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Extended supersymmetric multiparticle Euler–Calogero–Moser model

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Abstract. We review the construction of supersymmetric extension of the n-particle Euler–Calogero–Moser model within the Hamiltonian approach [1]. The main feature of the proposed supersymmetrization method is that it is automatically adapted for a model with an arbitrary even number of supersymmetries. It is shown that the number of fermions that must be used in this construction is $\frac{1}{2}\mathcal{N}n(n+1)$. We demonstrate that the resulting supersymmetric system is dynamically invariant with respect to the superconformal group $Osp(\mathcal{N}|2)$ and give the explicit realization of its generators in terms of the conserved currents. For the simplest case of the $\mathcal{N}=2$ supersymmetric n-particle Euler–Calogero–Moser model we provide its description in superspace by using the corresponding constrained superfields.

1. Introduction and Bosonic model

Recently, an interest in studying of supersymmetric extensions of matrix models has significantly increased. In many respects it is connected with a notable progress that was achieved in the supersymmetrization of the bosonic matrix models [2, 3, 4, 5, 6]. The matrix models were successfully used in constructing systems which preserve the conformal symmetry (see e.g. [7] and refs. therein). It is well known that the conformally invariant systems such as, for example, the Calogero model as well as its different extensions [8, 9, 10, 11, 12], can be obtained from the matrix models by a reduction procedure. In the supersymmetric case each element of a given matrix (unitary, Hermitian, symmetric, etc.) is replaced by a proper superfield, which may be constrained [2, 3, 4, 5]. However, the superfield approach is useful only for the first lowest values $\mathcal{N}$ of the extended supersymmetry, restricted by $\mathcal{N}\leq 4$, and it seems to be less efficient or even inapplicable for $\mathcal{N}>4$ supersymmetric cases.[1] In contrast, the Hamiltonian approach has no serious restriction on the number of supersymmetries, due to the absence of auxiliary components.

In the supersymmetrization of the bosonic matrix models, besides of the standard set of $\mathcal{N}\,n$ fermions accompanied n bosonic fields, it appears a large number of additional fermionic degrees of freedom, which are related with non-diagonal part of the supermatrices. Note that the number of these fermions depends on corresponding matrix model. So, as it was demonstrated

[1] Up to now unique example of a matrix system with $\mathcal{N}=8$ supersymmetry has appeared in [5] in $\mathcal{N}=4$ superspace.

in our recent paper [6], to construct a supersymmetric extension of Hermitian matrix models within the Hamiltonian approach, which admits an arbitrary number of supersymmetries, we introduced $\mathcal{N}\, n(n-1)$ additional fermions. Moreover, how it was shown in that paper, that, after the reduction procedure, an $\mathcal{N}$-extended n-particle supersymmetric Calogero model can be obtained. In this paper we apply the supersymmetrization procedure for the real symmetric matrix model [8] within the Hamiltonian approach.

We start with a basics of a spin generalization of the n-particle Calogero–Moser model, which is also known as the Euler–Calogero–Moser (ECM) model [8, 9]. As a bosonic system, this model is closely related to the free matrix models associated with real symmetric matrices (see e.g. [11]). The Hamiltonian, which depends on the coordinates $x_i(t)$ and momenta $p_i(t)$ of each particle and also on the internal degrees of freedom realized by the angular momenta $\ell_{ij} = -\ell_{ji}$, is given by

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \frac{1}{2}\sum_{i\neq j}^{n} \frac{\ell_{ij}^2}{(x_i - x_j)^2}\,. \tag{1.1}$$

The introduced variables satisfy the standard Poisson brackets

$$\big\{x_i, p_j\big\} = \delta_{ij}, \quad \big\{\ell_{ij}, \ell_{km}\big\} = \frac{1}{2}\big(\delta_{ik}\ell_{jm} + \delta_{jm}\ell_{ik} - \delta_{jk}\ell_{im} - \delta_{im}\ell_{jk}\big), \tag{1.2}$$

from which follows that the angular momenta form the $so(n)$ algebra.

It is well-known that the Euler–Calogero–Moser model with the Hamiltonian (1.1) is a conformally invariant system with respect to $SO(1,2)$ group. Besides the Hamiltonian, the rest set of its generators is defined as the conserved currents of dilatation D and conformal boost K as

$$D = -\frac{1}{2}\sum_{i=1}^{n} x_i p_i + tH \qquad \text{and} \qquad K = \frac{1}{2}\sum_{i=1}^{n} x_i^2 - t\sum_{i=1}^{n} x_i p_i + t^2 H \tag{1.3}$$

All together they form the one-dimensional conformal algebra $so(1,2)$

$$\big\{H, K\big\} = 2D, \quad \big\{H, D\big\} = H, \quad \big\{K, D\big\} = -K. \tag{1.4}$$

In this paper we review an $\mathcal{N}$-extended supersymmetric generalization of the Hamiltonian (1.1) and establish an $Osp(\mathcal{N}|2)$ invariance of the $\mathcal{N}$ supersymmetric ECM model. By considering of the simplest case of the model with $\mathcal{N} = 2$ supersymmetry, we provide its description in $\mathcal{N} = 2$ superspace in terms of constrained superfields. Finally, we give the $\mathcal{N} = 2$ supersymmetric version of a system, a crucial features of which is dependence on antisymmetric fermions, as well as on bosons, which enter the supercharges only through their sum.

