ the operator

$$\widehat{A} = P_t A P_t + (I - P_t). \tag{5.2}$$

We will occasionally abbreviate $Q_t := I - P_t$. In the following corollary, we explicitly exclude the case of f_t^∞, since $\widehat{A} \in \mathcal{L}(L_t^\infty)$ does not make any sense for $A \in \mathcal{L}(f_t^\infty)$.

Corollary 5.1.8. *Let $t > 0$. If $1 \leq p < \infty$ and $A \in \mathcal{T}^{p,t}$, then $\widehat{A} \in \mathrm{BDO}_t^p$. If $A \in \mathcal{T}^{\infty+,t}$, then $\widehat{A} \in \mathrm{BDO}_t^\infty$.*

Proof. Since $M_f \in \mathrm{BDO}_t^p$ and $P_t \in \mathrm{BDO}_t^p$, we obtain $\widehat{T_f^t} \in \mathrm{BDO}_t^p$ for every $f \in L^\infty(\mathbb{C}^n)$. Then take limits. □

Here is the reason why we are dealing with band-dominated operators:

Proposition 5.1.9. *a) Let $1 \leq p < \infty$ and $A \in \mathrm{BDO}_t^p$ such that $[A, P_t] = 0$. Assume there is a positive constant $M > 0$ satisfying the following:*

For every $s > 0$ there is an integer $j_0(s) > 0$ such that for all $j \geq j_0(s)$ there are operators $C_{j,s} \in \mathcal{L}(L_t^p)$ with

$$\|C_{j,s}\| \leq M$$

and

$$C_{j,s} A M_{\psi_{j,s}} = M_{\psi_{j,s}}.$$

Then, there is an operator $C \in \mathcal{L}(F_t^p)$ with $\|C + \mathcal{K}(F_t^p)\| \leq 2^{6n+1} \|P_t\| M$ such that $C + \mathcal{K}(F_t^p)$ is a left-inverse of $A|_{F_t^p} + \mathcal{K}(F_t^p)$ in $\mathcal{L}(F_t^p)/\mathcal{K}(F_t^p)$, i.e. $A|_{F_t^p} \in \Phi_l(F_t^p)$.

b) Let $1 < p < \infty$ and $A \in \mathrm{BDO}_t^p$ such that $[A, P_t] = 0$. Assume there is a positive constant $M > 0$ satisfying the following:

For every $s > 0$ there is an integer $j_0(s) > 0$ such that for all $j \geq j_0(s)$ there are operators $D_{j,s} \in \mathcal{L}(L_t^p)$ with

$$\|D_{j,s}\| \leq M$$

and

$$M_{\psi_{j,s}} A D_{j,s} = M_{\psi_{j,s}}.$$

Then, there is an operator $D \in \mathcal{L}(F_t^p)$ with $\|D + \mathcal{K}(F_t^p)\| \leq 2^{6n+1}\|P_t\|M$ such that $D + \mathcal{K}(F_t^p)$ is a right-inverse of $A|_{F_t^p} + \mathcal{K}(F_t^p)$ in $\mathcal{L}(F_t^p)/\mathcal{K}(F_t^p)$, i.e. $A|_{F_t^p} \in \Phi_r(F_t^p)$.

Proof. a) For $s > 0$ we define the operator

$$C_s := \sum_{j=j_0(s)}^{\infty} M_{\psi_{j,s}} C_{j,s} M_{\varphi_{j,s}}.$$

Imitating the proof of Lemma 5.0.5, we see that this series converges in the strong operator topology and we obtain $\|C_s\| \leq 2^{6n} M$. Since $\psi_{j,s} \equiv 1$ on $\operatorname{supp}(\varphi_{j,s})$ we get the following identity:

$$C_s A = \sum_{j=j_0(s)}^{\infty} M_{\psi_{j,s}} C_{j,s} A M_{\varphi_{j,s}} M_{\psi_{j,s}} + \sum_{j=j_0(s)}^{\infty} M_{\psi_{j,s}} C_{j,s} [M_{\varphi_{j,s}}, A].$$

Using Proposition 5.1.4 we deduce that $\sum_{j=j_0(s)}^{\infty} M_{\psi_{j,s}} C_{j,s} [M_{\varphi_{j,s}}, A]$ converges to 0 in operator norm as $s \to 0$. For the first term we obtain

$$\begin{aligned} \sum_{j=j_0(s)}^{\infty} M_{\psi_{j,s}} C_{j,s} A M_{\varphi_{j,s}} M_{\psi_{j,s}} &= \sum_{j=j_0(s)}^{\infty} M_{\psi_{j,s}} C_{j,s} A M_{\psi_{j,s}} M_{\varphi_{j,s}} \\ &= \sum_{j=j_0(s)}^{\infty} M_{\psi_{j,s}} M_{\psi_{j,s}} M_{\varphi_{j,s}} \\ &= \sum_{j=j_0(s)}^{\infty} M_{\varphi_{j,s}}. \end{aligned}$$

In particular,

$$\lim_{s \to 0} \left\| C_s A - \sum_{j=j_0(s)}^{\infty} M_{\varphi_{j,s}} \right\| = 0.$$

Since $\sum_{j=1}^{\infty} M_{\varphi_{j,s}} = I$ for every $s > 0$ by construction, we obtain

$$\sum_{j=1}^{\infty} P_t M_{\varphi_{j,s}}|_{F_t^p} = P_t \sum_{j=1}^{\infty} M_{\varphi_{j,s}}|_{F_t^p} = I \in \mathcal{L}(F_t^p).$$

Further,

$$\sum_{j=1}^{j_0(s)-1} P_t M_{\varphi_{j,s}}|_{F_t^p} \in \mathcal{K}(F_t^p)$$

by Proposition 2.3.14. Hence,

$$\sum_{j=j_0(s)}^{\infty} P_t M_{\varphi_{j,s}}|_{F_t^p} \in I + \mathcal{K}(F_t^p)$$

for every $s > 0$. We therefore get, using

$$\left\| P_t C_s A|_{F_t^p} - \sum_{j=j_0(s)}^{\infty} P_t M_{\varphi_{j,s}}|_{F_t^p} \right\| \leq \|P_t\| \left\| C_s A|_{F_t^p} - \sum_{j=j_0(s)}^{\infty} M_{\varphi_{j,s}}|_{F_t^p} \right\| \to 0, \quad s \to 0,$$

that $P_t C_s A|_{F_t^p} + \mathcal{K}(F_t^p) \to I + \mathcal{K}(F_t^p)$ in the Calkin algebra $\mathcal{L}(F_t^p)/\mathcal{K}(F_t^p)$. Let s small enough such that we have

$$\|(I + \mathcal{K}(F_t^p)) - (P_t C_s A|_{F_t^p} + \mathcal{K}(F_t^p))\| < \frac{1}{2}.$$

Then

$$P_t C_s A|_{F_t^p} + \mathcal{K}(F_t^p) = (I + \mathcal{K}(F_t^p)) - ((I + \mathcal{K}(F_t^p) - (P_t C_s A|_{F_t^p} + \mathcal{K}(F_t^p))$$

is invertible with inverse

$$\sum_{k=0}^{\infty}((I + \mathcal{K}(F_t^p)) - (P_t C_s A|_{F_t^p} + \mathcal{K}(F_t^p))^k.$$

Note that, by assumption on A, the range of $A|_{F_t^p}$ is contained in F_t^p. Hence, $P_t C_s A|_{F_t^p} = P_t C_s|_{F_t^p} A|_{F_t^p}$, i.e.

$$P_t C_s A|_{F_t^p} + \mathcal{K}(F_t^p) = (P_t C_s + \mathcal{K}(F_t^p))(A|_{F_t^p} + \mathcal{K}(F_t^p)).$$

Combining these facts, we get

$$\begin{aligned} &I + \mathcal{K}(F_t^p) \\ &= \left[\sum_{k=0}^{\infty}((I + \mathcal{K}(F_t^p)) - (P_t C_s A|_{F_t^p} + \mathcal{K}(F_t^p))^k\right] (P_t C_s A|_{F_t^p} + \mathcal{K}(F_t^p)) \\ &= \left[\sum_{k=0}^{\infty}((I + \mathcal{K}(F_t^p)) - (P_t C_s A|_{F_t^p} + \mathcal{K}(F_t^p))^k\right] (P_t C_s|_{F_t^p} + \mathcal{K}(F_t^p))(A|_{F_t^p} + \mathcal{K}(F_t^p)). \end{aligned}$$

Letting

$$C := \left[\sum_{k=0}^{\infty}((I+\mathcal{K}(F_t^p)) - (P_tC_sA|_{F_t^p} + \mathcal{K}(F_t^p))^k\right](P_tC_s|_{F_t^p} + \mathcal{K}(F_t^p)),$$

we obtain $CA|_{F_t^p} + \mathcal{K}(F_t^p) = I + \mathcal{K}(F_t^p)$ and

$$\|C + \mathcal{K}(F_t^p)\| \leq 2\|P_t\|\|C_s\| \leq 2^{6n+1}\|P_t\|M.$$

b) We set

$$D_s = \sum_{j=j_0(s)}^{\infty} M_{\varphi_{j,s}} D_{j,s} M_{\psi_{j,s}}.$$

Since $A^* \in \mathrm{BDO}_t^q$ (where $1/p + 1/q = 1$) satisfies the characterization in Proposition 5.1.4 2), we can proceed as in part a) to show that

$$\lim_{s\to 0}\left\| D_s^* A^* - \sum_{j=j_0(s)}^{\infty} M_{\varphi_{j,s}}\right\| = 0,$$

which of course yields

$$\lim_{s\to 0}\left\| AD_s - \sum_{j=j_0(s)}^{\infty} M_{\varphi_{j,s}}\right\| = 0.$$

Using $[A, P_t] = 0$, we therefore obtain

$$\lim_{s\to 0}\left\| AP_tD_s|_{F_t^p} - \sum_{j=j_0(s)}^{\infty} P_tM_{\varphi_{j,s}}|_{F_t^p}\right\| = 0$$

and conclude as in a). □

5.2 The Fredholm property for elements in $\mathcal{T}^{p,t}$

For the rest of this chapter, we will abbreviate $\mathcal{M} := \mathcal{M}(\mathrm{BUC}(\mathbb{C}^n))$.

Proposition 5.2.1. *Let (z_γ) be a net converging to $x \in \mathcal{M}\setminus\mathbb{C}^n$ and let $t > 0$. Further, let $f \in L^\infty(\mathbb{C}^n)$ have compact support.*

1) Assume $p \in [1,\infty)$. Let $A \in \mathcal{T}^{p,t}$ be such that A_x is left-invertible with left inverse B. Then, there is γ_0 such that for all $\gamma \geq \gamma_0$ there are operators $C_\gamma \in \mathcal{L}(L_t^p)$ satisfying

$$\|C_\gamma\| \leq 2(\|B\|\|P_t\| + \|Q_t\|)$$

and

$$C_\gamma \hat{A} M_{\alpha_{-z_\gamma}(f)} = M_{\alpha_{-z_\gamma}(f)}.$$

2) *Let $p \in (1, \infty)$ and assume $A \in \mathcal{T}^{p,t}$ is such that A_x is right-invertible with right inverse B. Then, there is a γ_0 such that for every $\gamma \geq \gamma_0$ there are operators $D_\gamma \in \mathcal{L}(F_t^p)$ with*

$$\|D_\gamma\| \leq 2(\|B\|\|P_t\| + \|Q_t\|)$$

and

$$M_{\alpha_{-z_\gamma}(f)} \hat{A} D_\gamma = M_{\alpha_{-z_\gamma}(f)}.$$

To prove this proposition, we need the following well-known fact. For completeness, we give the simple proof.

Lemma 5.2.2. *Let X be a Banach space, $K \in \mathcal{K}(X)$ and $(A_\gamma)_{\gamma \in \Gamma} \subset \mathcal{L}(X)$ a uniformly bounded net of operators converging to $A \in \mathcal{L}(X)$ in strong operator topology. Then, $A_\gamma K \to AK$ in operator norm topology. If further $A'_\gamma \to A' \in \mathcal{L}(X')$ in strong operator topology, then we also have $KA_\gamma \to KA$ in operator norm topology.*

Proof. We have

$$\|(A_\gamma - A)K\| = \sup_{x \in B} \|(A_\gamma - A)Kx\| = \sup_{y \in K(B)} \|(A_\gamma - A)y\| \leq \sup_{y \in \overline{K(B)}} \|(A_\gamma - A)y\|,$$

where B denotes the closed unit ball in X. Since K is compact, $K(B)$ is relatively compact, i.e. $\overline{K(B)}$ is a compact subset of X. Hence, for $\varepsilon > 0$ there are finitely many points $y_1, \ldots, y_n$ such that the open balls $B(y_j, \varepsilon)$ cover $\overline{K(B)}$. Choose γ_0 such that for $\gamma \geq \gamma_0$ we have $\|A_\gamma y_j - Ay_j\| < \varepsilon$ for every $j = 1, \ldots, n$. Then, for every $y \in B(y_j, \varepsilon)$ and $\gamma \geq \gamma_0$:

$$\|A_\gamma y - Ay\| \leq (\sup_\gamma \|A_\gamma\| + \|A\|)\varepsilon + \|A_\gamma y_j - Ay_j\| \leq \left(\sup_\gamma \|A_\gamma\| + 1\right)\varepsilon.$$

It follows that

$$\|(A_\gamma - A)K\| \leq \left(\sup_\gamma \|A_\gamma\| + 1\right)\varepsilon.$$

The second statement follows immediately from the first statement, since

$$\|K(A_\gamma - A)\| = \|(A'_\gamma - A')K'\|$$

and K' is compact. □

Proof of Proposition 5.2.1. 1) Assume A_x is left-invertible with left inverse B. Let $R > 0$ such that $\operatorname{supp}(f) \subset B(0, R)$. Recall that the operators W_z^t are also defined on the space L_t^p and have F_t^p as an invariant subspace. In particular, they commute with Q_t, hence $W_z^t Q_t W_{-z}^t = Q_t$ for every $z \in \mathbb{C}^n$. We obtain

$$\begin{aligned}
&\left\|\left(W_{z_\gamma}^t(AP_t+Q_t)W_{-z_\gamma}^t-(A_xP_t+Q_t)\right)M_{\chi_{B(0,R)}}\right\|\\
&\quad=\left\|\left(W_{z_\gamma}^tAP_tW_{-z_\gamma}^t-A_xP_t\right)M_{\chi_{B(0,R)}}\right\|\\
&\quad=\left\|(W_{z_\gamma}^tAW_{-z_\gamma}^t-A_x)P_tM_{\chi_{B(0,R)}}\right\|\\
&\quad\to 0
\end{aligned}$$

as $z_\gamma \to x$ by the previous lemma, since $P_tM_{\chi_{B(0,R)}}$ is compact by Proposition 2.3.14. Hence, there exists some γ_0 such that

$$\begin{aligned}
R_\gamma :&= (BP_t+Q_t)\left((W_{z_\gamma}^t(AP_t+Q_t)W_{-z_\gamma}^t-(A_xP_t+Q_t)\right)M_{\chi_{B(0,R)}}\\
&=(BP_t+Q_t)W_{z_\gamma}^t(AP_t+Q_t)W_{-z_\gamma}^tM_{\chi_{B(0,R)}}-M_{\chi_{B(0,R)}}
\end{aligned}$$

satisfies $\|R_\gamma\|<\frac{1}{2}$ for $\gamma\geq\gamma_0$. This in turn implies that $I+R_\gamma\in\mathcal{L}(L_t^p)$ is invertible for every $\gamma\geq\gamma_0$. We obtain

$$(BP_t+Q_t)W_{z_\gamma}^t(AP_t+Q_t)W_{-z_\gamma}^tM_f=(I+R_\gamma)M_f$$

and therefore

$$M_f=(I+R_\gamma)^{-1}(BP_t+Q_t)W_{z_\gamma}^t(AP_t+Q_t)W_{-z_\gamma}^tM_f.$$

Using the equality $W_z^tM_fW_{-z}^t=M_{\alpha_z(f)}$, which holds for any $z\in\mathbb{C}^n$, yields

$$W_{-z_\gamma}^t(I+R_\gamma)^{-1}(BP_t+Q_t)W_{z_\gamma}^t(AP_t+Q_t)M_{\alpha_{-z_\gamma}(f)}=M_{\alpha_{-z_\gamma}(f)}.$$

We now set $C_\gamma:=W_{-z_\gamma}^t(I+R_\gamma)^{-1}(BP_t+Q_t)W_{z_\gamma}^t$.

2) Let A_x be right-invertible with right inverse B. Similarly to above, we obtain

$$\begin{aligned}
&\left\|M_{\chi_{B(0,R)}}\left(W_{z_\gamma}^t(AP_t+Q_t)W_{-z_\gamma}^t-(BP_t+Q_t)\right)\right\|\\
&\quad=\left\|M_{\chi_{B(0,R)}}P_t(W_{z_\gamma}^tAW_{-z_\gamma}^t-B)P_t\right\|\\
&\quad\to 0,\quad z_\gamma\to x,
\end{aligned}$$

again by the previous lemma, since $M_{\chi_{B(0,R)}}P_t$ is compact by Proposition 2.3.14 and $(W_{z_\gamma}^tAW_{-z_\gamma}^t)^*\to A_x^*$ in the strong operator topology (here, $p\neq 1$ is important). Then, we set for γ sufficiently large

$$S_\gamma:=M_{\chi_{B(0,R)}}\left(W_{z_\gamma}^t(AP_t+Q_t)W_{-z_\gamma}^t-(A_xP_t+Q_t)\right)(BP_t+Q_t)$$

and

$$D_\gamma:=W_{-z_\gamma}^t(BP_t+Q_t)(I+S_\gamma)^{-1}W_{z_\gamma}^t,$$

which yields

$$M_{\alpha_{-z_\gamma}(f)}(AP_t+Q_t)D_\gamma=M_{\alpha_{-z_\gamma}(f)}.$$

□

In the following theorem, the operators B_x should not be understood as the limit operators of some $B \in \mathcal{L}(F_t^p)$. Instead, B_x simply denotes the left-/right-inverse to each A_x.

Theorem 5.2.3. *Let $t > 0$.*

1) Let $p \in [1, \infty)$. If $A \in \mathcal{T}^{p,t}$ is such that A_x is left-invertible for every $x \in \mathcal{M} \setminus \mathbb{C}^n$ with left inverse B_x and further

$$\sup_{x \in \mathcal{M} \setminus \mathbb{C}^n} \|B_x\| < \infty,$$

then A is left-Atkinson.

2) Let $p \in (1, \infty)$. If $A \in \mathcal{T}^{p,t}$ is such that A_x is right-invertible for every $x \in \mathcal{M} \setminus \mathbb{C}^n$ with right inverse B_x and further

$$\sup_{x \in \mathcal{M} \setminus \mathbb{C}^n} \|B_x\| < \infty,$$

then A is right-Atkinson.

Proof. 1) Assume A is not left-Atkinson. A simple computation verifies $[\hat{A}, P_t] = 0$. Hence, by Proposition 5.1.9, part a), there exists a strictly increasing sequence $(j_m)_{m \in \mathbb{N}}$ and some $s > 0$ such that

$$C\hat{A}M_{\psi_{j_m,s}} \neq M_{\psi_{j_m,s}} \tag{5.3}$$

for every $m \in \mathbb{N}$ and every $C \in \mathcal{L}(L_t^p)$ with

$$\|C\| \leq 2(\sup_{x \in \mathcal{M} \setminus \mathbb{C}^n} \|B_x\| \|P_t\| + \|Q_t\|).$$

By the definition of the $\psi_{j,s}$ and Lemma 5.1.2, we have $\operatorname{diam}(\operatorname{supp}(\psi_{j,s})) \leq \frac{12\sqrt{2n}}{s} =: R$ for every $j \in \mathbb{N}$. Therefore, there exists a sequence $(w_{j_m})_{m \in \mathbb{N}} \subset \mathbb{C}^n$ with $|w_{j_m}| \to \infty$ such that $\operatorname{supp}(\psi_{j_m,s}) \subseteq B(w_{j_m}, R)$. Since $\mathcal{M}$ is compact, there exists a subnet (w_γ) of (w_{j_m}) such that $(-w_\gamma)$ converges to some $y \in \mathcal{M} \setminus \mathbb{C}^n$. Part 1) of Proposition 5.2.1 now implies that there is some γ_0 such that for each $\gamma \geq \gamma_0$ there is an operator $C_\gamma \in \mathcal{L}(L_t^p)$ satisfying

$$\|C_\gamma\| \leq 2(\|B_y\| \|P_t\| + \|Q_t\|)$$

and

$$C_\gamma \widehat{A} M_{\chi_{B(w_\gamma),R)}} = M_{\chi_{B(w_\gamma,R)}},$$

which contradicts Equation (5.3).

2) Follows analogously to 1). □

In the following proposition, for the case $p = \infty+$ the limit operators of $A \in \mathcal{T}^{\infty+,t}$ are defined through their pre-adjoints, i.e. for $x \in \mathcal{M} \setminus \mathbb{C}^n$ we set $A_x := ((A|_{f_t^\infty})_x)^{**}$.

Proposition 5.2.4. *Let* $t > 0$*.*

1) Let $p \in [1, \infty+]$ *and assume* $A \in \mathcal{T}^{p,t}$ *is left-Atkinson such that there is* $B \in \mathcal{T}^{p,t}$ *with* $BA = I + K$ *and* $K \in \mathcal{K} \cap \mathcal{T}^{p,t}$*. Then,* A_x *is left-invertible for every* $x \in \mathcal{M} \setminus \mathbb{C}^n$ *with left-inverse* B_x*, which satisfies* $\|B_x\| \leq \|B + \mathcal{K} \cap \mathcal{T}^{p,t}\|$*.*

2) Let $p \in [1, \infty+]$ *and assume* $A \in \mathcal{T}^{p,t}$ *is right-Atkinson such that there is* $B \in \mathcal{T}^{p,t}$ *with* $AB = I + K$ *with* $K \in \mathcal{K} \cap \mathcal{T}^{p,t}$*. Then,* A_x *is right-invertible for every* $x \in \mathcal{M} \setminus \mathbb{C}^n$ *with right-inverse* B_x*, which satisfies* $\|B_x\| \leq \|B + \mathcal{K} \cap \mathcal{T}^{p,t}\|$*.*

3) Let $p \in [1, \infty+]$ *and assume* $A \in \mathcal{T}^{p,t}$ *is Fredholm. Then,* A_x *is invertible for any* $x \in \mathcal{M} \setminus \mathbb{C}^n$*. If* $B \in \mathcal{T}^{p,t}$ *is a Fredholm regularizer of* A*, then* $A_x^{-1} = B_x$ *for any* $x \in \mathcal{M} \setminus \mathbb{C}^n$ *and* $\|A_x^{-1}\| \leq \|(A + \mathcal{K} \cap \mathcal{T}^{p,t})^{-1}\|$*.*

Proof. Let us assume $p \in [1, \infty]$. For $p = \infty+$, the result follows by considering the pre-adjoints.

1) Since we assume $B \in \mathcal{T}^{p,t}$ and $BA - I \in \mathcal{K} \cap \mathcal{T}^{p,t}$, all the limit operators exist. By Lemma 4.2.5 we have

$$(BA)_x = B_x A_x = I,$$

i.e. B_x is the left-inverse of A_x. On the norm estimate: Observe that $\|B_x\| \leq \|B\|$, as B_x is the limit in strong operator topology of $\alpha_{z_\gamma}(B)$ with (z_γ) an appropriate net and $\|\alpha_{z_\gamma}(B)\| = \|B\|$. Further, for any $K \in \mathcal{K} \cap \mathcal{T}^{p,t}$ we also have $(B + K)A = I + \widetilde{K}$, some $\widetilde{K} \in \mathcal{K} \cap \mathcal{T}^{p,t}$, and the limit operators are the same, i.e. $B_x = (B + K)_x$. In particular, $\|B_x\| \leq \|B + \mathcal{K} \cap \mathcal{T}^{p,t}\|$.

2) Follows analogously to 1).

3) This is now a consequence of 1) and 2). Recall that the property of A being Fredholm is equivalent to the existence of a Fredholm regularizer in $\mathcal{T}^{p,t}$ by Corollary 3.3.19. □

We now summarize the results we obtained for the reflexive cases $p \in (1, \infty)$.

Theorem 5.2.5. *Let* $t > 0$*,* $p \in (1, \infty)$ *and* $A \in \mathcal{T}^{p,t}$*.*

1) (i) If A *is such that* A_x *is left-invertible with left-inverse* B_x *for every* $x \in \mathcal{M} \setminus \mathbb{C}^n$ *and*

$$\sup_{x \in \mathcal{M} \setminus \mathbb{C}^n} \|B_x\| < \infty,$$

then A *is left-Atkinson.*

(ii) If A *is left-invertible in* $\mathcal{T}^{p,t}$ *modulo* $\mathcal{K}$*, then* A_x *is left-invertible for every* $x \in \mathcal{M} \setminus \mathbb{C}^n$ *and the left-inverses* B_x *can be chosen such that*

$$\sup_{x \in \mathcal{M} \setminus \mathbb{C}^n} \|B_x\| < \infty.$$

2) (i) *If A is such that A_x is right-invertible with right-inverse B_x for every $x \in \mathcal{M} \setminus \mathbb{C}^n$ and*

$$\sup_{x \in \mathcal{M} \setminus \mathbb{C}^n} \|B_x\| < \infty,$$

then A is right-Atkinson.

(ii) *If A is right-invertible in $\mathcal{T}^{p,t}$ modulo $\mathcal{K}$, then A_x is right-invertible for every $x \in \mathcal{M} \setminus \mathbb{C}^n$ and the right-inverses B_x can be chosen such that*

$$\sup_{x \in \mathcal{M} \setminus \mathbb{C}^n} \|B_x\| < \infty.$$

3) *A is Fredholm if and only if A_x is invertible for every $x \in \mathcal{M} \setminus \mathbb{C}^n$ and the inverses satisfy*

$$\sup_{x \in \mathcal{M} \setminus \mathbb{C}^n} \|A_x^{-1}\| < \infty.$$

Proof. Everything follows from the statements presented before in this chapter. Note that $A \in \mathcal{T}^{p,t}$ is Fredholm if and only if it is invertible in $\mathcal{T}^{p,t}$ modulo $\mathcal{K}$ by Corollary 3.3.17. □

For the characterization of the one-sided invertibility modulo compact operators, there is still a small gap: Is $A \in \mathcal{T}^{p,t}$ left-/right-Atkinson if and only if is left-/right-invertible in $\mathcal{T}^{p,t}$ modulo $\mathcal{K}$? Since C^* algebras are closed under one-sided invertibility (see e.g. [54, Lemma 1.1]), the statement is true at least for $p = 2$. As the Toeplitz algebras $\mathcal{T}^{p,t}$ in general behave very much like C^* algebras, we would not be surprised if this would turn out to be true in general. Further, note that in the case of $p = 2$, the notions of an operator being left Atkinson and lower semi-Fredholm (respectively right Atkinson and upper semi-Fredholm) agree: the property of $\ker(A)$ (respectively $\operatorname{ran}(A)$) being complemented in Proposition 5.0.2, part 3) (part 4) respectively) is trivially satisfied in a Hilbert space. Therefore, we arrive at the following:

Proposition 5.2.6. *Let $A \in \mathcal{T}^{2,t}$.*

1) *A is left Atkinson if and only if A is lower semi-Fredholm if and only if A_x is left-invertible for every $x \in \mathcal{M} \setminus \mathbb{C}^n$.*

2) *A is right Atkinson if and only if A is upper semi-Fredholm if and only if A_x is right-invertible for every $x \in \mathcal{M} \setminus \mathbb{C}^n$.*

As we have already mentioned at the beginning of this chapter, our work in [73] was closely inspired by R. Hagger's paper [80], which in turn took its motivation from techniques which are well-established in the study of band-dominated operators on sequence spaces (see e.g. [44] for an introduction). Fredholm characterizations for such operators (similar to the results we have presented above) are well-known. For a long time, it has been an open problem if it is necessary to assume that the limit operators are uniformly invertible (i.e. if their inverses are uniformly bounded) or if this automatically follows if every limit operator is invertible. This problem has been resolved in [97], where one can also find a good historical overview on that issue. In the remaining part of this section, we will prove that the condition $\sup_{x\in\mathcal{M}\setminus\mathbb{C}^n}\|A_x^{-1}\|<\infty$ is actually redundant in the characterization of $A\in\mathcal{T}^{p,t}$ being Fredholm (and similarly in the sufficient and necessary criteria for A being left- or right-Atkinson). We will of course do this by following our presentation in [73], which in turn follows [80] closely (and borrows some details from there, which were omitted in [73]). Both presentations in [73, 80] have in common that they are closely inspired by Lindner's and Seidel's proof from [97], which we want to emphasize.

For $A\in\mathcal{L}(X)$, where X is any Banach space, we define the *lower bound* of A by

$$\nu(A) := \inf_{x\in X;\ \|x\|=1}\|Ax\|$$

and say that A is *bounded below* if $\nu(A)>0$. As is well-known [96, Lemma 2.32], A is bounded below if and only if it is injective with closed range.

Lemma 5.2.7. *Let X be a Banach space and $A\in\mathcal{L}(X)$ left-invertible. Then, the left inverse B of A satisfies $\|B\|=\frac{1}{\nu(A)}$.*

Proof. As in the proof of [96, Lemma 2.33] one sees that A is left-invertible if and only if it is bounded below and $\operatorname{ran}(A)$ is complemented. Further, a subspace complementing the range is given by $\ker(B)$: $X=\operatorname{ran}(A)\oplus\ker(B)$. Thus,

$$\begin{aligned}\|B\| &= \sup_{x\in X;\ \|x\|=1}\|Bx\| \\ &= \sup_{\substack{y+z\in\operatorname{ran}(A)\oplus\ker(B)\\ \|y+z\|=1}}\|B(y+z)\| \\ &= \sup_{\substack{y+z\in\operatorname{ran}(A)\oplus\ker(B)\\ \|y+z\|=1}}\|B(y)\| \\ &= \sup_{\substack{y\in\operatorname{ran}(A)\\ \|y\|\le 1}}\|By\| \\ &= \|B|_{\operatorname{ran}(A)}\|.\end{aligned}$$

Since A is injective, $A: X\to\operatorname{ran}(A)$ is bijective, hence invertible. Further, $B|_{\operatorname{ran}(A)}$ is a left-inverse, hence an inverse of $A\in\mathcal{L}(X,\operatorname{ran}(A))$. Now, one proves $\|B|_{\operatorname{ran}(A)}\|=\nu(A)^{-1}$ as in [96, Lemma 2.35]. □

We assume for the moment the following fact. The proof will be given in the following subsection.

Lemma 5.2.8. *Let $t>0$ and $p\in[1,\infty)$. Then, for each $A\in\mathcal{T}^{p,t}$ there exists $y\in\mathcal{M}\setminus\mathbb{C}^n$ such that*

$$\nu(\widehat{A_y}) = \inf\{\nu(\widehat{A_x});\ x\in\mathcal{M}\setminus\mathbb{C}^n\}.$$

Theorem 5.2.9. *Let $t > 0$, $p \in (1, \infty)$ and $A \in \mathcal{T}^{p,t}$.*

1) (i) If A is such that A_x is left-invertible for every $x \in \mathcal{M} \setminus \mathbb{C}^n$, then A is left-Atkinson.

(ii) If A is left-invertible in $\mathcal{T}^{p,t}$ modulo $\mathcal{K}$, then A_x is left-invertible for every $x \in \mathcal{M} \setminus \mathbb{C}^n$.

2) (i) If A is such that A_x is right-invertible for every $x \in \mathcal{M} \setminus \mathbb{C}^n$, then A is right-Atkinson.

(ii) If A is right-invertible in $\mathcal{T}^{p,t}$ modulo $\mathcal{K}$, then A_x is right-invertible for every $x \in \mathcal{M} \setminus \mathbb{C}^n$.

3) A is Fredholm if and only if A_x is invertible for every $x \in \mathcal{M} \setminus \mathbb{C}^n$.

Proof. 1) (i) Let A_x be left-invertible for every $x \in \mathcal{M} \setminus \mathbb{C}^n$ with left-inverse B_x. Then, $\widehat{A_x}$ is also left-invertible for every x with left-inverse $\widehat{B_x} = P_t B_x P_t + (I - P_t)$. By Lemmas 5.2.7 and 5.2.8,

$$\sup_{x \in \mathcal{M} \setminus \mathbb{C}^n} \|\widehat{B_x}\| = \sup_{x \in \mathcal{M} \setminus \mathbb{C}^n} \frac{1}{\nu(\widehat{A_x})} = \frac{1}{\inf_{x \in \mathcal{M} \setminus \mathbb{C}^n} \nu(\widehat{A_x})} = \frac{1}{\nu(\widehat{A_y})} < \infty.$$

Since $\widehat{B_x}|_{F_t^p} = B_x$, we obtain $\|B_x\| \le \|\widehat{B_x}\|$ and therefore also

$$\sup_{x \in \mathcal{M} \setminus \mathbb{C}^n} \|B_x\| < \infty,$$

i.e. the assumptions of Theorem 5.2.5 1)(i) are satisfied.

1) (ii) This follows from 5.2.5 1)(ii).

2) follows from 1) after considering the adjoint operator.

3) is an immediate consequence of 1) and 2). □

Corollary 5.2.10. *Let $t > 0$, $p \in (1, \infty)$ and $A \in \mathcal{T}^{p,t}$. Then,*

$$\sigma_{ess}(A) = \bigcup_{x \in \mathcal{M} \setminus \mathbb{C}^n} \sigma(A_x).$$

Proof. This follows immediately from the previous theorem: $A - \lambda$ is Fredholm if and only if $A_x - \lambda$ is invertible for every $x \in \mathcal{M} \setminus \mathbb{C}^n$ if and only if λ is in none of the spectra of the A_x. □

Proof of Lemma 5.2.8

We start by introducing the following notation: We will denote

$$r_s := \operatorname{diam}(\operatorname{supp}(\varphi_{j,s})) = \frac{8\sqrt{2n}}{s}.$$

For $F \subseteq \mathbb{C}^n$ and $A \in \mathcal{L}(L_t^p)$ set

$$\nu(A|_F) := \inf\{\|Af\|_{L_t^p};\ f \in L_t^p,\ \|f\|_{L_t^p} = 1,\ \operatorname{supp}(f) \subseteq F\}$$

and

$$\nu_s(A|_F) := \inf_{w \in \mathbb{C}^n} \nu(A|_{F \cap B(w,r_s)}).$$

Note that $\nu(A) = \nu(A|_{\mathbb{C}^n})$. We will also write $\nu_s(A) := \nu_s(A|_{\mathbb{C}^n})$.

Lemma 5.2.11. *Let $A, B \in \mathcal{L}(L_t^p)$ and $F \subseteq \mathbb{C}^n$. Then:*

1) $|\nu(A|_F) - \nu(B|_F)| \le \|(A-B)M_{\chi_F}\| \le \|A-B\|$;

2) $|\nu_s(A|_F) - \nu_s(B|_F)| \le \|A-B\|$.

Proof. 1) Let $\varepsilon > 0$ and pick $f \in L_t^p$ such that $\|f\| = 1$, $\operatorname{supp}(f) \subseteq F$ and $\|Bf\| \le \nu(B|_F) + \varepsilon$. Then, using the obvious estimate $\nu(A|_F) \le \|Af\|$:

$$\begin{aligned}\nu(A|_F) - \nu(B|_F) - \varepsilon &\le \nu(A|_F) - \|Bf\| \le \|Af\| - \|Bf\| \\ &\le \|(A-B)f\| \le \|(A-B)M_{\chi_F}\| \\ &\le \|A-B\|.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this gives $\nu(A|_F) - \nu(B|_F) \le \|(A-B)M_{\chi_F}\|$. The other estimate $\nu(B|_F) - \nu(A|_F) \le \|(B-A)M_{\chi_F}\|$ follows by symmetry.

2) Let again $\varepsilon > 0$ and $w \in \mathbb{C}^n$ such that $\nu(B|_{F \cap B(w,r_s)}) \le \nu_s(B|_F) + \varepsilon$. Then, also using part 1) of the lemma:

$$\begin{aligned}\nu_s(A|_F) - \nu_s(B|_F) - \varepsilon &\le \nu_s(A|_F) - \nu(B|_{F \cap B(w,r_s)}) \\ &\le \nu(A|_{F \cap B(w,r_s)}) - \nu(B|_{F \cap B(w,r_s)}) \\ &\le \|A-B\|.\end{aligned}$$

Again, the other estimate follows by symmetry. □

Lemma 5.2.12. *Let $t > 0$, $p \in [1,\infty)$ and $A \in \mathcal{T}^{p,t}$. For every $\varepsilon > 0$ there is some $s > 0$ such that for every $F \subseteq \mathbb{C}^n$ and every $B \in \{\widehat{A}\} \cup \{\widehat{A_x};\ x \in \mathcal{M} \setminus \mathbb{C}^n\}$ we have*

$$\nu(B|_F) \le \nu_s(B|_F) \le \nu(B|_F) + \varepsilon.$$

Proof. The first inequality is immediate from the definition, we only need to prove the second. By Corollary 5.1.8 we can choose a sequence of band operators $A_m \in \mathcal{L}(L_t^p)$ converging to $\widehat{A}$ as $m \to \infty$. Fix $\varepsilon > 0$ and let $m \in \mathbb{N}$ such that $\|\widehat{A} - A_m\| < \frac{\varepsilon}{4}$. Further, let $x \in \mathcal{M} \setminus \mathbb{C}^n$ and consider a net $(z_\gamma) \subset \mathbb{C}^n$ converging to x. Then, $(W_{z_\gamma}^t A_m W_{-z_\gamma}^t)_\gamma$ is a uniformly bounded net of operators in $\mathcal{L}(L_t^p)$, therefore it has a weakly convergent subnet, which we will also denote by $(W_{z_\gamma}^t A_m W_{-z_\gamma}^t)_\gamma$, and we will denote its limit by $(A_m)_x$. Recall that $\alpha_{z_\gamma}(A)$ converges strongly to A_x, and hence $W_{z_\gamma}^t \widehat{A} W_{-z_\gamma}^t$ converges strongly to $\widehat{A_x}$. In particular, $W_{z_\gamma}^t(\widehat{A} - A_m)W_{-z_\gamma}^t$ converges weakly to $\widehat{A_x} - (A_m)_x$. This implies

$$\|\widehat{A_x} - (A_m)_x\| \leq \sup_\gamma \|W_{z_\gamma}^t(\widehat{A} - A_m)W_{-z_\gamma}^t\| = \|\widehat{A} - A_m\| \leq \frac{\varepsilon}{4}.$$

Let $f, g \in L^\infty(\mathbb{C}^n)$ be such that $\operatorname{dist}(\operatorname{supp}(f), \operatorname{supp}(g)) > \omega(A_m)$. The identity

$$\operatorname{dist}(\operatorname{supp}(f), \operatorname{supp}(g)) = \operatorname{dist}(\operatorname{supp}(\alpha_{-z_\gamma}(f)), \operatorname{supp}(\alpha_{-z_\gamma}(g)))$$

implies

$$M_f W_{z_\gamma}^t A_m W_{-z_\gamma}^t M_g = W_{z_\gamma}^t M_{\alpha_{-z_\gamma}(f)} A_m M_{\alpha_{-z_\gamma}(g)} W_{-z_\gamma}^t = 0.$$

Using this, we obtain

$$\omega(W_{z_\gamma}^t A_m W_{-z_\gamma}^t) \leq \omega(A_m)$$

and therefore, by separate continuity of the weak operator topology,

$$M_f (A_m)_x M_g = 0.$$

After passing to the limit we receive

$$\omega((A_m)_x) \leq \omega(A_m).$$

Assume for the moment that there is some $s \in (0, 1)$ such that for all $F \subseteq \mathbb{C}^n$ and every $B \in \{A_m\} \cup \{(A_m)_x;\ x \in \mathcal{M} \setminus \mathbb{C}^n\}$ we have

$$\nu_s(B|_F) \leq \nu(B|_F) + \frac{\varepsilon}{2}.$$

Then, by the previous lemma,

$$\begin{aligned}
|\nu(\widehat{A}|_F) - \nu(A_m|_F)| &\leq \|\widehat{A} - A_m\| < \frac{\varepsilon}{4}, \\
|\nu(\widehat{A_x}|_F) - \nu((A_m)_x|_F)| &\leq \|A_x - (A_m)_x\| < \frac{\varepsilon}{4}, \\
|\nu_s(\widehat{A_x}|_F) - \nu_s((A_m)_x|_F)| &\leq \|A_x - (A_m)_x\| < \frac{\varepsilon}{4}, \\
|\nu_s(\widehat{A}|_F) - \nu_s(A_m|_F)| &\leq \|\widehat{A} - A_m\| < \frac{\varepsilon}{4}.
\end{aligned}$$

Combining all these estimates we obtain

$$\nu_s(B|_F) \leq \nu(B|_F) + \varepsilon$$

for every $B \in \{\widehat{A}\} \cup \{\widehat{A_x};\ x \in \mathcal{M} \setminus \mathbb{C}^n\}$.

