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Supersymmetric AdS_7 and AdS_6 vacua and their consistent truncations with vector multiplets

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Abstract: Using exceptional field theory we construct supersymmetric warped AdS_7 vacua of massive IIA and AdS_6 vacua of IIB, as well as their consistent truncations including vector multiplets. We show that there are no consistent truncations of massive IIA supergravity around its supersymmetric AdS_7 vacua with vector multiplets when the Romans mass is non-vanishing. For AdS_6 vacua of IIB supergravity, we find that in addition to the consistent truncation to pure F(4) gauged SUGRA, the only other half-maximal truncations that are consistent result in F(4) gauged SUGRA coupled to one or two Abelian vector multiplets, to three non-Abelian vector multiplets, leading to an ISO(3) gauged SUGRA, or to three non-Abelian plus one Abelian vector multiplet, leading to an $\mathrm{ISO}(3) \times \mathrm{U}(1)$ gauged SUGRA. These consistent truncations with vector multiplets exist when the two holomorphic functions that define the AdS_6 vacua satisfy certain differential conditions which we derive. We use these to deduce that no globally regular AdS_6 solutions admit a consistent truncation to F(4) gauged SUGRA with two vector multiplets, and show that the Abelian T-dual of the Brandhuber-Oz vacuum allows a consistent truncation to F(4) gauged SUGRA with a single vector multiplet.

Keywords: AdS-CFT Correspondence, Flux compactifications, Supergravity Models, Superstring Vacua

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1 Introduction

Supersymmetric AdS vacua of 10-/11-dimensional SUGRA play an important role in our modern understanding of theoretical physics. For example, they have led to many important insights into superconformal field theories via the AdS/CFT correspondence. For many holographic applications, it is useful to have a consistent truncation of 10-/11-dimensional SUGRA around a supersymmetric AdS vacuum. Such a consistent truncation allows us to uplift solutions of a lower-dimensional (usually gauged) SUGRA to solutions of 10-/11-dimensional SUGRA. This makes them a powerful tool in studying deformations of the AdS vacua, for example those breaking supersymmetry. Moreover, since the AdS radius of supersymmetric AdS vacua is typically of the same scale as the compactification radius, lower-dimensional SUGRA theories do not arise by integrating out the Kaluza-Klein tower of the compactification. Thus, consistent truncations are the only way to study AdS vacua via lower-dimensional supergravities.

However, constructing consistent truncations is a notoriously difficult task which has until recently largely eluded a systematic approach. For some purposes, it may even be enough to know that a consistent truncation of 10-/11-dimensional SUGRA exists, even without having the explicit truncation Ansätze. Yet, to date there is no classification of what consistent truncations exist around a given supersymmetric AdS vacuum, although it is conjectured that for every warped supersymmetric AdS$_D$ vacuum of 10-/11-dimensional SUGRA, there exists a "minimal" consistent truncation to D-dimensional gauged SUGRA keeping only the gravitational supermultiplet [1], which has been proven in some cases.

Powerful tools in constructing consistent truncations have recently come from exceptional field theory (ExFT) [2–5] and exceptional generalised geometry (EGG) [6–8], which reformulate 10-/11-dimensional SUGRA in a way which unifies the metric and flux degrees of freedom. In this framework, consistent truncations preserving all supersymmetries arise as "generalised Scherk-Schwarz" truncations [9–12], generalising consistent truncations on group manifolds [13] to the more general setting of "generalised (Leibniz) parallelisable spaces" [14], which includes certain homogeneous spaces. This has led to a proof of the consistency of the maximally supersymmetric S^5 truncation of IIB supergravity [14–16], and to new consistent truncations giving rise to compact and dyonic gaugings [17–23]. Moreover, all currently known maximally supersymmetric consistent truncations, including the truncations of 11-dimensional SUGRA on S^4 and S^7 [14, 15], first found in [24–26], and the truncation of massive IIA on S^6 [19, 20], first constructed in [27, 28], are nicely captured by the framework of generalised Scherk-Schwarz truncations.

Recently, [29, 30] has shown how use this framework to define consistent truncations breaking half of the supersymmetry. Such half-maximal truncations of type II/11-dimensional SUGRA then lead to a half-maximal gauged SUGRA in lower dimensions. Furthermore, [30] proved the half-maximal case of the conjecture of [1], i.e. that every half-maximally supersymmetric warped AdS_D vacuum of 10-/11-dimensional SUGRA admits a consistent truncation to half-maximal D-dimensional gauged SUGRA keeping only the gravitational supermultiplet.

Moreover, ExFT and EGG lead to a new geometric description of supersymmetric AdS vacua of 10-/11-dimensional SUGRA where the compactification manifold is characterised by "generalised holonomy", or a (weakly) integrable generalised G-structure, [30–35] in analogy to supersymmetric Minkowski vacua without fluxes arising from special holonomy compactifications [36]. Moreover, as showed in [30], once the generalised G-structure underyling the supersymmetric AdS vacuum is constructed, the "minimal" consistent truncation can be obtained immediately. Therefore, this framework is ideally suited to studying supersymmetric AdS vacua and their consistent truncations, which we will undertake in this paper.

In this work, we will focus on supersymmetric AdS_7 solutions of massive IIA SUGRA and supersymmetric AdS_6 solutions of IIB. Building on previous work [37–39], families of infinitely many such vacua have recently been constructed in the literature [40–44], where the AdS_7 solutions are characterised by a cubic function on an interval [45] and the AdS_6 solutions by two holomorphic functions on a Riemann surface.[1] These AdS vacua admit a "universal" consistent truncation to pure 7-dimensional SU(2) gauged SUGRA [47] and 6-dimensional F(4) gauged SUGRA [48], which takes the same form for any of the cubic functions / holomorphic functions defining the AdS vacua [49–51]. In a recent paper [51], we showed that in ExFT these infinite families of AdS solutions are described by the same universal generalised half-maximal structure and used this to explain the universal form of the AdS_7 consistent truncations and derive the consistent truncation around the AdS_6 vacua.