2. Supersymmetric spin ECM model in the Hamiltonian approach

2.1. Model with $\mathcal{N} = 2$ supersymmetry

The $\mathcal{N} = 2$ supersymmetric extension of the n-particle Euler–Calogero–Moser model is described by two supercharges $Q, \overline{Q}$ and Hamiltonian H, whose bosonic limit is (1.1), and which form $\mathcal{N} = 2$ super Poincaré algebra

$$\big\{Q, \overline{Q}\big\} = -2\mathrm{i}H, \quad \big\{Q, Q\big\} = \big\{\overline{Q}, \overline{Q}\big\} = 0. \tag{2.1}$$

In order to construct the supercharges Q and $\overline{Q}$, it is necessary to introduce a certain set of fermionic fields. For the n-particle case, we have to add $2n$ fermions $\psi_i(t), \bar{\psi}_i(t)$, $(i = 1, ..., n)$, for which the standard Poisson brackets must be satisfied

$$\big\{\psi_i, \bar{\psi}_j\big\} = -\mathrm{i}\delta_{ij}. \tag{2.2}$$

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These fermions can be considered as a superpartners of components x_i and, therefore, can be combined with them into a proper $\mathcal{N}=2$ supermultiplet. However, this set of fermions is not enough to realize the superchargers $Q, \overline{Q}$ so that their anticommutator can lead to the correct bosonic limit of the potential part in Hamiltonian (1.1). Indeed, to produce the potential $\sum_{i>j}^{n} \frac{\ell_{ij}^2}{(x_i-x_j)^2}$ in (1.1) the supercharges should, in particular, contain terms, which are linear in symmetric spinor fields $\rho_{ij}(t)$, $\bar{\rho}_{ji}(t)$ (due to antisymmetry of l_{ij}), of the following type

$$Q \sim \frac{\rho_{ij}\ell_{ij}}{x_i - x_j}, \qquad \overline{Q} \sim \frac{\bar{\rho}_{ij}\ell_{ij}}{x_i - x_j}. \tag{2.3}$$

A simple conjecture that these symmetric spinors can be constructed as a sum of the introduced fermionic components as

$$\rho_{ij} = \psi_i + \psi_j, \quad \bar{\rho}_{ij} = \bar{\psi}_i + \bar{\psi}_j \tag{2.4}$$

leads to inconsistency of the possible structure of supercharges with the basic relation (2.1) of the $\mathcal{N}=2$ super Poincaré algebra. Therefore, following the arguments which was proposed in [6], the spinors $\rho_{ij}(t), \bar{\rho}_{ji}(t)$ should be treated as new fields in addition to $\psi_i(t), \bar{\psi}_i(t)$. They also satisfy the condition $\rho_{ii} = \bar{\rho}_{ii} = 0$ for each of the indices i and obey the following Poisson brackets

$$\left\{\rho_{ij}, \bar{\rho}_{km}\right\} = -\frac{\mathrm{i}}{2}\left(1-\delta_{ij}\right)\left(1-\delta_{km}\right)\left(\delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk}\right). \tag{2.5}$$

Thus, a complete number of the fermionic degrees of freedom is follows: $n(n+1)$ fermionic fields in the model: $(\psi_i, \bar{\psi}_i) = 2n, (\rho_{ij}, \bar{\rho}_{ij}) = n(n-1)$. By using of these fermions, we can construct the composite object $\Pi_{ij} = -\Pi_{ji}$

$$\Pi_{ij} = -\mathrm{i}\left[\left(\psi_i - \psi_j\right)\bar{\rho}_{ij} + \left(\bar{\psi}_i - \bar{\psi}_j\right)\rho_{ij} + \sum_{k=1}^{n}\left(\rho_{ik}\bar{\rho}_{kj} - \rho_{jk}\bar{\rho}_{ki}\right)\right]. \tag{2.6}$$

It can be checked that with respect to the brackets (2.2), (2.5) the Π_{ij} also form the $so(n)$ algebra as it was for the operators ℓ_{ij}

$$\left\{\Pi_{ij}, \Pi_{km}\right\} = \frac{1}{2}\left(\delta_{ik}\Pi_{jm} + \delta_{jm}\Pi_{ik} - \delta_{jk}\Pi_{im} - \delta_{im}\Pi_{jk}\right). \tag{2.7}$$

Taking all these arguments into account, it is a matter of straightforward calculations to check that the supercharges Q and $\overline{Q}$ given by

$$Q = \sum_{i=1}^{n} p_i\psi_i - \sum_{i\neq j}^{n}\frac{\left(\ell_{ij}+\Pi_{ij}\right)\rho_{ij}}{x_i - x_j}, \qquad \overline{Q} = \sum_{i=1}^{n} p_i\bar{\psi}_i - \sum_{i\neq j}^{n}\frac{\left(\ell_{ij}+\Pi_{ij}\right)\bar{\rho}_{ij}}{x_i - x_j} \tag{2.8}$$

together with the Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \frac{1}{2}\sum_{i\neq j}^{n}\frac{\left(\ell_{ij}+\Pi_{ij}\right)^2}{\left(x_i - x_j\right)^2} \tag{2.9}$$

form $\mathcal{N}=2$ super Poincaré algebra (2.1) and describe therefore to the $\mathcal{N}=2$ supersymmetric extension of the n-particle Euler–Calogero–Moser model.