It remains to prove the existence of the above mentioned s. This will be done as in the proof of [80, Proposition 23].

Recall that by Equation (5.1) we have

$$\operatorname{dist}(\operatorname{supp}(\varphi_{j,s}), \operatorname{supp}(1-\psi_{j,s})) \geq \frac{1}{s}.$$

Choose $s \in (0,1)$ such that $\frac{1}{s} > \omega$. Then, for any $f \in L_t^p$ satisfying $\|f\| = 1$ and $\operatorname{supp}(f) \subseteq F$ we can estimate

$$\begin{aligned}
\left(\sum_{j=1}^{\infty} \left\|BM_{\varphi_{j,s}^{1/p}} f\right\|_{L_t^p}^p\right)^{1/p} &= \left(\sum_{j=1}^{\infty} \left\|BM_{\varphi_{j,s}^{1/p}} M_{\psi_{j,s}} f\right\|_{L_t^p}^p\right)^{1/p} \\
&\leq \left(\sum_{j=1}^{\infty} \left\|M_{\varphi_{j,s}^{1/p}} Bf\right\|_{L_t^p}^p\right)^{1/p} + \left(\sum_{j=1}^{\infty} \left\|M_{\varphi_{j,s}^{1/p}} BM_{1-\psi_{j,s}} f\right\|_{L_t^p}^p\right)^{1/p} + \\
&\quad + \left(\sum_{j=1}^{\infty} \left\|[B, M_{\varphi_{j,s}^{1/p}}] M_{\psi_{j,s}} f\right\|_{L_t^p}^p\right)^{1/p} \\
&= \left(\sum_{j=1}^{\infty} \left\|M_{\varphi_{j,s}^{1/p}} Bf\right\|_{L_t^p}^p\right)^{1/p} + \left(\sum_{j=1}^{\infty} \left\|[B, M_{\varphi_{j,s}^{1/p}}] M_{\psi_{j,s}} f\right\|_{L_t^p}^p\right)^{1/p},
\end{aligned}$$

where we used that B is a band operator with band-width $\leq \omega$. Since $\sum_{j=1}^{\infty} |\varphi_{j,s}(z)| = 1$ for every $z \in \mathbb{C}^n$ and $s > 0$, the first sum equals $\|Bf\|$. For the second sum, observe that

$$\begin{aligned}
\left(\sum_{j=1}^{\infty} \left\|[B, M_{\varphi_{j,s}^{1/p}}] M_{\psi_{j,s}} f\right\|_{L_t^p}^p\right)^{1/p} &\leq \sup_{j \in \mathbb{N}} \|[B, M_{\varphi_{j,s}^{1/p}}]\| \left(\sum_{j=1}^{\infty} \|M_{\psi_{j,s}} f\|_{L_t^p}^p\right)^{1/p} \\
&\leq (4^{2n})^{1/p} \sup_{j \in \mathbb{N}} \|[B, M_{\varphi_{j,s}^{1/p}}]\|.
\end{aligned}$$

Similarly to the proof of Proposition 5.1.4, the functions $\varphi_{j,s}^{1/p}$ satisfy the assumptions of Lemma 5.1.5. Therefore, given $\delta > 0$, by that lemma we can choose s small enough and independently of B (depending only on ω) such that $\sup_{j \in \mathbb{N}} \|[B, M_{\varphi_{j,s}^{1/p}}]\| \leq \frac{\delta}{(4^{2n})^{1/p}} \|B\|$. This yields

$$\left(\sum_{j=1}^{\infty} \left\|[B, M_{\varphi_{j,s}^{1/p}}] M_{\psi_{j,s}} f\right\|_{L_t^p}^p\right)^{1/p} \leq \delta \|B\|.$$

Since $\|B\| \leq \|\hat{A}\| + \frac{\varepsilon}{4}$ for any $B \in \{A_m\} \cup \{(A_m)_x;\ x \in \mathcal{M} \setminus \mathbb{C}^n\}$, choose δ such that $\delta\|B\| \leq \frac{\varepsilon}{4}$ for all such B. Choosing now f such that $\|f\|_{L_t^p} = 1$, $\operatorname{supp}(f) \subseteq F$ and $\|Bf\|_{L_t^p} \leq \nu(B|_F) + \frac{\varepsilon}{4}$, we can set the pieces together and obtain

$$\left(\sum_{j=1}^{\infty} \left\|BM_{\varphi_{j,s}^{1/p}} f\right\|_{L_t^p}^p\right)^{1/p} \leq \|Bf\|_{L_t^p} + \frac{\varepsilon}{4}$$

$$\leq \nu(B|_F) + \frac{\varepsilon}{2}$$
$$= \left(\nu(B|_F) + \frac{\varepsilon}{2}\right)\left(\sum_{j=1}^{\infty}\left\|M_{\varphi_{j,s}^{1/p}} f\right\|_{L_t^p}^p\right)^{1/p}.$$

Hence, for all $s > 0$ small enough there is some $j \in \mathbb{N}$ satisfying

$$\left\|BM_{\varphi_{j,s}^{1/p}} f\right\|_{L_t^p} \leq \left(\nu(B|_F) + \frac{\varepsilon}{2}\right)\left\|M_{\varphi_{j,s}^{1/p}} f\right\|_{L_t^p}.$$

Now, having $\operatorname{supp}(M_{\varphi_{j,s}^{1/p}} f) \subseteq \operatorname{supp}(\varphi_{j,s}) \subseteq B(w, r_s)$ for $w \in \mathbb{C}^n$ appropriately, we get

$$\nu_s(B|_F) \leq \nu(B|_F) + \frac{\varepsilon}{2}$$

for all $B \in \{A_m\} \cup \{(A_m)_x;\ x \in \mathcal{M} \setminus \mathbb{C}^n\}$. Since s was chosen independently of F and B (it depends only on the band width of A_m), the statement follows. □

Lemma 5.2.13. *For $t > 0$ and $p \in (1, \infty)$ let $A \in \mathcal{T}^{p,t}$, $w \in \mathbb{C}^n$ and $r > 0$. For any $f \in L_t^p$ satisfying $\operatorname{supp}(f) \subseteq B(w, r)$ and any $x \in \mathcal{M} \setminus \mathbb{C}^n$ there exist $g \in L_t^p$ and $y \in \mathcal{M} \setminus \mathbb{C}^n$ such that $\|g\| = \|f\|$, $\operatorname{supp}(g) \subseteq B(0, r)$ and $\|\widehat{A}_x f\| = \|\widehat{A}_y g\|$. Further, they satisfy $\nu(\widehat{A}_y|_{B(0,r+|w|)}) \leq \nu(\widehat{A}_x|_{B(0,r)})$.*

Proof. As already mentioned earlier, the α-invariance of $\mathrm{BUC}(\mathbb{C}^n)$ induces a natural action of $\mathbb{C}^n$ on $\mathcal{M}$, which leaves $\mathcal{M} \setminus \mathbb{C}^n$ invariant (and on $\mathbb{C}^n$ is simply $\alpha_w(z) = z - w$). For this action, it is easily verified that

$$A_{\alpha_w(x)} = \alpha_w(A_x).$$

Hence, let $y = \alpha_{-w}(x)$, i.e. if z_γ is a net in $\mathbb{C}^n$ converging to x, then (possibly after passing to a subnet) $z_\gamma + w$ converges to y. Since the Weyl operators also commute with the projection, one immediately obtains that $W_{-w}^t \widehat{A}_x W_w^t = \widehat{A}_y$. Consider now $f \in L_t^p$ such that $\operatorname{supp}(f) \subseteq B(w, r)$. Let $g = W_{-w}^t f$. Then, we have $\operatorname{supp}(g) \subseteq B(0, r)$ and, since the Weyl operators are isometric,

$$\|\widehat{A}_y g\|_{L_t^p} = \|W_{-w}^t \widehat{A}_x W_w^t g\|_{L_t^p} = \|W_{-w}^t \widehat{A}_x f\|_{L_t^p} = \|\widehat{A}_x f\|_{L_t^p},$$

which proves the first claim. Let $h \in L_t^p$ such that $\operatorname{supp}(h) \subseteq B(0, r)$. Then,

$$\operatorname{supp}(W_{-w}^t h) \subseteq B(-w, r) \subseteq B(0, r + |w|)$$

and, as above, $\|\widehat{A}_x h\|_{L_t^p} = \|\widehat{A}_y W_{-w}^t h\|_{L_t^p}$. □

Proof of Lemma 5.2.8. By Lemma 5.2.12 there is a sequence $(s_k)_{k \in \mathbb{N}} \in (0, \infty)$ converging to 0 such that

$$\nu_{s_k}(B|_F) \leq \nu(B|_F) + \frac{1}{2^{k+1}}$$

for any $k \in \mathbb{N}$, $F \subseteq \mathbb{C}^n$ and $B \in \{\widehat{A}\} \cup \{\widehat{A}_x;\ x \in \mathcal{M} \setminus \mathbb{C}^n\}$. Recall that $r_s = \frac{8\sqrt{2n}}{s} = \operatorname{diam}(\operatorname{supp}(\varphi_{j,s}))$ as defined earlier. Possibly after passing to a subsequence, we might further assume that $r_{s_{k+1}} > 2r_{s_k}$ for any $k \in \mathbb{N}$.

Let further $(x_j)_{j\in\mathbb{N}} \subset \mathcal{M} \setminus \mathbb{C}^n$ be a sequence such that

$$\nu(\widehat{A}_{x_j}) \to \inf\{\nu(\widehat{A}_x);\ x \in \mathcal{M} \setminus \mathbb{C}^n\}, \quad j \to \infty.$$

We claim that there exists a sequence $(y_j)_{j\in\mathbb{N}} \subset \mathcal{M} \setminus \mathbb{C}^n$ such that for any $l \in \mathbb{N}$ we have

$$\nu\left(\widehat{A}_{y_j}|_{B(0,4r_{s_l})}\right) \leq \nu(\widehat{A}_{x_j}) + \frac{1}{2^{l-1}}$$

whenever j is sufficiently large. Let us defer the construction of this sequence for a moment and finish the proof under the assumption that it exists. Since $x \mapsto A_x$, $\mathcal{M} \to \mathcal{T}^{p,t}$ is continuous in the strong operator topology and $\mathcal{M} \setminus \mathbb{C}^n$ is a compact subset of $\mathcal{M}$, $\{A_x;\ x \in \mathcal{M} \setminus \mathbb{C}^n\}$ is compact with respect to the strong operator topology. Hence, there exists a subnet $(A_{y_{j_\gamma}})_\gamma$ of $(A_{y_j})_j$ which strongly converges to A_y for some $y \in \mathcal{M} \setminus \mathbb{C}^n$. Then, for any $k \in \mathbb{N}$ we have

$$\left\|\left(\widehat{A}_{y_{j_\gamma}} - \widehat{A}_y\right) M_{\chi_{B(0,4r_{s_l})}}\right\| = \left\|\left(A_{y_{j_\gamma}} - A_y\right) P_t M_{\chi_{B(0,4r_{s_l})}}\right\| \to 0, \quad y_{j_\gamma} \to y$$

by Lemma 5.2.2, since $P_t M_{\chi_{B(0,4r_{s_l})}}$ is compact by Proposition 2.3.14. This in turn implies

$$\nu\left(\widehat{A}_{y_{j_\gamma}}|_{B(0,4r_{s_l})}\right) \to \nu\left(\widehat{A}_y|_{B(0,4r_{s_l})}\right)$$

by Lemma 5.2.11. Thus,

$$\begin{aligned}
\nu(\widehat{A}_y) &\leq \nu\left(\widehat{A}_y|_{B(0,4r_{s_l})}\right) \\
&= \lim_\gamma \nu\left(\widehat{A}_{y_{j_\gamma}}|_{B(0,4r_{s_l})}\right) \\
&\leq \lim_\gamma \nu(\widehat{A}_{x_{j_\gamma}}) + \frac{1}{2^{l-1}} \\
&= \lim_{j\to\infty} \nu(\widehat{A}_{x_j}) + \frac{1}{2^{l-1}} \\
&= \inf\{\nu(\widehat{A}_x);\ x \in \mathcal{M} \setminus \mathbb{C}^n\} + \frac{1}{2^{l-1}}.
\end{aligned}$$

Passing to the limit $l \to \infty$ proves the statement.

We now show the existence of the sequence (y_j). For the moment fix j. By the definition of ν_s, there is some $f_j^0 \in L_t^p$ with $\|f_j^0\| = 1$ such that $\operatorname{supp}(f_j^0) \subseteq B(w_j^0, r_{s_j})$ for some $w_j^0 \in \mathbb{C}^n$ and

$$\|\widehat{A}_{x_j} f_j^0\|_{L_t^p} \leq \nu_{s_j}(\widehat{A}_{x_j}) + \frac{1}{2^{j+1}} \leq \nu(\widehat{A}_{x_j}) + \frac{1}{2^j}.$$

Applying Lemma 5.2.13, there exist $y_j^0 \in \mathcal{M} \setminus \mathbb{C}^n$ and $g_j^0 \in L_t^p$ with $\|g_j^0\| = 1$ and $\operatorname{supp}(g_j^0) \subseteq B(0, r_{s_j})$ such that

$$\left\|\widehat{A}_{y_j^0} g_j^0\right\|_{L_t^p} = \left\|\widehat{A}_{x_j} f_j^0\right\|_{L_t^p} \leq \nu(\widehat{A}_{x_j}) + \frac{1}{2^j}.$$

For $k = 1, \dots, j$ we will now inductively define elements $y_j^k \in \mathcal{M} \setminus \mathbb{C}^n$ and f_j^k, $g_j^k \in L_t^p$ satisfying the following: Given y_j^{k-1}, there exists some $f \in L_t^p$, $\|f\|_{L_t^p} = 1$ with $\operatorname{supp}(f) \subseteq B(w_j^k, r_{s_{j-k}}) \cap B(0, r_{s_{j-k+1}})$ for some $w_j^k \in \mathbb{C}^n$ such that

$$\|\widehat{A}_{y_j^{k-1}} f_j^k\|_{L_t^p} \leq \nu_{s_{n-k}} \left(\widehat{A}_{y_j^{k-1}}|_{B(0, r_{s_{n-k+1}})}\right) + \frac{1}{2^{n-k+1}}.$$

Note that we necessarily have $|w_j^k| \leq r_{s_{j-k}} + r_{s_{j-k+1}}$, since otherwise we would have $B(w, r_{s_{j-k}}) \cap B(0, r_{s_{j-k+1}}) = \emptyset$. Applying Lemma 5.2.13 yields $y_j^k \in \mathcal{M} \setminus \mathbb{C}^n$ and $g_j^k \in L_t^p$, $\|g_j^k\| = 1$ with $\operatorname{supp}(g) \subseteq B(0, r_{s_{j-k}})$ satisfying

$$\left\|\widehat{A}_{y_j^k} g_j^k\right\|_{L_t^p} = \left\|\widehat{A}_{y_j^{k-1}} f_j^k\right\|_{L_t^p} \leq \nu\left(\widehat{A}_{y_j^{k-1}}|_{B(0, r_{s_{j-k+1}})}\right) + \frac{1}{2^{j-k}}.$$

From this, we obtain

$$\begin{aligned}
\nu\left(\widehat{A}_{y_j^k}|_{B(0, r_{s_{j-k}})}\right) &\leq \left\|\widehat{A}_{y_j^k} g_j^k\right\|_{L_t^p} \\
&\leq \nu(\widehat{A}_{y_j^{k-1}}|_{B(0, r_{s_{j-k+1}})}) + \frac{1}{2^{j-k}} \\
&\leq \left\|\widehat{A}_{y_j^{k-1}} g_j^{k-1}\right\|_{L_t^p} + \frac{1}{2^{j-k}} \\
&\leq \dots \\
&\leq \left\|\widehat{A}_{y_j^0} g_j^0\right\|_{L_t^p} + \frac{1}{2^{j-k}} + \dots + \frac{1}{2^{j-1}} \\
&\leq \nu(\widehat{A}_{x_j}) + \frac{1}{2^{j-k}} + \dots + \frac{1}{2^j} \\
&\leq \nu(\widehat{A}_{x_j}) + \frac{1}{2^{j-k-1}}.
\end{aligned}$$

Fix now $l \in \mathbb{N}$ and let $j > l$. Note that the assumption $r_{s_{l+1}} > 2r_{s_l}$ for all $l \in \mathbb{N}_0$ easily yields

$$r_{s_0} + 2r_{s_1} + \dots + 2r_{s_l} < 4r_{s_l}$$

for any $l \in \mathbb{N}$. In particular, we obtain

$$\nu(\widehat{A}_{y_j^j}|_{B(0, 4r_{s_l})}) \leq \nu(\widehat{A}_{y_j^j}|_{B(0, r_{s_0} + 2r_{s_1} + \dots + 2r_{s_l})}).$$

Further, using the second statement from Lemma 5.2.13 inductively, we have

$$\nu(\widehat{A}_{y_j^{j-l}}|_{B(0, r_{s_l})}) \geq \nu(\widehat{A}_{y_j^{j-l+1}}|_{B(0, r_{s_l} + |w_j^{j-l+1}|)})$$

$$\begin{aligned}
&\geq \nu(\widehat{A}_{y_j^{j-l+1}}|_{B(0,2r_{s_l}+r_{s_{l-1}})}) \\
&\geq \nu(\widehat{A}_{y_j^{j-l+2}}|_{B(0,2r_{s_l}+2r_{s_{l-1}}+r_{s_{l-2}})}) \\
&\geq \dots \\
&\geq \nu(\widehat{A}_{y_j^j}|_{B(0,2r_{s_l}+2r_{s_{l-1}}+\dots+2r_{s_1}+r_{s_0})}) \\
&\geq \nu(\widehat{A}_{y_j^j}|_{B(0,4r_{s_l})}).
\end{aligned}$$

Letting now $y_j := y_j^j$ and combining the above estimates we obtain for any $j \geq l$:

$$\nu(\widehat{A}_{y_j}|_{B(0,4r_{s_l})}) \leq \nu(\widehat{A}_{x_j}) + \frac{1}{2^{l-1}}. \qquad \square$$

5.3 Essential norm estimates

Methods similar to those presented above allow us to derive certain estimates for the essential norm. The result will be the following:

Theorem 5.3.1. *Let $t > 0$, $p \in (1,\infty)$ and $A \in \mathcal{T}^{p,t}$. Then, the following holds true:*

$$\frac{1}{\|P_t\|}\|A + \mathcal{K}(F_t^p)\| \leq \sup_{x \in \mathcal{M}\setminus\mathbb{C}^n} \|A_x\| \leq \|A + \mathcal{K}(F_t^p)\|.$$

For $p = 2$ the above supremum is attained by some limit operator and we have

$$\|A + \mathcal{K}(F_t^2)\| = \max_{x \in \mathcal{M}\setminus\mathbb{C}^n} \|A_x\|.$$

For $A \in \mathcal{L}(F_t^p)$ and $s > 0$, $F \subseteq \mathbb{C}^n$ we will denote

$$\begin{aligned}
\|AP_t|_F\| &:= \sup\{\|AP_tf\|_{L_t^p};\ f \in L_t^p,\ \|f\|_{L_t^p} = 1,\ \operatorname{supp}(f) \subseteq F\}, \\
\|AP_t|_F\|_s &:= \sup_{w \in \mathbb{C}^n} \|AP_t|_{F \cap B(w,r_s)}\|.
\end{aligned}$$

Observe that

$$\|AP_t|_F\| = \|AP_tM_{\chi_F}\|.$$

Lemma 5.3.2. *Let $t > 0$ and $p \in (1,\infty)$. For any $A \in \mathcal{T}^{p,t}$ and every $\varepsilon > 0$ there exists some $s > 0$ such that for all $F \subseteq \mathbb{C}^n$ and every $B \in \{A\} \cup \{A_x : x \in \mathcal{M} \setminus \mathbb{C}^n\}$ we have*

$$\|BP_t|_F\| \geq \|BP_t|_F\|_s \geq \|BP_t|_F\| - \varepsilon.$$

Proof. The proof is very similar to the proof of Lemma 5.2.12. The first estimate follows from the definition. On the second estimate: Let $A_m \in \mathcal{L}(L_t^p)$ be a band operator such that $\|AP_t - A_m\| < \frac{\varepsilon}{4}$. Following the lines of the proof of 5.2.12, exchanging the triangle inequality in $\ell^p(\mathbb{N})$ by the reverse triangle inequality, one shows that there exists $s > 0$ such that for all $F \subseteq \mathbb{C}^n$ and every $B \in \{A_m\} \cup \{(A_m)_x;\ x \in \mathcal{M} \setminus \mathbb{C}^n\}$ we have

$$\|BP_t|_F\| \leq \|BP_t|_F\|_s + \varepsilon.$$

Here, $(A_m)_x$ denotes the limit operators of A_m, which exist as limits in the weak operator topology as discussed in the proof of Lemma 5.2.12. Finally, passing from the estimates for the approximating band operator A_m to those of A works again as in the proof of Lemma 5.2.12, since the norms $\|BP_t|_F\|$ and $\|BP_t|_F\|_s$ obey similar continuity properties as $\nu(B|_F)$ and $\nu_s(B|_F)$. □

Proof of Theorem 5.3.1. The second estimate is readily established: Let $(z_\gamma)_\gamma \subset \mathbb{C}^n$ be a net converging to $x \in \mathcal{M} \setminus \mathbb{C}^n$ and $K \in \mathcal{K}(F_t^p)$. Since $\alpha_{z_\gamma}(K) \to 0$, a simple application of the Banach-Steinhaus principle shows

$$\|A_x\| = \|\lim_\gamma W_{z_\gamma}^t (A+K) W_{-z_\gamma}^t\| \leq \limsup_\gamma \|W_{z_\gamma}^t (A+K) W_{-z_\gamma}^t\| = \|A+K\|.$$

Since $K \in \mathcal{K}(F_t^p)$ was arbitrary, this shows $\|A_x\| \leq \|A + \mathcal{K}(F_t^p)\|$ for any $x \in \mathcal{M} \setminus \mathbb{C}^n$.

Let $K \in \mathcal{K}(L_t^p, F_t^p)$. Remarking the following estimate

$$\begin{aligned}\|AP_t + K\| &= \sup_{f \in L_t^p,\ \|f\|=1} \|(AP_t + K)f\| \geq \sup_{f \in F_t^p,\ \|f\|=1} \|(AP_t+K)f\| \\ &= \|A + K|_{F_t^p}\|,\end{aligned}$$

we therefore obtain

$$\|A + \mathcal{K}(F_t^p)\| \leq \inf_{K \in \mathcal{K}(L_t^p, F_t^p)} \|AP_t + K\|.$$

Therefore, it suffices to prove

$$\inf_{K \in \mathcal{K}(L_t^p, F_t^p)} \|AP_t + K\| \leq \sup_{x \in \mathcal{M} \setminus \mathbb{C}^n} \|A_x P_t\|.$$

We will show by contradiction that this statement holds true: Assume that

$$\inf_{K \in \mathcal{K}(L_t^p, F_t^p)} \|AP_t + K\| > \sup_{x \in \mathcal{M} \setminus \mathbb{C}^n} \|A_x P_t\| + \varepsilon.$$

We clearly have for any $r > 0$:

$$\|AP_t|_{\mathbb{C}^n \setminus B(0,r)}\| = \|AP_t M_{1-\chi_{B(0,r)}}\| = \|AP_t - AP_t M_{\chi_{B(0,r)}}\|.$$

Since $P_t M_{\chi_{B(0,r)}}$ is compact by Proposition 2.3.14, we necessarily have

$$\|AP_t|_{\mathbb{C}^n \setminus B(0,r)}\| > \sup_{x \in \mathcal{M} \setminus \mathbb{C}^n} \|A_x P_t\| + \varepsilon.$$

By the previous lemma, there exists some $s > 0$ such that

$$\|AP_t|_{\mathbb{C}^n \setminus B(0,r)}\|_s \geq \|AP_t|_{\mathbb{C}^n \setminus B(0,r)}\| - \frac{\varepsilon}{2} > \sup_{x \in \mathcal{M} \setminus \mathbb{C}^n} \|A_x P_t\| + \frac{\varepsilon}{2}.$$

Hence, for each $r > 0$ there necessarily is some $w_r \in \mathbb{C}^n$ such that

$$\begin{aligned}\|W_{-w_r}^t A W_{w_r}^t P_t M_{\chi_{B(0,r_s)}}\| &= \|AP_t M_{B(w_r,r_s)}\| \\ &\geq \|AP_t M_{\chi_{B(w_r,r_s)\setminus B(0,r)}}\| \\ &> \sup_{x \in \mathcal{M}\setminus\mathbb{C}^n} \|A_x P_t\| + \frac{\varepsilon}{2}.\end{aligned}$$

If w_r would remain bounded as $r \to \infty$, then $B(w_r, r_s) \setminus B(0, r)$ would be empty for r sufficiently large, violating the above estimates. Hence, possibly after passing to a subnet, we may assume that $-w_r$ converges to some $y \in \mathcal{M} \setminus \mathbb{C}^n$. Since $P_t M_{\chi_{B(0,r_s)}}$ is compact, this implies

$$W_{-w_r}^t A W_{w_r}^t P_t M_{\chi_{B(0,r_s)}} \to A_y P_t M_{\chi_{B(0,r_s)}}$$

in operator norm. But this in turn yields

$$\|A_y P_t M_{\chi_{B(0,r_s)}}\| > \sup_{x \in \mathcal{M}\setminus\mathbb{C}^n} \|A_x P_t\| + \frac{\varepsilon}{2},$$

which is a contradiction.

Regarding the statement for $p = 2$: Using the fact that $\|A_x P_t\| = \|A_x\|$ in the Hilbert space case and imitating the proof of Lemma 5.2.8 with $\nu(A|_F)$ and $\nu_s(A|_F)$ replaced by $\|AP_t|_F\|$ and $\|AP_t|_F\|_s$, one can show that there exists some $y \in \mathcal{M} \setminus \mathbb{C}^n$ such that

$$\|A_y\| = \|A_y P_t\| = \sup_{x \in \mathcal{M}\setminus\mathbb{C}^n} \|A_x P_t\| = \sup_{x \in \mathcal{M}\setminus\mathbb{C}^n} \|A_x\|,$$

which shows that the supremum is actually a maximum in this case. Further, since the projection P_t is orthogonal in the Hilbert space case, i.e. $\|P_t\| = 1$, we obtain equality from the norm estimates derived above. □

5.4 Essential spectra of Toeplitz operators with symbols of vanishing oscillation and vanishing mean oscillation

Let $f \in \mathrm{VO}_\partial(\mathbb{C}^n)$. As we have seen in Example 4.1.7.2), f_x is a constant function for any $x \in \mathcal{M}(\mathrm{VO}_\partial(\mathbb{C}^n)) \setminus \mathbb{C}^n$. This in turn yields that $(T_f^t)_x = T_{f_x}^t$ is simply a multiple of the identity. Patching things together, we obtain:

Proposition 5.4.1. *Let $t > 0$, $p \in (1, \infty)$ and $f \in \mathrm{VO}_\partial$. Then, we have*

$$\sigma_{ess}(T_f^t) = f(\partial\mathbb{C}^n),$$

where $f(\partial\mathbb{C}^n)$ denotes the set of limit points of $f(z)$ as $|z| \to \infty$.

Proof. As one easily sees (e.g. from the estimates in Example 4.1.7), $f(\partial\mathbb{C}^n)$ coincides with the values that the constant functions f_x can attain for $x \in \mathcal{M} \setminus \mathbb{C}^n$. Hence, every limit operator $(T_f^t)_x$ is of the form λI for some $\lambda \in f(\partial\mathbb{C}^n)$. The result is now imminent from Corollary 5.2.10. □

Corollary 5.4.2. *Let* $t > 0$, $p \in (1,\infty)$ *and* $f \in \mathrm{VMO}_\partial(\mathbb{C}^n)$. *Then,*

$$\sigma_{ess}(T_f^t) = \widetilde{f}^{(t)}(\partial\mathbb{C}^n).$$

Proof. For $f \in \mathrm{VMO}_\partial(\mathbb{C}^n)$ we have $\widetilde{f}^{(t)} \in \mathrm{VO}_\partial(\mathbb{C}^n)$ and $(\widetilde{f}^{(t)} - f)^{\sim(t)} \in C_0(\mathbb{C}^n)$ [137, Chapter 3.5]. By Corollary 3.3.10, $T^t_{\widetilde{f}^{(t)}-f}$ is therefore compact, i.e. $\sigma_{ess}(T_f^t) = \sigma_{ess}(T^t_{\widetilde{f}^{(t)}})$. Now, the result follows from the previous proposition. □

5.5 Remarks

As already mentioned earlier, the contents of this chapter originate from the author's joint paper with R. Hagger [73]. In contrast to that paper, we tried to extend certain results to the case $p = 1$. Since the methods do not give rise to a full Fredholm characterization in that case, we essentially do this to see where the method breaks down in the non-reflexive case. Another difference with the original paper [73] is that we emphasize how the Fredholm criterion can be turned into a criterion for left- and right-invertibility modulo compact operators, which eventually leads to the following open problem: Is $A \in \mathcal{T}^{p,t}$ left-invertible modulo $\mathcal{K}$ if and only if it is left-invertible in $\mathcal{T}^{p,t}$ modulo $\mathcal{K}$?

The paper [73] was closely inspired by R. Hagger's earlier work on the Fredholm property of operators on Bergman spaces over complex unit balls [80]. The idea of using limit operators for studying compactness and Fredholm properties initially goes back to the study of operators on sequence spaces [44, 96, 111]. The first occurrence of limit operator techniques on Bergman or Fock spaces was in the works [104, 122] on Bergman spaces and subsequently in [19] on Fock spaces, serving as a tool for the first proof of the compactness characterization, cf. Corollary 3.3.10 above. In particular, in [19] results in the spirit of the estimates in Section 5.3 were established, which we could slightly improve with the methods presented above. We also want to emphasize that the very recent paper [83] has proven that limit operator techniques are applicable in a very general geometric settings.

Results on the essential spectrum of Toeplitz operators with symbols of vanishing oscillation were already presented in [30]. Yet, it seems that those results were so far only discussed in the Hilbert space setting. Hence, our results in Section 5.4 contributed to extend the theory in the non-Hilbertian setting. Note that analogous results were proven independently in [1] with different methods, see also [89].

Chapter 6

The Resolvent Algebra

A cornerstone for quantum mechanics are the canonical commutation relations: Let (X, σ) be a symplectic space, $\mathcal{H}$ a Hilbert space and a ϕ real linear map from (X, σ) into the (unbounded) self-adjoint operators on $\mathcal{H}$ such that all $\phi(f)$ have a common core $\mathcal{D}$ on which they are essentially self-adjoint. Then, the canonical commutator relations are

$$[\phi(f), \phi(g)] = i\sigma(f, g), \quad f, g \in X. \tag{CCR}$$

Since dealing with algebraic expressions (such as (CCR)) can be quite cumbersome when dealing with unbounded operators (compare e.g. [113, Chapters VIII.5 and VIII.6]), one usually passes to a related C^*-algebra of bounded operators, which preserves the relations (CCR). Going back to Weyl, the most common approach is to consider the CCR-algebra generated by the unitary operators $\exp(i\phi(f))$, $f \in X$. Then, one obtains the modified CCR-relations

$$\exp(i\phi(f))\exp(i\phi(g)) = e^{-i\sigma(f,g)}\exp(i\phi(f+g))$$

and, replacing the self-adjointness,

$$\exp(i\phi(f))^* = \exp(-i\phi(f)).$$

Given a symplectic space (X, σ), one is hence interested in the C^* algebra $\mathrm{CCR}(X, \sigma)$ generated by the abstract relations

$$\begin{aligned} W(f)W(g) &= e^{-i\sigma(f,g)}W(f+g), \\ W(f)^* &= W(-f). \end{aligned}$$

Such C^*-algebras are usually called Weyl algebras (or CCR algebras).

Upon choosing for $t > 0$ the symplectic space $(\mathbb{C}^n, \sigma_t)$ with $\sigma_t(w, z) = \frac{\mathrm{Im}(w \cdot \overline{z})}{t}$, one obtains a representation of the Weyl algebra as operators acting on the Fock space F_t^2. Indeed, the unbounded operators realizing the relation (CCR) are obtained by the self-adjoint and unbounded Toeplitz operators

$$\phi(z) = T^t_{2\sigma_t(\cdot, z)},$$

or, more precisely, as the closure of $T^t_{2\sigma_t(\cdot,z)}|_{\mathcal{P}[z_1,\dots,z_n]}$, cf. Lemma 6.1.1 below. For keeping the notation simple, we will notationally not distinguish between $T^t_{2\sigma_t(\cdot,z)}$ and the closure of $T^t_{2\sigma_t(\cdot,z)}|_{\mathcal{P}[z_1,\dots,z_n]}$ in this chapter.

One can show (we will use such an argument below) that the corresponding Weyl algebra is generated by the unitary Weyl operators W_z^t, i.e. $iT^t_{2\sigma_t(\cdot,z)}$ is the generator of the unitary operator group $s \mapsto W^t_{sz}$. Hence, one obtains that

$$\mathrm{CCR}(\mathbb{C}^n, \sigma_t) \cong C^*(\{W_z^t;\ z \in \mathbb{C}^n\}). \tag{6.1}$$

The algebra $\mathrm{CCR}(\mathbb{C}^n) := \mathrm{CCR}(\mathbb{C}^n, \sigma_2)$ is a very classical object and various of its properties in a more general framework can be found in [40].

As already mentioned earlier, the representation of $\mathrm{CCR}(\mathbb{C}^n)$ on F_2^2 has already been studied by L. Coburn:

Theorem 6.0.1 ([49]). *It holds true that*

$$\mathrm{CCR}(\mathbb{C}^n) \cong \mathcal{T}_{lin}^{2,2}(\mathrm{TP}),$$

where

$$\mathrm{TP} := \mathrm{Span}\{w \mapsto \exp(i\,\mathrm{Im}(\langle w, z\rangle));\ z \in \mathbb{C}^n\} \subset L^\infty(\mathbb{C}^n).$$

As explained in Example 4.3.7, the CCR algebra is a prime example of a space that can be studied using Quantum Harmonic Analysis. In particular, it is easy to prove $\mathrm{CCR}(\mathbb{C}^n, \sigma_t) \cong \mathcal{T}_{lin}^{2,t}(\mathrm{AP})$ for any $t > 0$.

Unfortunately, there are certain drawbacks of using such CCR algebras for modeling quantum mechanics. The most significant problem is the fact that, when considering Hamiltonians in the standard Schrödinger representation (i.e. $-\Delta + V$ on $L^2(\mathbb{R}^n)$), time evolutions of such Hamiltonians do not give rise to *-automorphisms of $\mathrm{CCR}(\mathbb{C}^n)$ unless the potential V is trivial (cf. [68]).

Based upon this, it was suggested in [41, 42] to consider instead the C^* algebra generated by the resolvents of $\phi(f)$, which they named the *Resolvent Algebra*. This effectively provides a framework where the dynamics of the system can be explicitly studied as elements of a C^* algebra. Abstractly, the Resolvent Algebras are defined as follows:

Definition 6.0.2. For a symplectic space (X, σ), $\mathcal{R}_0(X, \sigma)$ is defined as the universal unital *-algebra generated by the set $\{R(\lambda, f);\ \lambda \in \mathbb{R} \setminus \{0\},\ f \in X\}$ and the relations

$$R(\lambda, 0) = -\frac{i}{\lambda} 1 \tag{6.2}$$

$$R(\lambda, f)^* = R(-\lambda, f) \tag{6.3}$$

$$\nu R(\nu\lambda, \nu f) = R(\lambda, f) \tag{6.4}$$

$$R(\lambda, f) - R(\mu, f) = i(\mu - \lambda) R(\lambda, f) R(\mu, f) \tag{6.5}$$

$$[R(\lambda, f), R(\mu, g)] = i\sigma(f, g) R(\lambda, f) R(\mu, g)^2 R(\lambda, f) \tag{6.6}$$

$$R(\lambda, f)R(\mu, g) = R(\lambda+\mu, f+g) \cdot \\ \cdot \ [R(\lambda, f) + R(\mu, g) + i\sigma(f,g)R(\lambda, f)^2 R(\mu, g)] \tag{6.7}$$

for $\lambda, \mu, \nu \in \mathbb{R} \setminus \{0\}$ and $f, g \in X$.