It has remained an interesting open problem to find any consistent truncations around the $\mathrm{AdS}_{6,7}$ vacua keeping more modes than just the gravitional supermultiplet. Supersymmetry implies that any extra modes kept will have to form vector multiplets of the 6- and

[1] An alternative characterisation of the AdS_6 vacua in terms of a real harmonic function is given in [46].

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7-dimensional gauged SUGRA obtained after truncation. Here we will use the framework of ExFT, and specifically the tools developed in [29, 30], to address this problem: we will classify all possible consistent truncations with vector multiplets around the supersymmetric $AdS_{6,7}$ vacua that are compatible with the Ansatz proposed in [29, 30]. Assuming the Ansatz of [29, 30] to be the most general Ansatz for consistent truncations with vector multiplets, our results give a full classification of the consistent truncations around supersymmetric $AdS_{6,7}$ vacua. We find that

- there are no consistent truncations with vector multiplets around the supersymmetric AdS_7 vacua of massive IIA SUGRA when the Romans mass is non-vanishing,
- supersymmetric AdS_6 vacua of IIB SUGRA admit consistent truncations with vector multiplets when the holomorphic functions characterising them admit certain differential conditions which we give explicitly. We construct the non-linear consistent truncation Ansätze that give rise to less than four vector multiplets.

Our paper is organised as follows. First, we give a summary of our results in 1.1. In section 2, we give a brief introduction to the relevant aspects of ExFT, while in section 3, we review the techniques developed in [29, 30] to describe supersymmetric AdS vacua of 10-/11-dimensional SUGRA and their minimal consistent truncations, in which only the gravitational supermultiplet is kept. In section 4, we review how to define consistent truncations with matter multiplets as described in [29, 30]. Next, we show how to compute the generalised metric from the half-maximal structure underlying the AdS vacua in section 5. In sections 6 and 7, we show how one can easily construct the supersymmetric AdS_7 vacua of massive IIA SUGRA and AdS_6 vacua of IIB SUGRA, respectively, using half-maximal structures of ExFT, before deriving their minimal consistent truncations in 8. Finally, in section 9 we show that there are no consistent truncation with vector multiplets around the supersymmetric AdS_7 vacua of massive IIA and in section 10 we classify all possible consistent truncations with vector multiplets around the supersymmetric AdS_6 vacua of IIB SUGRA. These consistent truncations require the holomorphic functions characterising the AdS_6 vacua to satisfy certain differential constraints which we derive. We also explicitly construct the non-linear consistent truncation Ansätze yielding less than four vector multiplets. We conclude with a discussion and outlook in section 11.

1.1 Summary of results

We summarise here our results. In sections 6 and 7 we construct and classify all supersymmetric AdS_7 vacua in mIIA theory and AdS_6 vacua in IIB, respectively. As we review in section 3.1, for each of them one can construct a consistent truncation to a minimal half-supersymmetric gauged supergravity with a gravitational supermultiplet, which we explicitly construct in section 8. Finally, in sections 9 and 10 we analyse the possibility of having consistent truncations with matter multiplets around these vacua, using the methods of [29, 30]. From the latter, it follows that we can have at most three (four) vector multiplets in consistent truncations around supersymmetric AdS_7 (AdS_6) vacua and, for the truncation to be consistent, the compactification space has to satisfy certain conditions. These extra conditions imply that there are no consistent truncations with vector multiplets around AdS_7 vacua for non-vanishing Romans mass. For the AdS_6 case, we find that only a small subset of 6-dimensional half-maximal gauged SUGRAs admitting supersymmetric AdS_6 vacua [52] can arise as a consistent truncation of IIB SUGRA, and we derive explicit differential constraints on the compactification space for the consistent truncations to exist. More concretely, our findings for each of the cases are the following:

$\mathbf{AdS_7}$ in mIIA. In section 6 we construct and classify all geometries in mIIA theory consisting of the warped product

$$\mathrm{AdS}_7 \times I \times S^2 \,, \tag{1.1}$$

with I an interval, that preserve supersymmetry, the minimal amount being 16 supercharges in seven dimensions. We encounter that they can be classified in terms of a function $t(z)$ on the interval I satisfying

$$\dddot{t} = -\frac{m}{2} \,, \qquad \text{and} \qquad t(z) \geq 0 \,, \tag{1.2}$$

where equality in the last condition holds on the endpoints of I and ensures that the total internal space has no boundaries. The parameter m is the Romans mass of mIIA. We study all possibilities of having consistent truncations with vector multiplets around these vacua and find that the only possibility is to keep a single vector multiplet in the truncation and only if $m = 0$. This consistent truncation is just a consistent subsector of the maximally supersymmetric consistent truncation around the $\mathrm{AdS}_7 \times S^4$ solution of 11-dimensional supergravity dimensionally reduced to IIA supergravity.

$\mathbf{AdS_6}$ in IIB. Similarly, in section 7 we construct and classify all geometries in IIB theory consisting of the warped product

$$\mathrm{AdS}_6 \times \Sigma \times S^2 \,, \tag{1.3}$$

where Σ is a Riemann surface (with boundaries), that preserve 16 supercharges. We find that they can be classified in terms of two holomorphic functions f^α, $\alpha = 1, 2$, on the Riemann surface. These functions have to satisfy the condition

$$i\,\partial f^\alpha \bar{\partial} \bar{f}_\alpha \geq 0 \,, \qquad r \geq 0 \,, \tag{1.4}$$

where equality holds on the boundary of Σ, ensuring that the total internal space has no boundaries. The function r is a real function of the Riemann surface defined up to an integration constant through the differential equation

$$dr = -p_\alpha \, dk^\alpha \,, \tag{1.5}$$

where p^α and k^α are the real/imaginary parts $f^\alpha = -p^\alpha + i\,k^\alpha$. We also study which consistent truncations with vector multiplets around these vacua exist, and our results are summarised in table 1. We explicitly construct the consistent truncations containing one, two and three vector multiplets.