Let us remind that the n-particle Euler–Calogero–Moser model is conformally invariant. So, we expect that the its $\mathcal{N}=2$ supersymmetric extension also possesses $\mathcal{N}=2$ superconformal symmetry. The superconformal symmetry is a dynamical one. It means that full set of generators which includes the supercharges $Q, \overline{Q}$ (2.8) , Hamiltonian H (2.9) and the following conserved currents

$$K = \frac{1}{2}\sum_{i=1}^{n} x_i^2 - t\sum_{i=1}^{n} x_i p_i + t^2 H, \quad D = -\frac{1}{2}\sum_{i=1}^{n} x_i p_i + tH, \quad U = -\frac{1}{2}\sum_{i=1}^{n}\psi_i\bar{\psi}_i - \frac{1}{2}\sum_{i\neq j}^{n}\rho_{ij}\bar{\rho}_{ij},$$
$$S = \sum_{i=1}^{n} x_i\psi_i - tQ, \quad \overline{S} = \sum_{i=1}^{n} x_i\bar{\psi}_i - t\overline{Q}, \tag{2.10}$$

form a superconformal algebra. The explicit calculation of (anti)commutators of these generators leads to the following relations

$$\begin{aligned}
&\{H,K\} = 2D, \quad \{H,D\} = H, \quad \{K,D\} = -K,\\
&\{U,Q\} = \frac{\mathrm{i}}{2}Q,\ \{U,\overline{Q}\} = -\frac{\mathrm{i}}{2}\overline{Q}, \quad \{U,S\} = \frac{\mathrm{i}}{2}S,\ \{U,\overline{S}\} = -\frac{\mathrm{i}}{2}\overline{S},\\
&\{D,Q\} = -\frac{1}{2}Q,\ \{D,\overline{Q}\} = -\frac{1}{2}\overline{Q}, \quad \{D,S\} = \frac{1}{2}S,\ \{D,\overline{S}\} = \frac{1}{2}\overline{S},\\
&\{H,S\} = -Q,\ \{H,\overline{S}\} = -\overline{Q}, \quad \{K,Q\} = S,\ \{K,\overline{Q}\} = \overline{S},\\
&\{Q,\overline{Q}\} = -2\mathrm{i}H,\ \{S,\overline{S}\} = -2\mathrm{i}K, \quad \{Q,\overline{S}\} = 2\mathrm{i}D + 2U,\ \{\overline{Q},S\} = 2\mathrm{i}D - 2U,
\end{aligned} \tag{2.11}$$

which assert that they form the $osp(2|2) \sim su(1,1|1)$ superconformal algebra.

To end this subsection, we make two comments concerning two possible modified sets of supercharges that still form, however, the $\mathcal{N}=2$ super Poincaré algebra. The first set relates to the following supercharges $\widetilde{Q}, \widetilde{\overline{Q}}$

$$\widetilde{Q} = Q + \mathrm{i}m\sum_{i\neq j}^{n}\frac{\psi_i - \psi_j}{x_i - x_j}, \quad \widetilde{\overline{Q}} = \overline{Q} - \mathrm{i}m\sum_{i\neq j}^{n}\frac{\bar{\psi}_i - \bar{\psi}_j}{x_i - x_j}, \quad m = \text{const}, \tag{2.12}$$

where $Q, \overline{Q}$ are given in (2.8). These supercharges together with the Hamiltonian

$$\widetilde{H} = H + \frac{m^2}{2}\sum_{j\neq i}^{n}\frac{1}{(x_i - x_j)^2} + \frac{m}{2}\sum_{j\neq i}\frac{(\psi_i - \psi_j)\left(\bar{\psi}_i - \bar{\psi}_j\right)}{(x_i - x_j)^2} - m\sum_{k\neq j\neq i}^{n}\frac{\rho_{ij}\bar{\rho}_{ij}}{x_i - x_j}\left(\frac{1}{x_i - x_k} - \frac{1}{x_j - x_k}\right) \tag{2.13}$$

form $\mathcal{N}=2$ super Poincaré algebra (2.1). Thus, the supercharges (2.12) and the Hamiltonian (2.13) provide a new $\mathcal{N}=2$ supersymmetric extension of the rational Calogero model with a modified *Calogero-like potential*.

The second case corresponds to the supercharges in the structure of which the symmetric spinor fields $\rho_{ij}, \bar{\rho}_{ij}$ are replaced by the antisymmetric spinor fields $\eta_{ij} = -\eta_{ji}, \bar{\eta}_{ij} = -\bar{\eta}_{ji}$ and the x dependent terms are represented as functions of a sum $x_i + x_j$. Having made such assumptions, it is possible to write down supecharges in the following form

$$\begin{aligned}
\hat{Q} &= \sum_{i=1}^{n} p_i\psi_i - \sum_{i\neq j}^{n}\frac{\ell_{ij}\,\eta_{ij}}{x_i + x_j} - \mathrm{i}\sum_{i\neq j}^{n}\frac{\psi_i + \psi_j}{x_i + x_j}\eta_{ij}\bar{\eta}_{ij} + \mathrm{i}\sum_{i\neq j\neq k}^{n}\frac{x_i + x_j}{(x_i + x_k)(x_j + x_k)}\eta_{ik}\eta_{jk}\bar{\eta}_{ij},\\
\hat{\overline{Q}} &= \sum_{i=1}^{n} p_i\bar{\psi}_i - \sum_{i\neq j}^{n}\frac{\ell_{ij}\,\bar{\eta}_{ij}}{x_i + x_j} + \mathrm{i}\sum_{i\neq j}^{n}\frac{\bar{\psi}_i + \bar{\psi}_j}{x_i + x_j}\eta_{ij}\bar{\eta}_{ij} - \mathrm{i}\sum_{i\neq j\neq k}^{n}\frac{x_i + x_j}{(x_i + x_k)(x_j + x_k)}\eta_{ij}\bar{\eta}_{ik}\bar{\eta}_{jk}.
\end{aligned} \tag{2.14}$$

The bosonic potential has an expected form

$$\hat{V} = \sum_{i \neq j} \frac{\ell_{ij}\ell_{ij}}{(x_i + x_j)^2}, \tag{2.15}$$

while the full Hamiltonian $\hat{H}$ has no such transparent structure as H (2.9).