Then, the Resolvent Algebra $\mathcal{R}(X, \sigma)$ is defined to be the closure of $\mathcal{R}_0$ with respect to a certain seminorm related to the GNS construction (cf. [42] for details).

Remark 6.0.3. 1) Equation (6.2) comes from $\phi(0) = 1$. Equation (6.3) encodes the self-adjointness of $\phi(f)$. Equations (6.4) and (6.7) are the $\mathbb{R}$-linearity of ϕ. Equation (6.5) is just the usual resolvent identity. Finally, (6.6) is the substitute for the (CCR) relations.

2) When one considers a representation of $\mathcal{R}_0$ as a concrete *-algebra generated by resolvents of self-adjoint operators on a Hilbert space, then passing from $\mathcal{R}_0$ to $\mathcal{R}$ is the same as taking the closure with respect to the operator norm.

We are interested in the Fock space representation of the Resolvent Algebra over the symplectic space $(\mathbb{C}^n, \sigma_t)$. Since the generators of the resolvents are just the Toeplitz operators $T^t_{2\sigma_t(\cdot, z)}$ as described above, this is just the C^*-algebra generated by the resolvents of these operators:

$$\mathcal{R}(\mathbb{C}^n, \sigma_t) \cong C^*\Big(\Big\{ (i\lambda - T^t_{2\sigma_t(\cdot,z)})^{-1};\ \lambda \in \mathbb{R} \setminus \{0\}, z \in \mathbb{C}^n \Big\} \Big). \tag{6.8}$$

We begin here with our analysis.

6.1 The Resolvent Algebra in the Bargmann representation

For readability, we will notationally not distinguish between the Resolvent Algebra $\mathcal{R}(\mathbb{C}^n, \sigma_t)$ and its representation on F^2_t. The key to studying $\mathcal{R}(\mathbb{C}^n, \sigma_t)$ consists of the following integral representations:

Lemma 6.1.1. *For $z \in \mathbb{C}^n$, $s \mapsto W^t_{sz}$ defines a strongly continuous unitary operator group. The set of holomorphic polynomials, $\mathcal{P}[z_1, \dots, z_n]$, is a core for the generator of the operator group and the generator is given by the closure of $iT_{2\sigma_t(\cdot,z)}|_{\mathcal{P}[z_1,\dots,z_n]}$, which we abbreviate for simplicity by $iT_{2\sigma_t(\cdot,z)}$. In particular, for $\lambda > 0$ the following integral representations for the resolvents hold true, where the integrals are understood to converge in strong operator topology:*

$$(T^t_{2\sigma_t(\cdot,z)} + i\lambda)^{-1} = (T^t_{2\sigma_t(\cdot,z)+i\lambda})^{-1} = -i \int_0^\infty e^{-\lambda s} W^t_{sz}\, ds,$$
$$(T^t_{2\sigma_t(\cdot,z)} - i\lambda)^{-1} = (T^t_{2\sigma_t(\cdot,z)-i\lambda})^{-1} = i \int_0^\infty e^{-\lambda s} W^t_{-sz}\, ds.$$

Proof. As we have used sufficiently often by now, $s \mapsto W^t_{sz}$ is continuous with respect to the strong operator topology. Since each W^t_{sz} is unitary, this is indeed a strongly continuous unitary group. By the general theory on such groups (cf. [56]), there exists some self-adjoint (in general unbounded) operator A on F_t^2 which generates this group: $W^t_{sz} = e^{isA}$. The fact that the polynomials form a core for the generator and the precise form of the generator is not important for the following discussions. This can be discussed as in [85, Chapter 14.4].

Since each W^t_{sz} is unitary, we of course have

$$\|W^t_{sz}\| = 1 \leq e^{s\cdot 0},$$

i.e. the group has growth bound $\omega_0 = 0$. For any $\lambda > 0$, this implies, by the general theory on operator (semi-)groups (cf. [56, Theorem I.1.10]), that

$$(\lambda - iT^t_{2\sigma_t(\cdot,z)})^{-1} = \int_0^\infty e^{-\lambda s} W^t_{sz}\, ds,$$

where the integral converges in strong operator topology. Hence

$$(T^t_{2\sigma_t(\cdot,z)} + i\lambda)^{-1} = -i \int_0^\infty e^{-\lambda s} W^t_{sz}\, ds,$$

and since $(W^t_{sz})^* = W^t_{-sz}$,

$$(T^t_{2\sigma_t(\cdot,z)} - i\lambda)^{-1} = \left((T^t_{2\sigma_t(\cdot,z)} + i\lambda)^{-1}\right)^* = i \int_0^\infty e^{-\lambda s} W^t_{-sz}\, ds,$$

proving the claim. □

In what follows, we will denote the resolvents by

$$R(\lambda, z) := (T^t_{2\sigma_t(\cdot,z)} - i\lambda)^{-1}, \quad \lambda \in \mathbb{C} \setminus i\mathbb{R},\ z \in \mathbb{C}^n.$$

Hence, for $\lambda > 0$ the previous lemma states

$$R(\lambda, z) = i \int_0^\infty e^{-\lambda s} W^t_{-sz}\, ds,$$
$$R(-\lambda, z) = -i \int_0^\infty e^{-\lambda s} W^t_{sz}\, ds.$$

Let us recall the following standard result on resolvents [129, Theorem 5.14].

Lemma 6.1.2. *Let $\lambda, \lambda_0 \in \mathbb{C} \setminus i\mathbb{R}$ such that $|\lambda_0 - \lambda| < |\operatorname{Re}(\lambda_0)|$. Then, we have*

$$R(\lambda, z) = \sum_{k=0}^\infty (\lambda_0 - \lambda)^k i^k R(\lambda_0, z)^{k+1},$$

where the series converges in operator norm. In particular, we have

$$R(\lambda_0, z)^k = \frac{i^{k-1}}{(k-1)!} \frac{d^{k-1}}{d\lambda^{k-1}} R(\lambda, z)|_{\lambda=\lambda_0}.$$

It will be useful to consider resolvents $R(\lambda, z)$ for the larger class $\lambda \in \mathbb{C} \setminus i\mathbb{R}$. An immediate consequence of the previous lemma is that the resulting C^* algebra remains the same.

Lemma 6.1.3. *The following holds true:*

$$\mathcal{R}(\mathbb{C}^n, \sigma_t) = C^*(\{R(\lambda, z);\ z \in \mathbb{C}^n,\ \lambda \in \mathbb{C} \setminus i\mathbb{R}\}).$$

It is important to note that the same integral formulas for $R(\lambda, z)$ derived in Lemma 6.1.1 hold for any $\lambda \in \mathbb{C} \setminus i\mathbb{R}$:

Lemma 6.1.4. *Let $z \in \mathbb{C}^n$ and $\lambda \in \mathbb{C} \setminus i\mathbb{R}$. Then,*

$$R(\lambda, z) = i \int_0^\infty e^{-\lambda s} W^t_{-sz}\ ds, \quad \mathrm{Re}(\lambda) > 0,$$
$$R(\lambda, z) = -i \int_0^\infty e^{\lambda s} W^t_{sz}\ ds, \quad \mathrm{Re}(\lambda) < 0.$$

Proof. The proof is identical to the case $\lambda \in \mathbb{R} \setminus \{0\}$. □

Let us consider the following class of bounded functions on $\mathbb{C}^n$:

$$\mathrm{FR} := \{(\lambda - 2i\sigma_t(\cdot, z))^{-(2k+1)};\ z \in \mathbb{C}^n, \lambda \in \mathbb{R} \setminus \{0\},\ k \in \mathbb{N}_0\}.$$

Obviously, the class FR is independent of $t > 0$.

Proposition 6.1.5. *We have*

$$\mathcal{R}(\mathbb{C}^n, \sigma_t) \subseteq \mathcal{T}^{2,t}_*(\mathrm{FR}).$$

Proof. We need to prove that the generators of $\mathcal{R}(\mathbb{C}^n, \sigma_t)$ are contained in $\mathcal{T}^{2,t}_*(\mathrm{FR})$, i.e. that $R(\lambda, z) \in \mathcal{T}^{2,t}_*(\mathrm{FR})$ for any $z \in \mathbb{C}^n$, $\lambda \in \mathbb{R} \setminus \{0\}$. For $\lambda > 0$ we have

$$\begin{aligned} iR(-\lambda, z) &= \int_0^\infty e^{-\lambda s} W^t_{sz}\ ds \\ &= \int_0^\infty e^{-\lambda s} e^{\frac{s^2}{2t}|z|^2} T^t_{e^{2is\sigma_t(\cdot, z)}}\ ds \\ &= \int_0^\infty e^{-\lambda s} \sum_{k=0}^\infty \frac{1}{k!} \left(\frac{s^2|z|^2}{2t}\right)^k T^t_{e^{2is\sigma_t(\cdot, z)}}\ ds. \end{aligned}$$

Since the Weyl operators have norm one, we conclude

$$\|T^t_{e^{2i\sigma_t(\cdot, z)}}\| = e^{-\frac{|z|^2}{2t}} \|T^t_{g^t_z}\| = e^{-\frac{|z|^2}{2t}} \|W^t_z\| = e^{-\frac{|z|^2}{2t}}.$$

Therefore, we obtain

$$\left\| iR(-\lambda, z) - \sum_{k=0}^m \frac{|z|^{2k}}{k! t^k 2^k} \int_0^\infty s^{2k} e^{-\lambda s} T^t_{e^{2is\sigma_t(\cdot, z)}}\ ds \right\|$$

$$\leq \int_0^\infty e^{-\lambda s} \left(e^{\frac{s^2}{2t}|z|^2} - \sum_{k=0}^m \frac{s^{2k}}{k! t^k 2^k} |z|^{2k} \right) e^{-\frac{s^2}{2t}|z|^2} \, ds$$

$$\to 0, \quad m \to \infty$$

by the Dominated Convergence Theorem. Thus,

$$iR(-\lambda, z) = \sum_{k=0}^\infty \frac{1}{k!} \frac{|z|^{2k}}{2^k t^k} \int_0^\infty T^t_{e^{-\lambda s} s^{2k} e^{2si\sigma_t(\cdot, z)}} \, ds,$$

where the infinite series converges in operator norm. Let $N > 0$ be fixed. Note that the integral $\int_0^N T^t_{e^{-\lambda s} s^{2k} e^{2si\sigma_t(\cdot,z)}} \, ds$ converges in the strong operator topology, which is a simple consequence of the continuity of $s \mapsto W^t_{sz}$. Hence, upon applying the Berezin transform to this operator integral, we obtain

$$\begin{aligned} \left(\int_0^N T^t_{e^{-\lambda s} s^{2k} e^{2si\sigma_t(\cdot,z)}} \, ds \right)^\sim (u) &= \left\langle \int_0^N e^{-\lambda s} s^{2k} e^{2si\sigma_t(\cdot,z)} k_u^t \, ds, k_u^t \right\rangle_t \\ &= \left\langle \int_0^N e^{-\lambda s} s^{2k} e^{2si\sigma_t(\cdot,z)} \, ds \; k_u^t, k_u^t \right\rangle_t . \end{aligned}$$

Therefore, injectivity of the Berezin transform gives

$$\int_0^N T^t_{e^{-\lambda s} s^{2k} e^{2si\sigma_t(\cdot,z)}} \, ds = T^t_{\int_0^N e^{-\lambda s} s^{2k} e^{2si\sigma_t(\cdot,z)} \, ds}.$$

Since

$$\int_0^N e^{-\lambda s} s^{2k} e^{2si\sigma_t(\cdot,z)} \, ds \overset{N \to \infty}{\longrightarrow} \int_0^\infty e^{-\lambda s} s^{2k} e^{2si\sigma_t(\cdot,z)} \, ds$$

uniformly on $\mathbb{C}^n$ as a function of w, we obtain

$$iR(-\lambda, z) = \sum_{k=0}^\infty \frac{1}{k!} \frac{|z|^{2k}}{2^k t^k} T^t_{\int_0^\infty e^{-\lambda s} s^{2k} e^{2si\sigma_t(\cdot,z)} \, ds}.$$

The symbol of the Toeplitz operator in the series expansion can be computed explicitly, using properties of the Laplace transform:

$$\begin{aligned} \int_0^\infty e^{-\lambda s} s^{2k} e^{2si\sigma_t(w,z)} \, ds &= \frac{d^{2k}}{d\mu^{2k}} \left[\int_0^\infty e^{-\mu s} e^{2si\sigma_t(w,z)} \, ds \right] |_{\mu=\lambda} \\ &= (\lambda - 2i\sigma_t(w,z))^{-(2k+1)}. \end{aligned}$$

Thus,

$$iR(-\lambda, z) = \sum_{k=0}^\infty \frac{1}{k!} \frac{|z|^{2k}}{2^k t^k} T^t_{(\lambda - 2i\sigma_t(\cdot,z))^{-(2k+1)}}.$$

Since the sum converges in operator norm, as already mentioned above, we therefore obtain $R(-\lambda, z) \in \mathcal{T}_*^{2,t}(\mathrm{FR})$. Furthermore, using the identity $R(-\lambda, z) = -R(\lambda, -z)$, we obtain the same statement for those resolvents. $\square$

Let us compute the shifts of a resolvent: For $w \in \mathbb{C}^n$ and $\lambda > 0$ we have, using the standard relations for products of Weyl operators,

$$\begin{aligned} W_w^t R(-\lambda, z) W_{-w}^t &= -i \int_0^\infty e^{-\lambda s} W_w^t W_{sz}^t W_{-w}^t \, ds \\ &= -i \int_0^\infty e^{-\lambda s} e^{-2is\sigma_t(w,z)} W_{sz}^t \, ds \\ &= R(\lambda + 2i\sigma_t(w,z), z). \end{aligned}$$

Since, by Lemma 6.1.3, the C^* algebra generated by those $R(\lambda, z)$ with $\lambda \in \mathbb{R} \setminus \{0\}$ is the same as the C^* algebra generated by all $R(\lambda, z)$ with $\lambda \in \mathbb{C} \setminus i\mathbb{R}$ we deduce:

Proposition 6.1.6. *$\mathcal{R}(\mathbb{C}^n, \sigma_t)$ is an α-invariant C^* subalgebra of $\mathcal{T}^{2,t}$.*

Hence, applying Correspondence Theory to $\mathcal{R}(\mathbb{C}^n, \sigma_t)$, we know that for each $t > 0$ there is some α-invariant and closed subspace $\mathcal{D}_0^t$ of $\mathrm{BUC}(\mathbb{C}^n)$ such that

$$\mathcal{R}(\mathbb{C}^n, \sigma_t) = \mathcal{T}_{lin}^{2,t}(\mathcal{D}_0^t).$$

Lemma 6.1.7. *Let $\lambda \in \mathbb{C} \setminus i\mathbb{R}$ with $\mathrm{Re}(\lambda) > 0$ and $z, w \in \mathbb{C}^n$. For each $k \in \mathbb{N}$ the Berezin transform of $R(\lambda, z)^k$ is given by*

$$(R(\lambda, z)^k)^\sim(w) = \frac{(-1)^{k-1} i^k}{(k-1)!} \int_0^\infty s^{k-1} e^{-\lambda s + 2is\sigma_t(z,w) - \frac{s^2}{2t}|z|^2} \, ds.$$

A similar formula holds for $\mathrm{Re}(\lambda) < 0$ by applying the relation

$$R(\lambda, z)^k = (-1)^k R(-\lambda, -z)^k.$$

Proof. Recall that we have, according to Lemma 6.1.2:

$$\begin{aligned} (R(\lambda, z)^k)^\sim(w) &= \frac{i^{k-1}}{(k-1)!} \left\langle \frac{d^{k-1}}{d\mu^{k-1}} R(\mu, z) k_w^t, k_w^t \right\rangle_t |_{\mu=\lambda} \\ &= \frac{i^{k-1}}{(k-1)!} \frac{d^{k-1}}{d\mu^{k-1}} \left\langle W_{-w}^t R(\mu, z) W_w^t 1, 1 \right\rangle_t |_{\mu=\lambda}. \end{aligned}$$

Applying the integral representation for $R(\mu, z)$ from the case $\mathrm{Re}(\mu) > 0$ we obtain

$$\begin{aligned} (R(\lambda, z)^k)^\sim(w) &= \frac{i^k}{(k-1)!} \frac{d^{k-1}}{d\mu^{k-1}} \int_0^\infty e^{-\mu s} \langle W_{-w}^t W_{-sz}^t W_w^t 1, 1 \rangle_t \, ds|_{\mu=\lambda} \\ &= \frac{i^k}{(k-1)!} \frac{d^{k-1}}{d\mu^{k-1}} \int_0^\infty e^{-\mu s + 2is\sigma_t(z,w)} \langle W_{-sz}^t 1, 1 \rangle_t \, ds|_{\mu=\lambda}. \end{aligned}$$

Since $\langle W_{-sz}^t 1, 1 \rangle_t = k_{sz}^t(0) = e^{-\frac{s^2|z|^2}{2t}}$ we obtain

$$(R(\lambda, z)^k)^\sim(w) = \frac{i^k}{(k-1)!} \frac{d^{k-1}}{d\mu^{k-1}} \int_0^\infty e^{-\mu s + 2is\sigma_t(z,w) - \frac{s^2|z|^2}{2t}} \, ds|_{\mu=\lambda}.$$

Exchanging derivation and integration we obtain the claim. □

Proposition 6.1.8. *For any $t > 0$ we have $\mathcal{R}(\mathbb{C}^n, \sigma_t) = \mathcal{T}_*^{2,t}(\mathrm{FR})$.*

Proof. Let $\lambda > 0, k \in \mathbb{N}, z \in \mathbb{C}^n$ and set

$$g(w) = (\lambda - 2i \operatorname{Im} \sigma_t(w,z))^{-(2k+1)} = \int_0^\infty s^{2k} e^{-\lambda s + 2si\sigma_t(w,z)} \, ds \in \mathrm{FR}\,.$$

Imitating the arguments from the proof of Proposition 6.1.5, we can express the Toeplitz operator T_g^t as an operator-valued integral:

$$\begin{aligned} T_g^t &= \int_0^\infty T^t_{s^{2k}e^{-\lambda s + 2is\sigma_t(\cdot,z)}} \, ds \\ &= \int_0^\infty e^{\frac{s^2|z|^2}{2t}} T^t_{s^{2k}e^{-\lambda s + 2is\sigma_t(\cdot,z) - \frac{s^2}{2t}|z|^2}} \, ds \\ &= \sum_{l=0}^\infty \frac{|z|^{2l}}{2^l t^l l!} T^t_{\int_0^\infty s^{2l+2k}e^{-\lambda s + 2is\sigma_t(\cdot,z) - \frac{s^2}{2t}|z|^2} \, ds} . \end{aligned}$$

Note that this series converges again with respect to the operator norm. By the previous lemma, we have

$$T^t_{\int_0^\infty s^{2l+2k}e^{-\lambda s + 2is\sigma_t(\cdot,z) - \frac{s^2}{2t}|z|^2} \, ds} = i^{2k+2l+1}(2k+2l)! T^t_{(R(\lambda,z)^k)^\sim} .$$

Since

$$T^t_{(R(\lambda,z)^k)^\sim} = f_t * R(\lambda,z)^k,$$

where f_t denotes as usual the Gaussian $f_t(z) = \frac{1}{(\pi t)^n} e^{-\frac{|z|^2}{t}}$, and $\mathcal{R}(\mathbb{C}^n, \sigma_t)$ is an α-invariant closed subalgebra of $\mathcal{T}^{2,t}$, we obtain

$$T^t_{\int_0^\infty s^{2l+2k}e^{-\lambda s + 2is\sigma_t(\cdot,z) - \frac{s^2}{2t}|z|^2} \, ds} \in \mathcal{R}(\mathbb{C}^n, \sigma_t).$$

The standard estimate

$$\begin{aligned} &\left\| T_g^t - \sum_{l=0}^N \frac{|z|^{2l}}{2^l t^l l!} T^t_{\int_0^\infty s^{2l+2k}e^{-\lambda s + 2is\sigma_t(\cdot,z) - \frac{s^2}{2t}|z|^2} \, ds} \right\| \\ &\quad \leq \int_0^\infty \left(e^{\frac{s^2|z|^2}{2t}} - \sum_{l=0}^N \frac{|z|^{2l} s^{2l}}{2^l t^l l!} \right) s^{2k} e^{-\lambda s - \frac{s^2|z|^2}{2t}} \, ds \to 0, \quad N \to \infty \end{aligned}$$

therefore yields $T_g^t \in \mathcal{R}(\mathbb{C}^n, \sigma_t)$. Since g was a generic function from FR, we obtain $\mathcal{T}_*^{2,t}(\mathrm{FR}) \subseteq \mathcal{R}(\mathbb{C}^n, \sigma_t)$. Combining this with Proposition 6.1.5 proves the claim. □

Since $\mathcal{R}(\mathbb{C}^n, \sigma_t)$ is α-invariant, we also obtain

$$\mathcal{R}(\mathbb{C}^n, \sigma_t) = \mathcal{T}_*^{2,t}(\overline{\mathrm{Span}}(\alpha(\mathrm{FR}))),$$

where

$$\alpha(\mathrm{FR}) = \{\alpha_z(f);\ f \in \mathrm{FR},\ z \in \mathbb{C}^n\}.$$

On the level of symbols one might suspect that the "Classical Resolvent Algebra",

$$\mathcal{R} := C^*(\{(\lambda - i\sigma_t(\cdot, z))^{-1};\ \lambda \in \mathbb{R} \setminus \{0\},\ z \in \mathbb{C}^n\})$$

agrees with $\mathcal{D}_0^t$ (the function space corresponding to $\mathcal{R}(\mathbb{C}^n, \sigma_t)$). Indeed, it is not difficult to verify that $\mathcal{R}$ is an α- and U-invariant C^* subalgebra of $\mathrm{BUC}(\mathbb{C}^n)$, which is also independent of the choice of $t > 0$. Hence, $\mathcal{T}_{lin}^{2,t}(\mathcal{R})$ is indeed a C^* algebra by Theorem 4.3.3 and therefore satisfies $\mathcal{T}_*^{2,t}(\mathcal{R}) = \mathcal{T}_{lin}^{2,t}(\mathcal{R})$ for any $t > 0$. Moreover, it is relatively simple to verify that

$$\mathcal{R} = C^*(\mathrm{FR}).$$

An application of Proposition 4.3.8 yields

$$\overline{\mathrm{Span}}(\alpha(\mathrm{FR})) \subseteq \mathcal{D}_0^t \subseteq \mathcal{R}$$

for any $t > 0$. So far we do not know if $\mathcal{D}_0^t$ agrees with either $\overline{\mathrm{Span}}(\alpha(\mathrm{FR}))$ or $\mathcal{R}$ or if possibly even both inclusions above are equalities. We will now work towards a description of $\mathcal{D}_0^t$, which will nevertheless not settle this question.

Lemma 6.1.9. *Let $z_1, \dots, z_m \in \mathbb{C}^n,\ k_1, \dots, k_m \in \mathbb{N},\ \lambda_1, \dots, \lambda_m \in \mathbb{C} \setminus i\mathbb{R}$ such that $\mathrm{Re}(\lambda_j) > 0$ for every j. Then, we have*

$$\begin{aligned}&\frac{(k_1-1)!\dots(k_m-1)!}{i^{k_1+\dots+k_m}}(-1)^{k_1+\dots+k_m-m}(R(\lambda_1,z_1)^{k_1}\dots R(\lambda_k,z_k)^{k_m})^\sim(w)\\ &= \int_{(0,\infty)^m} \mathbf{s}^{\mathbf{k-1}} e^{-(\Lambda\cdot\mathbf{s})-2i\sigma_t(w,\mathbf{s}\cdot z)-i(\sum_{j<\ell}s_js_\ell\sigma_t(z_j,z_\ell))-\frac{1}{2t}|\mathbf{s}\cdot z|^2}\ d\mathbf{s}.\end{aligned}$$

Here, we used the abbreviations $\mathbf{s} = (s_1, \dots, s_m),\ \mathbf{k} = (k_1, \dots, k_m),\ \mathbf{1} = (1, \dots, 1)$, $\mathbf{s} \cdot z = s_1 z_1 + \dots + s_m z_m$.

As before, the Berezin transform for the general case (i.e. arbitrary $\lambda_j \in \mathbb{C} \setminus i\mathbb{R}$) can by deduced from this using the relation $R(-\lambda, z) = -R(\lambda, -z)$.

Proof. By Lemma 6.1.2, we have

$$\begin{aligned}&(R(\lambda_1,z_1)^{k_1}\dots R(\lambda_k,z_k)^{k_m})^\sim(w)\\ &= \frac{i^{k_1+\dots+k_m-m}}{(k_1-1)!\dots(k_m-1)!}\times\\ &\quad\times\Big\langle \frac{\partial^{k_1-1}}{\partial\mu_1^{k_1-1}}\cdots\frac{\partial^{k_m-1}}{\partial\mu_m^{k_m-1}}|_{\mu_1=\lambda_1,\dots,\mu_m=\lambda_m} R(\mu_1,z_1)\dots R(\mu_m,z_m)k_w^t, k_w^t\Big\rangle_t.\end{aligned}$$

In the following lines, we will use the abbreviation

$$\frac{\partial^{\mathbf{k-1}}}{\partial \mu^{\mathbf{k-1}}}|_{\mu=\Lambda} = \frac{\partial^{k_1-1}}{\partial \mu_1^{k_1-1}} \cdots \frac{\partial^{k_m-1}}{\partial \mu_m^{k_m-1}}|_{\mu_1=\lambda_1,\dots,\mu_m=\lambda_m}.$$

Since the resolvent maps $\mu_j \mapsto R(\mu_j, z_j)$ are analytic in $\mathcal{L}(F_t^2)$, the difference quotients converge in operator norm. Therefore, differentiation can be exchanged with the inner product. This gives

$$\begin{aligned}
&\frac{(k_1-1)!\dots(k_m-1)!}{i^{k_1+\dots+k_m-m}}(R(\lambda_1,z_1)^{k_1}\dots R(\lambda_k,z_k)^{k_m})^{\sim}(w)\\
&= \frac{\partial^{\mathbf{k-1}}}{\partial \mu^{\mathbf{k-1}}}|_{\mu=\Lambda}\langle R(\mu_1,z_1)\dots R(\mu_m,z_m)k_w^t, k_w^t\rangle_t\\
&= \frac{\partial^{\mathbf{k-1}}}{\partial \mu^{\mathbf{k-1}}}|_{\mu=\Lambda}\langle W_{-w}^t R(\mu_1,z_1)W_w^t \dots W_{-w}^t R(\mu_m,z_m)W_w^t 1, 1\rangle_t\\
&= i^m \frac{\partial^{\mathbf{k-1}}}{\partial \mu^{\mathbf{k-1}}}|_{\mu=\Lambda}\int_{(0,\infty)^m} e^{-(\mu_1 s_1+\dots+\mu_m s_m)}\langle W_{-w}^t W_{s_1z_1}^t W_w^t \dots W_{-w}^t W_{s_m z_m}^t W_w^t 1, 1\rangle_t \, ds\\
&= i^m \frac{\partial^{\mathbf{k-1}}}{\partial \mu^{\mathbf{k-1}}}|_{\mu=\Lambda}\int_{(0,\infty)^m} e^{-(\mu_1 s_1+\dots+\mu_m s_m)-2i\sum_j s_j\sigma_t(w,z_j)}\langle W_{s_1z_1}^t \dots W_{s_m z_m}^t 1, 1\rangle_t \, ds\\
&= i^m \frac{\partial^{\mathbf{k-1}}}{\partial \mu^{\mathbf{k-1}}}|_{\mu=\Lambda}\int_{(0,\infty)^m} e^{-(\mu_1 s_1+\dots+\mu_m s_m)-2i(\sum_j s_j\sigma_t(w,z_j))}\times\\
&\qquad\qquad \times e^{-i(\sum_{j<\ell} s_j s_\ell \sigma_t(z_j,z_\ell))}\langle W_{s_1z_1+\dots+s_m z_m}^t 1, 1\rangle_t \, ds\\
&= i^m \frac{\partial^{\mathbf{k-1}}}{\partial \mu^{\mathbf{k-1}}}|_{\mu=\Lambda}\int_{(0,\infty)^m} e^{-(\mu_1 s_1+\dots+\mu_m s_m)-2i(\sum_j s_j\sigma_t(w,z_j))}\times\\
&\qquad\qquad \times e^{-i(\sum_{j<\ell} s_j s_\ell \sigma_t(z_j,z_\ell))}e^{-\frac{1}{2t}|\sum_j s_j z_j|^2} \, ds\\
&= i^m(-1)^{k_1+\dots+k_m-m}\int_{(0,\infty)^m} s_1^{k_1-1}\dots s_m^{k_m-1}e^{-(\lambda_1 s_1+\dots+\lambda_m s_m)-2i(\sum_j s_j\sigma_t(w,z_j))}\times\\
&\qquad\qquad \times e^{-i(\sum_{j<\ell} s_j s_\ell \sigma_t(z_j,z_\ell))}e^{-\frac{1}{2t}|\sum_j s_j z_j|^2} \, ds,
\end{aligned}$$

which proves the statement. □

The following result is already known [42]. Yet, we give a proof which works directly in our Fock space setting.

Lemma 6.1.10. *For any $t > 0$ we have $\mathcal{K}(F_t^2) \subset \mathcal{R}(\mathbb{C}^n, \sigma_t)$.*

Proof. Since $\mathcal{R}(\mathbb{C}^n, \sigma_t)$ is irreducible [42], it suffices by some well-known C^* algebraic argument to prove that there is one compact operator contained in $\mathcal{R}(\mathbb{C}^n, \sigma_t)$. We claim that for any $\lambda > 0$, the operator

$$\begin{aligned}
&R(\lambda,(1+i,0,\dots,0))R(\lambda,(1-i,0,\dots,0))\circ\\
&\qquad R(\lambda,(0,1+i,\dots,0))\dots R(\lambda,(0,\dots,0,1-i))
\end{aligned}$$

is compact. We prove this only for $n = 1$, the general case being proved identically (up to a messier notation). We will do this by proving that the Berezin transform of $R(\lambda, 1)R(\lambda, i)$ vanishes at infinity, which will then imply the result by Corollary 3.3.10. By the previous lemma, the Berezin transform of $R(\lambda, 1+i)R(\lambda, 1-i)$ at $w \in \mathbb{C}$ is given, up to some constant, by

$$\int_{(0,\infty)^2} e^{-\lambda(s_1+s_2)-2i(s_1\sigma_t(w,1+i)+s_2\sigma_t(w,1-i))-is_1s_2\sigma_t(1+i,1-i)-\frac{1}{2t}|s_1(1+i)+s_2(1-i)|^2}\,ds$$
$$= \int_{(0,\infty)^2} e^{-\lambda(s_1+s_2)-\frac{2i}{t}(s_1(\mathrm{Im}(w)+\mathrm{Re}(w))+s_2(\mathrm{Im}(w)-\mathrm{Re}(w)))-\frac{2i}{t}s_1s_2-\frac{1}{t}(s_1^2+s_2^2)}\,ds.$$

Fix $w \in \mathbb{C}$ such that $|w| = 1$ and let $\alpha > 0$. Then, the Berezin transform at αw is given by

$$I_{\alpha,w} := \int_{(0,\infty)^2} e^{-\lambda(s_1+s_2)-\frac{2i}{t}\alpha(s_1(\mathrm{Im}(w)+\mathrm{Re}(w))+s_2(\mathrm{Im}(w)-\mathrm{Re}(w)))-\frac{2i}{t}s_1s_2-\frac{1}{t}(s_1^2+s_2^2)}\,ds.$$

We claim that $I_{\alpha,w} \to 0$ as $\alpha \to \infty$. To see this, let

$$g(s_1, s_2) := e^{-\lambda(s_1+s_2)-\frac{2i}{t}s_1s_2-\frac{1}{t}(s_1^2+s_2^2)}$$

such that

$$I_{\alpha,w} = \int_{(0,\infty)^2} g(s_1, s_2)e^{-\frac{2i}{t}\alpha(s_1(\mathrm{Im}(w)+\mathrm{Re}(w))+s_2(\mathrm{Im}(w)-\mathrm{Re}(w)))}\,ds.$$

For $\varepsilon > 0$ let $\chi_\varepsilon \in C^\infty(\mathbb{R}^2)$ such that $0 \le \chi_\varepsilon(s) \le 1$ everywhere, $\chi_\varepsilon|_{(0,\infty)^2} \equiv 1$ and $\chi_\varepsilon|_{\mathbb{R}^2\setminus(-\varepsilon,\infty)^2} \equiv 0$. Since

$$|g(s_1, s_2)| \lesssim e^{-\frac{1}{2t}s_1^2}$$

on $(-1, \infty) \times (-1, 1)$ and

$$|g(s_1, s_2)| \lesssim e^{-\frac{1}{2t}s_2^2}$$

on $(-1, 1) \times (-1, \infty)$, it is not difficult to verify that

$$\int_{\mathbb{R}^2} \chi_\varepsilon(s_1, s_2)g(s_1, s_2)e^{-\frac{2i}{t}\alpha(s_1(\mathrm{Im}(w)+\mathrm{Re}(w))+s_2(\mathrm{Im}(w)-\mathrm{Re}(w)))}\,ds = I_{\alpha,w} + O(\varepsilon)$$

as $\varepsilon \to 0$. Note that $\chi_\varepsilon(s_1, s_2)g(s_1, s_2)$ is exponentially decaying at infinity and since $w \neq 0$, $(s_1, s_2) \mapsto s_1(\mathrm{Im}(w) + \mathrm{Re}(w)) + s_2(\mathrm{Im}(w) - \mathrm{Re}(w))$ has no stationary points in $\mathbb{R}^2$. Therefore, the method of stationary phase yields

$$\int_{\mathbb{R}^2} \chi_\varepsilon(s_1, s_2)g(s_1, s_2)e^{-\frac{2i}{t}\alpha(s_1(\mathrm{Im}(w)+\mathrm{Re}(w))+s_2(\mathrm{Im}(w)-\mathrm{Re}(w)))}\,ds \to 0, \quad \alpha \to \infty.$$

Putting the pieces together, we obtain

$$(R(\lambda, 1+i)R(\lambda, 1-i))^\sim(\alpha w) \to 0, \quad \alpha \to \infty$$

for any $w \neq 0$, which proves the result. $\square$

Let us come back to the description of $\mathcal{D}_0^t$. Since products of the resolvents $R(\lambda, z)$ span a dense subspace of $\mathcal{R}(\mathbb{C}^n, \sigma_t)$, Theorem 3.3.7 tells us that

$$\operatorname{Span}\{\mathcal{B}_t(R(\lambda_1, z_1)^{k_1} \dots R(\lambda_m, z_m)^{k_m});\ k_j \in \mathbb{N}_0,\ \lambda_j \in \mathbb{C} \setminus i\mathbb{R}, z_j \in \mathbb{C}^n\}$$

is dense in $\mathcal{D}_0^t$. We have already identified these Berezin transforms in Lemma 6.1.9 (up to a constant) as

$$res_{\Lambda,z,\mathbf{k}}^t(w) := \int_{(0,\infty)^m} \mathbf{s}^{\mathbf{k}-\mathbf{1}} e^{-(\tilde{\Lambda}\cdot\mathbf{s})-2i\sigma_t(w,\mathbf{s}\cdot z)-i(\sum_{j<\ell} s_j s_\ell \sigma_t(z_j,z_\ell)-\frac{1}{2t}|\mathbf{s}\cdot z|^2}\, d\mathbf{s},$$

where $\tilde{\Lambda} = (\operatorname{sgn}(\operatorname{Re}(\lambda_1))\lambda_1, \dots, \operatorname{sgn}(\operatorname{Re}(\lambda_m))\lambda_m)$, $\mathbf{k} = (k_1, \dots, k_m)$, $\mathbf{s} = (s_1, \dots, s_m)$, $\mathbf{1} = (1, \dots, 1)$ and $\mathbf{s} \cdot z = s_1 z_1 \dots s_m z_m$. Hence,

$$\mathcal{D}_0^t = \overline{\operatorname{Span}}\{res_{\Lambda,z,\mathbf{k}}^t;\ m \in \mathbb{N},\ \lambda_1, \dots, \lambda_m \in \mathbb{C} \setminus i\mathbb{R},\ k_1, \dots, k_m \in \mathbb{N},\ z_1, \dots, z_m \in \mathbb{C}^n\}.$$

As we have already stated, we have $\overline{\operatorname{Span}}(\alpha(\mathrm{FR})) \subseteq \mathcal{D}_0^t \subseteq \mathcal{R}$ for any $t > 0$ but so far cannot say anything else about $\mathcal{D}_0^t$. We have the following weak result:

Proposition 6.1.11. *The following holds true:*

$$\overline{\bigcup_{t>0} \mathcal{D}_0^t} = \mathcal{R}.$$

Proof. As we have seen earlier, for any $\lambda \in \mathbb{C} \setminus i\mathbb{R}$, $z \in \mathbb{C}^n$ we have

$$T_{(\lambda - 2i\sigma_t(\cdot, z))^{-1}} \in \mathcal{R}(\mathbb{C}^n, \sigma_t)$$

and therefore, by re-scaling z

$$T_{(\lambda - 2i\sigma_1(\cdot, z))^{-1}} \in \mathcal{R}(\mathbb{C}^n, \sigma_t).$$

Since $\mathcal{R}(\mathbb{C}^n, \sigma_t)$ is a C^* algebra,

$$T^t_{(\lambda_1 - 2i\sigma_1(\cdot,z_1))^{-1}} T^t_{(\lambda_2 - 2i\sigma_1(\cdot,z_2))^{-1}} \dots T^t_{(\lambda_m - 2i\sigma_1(\cdot,z_m))^{-1}} \in \mathcal{R}(\mathbb{C}^n, \sigma_t) = \mathcal{T}_{lin}^{2,t}(\mathcal{D}_0^t)$$

for any $\lambda_j \in \mathbb{C} \setminus i\mathbb{R}$ and $z_j \in \mathbb{C}^n$. As in the proof of Theorem 4.3.5, one obtains now

$$\begin{aligned} &\mathcal{B}_t(T^t_{(\lambda_1 - 2i\sigma_1(\cdot,z_1))^{-1}} T^t_{(\lambda_2 - 2i\sigma_1(\cdot,z_2))^{-1}} \dots T^t_{(\lambda_m - 2i\sigma_1(\cdot,z_m))^{-1}}) \\ &\quad \to (\lambda_1 - 2i\sigma_1(\cdot, z_1))^{-1} \dots (\lambda_m - 2i\sigma_1(\cdot, z_m))^{-1} \end{aligned}$$

uniformly as $t \to 0$. Since $\mathcal{B}_t(T^t_{(\lambda_1 - 2i\sigma_1(\cdot,z_1))^{-1}} \cdot \dots \cdot T^t_{(\lambda_m - 2i\sigma_1(\cdot,z_m))^{-1}}) \in \mathcal{D}_0^t$ for any $t > 0$, the result follows. $\square$

6.2 Remarks

As we have mentioned, this chapter is in principle only a report on work-in-progress. Nevertheless, it already fits in here quite nicely, as the Resolvent Algebra serves as an example of a Toeplitz algebra with non-trivial Correspondence Theory. Besides the obvious task, i.e. completing our understanding of the correspondence (is $\mathcal{D}_0^t = \mathcal{R}$ or a strict subset of $\mathcal{R}$ for fixed t?), we are planning to investigate the Toeplitz representation of the Resolvent Algebra on infinite-dimensional symplectic spaces. Further, even though many properties of the Resolvent Algebras $\mathcal{R}(\mathbb{C}^n, \sigma_t)$ are already known due to the works [41, 42], it seems tempting to study the structure of the Resolvent Algebra, e.g. the structure of its ideals, in the setting of its Bargmann representation. In particular, it seems interesting to see if the methods of Correspondence Theory are of any use here.