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N	$\mathrm{SU}(2)_R$ rep	Consistent truncations with vect. mult.	Gauging
1	$\mathbf{1}$	Only if $\exists\, g \in \mathrm{U}(1)$ s.t. $\partial(g\,\partial f^\alpha) \in$ real functions on Σ	$\mathrm{SU}(2)\times\mathrm{U}(1)$
2	$\mathbf{1}\oplus\mathbf{1}$	NO (due to global issues)	$\mathrm{SU}(2)\times\mathrm{U}(1)^2$
3	$\mathbf{1}\oplus\mathbf{1}\oplus\mathbf{1}$	NO	N/A
3	$\mathbf{3}$	Only if $r\,d\pi^\alpha = p^\alpha \pi^\beta \wedge \pi_\beta$	ISO(3)
4	$\mathbf{1}\oplus\mathbf{1}\oplus\mathbf{1}\oplus\mathbf{1}$	NO	N/A
4	$\mathbf{3}\oplus\mathbf{1}$	Only if $\exists\,\mathbf{3}$ and $\exists\,\mathbf{1}$ with $g = \pm\frac{1}{2}\left(\frac{p_\alpha \bar{\partial}\bar{f}^\alpha}{p_\beta \partial f^\beta}\right)$	$\mathrm{ISO}(3)\times\mathrm{U}(1)$

Table 1. Possible consistent truncations with N vector multiplets around supersymmetric $\mathrm{AdS}_6 \times S^2$ vacua in IIB and the resulting gauging of the gauged SUGRA. Consistency requires that $N \leq 4$ and that the vector multiplets form representations of $\mathrm{SU}(2)_R$, the R-symmetry of the AdS_6 vacua. The one-forms π^α are explicitly defined in terms of the background functions f^α, see equations (10.63) and (10.64).

2 Review of exceptional field theories

In this section, we review the structure of the relevant exceptional field theories. Exceptional field theories (ExFTs) are the manifestly duality covariant formulations of maximal higher-dimensional supergravity theories [5, 53, 54]. For our purposes, we will need the ExFTs built on the groups $\mathrm{E}_{5(5)} \equiv \mathrm{SO}(5,5)$ [55], and $\mathrm{E}_{4(4)} \equiv \mathrm{SL}(5)$ [56], respectively.

The reformulation of the higher-dimensional supergravities is based on the split of their coordinates into D *external* coordinates x^μ and the remaining *internal* coordinates y^i , with the latter embedded into a set of generalised internal coordinates Y^M transforming in a representation R_1 of the duality group $\mathrm{E}_{d(d)}$, with $d = 11 - D$. Internal diffeomorphisms and tensor gauge transformations of the higher-dimensional supergravity combine into a single symmetry structure of *generalised diffeomorphisms* in the coordinates Y^M [4, 7]. Parametrised by a gauge parameter ξ^M in R_1, the generalised Lie derivative of a generalised vector field V^M in R_1 reads

$$\mathcal{L}_\xi V^M = \xi^N \partial_N V^M - \partial_N \xi^M\, V^N + Y^{MN}{}_{KL}\, \partial_N \xi^K\, V^L\,. \tag{2.1}$$

The constant $\mathrm{E}_{d(d)}$-invariant tensor $Y^{MN}{}_{KL}$ encodes the deviation from standard diffeomorphisms. Its presence implies that the transformations (2.1) close into an algebra only after imposing the *section constraints*

$$Y^{MN}{}_{KL}\, \partial_M \otimes \partial_N = 0\,, \tag{2.2}$$

where the internal derivatives act on any pair of fields or gauge parameters. Solutions of the section constraints restrict the internal coordinate dependence of all fields to linear subspaces of R_1 upon which one recovers the standard supergravity theories. The action (2.1) can be rewritten as

$$\mathcal{L}_\xi V^M = \left(\xi^N \partial_N + \frac{1}{9-d}\, \partial_N \xi^N - a_d\, (t^\alpha)^K{}_L\, \partial_K \xi^L\, t_\alpha \cdot\right) V^M\,, \tag{2.3}$$

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with constant a_d, and t_α labelling the generators of $\mathrm{E}_{d(d)}$. From this formula one also reads off the action of generalised Lie derivatives on different representations. Modulo the section constraints (2.2), the transformations (2.1) close into an algebra defining the E-bracket

$$[\xi_1, \xi_2]^M \equiv 2\, \xi_{[1}^N \partial_N \xi_{2]}^M - Y^{MN}{}_{KL}\, \xi_{[1}^K \partial_N \xi_{2]}^L \,. \tag{2.4}$$

The presence of the tensor $Y^{MN}{}_{KL}$ implies the existence of trivial gauge parameters and non-associativity of the algebra. Generalised diffeomorphisms are realised as local symmetries of ExFT (i.e. with parameters ξ^M depending on internal and external coordinates) by introducing covariant external derivatives $\mathcal{D}_\mu \equiv \partial_\mu - \mathcal{L}_{\mathcal{A}_\mu}$ with the ExFT vector fields $\mathcal{A}_\mu$ in the R_1 representation. Non-associativity of the algebra (2.4) implies that the standard Yang-Mills field strength based on (2.4) is not a tensor w.r.t. the generalised Lie derivative (2.1). Rather it has to be completed by a coupling to the two-forms $\mathcal{B}_{\mu\nu}$ of the theory following the structure of the tensor hierarchy [57]

$$\mathcal{F}_{\mu\nu} = 2\, \partial_{[\mu}\mathcal{A}_{\nu]} - [\mathcal{A}_\mu, \mathcal{A}_\nu] + d\mathcal{B}_{\mu\nu} \,. \tag{2.5}$$