2.2. Model with even number of supersymmetries

The $\mathcal{N} = 2$ supersymmetric n-particle Euler–Calogero–Moser model admits a generalization to those which are invariant under the supersymmetry with arbitrary even number of supercharges. This generalization provides by supercharges which have extra $su(M)$ indices a, b and form the $\mathcal{N} = 2M$ super Poincaré algebra

$$\left\{Q^a, \overline{Q}_b\right\} = -2\mathrm{i}\delta^a_b H, \quad \left\{Q^a, Q^b\right\} = \left\{\overline{Q}_a, \overline{Q}_b\right\} = 0, \quad a, b = 1, ..., M. \tag{2.16}$$

Indeed, we can consider the following set of $\frac{1}{2}\mathcal{N}n(n+1)$ fermions $\psi^a_i, \bar{\psi}_{i\,a}$ and $\rho^a_{ij}, \bar{\rho}_{ij\,a}$ that satisfy the Poisson brackets

$$\left\{\psi^a_i, \bar{\psi}_{b\,j}\right\} = -\mathrm{i}\delta^a_b \delta_{ij}, \quad \left\{\rho^a_{ij}, \bar{\rho}_{km\,b}\right\} = -\frac{\mathrm{i}}{2}\delta^a_b\big(1-\delta_{ij}\big)\big(1-\delta_{km}\big)\big(\delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk}\big). \tag{2.17}$$

Then, by analogy with the $\mathcal{N} = 2$ supersymmetric case, it is possible to construct a composite object $\Pi_{ij} = -\Pi_{ji}$

$$\Pi_{ij} = -\mathrm{i}\sum_{a=1}^{\mathcal{N}} \Big[\big(\psi^a_i - \psi^a_j\big)\bar{\rho}_{ij\,a} + \big(\bar{\psi}_{i\,a} - \bar{\psi}_{j\,a}\big)\rho^a_{ij} + \sum_{k=1}^{n}\big(\rho^a_{ik}\bar{\rho}_{kj\,a} - \rho^a_{jk}\bar{\rho}_{ki\,a}\big)\Big], \tag{2.18}$$

that satisfies, as before, the commutation relations of the $so(n)$ algebra (2.7). Using (2.18), the supercharges $Q^a, \overline{Q}_a$, which correspond to the extended $\mathcal{N} = 2M$ supersymmetry, can be written as follows

$$Q^a = \sum_{i=1}^{n} p_i\psi^a_i - \sum_{i\neq j}^{n} \frac{(\ell_{ij} + \Pi_{ij})\,\rho^a_{ij}}{x_i - x_j}, \quad \overline{Q}_a = \sum_{i=1}^{n} p_i\bar{\psi}_{i\,a} - \sum_{i\neq j}^{n} \frac{(\ell_{ij} + \Pi_{ij})\,\bar{\rho}_{ij\,a}}{x_i - x_j}. \tag{2.19}$$

They form together with the Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \frac{1}{2}\sum_{i\neq j}^{n} \frac{(\ell_{ij} + \Pi_{ij})^2}{(x_i - x_j)^2} \tag{2.20}$$

$\mathcal{N} = 2M$ super Poincaré algebra (2.16) and describe the $\mathcal{N} = 2M$ supersymmetric extension of the n-particle Euler–Calogero–Moser model.

It is rather easy to check that the supercharges $Q^a, \overline{Q}_a$ (2.19), Hamiltonian H (2.20) and the following conserved currents

$$\begin{aligned}
&K = \frac{1}{2}\sum_{i=1}^{n} x_i^2 - t\sum_{i=1}^{n} x_i p_i + t^2 H, \quad D = -\frac{1}{2}\sum_{i=1}^{n} x_i p_i + tH, \quad J^a{}_b = -\sum_{i=1}^{n}\psi^a_i\bar{\psi}_{i\,b} - \sum_{i\neq j}^{n}\rho^a_{ij}\bar{\rho}_{ij\,b}, \\
&I^{ab} = -\sum_{i=1}^{n}\psi^a_i\psi^b_i - \sum_{i\neq j}^{n}\rho^a_{ij}\rho^b_{ij}, \quad \overline{I}_{ab} = \sum_{i=1}^{n}\bar{\psi}_{i\,a}\bar{\psi}_{i\,b} + \sum_{i\neq j}^{n}\bar{\rho}_{ij\,a}\bar{\rho}_{ij\,b}, \\
&S^a = \sum_{i=1}^{n} x_i\psi^a_i - tQ^a, \quad \overline{S}_a = \sum_{i=1}^{n} x_i\bar{\psi}_{i\,a} - t\overline{Q}_a,
\end{aligned} \tag{2.21}$$

form the superalgebra $osp(\mathcal{N}|2)$ [1]. Note that the generators $J^a{}_b$ form $u(M)$ subalgebra, while together with the generators I^{ab} and $\overline{I}_{ab}$ they form $so(2M)$ subalgebra of $osp(\mathcal{N}|2)$ superalgebra.