Chapter 7

The Berger-Coburn Theorem

The trivial estimate $\|T_f^t\| \lesssim \|f\|_\infty$, which holds for any $t > 0$ and $1 \leq p \leq \infty$, shows that a Toeplitz operator with bounded symbol is bounded. The converse statement is false: Indeed, there are many bounded Toeplitz operators with unbounded symbols. It is one of the most important open problems on Toeplitz operators over Fock spaces (and also over other spaces) to characterize their boundedness in terms of properties of the symbol. In the setting of Fock spaces, the best-known result in that direction is the following theorem due to C. A. Berger and L. A. Coburn:

Theorem 7.0.1 ([31]). *Let $f : \mathbb{C}^n \to \mathbb{C}$ be measurable such that $fK_z^t \in L_t^2$ for any $z \in \mathbb{C}^n$. Then, the following norm estimates hold true for the densely defined operator T_f^t on F_t^2:*

$$C(s)\|T_f^t\| \geq \|\widetilde{f}^{(s)}\|_\infty, \quad 2t > s > t/2,$$
$$c(s)\|\widetilde{f}^{(s)}\|_\infty \geq \|T_f^t\|, \quad t/2 > s > 0.$$

Here, $C(s),\ c(s) > 0$ are universal constants depending only on s, t and n.

We also want to mention the recent results in [50, 51], which head in a similar direction. Note that Berger and Coburn provided the proof for the above theorem only for $t = 2$, but it carries over to any $t > 0$. It is the aim of this chapter to derive analogous estimates for any $p \in [1, \infty]$. Further, we will derive similar estimates for the Schatten class norm of T_f^t in terms of the L^{p_0} norms of the Berezin transforms of f. Finally, we show a connection between results related to the Berger-Coburn estimates and the Correspondence Theory we have established earlier.

7.1 The first estimate

Let us first add a remark on the assumption that $f : \mathbb{C}^n \to \mathbb{C}$ satisfies $fK_z^t \in L_t^2$ for any $z \in \mathbb{C}^n$. Elementary computations show that this implies that $\widetilde{f}^{(s)}(z)$ exists for any $s < 2t$. We will use this fact without mentioning it.

Berger and Coburn proved the first estimate $\|\widetilde{f}^{(s)}\|_\infty \lesssim \|T_f^t\|$ by constructing a trace class operator $T_0^{(s)}$ such that

$$\operatorname{Tr}(T_f^t T_0^{(s)}) = \widetilde{f}^{(s)}(0)$$

and afterwards shifted the operators and used standard trace estimates. In principle, we will do the same thing and the operator $T_0^{(s)}$ will even turn out to be the same operator as in the Hilbert space case. Yet, some additional care must be spent for getting all the techniques working in the setting of $p \neq 2$.

Recall that $\{e_\alpha^t;\ \alpha \in \mathbb{N}_0^n\}$ denotes the standard Schauder basis of F_t^p (being orthonormal for $p = 2$). Then, denote for $k \in \mathbb{N}_0$ by P_k the finite rank operator

$$P_k := \sum_{|\alpha|=k} e_\alpha^t \otimes e_\alpha^t.$$

For $s > t/2$ we define

$$T_0^{(s)} := \sum_{k=0}^\infty \left(1 - \frac{t}{s}\right)^k P_k.$$

In particular, observe that by our choice of s we have $|1 - t/s| < 1$. The key fact for establishing the estimate will be the following:

Lemma 7.1.1. *Let $s > t/2$. Then, the infinite series defining $T_0^{(s)}$ converges in nuclear norm. In particular, $T_0^{(s)} \in \mathcal{N}(F_t^p)$ for any $p \in [1, \infty)$ and also $T_0^{(s)} \in \mathcal{N}(f_t^\infty)$.*

For proving this, we will need the following:

Lemma 7.1.2. *Let $p \in [1, \infty]$ and $t > 0$. If $q \in [1, \infty]$ denotes the exponent conjugate to p, i.e. $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\sup_{\alpha \in N_0^n} \|e_\alpha^t\|_{F_t^p} \|e_\alpha^t\|_{F_t^q} < \infty.$$

Proof. Using the product structure of the basis, it is easy to see that we only need to consider the case $n = 1$. For $p = \infty$ and $k \in \mathbb{N}_0$ we compute

$$\begin{aligned}
\|e_k^t\|_{F_t^\infty} &= \sup_{z \in \mathbb{C}} \frac{1}{\sqrt{k! t^k}} |z|^k e^{-\frac{|z|^2}{2t}} \\
&= \frac{1}{\sqrt{k! t^k}} \sup_{r \geq 0} r^k e^{-\frac{r^2}{2t}}.
\end{aligned}$$

It is a matter of elementary calculus to find this supremum in r as

$$\|e_k^t\|_{F_t^\infty} = \frac{1}{\sqrt{k!}} k^{k/2} e^{-\frac{k}{2}}.$$

On the other hand, for $p = 1$ we compute

$$\|e_k^t\|_{F_t^1} = \frac{1}{2\pi t} \int_{\mathbb{C}} \frac{|z|^k}{\sqrt{k! t^k}} e^{-\frac{|z|^2}{2t}}\ dz$$

$$
\begin{aligned}
&= \frac{1}{t\sqrt{k!t^k}} \int_0^\infty r^{k+1} e^{-\frac{r^2}{2t}} \, dr \\
&= \frac{1}{\sqrt{k!t^k}} (2t)^{k/2} \Gamma\left(\frac{k}{2}+1\right) \\
&= \frac{2^{k/2}}{\sqrt{k!}} \Gamma\left(\frac{k}{2}+1\right),
\end{aligned}
$$

and putting everything together we obtain

$$
\|e_k^t\|_{F_t^1} \|e_k^t\|_{F_t^\infty} = \frac{(2k)^{k/2}}{k!} \Gamma\left(\frac{k}{2}+1\right) e^{-\frac{k}{2}}. \tag{7.1}
$$

Recall now Stirling's approximation

$$
\Gamma(x) = \sqrt{\frac{2\pi}{x}} \left(\frac{x}{e}\right)^x \left(1+\mathcal{O}\left(\frac{1}{x}\right)\right) \qquad \text{as } x \to \infty.
$$

Applying this to Equation (7.1) yields

$$
\begin{aligned}
\|e_k^t\|_{F_t^1} \|e_k^t\|_{F_t^\infty} &= \frac{(2k)^{k/2}}{\sqrt{\frac{2\pi}{k+1}} \left(\frac{k+1}{e}\right)^{k+1} \left(1+\mathcal{O}\left(\frac{1}{k+1}\right)\right)} e^{-\frac{k}{2}} \\
&\quad \times \sqrt{\frac{2\pi}{\frac{k}{2}+1}} \left(\frac{\frac{k}{2}+1}{e}\right)^{\frac{k}{2}+1} \left(1+\mathcal{O}\left(\frac{1}{\frac{k}{2}+1}\right)\right) \\
&= \frac{(k(k+2))^{\frac{k}{2}} \left(\frac{k}{2}+1\right)^{1/2}}{(k+1)^{k+\frac{1}{2}}} \cdot \frac{1+\mathcal{O}\left(\frac{1}{\frac{k}{2}+1}\right)}{1+\mathcal{O}\left(\frac{1}{k+1}\right)} \\
&\longrightarrow \frac{1}{\sqrt{2}}, \quad k \to \infty,
\end{aligned}
$$

proving the claim for $p = 1, \infty$. For $p \in (1, \infty)$ this follows now immediately from Littlewood's inequality (Lemma 2.2.7):

$$
\begin{aligned}
\|e_k^t\|_{F_t^p} \|e_k^t\|_{F_t^q} &\le p^{\frac{n}{p}} q^{\frac{n}{q}} \|e_k^t\|_{F_t^1}^{\frac{1}{p}} \|e_k^t\|_{F_t^\infty}^{1-\frac{1}{q}} \|e_k^t\|_{F_t^1}^{\frac{1}{q}} \|e_k^t\|_{F_t^\infty}^{1-\frac{1}{p}} \\
&= p^{\frac{n}{p}} q^{\frac{n}{q}} \|e_k^t\|_{F_t^1} \|e_k^t\|_{F_t^\infty}.
\end{aligned}
$$

□

Proof of Lemma 7.1.1. Let $p \in [1, \infty)$ and C a suitable constant for the norm equivalence of F_t^q and $(F_t^p)'$. Then, we obtain

$$
\begin{aligned}
\sum_{k=0}^\infty \left|1 - \frac{t}{s}\right|^k \|P_k\|_{\mathcal{N}} &\le \sum_{k=0}^\infty \left|1 - \frac{t}{s}\right|^k \sum_{|\alpha|=k} \|e_\alpha^t\|_{F_t^p} \|e_\alpha^t\|_{(F_t^p)'} \\
&\le C \sum_{k=0}^\infty \left|1 - \frac{t}{s}\right|^k \sum_{|\alpha|=k} \|e_\alpha^t\|_{F_t^p} \|e_\alpha^t\|_{F_t^q}.
\end{aligned}
$$

Let us denote by $C_{p,t}$ the supremum obtained in the previous lemma. Then, we have just shown

$$\sum_{k=0}^{\infty} \left|1 - \frac{t}{s}\right|^k \|P_k\|_{\mathcal{N}} \leq CC_{p,t} \sum_{k=0}^{\infty} \left|1 - \frac{t}{s}\right|^k \cdot (\#\{\alpha \in \mathbb{N}_0^n;\ |\alpha| = k\}),$$

where # denotes the cardinality of the set. It is basic combinatorics to show that

$$\#\{\alpha \in \mathbb{N}_0^n;\ |\alpha| = k\} = \binom{k-1+n}{k}.$$

Therefore, we arrive at

$$\sum_{k=0}^{\infty} \left|1 - \frac{t}{s}\right|^k \|P_k\|_{\mathcal{N}} \leq CC_{p,t} \sum_{k=0}^{\infty} \left|1 - \frac{t}{s}\right|^k \binom{k-1+n}{k}.$$

The quotient test now yields that the right-hand side converges. Since $\mathcal{N}(F_t^p)$ is complete, we therefore obtain that the series

$$\sum_{k=0}^{\infty} \left(1 - \frac{t}{s}\right)^k P_k$$

converges in $\mathcal{N}(F_t^p)$ and hence $T_0^{(s)} \in \mathcal{N}(F_t^p)$.

The same arguments work over f_t^∞. □

At this point, Berger and Coburn applied what they called the *Berezin model* to compare $\widetilde{f}^{(s)}(z)$ with the trace $\mathrm{Tr}(T_0^{(s)} W_{-z}^t T_f^t W_z^t)$. This can be done more directly:

Lemma 7.1.3. *Let $s \in (\frac{t}{2}, t]$, $p \in [1, \infty)$ and $z \in \mathbb{C}^n$ such that $fK_z^t \in L_t^2$. Further, assume that $\alpha_{-z}(T_f^t)$ is bounded on either F_t^p or f_t^∞. Then, there is a constant C depending only on s, t and n such that*

$$\widetilde{f}^{(s)}(z) = C\,\mathrm{Tr}(T_0^{(s)} W_{-z}^t T_f^t W_z^t).$$

Proof. Without loss of generality, we may assume $z = 0$. Since F_t^p for $p \in [1, \infty)$ and also f_t^∞ has a Schauder basis, the spaces have the approximation property. In particular, the nuclear trace is well-defined. Since the nuclear operators form an ideal, we have $T_0^{(s)} T_f^t \in \mathcal{N}(F_t^p)$ or $\in \mathcal{N}(f_t^\infty)$. Then,

$$\begin{aligned}
\mathrm{Tr}(T_0^{(s)} T_f^t) &= \sum_{k=0}^{\infty} \left(1 - \frac{t}{s}\right)^k \sum_{|\alpha|=k} \mathrm{Tr}(P_k T_f^t) \\
&= \sum_{k=0}^{\infty} \left(1 - \frac{t}{s}\right)^k \sum_{|\alpha|=k} \langle T_f^t e_\alpha^t, e_\alpha^t \rangle_{F_t^2}
\end{aligned}$$

$$= \sum_{k=0}^{\infty} \left(1 - \frac{t}{s}\right)^k \sum_{|\alpha|=k} \langle f e_\alpha^t, e_\alpha^t \rangle_{F_t^2}$$
$$= \frac{1}{(\pi t)^n} \sum_{\alpha \in \mathbb{N}_0^n} \int_{\mathbb{C}^n} f(z) \frac{|z_1|^{2\alpha_1} \dots |z_n|^{2\alpha_n}}{\alpha!} \left(\frac{1}{t} - \frac{1}{s}\right)^{|\alpha|} e^{-\frac{|z|^2}{t}} \, dz.$$

For $s = t$ we obtain $T_0^{(s)} = P_{\mathbb{C}}$ and therefore

$$\operatorname{Tr}(T_0^{(s)} T_f^t) = \operatorname{Tr}(P_{\mathbb{C}} T_f^t) = P_{\mathbb{C}} * T_f^t(0) = \widetilde{T_f^t}(0) = \widetilde{f}^{(t)}(0).$$

Note that $f \in L_t^2$ implies $f \in L_s^1$ for $s \in (t/2, t)$. Hence, by the Dominated Convergence Theorem, we obtain for this case

$$\operatorname{Tr}(T_0^{(s)} T_f^t) = \frac{1}{(\pi t)^n} \int_{\mathbb{C}^n} f(z) \sum_{\alpha \in \mathbb{N}_0^n} \frac{|z_1|^{2\alpha_1} \dots |z_n|^{2\alpha_n}}{\alpha!} \left(\frac{1}{t} - \frac{1}{s}\right)^{|\alpha|} e^{-\frac{|z|^2}{t}} \, dz$$
$$= \left(\frac{s}{t}\right)^n \frac{1}{(\pi s)^n} \int_{\mathbb{C}^n} f(z) e^{-\frac{|z|^2}{s}} \, dz$$
$$= \left(\frac{s}{t}\right)^n \widetilde{f}^{(s)}(0).$$

□

Theorem 7.1.4. *Let $s \in \left(\frac{t}{2}, 2t\right)$, $p \in [1, \infty)$ and f such that $f K_z^t \in L_t^2$ for every $z \in \mathbb{C}^n$. Then,*

$$\|\widetilde{f}^{(s)}\|_\infty \leq C_{s,t,p,n} \|T_f^t\|_{F_t^p \to F_t^p}$$

and

$$\|\widetilde{f}^{(s)}\|_\infty \leq C_{s,t,\infty,n} \|T_f^t\|_{f_t^\infty \to f_t^\infty}.$$

Proof. Without loss of generality we may assume that T_f^t is bounded. Since the Weyl operators are isometric, this implies that $\alpha_{-z}(T_f^t)$ is bounded for any z. For $s \in \left(\frac{t}{2}, t\right]$ the result follows from the previous lemma: We have, by standard estimates for the trace,

$$|\widetilde{f}^{(s)}(z)| = C |\operatorname{Tr}(T_0^{(s)} \alpha_{-z}(T_f^t))| \leq C \|T_0^{(s)}\|_{\mathcal{N}} \|T_f^t\|.$$

If $s \in (t, 2t)$, then the semigroup property of the heat transform easily yields

$$\|\widetilde{f}^{(s)}\|_\infty \leq \|\widetilde{f}^{(t)}\|_\infty,$$

cf. also Equation (7.2) below. □

7.2 The second estimate

This section is based on joint work with Wolfram Bauer [15]. The standing assumption in this part is again that $f : \mathbb{C}^n \to \mathbb{C}$ is measurable with $fK_z^t \in L_t^2$ for any $z \in \mathbb{C}^n$. In particular, $\widetilde{f}^{(t)}(z) = \langle fk_z^t, k_z^t\rangle_{L_t^2}$ exists for any $z \in \mathbb{C}^n$. Let us first study the Berezin transform of f. Indeed, under above assumptions, for any $s \in (0,t)$ we have $fK_z^s \in L_s^2$ as a simple computation shows. Hence, $\widetilde{f}^{(s)}(z)$ exists for any such s. Since $\widetilde{f}^{(t)}$ is simply the *heat transform* of f (at time $t/4$), it enjoys the same semigroup property: For any $s \in (0,t)$ we have

$$\langle fk_z^t, k_z^t\rangle_t = \widetilde{f}^{(t)}(z) = \widetilde{\widetilde{f}^{(s)}}^{(t-s)}(z) = \langle \widetilde{f}^{(s)} k_z^{t-s}, k_z^{t-s}\rangle_{t-s}. \tag{7.2}$$

Getting rid of the normalizing factors, this is the same as

$$\langle fK_z^t, K_z^t\rangle_t = e^{-\frac{s}{t(t-s)}|z|^2} \langle \widetilde{f}^{(s)} K_z^{t-s}, K_z^{t-s}\rangle_{t-s}. \tag{7.3}$$

Recall that the bivariate Berezin transform is defined by

$$\widetilde{f}^{(t)}(z,w) := \langle fk_z^t, k_w^t\rangle_t.$$

Under our assumptions on f, this exists for any $z, w \in \mathbb{C}^n$. Equation (7.2) now extends to the off-diagonal values of $\widetilde{f}^{(t)}$ in the following way:

Lemma 7.2.1. *For $z, w \in \mathbb{C}^n$ and $0 < s < t$ the following holds true:*

$$\widetilde{f}^{(t)}(z,w) = e^{\frac{s}{2t(t-s)}|z-w|^2 - \frac{is}{t(t-s)}\operatorname{Im}(w\cdot\overline{z})} \langle \widetilde{f}^{(s)} k_z^{t-s}, k_w^{t-s}\rangle_{t-s}.$$

Proof. One readily verifies that $\langle fK_z^t, K_w^t\rangle_t$ is anti-holomorphic in z and holomorphic in w. The same holds true for

$$e^{-\frac{s}{t(t-s)}w\cdot\overline{z}} \langle \widetilde{f}^{(s)} K_z^{t-s}, K_w^{t-s}\rangle_{t-s}.$$

Both functions agree on the diagonal $z = w$ by Equation (7.3). Hence, a well-known theorem [69, Proposition 1.69] shows that the functions agree everywhere, i.e.

$$\langle fK_z^t, K_w^t\rangle_t = e^{-\frac{s}{t(t-s)}w\cdot\overline{z}} \langle \widetilde{f}^{(s)} K_z^{t-s}, K_w^{t-s}\rangle_{t-s}$$

for any $z, w \in \mathbb{C}^n$. Up to the normalizing factors, this is just the equation we wanted to prove. ☐

Lemma 7.2.2. *Let $t > 0$ and $g \in L^\infty(\mathbb{C}^n)$. Then, for any $z, w \in \mathbb{C}^n$ the following holds true:*

$$\left|\langle gk_z^t, k_w^t\rangle_t\right| \le \|g\|_\infty e^{-\frac{1}{4t}|w-z|^2}.$$

Proof. We obtain the estimates directly from the following calculations:

$$\begin{aligned}
|\langle g k_z^t, k_w^t \rangle_t| &= \frac{1}{\sqrt{K^t(z,z)K^t(w,w)}} \left| \int_{\mathbb{C}^n} g(u) e^{\frac{1}{t}(u\cdot\overline{w} + z\cdot\overline{u})} \, d\mu_t(u) \right| \\
&\leq e^{-\frac{1}{2t}(|z|^2+|w|^2)} \|g\|_\infty \int_{\mathbb{C}^n} e^{\frac{1}{t}\operatorname{Re}(u\cdot\overline{w}+z\cdot\overline{u})} \, d\mu_t(u) \\
&= e^{-\frac{1}{2t}(|z|^2+|w|^2)} \|g\|_\infty \int_{\mathbb{C}^n} e^{\frac{1}{2t}u\cdot\overline{(w+z)} + \frac{1}{2t}\overline{u}\cdot(w+z)} \, d\mu_t(u) \\
&= e^{-\frac{1}{2t}(|z|^2+|w|^2)} \|g\|_\infty \langle K^t_{(w+z)/2}, K^t_{(w+z)/2} \rangle_t \\
&= e^{-\frac{1}{2t}(|z|^2+|w|^2)} \|g\|_\infty K^t((w+z)/2, (w+z)/2) \\
&= e^{-\frac{1}{2t}(|z|^2+|w|^2) + \frac{1}{4t}|w+z|^2} \|g\|_\infty \\
&= e^{-\frac{1}{4t}|z-w|^2} \|g\|_\infty. \qquad \square
\end{aligned}$$

Let us go back to Toeplitz operators. Recall that we assumed $fK_z^t \in L_t^2$ for any $z \in \mathbb{C}^n$. For $p = 2$, this means $K_z^t \in D(T_f^t)$ for any z. Since the reproducing kernels span a dense subspace of F_t^2, the Toeplitz operator T_f^t is of course densely defined and we can consider its adjoint $(T_f^t)^*$. Recall the following characterization for the domain of the adjoint:

$$h \in D((T_f^t)^*) \Leftrightarrow \exists C > 0 : \ |\langle T_f^t g, h \rangle_t| \leq C \|g\|_{F_t^2} \text{ for every } g \in F_t^2.$$

Clearly, we have for any $g \in D(T_f^t) \subset F_t^2$ and any $z \in \mathbb{C}^n$:

$$|\langle T_f^t g, K_z^t \rangle_t| = |\langle g, \overline{f} K_z^t \rangle_t| \leq \|g\|_{F_t^2} \|f K_z^t\|_{F_t^2},$$

hence $K_z^t \in D((T_f^t)^*)$ for any $z \in \mathbb{C}^n$, which in particular means that the adjoint of T_f^t is also densely defined. From this, we obtain

$$\begin{aligned}
\overline{((T_f^t)^* K_z^t)(w)} &= \overline{\langle (T_f^t)^* K_z^t, K_w^t \rangle_t} \\
&= \langle T_f^t K_w^t, K_z^t \rangle_t \\
&= \langle f K_w^t, K_z^t \rangle_t \\
&= \sqrt{K^t(w,w)K^t(z,z)} \langle f k_w^t, k_z^t \rangle_t \\
&= e^{\frac{1}{2t}(|z|^2+|w|^2)} \widetilde{f}^{(t)}(w,z) \\
&= e^{\frac{1}{2t}|z-w|^2 + \frac{1}{t}\operatorname{Re}(z\cdot\overline{w})} \widetilde{f}^{(t)}(w,z).
\end{aligned}$$

This identity can be used to obtain the following integral representation for T_f^t: For any $g \in D(T_f^t)$ we have

$$\begin{aligned} T_f^t g(z) &= \langle T_f^t g, K_z^t \rangle_t \\ &= \langle g, (T_f^t)^* K_z^t \rangle_t \\ &= \int_{\mathbb{C}^n} g(w) \overline{((T_f^t)^* K_z^t)(w)} \, d\mu_t(w) \\ &= \int_{\mathbb{C}^n} e^{\frac{1}{2t}|z-w|^2 + \frac{1}{t}\operatorname{Re}(z\cdot\overline{w})} \widetilde{f}^{(t)}(w,z) g(w) \, d\mu_t(w). \end{aligned}$$

Recall that for $p \neq 2$ the Toeplitz operator T_f^t is by definition the same integral operator as for $p = 2$. It is not a priori clear that the operator is densely defined for $p < 2$, as it may not map $\operatorname{Span}\{K_t^t;\ z \in \mathbb{C}^n\}$ into F_t^p. If we assume that $\widetilde{f}^{(s)}$ is bounded for some $s \in (0, t/2)$, which is the assumption of the second estimate of the Berger-Coburn Theorem, this will follow from the next lemma. Here, we will always denote by $I_f^t g$ the integral operator

$$I_f^t g(z) := \int_{\mathbb{C}^n} e^{\frac{1}{2t}|z-w|^2 + \frac{1}{t}\operatorname{Re}(z\cdot\overline{w})} \widetilde{f}^{(t)}(w,z) g(w) \, d\mu_t(w),$$

which we formally define on all g such that the above integral exists. Further, we will denote by $\gamma_{s,t}$ the constant

$$\gamma_{s,t} := \frac{1}{4(t-s)} - \frac{s}{2t(t-s)}.$$

We need to assume $s < t/2$ in the following because $\gamma_{s,t} > 0$ if and only if $s < t/2$.

Lemma 7.2.3. *Let $t > 0$ and $f : \mathbb{C}^n \to \mathbb{C}$ measurable such that $fK_z^t \in L_t^2$ for any $z \in \mathbb{C}^n$ and $\widetilde{f}^{(s)} \in L^\infty(\mathbb{C}^n)$ for some $s \in (0, t/2)$. Then, for any $g \in L_t^1$ we have*

$$\|I_f^t g\|_{L_t^1} \leq \left(\frac{1}{\gamma_{s,t} t}\right)^n \|\widetilde{f}^{(s)}\|_\infty \|g\|_{L_t^1},$$

i.e. the integral operator I_f^t is bounded on L_t^1 with $\|I_f^t\|_{L_t^1 \to L_t^1} \leq \left(\frac{1}{\gamma_{s,t} t}\right)^n \|\widetilde{f}^{(s)}\|_\infty$.

Proof. Using the simple identity $e^{\frac{1}{2t}|z-w|^2} = e^{\frac{1}{2t}(|z|^2 + |w|^2 - 2\operatorname{Re}(z\cdot\overline{w}))}$, we derive the estimate as follows:

$$\begin{aligned} &\|I_f^t g\|_{L_t^1} \\ &= \int_{\mathbb{C}^n} |I_f^t g(z)| \, d\mu_{2t}(z) \\ &= \left(\frac{1}{2t^2\pi^2}\right)^n \int_{\mathbb{C}^n} \left| \int_{\mathbb{C}^n} e^{\frac{1}{2t}|z-w|^2 + \frac{1}{t}\operatorname{Re}(z\cdot\overline{w}) - \frac{1}{t}|w|^2 - \frac{1}{2t}|z|^2} \widetilde{f}^{(t)}(w,z) g(w) \, dw \right| dz \\ &\leq \left(\frac{1}{2t^2\pi^2}\right)^n \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} |g(w)| |\widetilde{f}^{(t)}(w,z)| e^{-\frac{1}{2t}|w|^2} \, dw \, dz. \end{aligned}$$

Combining Lemmas 7.2.1 and 7.2.2 we obtain the estimate

$$|\widetilde{f}^{(t)}(w,z)| \leq \|\widetilde{f}^{(s)}\|_\infty e^{-\gamma_{s,t}|z-w|^2} \tag{7.4}$$

and therefore

$$\begin{aligned}
\|I_f^t g\|_{L_t^1} &\leq \left(\frac{1}{2t^2\pi^2}\right)^n \|\widetilde{f}^{(s)}\|_\infty \int_{\mathbb{C}^n}\int_{\mathbb{C}^n} e^{-\gamma_{s,t}|w-z|^2} e^{-\frac{1}{2t}|w|^2}|g(w)|\ dw\ dz \\
&= \|\widetilde{f}^{(s)}\|_\infty \|g\|_{L_t^1} \left(\frac{1}{\pi t}\right)^n \int_{\mathbb{C}^n} e^{-\gamma_{s,t}|z|^2}\ dz \\
&= \left(\frac{1}{\gamma_{s,t}t}\right)^n \|\widetilde{f}^{(s)}\|_\infty \|g\|_{L_t^1}.
\end{aligned}$$

□

Note that I_f^t actually maps L_t^1 to F_t^1: It is still the same integral expression as for T_f^t, hence a standard application of Morera's Theorem yields that functions in the image are always holomorphic.

By the above estimate, we know that under the assumptions of above lemma, T_f^t actually maps $\mathrm{Span}\{K_z^t;\ z \in \mathbb{C}^n\}$ into F_t^1. We obtain that for any $p \in [1,\infty]$, T_f^t is a densely defined operator on F_t^p.

Theorem 7.2.4. *Under the assumptions of Lemma 7.2.3, I_f^t is a bounded operator on L_t^p for any $p \in [1,\infty]$ with norm*

$$\|I_f^t\|_{L_t^p \to L_t^p} \leq \|\widetilde{f}^{(s)}\|_\infty \left(\frac{1}{\gamma_{s,t}t}\right)^n.$$

Proof. We have already seen that the statement is true for $p = 1$. Let us show the norm estimate for $p = \infty$. For $g \in L_t^\infty$ we have, using again Estimate (7.4),

$$\begin{aligned}
&\|I_f^t g\|_{L_t^\infty} \\
&= \sup_{z\in\mathbb{C}^n} |I_f^t g(z)| e^{-\frac{|z|^2}{2t}} \\
&= \left(\frac{1}{\pi t}\right)^n \sup_{z\in\mathbb{C}^n} \left| \int_{\mathbb{C}^n} e^{\frac{1}{2t}|z-w|^2 + \frac{1}{t}\operatorname{Re}(z\cdot\overline{w}) - \frac{1}{t}|w|^2 - \frac{1}{2t}|z|^2} \widetilde{f}^{(t)}(w,z) g(w)\ dw \right| \\
&\leq \left(\frac{1}{\pi t}\right)^n \|\widetilde{f}^{(s)}\|_\infty \sup_{z\in\mathbb{C}^n} \|\widetilde{f}^{(s)}\|_\infty \int_{\mathbb{C}^n} e^{(\frac{1}{2t}-\gamma_{s,t})|z-w|^2 + \frac{1}{t}\operatorname{Re}(z\cdot\overline{w}) - \frac{1}{t}|w|^2 - \frac{1}{2t}|z|^2} |g(w)|\ dw \\
&\leq \left(\frac{1}{\pi t}\right)^n \|\widetilde{f}^{(s)}\|_\infty \|g\|_{L_t^\infty} \sup_{z\in\mathbb{C}^n} \int_{\mathbb{C}^n} e^{-\gamma_{s,t}|z-w|^2}\ dw \\
&= \|\widetilde{f}^{(s)}\|_\infty \|g\|_{L_t^\infty} \left(\frac{1}{\gamma_{s,t}t}\right)^n.
\end{aligned}$$

Thus, we also have the estimate

$$\|I_f^t\|_{L_t^\infty \to L_t^\infty} \leq \|\widetilde{f}^{(s)}\|_\infty \left(\frac{1}{\gamma_{s,t}t}\right)^n.$$

Combining this with the L_t^1-estimate from the previous lemma and applying the Complex Interpolation Method, we obtain

$$\|I_f^t\|_{L_t^p \to L_t^p} \leq \|\widetilde{f}^{(s)}\|_\infty \left(\frac{1}{\gamma_{s,t} t}\right)^n$$

for any $p \in [1, \infty]$. □

Corollary 7.2.5. *Under the assumptions of Lemma 7.2.3, T_f^t is bounded on F_t^p for any $p \in [1, \infty]$ and on f_t^∞ with*

$$\|T_f^t\| \leq \|\widetilde{f}^{(s)}\|_\infty \left(\frac{1}{\gamma_{s,t} t}\right)^n$$

on any of these spaces.

Proof. As for the case $p = 1$, one easily sees that the image of I_f^t is holomorphic for any p. Further, T_f^t is simply the restriction of I_f^t to F_t^p, hence by the theorem we have

$$\|T_f^t\|_{F_t^p \to F_t^p} \leq \|\widetilde{f}^{(s)}\|_\infty \left(\frac{1}{\gamma_{s,t} t}\right)^n$$

for any $p \in [1, \infty]$. Using the Complex Interpolation Method we have $(F_t^1, F_t^\infty)_{[1]} = f_t^\infty$ as already mentioned earlier, hence $T_f^t \in \mathcal{L}(f_t^\infty)$ with the same norm estimate. □

7.3 $L^{p_0} - \mathcal{S}^{p_0}$ versions of the estimates

In [14], estimates related to the Berger-Coburn estimates were obtained. In particular, the following results were shown:

Theorem 7.3.1 ([14])**.** *Let $f : \mathbb{C}^n \to \mathbb{C}$ measurable be such that $fK_z^t \in L_t^2$ for any $z \in \mathbb{C}^n$. Then, considering T_f^t as an operator on F_t^2, we have:*

1) If $s \in (0, \frac{t}{2})$ and $\widetilde{f}^{(s)} \in C_0(\mathbb{C}^n)$, then T_f^t is compact.

2) If $s \in (\frac{t}{2}, 2t)$ and T_f^t is compact, then $\widetilde{f}^{(s)} \in C_0(\mathbb{C}^n)$.

3) If $s \in (0, \frac{t}{2})$ and $p_0 \in [1, \infty)$, then there is a constant $C = C(s, t, N, p_0) > 0$ such that $\|T_f^t\|_{\mathcal{S}^{p_0}(F_t^2)} \leq C\|\widetilde{f}^{(s)}\|_{L^{p_0}}$[1].

The authors of [14] also conjectured an estimate of the form

$$\|\widetilde{f}^{(s)}\|_{L^{p_0}} \leq C\|T_f^t\|_{\mathcal{S}^{p_0}}, \quad s \in (t/2, 2t),$$

but could not prove it (cf. "Question 1" in the paper). We can now fill this gap.

[1]Actually, the authors of [14] only showed that $\widetilde{f}^{(s)} \in L^{p_0}$ implies $T_f^t \in \mathcal{S}^{p_0}(F_t^2)$. An estimate of this form can be obtained by combining their reasoning with the results in e.g. [4]

Since the convolutions discussed in Chapter 3 are only well-behaved on F_t^p for $p \in (1, \infty)$ and on f_t^∞, we formulate the following proposition only for those cases. Nevertheless, it should be possible to include the case of F_t^1, since the operators involved are "nice" also in this case (i.e. $z \mapsto \alpha_z(T_0^{(s)})$ is $\mathcal{N}$-continuous even in this case).

Proposition 7.3.2. *Let $s \in (\frac{t}{2}, 2t)$ and $f : \mathbb{C}^n \to \mathbb{C}$ measurable such that $fK_z^t \in L_t^2$ for any $z \in \mathbb{C}^n$. Further, let $p_0 \in [1, \infty)$. Then, for every $p \in (1, \infty)$ there is are constant $C_{n,s,t,p,p_0} > 0$ such that we have*

$$\|\widetilde{f}^{(s)}\|_{L^{p_0}} \leq C_{n,s,t,p,p_0} \|T_f^t\|_{\mathcal{S}^{p_0}(F_t^p)}.$$

Further, there is a constant $C_{n,s,t,\infty,p_0} > 0$ such that

$$\|\widetilde{f}^{(s)}\|_{L^{p_0}} \leq C_{n,s,t,\infty,p_0} \|T_f^t\|_{\mathcal{S}^{p_0}(f_t^\infty)}.$$

Proof. Assume that $s \in (\frac{t}{2}, t]$. Further, assume that $\|T_f^t\|_{\mathcal{S}^{p_0}}$ is finite, otherwise the statement is trivial. The operator $T_0^{(s)}$ introduced in Section 7.1 is easily seen to be U-invariant, i.e. $T_0^{(s)} = UT_0^{(s)}U$. Further, we have seen in Lemma 7.1.3 that

$$\widetilde{f}^{(s)}(z) = C \operatorname{Tr}(T_0^{(s)} W_{-z}^t T_f^t W_z^t).$$

By U-invariance of $T_0^{(s)}$ we now obtain

$$\operatorname{Tr}(T_0^{(s)} W_{-z}^t T_f^t W_z^t) = \operatorname{Tr} UT_0^{(s)} UW_{-z} T_f^t W_z^t = \operatorname{Tr}(T_0^{(s)} W_z^t U T_f^t U W_{-z}^t) = T_0^{(s)} * T_f^t(z).$$

Now, Lemma 3.1.15 implies

$$\int_{\mathbb{C}^n} |\widetilde{f}^{(s)}(z)|^{p_0} \, dz = C \|T_0^{(s)} * T_f^t\|_{L^{p_0}}^{p_0} \lesssim \|T_0^{(s)}\|_{\mathcal{N}} \|T_f^t\|_{\mathcal{S}^{p_0}}.$$

For $s \in (t, 2t)$ the result follows from simply from the estimates

$$\|\widetilde{f}^{(s)}\|_{L^{p_0}} \lesssim \|\widetilde{f}^{(t)}\|_{L^{p_0}},$$

which is either a consequence of the contractivity of the heat semigroup or follows immediately from properties of the convolutions, using $\widetilde{f}^{(s)} = \widetilde{f}^{(t)} * f_{s-t}$, where f_{s-t} is as usual the appropriate Gaussian. ☐

7.4 Berger-Coburn type results and their connection to Correspondence Theory

Recalling Theorem 7.3.1, there is a necessary and sufficient criterion for the membership of a Toeplitz operator with unbounded symbol in the class of compact operators. Indeed, a generalization of these results can easily be obtained using Correspondence Theory, as we shall outline now. While we are confident that these result do not hinge on the restriction $p \in (1, \infty)$, we shall only formulate the results for this case. The reason for this is that certain key facts on sufficiently localized operators, that we will introduce in Chapter 8, are so far only available for the reflexive case.