Here, the bracket $[\mathcal{A}_\mu, \mathcal{A}_\nu]$ refers to (2.4) while the d operator in the last term denotes a covariant differential operator from the R_2 representation of two-forms into R_1. Explicitly, it takes the form

$$(d\mathcal{B}_{\mu\nu})^M = Y^{MN}{}_{KL}\, \partial_N \mathcal{B}_{\mu\nu}{}^{KL} \,, \tag{2.6}$$

with the two-forms $\mathcal{B}_{\mu\nu}{}^{KL}$ living in (a sub-representation of) the symmetric tensor product $R_2 \subset (R_1 \otimes R_1)_{\mathrm{sym}}$. Continuing the tensor hierarchy gives rise to the couplings of three-forms $\mathcal{C}_{\mu\nu\rho} \subset R_3$, four-forms $\mathcal{D}_{\mu\nu\rho\sigma} \subset R_4$, etc., with the lowest non-abelian field strengths given by

$$\begin{aligned}
\mathcal{H}_{\mu\nu\rho} &= 3\, \mathcal{D}_{[\mu}\mathcal{B}_{\nu\rho]} - 3\partial_{[\mu}\mathcal{A}_\nu \wedge \mathcal{A}_{\rho]} + \mathcal{A}_{[\mu} \wedge \big[\mathcal{A}_\nu, \mathcal{A}_{\rho]}\big] + d\mathcal{C}_{\mu\nu\rho} \,, \\
\mathcal{J}_{\mu\nu\rho\sigma} &= 4\, \mathcal{D}_{[\mu}\mathcal{C}_{\nu\rho\sigma]} + 2\, \mathcal{F}_{[\mu\nu} \wedge \mathcal{B}_{\rho\sigma]} - d\mathcal{B}_{[\mu\nu} \wedge \mathcal{B}_{\rho\sigma]} - \frac{4}{3}\, \mathcal{A}_{[\mu} \wedge \big(\mathcal{A}_\nu \wedge \partial_\rho \mathcal{A}_{\sigma]}\big) \\
&\quad + \frac{1}{3}\, \mathcal{A}_{[\mu} \wedge \big(\mathcal{A}_\nu \wedge [\mathcal{A}_\rho, \mathcal{A}_{\sigma]}]\big) + d\mathcal{D}_{\mu\nu\rho\sigma} \,.
\end{aligned} \tag{2.7}$$

Again, the d operator denotes the covariant internal differential operators mapping $R_p \longrightarrow R_{p-1}$, while the wedge $\wedge$ represents algebraic maps

$$(R_1 \otimes R_1)_{\mathrm{sym}} \longrightarrow R_2 \,, \qquad R_1 \otimes R_2 \longrightarrow R_3 \,, \tag{2.8}$$

etc. Just as p-form field strengths are tensor with respect to the Lie derivative, the field strengths (2.5), (2.7), are tensors with respect to the generalised Lie derivative.

Let us now make these structures explicit for the theories we will be using in the following. For $d = 4$, the $\mathrm{E}_{4(4)} = \mathrm{SL}(5)$ ExFT is based on coordinates $Y^{ab} = Y^{[ab]}$, in the $R_1 = \mathbf{10}$ representation of SL(5), with $a, b = 1, \ldots, 5$ labelling the fundamental representation. The Y-tensor in (2.1) is given by $Y^{ef,gh}{}_{ab,cd} = 6\, \delta^{efgh}_{abcd}$, and induces a tensor hierarchy of p-forms living in representations R_p as

$$\mathcal{A}_\mu{}^{ab} : \mathbf{10} \,, \quad \mathcal{B}_{\mu\nu\, a} : \overline{\mathbf{5}} \,, \quad \mathcal{C}_{\mu\nu\rho}{}^a : \mathbf{5} \,, \quad \mathcal{D}_{\mu\nu\rho\sigma\, ab} : \overline{\mathbf{10}} \,. \tag{2.9}$$

The relevant $\wedge$ products (2.8) and the d operators in (2.5), (2.7) are explicitly given by

$$(\mathcal{A}_1 \wedge \mathcal{A}_2)_a = \frac{1}{4}\epsilon_{abcde}\,\mathcal{A}_1{}^{bc}\mathcal{A}_2{}^{de}\,, \qquad (\mathcal{A}\wedge\mathcal{B})^a = \mathcal{A}^{ab}\mathcal{B}_b\,, \tag{2.10}$$

$$(d\mathcal{B})^{ab} = \frac{1}{2}\epsilon^{abcde}\,\partial_{cd}\mathcal{B}_e\,, \qquad (d\mathcal{C})_a = \partial_{ba}\mathcal{C}^b\,, \qquad (d\mathcal{D})^a = \frac{1}{2}\epsilon^{abcde}\partial_{bc}\mathcal{D}_{de}\,. \tag{2.11}$$

For what follows, it will be similarly useful to define $\wedge : R_1\otimes R_3 \longrightarrow R_4$ and $\wedge : R_2\otimes R_3 \longrightarrow \mathbf{1}$ as

$$(\mathcal{A}\wedge\mathcal{C})_{ab} = \frac{1}{4}\epsilon_{abcde}\,\mathcal{A}^{cd}\mathcal{C}^e\,, \qquad \mathcal{B}\wedge\mathcal{C} = \mathcal{B}_a\mathcal{C}^a\,. \tag{2.12}$$

Moreover, the theory features 14 scalar fields, parameterising the coset space SL(5)/SO(5), which are most conveniently described by a group-valued generalised metric $\mathcal{M}_{ab}$. The ExFT dynamics comes from an action [56], giving rise to standard second order field equations. In particular, the 4-form field strength is dual to the 3-form field strength via the first order equation

$$\mathcal{J}_{\mu\nu\rho\sigma}{}^a = \frac{1}{3!}\sqrt{|\mathcal{G}|}\epsilon_{\mu\nu\rho\sigma\kappa\lambda\tau}\,\mathcal{M}^{ab}\,\mathcal{H}^{\kappa\lambda\tau}{}_b\,, \tag{2.13}$$

with the scalar matrix $\mathcal{M}^{ab}$, and where $|\mathcal{G}|$ is the determinant of the external metric, $\mathcal{G}_{\mu\nu}$, of the ExFT, which is used to raise/lower the external indices on the field strengths.