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3. $\mathcal{N}=2$ supersymmetric Euler–Calogero–Moser model in superspace

The $\mathcal{N}$-extended supersymmetric ECM model, that has been constructed within the Hamiltonian methods, admits a nice superfield description for the simplest case with $\mathcal{N}=2$ supersymmetry. An importance of the $\mathcal{N}=2$ superfield approach is that it can make more clear the meaning for introducing the new fermionic fields of the ρ-type and the role played by the additional currents ℓ_{ij}. To obtain the $\mathcal{N}=2$ supersymmetric ECM model in superspace, defined by the supercharges $Q,\overline{Q}$ (2.8) and the Hamiltonian (2.9), one needs to solve two tasks:

- assemble the physical components $x_i, \psi_i, \bar\psi_i, \rho_{ij}$ and $\bar\rho_{ij}$ into appropriate $\mathcal{N}=2$ superfields
- introduce auxiliary bosonic superfields $v_i, \bar v_i$ whose leading components realize ℓ_{ij} via bilinear combinations.

Let us start with the first task. From the structure of the supercharges $Q,\overline{Q}$ (2.8) it is clear that under the $\mathcal{N}=2$ supersymmetry transformations, defined as

$$\delta_{susy} z(t) = \mathrm{i}\big\{ z(t), \bar\varepsilon Q + \varepsilon \overline{Q}\big\}, \tag{3.1}$$

follows that the coordinates x_i transform through fermions $\psi_i, \bar\psi_i$. So, one have to introduce n bosonic $\mathcal{N}=2$ superfields $\boldsymbol{x}_i$ with the following components,

$$x_i = \boldsymbol{x}_i|, \quad \psi_i = -\mathrm{i}D\boldsymbol{x}_i|, \quad \bar\psi_i = -\mathrm{i}\overline{D}\boldsymbol{x}_i|, \quad A_i = \frac{1}{2}\left[\overline{D}, D\right]\boldsymbol{x}_i|, \tag{3.2}$$

where $|$ denotes the $\theta=\bar\theta=0$ projection. As usual, D and $\overline{D}$ are $\mathcal{N}=2$ covariant derivatives whose anicommutators are given by

$$\big\{D,\overline{D}\big\} = 2\mathrm{i}\partial_t \qquad \text{and} \qquad \big\{D,D\big\} = \big\{\overline{D},\overline{D}\big\} = 0. \tag{3.3}$$

The fermions $\rho_{ij},\ \bar\rho_{ij}$ can be embedded as the first components into $n(n-1)$ fermionic superfields $\boldsymbol{\rho}_{ij},\ \bar{\boldsymbol{\rho}}_{ij}$, symmetric and of zero diagonal in the indices i,j, i.e.

$$\boldsymbol{\rho}_{ij} = \boldsymbol{\rho}_{ji}, \quad \bar{\boldsymbol{\rho}}_{ij} = \bar{\boldsymbol{\rho}}_{ji}, \qquad \boldsymbol{\rho}_{ii} = \bar{\boldsymbol{\rho}}_{ii} = 0 \quad (\text{no sum})\,. \tag{3.4}$$

As $\mathcal{N}=2$ superfields the $\boldsymbol{\rho}_{ij}$ and $\bar{\boldsymbol{\rho}}_{ij}$ contain a lot of components. However, their leading components ρ_{ij} and $\bar\rho_{ij}$ transform under the $\mathcal{N}=2$ supersymmetry generated by Q and $\overline{Q}$ (2.8) as follows,

$$\begin{aligned}
\delta_Q \rho_{ij} &\sim \mathrm{i}\bar\epsilon\left[\frac{\psi_i-\psi_j}{x_i-x_j}\rho_{ij} - \sum_{k\neq i,j}^{n}\frac{x_i-x_j}{(x_i-x_k)(x_j-x_k)}\rho_{ik}\rho_{jk}\right],\\
\delta_{\overline{Q}} \bar\rho_{ij} &\sim \mathrm{i}\epsilon\left[\frac{\bar\psi_i-\bar\psi_j}{x_i-x_j}\bar\rho_{ij} - \sum_{k\neq i,j}^{n}\frac{x_i-x_j}{(x_i-x_k)(x_j-x_k)}\bar\rho_{ik}\bar\rho_{jk}\right].
\end{aligned} \tag{3.5}$$

To realize these transformations in superspace we are forced to impose the following nonlinear chirality conditions,

$$\begin{aligned}
D\boldsymbol{\rho}_{ij} &= \mathrm{i}\left[\frac{\boldsymbol{\psi}_i-\boldsymbol{\psi}_j}{\boldsymbol{x}_i-\boldsymbol{x}_j}\boldsymbol{\rho}_{ij} - \sum_{k\neq i,j}^{n}\frac{\boldsymbol{x}_i-\boldsymbol{x}_j}{(\boldsymbol{x}_i-\boldsymbol{x}_k)(\boldsymbol{x}_j-\boldsymbol{x}_k)}\boldsymbol{\rho}_{ik}\boldsymbol{\rho}_{jk}\right],\\
\overline{D}\bar{\boldsymbol{\rho}}_{ij} &= \mathrm{i}\left[\frac{\bar{\boldsymbol{\psi}}_i-\bar{\boldsymbol{\psi}}_j}{\boldsymbol{x}_i-\boldsymbol{x}_j}\boldsymbol{\rho}_{ij} - \sum_{k\neq i,j}^{n}\frac{\boldsymbol{x}_i-\boldsymbol{x}_j}{(\boldsymbol{x}_i-\boldsymbol{x}_k)(\boldsymbol{x}_j-\boldsymbol{x}_k)}\bar{\boldsymbol{\rho}}_{ik}\bar{\boldsymbol{\rho}}_{jk}\right].
\end{aligned} \tag{3.6}$$