Proposition 7.4.1. *Assume $p \in (1, \infty)$ and $t > 0$. Let $\mathcal{D}_0 \subseteq \mathrm{BUC}(\mathbb{C}^n)$ be an α-invariant and closed subspace. If $f : \mathbb{C}^n \to \mathbb{C}$ is such that $fK_z^t \in L_t^2$ for any $z \in \mathbb{C}^n$ and further $\widetilde{f}^{(s)} \in \mathcal{D}_0$ for some $s \in (0, t/2)$, then $T_f^t \in \mathcal{T}_{lin}^{p,t}(\mathcal{D}_0)$.*

Proof. We already know that T_f^t is a bounded operator in these cases. From Remark 8.0.7 we will see that we even have $T_f^t \in \mathcal{T}^{p,t}$. We defer this argument for a moment, as it will be based on arguments using *sufficiently localized operators* that we will only introduce later. Now, assuming $\widetilde{f}^{(s)} \in \mathcal{D}_0$, we of course obtain $\widetilde{T_f^t} = \widetilde{f}^{(t)} = \widetilde{f}^{(s)} * f_{t-s} \in \mathcal{D}_0$, where f_{t-s} is the appropriate Gaussian. The Correspondence Theorem 3.3.7 therefore yields $T_f^t \in \mathcal{T}_{lin}^{p,t}(\mathcal{D}_0)$. □

Proposition 7.4.2. *Assume $p \in (1, \infty)$ and $t > 0$. Let $\mathcal{D}_0 \subseteq \mathrm{BUC}(\mathbb{C}^n)$ be an α-invariant and closed subspace. If $f : \mathbb{C}^n \to \mathbb{C}$ is such that $fK_z^t \in L_t^2$ for any $z \in \mathbb{C}^n$ and further $T_f^t \in \mathcal{T}_{lin}^{p,t}(\mathcal{D}_0)$, then $\widetilde{f}^{(s)} \in \mathcal{D}_0$ for any $s \in (t/2, 2t)$.*

Proof. Assuming T_f^t is bounded, we have seen that $\widetilde{f}^{(s)} = T_0^{(s)} * T_f^t$. We claim that $T_0^{(s)} * A \in \mathcal{D}_0$ for any $A \in \mathcal{T}_{lin}^{p,t}(\mathcal{D}_0)$. Indeed, it actually holds true that $N * A \in \mathcal{D}_0$ for any $A \in \mathcal{T}_{lin}^{p,t}(\mathcal{D}_0)$ and $N \in \mathcal{N}(F_t^p)$. Once we have proven this, the result follows since $T_0^{(s)}$ is nuclear.

The Correspondence Theorem 3.3.7 in particular states that $\widetilde{A} = P_{\mathbb{C}} * A \in \mathcal{D}_0$ for any $A \in \mathcal{T}_{lin}^{p,t}(\mathcal{D}_0)$. Since $\mathcal{D}_0$ is α-invariant, we also obtain $\alpha_z(P_{\mathbb{C}}) * A = \alpha_z(\widetilde{A}) \in \mathcal{D}_0$. Recall that, under trace duality, $\mathcal{N}(F_t^p)' \cong \mathcal{L}(F_t^p)$. If we have $B \in \mathcal{L}(F_t^p)$ such that

$$\mathrm{Tr}(\alpha_z(P_{\mathbb{C}})B) = P_{\mathbb{C}} * B(z) = \widetilde{B}(z) = 0$$

for every $z \in \mathbb{C}^n$, then this implies by the injectivity of the Berezin transform that $B = 0$. In particular, we obtain

$$\langle N, B \rangle_{tr} = 0 \text{ for every } N \in \mathrm{Span}\{\alpha_z(P_{\mathbb{C}});\ z \in \mathbb{C}^n\} \Longrightarrow B = 0$$

for every $B \in \mathcal{L}(F_t^p) \cong \mathcal{N}(F_t^p)'$. By the Hahn-Banach Theorem, this immediately implies that $\mathrm{Span}\{\alpha_z(P_{\mathbb{C}});\ z \in \mathbb{C}^n\}$ is dense in $\mathcal{N}(F_t^p)$. By continuity of the convolution, this gives $N * A \in \mathcal{D}_0$ for every $N \in \mathcal{N}(F_t^p)$ and $A \in \mathcal{T}_{lin}^{p,t}(\mathcal{D}_0)$. □

Letting $\mathcal{D}_0 = C_0(\mathbb{C}^n)$ and $p = 2$, we of course obtain the statements 1) and 2) in Theorem 7.3.1.

7.5 Remarks

The possibly most important task concerning the topic is of course achieving a proof or finding a counterexample to the Berger-Coburn conjecture, which states that T_f^t is bounded if and only if $\widetilde{f}^{(t/2)}$ is bounded. There are several hints that this might indeed be true and the conjecture has been proven in several particular cases (e.g. $f \geq 0$ or $f \in \mathrm{BMO}(\mathbb{C}^n)$, cf. our discussion in [15]). Recently, there have been advances on the problem for a particular class of symbols by applying methods of Fourier Integral Operators on $\mathbb{C}^n$, cf. [50, 51]. Nevertheless, a proof of the full conjecture seems to be out of reach as of now.

Since we have seen that the original Berger-Coburn estimates are not related to the Hilbert space structure, we expect that the statement of Theorem 7.3.1, part 3) also do not depend on $p = 2$. The proof of part 3) of that theorem, as given by the authors of [14], depends on rather strong results on Schatten class properties of Weyl Pseudodifferential Operators, which is used after adjoining the Bargmann transform. It could be interesting to search for a proof which works directly in the Fock space setting and is independent of the Hilbert space structure.

There are of course also other interesting questions in the same spirit. For example, it is an open problem to give a criterion for the boundedness of a product of two Toeplitz operators. There are some results for very particular symbols [45], but the general case seems to be wide open.

Chapter 8

Characterizations of the Toeplitz algebra

During this section, we will restrict ourselves to the reflexive cases: $p \in (1, \infty)$. We want to discuss ways to check if a bounded operator belongs to the Toeplitz algebra $\mathcal{T}^{p,t}$ over F_t^p, i.e. we are looking for different characterizations of $\mathcal{T}^{p,t}$. We have already given several results in that direction, cf. Corollary 3.3.4:

$$\mathcal{T}^{p,t} = \mathcal{T}_{lin}^{p,t}(\mathrm{BUC}(\mathbb{C}^n)) = \mathcal{C}_1.$$

Since convolution by the Gaussians

$$f_s(z) = \frac{1}{(\pi s)^n} e^{-\frac{|z|^2}{s}}$$

was shown to be an approximate identity of $\mathcal{C}_1$ in Lemma 3.3.1 and any convolution (by f_s) is contained in $\mathcal{C}_1$ by Proposition 3.1.13, we also know that

$$\mathcal{T}^{p,t} = \{A \in \mathcal{L}(F_t^p);\ f_s * A \to A \text{ in operator norm as } s \to 0\}.$$

Finally, an application of the Cohen-Hewitt Factorization Theorem (see [55]) yields

$$\mathcal{T}^{p,t} = \{g * B;\ g \in L^1(\mathbb{C}^n), B \in \mathcal{L}(F_t^p)\}.$$

In this short chapter, we will discuss a few additional characterizations of $\mathcal{T}^{p,t}$.

In [132], the notion of *sufficiently localized operators* on Fock spaces was introduced. Let $A \in \mathcal{L}(F_t^p)$. We say that A is sufficiently localized if

$$|\langle Ak_z^t, k_w^t\rangle_t| \leq \frac{C}{(1 + |z - w|)^\beta}$$

for some constants $C > 0$, $\beta > 2n$. We denote the set of sufficiently localized operators on F_t^p by $\mathcal{A}_{sl}^{p,t}$. In [90], the notion was further generalized: $A \in \mathcal{L}(F_t^p)$ is said to be *weakly localized* if, for $B \in \{A, A^*\}$ we have

$$\sup_{z \in \mathbb{C}^n} \int_{\mathbb{C}^n} |\langle Bk_z^t, k_w^t\rangle_t|\ dw < \infty$$

as well as

$$\lim_{R\to\infty} \sup_{z\in\mathbb{C}^n} \int_{\mathbb{C}^n\setminus B(z,R)} |\langle Bk_z^t, k_w^t\rangle_t| \ dw = 0.$$

Here, A^* is of course understood as an operator on F_t^q with $\frac{1}{p} + \frac{1}{q} = 1$. The set of weakly localized operators is denoted by $\mathcal{A}_{wl}^{p,t}$. It has been known for a while that $\mathcal{T}^{p,t} \subseteq \overline{\mathcal{A}_{sl}^{p,t}} \subseteq \overline{\mathcal{A}_{wl}^{p,t}}$, where closures are taken in operator norm [90]. In [131], J. Xia surprisingly managed to prove that

$$\mathcal{T}^{2,t} = \mathcal{T}_{lin}^{2,t}(L^\infty(\mathbb{C}^n)) = \overline{\mathcal{A}_{sl}^{2,t}} = \overline{\mathcal{A}_{wl}^{2,t}}.$$

Our results from Corollary 3.3.4 are therefore extending parts of Xia's result to the case $p \neq 2$. The full extension of Xia's result to the case $p \neq 2$ was recently proven, based on our characterization $\mathcal{T}^{p,t} = \mathcal{C}_1$, by R. Hagger, who also added two further characterizations:

Theorem 8.0.1 ([79])**.** *For any $p \in (1,\infty)$ and $t > 0$ the following holds true:*

$$\begin{aligned}\mathcal{T}^{p,t} &= \overline{A_{sl}}\\ &= \overline{A_{wl}}\\ &= P_t \operatorname{BDO}_t^p P_t\\ &= \{A \in \mathcal{L}(F_t^p);\ [A, T_f^t] \in \mathcal{K}(F_t^p) \text{ for all } f \in \operatorname{VO}_\partial(\mathbb{C}^n)\}.\end{aligned}$$

Here, we used the (somewhat imprecise) notation

$$P_t \operatorname{BDO}_t^p P_t := \{A \in \mathcal{L}(F_t^p);\ AP_t \in \operatorname{BDO}_t^p\}.$$

To all the characterizations mentioned so far, we will add a few more. Recall that

$$\begin{aligned}L_t^p &= \{f : \mathbb{C}^n \to \mathbb{C};\ \int_{\mathbb{C}^n} |f(z)|^p e^{-\frac{p|z|^2}{2t}} \ dz < \infty\}\\ &= \{f : \mathbb{C}^n \to \mathbb{C};\ fe^{-\frac{|\cdot|^2}{2t}} \in L^p(\mathbb{C}^n)\}.\end{aligned}$$

Consider the inductively defined sequence

$$c_0^t = 0, \quad c_{k+1}^t = \frac{1}{4t(1 - c_k^t t)}.$$

Then, for all k we have $0 \leq c_k^t < \frac{1}{2t}$ and c_k^t is strictly increasing towards $\frac{1}{2t}$. Define

$$\mathcal{D}_t^k := \{f : \mathbb{C}^n \to \mathbb{C};\ \|f\|_{\mathcal{D}_t^k} := \operatorname*{ess\,sup}_{z\in\mathbb{C}^n} |f(z)e^{-c_k^t|z|^2}| < \infty\}.$$

$\mathcal{D}_t^k$ is easily seen to be a Banach space (in principle, $\mathcal{D}_t^k = L_{1/(2c_k^t)}^\infty$). Further, $\mathcal{D}_t^k \subset L_t^p$. Set

$$\mathcal{D}_t := \bigcup_{k\in\mathbb{N}_0} \mathcal{D}_t^k.$$

Then, we obtain the following scale of Banach spaces in L_t^p:

$$L^\infty(\mathbb{C}^n) = \mathcal{D}_t^0 \subset \mathcal{D}_t^1 \subset \cdots \subset \mathcal{D}_t^k \subset \cdots \subset \mathcal{D}_t \subset L_t^p.$$

This scale of Banach spaces was seemingly first considered (in the Hilbert space case) in [16]. Since we are mainly interested in operators acting on spaces of holomorphic functions, we also introduce

$$\mathcal{H}_t^k := \mathcal{D}_t^k \cap \operatorname{Hol}(\mathbb{C}^n), \quad \mathcal{H}_t = \bigcup_{k \in \mathbb{N}_0} \mathcal{H}_t^k = \mathcal{D}_t \cap \operatorname{Hol}(\mathbb{C}^n).$$

Then, we obtain again a scale of Banach spaces:

$$\mathbb{C} \cong \mathcal{H}_t^0 \subset \mathcal{H}_t^1 \subset \cdots \subset \mathcal{H}_t^k \subset \cdots \subset \mathcal{H}_t \subset F_t^p. \tag{8.1}$$

Here is an important fact regarding this scale of Banach spaces and Toeplitz operators:

Proposition 8.0.2 ([16]). *Let $g \in L^\infty(\Omega)$. Then, for any $f \in \mathcal{H}_t^k$ we have*

$$\|T_g^t f\|_{\mathcal{H}_t^{k+1}} \leq \frac{\|g\|_\infty \|f\|_{\mathcal{H}_t^k}}{(1 - c_k^t t)^n}.$$

Since the computations in [16] are pretty straightforward and explain the concept quite well, we decided to include them here for completeness:

Proof. For any $z \in \mathbb{C}^n$ we have

$$\begin{aligned}
|T_g^t f(z)| &\leq \frac{1}{(\pi t)^n} \int_{\mathbb{C}^n} |f(w)||g(w)| e^{\frac{\operatorname{Re}(z \cdot \overline{w})}{t} - \frac{|w|^2}{t}} \, dw \\
&\leq \frac{1}{(\pi t)^n} \|g\|_\infty \|f\|_{\mathcal{H}_t^k} \int_{\mathbb{C}^n} e^{\frac{\operatorname{Re}(z \cdot \overline{w})}{t} - \left(\frac{1}{t} - c_k^t\right)|w|^2} \, dw \\
&= \frac{1}{(\pi t)^n} \frac{1}{(1 - c_k^t t)^n} \|g\|_\infty \|f\|_{\mathcal{H}_t^k} \int_{\mathbb{C}^n} e^{2\frac{\operatorname{Re}(z \cdot \overline{w})}{2t\sqrt{1 - c_k^t t}} - \frac{1}{t}|w|^2} \, dw \\
&= \frac{\|g\|_\infty \|f\|_{\mathcal{H}_t^k}}{(1 - c_k^t t)^n} \left\langle K^t_{\frac{z}{2\sqrt{1 - c_k^t t}}}, K^t_{\frac{z}{2\sqrt{1 - c_k^t t}}} \right\rangle_t \\
&= \frac{\|g\|_\infty \|f\|_{\mathcal{H}_t^k}}{(1 - c_k^t t)^n} e^{\frac{|z|^2}{4t(1 - c_k^t t)}} \\
&= \frac{\|g\|_\infty \|f\|_{\mathcal{H}_t^k}}{(1 - c_k^t t)^n} e^{c_{k+1}^t |z|^2}.
\end{aligned}$$

Hence, for any $z \in \mathbb{C}^n$:

$$|T_g^t f(z)| e^{-c_{k+1}^t |z|^2} \leq \frac{\|g\|_\infty \|f\|_{\mathcal{H}_t^k}}{(1 - c_k^t t)^n},$$

which is exactly the statement we wanted to prove. □

In particular, this implies that any Toeplitz operator with bounded symbol leaves $\mathcal{H}_t$ invariant. Since the Weyl operators W_z^t are themselves Toeplitz operators with bounded symbols, we obtain the following: Whenever $A \in \mathcal{L}(F_t^p)$ is such that it leaves $\mathcal{H}_t$ invariant, then $\alpha_z(A)$ leaves $\mathcal{H}_t$ invariant for any $z \in \mathbb{C}^n$. This justifies the following definition taken from [16]:

Definition 8.0.3. A linear operator $A \in \mathcal{L}(F_t^p)$ satisfying $A(\mathcal{H}_t) \subseteq \mathcal{H}_t$ is said to act *uniformly continuously* on the scale (8.1) if for any $k_1 \in \mathbb{N}_0$ there exist $k_2 \geq k_1,\ d > 0$ such that for all $z \in \mathbb{C}^n,\ f \in \mathcal{H}_t^{k_1}$:

$$\|\alpha_z(A)f\|_{\mathcal{H}_t^{k_2}} \leq d\|f\|_{\mathcal{H}_t^{k_1}}.$$

We will denote by $\mathcal{A}_{uc}^{p,t}$ the set of all bounded operators on F_t^p which act uniformly continuously on (8.1). Since the product of two uniformly continuously acting operators is easily seen to act uniformly continuously again, this is indeed an algebra.

Proposition 8.0.2 yields that the sum of products of Toeplitz operators with bounded symbols is contained in $\mathcal{A}_{uc}^{p,t}$. Thus, we clearly have

$$\mathcal{T}^{p,t} \subseteq \overline{\mathcal{A}_{uc}^{p,t}},$$

where the closure is taken with respect to the operator norm. In [15], W. Bauer and the author presented the following result for the case $p = 2$, $t = 1$. The proof carries over to arbitrary p and t.

Proposition 8.0.4. *Let $t > 0$ and $p \in (1, \infty)$. Then, we have the equality*

$$\mathcal{T}^{p,t} = \overline{\mathcal{A}_{uc}^{p,t}}.$$

Proof. Having Theorem 8.0.1 at hand, it suffices to prove that any $A \in \mathcal{A}_{uc}^{p,t}$ is sufficiently localized. This can be seen as follows:

$$\begin{aligned}
\langle Ak_z^t, k_w^t\rangle_t &= \langle AW_z^t 1, W_z^t W_{-z}^t W_w^t 1\rangle_t \\
&= \langle \alpha_{-z}(A)1, W_{-z}^t W_w^t 1\rangle_t \\
&= \langle \alpha_{-z}(A)1, W_{w-z}^t 1\rangle_t e^{-\frac{i\operatorname{Im}(z\cdot\overline{w})}{t}} \\
&= \langle \alpha_{-z}(A)1, k_{w-z}^t\rangle_t e^{-\frac{i\operatorname{Im}(z\cdot\overline{w})}{t}}.
\end{aligned}$$

This gives us, for an appropriate choice of $k \in \mathbb{N}$ and $d > 0$ such that $\|\alpha_z(A)1\|_{\mathcal{H}_t^k} \leq d\|1\|_{H_t^0} = d$:

$$\begin{aligned}
|\langle Ak_z^t, k_w^t\rangle_t| &\leq \frac{1}{(\pi t)^n}\int_{\mathbb{C}^n} |[\alpha_{-z}(A)1](u)k_{w-z}^t(u)|e^{-\frac{|u|^2}{t}}\, du \\
&\leq \frac{1}{(\pi t)^n}\int_{\mathbb{C}^n} \|\alpha_{-z}(A)1\|_{\mathcal{H}_t^k} e^{\frac{\operatorname{Re}((w-z)\cdot\overline{u})}{t} - \frac{|w-z|^2}{2t} - (\frac{1}{t} - c_k^t)|u|^2}\, du \\
&\leq \frac{d}{(\pi t)^n} e^{-\frac{|z-w|^2}{2t}} \int_{\mathbb{C}^n} e^{\frac{\operatorname{Re}((w-z)\cdot\overline{u})}{t} - (\frac{1}{t} - c_k^t)|u|^2}\, du
\end{aligned}$$

$$= de^{-\frac{|z-w|^2}{2t}} \frac{1}{(1-c_k^t t)^n} e^{\frac{|z-w|^2}{4t(1-c_k^t t)}}$$
$$= \frac{d}{(1-c_k^t t)^n} e^{-\left(\frac{1}{2t}-c_{k+1}^t\right)|z-w|^2}.$$

Since $c_{k+1}^t < \frac{1}{2t}$, A is clearly sufficiently localized. □

Let us recall the following result:

Theorem 8.0.5 ([14, Theorem 6])**.** *Let $f \in \mathrm{BMO}(\mathbb{C}^n)$. Then, $\widetilde{f}^{(t)}$ is bounded for one $t > 0$ if and only if it is bounded for all $t > 0$.*

As in [15, Lemma 4.11] we obtain the following:

Lemma 8.0.6. *Let $f \in \mathrm{BMO}(\mathbb{C}^n)$ such that T_f^t is bounded on F_t^p. Then, we have $T_f^t \in \mathcal{A}_{sl}$.*

Proof. From Lemma 7.2.1 we get

$$\langle T_f^t k_z^t, k_w^t\rangle_t = \langle f k_z^t, k_w^t\rangle_t = e^{\frac{s}{2t(t-s)}|z-w|^2 - \frac{is}{t(t-s)}\operatorname{Im}(w\cdot\overline{z})}\langle \tilde{f}^{(s)} k_z^{t-s}, k_w^{t-s}\rangle_{t-s}.$$

Boundedness of T_f^t clearly implies that $\widetilde{f}^{(t)}$ is bounded. Letting $s < \frac{t}{2}$ we obtain from the above theorem that $\widetilde{f}^{(s)}$ is also bounded. Lemma 7.2.2 now gives

$$|\langle \widetilde{f}^{(s)} k_z^{t-s}, k_w^{t-s}\rangle_{t-s}| \le \|\widetilde{f}^{(s)}\|_\infty e^{\frac{1}{4(t-s)}|z-w|^2}$$

and therefore

$$|\langle f k_z^t, k_w^t\rangle_t| \le \|\widetilde{f}^{(s)}\|_\infty e^{\frac{1}{2(t-s)}\left(\frac{s}{t}-\frac{1}{2}\right)|z-w|^2}$$

with $\frac{s}{t} - \frac{1}{2} < 0$. Hence, T_f^t is sufficiently localized. □

Remark 8.0.7. Let us assume that $f : \mathbb{C}^n \to \mathbb{C}$ is measurable with $fK_z^t \in F_t^2$ for every $z \in \mathbb{C}^n$ such that $\widetilde{f}^{(s)}$ is bounded for some $s \in (0, t/2)$. Corollary 7.2.5 proves that T_f^t is bounded under these assumptions. All the estimates from the proof of the previous lemma can be carried out in the same way, proving that under the assumptions of Corollary 7.2.5 (and $p \in (1,\infty)$) we even obtain $T_f^t \in \mathcal{T}^{p,t}$.

Proposition 8.0.8. *For any $p \in (1,\infty)$, $t > 0$ we have*

$$\mathcal{T}^{p,t} = \overline{\{T_f^t;\ f \in \mathrm{BMO}(\mathbb{C}^n) \text{ such that } T_f^t \in \mathcal{L}(F_t^p)\}}.$$

Proof. Follows from the previous lemma and the inclusion $L^\infty(\mathbb{C}^n) \subset \mathrm{BMO}(\mathbb{C}^n)$. □

Summarizing the results, we now have the following characterizations of the Toeplitz algebra at hand:

Theorem 8.0.9. *For any $p \in (1,\infty)$ and $t > 0$ we have*

$$\begin{aligned}\mathcal{T}^{p,t} &= \mathcal{T}^{p,t}_{lin}(\mathrm{BUC}(\mathbb{C}^n)) = \mathcal{C}_1\\ &= \{A \in \mathcal{L}(F^p_t);\ f_s * A \to A \ \text{as } s \to 0\}\\ &= \{g * B;\ g \in L^1(\mathbb{C}^n),\ B \in \mathcal{L}(F^p_t)\}\\ &= \overline{\mathcal{A}_{sl}} = \overline{\mathcal{A}_{wl}} = \overline{\mathcal{A}^{p,t}_{uc}}\\ &= P_t\,\mathrm{BDO}^p_t\,P_t\\ &= \{A \in \mathcal{L}(F^p_t);\ [A, T^t_f] \in \mathcal{K}(F^p_t) \ \text{for all } f \in \mathrm{VO}_\partial(\mathbb{C}^n)\}.\end{aligned}$$

8.1 Remarks

Those characterizations of $\mathcal{T}^{p,t}$ which are consequences of Quantum Harmonic Analysis hold, as we have discussed in previous chapters, also in the non-reflexive cases. The characterizations presented in [79], together with their proofs, should not depend on reflexivity and carry over to the cases $p = 1,\ \infty$. Hence, in principle we expect that all the characterizations mentioned in this chapter work for all $p \in [1,\infty]$. Since this is not the right place to digress on the p-(in)dependence of the arguments presented in [79], we preferred to place our discussion only in the reflexive setting.

Let us alter the definition of band operators in the following way: We say that $A \in \mathcal{L}(L^p_t)$ is an essentially band operator, $A \in \mathrm{BO}_{ess}$, if

$$\exists \omega > 0 : \forall f, g \in L^\infty(\mathbb{C}^n) \text{ with } \mathrm{dist}(\mathrm{supp}(f), \mathrm{supp}(g)) > \omega : M_f A M_g \in \mathcal{K}(L^p_t).$$

Further, let us define the essentially band-dominated operators as

$$\mathrm{BDO}^p_{ess,t} := \overline{\mathrm{BO}_{ess}}.$$

N. Vasilevski asked if $P_t\,\mathrm{BDO}^p_t\,P_t$ and $P_t\,\mathrm{BDO}^p_{ess,t}\,P_t$, are the same algebra, i.e. if

$$\mathcal{T}^{p,t} = P_t\,\mathrm{BDO}^p_{ess,t}\,P_t$$

holds true. Clearly, we have the inclusion $\mathcal{T}^{p,t} \subseteq P_t\,\mathrm{BDO}^p_{ess,t}\,P_t$. Since $\mathcal{K}(L^p_t)$ is contained in BDO^p_t, one might expect that $\mathrm{BDO}^p_t = \mathrm{BDO}^p_{ess,t}$. This seems to be an interesting problem for future work.

Chapter 9

Toeplitz operators on pluriharmonic Fock and Bergman spaces

9.1 Bergman spaces on bounded symmetric domains and their operators

Let $\Omega \subset \mathbb{C}^n$ be a bounded symmetric domain in its Harish-Chandra realization, cf. [21, 58, 86, 101, 124]. Since we will not need much of the theory of such domains, we do not give a detailed introduction and mention only several facts. Further information can easily be found in the literature.

Let us denote by g the genus of Ω and by (r, a, b) its type, all of which are certain numerical invariants. Note that the genus is usually denoted by p, which we will not do, as we will reserve the use of p for L^p-spaces in consistency with the rest of this work. $h : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ will denote the Jordan triple determinant of Ω, which is a certain polynomial function being holomorphic in the first and anti-holomorphic in the second variable. For any $\lambda > -1$, it is well-known that the measure ν_λ on Ω,

$$d\nu_\lambda(z) = c_\lambda h(z, z)^\lambda dz,$$

is finite. We choose the constant c_λ such that ν_λ is a probability measure. We will also consider ν_{-g}, i.e.

$$d\nu_{-g}(z) = h(z, z)^{-g} dz,$$

which is well-known to be invariant under holomorphic automorphisms of Ω. In particular, ν_{-g} is invariant under every φ_z, the geodesic symmetry of Ω exchanging z and 0. With respect to the automorphisms φ_z, the Jordan triple determinant satisfies the following important identity:

$$h(\varphi_z(w), \varphi_z(w)) = \frac{h(z, z)h(w, w)}{|h(z, w)|^2}.$$

Since

$$h(z,w) = \overline{h(w,z)},$$

this yields

$$h(\varphi_z(w), \varphi_z(w)) = h(\varphi_w(z), \varphi_w(z)).$$

For any $p \in (1,\infty)$ we define the standard weighted Bergman spaces $\mathcal{A}^p_\lambda(\Omega)$ as

$$\mathcal{A}^p_\lambda(\Omega) := L^p(\Omega, \nu_\lambda) \cap \mathrm{Hol}(\Omega).$$

Similarly to the case of the Fock space, one sees that this is always a closed subspace of the enveloping Lebesgue space $L^p(\Omega, \nu_\lambda)$. Of course, for $p = 2$ we are in the Hilbert space setting. $\mathcal{A}^2_\lambda(\Omega)$ turns again out to be a reproducing kernel Hilbert space, the kernel functions being given by

$$K^\lambda_z(w) = K^\lambda(w,z) = h(w,z)^{-\lambda-g}.$$

Since we assume Ω to be in its Harish-Chandra realization, we always have $0 \in \Omega$ and Ω is circular around the origin. This can be seen to imply $K^\lambda(w,0) = 1$ for each $w \in \Omega$, which we will occasionally use later. As in the case of the Fock space, the $\mathcal{A}^2_\lambda$ inner product induces the standard duality between $\mathcal{A}^p_\lambda(\Omega)$ and $\mathcal{A}^q_\lambda(\Omega)$, where $\frac{1}{p} + \frac{1}{q} = 1$. We will denote by $d(z,w) := \beta(z,w)$ the Bergman distance function on Ω, which is the distance associated with the Hermitian metric with tensor

$$g_{jk}(z) = \frac{\partial^2}{\partial z_j \partial \overline{z}_k} \log K^0(z,z).$$

By P_λ we will denote the orthogonal projection from $L^2(\Omega, \nu_\lambda)$ onto $\mathcal{A}^2_\lambda(\Omega)$. The projection is given by the integral operator

$$P_\lambda f(z) = \int_\Omega f(w) K^\lambda(z,w) d\nu_\lambda(w) = \int_\Omega f(w) h(z,w)^{-\lambda-g} d\nu_\lambda(w). \tag{9.1}$$

As in the case of the Fock space, we would like to consider the same integral operator as a projection from $L^p(\Omega, \nu_\lambda)$ to $\mathcal{A}^p_\lambda(\Omega)$. Unfortunately, the projection is in general unbounded for $p \neq 2$. We always assume in the following that p and λ are such that P_λ defines a bounded projection from $L^p(\Omega, \nu_\lambda)$ to $\mathcal{A}^p_\lambda(\Omega)$. A sufficient condition for this is given by the following result in terms of the numerical invariants of Ω.

Proposition 9.1.1 ([58, Lemma 9])**.** *Let $\lambda > g - 1$ and*

$$1 + \frac{(r-1)a}{(r-1)a + 2(\lambda - g + 1)} < p < 1 + \frac{(r-1)a + 2(\lambda - g + 1)}{(r-1)a}.$$

Then, the integral operator P_λ from Equation (9.1) *defines a bounded projection from $L^p(\Omega, \nu_\lambda)$ onto $\mathcal{A}^p_\lambda(\Omega)$.*

Using the projection P_λ, we can define Toeplitz operators on $\mathcal{A}^p_\lambda(\Omega)$: For any $f \in L^\infty(\Omega)$ we let

$$T_f^\lambda : \mathcal{A}^p_\lambda(\Omega) \to \mathcal{A}^p_\lambda(\Omega), \quad T_f^\lambda(g) = P_\lambda(fg).$$

For a suitable symbol $f : \Omega \to \mathbb{C}$, the Hankel operator H_f^λ is defined as

$$H_f^\lambda = (I - P_\lambda)M_f : \mathcal{A}^p_\lambda \to L^p(\Omega, \nu_\lambda).$$

The Banach space adjoint of H_f^λ can always be identified, under the standard dual pairing, with

$$(H_f^\lambda)^* = P_\lambda M_{\overline{f}}(I - P_\lambda) : L^q(\Omega, \nu_\lambda) \to \mathcal{A}^q_\lambda.$$

In particular, we always have the following well-known relation between Toeplitz and Hankel operators for suitable symbols:

$$T_f^\lambda T_g^\lambda - T_{fg}^\lambda = -(H_{\overline{f}}^\lambda)^* H_g^\lambda.$$

We define the Berezin transform of $f \in L^\infty(\Omega)$ for $\lambda > -1$ as

$$\begin{aligned}\mathcal{B}_\lambda(f)(z) &:= \left\langle f\frac{K_z^\lambda}{\|K_z^\lambda\|_{\mathcal{A}^2_\lambda}}, \frac{K_z^\lambda}{\|K_z^\lambda\|_{\mathcal{A}^2_\lambda}} \right\rangle_{\mathcal{A}^2_\lambda} \\ &= h(z,z)^{\lambda+g} \int_\Omega f(w) \frac{1}{|h(w,z)|^{2(\lambda+g)}} d\nu_\lambda(w).\end{aligned}$$

Let us mention some important results:

Proposition 9.1.2 ([58, 82])**.** *Assume either $p = 2$ or*

$$1 + \frac{(r-1)a}{2(\lambda+1)} < p < 1 + \frac{2(\lambda+1)}{(r-1)a}.$$

For $f \in L^\infty(\Omega)$ we have

$$T_f^\lambda \text{ is compact } \iff \mathcal{B}_\lambda(f) \in C_0(\Omega).$$

A bounded function $f : \Omega \to \mathbb{C}$ is said to be of *vanishing mean oscillation*, we write $f \in L^\infty(\Omega) \cap \mathrm{VMO}_\partial^\lambda(\Omega)$, if

$$\mathrm{MO}_\lambda(f)(z) := \mathcal{B}_\lambda(|f|^2)(z) - |\mathcal{B}_\lambda(f)(z)|^2(z) \to 0, \quad \beta(0,z) \to \infty.$$

As in the Fock space setting, one can see that for bounded functions the membership in $\mathrm{VMO}_\partial^\lambda(\Omega)$ is indeed independent of λ [84]. Therefore, we will write

$$\mathrm{VMO}_\partial(\Omega) := L^\infty(\Omega) \cap \mathrm{VMO}_\partial^\lambda(\Omega).$$

Proposition 9.1.3 ([21, 82, 84]). *Assume either $p = 2$ or*

$$1 + \frac{(r-1)a}{2(\lambda+1)} < p < 1 + \frac{2(\lambda+1)}{(r-1)a}.$$

If $f \in \mathrm{VMO}_\partial(\Omega)$, then H_f^λ is compact. Furthermore,

$$\sigma_{ess}(T_f^\lambda) = \bigcap_{R>0} \mathcal{B}_\lambda(f)(\Omega \setminus E(0,R)).$$

Here, $E(0,R)$ denotes the metric balls around 0 with respect to the Bergman metric.

The map $f \mapsto T_f^\lambda$ also serves as a model for *strict quantization* on the bounded symmetric domain, i.e. for a suitable class of symbols the following asymptotics hold true:

$$\lim_{\lambda\to\infty} \|T_f^\lambda\| = \|f\|_\infty, \tag{9.2}$$

$$\lim_{\lambda\to\infty} \|T_f^\lambda T_g^\lambda - T_{fg}^\lambda\| = 0, \tag{9.3}$$

$$\lim_{\lambda\to\infty} \left\| \frac{\lambda}{i}[T_f^\lambda, T_g^\lambda] - T_{\{f,g\}}^\lambda \right\| = 0. \tag{9.4}$$

We will now present some results on these asymptotics, all of which are in the Hilbert space setting $p = 2$.

The following result is well-known for certain classes of functions (see e.g. [12, Proposition 4.4] for functions which are uniformly continuous with respect to the Bergman metric). The general case was proven by the author in [71]. We will later provide a proof different from the one shown there.

Proposition 9.1.4. *Let $f \in L^\infty(\Omega)$. Then, we have*

$$\lim_{\lambda\to\infty} \|\mathcal{B}_\lambda(f)\|_\infty = \lim_{\lambda\to\infty} \|T_f^\lambda\| = \|f\|_\infty.$$

By $\mathrm{UC}(\Omega)$ we denote those functions from Ω to $\mathbb{C}$ which are uniformly continuous (not necessarily bounded) with respect to the Bergman metric β. When considering Toeplitz operators with such symbols, they are in general unbounded. Yet, compositions of such Toeplitz operators are well-defined: For each λ there is a dense subspace $\mathcal{D}_\lambda$ of $L^2(\Omega, \nu_\lambda)$ (the construction of which is similar to the space $\mathcal{D}_t$ from Chapter 8) which is self-adjoint and an invariant subspace of both P_λ and M_f for $f \in \mathrm{UC}(\Omega)$, cf. [17] for details. Hence, $\mathcal{D}_\lambda \cap \mathcal{A}_\lambda^2(\Omega)$ is a dense subspace of $\mathcal{A}_\lambda^2(\Omega)$ being invariant under T_f^λ for f uniformly continuous. In particular, we may form compositions of Toeplitz operators with such symbols on this dense subspace.

While unbounded symbols in general give rise to unbounded Hankel operators, they are bounded if the symbol is uniformly continuous, at least for $p = 2$. We have the following even stronger result, which relates Hankel operators to the quantization estimates:

Proposition 9.1.5 ([17]). *For $p = 2$ and $f \in \mathrm{UC}(\Omega)$ we have*

$$\|H_f^\lambda\| \to 0, \quad \lambda \to \infty.$$

In particular, for any $g \in \mathrm{UC}(\Omega)$ or $g \in L^\infty(\Omega)$ the following holds true:

$$\|T_f^\lambda T_g^\lambda - T_{fg}^\lambda\| \to 0, \quad \lambda \to \infty.$$

If $\Omega = \mathbb{B}_n$, the unit ball of $\mathbb{C}^n$, the following result can be shown to hold true.

Proposition 9.1.6 ([17]). *For $f \in \mathrm{VMO}_b(\mathbb{B}_n)$ we have*

$$\|H_f^\lambda\| \to 0, \quad \lambda \to \infty.$$

In particular, for any $g \in L^\infty(\mathbb{B}_n)$ we obtain

$$\|T_f^\lambda T_g^\lambda - T_{fg}^\lambda\| \to 0, \quad \lambda \to \infty.$$

Here, $\mathrm{VMO}_b(\mathbb{B}_n)$ denotes the class of all $f \in L^\infty(\mathbb{B}_n)$ for which

$$\lim_{\rho \to 0} A_2(f, z, \rho) = 0 \quad \text{uniformly on } \mathbb{B}_n.$$

In this, we used the notations

$$A_2(f, z, \rho) := \frac{1}{|E(z,\rho)|} \int_{E(z,\rho)} |f(w) - f_{E(z,\rho)}|^2 \, dV(w)$$

and

$$f_{E(z,\rho)} = \frac{1}{|E(z,\rho)|} \int_{E(z,\rho)} f(w) \, dV(w)$$

and $E(z, \rho)$ is the metric ball of radius ρ around z with respect to the metric β.

As we will not need any results on the quantization estimate (9.4), we do not discuss any advances in that direction and only refer to the literature [35, 61], cf. also [81] for related results in the Fock space setting.

Let us end this section by providing a proof of Proposition 9.1.4. We already gave a proof of that statement in [71]. Here, we will present a different proof, which is more in the spirit of the duality arguments used earlier in this thesis. The following fact is probably well-known, yet we could not locate it in the literature.