For $d=5$, the $\mathrm{E}_{5(5)} = \mathrm{SO}(5,5)$ ExFT is based on coordinates Y^M, in the $R_1 = \mathbf{16}$ spinor representation of SO(5,5), with $M = 1,\ldots,16$. The Y-tensor in (2.1) is given by $Y^{PQ}{}_{MN} = \frac{1}{2}(\gamma^I)_{MN}(\gamma_I)^{PQ}$ in terms of the SO(5,5) gamma matrices, with the index $I = 1,\ldots 10$, labelling the vector representation, raised and lowered by the constant SO(5,5) invariant metric η_{IJ}. It induces a tensor hierarchy of p-forms living in representations R_p as

$$\mathcal{A}_\mu{}^M : \mathbf{16}\,, \quad \mathcal{B}_{\mu\nu}{}^I : \mathbf{10}\,, \quad \mathcal{C}_{\mu\nu\rho\,M} : \overline{\mathbf{16}}\,, \quad \mathcal{D}_{\mu\nu\rho\sigma}{}^{[IJ]} : \mathbf{45}\,. \tag{2.14}$$

Strictly speaking, the theory also carries additional covariantly constrained 4-forms $\mathcal{D}_{\mu\nu\rho\sigma\,M}$, but for our purposes we will only consider equations in which all four-forms drop out. The relevant $\wedge$ products (2.8) and the d operators in (2.5), (2.7) are explicitly given by

$$(\mathcal{A}\wedge\mathcal{A})^I = \frac{1}{2}(\gamma^I)_{MN}\,\mathcal{A}^M\mathcal{A}^N\,, \qquad (\mathcal{A}\wedge\mathcal{B})_M = \frac{1}{2}(\gamma_I)_{MN}\,\mathcal{A}^N\mathcal{B}^I\,, \tag{2.15}$$

$$(d\mathcal{B})^M = (\gamma_I)^{MN}\,\partial_N\mathcal{B}^I\,, \qquad (d\mathcal{C})_I = \frac{1}{2}(\gamma_I)^{MN}\partial_M\mathcal{C}_N\,. \tag{2.16}$$

Once again, it will be useful to also define $\wedge : R_2\otimes R_2 \longrightarrow \mathbf{1}$ as

$$\mathcal{B}_1\wedge\mathcal{B}_2 = \eta_{IJ}\,\mathcal{B}_1{}^I\mathcal{B}_2{}^J\,. \tag{2.17}$$

Moreover, the theory features 25 scalar fields, parameterising the coset space SO(5,5)/ (SO(5) $\times$ SO(5)), which are most conveniently described by a group-valued generalised metric $\mathcal{M}_{MN}$ in the spinor representation, or by a group-valued generalised metric $\mathcal{M}_{IJ}$ in the vector representation. The ExFT dynamics comes from a pseudo-action [55], which has to be supplemented by first order duality and self-duality equations among the p-form field strengths

$$\begin{aligned} \mathcal{J}_{\mu\nu\rho\sigma\, M} &= \frac{1}{2}\sqrt{|\mathcal{G}|}\epsilon_{\mu\nu\rho\sigma\kappa\lambda}\mathcal{M}_{MN}\,\mathcal{F}^{\kappa\lambda N}\,, \\ \mathcal{H}_{\mu\nu\rho\, I} &= -\frac{1}{3!}\sqrt{|\mathcal{G}|}\epsilon_{\mu\nu\rho\sigma\kappa\lambda}\eta_{IJ}\mathcal{M}^{JK}\,\mathcal{H}^{\sigma\kappa\lambda}{}_{K}\,, \end{aligned} \tag{2.18}$$

where $\mathcal{G}_{\mu\nu}$ is the external metric which is used to raise/lower the external indices on the field strengths and $|\mathcal{G}|$ is its determinant.

For details about the ExFT actions and field equations, we refer to [55, 56]. In appendices A and B, we collect/derive the details of the dictionaries between the ExFT fields and the original IIA/IIB supergravity fields.

3 Half-maximal AdS vacua from ExFT

Generic supersymmetric AdS vacua of 10-/11-dimensional SUGRA have non-trivial fluxes. Since ExFT unifies fluxes and geometry into generalised tensor fields, it leads to a natural description of supersymmetric AdS vacua that is largely analogous to special holonomy spaces in Riemannian geometry, as shown in [30] for the case of 16 supercharges, and in [33] for 8 supercharges. Thus, having a supersymmetric $\mathrm{AdS}_D \times M$ vacuum is equivalent to the existence of a nowhere vanishing set of generalised tensor fields on M subject to certain algebraic compatibility conditions and differential conditions. These conditions ensure that M admits appropriate Killing spinors for the AdS_D vacuum. As shown in [30], for supersymmetric AdS_6 and AdS_7 vacua the relevant generalised tensors are $d-1$ generalised vector fields $J_u \in \Gamma\left(\mathcal{R}_1\right)$, with $u = 1, \ldots, d-1$, and a generalised tensor field $\hat{K} \in \Gamma\left(\mathcal{R}_{D-4}\right)$, where $d = 11 - D$ and D denotes the dimension of the AdS vacuum. Here we denote by $\mathcal{R}_p$ the generalised vector bundle whose fibres are R_p, as listed in (2.9) and (2.14). These generalised tensors must satisfy the algebraic conditions

$$\begin{aligned} J_u \wedge J_v - \frac{1}{d-1}\delta_{uv} J_w J^w &= 0\,, \\ J_u \wedge J^u \wedge \hat{K} &> 0\,, \\ \hat{K} \otimes \hat{K}|_{R_c} &= 0\,, \end{aligned} \tag{3.1}$$

with the ExFT $\wedge$ product defined in (2.10), (2.12) and (2.15), (2.17), the $u, v = 1, \ldots, d-1$ indices raised and lowered by δ_{uv} and $R_c = \emptyset$ for $D = 7$ and $R_c = \mathbf{1}$ for $D = 6$. This set of generalised tensors J_u and $\hat{K}$ defines a $\mathrm{G}_{\mathrm{half}} = \mathrm{SO}(d-1)$ structure, because it is stabilised by $\mathrm{SO}(d-1) \subset E_{d(d)}$. This ensures the existence of well-defined spinors on M_{int} carrying 16 supercharges,[2] and we will therefore also call the set J_u, $\hat{K}$ satisfying (3.1) a "half-maximal structure". The commutant of $\mathrm{G}_{\mathrm{half}}$ within the maximal compact subgroup of $E_{d(d)}$ is itself given by $\mathrm{SO}(d-1)_R$ which acts as a R-symmetry group, rotating the