These conditions are self-consistent and leave in the superfields $\boldsymbol{\rho}_{ij}$ and $\bar{\boldsymbol{\rho}}_{ij}$ only the components

$$\rho_{ij} = \boldsymbol{\rho}_{ij}|, \quad B_{ij} = \overline{D}\boldsymbol{\rho}_{ij}|, \qquad \bar{\rho}_{ij} = \bar{\boldsymbol{\rho}}_{ij}|, \quad \overline{B}_{ij} = D\bar{\boldsymbol{\rho}}_{ij}| \,. \tag{3.7}$$

To get the correct Poisson brackets for $\psi_i, \bar{\psi}_i$ and $\rho_{ij}, \bar{\rho}_{ij}$ (2.2) after passing to the Hamiltonian formalism, the kinetic terms for these fermionic components must read

$$\mathcal{L}^{\psi}_{kin} = \frac{\mathrm{i}}{2}\sum_{i=1}^{n}\left(\dot{\psi}_i\bar{\psi}_i - \psi_i\dot{\bar{\psi}}_i\right) \qquad \text{and} \qquad \mathcal{L}^{\rho}_{kin} = \frac{\mathrm{i}}{2}\sum_{i,j}^{n}\left(\dot{\rho}_{ij}\bar{\rho}_{ij} - \rho_{ij}\dot{\bar{\rho}}_{ij}\right). \tag{3.8}$$

Altogether, we arrive at the following superfield action for the purely $\mathcal{N}=2$ supersymmetric system with $l_{ij}=0$,

$$S_0 = \int \mathrm{d}t\, \mathrm{d}^2\,\theta\Big[-\frac{1}{2}\sum_{i=1}^{n} D\boldsymbol{x}_i\,\overline{D}\boldsymbol{x}_i + \frac{1}{2}\sum_{i,j}^{n}\boldsymbol{\rho}_{ij}\bar{\boldsymbol{\rho}}_{ij}\Big], \qquad d^2\theta \equiv D\overline{D}. \tag{3.9}$$

To resolve the second task, one has to realize the ℓ_{ij} in terms of auxiliary semi-dynamical variables. As $so(n)$ generators the ℓ_{ij} possess the standard realization

$$\hat{\ell}_{ij} = \frac{\mathrm{i}}{2}\left(v_i\bar{v}_j - v_j\bar{v}_i\right) \tag{3.10}$$

in terms of $2n$ bosonic variables $v_i, \bar{v}_i$ subject to

$$\left\{v_i, \bar{v}_j\right\} = -\mathrm{i}\delta_{ij}. \tag{3.11}$$

To implement these new semi-dynamical variables $v_i, \bar{v}_i$ at the superfield level, we have to introduce $2n$ bosonic superfields $\boldsymbol{v}_i, \bar{\boldsymbol{v}}_i$. Additional information about these superfields again comes from the transformation of their first components under $\mathcal{N}=2$ supersymmetry. These transformations can be learned from the explicit structure of the supercharges Q, $\overline{Q}$ (2.8), with the ℓ_{ij} being replaced by their realization $\hat{\ell}_{ij}$ (3.10):

$$\delta_Q v_i \sim \mathrm{i}\,\bar{\epsilon}\sum_{j\neq i}^{n}\frac{\rho_{ij}v_j}{x_i - x_j} \qquad \text{and} \qquad \delta_{\overline{Q}}\bar{v}_i \sim \mathrm{i}\,\epsilon\sum_{j\neq i}^{n}\frac{\bar{\rho}_{ij}\bar{v}_j}{x_i - x_j}. \tag{3.12}$$

This form of transformations implies that, similarly to $\boldsymbol{\rho}_{ij}$ and $\bar{\boldsymbol{\rho}}_{ij}$, the superfields $\boldsymbol{v}_i$ and $\bar{\boldsymbol{v}}_i$ are subjected to the nonlinear chirality conditions,

$$D\boldsymbol{v}_i = \mathrm{i}\sum_{j\neq i}^{n}\frac{\boldsymbol{\rho}_{ij}\boldsymbol{v}_j}{\boldsymbol{x}_i - \boldsymbol{x}_j} \qquad \text{and} \qquad \overline{D}\bar{\boldsymbol{v}}_i = \mathrm{i}\sum_{j\neq i}^{n}\frac{\bar{\boldsymbol{\rho}}_{ij}\bar{\boldsymbol{v}}_j}{\boldsymbol{x}_i - \boldsymbol{x}_j}. \tag{3.13}$$

Due to these constraints, there are the following independent components in superfields $\boldsymbol{v}_i$ and $\bar{\boldsymbol{v}}_i$

$$v_i = \boldsymbol{v}_i|, \quad C_i = -\mathrm{i}\overline{D}\boldsymbol{v}_i|, \qquad \bar{v}_i = \bar{\boldsymbol{v}}_i|, \quad \overline{C}_i = -\mathrm{i}D\bar{\boldsymbol{v}}_i|. \tag{3.14}$$

Finally, to have the brackets (3.11), the kinetic terms for $v_i, \bar{v}_i$ must take the form

$$\mathcal{L}^{v}_{kin} = -\frac{\mathrm{i}}{2}\sum_{i=1}^{n}\left(\dot{v}_i\bar{v}_i - v_i\dot{\bar{v}}_i\right). \tag{3.15}$$