Lemma 9.1.7. *Let $f \in L^1(\Omega, \nu_{-g})$. Then, $\mathcal{B}_\lambda(f) \in L^1(\Omega, \nu_{-g})$ for $\lambda > -1$ and $\mathcal{B}_\lambda(f) \to f$ in $L^1(\Omega, \nu_{-g})$ as $\lambda \to \infty$.*

Proof. A key fact in the following computations will be that $h(z, z) \leq 1$ and hence $h(z, z)^{-g} \geq 1$ for all $z \in \Omega$. We first prove that $\mathcal{B}_\lambda(f) \in L^1(\Omega, v_{-g})$:

$$\int_\Omega |\mathcal{B}_\lambda(f)(z)| \, d\nu_{-g}(z) \leq c_\lambda \int_\Omega \int_\Omega |f(w)| \frac{h(z,z)^{\lambda+g} h(w,w)^{\lambda+g}}{|h(z,w)|^{2(\lambda+g)}} \, d\nu_{-g}(w) \, d\nu_{-g}(z)$$

$$\begin{aligned}
&= \int_\Omega |f(w)| h(w,w)^{\lambda+g} c_\lambda \int_\Omega \frac{h(z,z)^{\lambda+g}}{|h(z,w)|^{2(\lambda+g)}} \, d\nu_{-g}(z) \, d\nu_{-g}(w) \\
&= \int_\Omega |f(w)| h(w,w)^{\lambda+g} \langle K_w^\lambda, K_w^\lambda \rangle_\lambda \, d\nu_{-g}(w) \\
&= \int_\Omega |f(w)| \, d\nu_{-g}(w) \\
&= \|f\|_{L^1(\Omega,\nu_{-g})}.
\end{aligned}$$

Recall that the Bergman length metric β is invariant under holomorphic automorphisms of Ω. In particular, $\beta(\varphi_z(u), \varphi_z(v)) = \beta(u,v)$ for any $z, u, v \in \Omega$. Hence, if $K_1, K_2 \subset \Omega$ are compact, then we have for $z \in K_1, w \in K_2$:

$$\beta(0, \varphi_z(w)) = \beta(\varphi_z(0), w) = \beta(z,w) \leq \beta(z,0) + \beta(0,w),$$

which is uniformly bounded in z and w by compactness of K_1 and K_2. Therefore,

$$\bigcup_{z \in K_1} \varphi_z(K_2)$$

is relatively compact in Ω. Now, let $g \in C_c(\Omega)$ and $K := \operatorname{supp}(g)$. For some $\delta > 0$ let

$$\widetilde{K} := \overline{\bigcup_{w \in K} \varphi_w(\overline{E(0,\delta)})},$$

which is compact with $\varphi_w(\widetilde{K}) \supseteq E(0,\delta)$ for any $w \in K$, where $E(0,\delta)$ denotes the ball with respect to the Bergman metric β. Then:

$$\begin{aligned}
\int_{\Omega \setminus \widetilde{K}} |\mathcal{B}_\lambda(g)(z)| d\nu_{-g}(z) &\leq \int_K |g(w)| c_\lambda \int_{\Omega \setminus \widetilde{K}} \frac{h(z,z)^{\lambda+g} h(w,w)^{\lambda+g}}{|h(z,w)|^{2(\lambda+g)}} \, d\nu_{-g}(z) d\nu_{-g}(w) \\
&= \int_K |g(w)| c_\lambda \int_{\Omega \setminus \widetilde{K}} h(\varphi_w(z), \varphi_w(z))^{\lambda+g} d\nu_{-g}(z) d\nu_{-g}(w) \\
&= \int_K |g(w)| c_\lambda \int_{\varphi_w(\Omega \setminus \widetilde{K})} h(z,z)^{\lambda+g} d\nu_{-g}(z) d\nu_{-g}(w) \\
&= \int_K |g(w)| c_\lambda \int_{\Omega \setminus \varphi_w(\widetilde{K})} h(z,z)^{\lambda+g} d\nu_{-g}(z) d\nu_{-g}(w) \\
&\leq \int_K |g(w)| c_\lambda \int_{\Omega \setminus E(0,\delta)} h(z,z)^{\lambda+g} d\nu_{-g}(z) d\nu_{-g}(w) \\
&\leq \|g\|_{L^1(\Omega,\nu_{-g})} \nu_\lambda(\Omega \setminus B(0,\delta)).
\end{aligned}$$

As is well known, $\nu_\lambda(\Omega \setminus E(0,\delta)) \to 0$ as $\lambda \to \infty$ (since $h(z,z)$ is strictly less than 1 outside of $E(0,\delta)$).

For $g \in C_c(\Omega)$ and $\varepsilon > 0$ arbitrary we can therefore find $K \subset \Omega$ compact such that for all λ large enough we have

$$\|g - \mathcal{B}_\lambda(g)\|_{L^1(\Omega,\nu_{-g})} < \varepsilon + \|g - \mathcal{B}_\lambda(g)\|_{L^1(K,\nu_{-g})}. \tag{9.5}$$

Furthermore, as is well-known (see e.g. [12, Theorem 4.10]), $\|g - \mathcal{B}_\lambda(g)\|_\infty \to 0$ as $\lambda \to \infty$ for any $g \in C_c(\Omega)$. Now, we can put the pieces together: Let $f \in L^1(\Omega, \nu_{-g})$ and $\varepsilon > 0$ arbitrary. Then, we can find $g \in C_c(\Omega)$ with $\|f - g\|_{L^1(\Omega,\nu_{-g})} < \varepsilon$. This gives

$$\begin{aligned}&\|f - \mathcal{B}_\lambda(f)\|_{L^1(\Omega,\nu_{-g})}\\ &\quad \leq \|f - g\|_{L^1(\Omega,\nu_{-g})} + \|g - \mathcal{B}_\lambda(g)\|_{L^1(\Omega,\nu_{-g})} + \|\mathcal{B}_\lambda(g) - \mathcal{B}_\lambda(f)\|_{L^1(\Omega,\nu_{-g})}.\end{aligned}$$

By assumption, the first term is less that ε. Moreover, by linearity and the $L^1(\Omega, \nu_{-g})$ estimate for the Berezin transform, the third term is also less than ε. For the second term, let us pick $K \subset \Omega$ compact according to Equation (9.5). Furthermore, let us choose λ large enough such that $\|g - \mathcal{B}_\lambda(g)\|_\infty < \frac{\varepsilon}{\nu_{-g}(K)}$. Then,

$$\|g - \mathcal{B}_\lambda(g)\|_{L^1(\Omega,\nu_{-g})} \leq \varepsilon + \|g - \mathcal{B}_\lambda(g)\|_{L^1(K,\nu_{-g})} \leq 2\varepsilon,$$

which finishes the proof. □

Proof of Proposition 9.1.4. We want to emphasize that the very simple idea of using the duality $(L^1)' = L^\infty$ in this proof was kindly communicated to us by Raffael Hagger (private communications), who in turn received this hint from Christian Seifert.

Since $L^1(\Omega, \nu_{-g})' = L^\infty(\Omega)$ isometrically under the $L^2(\Omega, \nu_{-g})$ dual pairing, we know that for any $f \in L^\infty(\Omega)$:

$$\|f\|_\infty = \sup_{g \in L^1(\Omega,\nu_{-g}),\ \|g\|_{L^1} = 1} |\langle f, g\rangle_{L^2(\Omega,\nu_{-g})}|.$$

An easy application of Fubini's Theorem shows that for any $f \in L^\infty(\Omega)$ and $g \in L^1(\Omega, \nu_{-g})$:

$$\langle \mathcal{B}_\lambda(f), g\rangle_{L^2(\Omega,\nu_{-g})} = \langle f, \mathcal{B}_\lambda(g)\rangle_{L^2(\Omega,\nu_{-g})}.$$

By the previous lemma, we obtain for any such f, g:

$$\langle \mathcal{B}_\lambda(f), g\rangle_{L^2(\Omega,\nu_{-g})} \to \langle f, g\rangle_{L^2(\Omega,\nu_{-g})}, \quad \lambda \to \infty.$$

Thus, for $\|g\|_{L^1} = 1$:

$$\|\mathcal{B}_\lambda(f)\|_\infty \geq |\langle \mathcal{B}_\lambda(f), g\rangle_{L^2(\Omega,\nu_{-g})}| \to |\langle f, g\rangle_{L^2(\Omega,\nu_{-g})}|$$

and hence

$$\liminf_{\lambda \to \infty} \|\mathcal{B}_\lambda(f)\|_\infty \geq \|f\|_\infty.$$

On the other hand,

$$\|\mathcal{B}_\lambda(f)\|_\infty \leq \|T_f^\lambda\|_{op} \leq \|f\|_\infty,$$

which proves the result. □

9.2 Bergman and Fock spaces of pluriharmonic functions

For the rest of this chapter, we will try to notationally not distinguish between Fock and Bergman spaces. We will always make the following assumptions on p and λ:

Assumption 1. If $\Omega = \mathbb{C}^n$, we assume $p \in (1, \infty)$ and $\lambda > 0$. If Ω is a bounded symmetric domain, we assume either $p = 2$ and $\lambda > -1$ arbitrary or $p \in (1, \infty)$ and $\lambda > -1$ such that

$$1 + \frac{(r-1)a}{2(\lambda+1)} < p < 1 + \frac{2(\lambda+1)}{(r-1)a}.$$

We want to emphasize that this assumption is satisfied for p, λ if and only if it is satisfied for q, λ with $1 = 1/p + 1/q$.

In this and the following sections we will try to use a unified notation for both the Fock space and Bergman spaces on bounded symmetric domain. This necessarily causes some inconveniences related to one of the parameters t or λ. This can be best seen when dealing with the limits of strict quantization: In the Fock space setting this corresponds to letting $t \to 0$, whereas on bounded symmetric domains the limit one takes is $\lambda \to \infty$. That is, $t \approx \hbar$ but $\lambda \approx 1/\hbar$. To resolve this, one has to pass to the reciprocal of one of the parameters t or λ. Which of the two possibilities is chosen is probably a matter of personal taste. Since the literature on bounded symmetric domains always consideres the parameter λ, but it is not entirely uncommon to consider $1/t$ on the Fock space (for example K. Zhu uses $\alpha = 1/t$ in [137]), we decided to stick with the usual conventions on bounded symmetric domains and break with our earlier conventions on Fock spaces. This has also the advantage that we will use the "right" convention in Section 9.5, where we solely deal with statements on Bergman spaces over the unit ball.

For Ω a bounded symmetric domain in its Harish-Chandra realization (we refer again to e.g. [58] for an explanation of that term) we set

$$L^p_\lambda(\Omega) := L^p(\Omega, \nu_\lambda)$$

and further

$$L^p_\lambda(\mathbb{C}^n) := L^p(\mathbb{C}^n, \mu_{2/(\lambda p)})$$

for those values of λ, p which satisfy Assumption 1. In comparison with the standard notation of spaces on $\mathbb{C}^n$ introduced in Chapter 2, we have

$$L^p_\lambda(\mathbb{C}^n) = L^p_{1/\lambda}.$$

For Ω a bounded symmetric domain or $\Omega = \mathbb{C}^n$ we let

$$\mathcal{A}^p_\lambda(\Omega) := L^p_\lambda(\Omega) \cap \mathrm{Hol}(\Omega).$$

In particular,

$$\mathcal{A}^p_\lambda(\mathbb{C}^n) = F^p_{1/\lambda}.$$

We might occasionally write ν_λ instead of $\mu_{1/\lambda}$ for the Gaussian measure on $\mathbb{C}^n$. Finally, we will use the notation T_f^λ for the Toeplitz operators acting on $\mathcal{A}_\lambda^p(\Omega)$. In the Fock space setting, this is not in accordance with our earlier conventions (the Toeplitz operators on $\mathcal{A}_\lambda^p(\mathbb{C}^n)$ would have been denoted $T_f^{1/\lambda}$). For the rest of this chapter, we prefer to "forget" about this particular Fock space convention used in earlier chapters to have a unified and more compact notation for all occurring cases. This is a price we are willing the pay for the sake of simpler notation.

We will denote by $d(z, w)$ the appropriate distance between $z, w \in \Omega$, i.e. for $\Omega = \mathbb{C}^n$ this will denote the Euclidean distance and for Ω a bounded symmetric domain this means the distance considered with respect to the Bergman length metric. Further, by $E(z, \rho)$ we will denote the open ball of radius $\rho > 0$ around $z \in \Omega$ with respect to the metric d.

Recall that anti-holomorphic functions are, simply speaking, complex conjugates of holomorphic functions (or equivalently functions $f \in C^1(\Omega)$ satisfying $\frac{\partial f}{\partial z_j} = 0$ on Ω for all $j = 1, \dots, n$) and pluriharmonic functions are those $f \in C^2(\Omega)$ such that $\frac{\partial^2 f}{\partial z_j \partial \overline{z}_k} = 0$ for all $j, k = 1, \dots, n$, cf. also Appendix A.1. In analogy to the spaces of holomorphic functions, we define the spaces of anti-holomorphic and pluriharmonic functions as

$$\mathcal{A}_{\lambda,\mathrm{ah}}^p(\Omega) := \{f \in L_\lambda^p(\Omega);\ f \text{ is anti-holomorphic}\},$$
$$\mathcal{A}_{\lambda,\mathrm{ph}}^p(\Omega) := \{f \in L_\lambda^p(\Omega);\ f \text{ is pluriharmonic}\}.$$

The map $C : f \mapsto \overline{f}$ is an anti-linear isometric bijection from $\mathcal{A}_\lambda^p(\Omega)$ to $\mathcal{A}_{\lambda,\mathrm{ah}}^p(\Omega)$. Since the holomorphic polynomials are dense in $\mathcal{A}_\lambda^p(\Omega)$, the anti-holomorphic polynomials are dense in $\mathcal{A}_{\lambda,\mathrm{ah}}^p(\Omega)$. Denote by P_λ the orthogonal projection from $L_\lambda^2(\Omega)$ to $\mathcal{A}_\lambda^2(\Omega)$. Our Assumption 1 implies that P_λ extends to a bounded projection from $L_\lambda^p(\Omega)$ to $\mathcal{A}_\lambda^p(\Omega)$ [80].

Recall that P_λ is the integral operator

$$P_\lambda(f)(z) = \int_\Omega f(w) K^\lambda(z, w)\ d\nu_\lambda(w),$$

where $K^\lambda(z, w)$ is the reproducing kernel function from $\mathcal{A}_\lambda^2(\Omega)$. In particular, reminding ourselves of the changed conventions, we have on the Fock space:

$$K^\lambda(z, w) = e^{\lambda w \cdot \overline{z}}, \quad z, w \in \mathbb{C}^n.$$

The projection with the complex conjugate integral kernel,

$$P_{\lambda,\mathrm{ah}}(f)(z) := \int_\Omega f(w) K^\lambda(w, z)\ d\nu_\lambda(w)$$

defines the orthogonal projection from $L_\lambda^2(\Omega)$ to $\mathcal{A}_{\lambda,\mathrm{ah}}^2(\Omega)$. Since they are related by $P_\lambda = C P_{\lambda,\mathrm{ah}} C$, P_λ extends boundedly from $L_\lambda^p(\Omega)$ to $\mathcal{A}_\lambda^p(\Omega)$ if and only if $P_{\lambda,\mathrm{ah}}$ extends boundedly from $L_\lambda^p(\Omega)$ to $\mathcal{A}_{\lambda,\mathrm{ah}}^p(\Omega)$.

We will also need

$$\mathcal{A}^p_{\lambda,\mathrm{ah}\ominus\mathbb{C}}(\Omega) := \{f \in \mathcal{A}^p_{\lambda,\mathrm{ah}}(\Omega);\ f(0) = 0\}.$$

One readily verifies that $P_{\lambda,\mathrm{ah}} - P_{\lambda,\mathrm{ah}}P_\lambda$ is a bounded projection onto $\mathcal{A}^p_{\lambda,\mathrm{ah}\ominus\mathbb{C}}(\Omega)$, in particular it is also a closed subspace of $L^2_\lambda(\Omega)$.

Recall that by Lemma A.1.5, every pluriharmonic function $f : \Omega \to \mathbb{C}$ can be written as

$$f = g + h,$$

where $g : \Omega \to \mathbb{C}$ is holomorphic and $h : \Omega \to \mathbb{C}$ is anti-holomorphic with $h(0) = 0$. Since both g and h can be approximated uniformly on compact subsets of Ω by their power series expansion around the origin, the sum of holomorphic and anti-holomorphic polynomials can be seen to be dense in $\mathcal{A}^p_{\lambda,\mathrm{ph}}(\Omega)$ (the argument is analogous to the density of holomorphic polynomials in $\mathcal{A}^p_\lambda(\Omega)$).

Let p_1 be a holomorphic polynomial on Ω, i.e. a polynomial in $z = (z_1, \ldots, z_n)$, and p_2 an anti-holomorphic polynomial, i.e. in $\overline{z} = (\overline{z_1}, \ldots, \overline{z_n})$, such that $p_2(0) = 0$. Then, using the notation

$$p_2^*(z) := \overline{p_2(\overline{z})},$$

which is a holomorphic polynomial, we have

$$\begin{aligned}\langle p_1, p_2\rangle_{L^2_\lambda(\Omega)} &= \langle p_1 p_2^*, 1\rangle_{L^2_\lambda(\Omega)}\\ &= \langle p_1 p_2^*, K_0^\lambda\rangle_{L^2_\lambda(\Omega)}\\ &= p_1(0)p_2^*(0)\\ &= 0.\end{aligned}$$

Using density of these polynomials, this implies that we have the orthogonal decomposition

$$\mathcal{A}^2_{\lambda,\mathrm{ph}}(\Omega) = \mathcal{A}^2_\lambda(\Omega) \oplus \mathcal{A}^2_{\lambda,\mathrm{ah}\ominus\mathbb{C}}(\Omega).$$

For $p \neq 2$, this still holds true as a direct sum decomposition: The only nontrivial fact about this is that for

$$\mathcal{A}^p_{\lambda,\mathrm{ph}}(\Omega) \ni f = g + h$$

with g holomorphic and h anti-holomorphic with $h(0) = 0$, g and h satisfy the integrability conditions. If we assume f to be a pluriharmonic polynomial, then of course both g and h are polynomials, hence

$$P_\lambda f = g, \quad (P_{\lambda,\mathrm{ah}} - P_{\lambda,\mathrm{ah}}P_\lambda)f = h$$

by what we have seen above. Using the density of polynomials, we obtain

$$P_\lambda f = g,\ (P_{\lambda,\mathrm{ah}} - P_{\lambda,\mathrm{ah}}P_\lambda)f = h$$

for arbitrary $f \in \mathcal{A}^p_{\lambda,\mathrm{ph}}(\Omega)$ and therefore in particular $g \in \mathcal{A}^p_\lambda(\Omega)$, $h \in \mathcal{A}^p_{\lambda,\mathrm{ah}\ominus\mathbb{C}}(\Omega)$ by the boundedness of the projections. This proves that we can always decompose $\mathcal{A}^p_{\lambda,\mathrm{ph}}(\Omega)$ into the direct sum

$$\mathcal{A}^p_{\lambda,\mathrm{ph}}(\Omega) = \mathcal{A}^p_\lambda(\Omega) \oplus \mathcal{A}^p_{\lambda,\mathrm{ah}\ominus\mathbb{C}}(\Omega).$$

For future notation, we also introduce the projection

$$P_{\lambda,\mathbb{C}} := P_{\lambda,\mathrm{ah}} P_\lambda = P_\lambda P_{\lambda,\mathrm{ah}},$$

which has the range

$$\operatorname{ran}(P_{\lambda,\mathbb{C}}) = \mathcal{A}^p_\lambda(\Omega) \cap \mathcal{A}^p_{\lambda,\mathrm{ah}}(\Omega).$$

Therefore, the projection onto $\mathcal{A}^p_{\lambda,\mathrm{ah}\ominus\mathbb{C}}(\Omega)$ is given by $P_{\lambda,\mathrm{ah}} - P_{\lambda,\mathbb{C}}$. We define a collection of new Toeplitz operators:

$$\begin{aligned} T_f^{\lambda,\mathrm{ah}} &: \mathcal{A}^p_{\lambda,\mathrm{ah}}(\Omega) \to \mathcal{A}^p_{\lambda,\mathrm{ah}}(\Omega), & T_f^{\lambda,\mathrm{ah}} &= P_{\lambda,\mathrm{ah}} M_f \\ T_f^{\lambda,\mathrm{ph}} &: \mathcal{A}^p_{\lambda,\mathrm{ph}}(\Omega) \to \mathcal{A}^p_{\lambda,\mathrm{ph}}(\Omega), & T_f^{\lambda,\mathrm{ph}} &= P_{\lambda,\mathrm{ph}} M_f \\ T_f^{\lambda,\mathrm{ah}\ominus\mathbb{C}} &: \mathcal{A}^p_{\lambda,\mathrm{ah}\ominus\mathbb{C}}(\Omega) \to \mathcal{A}^p_{\lambda,\mathrm{ah}\ominus\mathbb{C}}(\Omega), & T_f^{\lambda,\mathrm{ah}\ominus\mathbb{C}} &= (P_{\lambda,\mathrm{ah}} - P_{\lambda,\mathbb{C}}) M_f. \end{aligned}$$

The properties of the operators $T_f^{\lambda,\mathrm{ah}}$ are closely related to those of the usual Toeplitz operators T_f^λ, since they are connected through

$$C T_f^\lambda C = T_{\overline{f}}^{\lambda,\mathrm{ah}}.$$

We will also encounter the following variation of the usual Hankel operator:

$$H_f^{\lambda,\mathrm{ah}} = (I - P_{\lambda,\mathrm{ah}}) M_f : \mathcal{A}^p_{\lambda,\mathrm{ah}}(\Omega) \to L^p_\lambda(\Omega).$$

Note that $C H_f^\lambda C = H_{\overline{f}}^{\lambda,\mathrm{ah}}$, hence

$$\|H_f^{\lambda,\mathrm{ah}}\| \to 0, \quad \lambda \to \infty \quad \iff \quad \|H_{\overline{f}}^\lambda\| \to 0, \quad \lambda \to \infty$$

and

$$H_f^{\lambda,\mathrm{ah}} \text{ is compact} \quad \iff \quad H_{\overline{f}}^\lambda \text{ is compact}.$$

Our goal is to study the operators $T_f^{\lambda,\mathrm{ph}}$. Toeplitz operators on pluriharmonic function spaces have been a subject of constant interest in the past years, cf. [16, 46, 47, 63, 78, 98–100, 128]. Surprisingly, it seems that the approach we are following for studying these operators has been widely ignored in most works on the matter. We could only locate it in [16]. To be precise, we will study the properties of these operators through the following matrix decomposition, which we fix for later reference as a lemma:

Lemma 9.2.1. *Let $f \in L^\infty(\Omega)$. Then, with respect to the direct sum decomposition*

$$\mathcal{A}^p_{\lambda,ph}(\Omega) = \mathcal{A}^p_\lambda(\Omega) \oplus \mathcal{A}^p_{\lambda,ah\ominus\mathbb{C}}(\Omega),$$

we can represent $T_f^{\lambda,ph}$ as the operator matrix

$$T_f^{\lambda,ph} = \left(\begin{array}{c|c} T_f^\lambda & A_f^\lambda \\ \hline B_f^\lambda & T_f^{\lambda,ah\ominus\mathbb{C}} \end{array}\right),$$

where

$$\begin{aligned} A_f^\lambda &= P_\lambda M_f : \mathcal{A}^p_{\lambda,ah\ominus\mathbb{C}}(\Omega) \to \mathcal{A}^p_\lambda(\Omega) \\ B_f^\lambda &= (P_{\lambda,ah} - P_{\lambda,\mathbb{C}})M_f : \mathcal{A}^p_\lambda(\Omega) \to \mathcal{A}^p_{\lambda,ah\ominus\mathbb{C}}(\Omega). \end{aligned}$$

Since we have the two complementary projections $(P_{\lambda,\mathrm{ah}} - P_{\lambda,\mathbb{C}})$ and $P_{\lambda,\mathbb{C}}$ on $\mathcal{A}^p_{\lambda,\mathrm{ah}}(\Omega)$, we obtain the direct sum decomposition

$$\mathcal{A}^p_{\lambda,\mathrm{ah}}(\Omega) = \mathcal{A}^p_{\lambda,\mathrm{ah}\ominus\mathbb{C}}(\Omega) \oplus \mathcal{A}^p_{\lambda,\mathbb{C}}(\Omega), \tag{9.6}$$

where

$$\mathcal{A}^p_{\lambda,\mathbb{C}}(\Omega) = \mathrm{ran}(P_{\lambda,\mathbb{C}}) = \{f \in \mathcal{A}^p_{\lambda,\mathrm{ah}}(\Omega);\ f \equiv \text{ const.}\}.$$

It will also turn out useful to write $T_f^{\lambda,\mathrm{ah}}$ with respect to this decomposition.

Lemma 9.2.2. *For $f \in L^\infty(\Omega)$ we have with respect to the decomposition* (9.6)*:*

$$T_f^{\lambda,ah} = \left(\begin{array}{c|c} T_f^{\lambda,ah\ominus\mathbb{C}} & E_f^\lambda \\ \hline G_f^\lambda & P_{\lambda,\mathbb{C}}M_f : \mathcal{A}^p_{\lambda,\mathbb{C}}(\Omega) \to \mathcal{A}^p_{\lambda,\mathbb{C}}(\Omega) \end{array}\right).$$

Here,

$$\begin{aligned} E_f^\lambda &:= (P_{\lambda,ah} - P_{\lambda,\mathbb{C}})M_f : \mathcal{A}^p_{\lambda,\mathbb{C}}(\Omega) \to \mathcal{A}^p_{\lambda,ah\ominus\mathbb{C}}(\Omega), \\ G_f^\lambda &:= P_{\lambda,\mathbb{C}}M_f : \mathcal{A}^p_{\lambda,ah\ominus\mathbb{C}}(\Omega) \to \mathcal{A}^p_{\lambda,\mathbb{C}}(\Omega). \end{aligned}$$

9.3 Spectral theory for VMO$_\partial$ symbols

The essential spectrum of Toeplitz operators on holomorphic function spaces with symbols of vanishing mean oscillation is well understood, cf. Corollary 5.4.2 or Proposition 9.1.3. We want to derive the same result for Toeplitz operators on $\mathcal{A}^p_{\lambda,\mathrm{ph}}(\Omega)$:

Proposition 9.3.1. *Let p, λ satisfy Assumption 1 and $f \in \mathrm{VMO}_\partial(\Omega)$. Then, we have*

$$\sigma_{ess}(T_f^{\lambda,ph}) = \bigcap_{R>0} (\mathcal{B}_\lambda(f)(\Omega \setminus E(0,R))).$$

Here, $E(0,R)$ denotes the Euclidean ball for $\Omega = \mathbb{C}^n$ or the metric ball with respect to β for Ω a bounded symmetric domain and $\mathcal{B}_\lambda(f)$ denotes the "usual" Berezin transform of f, i.e. the Berezin transform arising from the holomorphic Bergman/Fock space $\mathcal{A}^2_\lambda(\Omega)$.

Proof. Note that

$$B_f^\lambda = (P_{\lambda,\mathrm{ah}} - P_{\lambda,\mathbb{C}})H_f^\lambda$$

and

$$A_f^\lambda = (B_{\overline{f}}^\lambda)^*,$$

where we consider $B_{\overline{f}}^\lambda$ as an operator from $\mathcal{A}_\lambda^q(\Omega)$ to $\mathcal{A}_{\lambda,\mathrm{ah}\ominus\mathbb{C}}^q(\Omega)$. By Proposition 9.1.3 and Theorem 2.3.5, both A_f^λ and B_f^λ are compact. By Lemma 9.2.1, $T_f^{\lambda,\mathrm{ph}}$ is Fredholm if and only if

$$\left(\begin{array}{c|c} T_f^\lambda & 0 \\ \hline 0 & T_f^{\lambda,\mathrm{ah}\ominus\mathbb{C}} \end{array}\right)$$

is Fredholm. By simply considering the definition of what it means to be Fredholm, this clearly is equivalent to both T_f^λ and $T_f^{\lambda,\mathrm{ah}\ominus\mathbb{C}}$ being Fredholm.

Let us check when $T_f^{\lambda,\mathrm{ah}\ominus\mathbb{C}}$ is Fredholm using Lemma 9.2.2. For the operator E_f^λ we get

$$E_f^\lambda = (P_{\lambda,\mathrm{ah}} - P_{\lambda,\mathbb{C}})(I - P_\lambda)M_f = (P_{\lambda,\mathrm{ah}} - P_{\lambda,\mathbb{C}})H_f^\lambda|_{\mathcal{A}_{\lambda,\mathrm{ah}\ominus\mathbb{C}}^p(\Omega)}.$$

Hence, E_f^λ is also compact for $f \in \mathrm{VMO}_\partial(\Omega)$. Furthermore,

$$G_f^\lambda = (E_{\overline{f}}^\lambda)^*$$

in the same sense as $A_f^\lambda = (B_{\overline{f}}^\lambda)^*$, yielding compactness of G_f^λ. By Lemma 9.2.2 we obtain that $T_f^{\lambda,\mathrm{ah}}$ is Fredholm precisely if $T_f^{\lambda,\mathrm{ah}\ominus\mathbb{C}}$ and $P_{\lambda,\mathbb{C}}M_f : \mathcal{A}_{\lambda,\mathbb{C}}^p(\Omega) \to \mathcal{A}_{\lambda,\mathbb{C}}^p(\Omega)$ are both Fredholm. But the latter operator is always Fredholm, as it acts on a one-dimensional space. Hence, we have obtained

$$T_f^{\lambda,\mathrm{ph}} \text{ is Fredholm } \iff T_f^\lambda, T_f^{\lambda,\mathrm{ah}} \text{ are both Fredholm,}$$

i.e.

$$\sigma_{ess}(T_f^{\lambda,\mathrm{ph}}) = \sigma_{ess}(T_f^\lambda) \cup \sigma_{ess}(T_f^{\lambda,\mathrm{ah}}).$$

Recall that

$$T_f^{\lambda,\mathrm{ah}} = CT_{\overline{f}}^\lambda C.$$

Since C is isometric and anti-linear, this yields

$$\sigma_{ess}(T_f^{\lambda,\mathrm{ah}}) = \overline{\sigma_{ess}(T_{\overline{f}}^\lambda)} = \sigma_{ess}(T_f^\lambda).$$

Together with Proposition 9.1.3 and Corollary 5.4.2 this completes the proof. □

Proposition 9.3.2. *Let p, λ satisfy Assumption 1 and $f \in \mathrm{VMO}_\partial(\Omega)$. Then, $T_f^{\lambda,ph}$ is compact if and only if*

$$B_\lambda(f)(z) \to 0, \quad d(0,z) \to \infty,$$

where $B_\lambda(f)$ is again the usual Berezin transform coming from the reproducing kernels of $\mathcal{A}_\lambda^2(\Omega)$.

Proof. The proof follows arguments completely analogous to those of the previous proof, using the compactness characterization of T_f^λ in terms of the Berezin transform, Proposition 9.1.2 and Corollary 3.3.10. □

Remark 9.3.3. A more general compactness criterion for T_f^λ acting on $\mathcal{A}^2_{\lambda,ph}(\mathbb{C}^n)$, where $f \in L^\infty(\mathbb{C}^n)$, has been obtained in [16]: T_f^λ is compact if and only if the *pluriharmonic* Berezin transform (i.e. the Berezin transform induced from the reproducing kernels of $\mathcal{A}^2_{\lambda,ph}(\mathbb{C}^n)$, see also the next section) vanishes at infinity.

We want to end this short section by a brief discussion of the Fredholm index of Toeplitz operators on pluriharmonic function spaces, without going too much into technical details. As is well-known, an interesting Fredholm theory needs matrix-valued symbols. As results on the Fredholm index on the holomorphic function spaces are mainly available in the Hilbert space setting, we restrict ourselves also to the case $p = 2$. For $f \in L^\infty(\Omega, M_n(\mathbb{C}))$, the Toeplitz operator $T_F^{\lambda,n}$ is defined as

$$T_F^{\lambda,n} g = P_\lambda(Fg),$$

where

$$g = \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix} \in \mathcal{A}_\lambda^2(\Omega)^n,$$

and the projection P_λ acts componentwise as the projection. The anti-holomorphic and pluriharmonic Toeplitz operators $T_F^{\lambda,\mathrm{ah},n}$ and $T_F^{\lambda,\mathrm{ph},n}$ are defined analogously. When the matrix symbol is in $\mathrm{VMO}_\partial(\Omega)$ componentwise, one can show, very similarly to the proof of Proposition 9.3.1, that we have

$$T_F^{\lambda,\mathrm{ph},n} \cong \left(\begin{array}{c|c} T_F^{\lambda,n} & 0 \\ \hline 0 & T_f^{\lambda,\mathrm{ah},n} \end{array}\right) + K$$

for some K compact, hence

$$\mathrm{ind}(T_F^{\lambda,\mathrm{ph},n}) = \mathrm{ind}(T_F^{\lambda,n}) + \mathrm{ind}(T_F^{\lambda,\mathrm{ah},n}).$$

Further, by adjoining the operator C (considered as acting on $L_\lambda^2(\Omega)^n$ in the obvious way), we have

$$T_F^{\lambda,\mathrm{ah},n} = C T_{\overline{F}}^{\lambda,N} C,$$

and therefore, since C is an (anti-linear) isomorphism,

$$\operatorname{ind}(T_F^{\lambda,\mathrm{ah},n}) = \operatorname{ind}(T_{\overline{F}}^{\lambda,n}),$$

where $\overline{F}$ is the matrix symbol obtained from F by entrywise complex conjugation. Hence,

$$\operatorname{ind}(T_F^{\lambda,\mathrm{ph},n}) = \operatorname{ind}(T_F^{\lambda,n}) + \operatorname{ind}(T_{\overline{F}}^{\lambda,n}).$$

If we consider the particular case of the unit ball

$$\Omega = \mathbb{B}_n := \{z \in \mathbb{C}^n;\ |z| < 1\},$$

then we can say even more. Here, we have (at least for the case $\lambda = 0$, most likely even for every λ) the Venugopalkrishna Index Theorem [126] and its extension to VMO_∂-symbols [133], expressing the Fredholm index of $T_F^{\lambda,n}$ in terms of the topological mapping degree of the first column of F. Since passing from the first column of F to the first column of $\overline{F}$ geometrically corresponds to applying n reflections with respect to (real) hyperplanes, we obtain from standard facts on the topological degree that

$$\operatorname{ind}(T_{\overline{F}}^{0,n}) = (-1)^n \operatorname{ind}(T_F^{0,n}),$$

yielding

$$\operatorname{ind}(T_F^{0,\mathrm{ph},n}) = \begin{cases} 2\operatorname{ind}(T_F^{0,n}), & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

for this particular case. We expect the same formula to be valid for $\Omega = \mathbb{C}^n$ and any λ.

9.4 Quantization estimates

In this section, we will investigate the quantization asymptotics (9.2)-(9.4) in the setting of pluriharmonic Toeplitz operators. Since the quantization estimates for the holomorphic cases have so far only been studied in the Hilbert space setting $p = 2$, we will impose the same restriction throughout this section. The first estimate can be easily derived from the holomorphic case:

Proposition 9.4.1. *For any $f \in L^\infty(\Omega)$ we have*

$$\|T_f^{\lambda,ph}\| \to \|f\|_\infty, \quad \lambda \to \infty.$$

Proof. Lemma 9.2.1 immediately yields the estimate

$$\|T_f^{\lambda,\mathrm{ph}}\| \geq \|T_f^\lambda\|.$$

Combining this with the trivial estimate $\|T_f^{\lambda,\mathrm{ph}}\| \leq \|f\|_\infty$ and the quantization results of Proposition 9.1.4 and Theorem 2.3.6 proves the result. □

Remark 9.4.2. The proof of the previous proposition only hinges on the fact that $\mathcal{A}^2_\lambda(\Omega)$ is a subspace of $\mathcal{A}^2_{\lambda,\mathrm{ph}}(\Omega)$. Hence, if A_λ is any closed subspace of $L^2_\lambda(\Omega)$ containing $\mathcal{A}^2_\lambda(\Omega)$ and P_{A_λ} is the orthogonal projection onto A_λ, then

$$\|T_f^{\lambda,A_\lambda}\| \to \|f\|_\infty, \quad \lambda \to \infty$$

for any $f \in L^\infty(\Omega)$, where

$$T_f^{\lambda,A_\lambda} = P_{A_\lambda} M_f : A_\lambda \to A_\lambda.$$

Indeed, there is more we can say on the first order of the pluriharmonic quantization than just the previous proposition. For example, one can consider the convergence of the *pluriharmonic Berezin transform*, which we are going to define now.