[2] In 6 dimensions, the above description is equivalent to having 16 non-chiral supercharges. It is also possible to have a chiral set of 16 supercharges in 6 dimensions, which requires having a different set of generalised tensors [30]. However, there are no chirally supersymmetric AdS_6 vacua, and so we will not comment further on this possibility.

well-defined spinors into each other, and similarly the generalised vector fields J_u. As we will show in section 5, one can express the generalised metric, i.e. the scalar fields on M, in terms of a $\mathrm{SO}(d-1)_R$-invariant combination of the half-maximal structure, J_u and $\hat{K}$.

To ensure that the well-defined spinors are Killing spinors of the supersymmetric $\mathrm{AdS}_{6,7}$ vacua, the half-maximal structure J_u, $\hat{K}$ must satisfy the following differential constraints [30]

$$\begin{aligned}
\mathcal{L}_{J_u} J_v &= -\Lambda_{uvw} J^w \,, \\
\mathcal{L}_{J_u} \hat{K} &= 0 \,, \\
d\hat{K} &= \begin{cases} \frac{1}{3!3\sqrt{2}} \epsilon^{uvw} \Lambda_{uvw} J_x \wedge J^x \,, & \text{when } \mathrm{D} = 7 \,, \\ \frac{1}{9} \epsilon_{uvwx} \Lambda^{uvw} J^x \,, & \text{when } \mathrm{D} = 6 \,, \end{cases}
\end{aligned} \tag{3.2}$$

where the generalised Lie derivatives, $\mathcal{L}_{J_u} J_v$, $\mathcal{L}_{J_u} \hat{K}$, and the $d\hat{K}$ operator are as defined in equations (2.1), (2.3), (2.11) and (2.16). For $D = 7$, i.e. in the SL(5) ExFT, the explicit expressions for the generalised Lie derivatives appearing in the first two equations of (3.2) are

$$\begin{aligned}
\mathcal{L}_{J_u} J_v{}^{ab} &= \frac{1}{2} J_u{}^{cd} \partial_{cd} J_v{}^{ab} - 2 J_v{}^{c[b} \partial_{cd} J_u{}^{a]d} + \frac{1}{2} J_v{}^{ab} \partial_{cd} J_u{}^{cd} \,, \\
\mathcal{L}_{J_u} \hat{K}^a &= \frac{1}{2} J_u{}^{bc} \partial_{bc} \hat{K}^a - \hat{K}^b \partial_{bc} J_u{}^{ac} + \frac{1}{2} \hat{K}^a \partial_{bc} J_u{}^{bc} \,,
\end{aligned} \tag{3.3}$$

while for $D = 6$, i.e. in the SO(5,5) ExFT, they are

$$\begin{aligned}
\mathcal{L}_{J_u} J_v{}^M &= J_u{}^N \partial_N J_v{}^M - J_v{}^N \partial_N J_u{}^M + \frac{1}{2} (\gamma_I)^{MN} \left(\gamma^I\right)_{PQ} J_v{}^P \partial_N J_u{}^Q \,, \\
\mathcal{L}_{J_u} \hat{K}^I &= J_u{}^M \partial_M \hat{K}^I + \frac{1}{2} \hat{K}^J (\gamma_J)_{MN} \left(\gamma^I\right)^{NP} \partial_P J_u{}^M \,.
\end{aligned} \tag{3.4}$$

The objects Λ_{uvw} appearing in (3.2) are totally antisymmetric constants which imply that the J_u's generate a $\mathrm{SU}(2)_R$ algebra with respect to the generalised Lie derivative and that the $\hat{K}$ is invariant under this $\mathrm{SU}(2)_R$ symmetry [30]. The cosmological constant, Λ, of the $\mathrm{AdS}_{6,7}$ vacuum is encoded in Λ_{uvw} as

$$\Lambda_{uvw} \Lambda^{uvw} \sim -\Lambda \,, \tag{3.5}$$

up to numerical factors which we will fix in sections 6 and 7 by comparing with known supersymmetric $\mathrm{AdS}_{6,7}$ vacua. From (3.2), we see that Λ_{uvw} breaks the $\mathrm{SO}(d-1)_R$ symmetry of the half-maximal structure to $\mathrm{SU}(2)_R$, the R-symmetry of the supersymmetric $\mathrm{AdS}_{6,7}$ vacua.

Moreover, (3.2) implies that the vector fields, J_u, generate, via the generalised Lie derivative, a $\mathrm{SU}(2)_R \subset \mathrm{SO}(d-1)_R$ rotation on the J_u's themselves and leave $\hat{K}$ invariant. As we will make explicit in the next section, the generalised metric $\mathcal{M}$ is constructed from $\mathrm{SO}(d-1)_R$-invariant combinations of J_u and $\hat{K}$ and thus

$$\mathcal{L}_{J_u} \mathcal{M} = 0 \,. \tag{3.6}$$

Thefore, the J_u are generalised Killing vector fields of the background. As made explicit in appendices A and B generalised vector fields consist of formal sums of spacetime vector fields and differential forms. Equation (3.6) implies that either the spacetime vector fields in J_u are Killing vector fields of the spacetime metric and leave the SUGRA field strengths invariant [30], or that some of the J_u contain a vanishing spacetime vector field component and consist of only exact differential forms. We call a generalised Killing vector of the latter type a trivial Killing vector field. As discussed in more detail in [30], for AdS_7 vacua we see that the $SU(2)_R$ symmetry must be generated by the three spacetime vector fields of J_u, $u = 1, \ldots, 3$. On the other hand, for AdS_6 vacua three of the J_u's contain spacetime Killing vector fields that generate the $SU(2)_R$ symmetry, while the fourth generalised vector field

$$J_T \equiv \frac{1}{3!}\epsilon^{uvwx}\Lambda_{uvw}J_x \,, \tag{3.7}$$

is given by

$$J_T = \frac{3}{2}d\hat{K} \,, \tag{3.8}$$

which implies that it is a trivial generalised Killing vector field. In fact, it satisfies

$$\mathcal{L}_{J_T} = 0 \,, \tag{3.9}$$

when acting on any generalised tensor. We will make use of these general properties of J_u and $\hat{K}$ when constructing supersymmetric $AdS_{6,7}$ vacua in section 6 and 7.