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Therefore, the interaction part ($l_{ij} \neq 0$) of the superfield action reads

$$S_1 = -\frac{1}{2} \int \mathrm{d}t \; \mathrm{d}^2\theta \; \sum_{i=1}^{n} \boldsymbol{v}_i \bar{\boldsymbol{v}}_i. \tag{3.16}$$

Combining everything together, we conclude that the superfield action should have the form

$$S = S_0 + S_1 = \int \mathrm{d}t \; \mathrm{d}^2\theta \left[-\frac{1}{2} \sum_{i=1}^{n} D\boldsymbol{x}_i \, \overline{D}\boldsymbol{x}_i + \frac{1}{2} \sum_{i,j}^{n} \boldsymbol{\rho}_{ij} \bar{\boldsymbol{\rho}}_{ij} - \frac{1}{2} \sum_{i=1}^{n} \boldsymbol{v}_i \bar{\boldsymbol{v}}_i \right], \tag{3.17}$$

where the superfields $\boldsymbol{\rho}_{ij}, \bar{\boldsymbol{\rho}}_{ij}, \boldsymbol{v}_i$ and $\bar{\boldsymbol{v}}_i$ are subject to the constraints (3.6) and (3.13), respectively.

Despite the extremely simple form of the superfield action (3.17), its *off-shell* component version looks quite complicated due to the nonlinear chirality constraints (3.6) and (3.13). We omit here the detailed calculation and give the final result for the on-shell component Lagrangian. Its expression follows after integration over the Grassmann variables and exclusion the auxiliary components A_i, B_{ij}, $\overline{B}_{ij}$, C_i and $\overline{C}_i$ by their equations of motion in (3.17)

$$\mathcal{L} = \frac{1}{2} \sum_{i=1}^{n} \dot{x}_i \dot{x}_i + \frac{\mathrm{i}}{2} \sum_{i=1}^{n} \left(\dot{\psi}_i \bar{\psi}_i - \psi_i \dot{\bar{\psi}}_i \right) + \frac{\mathrm{i}}{2} \sum_{i,j}^{n} \left(\dot{\rho}_{ij} \bar{\rho}_{ij} - \rho_{ij} \dot{\bar{\rho}}_{ij} \right) - \frac{\mathrm{i}}{2} \sum_{i=1}^{n} \left(\dot{v}_i \bar{v}_i - v_i \dot{\bar{v}}_i \right) - \sum_{i \neq j}^{n} \frac{(\hat{\ell}_{ij} + \Pi_{ij})^2}{2\,(x_i - x_j)^2}. \tag{3.18}$$

In (3.18) Π_{ij} is still defined as in (2.6) and $\hat{\ell}_{ij}$ is expressed in terms of semi-dynamical variables as in (3.10). Thus, the superfield action (3.17), with the superfields $\boldsymbol{\rho}_{ij}, \bar{\boldsymbol{\rho}}_{ij}, \boldsymbol{v}_i$ and $\bar{\boldsymbol{v}}_i$ being nonlinearly constrained by (3.6) and (3.13), indeed describes the $\mathcal{N} = 2$ supersymmetric Euler–Calogero–Moser model.

To conclude, let us make a few comments:

- The nonlinear chirality conditions (3.6) can be slightly simplified by passing to the superfields $\boldsymbol{\xi}_{ij}, \bar{\boldsymbol{\xi}}_{ij}$:

  $$\boldsymbol{\xi}_{ij} \equiv \frac{\boldsymbol{\rho}_{ij}}{\boldsymbol{x}_i - \boldsymbol{x}_j}, \quad \bar{\boldsymbol{\xi}}_{ij} \equiv \frac{\bar{\boldsymbol{\rho}}_{ij}}{\boldsymbol{x}_i - \boldsymbol{x}_j} \quad \Rightarrow \quad D\boldsymbol{\xi}_{ij} + \mathrm{i} \sum_{k=1}^{n} \boldsymbol{\xi}_{ik} \boldsymbol{\xi}_{jk} = 0, \quad \overline{D}\bar{\boldsymbol{\xi}}_{ij} + \mathrm{i} \sum_{k=1}^{n} \bar{\boldsymbol{\xi}}_{ik} \bar{\boldsymbol{\xi}}_{jk} = 0.$$

  However, the Lagrangian, Hamiltonian and the Poisson brackets will look more complicated, being written in terms of $\boldsymbol{\xi}_{ij}$ and $\bar{\boldsymbol{\xi}}_{ij}$ despite the fact that these superfields now are defined independently of the superfields $\boldsymbol{x}_i$.
- It turns out that the auxiliary superfields $\boldsymbol{v}_i, \bar{\boldsymbol{v}}_i$ cannot be re-defined in a similar manner. Thus, the nonlinear chirality constraints (3.13) which relate these superfields with the $\boldsymbol{x}_i$ ones are crucial for the superfields description.
- It should be noted that the semi-dynamical variables $v_i, \bar{v}_i$ obeying the brackets (3.11) can be used for the construction of $su(n)$ generators. Clearly, the kinetic Lagrangian $\mathcal{L}^v_{kin}$ (3.15) possesses $su(n)$ symmetry. However, this $su(n)$ symmetry is reduced to the $so(n)$ one upon using the nonlinear chirality constraints (3.13).
- In $\mathcal{N} = 2$ superspace a system defined by supercharges (2.14) is described by superfields $\boldsymbol{\eta}_{ij}, \bar{\boldsymbol{\eta}}_{ij}, \boldsymbol{v}_i$ and $\bar{\boldsymbol{v}}_i$ subjected to modified nonlinear chirality constraints