$\mathcal{A}^2_{\lambda,\mathrm{ph}}(\Omega)$, being the orthogonal sum of the two reproducing kernel Hilbert spaces $\mathcal{A}^2_\lambda(\Omega)$ and $\mathcal{A}^2_{\lambda,\mathrm{ah}\ominus\mathbb{C}}(\Omega)$, is itself a reproducing kernel Hilbert space, the reproducing kernel being the sum of the two kernels of $\mathcal{A}^2_\lambda(\Omega)$ and $\mathcal{A}^2_{\lambda,\mathrm{ah}\ominus\mathbb{C}}(\Omega)$. As C maps $\mathcal{A}^2_\lambda(\Omega)$ bijectively and isometrically to $\mathcal{A}^2_{\lambda,\mathrm{ah}}(\Omega)$, the reproducing kernel of $\mathcal{A}^2_{\lambda,\mathrm{ah}}(\Omega)$, being anti-holomorphic in the first and holomorphic in the second variable, is given by

$$K^{\lambda,\mathrm{ah}}(w,z) = \overline{K^\lambda(w,z)} = K^\lambda(z,w).$$

Since the reproducing kernel of $\mathcal{A}^2_{\lambda,\mathrm{ah}}(\Omega)$ at $z = 0$ is constantly 1, the reproducing kernel of $\mathcal{A}^2_{\lambda,\mathrm{ah}\ominus\mathbb{C}}(\Omega)$ is

$$K^{\lambda,\mathrm{ah}\ominus\mathbb{C}}(w,z) = K^{\lambda,\mathrm{ah}}(w,z) - 1 = \overline{K^\lambda(w,z)} - 1.$$

In total, the reproducing kernel of $\mathcal{A}^2_{\lambda,\mathrm{ph}}(\Omega)$ is given by

$$K^{\lambda,\mathrm{ph}}(w,z) = K^\lambda(w,z) + K^{\lambda,\mathrm{ah}}(w,z) - 1 = K^\lambda(w,z) + \overline{K^\lambda(w,z)} - 1.$$

We denote the normalized kernels by

$$k^\lambda(w,z) := \frac{K^\lambda(w,z)}{\|K^\lambda(\cdot,z)\|_{L^2_\lambda(\Omega)}},$$
$$k^{\lambda,\mathrm{ah}}(w,z) := \frac{K^{\lambda,\mathrm{ah}}(w,z)}{\|K^{\lambda,\mathrm{ah}}(\cdot,z)\|_{L^2_\lambda(\Omega)}},$$
$$k^{\lambda,\mathrm{ph}}(w,z) := \frac{K^{\lambda,\mathrm{ph}}(w,z)}{\|K^{\lambda,\mathrm{ph}}(\cdot,z)\|_{L^2_\lambda(\Omega)}}.$$

Then, the antiholomorphic and the pluriharmonic Berezin transforms of $f \in L^\infty(\Omega)$ are defined by

$$\mathcal{B}^{\mathrm{ah}}_\lambda(f)(z) := \langle f k^{\lambda,\mathrm{ah}}(\cdot,z), k^{\lambda,\mathrm{ah}}(\cdot,z)\rangle_{L^2_\lambda(\Omega)},$$
$$\mathcal{B}^{\mathrm{ph}}_\lambda(f)(z) := \langle f k^{\lambda,\mathrm{ph}}(\cdot,z), k^{\lambda,\mathrm{ph}}(\cdot,z)\rangle_{L^2_\lambda(\Omega)}.$$

Since $k^{\lambda,\mathrm{ah}}$ is simply the complex conjugate of k^λ, we have

$$|k^{\lambda,\mathrm{ah}}(w,z)|^2 = |k^\lambda(w,z)|^2$$

and therefore evidently

$$\mathcal{B}_\lambda(f)(z) = \mathcal{B}_\lambda^{\mathrm{ah}}(f)(z).$$

Lemma 9.4.3. *Let $f \in L^\infty(\Omega)$ be such that $\|H_f^\lambda\|$, $\|H_f^{\lambda,ah}\| \to 0$ as $\lambda \to \infty$. Further, assume $z \in \Omega$ is such that*

$$\mathcal{B}_\lambda(f)(z) \to c, \quad \lambda \to \infty$$

for some $c \in \mathbb{C}$. Then,

$$\mathcal{B}_\lambda^{ph}(f)(z) \to c, \quad \lambda \to \infty.$$

Proof. Observe that we have

$$k^{\lambda,\mathrm{ph}}(\cdot,0) = 1 = k^\lambda(\cdot,0)$$

and therefore

$$\mathcal{B}_\lambda^{\mathrm{ph}}(f)(0) = \mathcal{B}_\lambda(f)(0)$$

for all λ. We may thus assume $z \neq 0$. Then,

$$\|K^\lambda(\cdot,z)\| \to \infty, \quad \lambda \to \infty.$$

By orthogonality, we have

$$\begin{aligned}\|K^\lambda(\cdot,z) + K^{\lambda,\mathrm{ah}}(\cdot,z) - 1\|^2 &= \|K^\lambda(\cdot,z)\|^2 + \|K^{\lambda,\mathrm{ah}}(\cdot,z) - 1\|^2 \\ &= \|K^\lambda(\cdot,z)\|^2 \cdot \left(1 + \frac{\|K^{\lambda,\mathrm{ah}}(\cdot,z) - 1\|^2}{\|K^\lambda(\cdot,z)\|^2}\right).\end{aligned}$$

Since

$$\begin{aligned}\langle K^{\lambda,\mathrm{ah}}(\cdot,z), 1\rangle_\lambda &= \langle K^{\lambda,\mathrm{ah}}(\cdot,z), K^{\lambda,\mathrm{ah}}(\cdot,0)\rangle_\lambda \\ &= K^{\lambda,\mathrm{ah}}(0,z) \\ &= 1,\end{aligned}$$

we obtain

$$\begin{aligned}\|K^{\lambda,\mathrm{ah}}(\cdot,z) - 1\|^2 &= \|K^{\lambda,\mathrm{ah}}(\cdot,z)\|^2 - 1 \\ &= \|K^\lambda(\cdot,z)\|^2 - 1.\end{aligned}$$

Hence,

$$1 + \frac{\|K^{\lambda,\mathrm{ah}}(\cdot,z) - 1\|^2}{\|K^\lambda(\cdot,z)\|^2} \to 2, \quad \lambda \to \infty.$$

Combining these computations, we have shown that

$$\frac{\|K^\lambda(\cdot,z)+K^{\lambda,\mathrm{ah}}(\cdot,z)-1\|^2}{\|K^\lambda(\cdot,z)\|^2}\to 2,\quad \lambda\to\infty.$$

In particular, if one of the two limits below exists, we have the equality

$$\begin{aligned}&\lim_{\lambda\to\infty}\frac{\langle f(K^\lambda(\cdot,z)+K^{\lambda,\mathrm{ah}}(\cdot,z)-1),K^\lambda(\cdot,z)+K^{\lambda,\mathrm{ah}}(\cdot,z)-1\rangle_\lambda}{\|K^\lambda(\cdot,z)+K^{\lambda,\mathrm{ah}}(\cdot,z)-1\|^2}\\ &\quad=\lim_{\lambda\to\infty}\frac{\langle f(K^\lambda(\cdot,z)+K^{\lambda,\mathrm{ah}}(\cdot,z)-1),K^\lambda(\cdot,z)+K^{\lambda,\mathrm{ah}}(\cdot,z)-1\rangle_\lambda}{2\|K^{\lambda,\mathrm{ah}}(\cdot,z)-1\|^2},\end{aligned}$$

where the left-hand side of that equation is simply the limit of $\mathcal{B}_\lambda^{\mathrm{ph}}(f)(z)$. Using sesquilinearity, we will split the right-hand side of that equation into simpler terms, the limits of which we can compute. Let us first deal with those terms which contribute to the limit:

$$\begin{aligned}\frac{\langle fK^\lambda(\cdot,z),K^\lambda(\cdot,z)\rangle_\lambda}{2\|K^\lambda(\cdot,z)\|^2}&=\frac{1}{2}\mathcal{B}_\lambda(f)(z)\to\frac{1}{2}c,\quad\lambda\to\infty\\ \frac{\langle fK^{\lambda,\mathrm{ah}}(\cdot,z),K^{\lambda,\mathrm{ah}}(\cdot,z)\rangle_\lambda}{2\|K^\lambda(\cdot,z)\|^2}&=\frac{1}{2}\mathcal{B}_\lambda^{\mathrm{ah}}(f)(z)=\frac{1}{2}\mathcal{B}_\lambda(f)(z)\to\frac{1}{2}c,\quad\lambda\to\infty.\end{aligned}$$

Since the measure ν_λ was normalized to one we have $\|f\|_{L^2_\lambda}\le\|f\|_\infty$, hence

$$\frac{1}{\|K^\lambda(\cdot,z)\|^2}|\langle f,1\rangle_\lambda|\le\frac{\|f\|_\infty}{\|K^\lambda(\cdot,z)\|^2}\to 0,\quad\lambda\to\infty.$$

Further,

$$\begin{aligned}\frac{|\langle fK^\lambda(\cdot,z),K^{\lambda,\mathrm{ah}}(\cdot,z)-1\rangle_\lambda|}{\|K^\lambda(\cdot,z)\|^2}&=\frac{|\langle(I-P_\lambda)(fK^\lambda(\cdot,z)),K^{\lambda,\mathrm{ah}}-1\rangle_\lambda|}{\|K^\lambda(\cdot,z)\|^2}\\ &=\frac{|\langle H_f^\lambda(K^\lambda(\cdot,z)),K^{\lambda,\mathrm{ah}}(\cdot,z)-1\rangle_\lambda|}{\|K^\lambda(\cdot,z)\|^2}\\ &\le\|H_f^\lambda\|\frac{\|K^{\lambda,\mathrm{ah}}(\cdot,z)-1\|}{\|K^\lambda(\cdot,z)\|}.\end{aligned}$$

As we have noted earlier, $\frac{\|K^{\lambda,\mathrm{ah}}(\cdot,z)-1\|}{\|K^\lambda(\cdot,z)\|}\to 1$ as $\lambda\to\infty$. Since we assumed $\|H_f^\lambda\|\to 0$ when $\lambda\to\infty$, the initial expression converges to 0. The same reasoning shows that

$$\frac{\langle fK^{\lambda,\mathrm{ah}}(\cdot,z),K^\lambda(\cdot,z)-1\rangle_\lambda}{\|K^\lambda(\cdot,z)\|^2}\to 0,\quad\lambda\to\infty.$$

Finally,

$$\frac{|\langle f,K^\lambda(\cdot,z)\rangle_\lambda|}{\|K^\lambda(\cdot,z)\|^2}\le\frac{\|f\|_\infty}{\|K^\lambda(\cdot,z)\|}\to 0,\quad\lambda\to\infty$$

and the same holds true for

$$\frac{\langle f, K^{\lambda,\mathrm{ah}}(\cdot,z)\rangle_\lambda}{\|K^\lambda(\cdot,z)\|^2}.$$

Combining all the pieces, we obtain

$$\mathcal{B}_\lambda^{\mathrm{ph}}(f)(z) \to c$$

as $\lambda \to \infty$. □

Proposition 9.4.4. *For $f \in C_b(\Omega)$ we have*

$$\mathcal{B}_\lambda^{ph}(f)(z) \to f(z), \quad \lambda \to \infty$$

for all $z \in \Omega$.

Proof. Fix $z \in \Omega$ and let $\varepsilon > 0$ arbitrary. Let $\delta > 0$ be such that $w \in E(z,\delta)$ implies $|f(z) - f(w)| < \varepsilon$. Then,

$$\begin{aligned}|\mathcal{B}_\lambda^{\mathrm{ph}}(f)(z) - f(z)| &\leq \int_\Omega |f(w) - f(z)| \frac{|K^{\lambda,\mathrm{ph}}(z,w)|^2}{K^{\lambda,\mathrm{ph}}(z,z)} d\nu_\lambda(w)\\ &\leq \varepsilon + 2\|f\|_\infty \int_{\Omega\setminus E(z,\delta)} \frac{|K^{\lambda,\mathrm{ph}}(z,w)|^2}{K^{\lambda,\mathrm{ph}}(z,z)}\, d\nu_\lambda(w).\end{aligned}$$

Pick $\chi \in C(\Omega)$ such that $\chi|_{\Omega\setminus E(z,\delta)} \equiv 1$, $0 \leq \chi \leq 1$ and $\chi(z) = 0$. Such a function is clearly uniformly continuous. Hence, by Proposition 9.1.5 and Theorem 2.3.7, χ satisfies the assumptions of Lemma 9.4.3. Thus,

$$\begin{aligned}\int_{\Omega\setminus E(z,\delta)} \frac{|K^{\lambda,\mathrm{ph}}(z,w)|^2}{K^{\lambda,\mathrm{ph}}(z,z)}\, d\nu_\lambda(w) &= \int_{\Omega\setminus E(z,\delta)} [\chi(w) - \chi(z)] \frac{|K^{\lambda,\mathrm{ph}}(z,w)|^2}{K^{\lambda,\mathrm{ph}}(z,z)}\, d\nu_\lambda(w)\\ &\leq \int_\Omega [\chi(w) - \chi(z)] \frac{|K^{\lambda,\mathrm{ph}}(z,w)|^2}{K^{\lambda,\mathrm{ph}}(z,z)}\, d\nu_\lambda(w)\\ &= \mathcal{B}_\lambda^{\mathrm{ph}}(\chi)(z) - \chi(z) \to 0, \quad \lambda \to \infty.\end{aligned}$$

Therefore,

$$\limsup_{\lambda\to\infty} |\mathcal{B}_\lambda^{\mathrm{ph}}(f)(z) - f(z)| \leq \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, the result follows. □

Let us come to the second quantization result for pluriharmonic Toeplitz operators, i.e.

$$\|T_f^{\lambda,\mathrm{ph}} T_g^{\lambda,\mathrm{ph}} - T_{fg}^{\lambda,\mathrm{ph}}\| \to 0, \quad \lambda \to \infty.$$

As mentioned earlier, there is a dense, self-adjoint subspace $\mathcal{D}_\lambda$ of $L^2_\lambda(\Omega)$ which is invariant under P_λ and M_f for $f \in \mathrm{UC}(\Omega)$. Then, it is also invariant under $P_{\lambda,\mathrm{ah}}$, since

$$P_{\lambda,\mathrm{ah}}(f) = \overline{P_\lambda(\overline{f})}.$$

In particular, we obtain that $\mathcal{D}_\lambda \cap \mathcal{A}^2_{\lambda,\mathrm{ph}}(\Omega)$ is invariant under $T_f^{\lambda,\mathrm{ph}}$ for $f \in \mathrm{UC}(\Omega)$. Hence, we can form the product of $T_f^{\lambda,\mathrm{ph}}$ with $T_g^{\lambda,\mathrm{ph}}$ for either $g \in L^\infty(\Omega)$ or $g \in \mathrm{UC}(\Omega)$.

Proposition 9.4.5. *Let $f \in \mathrm{UC}(\Omega)$. Then, for any $g \in L^\infty(\Omega)$ or $g \in \mathrm{UC}(\Omega)$ we have*

$$\|T_f^{\lambda,ph} T_g^{\lambda,ph} - T_{fg}^{\lambda,ph}\| \to 0, \quad \lambda \to \infty.$$

Proof. Applying Lemma 9.2.1, it is immediate to see that, with respect to the orthogonal decomposition

$$\mathcal{A}^2_{\lambda,\mathrm{ph}}(\Omega) = \mathcal{A}^2_\lambda(\Omega) \oplus \mathcal{A}^2_{\lambda,\mathrm{ah}\ominus\mathbb{C}}(\Omega)$$

we obtain the matrix representation

$$T_f^{\lambda,\mathrm{ph}} T_g^{\lambda,\mathrm{ph}} - T_{fg}^{\lambda,\mathrm{ph}} = \left(\begin{array}{c|c} (1,1) & (1,2) \\ \hline (2,1) & (2,2) \end{array}\right),$$

where

$$\begin{aligned}
(1,1) &= T_f^\lambda T_g^\lambda - T_{fg}^\lambda + A_f^\lambda B_f^\lambda, \\
(1,2) &= T_f^\lambda A_g^\lambda + A_f^\lambda T_g^{\lambda,\mathrm{ah}\ominus\mathbb{C}} - A_{fg}^\lambda, \\
(2,1) &= B_f^\lambda T_g^\lambda + T_f^{\lambda,\mathrm{ah}\ominus\mathbb{C}} B_g^\lambda - B_{fg}^\lambda, \\
(2,2) &= B_f^\lambda A_g^\lambda + T_f^{\lambda,\mathrm{ah}\ominus\mathbb{C}} T_g^{\lambda,\mathrm{ah}\ominus\mathbb{C}} - T_{fg}^{\lambda,\mathrm{ah}\ominus\mathbb{C}}.
\end{aligned}$$

We need to show that the operator norm of all four components of that matrix converge to zero as $\lambda \to \infty$.

Recalling that

$$A_f^\lambda = (B_{\overline{f}}^\lambda)^*$$

and

$$B_f^\lambda = (P_{\lambda,\mathrm{ah}} - P_{\lambda,\mathbb{C}}) H_f^\lambda,$$

we immediately obtain that $\|(1,1)\| \to 0$ as $\lambda \to \infty$ by a simple application of Proposition 9.1.5 or Theorem 2.3.7.

In regard to $\|(1,2)\| \to 0$ and $\|(2,1)\| \to 0$, observe that both blocks factorize as

$$(1,2) = -D_f^{1,\lambda} C_g^{2,\lambda}, \quad (2,1) = -D_f^{2,\lambda} C_g^{1,\lambda},$$

where

$$\begin{aligned}
C_g^{1,\lambda} &= (I - P_{\lambda,\mathrm{ph}})M_f : \mathcal{A}_\lambda^2(\Omega) \to \mathcal{A}_{\lambda,\mathrm{ph}}^2(\Omega)^\perp, \\
C_g^{2,\lambda} &= (I - P_{\lambda,\mathrm{ph}})M_f : \mathcal{A}_{\lambda,\mathrm{ah}\ominus\mathbb{C}}^2(\Omega) \to \mathcal{A}_{\lambda,\mathrm{ph}}^2(\Omega)^\perp, \\
D_f^{1,\lambda} &= P_\lambda M_f : \mathcal{A}_{\lambda,\mathrm{ph}}^2(\Omega)^\perp \to \mathcal{A}_\lambda^2(\Omega), \\
D_f^{2,\lambda} &= (P_{\lambda,\mathrm{ah}} - P_{\lambda,\mathbb{C}})M_f : \mathcal{A}_{\lambda,\mathrm{ph}}^2(\Omega)^\perp \to \mathcal{A}_{\lambda,\mathrm{ah}\ominus\mathbb{C}}^2(\Omega).
\end{aligned}$$

Now, observe that

$$\begin{aligned}
C_g^{1,\lambda} &= (I - P_{\lambda,\mathrm{ph}})H_g^\lambda, \\
C_g^{2,\lambda} &= (I - P_{\lambda,\mathrm{ph}})H_f^{\lambda,\mathrm{ah}}|_{\mathcal{A}_{\lambda,\mathrm{ah}\ominus\mathbb{C}}^2(\Omega)}
\end{aligned}$$

and

$$D_f^{1,\lambda} = (C_{\overline{f}}^{1,\lambda})^*, \quad D_f^{2,\lambda} = (C_{\overline{f}}^{2,\lambda})^*.$$

Thus, we obtain $\|(1,2)\| \to 0$ and $\|(2,1)\| \to 0$ by another application of Proposition 9.1.5 or Theorem 2.3.7.

Finally, since we already know that $\|B_f^\lambda A_g^\lambda\| \to 0$ as $\lambda \to \infty$, we obtain $\|(2,2)\| \to 0$ if we can show that

$$\|T_f^{\lambda,\mathrm{ah}\ominus\mathbb{C}} T_g^{\lambda,\mathrm{ah}\ominus\mathbb{C}} - T_{fg}^{\lambda,\mathrm{ah}\ominus\mathbb{C}}\| \to 0.$$

Since

$$T_f^{\lambda,\mathrm{ah}} T_g^{\lambda,\mathrm{ah}} - T_{fg}^{\lambda,\mathrm{ah}} = C[T_{\overline{f}}^\lambda T_{\overline{g}}^\lambda - T_{\overline{fg}}^\lambda]C$$

and C is isometric, we know that

$$\|T_f^{\lambda,\mathrm{ah}} T_g^{\lambda,\mathrm{ah}} - T_{fg}^{\lambda,\mathrm{ah}}\| \to 0, \quad \lambda \to \infty.$$

Applying Lemma 9.2.2, we obtain that the upper left entry of $T_f^{\lambda,\mathrm{ah}} T_g^{\lambda,\mathrm{ah}} - T_{fg}^{\lambda,\mathrm{ah}}$ with respect to the decomposition (9.6) is given by

$$T_f^{\lambda,\mathrm{ah}\ominus\mathbb{C}} T_g^{\lambda,\mathrm{ah}\ominus\mathbb{C}} - T_{fg}^{\lambda,\mathrm{ah}\ominus\mathbb{C}} + E_f^\lambda G_f^\lambda.$$

Recalling that

$$E_f^\lambda = (P_{\lambda,\mathrm{ah}} - P_{\lambda,\mathbb{C}})H_f^\lambda|_{\mathcal{A}_{\lambda,\mathrm{ah}\ominus\mathbb{C}}^2(\Omega)}, \quad G_g^\lambda = (E_{\overline{g}}^\lambda)^*,$$

we deduce that $\|E_f^\lambda G_g^\lambda\| \to 0$. Since the norm of the full operator matrix representing $T_f^{\lambda,\mathrm{ah}} T_g^{\lambda,\mathrm{ah}} - T_{fg}^{\lambda,\mathrm{ah}}$ converges to zero, also the norm of the upper left entry has to converge to zero. Thus, we get

$$\|T_f^{\lambda,\mathrm{ah}\ominus\mathbb{C}} T_g^{\lambda,\mathrm{ah}\ominus\mathbb{C}} - T_{fg}^{\lambda,\mathrm{ah}\ominus\mathbb{C}}\| \to 0, \quad \lambda \to \infty.$$

This now yields $\|(2,2)\| \to 0$ as $\lambda \to \infty$. □

For $\Omega = \mathbb{C}^n$ or $\Omega = \mathbb{B}_n$, the same proof yields a $\mathrm{VMO}_b(\Omega)$-version of the result, using Proposition 9.1.6 or the VMO_b part of Theorem 2.3.7:

Proposition 9.4.6. *Let $\Omega = \mathbb{C}^n$ or $\Omega = \mathbb{B}_n$ and $f \in \mathrm{VMO}_b(\Omega)$. Then, for any $g \in L^\infty(\Omega)$ we have*

$$\|T_f^{\lambda,ph} T_g^{\lambda,ph} - T_{fg}^{\lambda,ph}\| \to 0, \quad \lambda \to \infty.$$

We want to end this section by explaining why (9.4) fails on pluriharmonic function spaces. This has already been noted by M. Engliš in [62]. We reproduce the argument presented there, but arrive at a slightly more detailed result.

Note that $P_{\lambda,\mathrm{ph}}$ is an integral operator with real-valued integral kernel, hence

$$\overline{P_{\lambda,\mathrm{ph}} h} = P_{\lambda,\mathrm{ph}} \overline{h}$$

for any $h \in L^2_\lambda(\Omega)$. This implies

$$\overline{T_f^{\lambda,\mathrm{ph}} h} = T_{\overline{f}}^{\lambda,\mathrm{ph}} \overline{h}$$

for any $h \in \mathcal{A}^2_{\lambda,\mathrm{ph}}(\Omega)$. Therefore, for $f, g \in L^\infty(\Omega)$ and $h \in \mathcal{A}^2_{\lambda,\mathrm{ph}}(\Omega)$ we obtain

$$\begin{aligned}
\overline{[T_f^{\lambda,\mathrm{ph}}, T_g^{\lambda,\mathrm{ph}}]^*(h)} &= \overline{[T_{\overline{g}}^{\lambda,\mathrm{ph}}, T_{\overline{f}}^{\lambda,\mathrm{ph}}](h)} \\
&= [T_g^{\lambda,\mathrm{ph}}, T_f^{\lambda,\mathrm{ph}}](\overline{h}) \\
&= -[T_f^{\lambda,\mathrm{ph}}, T_g^{\lambda,\mathrm{ph}}](\overline{h}).
\end{aligned}$$

The pluriharmonic Berezin transform of an operator acting on $\mathcal{A}^2_{\lambda,\mathrm{ph}}(\Omega)$ is of course defined as

$$\mathcal{B}_\lambda^{\mathrm{ph}}(A)(z) = \langle A k^{\lambda,\mathrm{ph}}(z,\cdot), k^{\lambda,\mathrm{ph}}(z,\cdot)\rangle.$$

On the level of Toeplitz operators, this of course agrees with the Berezin transform of the symbol:

$$\mathcal{B}_\lambda^{\mathrm{ph}}(T_f^{\lambda,\mathrm{ph}}) = \mathcal{B}_\lambda^{\mathrm{ph}}(f).$$

Since the pluriharmonic reproducing kernel is real-valued, this implies

$$\begin{aligned}
\mathcal{B}_\lambda^{\mathrm{ph}}([T_f^{\lambda,\mathrm{ph}}, T_g^{\lambda,\mathrm{ph}}])(z) &= \frac{\langle [T_f^{\lambda,\mathrm{ph}}, T_g^{\lambda,\mathrm{ph}}] K^{\lambda,\mathrm{ph}}(z,\cdot), K^{\lambda,\mathrm{ph}}(z,\cdot)\rangle}{K^{\lambda,\mathrm{ph}}(z,z)} \\
&= \frac{\langle \overline{[T_f^{\lambda,\mathrm{ph}}, T_g^{\lambda,\mathrm{ph}}]^* K^{\lambda,\mathrm{ph}}(z,\cdot)}, \overline{K^{\lambda,\mathrm{ph}}(z,\cdot)}\rangle}{K^{\lambda,\mathrm{ph}}(z,z)} \\
&= -\frac{\langle [T_f^{\lambda,\mathrm{ph}}, T_g^{\lambda,\mathrm{ph}}] K^{\lambda,\mathrm{ph}}(z,\cdot), K^{\lambda,\mathrm{ph}}(z,\cdot)\rangle}{K^{\lambda,\mathrm{ph}}(z,z)}
\end{aligned}$$

$$= -\mathcal{B}_\lambda^{\mathrm{ph}}([T_f^{\lambda,\mathrm{ph}}, T_g^{\lambda,\mathrm{ph}}])(z)$$

and hence $\mathcal{B}_\lambda^{\mathrm{ph}}([T_f^{\lambda,\mathrm{ph}}, T_g^{\lambda,\mathrm{ph}}])(z) = 0$ for all $z \in \Omega$. If we now let $f, g \in L^\infty(\Omega)$ and $h \in C_b(\Omega)$, then

$$\left\| \frac{\lambda}{i}[T_f^{\lambda,\mathrm{ph}}, T_g^{\lambda,\mathrm{ph}}] - T_h^{\lambda,\mathrm{ph}} \right\| \to 0, \quad \lambda \to \infty$$

and

$$\begin{aligned} \left\| \frac{\lambda}{i}[T_f^{\lambda,\mathrm{ph}}, T_g^{\lambda,\mathrm{ph}}] - T_h^{\lambda,\mathrm{ph}} \right\| &\geq \left\| \mathcal{B}_\lambda^{\mathrm{ph}} \left(\frac{\lambda}{i}[T_f^{\lambda,\mathrm{ph}}, T_g^{\lambda,\mathrm{ph}}] - T_h^{\lambda,\mathrm{ph}} \right) \right\|_\infty \\ &= \|\mathcal{B}_\lambda^{\mathrm{ph}}(T_h^{\lambda,\mathrm{ph}})\|_\infty = \|\mathcal{B}_\lambda^{\mathrm{ph}}(h)\|_\infty \geq 0 \end{aligned}$$

together imply that $\|\mathcal{B}_\lambda^{\mathrm{ph}}(h)\|_\infty \to 0$. By Proposition 9.4.4 this yields $h = 0$. Hence, we obtain:

Proposition 9.4.7. *Let $f, g \in L^\infty(\Omega)$ and $h \in C_b(\Omega)$. Then,*

$$\left\| \frac{\lambda}{i}[T_f^{\lambda,ph}, T_g^{\lambda,ph}] - T_h^{\lambda,ph} \right\| \to 0, \quad \lambda \to \infty$$

holds if and only if $h \equiv 0$ and

$$\|[T_f^{\lambda,ph}, T_g^{\lambda,ph}]\| \in o(1/\lambda) \text{ as } \lambda \to \infty.$$

In particular, there cannot be any Poisson structure $\{\cdot,\cdot\}$ on Ω such that

$$\left\| \frac{\lambda}{i}[T_f^{\lambda,ph}, T_g^{\lambda,ph}] - T_{\{f,g\}}^{\lambda,ph} \right\| \to 0, \quad \lambda \to \infty$$

holds for all $f, g \in C_c^\infty(\Omega)$.

9.5 Applying quantization estimates in spectral theory

As we have just seen, Toeplitz quantization on pluriharmonic function spaces is not a "good" quantization in the sense that the property (9.4) fails entirely. Yet, it is not pointless to study the remaining properties (9.2), (9.3) in the pluriharmonic setting, as suitable understanding of these quantization estimates has applications in spectral theory. We will show one of such possible applications, which was strongly motivated by the work done in [18, 20]. Those two papers dealt with the representation and spectral theory of Toeplitz algebras on $\mathcal{A}_\lambda^2(\mathbb{B}_n)$ for symbols with certain product structures. Let us introduce a similar setting. Note that we will only deal with the lowest dimensional case (that of $\mathbb{B}_2$) in which these methods work. Generalizations to $\mathbb{B}_n$ for $n > 2$ can be concluded very analogously to [18].

Recall that the measure ν_λ on $\mathbb{B}_1 := \{z \in \mathbb{C};\ |z| < 1\}$ is given by

$$d\nu_\lambda(z) = \frac{\Gamma(2+\lambda)}{\pi\Gamma(\lambda+1)}(1-|z|^2)^\lambda dV(z).$$

The standard orthonormal basis of $\mathcal{A}^2_\lambda(\mathbb{B}_1)$ is

$$e_a^\lambda(z) = \sqrt{\frac{\Gamma(a+\lambda+2)}{a!\Gamma(\lambda+2)}}z^a, \quad z \in \mathbb{B}_1,\ a \in \mathbb{N}_0.$$

Hence, we obtain the "standard" orthonormal basis of $\mathcal{A}^2_{\lambda,\mathrm{ph}}(\mathbb{B}_1)$ by adding the orthonormal family

$$\overline{e}_b^\lambda(z) = \sqrt{\frac{\Gamma(b+\lambda+2)}{b!\Gamma(\lambda+2)}}\overline{z}^b, \quad z \in \mathbb{B}_1,\ b \in \mathbb{N},$$

i.e.

$$\mathcal{A}^2_{\lambda,\mathrm{ph}}(\mathbb{B}_1) = \overline{\mathrm{Span}}\{e_a^\lambda;\ a \in \mathbb{N}_0\} \oplus \overline{\mathrm{Span}}\{\overline{e}_b^\lambda;\ b \in \mathbb{N}\}.$$

On $\mathbb{B}_2 := \{(z_1,z_2) \in \mathbb{C}^2;\ |z_1|^2 + |z_2|^2 < 1\}$ the measure ν_λ (which we will in our notation, somewhat imprecisely, not distinguish from the measure ν_λ on $\mathbb{B}_1$) is defined as

$$d\nu_\lambda(z_1,z_2) = \frac{\Gamma(3+\lambda)}{\pi^2\Gamma(\lambda+1)}(1-|z_1|^2-|z_2|^2)^\lambda dV(z_1,z_2).$$

We now introduce for $\lambda > -1$ the Bergman spaces $\mathcal{A}^2_{\lambda,\text{ph-h}}(\mathbb{B}_2)$ as the closed subspace of $L^2_\lambda(\mathbb{B}_2) = L^2(\mathbb{B}_2,\nu_\lambda)$ specified by the following orthonormal basis:

$$\mathcal{A}^2_{\lambda,\text{ph-h}}(\mathbb{B}_2) := \overline{\mathrm{Span}}\{\mathfrak{e}^{\lambda,+}_{(a_1,a_2)},\ \mathfrak{e}^{\lambda,-}_{(b_1,b_2)};\ (a_1,a_2) \in \mathbb{N}_0^2,\ (b_1,b_2) \in \mathbb{N}\times\mathbb{N}_0\}.$$

Here, we define the basis functions as

$$\mathfrak{e}^{\lambda,+}_{(a_1,a_2)}(z_1,z_2) = \sqrt{\frac{\Gamma(a_1+a_2+\lambda+3)}{a_1!a_2!\Gamma(\lambda+3)}}z_1^{a_1}z_2^{a_2}, \quad (a_1,a_2) \in \mathbb{N}_0^2,$$
$$\mathfrak{e}^{\lambda,-}_{(b_1,b_2)}(z_1,z_2) = \sqrt{\frac{\Gamma(b_1+b_2+\lambda+3)}{b_1!b_2!\Gamma(\lambda+3)}}\overline{z_1}^{b_1}z_2^{b_2}, \quad (b_1,b_2) \in \mathbb{N}\times\mathbb{N}_0.$$

Hence, $\mathcal{A}^2_{\lambda,\text{ph-h}}(\mathbb{B}_2)$ consists of all C^2 functions f on $\mathbb{B}_2$ satisfying

$$\frac{\partial^2 f}{\partial z_1 \partial \overline{z_1}} = 0, \quad \frac{\partial f}{\partial \overline{z_2}} = 0,$$

which also satisfy the integrability condition of $L^2_\lambda(\mathbb{B}_2)$. In particular, this means that $\mathcal{A}^2_{\lambda,\text{ph-h}}(\mathbb{B}_2)$ consists of those functions from $L^2_\lambda(\mathbb{B}_2)$ which are (pluri-)harmonic in z_1

and holomorphic in z_2. In particular, every such function can be written as a power series on $\mathbb{B}_2$:

$$\begin{aligned} f(z_1, z_2) &= \sum_{j=0}^{\infty} \left(\sum_{k=0}^{\infty} c_{j,k} z_1^k z_2^j + \sum_{l=1}^{\infty} d_{j,l} \overline{z_1}^l z_j^2 \right) \\ &= \sum_{j=0}^{\infty} \left(\sum_{k=0}^{\infty} c'_{j,k} \mathfrak{e}^{\lambda,+}_{(k,j)}(z_1, z_2) + \sum_{l=1}^{\infty} d'_{j,l} \mathfrak{e}^{\lambda,-}_{(l,j)}(z_1, z_2) \right). \end{aligned}$$

An obvious factorization yields

$$\begin{aligned} \mathfrak{e}^{\lambda,+}_{(a_1,a_2)}(z_1, z_2) &= e^{a_2+\lambda+1}_{a_1}(z_1) e^{\lambda+1}_{a_2}(z_2), \\ \mathfrak{e}^{\lambda,-}_{(b_1,b_2)}(z_1, z_2) &= \overline{e}^{b_2+\lambda+1}_{b_1}(z_1) e^{\lambda+1}_{b_2}(z_2). \end{aligned}$$

Setting for every $a_2 \in \mathbb{N}_0$

$$H_{a_2} := \overline{\mathrm{Span}}\{\mathfrak{e}^{\lambda,+}_{(a_1,a_2)},\ \mathfrak{e}^{\lambda,-}_{(b_1,a_2)};\ a_1 \in \mathbb{N}_0,\ b_1 \in \mathbb{N}\}$$

gives the following orthogonal decomposition:

$$\mathcal{A}^2_{\lambda,\text{ph-h}}(\mathbb{B}_2) = \bigoplus_{a_2 \in \mathbb{N}_0} H_{a_2}. \tag{9.7}$$

A simple calculation shows that every $f \in H_{a_2}$ can be written as

$$f(z_1, z_2) = f_{a_2}(z_1) e^{\lambda+1}_{a_2}(z_2)$$

for some unique $f_{a_2} \in \mathcal{A}^2_{a_2+\lambda+1,\text{ph}}(\mathbb{B}_1)$. In particular, every $f \in \mathcal{A}^2_{\lambda,\text{ph-h}}(\mathbb{B}_2)$ can be expanded into a series

$$f(z_1, z_2) = \sum_{a_2 \in \mathbb{N}_0} f_{a_2}(z_1) e^{\lambda+1}_{a_2}(z_2)$$

satisfying

$$\|f\|^2_{\mathcal{A}^2_{\lambda,\text{ph-h}}(\mathbb{B}_2)} = \sum_{a_2 \in \mathbb{N}_0} \|f_{a_2}\|^2_{\mathcal{A}^2_{a_2+\lambda+1}(\mathbb{B}_1)}.$$

We denote by $u_{a_2} : H_{a_2} \to \mathcal{A}^2_{a_2+\lambda+1,\text{ph}}(\mathbb{B}_2)$ the operator

$$u_{a_2}(f) = f_{a_2}.$$

Then, we obtain an isometric isomorphism

$$U = \bigoplus_{a_2 \in \mathbb{N}_0} u_{a_2} : \mathcal{A}^2_{\lambda,\text{ph-h}}(\mathbb{B}_2) = \bigoplus_{a_2 \in \mathbb{N}_0} H_{a_2} \to \bigoplus_{a_2 \in \mathbb{N}_0} \mathcal{A}^2_{a_2+\lambda+1,\text{ph}}(\mathbb{B}_1). \tag{9.8}$$

By $P_{\lambda,\text{ph-h}}$ we will denote the orthogonal projection from $L^2_\lambda(\mathbb{B}_2)$ onto $\mathcal{A}^2_{\lambda,\text{ph-h}}(\mathbb{B}_2)$. For $f \in L^\infty(\mathbb{B}_2)$ we denote by $T_f^{\lambda,\text{ph-h}}$ the Toeplitz operator

$$T_f^{\lambda,\text{ph-h}} = P_{\lambda,\text{ph-h}} M_f : \mathcal{A}^2_{\lambda,\text{ph-h}}(\mathbb{B}_2) \to \mathcal{A}^2_{\lambda,\text{ph-h}}(\mathbb{B}_2).$$

Our goal will be to understand the essential spectrum of $T_f^{\lambda,\text{ph-h}}$ whenever the symbol f has a particularly nice structure. In principle, one could allow the same product structure as for symbols of Toeplitz operators on $\mathcal{A}^2_\lambda(\mathbb{B}_2)$ studied in [20], i.e. for $g \in L^\infty(\mathbb{B}_1 \times (0,1))$ consider

$$f(z_1, z_2) := g\left(z_1, \frac{r_1}{\sqrt{1-r_2^2}}\right) \in L^\infty(\mathbb{B}_2),$$

where

$$z_1 = r_1 e^{i\theta_1}, \; z_2 = r_2 e^{i\theta_2}$$

are the coordinatewise polar coordinates. Then, one could, up to some obvious changes, imitate the approach taken in [20] by studying the representation theory of algebras generated by such Toeplitz operators. Upon assuming certain continuity assumptions on the symbol, one could then obtain a description of the essential spectrum. Nevertheless, we prefer to take a simpler symbol structure: For $g \in L^\infty(\mathbb{B}_1)$ we consider

$$\widetilde{g}(z_1, z_2) := g(z_1) \in L^\infty(\mathbb{B}_2).$$

Considering this simpler symbol structure has the advantage that we can give a significantly shorter proof for the characterization of the essential spectrum of $T_{\widetilde{g}}^{\lambda,\text{ph-h}}$ which does not simply follow the lines of [20]. That is, we will prove the following result:

Proposition 9.5.1. *Let $g \in \mathrm{VO}_\partial(\mathbb{B}_1)$. Then, $T_{\widetilde{g}}^{\lambda,ph\text{-}h}$ is Fredholm if and only if there is some $c > 0$ such that $|g(z_1)| \geq c$ for all $z_1 \in \mathbb{B}_1$. In particular,*

$$\sigma_{ess}(T_{\widetilde{g}}^{\lambda,ph\text{-}h}) = \overline{g(\mathbb{B}_1)}.$$

Here is the key fact that relates the action of $T_{\widetilde{g}}^{\lambda,\text{ph-h}}$ with the decomposition (9.7).