Finally, one can define the following generalised tensor fields from the half-maximal structure J_u and $\hat{K}$ which will be useful to us

$$J_u \wedge J_v = \delta_{uv} K \,, \qquad K \wedge \hat{K} = \kappa^{D-2} \,, \qquad \hat{J}_u = J_u \wedge \hat{K} \,, \tag{3.10}$$

where $K \in \Gamma(\mathcal{R}_2)$ and κ is a scalar density of weight $\frac{1}{D-2}$.

3.1 Minimal consistent truncation

One benefit of constructing or describing half-maximal AdS_D vacua by the structures J_u and $\hat{K}$ is that we immediately obtain a "minimal" consistent truncation around the vacuum to a half-maximal D-dimensional gauged SUGRA containing only the gravitational supermultiplet [30]. This is therefore a proof and an explicit realisation of the (half-maximal subcase of the) conjecture that such a consistent truncation exists for any supersymmetric warped AdS vacuum of 10-/11-dimensional SUGRA [1]. Moreover, the usually highy non-linear truncation Ansatz is given by a simple linear factorisation Ansatz on the ExFT structures. If we denote by Y^M the internal coordinates and by x^μ the D-dimensional external coordinates, then the truncation Ansatz (of the purely internal fields from the D-dimensional perspective) is given by [29, 30]

$$\begin{aligned} \mathcal{J}_u(x,Y) &= X^{-1}(x)\, J_u(Y) \,, \\ \hat{\mathcal{K}}(x,Y) &= X^2(x)\, \hat{K}(Y) \,. \end{aligned} \tag{3.11}$$

Here $X(x)$ is the scalar field of the D-dimensional half-maximal SUGRA. For each value of the scalar field $X(x) > 0$, $\mathcal{J}_u(x,Y)$ and $\hat{K}(x,Y)$ satisfy the algebraic conditions (3.1) and

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thus a half-maximal structure. This guarantees that the theory obtained after truncation is half-maximally supersymmetric. However, for $X \neq 1$, the differential conditions (3.2) defining the AdS vacuum are no longer satisfied. Therefore, at $X \neq 1$, the theory will not have a supersymmetric AdS vacuum. Finally, as shown in [30], the differential conditions (3.2) ensure that the truncation Ansatz (3.11) is consistent.

We will show in section 5 how to construct the generalised metric from the half-maximal structure. By constructing the generalised metric of $\mathcal{J}_u$ and $\hat{\mathcal{K}}$ and using the dictionary between ExFT and SUGRA, given in appendices A.2 and B.2, we thus obtain the non-linear truncation Ansatz for the internal supergravity fields.

For the fields of the ExFT tensor hierarchy, the truncation Ansatz is as follows [30]. For the ExFT vector fields, we have

$$\mathcal{A}_\mu(x,Y) = A_\mu{}^u(x)\, J_u(Y)\,. \tag{3.12}$$

In $D=7$, the truncation Ansatz for the remaining fields is

$$\begin{aligned} \mathcal{B}_{\mu\nu}(x,Y) &= -B_{\mu\nu}(x)\, K(Y)\,, \\ \mathcal{C}_{\mu\nu\rho}(x,Y) &= C_{\mu\nu\rho}(x)\, \hat{K}(Y)\,, \\ \mathcal{D}_{\mu\nu\rho\sigma}(x,Y) &= D_{\mu\nu\rho\sigma}{}^u(x)\, \hat{J}_u(Y)\,, \end{aligned} \tag{3.13}$$

where $A_\mu{}^u(x)$, $B_{\mu\nu}(x)$, $C_{\mu\nu\rho}(x)$ and $D_{\mu\nu\rho\sigma}{}^u(x)$ are the fields of the 7-dimensional half-maximal gravitational supermultiplet. In particular, $A_\mu{}^u$ are the 3 vector fields, $B_{\mu\nu}$ are the 2-forms, $C_{\mu\nu\rho}$ are the 3-forms dual to $B_{\mu\nu}$, and $D_{\mu\nu\rho\sigma}{}^u$ are the 4-forms dual to the vector fields. The duality relations between these half-maximal gauged SUGRA fields comes from the duality relations (2.13) between the ExFT field strengths (2.5), (2.7). Finally, the truncation Ansatz for the external 7-D ExFT metric is

$$\mathcal{G}_{\mu\nu}(x,Y) = G_{\mu\nu}(x)\, \kappa^2(Y)\,, \tag{3.14}$$

with $G_{\mu\nu}(x)$ the metric of the half-maximal gauged SUGRA.