$$D\boldsymbol{\eta}_{ij} = \mathrm{i} \left[\frac{\boldsymbol{\psi}_i + \boldsymbol{\psi}_j}{\boldsymbol{x}_i + \boldsymbol{x}_j} \boldsymbol{\eta}_{ij} - \sum_{k \neq i,j}^{n} \frac{\boldsymbol{x}_i + \boldsymbol{x}_j}{(\boldsymbol{x}_i + \boldsymbol{x}_k)\,(\boldsymbol{x}_j + \boldsymbol{x}_k)} \boldsymbol{\eta}_{ik} \boldsymbol{\eta}_{jk} \right],$$
$$\overline{D}\bar{\boldsymbol{\eta}}_{ij} = \mathrm{i} \left[\frac{\bar{\boldsymbol{\psi}}_i + \bar{\boldsymbol{\psi}}_j}{\boldsymbol{x}_i + \boldsymbol{x}_j} \boldsymbol{\eta}_{ij} - \sum_{k \neq i,j}^{n} \frac{\boldsymbol{x}_i + \boldsymbol{x}_j}{(\boldsymbol{x}_i + \boldsymbol{x}_k)\,(\boldsymbol{x}_j + \boldsymbol{x}_k)} \bar{\boldsymbol{\eta}}_{ik} \bar{\boldsymbol{\eta}}_{jk} \right], \tag{3.19}$$

and

$$D\boldsymbol{v}_i = \mathrm{i}\sum_{j\neq i}^{n}\frac{\boldsymbol{\eta}_{ij}\boldsymbol{v}_j}{\boldsymbol{x}_i+\boldsymbol{x}_j}, \qquad \overline{D}\bar{\boldsymbol{v}}_i = \mathrm{i}\sum_{j\neq i}^{n}\frac{\bar{\boldsymbol{\eta}}_{ij}\bar{\boldsymbol{v}}_j}{\boldsymbol{x}_i+\boldsymbol{x}_j}. \tag{3.20}$$

However, the superfield action for the system still has the same form as in (3.17)

$$S = \int dt d^2\theta \Big[-\frac{1}{2}\sum_{i=1}^{n} D\boldsymbol{x}_i\,\overline{D}\boldsymbol{x}_i + \frac{1}{2}\sum_{i,j}^{n}\boldsymbol{\eta}_{ij}\bar{\boldsymbol{\eta}}_{ij} - \frac{1}{2}\sum_{i=1}^{n}\boldsymbol{v}_i\bar{\boldsymbol{v}}_i \Big], \tag{3.21}$$

This system is needed to be further analyzed.

4. Conclusion

We reviewed a new $\mathcal{N}$-extended supersymmetric $so(n)$ spin-Calogero model by a direct supersymmetrization of the bosonic Euler–Calogero–Moser system [8]. The crucial feature of a given construction, besides the standard $\mathcal{N}\, n$ fermions ψ_i^a and $\bar{\psi}_{i\,a}$ accompanying the bosonic fields x_i, is the presence of an additional set of fermionic degrees of freedom, namely, $\frac{1}{2}\mathcal{N}\times n(n-1)$ symmetric fermions $\rho_{ij}^a = \rho_{ji}^a$, which originate from the off-diagonal part of the symmetric supermatrices.

We obtained the supercharges Q^a and $\overline{Q}_a$ and the Hamiltonian which form an $\mathcal{N}$-extended super Poincaré algebra. As it was shown, the supercharges have the standard structure, cubic in the fermions involved. We realized in term of all coordinates the generators of a dynamical $osp(\mathcal{N}|2)$ superconformal algebra as the conserved currents and demonstrated the invariance of ECM model with respect to this supergroup.

In the simplest case of $\mathcal{N} = 2$ supersymmetric extension of the ECM model, we provided its description in terms of $\mathcal{N} = 2$ superfields. The peculiarity of this construction is reflected in following facts:

- the coordinates x_i and fermions $\psi_i, \bar{\psi}_j$ forming standard unconstrained bosonic superfields,
- fermionic symmetric matrices $\rho_{ij}, \bar{\rho}_{ij}$ (with vanishing diagonal), subject to the nonlinear chirality constraints,
- $2n$ bosonic $\mathcal{N} = 2$ semi-dynamical superfields $v_i, \bar{v}_i$ also obeying the nonlinear chirality constraints.

It is shown that the $\mathcal{N} = 2$ superspace action is written as a sum of the standard kinetic terms for all superfields. At the component level, the off-shell action has rather complicated structure due to the nonlinear constraints. However, after eliminating the auxiliary components via their equations of motion, the action acquires quite a simple form again, with an interaction quadratic and quartic in the fermions.

However, the presented $\mathcal{N} = 2$ supersymmetric case is not too instructive, since it can also be constructed without additional fermions $\rho_{ij}$and $\bar{\rho}_{ij}$, in analogy with the $\mathcal{N} = 2$ supersymmetric Calogero model [13, 14] if the terms quadratic in ρ_{ij} and $\bar{\rho}_{ij}$ in the nonlinear chirality constraints (3.6) will be discarded. Thus, the generic superfield structure of the $\mathcal{N}$-extended ECM model becomes visible at $\mathcal{N} = 4$ only. We are planning to address this elsewhere.

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