Lemma 9.5.2. *$T_{\widetilde{g}}^{\lambda,ph\text{-}h}$ acts as follows:*

$$\langle T_{\widetilde{g}}^{\lambda,ph\text{-}h} \mathfrak{e}^{\lambda,+}_{(a_1,a_2)}, \mathfrak{e}^{\lambda,+}_{(\tilde{a}_1,\tilde{a}_2)} \rangle = \begin{cases} 0, & a_2 \neq \tilde{a}_2 \\ \langle T_g^{a_2+\lambda+1,ph} e_{a_1}^{a_2+\lambda+1}, e_{\tilde{a}_1}^{a_2+\lambda+1} \rangle, & a_2 = \tilde{a}_2 \end{cases},$$

$$\langle T_{\widetilde{g}}^{\lambda,ph\text{-}h} \mathfrak{e}^{\lambda,+}_{(a_1,a_2)}, \mathfrak{e}^{\lambda,-}_{(b_1,b_2)} \rangle = \begin{cases} 0, & a_2 \neq b_2 \\ \langle T_g^{a_2+\lambda+1,ph} e_{a_1}^{a_2+\lambda+1}, \overline{e}_{b_1}^{a_2+\lambda+1} \rangle, & a_2 = b_2 \end{cases},$$

$$\langle T_{\widetilde{g}}^{\lambda,ph\text{-}h} \mathfrak{e}_{(b_1,b_2)}^{\lambda,-}, \mathfrak{e}_{(a_1,a_2)}^{\lambda,+}\rangle = \begin{cases} 0, & b_2 \neq a_2 \\ \langle T_g^{a_2+\lambda+1,ph} \overline{e_{b_1}^{a_2+\lambda+1}}, e_{a_1}^{a_2+\lambda+1}\rangle, & b_2 = a_2 \end{cases},$$

$$\langle T_{\widetilde{g}}^{\lambda,ph\text{-}h} \mathfrak{e}_{(b_1,b_2)}^{\lambda,-}, \mathfrak{e}_{(\tilde{b}_1,\tilde{b}_2)}^{\lambda,-}\rangle = \begin{cases} 0, & b_2 \neq \tilde{b}_2 \\ \langle T_g^{b_2+\lambda+1,ph} \overline{e_{b_1}^{b_2+\lambda+1}}, \overline{e_{\tilde{b}_1}^{b_2+\lambda+1}}\rangle, & b_2 = \tilde{b}_2 \end{cases}.$$

In particular, $T_{\widetilde{g}}^{\lambda,ph\text{-}h}$ *leaves the decomposition* (9.7) *invariant.*

Proof. The computations are identical to those in the proof of [20, Lemma 2.2]. We repeat them here to prove the first identity, the remaining cases can be worked out entirely analogously.

Let $(a_1, a_2), (\tilde{a}_1, \tilde{a}_2) \in \mathbb{N}_0^2$. Then,

$$\begin{aligned}
&\langle T_{\widetilde{g}}^{\lambda,\text{ph-h}} \mathfrak{e}_{(a_1,a_2)}^{\lambda,+}, \mathfrak{e}_{(\tilde{a}_1,\tilde{a}_2)}^{\lambda,+}\rangle \\
&= \sqrt{\frac{\Gamma(a_1+a_2+\lambda+3)}{a_1!a_2!\Gamma(\lambda+3)} \frac{\Gamma(\tilde{a}_1+\tilde{a}_2+\lambda+3)}{\tilde{a}_1!\tilde{a}_2!\Gamma(\lambda+3)}} \frac{\Gamma(\lambda+3)}{\pi^2\Gamma(\lambda+1)} \\
&\times \int_{\mathbb{B}^2} g(z_1) z_1^{a_1} z_2^{a_2} \overline{z_1}^{\tilde{a}_1} \overline{z_2}^{\tilde{a}_2} (1-(|z_1|^2+|z_2|^2))^\lambda dV(z_1,z_2).
\end{aligned}$$

Introducing polar coordinates $z_1 = r_1 e^{i\theta_1}, z_2 = r_2 e^{i\theta_2}$, we obtain

$$\begin{aligned}
&= \sqrt{\frac{\Gamma(a_1+a_2+\lambda+3)\Gamma(\tilde{a}_1+\tilde{a}_2+\lambda+3)}{a_1!a_2!\tilde{a}_1!\tilde{a}_2!}} \frac{1}{\pi^2\Gamma(\lambda+1)} \int_0^{2\pi} e^{i\theta_2(a_2-\tilde{a}_2)} d\theta_2 \\
&\times \int_{\{r_1,r_2>0; r_1^2+r_2^2<1\}} \int_0^{2\pi} g(r_1 e^{i\theta_1}) r_1^{a_1+\tilde{a}_1+1} r_2^{a_2+\tilde{a}_2+1} e^{i\theta_1(a_1-\tilde{a}_1)} \\
&\times (1-r_1^2-r_2^2)^\lambda d\theta_1 dr_2 dr_1.
\end{aligned}$$

Of course, the first integral in this expression equals 0 for $a_2 \neq \tilde{a}_2$ and 2π for $a_2 = \tilde{a}_2$. For the latter case, we get

$$\begin{aligned}
&= \sqrt{\frac{\Gamma(a_1+a_2+\lambda+3)\Gamma(\tilde{a}_1+a_2+\lambda+3)}{a_1!(a_2!)^2\tilde{a}_1!}} \frac{2}{\pi\Gamma(\lambda+1)} \\
&\times \int_{\{r_1,r_2>0; r_1^2+r_2^2<1\}} \int_0^{2\pi} g(r_1 e^{i\theta_1}) r_1^{a_1+\tilde{a}_1+1} r_2^{2a_2+1} e^{i\theta_1(a_1-\tilde{a}_1)} \\
&\times (1-r_1^2-r_2^2)^\lambda d\theta_1 dr_2 dr_1.
\end{aligned}$$

Using the substitution $s = \frac{r_2}{\sqrt{1-r_1^2}}$ in the r_2 integral we get

$$= \sqrt{\frac{\Gamma(a_1+a_2+\lambda+3)\Gamma(\tilde{a}_1+a_2+\lambda+3)}{a_1!(a_2!)^2\tilde{a}_1!}} \frac{2}{\pi\Gamma(\lambda+1)}$$

$$\begin{aligned}
&\times \int_0^{2\pi}\int_0^1 g(r_1e^{i\theta_1})r_1^{a_1+\tilde{a}_1+1}e^{i\theta_1(a_1-\tilde{a}_1)}(1-r_1^2)^{a_2+\lambda+1}d\theta_1 dr_1\\
&\times \int_0^1 s^{2a_2+1}(1-s^2)^\lambda ds\\
&= \sqrt{\frac{\Gamma(a_1+a_2+\lambda+3)\Gamma(\tilde{a}_1+a_2+\lambda+3)}{a_1!(a_2!)^2\tilde{a}_1!}}\frac{1}{\pi\Gamma(\lambda+1)}\\
&\times \int_{\mathbb{B}_1} g(z_1)z_1^{a_1}\overline{z}_1^{\tilde{a}_1}(1-|z_1|^2)^{a_2+\lambda+1}dv(z)\int_0^1 s^{a_2}(1-s)^\lambda ds.
\end{aligned}$$

Using the beta function $B(x,y)=\int_0^1 s^{x-1}(1-s)^{y-1}ds$ and the well-known identity $B(x,y)=\dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ we obtain

$$\begin{aligned}
&= \frac{\Gamma(a_2+\lambda+2)}{a_2!\Gamma(\lambda+1)}B(a_2+1,\lambda+1)\\
&\times \int_{\mathbb{B}} g(z_1)e_{a_1}^{a_2+\lambda+1}(z_1)\overline{e}_{\tilde{a}_1}^{a_2+\lambda+1}(z_1)\frac{\Gamma(a_2+\lambda+3)}{\pi\Gamma(a_2+\lambda+2)}(1-|z_1|^2)^{a_2+\lambda+1}dz_1\\
&= \langle T_g^{a_2+\lambda+1}e_{a_1}^{a_2+\lambda+1}, e_{\tilde{a}_1}^{a_2+\lambda+1}\rangle_{a_2+\lambda+1}.
\end{aligned}$$

Since

$$\langle T_g^{a_2+\lambda+1}e_{a_1}^{a_2+\lambda+1}, e_{\tilde{a}_1}^{a_2+\lambda+1}\rangle_{a_2+\lambda+1} = \langle T_g^{a_2+\lambda+1,\mathrm{ph}}e_{a_1}^{a_2+\lambda+1}, e_{\tilde{a}_1}^{a_2+\lambda+1}\rangle_{a_2+\lambda+1},$$

this finishes the proof. □

Recalling the operator U from (9.8), we have proven:

Corollary 9.5.3. *When adjoining by the operator U, we obtain the unitary equivalence*

$$\begin{aligned}
&T_{\widetilde{g}}^{\lambda,ph\text{-}h} : \mathcal{A}^2_{\lambda,ph\text{-}h}(\mathbb{B}_2) \to \mathcal{A}^2_{\lambda,ph\text{-}h}(\mathbb{B}_2)\\
\cong &\bigoplus_{a_2\in\mathbb{N}_0} T_g^{a_2+\lambda+1,ph} : \bigoplus_{a_2\in\mathbb{N}_0}\mathcal{A}^2_{a_2+\lambda+1,ph}(\mathbb{B}_1) \to \bigoplus_{a_2\in\mathbb{N}_0}\mathcal{A}^2_{a_2+\lambda+1,ph}(\mathbb{B}_1).
\end{aligned}$$

Before presenting the proof of Proposition 9.5.1, we need one more purely functional analytic fact:

Lemma 9.5.4. *Let $H_k,\ k\in\mathbb{N}_0$ be a family of Hilbert spaces and let $\bigoplus_k H_k$ be their direct orthogonal sum. For a family $A_k\in\mathcal{L}(H_k)$ let $A:=\bigoplus_k A_k$ act diagonally on H. Then, A is Fredholm if and only if each A_k is Fredholm and there are $B_k^1, B_k^2\in\mathcal{L}(H_k)$ with*

$$A_kB_k^1 - I\in\mathcal{K}(H_k),\quad B_k^2A_k - I\in\mathcal{K}(H_k)$$

such that

$$\begin{aligned}
\|A_kB_k^1 - I\| \to 0,\quad k\to\infty,\\
\|B_k^2A_k - I\| \to 0,\quad k\to\infty.
\end{aligned}$$

While the lemma is certainly well-known, we provide a proof for completeness.

Proof. Let $K = (K_k)_k$ with $K_k \in \mathcal{L}(H_k)$ act diagonally on H. We claim that K is compact if and only if each K_k is compact and

$$\|K_k\| \to 0, \quad k \to \infty. \tag{9.9}$$

That every K_k has to be compact for K to be compact is of course necessary. Let us assume that K does not satisfy (9.9). Pick a sequence $(m_n)_{m\in\mathbb{N}} \subset (\mathbb{N}_0)$ such that

$$\|K_{m_n}\| \to \limsup_{k\to\infty} \|K_k\|, \quad n \to \infty.$$

For every $n \in \mathbb{N}_0$ let $e_n \in H_{m_n}$ be such that $\|e_n\|_{H_{m_n}} = 1$ and

$$\|K_{m_n} e_n\|_{H_{m_n}} \geq \|K_{m_n}\|_{\mathcal{L}(H_{m_n})} - \frac{1}{n}.$$

Then, the sequence

$$f_n := (\delta_{k,m_n} e_n)_{k\in\mathbb{N}_0} \in \bigoplus_{k=0}^{\infty} H_k$$

is clearly of norm one and converges weakly to 0. Further,

$$\|Kf_n\| = \|K_{m_n} e_n\| \to \limsup_{k\to\infty} \|K_k\| > 0, \quad n \to \infty$$

hence Kf_n does not converge strongly and K cannot be compact.

On the other hand, assume that (9.9) holds true. Let $\varepsilon > 0$ be arbitrary and fix $N \in \mathbb{N}$ such that $\|K_k\| < \varepsilon$ for $k > N$. For $k = 1, \dots, N$ let S_k be finite rank such that $\|K_k - S_k\|_{\mathcal{L}(H_k)} < \varepsilon$ and let

$$S = S_1 \oplus S_2 \oplus \cdots \oplus S_N \oplus 0 \oplus 0 \oplus \cdots \in \mathcal{L}\left(\bigoplus_{k=1}^{\infty} H_k\right).$$

Then,

$$\|K - S\|_H \leq \varepsilon.$$

Hence, K can be approximated arbitrarily well by finite rank operators and is therefore compact.

From this, the characterization of Fredholm operators on $\mathcal{L}(H)$ is now immediate by Atkinson's Theorem. □

Proof of Proposition 9.5.1. Assume that $|g(z_1)| \geq c > 0$ for all $z \in \mathbb{B}_1$. Since

$$\mathcal{B}_{a_2+\lambda+1}(g) - g \in C_0(\mathbb{B}_1)$$

for $g \in \mathrm{VO}_\partial(\mathbb{B}_1)$, it follows by Proposition 9.3.1 that $T_g^{a_2+\lambda+1,\mathrm{ph}} \in \mathcal{L}(\mathcal{A}^2_{a_2+\lambda+1,\mathrm{ph}}(\mathbb{B}_1))$ is Fredholm for every $a_2 \in \mathbb{N}_0$. Furthermore, since $\mathrm{VO}_\partial(\mathbb{B}_1)$ is contained in $\mathrm{UC}(\mathbb{B}_1)$, we know from Proposition 9.4.5 that

$$\|T_g^{a_2+\lambda+1,\mathrm{ph}} T_{1/g}^{a_2+\lambda+1,\mathrm{ph}} - I\| \to 0, \quad a_2 \to \infty.$$

Hence, combining Corollary 9.5.3 with the previous lemma we obtain that $T_g^{\lambda,\text{ph-h}}$ is Fredholm.

Now, assume that $\inf_{z_1 \in \mathbb{B}_1} |g(z_1)| = 0$. There are two possible cases:

1. There is a sequence $(z_1^j)_j \in \mathbb{B}_1$ with $z_1^j \to \partial\mathbb{B}_1$ such that $g(z_1^j) \to 0$.

2. There is some $z_1 \in \mathbb{B}_1$ with $g(z_1) = 0$.

In the first case, none of the operators $T_g^{a_2+\lambda+1,\mathrm{ph}}$ on $\mathcal{A}^2_{a_2+\lambda+1}(\mathbb{B}_1)$ can be Fredholm (recall again that $\mathcal{B}_{a_2+\lambda+1}(g) - g \in C_0(\mathbb{B}_1)$ for $g \in \mathrm{VO}_\partial(\mathbb{B}_1)$) by Proposition 9.3.1, hence $T_g^{\lambda,\text{ph-h}}$ is not Fredholm by the previous Lemma.

In the second case, one can do the following: Consider the sequence $(f_j)_{j\in\mathbb{N}} \subseteq \mathcal{A}^2_{\lambda,\text{ph-h}}(\mathbb{B}_2)$ defined using the decomposition (9.7) as

$$f_j = (\delta_{a_2,j} k^{a_2+\lambda+1}(\cdot, z_1))_{a_2\in\mathbb{N}_0} \in \bigoplus_{a_2\in\mathbb{N}_0} \mathcal{A}^2_{a_2+\lambda+1,\mathrm{ph}}(\mathbb{B}_1),$$

where

$$k^{a_2+\lambda+1}(\cdot, z_1) = k_{z_1}^{a_2+\lambda+1}$$

is the normalized reproducing kernel of $\mathcal{A}^2_{a_1+\lambda+1}(\mathbb{B}_1)$ (which is of course also contained in $\mathcal{A}^2_{a_2+\lambda+1,\mathrm{ph}}(\mathbb{B}_1)$). Since each f_j has norm one and the f_j are pairwise orthogonal, we of course have $f_j \to 0$ weakly as $j \to \infty$. Now,

$$\begin{aligned}
\|T_{\widetilde{g}}^{\lambda,\text{ph-h}} f_j\|^2 &\leq \langle g k_{z_1}^{j+\lambda+1}, g k_{z_1}^{j+\lambda+1}\rangle_{\mathcal{A}^2_{j+\lambda+1}(\mathbb{B}_1)}\\
&= \langle |g|^2 k_{z_1}^{j+\lambda+1}, k_{z_1}^{j+\lambda+1}\rangle_{\mathcal{A}^2_{j+\lambda+1}(\mathbb{B}_1)}\\
&= \mathcal{B}_{j+\lambda+1}(|g|^2)(z_1),
\end{aligned}$$

which is simply the Berezin transform of $|g|^2$ at z_1. Since $g \in \mathrm{VO}_\partial(\mathbb{B}_1)$, we in particular have $g \in C_b(\mathbb{B}_1)$ and hence $|g|^2 \in C_b(\mathbb{B}_1)$. Thus,

$$\mathcal{B}_{j+\lambda+1}(|g|^2)(z_1) \to |g|^2(z_1) = 0, \quad j \to \infty.$$

This yields that $(T_{\widetilde{g}}^{\lambda,\text{ph-h}} f_j)_{j\in\mathbb{N}}$ converges strongly to zero. Therefore, the operator cannot be Fredholm, as no Fredholm operator can map sequence, weakly but not strongly convergent to 0, to a sequence strongly convergent to 0. □

9.6 Remarks

In principle, the methods presented in this chapter, which boil down to applying the matrix representation of T_f^t with respect to the decomposition into holomorphic and anti-holomorphic part, do not hinge on Ω being $\mathbb{C}^n$ or a bounded symmetric domain. All that is needed is a good understanding of the holomorphic situation (and Ω should be at least simply connected). In particular, the method should also be applicable to obtain results if, say, Ω is a strictly pseudoconvex domain. Nevertheless, we restricted ourselves to the cases discussed here, since the properties we had in mind are best understood in these settings.

Appendices

A.1 Holomorphic and pluriharmonic functions of several complex variables

The theory of holomorphic functions of several complex variables is certainly a field of deep and interesting studies. We refer to the well-known textbooks [87, 93] for introductions to this topic. For our purpose, it suffices to know the following simple facts. For $\Omega \subset \mathbb{C}^n$ open, we say that $f : \Omega \to \mathbb{C}$ is holomorphic if it is holomorphic in every variable separately. Every holomorphic function can be locally expanded into a power series in $z_1, \ldots, z_n$. We will denote by

$$D(z, r) := \{w \in \mathbb{C};\ |z - w| < r\}$$

the disc around $z \in \mathbb{C}$ with radius $r > 0$ and, more generally, for $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, $r > 0$ by $P(z, r)$ the polydisc

$$\begin{aligned} P(z, r) :&= \{w = (w_1, \ldots, w_n) \in \mathbb{C}^n; |z_j - w_j| < r \text{ for every } j = 1, \ldots, n\} \\ &= \prod_{j=1}^{n} D(z_j, r). \end{aligned}$$

Let us cite the following fact:

Proposition A.1.1 ([87, Theorem 2.2.6])**.** *Assume $\Omega \subset \mathbb{C}^n$ is open and $f : \Omega \to \mathbb{C}^n$ is holomorphic. If $\overline{P(z, r)} \subset \Omega$, then f can be uniformly approximated by its power series on $P(z, r)$.*

Lemma A.1.2. *Let $\Omega \subset \mathbb{C}$ be open. Further, let $f : \Omega \to \mathbb{C}$ be a holomorphic function. Then, for each $z \in \Omega$ and $r > 0$ with $\overline{D(z, r)} \subset \Omega$ the following holds true:*

$$f(z) = \frac{1}{\pi r^2} \int_{D(z,r)} f(w)\ dw. \tag{A.10}$$

Proof. By Proposition A.1.1 it suffices to verify the identity for monomials in $w - z$. Using polar coordinates, one easily sees that

$$\begin{aligned}\int_{D(z,r)} (w-z)^k \, dw &= \int_{D(0,r)} w^k \, dw \\ &= \int_0^r s^{k+1} ds \int_0^{2\pi} e^{i\theta k} d\theta \\ &= \begin{cases} \pi r^2, & k=0 \\ 0, & k \geq 1. \end{cases}\end{aligned}$$

□

The following is the corresponding version of the previous lemma for several variables:

Corollary A.1.3. *Let $\Omega \subset \mathbb{C}^n$ be open and $f : \Omega \to \mathbb{C}$ holomorphic. For each $z = (z_1, \ldots, z_n) \in \Omega$ and $r > 0$ such that $\overline{P(z,r)} \subset \Omega$ we have*

$$f(z) = \frac{1}{(\pi r^2)^n} \int_{P(z,r)} f(w) \, dw.$$

Proof. Follows from an iterated use of Lemma A.1.2 for each variable. □

Corollary A.1.4. *Let $\Omega \subset \mathbb{C}^n$ be open and $\omega : \Omega \to (0,\infty)$ continuous. If $K \subset \Omega$ is compact, there is $K' \subset \Omega$ compact such that $K \subseteq K'$ and a constant $C_K > 0$ such that for all holomorphic $f : \Omega \to \mathbb{C}$ the following holds true:*

$$\sup_{z \in K} |f(z)| \leq C_K \int_{K'} |f(w)| \omega(w) \, dw.$$

Proof. Let $\varepsilon = \min\left\{\frac{\operatorname{dist}(K,\partial\Omega)}{2}, 1\right\}$, where dist is the usual distance between compact and closed sets in $\mathbb{C}^n \cong \mathbb{R}^{2n}$. Let $K' = \{w \in \mathbb{C}^n;\ \operatorname{dist}(w,K) \leq \varepsilon\}$. It is standard to prove that the Euclidean ball

$$B(z,r) := \{w \in \mathbb{C}^n;\ \left(\sum_{j=1}^n |z_j - w_j|^2\right)^{1/2} < r\}$$

contains certain polycylinders:

$$P(z, r/\sqrt{n}) \subset B(z,r).$$

In particular, for each $z \in K$ we have $P(z, \varepsilon/\sqrt{n}) \subset K'$. This yields, using Corollary A.1.3:

$$\begin{aligned}|f(z)| &\leq \left(\frac{n}{\pi\varepsilon^2}\right)^n \int_{P(z,\varepsilon/\sqrt{n})} |f(w)| \, dw \\ &\leq \left(\frac{n}{\pi\varepsilon^2}\right)^n \int_{K'} |f(w)| \, dw.\end{aligned}$$

Since ω is continuous, there is some $c > 0$ such that $\omega(z) \geq c$ for all $z \in K'$. Then, for all $z \in K$:

$$\begin{aligned} |f(z)| &\leq \frac{1}{c}\left(\frac{n}{\pi\varepsilon^2}\right)^n \int_{K'} |f(w)|c\, dw \\ &\leq \frac{1}{c}\left(\frac{n}{\pi\varepsilon^2}\right)^n \int_{K'} |f(w)|\omega(w)\, dw. \end{aligned}$$ □

Let $\Omega \subseteq \mathbb{C}^n$ be open and connected. We say that $f : \Omega \to \mathbb{C}$ is *pluriharmonic* if f is twice continuously differentiable and

$$\frac{\partial^2 f}{\partial z_j \partial \overline{z}_k} = 0 \tag{A.11}$$

for all $j, k = 1, \ldots, n$. The following fact about pluriharmonic functions will be of crucial importance to us:

Lemma A.1.5. *Let $\Omega \subseteq \mathbb{C}^n$ be open and simply connected and $x_0 \in \Omega$ a fixed point. Then, a C^2-function $f : \Omega \to \mathbb{C}$ is pluriharmonic if and only if there are holomorphic functions $g, h : \Omega \to \mathbb{C}$ such that*

$$f = g + \overline{h}.$$

If we further assume that $h(x_0) = 0$, then g and h are uniquely determined.

Proof. If g and h are assumed to exist, then it is obvious that f satisfies Equation (A.11) and thus f is pluriharmonic.

Hence, let us assume that f is pluriharmonic. Since we assume f to be C^2, the order of differentiation in Equation (A.11) does not matter. Furthermore, it is easy to check by the definitions of the Wirtinger derivatives $\frac{\partial}{\partial z_j}$ and $\frac{\partial}{\partial \overline{z}_k}$ that we have

$$\frac{\partial \overline{f}}{\partial z_j} = \overline{\left(\frac{\partial f}{\partial \overline{z}_j}\right)}, \quad \frac{\partial \overline{f}}{\partial \overline{z}_k} = \overline{\left(\frac{\partial f}{\partial z_k}\right)}.$$

Combining these statements, we obtain

$$\frac{\partial^2 \overline{f}}{\partial z_j \partial \overline{z}_k} = \frac{\partial^2 \overline{f}}{\partial \overline{z}_k \partial z_j} = \overline{\left(\frac{\partial^2 f}{\partial z_k \partial \overline{z}_j}\right)} = 0.$$

In particular, f is pluriharmonic if and only if $\overline{f}$ is. Since the differential expression defining pluriharmonicity is linear, this implies that $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are pluriharmonic. It is well-known that every real-valued pluriharmonic function on a simply-connected domain is the real part of a holomorphic function [77, Theorem K.3]. Hence, there are unique holomorphic functions g_0, h_0 such that

$$g_0 = \mathrm{Re}(f) + i\,\mathrm{Im}(g_0), \quad h_0 = \mathrm{Im}(f) + i\,\mathrm{Im}(h_0).$$

Solving these for $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ gives

$$\mathrm{Re}(f) = \frac{1}{2}(g_0 + \overline{g_0}), \quad \mathrm{Im}(f) = \frac{1}{2}(h_0 + \overline{h_0}),$$

hence

$$f = \mathrm{Re}(f) + i\,\mathrm{Im}(f) = \frac{1}{2}(g_0 + ih_0) + \frac{1}{2}\overline{(g_0 - ih_0)}$$

Letting $g = g_0 + ih_0$ and $h = g_0 - ih_0$ yields the representation for f.

Regarding the uniqueness, expanding g and $\overline{h}$ in a power series in z and $\overline{z}$, respectively, around x_0 shows that the only ambivalence of the representation comes from the constant part of those power series expansions, which can be made unique by letting $h(x_0) = 0$. □

A.2 The Complex Interpolation Method

Interpolation theory is a mathematical field of outstanding importance in functional analysis. Among the many interpolation methods, the Complex Interpolation Method is among the best studied. A section in an appendix like this certainly doesn't offer enough space to treat interpolation as a whole, so we restrict ourselves to introducing notation and mentioning the most important results. For a comprehensive treatment of the topic, we refer to the classical monograph [33], whose presentation and notation we also follow here.

Let X_0, X_1 be Banach spaces (endowed with the norms $\|\cdot\|_{X_0},\ \|\cdot\|_{X_1}$). We say that they are *compatible* if there exists a Hausdorff topological vector space X such that both X_0 and X_1 are subspaces of X. In X we can consider the subspaces

$$\begin{aligned} X_0 \cap X_1 &= \{f \in X;\ f \in X_0 \text{ and } f \in X_1\}, \\ X_0 + X_1 &= \{f \in X;\ \exists f_0 \in X_0,\ f_1 \in X_1 : f = f_0 + f_1\}. \end{aligned}$$

On $X_0 \cap X_1$ and $X_0 + X_1$ we have the following natural norms:

$$\begin{aligned} \|f\|_{X_0 \cap X_1} &= \max\{\|f\|_{X_0}, \|f\|_{X_1}\}, \\ \|f\|_{X_0 + X_1} &= \inf\{\|f_0\|_{X_0} + \|f_1\|_{X_1};\ f_0 \in X_0,\ f_1 \in X_1 \text{ s.th. } f = f_0 + f_1\}. \end{aligned}$$

With these norms, $X_0 \cap X_1$ and $X_0 + X_1$ can be seen to be complete [33, Lemma 2.3.1]. We will occasionally write such compatible couples of Banach spaces as pairs $\overline{X} = (X_0, X_1)$. If (X_0, X_1) and (Y_0, Y_1) are compatible couples of Banach spaces, the "well behaved operators" $T : (X_0, X_1) \to (Y_0, Y_1)$ (i.e. the morphisms of the category of compatible couples of Banach spaces) are bounded linear operators

$$T : X_0 + X_1 \to Y_0 + Y_1$$

such that

$$T|_{X_0} : X_0 \to Y_0, \quad T|_{X_1} : X_1 \to Y_1$$

continuously. Here, $T|_A$ denotes the restriction to the subspace A. It is easy to see that such an operator satisfies

$$\|Tf\|_{Y_0+Y_1} \le \|T\|_{X_0 \to Y_0} \|\|f_0\|_{X_0} + \|T\|_{X_1 \to Y_1} \|f_1\|_{X_1}$$

for all $f = f_0 + f_1 \in X_0 + X_1$, which yields

$$\|T\|_{X_0+X_1 \to Y_0+Y_1} \leq \max\{\|T\|_{X_0 \to Y_0}, \|T\|_{X_1 \to Y_1}\}.$$

For a couple $\overline{X} = (X_0, X_1)$ we will also write

$$\Delta(\overline{X}) = X_0 \cap X_1,$$
$$\Sigma(\overline{X}) = X_0 + X_1.$$

Another Banach space B will be said to be an *intermediate space* between X_0 and X_1 if

$$\Delta(\overline{X}) \subset B \subset \Sigma(\overline{X})$$

and the inclusions are continuous. Such an intermediate space B is further called an *interpolation space* if

$$T : \overline{X} \to \overline{X}$$

implies

$$T : B \to B.$$

More generally, if $\overline{X} = (X_0, X_1)$ and $\overline{Y} = (Y_0, Y_1)$ are two compatible couples, then two Banach spaces B, C are said to be interpolation spaces for the couples $\overline{X}$, $\overline{Y}$ if they are intermediate spaces with respect to $\overline{X}$, $\overline{Y}$ and

$$T : \overline{X} \to \overline{Y}$$

implies

$$T : B \to C.$$

Examples of such interpolation spaces are $\Delta(\overline{X})$, $\Delta(\overline{Y})$ and $\Sigma(\overline{X})$, $\Sigma(\overline{Y})$. For $0 \leq \theta \leq 1$ we say that the interpolation spaces B, C for the couples $\overline{X}, \overline{Y}$ are *exact of exponent θ* if

$$T : \overline{X} \to \overline{Y}$$

implies

$$\|T\|_{B \to C} \leq \|T\|_{X_0 \to Y_0}^{1-\theta} \|T\|_{X_1 \to Y_1}^{\theta}.$$

We now define the Complex Interpolation Method. Let $S \subset \mathbb{C}$ be the strip $0 \leq \mathrm{Re}(z) \leq 1$ and S_0 its interior $0 < \mathrm{Re}(z) < 1$. Given a compatible couple $\overline{X}$ set $\mathcal{F}(\overline{X})$ to be the set of all functions $f : S \to \Sigma(\overline{X})$ satisfying the following properties:

- f is continuous;

- $f|_{S_0}$ is analytic;
- $t \mapsto f(it)$ is a continuous function from $\mathbb{R}$ to X_0 vanishing at infinity;
- $t \mapsto f(1+it)$ is a continuous function from $\mathbb{R}$ to X_1 vanishing at infinity.

Endowed with the norm

$$\|f\|_{\mathcal{F}} = \max\{\sup_{t\in\mathbb{R}} \|f(it)\|_{X_0}, \sup_{t\in\mathbb{R}} \|f(1+it)\|_{X_1}\},$$

$\mathcal{F}(\overline{X})$ can be seen to be a Banach space [33, Lemma 4.1.1]. For $0 \le \theta \le 1$ we let

$$\overline{X}_{[\theta]} := \{x \in \Sigma(\overline{X});\ x = f(\theta) \text{ for some } f \in \mathcal{F}(\overline{X})\},$$

endowed with the norm

$$\|x\|_{[\theta]} := \inf\{\|f\|_{\mathcal{F}};\ f \in \mathcal{F}(\overline{X}) \text{ with } f(\theta) = x\}.$$

$(\overline{X}_{[\theta]},\ \|\cdot\|_{[\theta]})$ turns out to be an interpolation space for the couple $\overline{X}$. Moreover, this interpolation method is functorial, i.e. if $\overline{X}$ and $\overline{Y}$ are compatible couples, then $\overline{X}_\theta$ and $\overline{Y}_\theta$ are interpolation spaces with respect to those couples which are exact of exponent θ [33, Theorem 4.1.2]. We list further important properties:

Theorem A.2.1 ([33, Theorem 4.2.1, Theorem 4.2.2]). *Let $\overline{X} = (X_0, X_1)$ be a couple of compatible Banach spaces. Then, the following hold true:*

1) $(X_0, X_1)_{[\theta]} = (X_1, X_0)_{[1-\theta]}$ for $0 \le \theta \le 1$;

2) $X_0 \subset X_1 \Rightarrow (X_0, X_1)_{[\theta_1]} \subset (X_0, X_1)_{[\theta_2]}$ for $\theta_1 \le \theta_2$;

3) $\Delta(\overline{X})$ is dense in $(X_0, X_1)_{[\theta]}$ for $0 \le \theta \le 1$;

4) Let X_j° denote the closure of $\Delta(\overline{X})$ in X_j $(j = 0, 1)$. Then, we have for all $0 \le \theta \le 1$:

$$(X_0, X_1)_{[\theta]} = (X_0^\circ, X_1)_{[\theta]} = (X_0, X_1^\circ)_{[\theta]} = (X_0^\circ, X_1^\circ)_{[\theta]}.$$

A.3 Nuclear operators and operator ideals

We want to add some details on certain operator ideals. We closely follow the author's presentation of that topic in [72, Section 2.1] and copy certain passages from there verbatim.

Let X be a complex Banach space. Recall that an operator $A \in \mathcal{L}(X)$ is *nuclear*, if there are sequences $(x_j) \subset X, (y_j) \subset X'$ with $\sum_{j=1}^{\infty} \|y_j\|_{X'}\|x_j\|_X < \infty$ such that

$$A = \sum_{j=1}^{\infty} y_j \otimes x_j. \tag{A.12}$$

For such an operator we define

$$\|A\|_{\mathcal{N}} := \inf \sum_{j=1}^{\infty} \|y_j\|_{X'} \|x_j\|_X,$$

where the infimum is taken over all possible representations (A.12). For rank one operators

$$A = y \otimes x$$

one can indeed show that

$$\|A\|_{\mathcal{N}} = \|y\|_{X'} \|x\|_X.$$

We denote by $\mathcal{N}(X)$ the set of all nuclear operators on X. Together with the norm $\|\cdot\|_{\mathcal{N}}$, this is well-known to be a Banach ideal in $\mathcal{L}(X)$. If the underlying Banach space X has the approximation property, which we always assume in the following, we can define the nuclear trace for $A \in \mathcal{N}(X)$ as

$$\mathrm{Tr}(A) = \sum_{j=1}^{\infty} y_j(x_j),$$

where the trace is independent of the choice of representation (A.12), cf. [75, Theorem V.1.2]. If X is even reflexive, one can show that the duality relations

$$(\mathcal{K}(X))' = \mathcal{N}(X), \quad (\mathcal{N}(X))' = \mathcal{L}(X)$$

hold true isometrically, where the duality is induced by the trace map:

$$\langle A, B \rangle = \mathrm{Tr}(AB).$$

If X is not reflexive, we can still identify $\mathcal{L}(X)$ isometrically with a subspace of $(\mathcal{N}(X))'$ via the trace duality pairing. For details on the general theory of operator ideals, we refer to the books [53, 75, 108].

We now want to interpolate between the spaces $\mathcal{N}(X)$ and $\mathcal{L}(X)$. Since $\mathcal{N}(X) \subset \mathcal{L}(X)$, we can use the Complex Interpolation Method to obtain new ideals between $\mathcal{N}(X)$ and $\mathcal{L}(X)$. Using $\mathcal{L}(X)$ as the ambient Hausdorff topological vector space, in which we embed the compatible couple $\overline{A} := (\mathcal{L}(X), \mathcal{N}(X))$, we get

$$\Delta(\overline{A}) := \mathcal{N}(X) \cap \mathcal{L}(X) = \mathcal{N}(X) \quad \text{and} \quad \Sigma(\overline{A}) := \mathcal{N}(X) + \mathcal{L}(X) = \mathcal{L}(X),$$

where equalities are understood as normed vector spaces. Using the Complex Interpolation Method, we obtain a family of subspaces of $\mathcal{L}(X)$:

$$(\mathcal{N}(X), \mathcal{L}(X))_{[\theta]}, \quad 0 \le \theta \le 1.$$

Since $\mathcal{N}(X) \subset \mathcal{L}(X)$, the family of interpolation spaces is increasing by Theorem A.2.1, part 2):

$$\theta_1 \le \theta_2 : (\mathcal{N}(X), \mathcal{L}(X))_{[\theta_1]} \subset (\mathcal{N}(X), \mathcal{L}(X))_{[\theta_2]}.$$

Further, since $\Delta(\overline{A}) = \mathcal{N}(X)$ is dense in $(\mathcal{N}(X), \mathcal{L}(X))_{[1]}$ by Theorem A.2.1 3), we obtain by the approximation property:

$$(\mathcal{N}(X), \mathcal{L}(X))_{[1]} = \mathcal{K}(X).$$

We also have

$$(\mathcal{N}(X), \mathcal{L}(X))_{[0]} = \mathcal{N}(X).$$

With Theorem A.2.1 4) we obtain

$$(\mathcal{N}(X), \mathcal{L}(X))_{[\theta]} = (\mathcal{N}(X), \mathcal{K}(X))_{[\theta]},$$

i.e. each interpolation space consists of compact operators. Further, since $\mathcal{L}(X)$ and $\mathcal{N}(X)$ are ideals, for each $A \in \mathcal{L}(X)$ we obtain maps (which we denote by the same symbol):

$$\begin{aligned} L_A &: \mathcal{N}(X) \to \mathcal{N}(X), \quad B \mapsto AB, \\ L_A &: \mathcal{L}(X) \to \mathcal{L}(X), \quad B \mapsto AB. \end{aligned}$$

Interpolating this map, we obtain

$$L_A : (\mathcal{N}(X), \mathcal{L}(X))_{[\theta]} \to (\mathcal{N}(X), \mathcal{L}(X))_{[\theta]}, \quad B \mapsto AB,$$

i.e. the interpolated spaces are left ideals of $\mathcal{L}(X)$. Analogously, they are right ideals. For $1 \leq p_0 < \infty$, we define the ideals of compact operators $\mathcal{S}^{p_0}(X)$ by

$$S^{p_0}(X) := (\mathcal{N}(X), \mathcal{L}(X))_{[1-1/p_0]}.$$

In particular,

$$\mathcal{S}^1(X) = \mathcal{N}(X).$$

One can show the norm inequalities

$$\|A\|_{op} \leq \|A\|_{\mathcal{S}^{p_0}} \leq \|A\|_{\mathcal{S}^{q_0}}$$

for $p_0 \geq q_0$, where $\|\cdot\|_{op}$ denotes the operator norm on $\mathcal{L}(X)$. If X is a Hilbert space, these interpolated ideals are just the usual Schatten-von Neumann ideals [110, 120]. Surprisingly, it seems that no concrete description of the ideals $\mathcal{S}^{p_0}(X)$ is available if X is not a Hilbert space [109, Section 6.6.6.1]. Finally, let us note the following duality result:

Theorem A.3.1 ([112])**.** *Let X be a reflexive complex Banach space with the approximation property and $p_0 \in (1, \infty)$. Then, the dual of $\mathcal{S}^{p_0}(X)$ can be isometrically identified with $\mathcal{S}^{q_0}(X')$ via the trace duality*

$$\langle A, B \rangle = \mathrm{Tr}(AB').$$

Here, q_0 is the conjugate exponent to p_0: $1 = 1/p_0 + 1/q_0$.

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