Similarly, in $D=6$, the truncation Ansatz for the tensor hierarchy field is

$$\begin{aligned} \mathcal{B}_{\mu\nu}(x,Y) &= B_{\mu\nu}(x)\, \hat{K}(Y) - \tilde{B}_{\mu\nu}(x)\, K(Y)\,, \\ \mathcal{C}_{\mu\nu\rho}(x,Y) &= C_{\mu\nu\rho}{}^u(x)\, \hat{J}_u(Y)\,. \end{aligned} \tag{3.15}$$

Now, $A_\mu{}^u$ are the 4 vector fields of the gravitational supermultiplet, while $B_{\mu\nu}$ is its 2-form. $\tilde{B}_{\mu\nu}$ is the dual 2-form and $C_{\mu\nu\rho}{}^u$ are the 3-forms dual to the vector fields. Once again, the relationship between these objects arises from the duality relation (2.18) between the ExFT field strengths (2.5), (2.7). Using the truncation Ansatz (3.15) and differential conditions (3.2), we find that the field strengths factorise as

$$\begin{aligned} \mathcal{F}_{\mu\nu}(x,Y) &= \tilde{F}_{\mu\nu}{}^u(x)\, J_u(Y)\,, \\ \mathcal{H}_{\mu\nu\rho}(x,Y) &= \tilde{F}_{\mu\nu\rho}(x)\, \hat{K}(Y) - \tilde{G}_{\mu\nu\rho}(x)\, K(Y)\,, \end{aligned} \tag{3.16}$$

where

$$
\begin{aligned}
\tilde{F}_{\mu\nu}{}^{u} &= 2\partial_{[\mu}A_{\nu]}{}^{u} + \Lambda^{uvw}A_{\mu\, v}\, A_{\nu\, w} - \frac{1}{9}\epsilon^{uvwx}\Lambda_{vwx}\, B_{\mu\nu}\,, \\
\tilde{F}_{\mu\nu\rho} &= 3\partial_{[\mu}B_{\nu\rho]}\,, \\
\tilde{G}_{\mu\nu\rho} &= 3\partial_{[\mu}\tilde{B}_{\nu\rho]} + 3A_{[\mu}{}^{u}\partial_{\nu}A_{\rho]\, u} + \Lambda_{uvw}\, A_{\mu}{}^{u}A_{\nu}{}^{v}A_{\rho}{}^{w} - \frac{1}{9}\epsilon_{uvwx}\Lambda^{uvw}\, C_{\mu\nu\rho}{}^{x}\,,
\end{aligned}
\tag{3.17}
$$

and similarly for the higher field strengths of the ExFT. We will use this to derive the duality relations between $B_{\mu\nu}$ and $\tilde{B}_{\mu\nu}$ explicitly in section 5.2. Similar to $D = 7$, the truncation Ansatz for the external ExFT metric is

$$
\mathcal{G}_{\mu\nu}(x, Y) = \sqrt{2}\, G_{\mu\nu}(x)\, \kappa^2(Y)\,, \tag{3.18}
$$

with $G_{\mu\nu}(x)$ the 6-dimensional gauged SUGRA metric.

4 Consistent truncations with matter multiplets

As shown in [30], half-maximal consistent truncations with N vector multiplets require a further reduction of the structure group to $\mathrm{SO}(d-1-N) \subset \mathrm{SO}(d-1) \subset \mathrm{E}_{d(d)}$, as well as differential conditions on the tensors defining the $\mathrm{SO}(d-1-N)$ structure. In order to have a $\mathrm{SO}(d-1-N)$ structure, we require $d-1+N$ generalised vector fields, J_a, where $a = 1, \ldots, d-1+N$, satisfying

$$
J_a \wedge J_b = \eta_{ab} K\,, \tag{4.1}
$$

in addition to the $\hat{K}$ as in (3.1). Here η_{AB} is a constant $\mathrm{SO}(d-1, N)$ invariant metric and K is defined as in (3.10),

$$
K = \frac{1}{d-1} J_u \wedge J^u\,. \tag{4.2}
$$

Therefore, given the $d-1$ generalised vector fields defining the half-maximal structure of the AdS vacuum (3.1), we must have a further N generalised vector fields, one for each vector multiplet. Labelling these extra generalised vector fields by $\bar{u} = 1, \ldots, N$, the algebraic conditions (4.1) become

$$
\begin{aligned}
J_{\bar{u}} \wedge J_u &= 0\,, \\
J_{\bar{u}} \wedge J_{\bar{v}} &= -\delta_{\bar{u}\bar{v}} K\,.
\end{aligned}
\tag{4.3}
$$

Since these algebraic conditions must hold point-wise, it is easy to show that we can only have $N \leq d-1$ vector multiplets in a consistent truncation.

Moreover, for the truncation around the supersymmetric AdS vacuum to be consistent, the $\mathrm{SO}(d-1-N)$ structure must satisfy the differential conditions

$$
\begin{aligned}
\mathcal{L}_{J_a} J_b &= -f_{ab}{}^{c}\, J_c\,, \\
\mathcal{L}_{J_a} \hat{K} &= 0\,,
\end{aligned}
\tag{4.4}
$$

where $f_{abc} = f_{ab}{}^{d}\eta_{dc}$ are totally antisymmetric structure constants with $f_{uvw} = \Lambda_{uvw}$ and $f_{uv\bar{w}} = 0$. Here we are considering a special case of the more general conditions given in [30]

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because we want to ensure that the truncation contains a supersymmetric AdS vacuum. The differential conditions (4.4) can be thought of as the higher-dimensional analogue of the conditions imposed on the embedding tensor of 6-/7-dimensional half-maximal gauged SUGRA in [52, 58].

For what follows, it's useful to note that the first condition of (4.4) implies that the extra generalised vector fields form a representation under the R-symmetry group

$$\mathcal{L}_{J_u} J_{\bar{v}} = -f_{u\bar{v}}{}^{\bar{w}} J_{\bar{w}} \,. \tag{4.5}$$

Together with the fact that there can be only $N \leq d-1$ vector multiplets, this will allow us to fully classify the possible consistent truncations with vector multiplets in sections 9 and 10.

4.1 Truncation Ansatz

As shown in [30], given the $d-1+N$ vector fields satisfying (4.1) and (4.4), we obtain a consistent truncation by expanding the fields of the ExFT as follows.

For the scalar sector, we expand the background $\mathrm{SO}(d-1)$ structure in terms of the $\mathrm{SO}(d-1-N)$ structure as

$$\begin{aligned} \mathcal{J}_u(x,Y) &= X^{-1}(x)\, b_u{}^a(x)\, J_a(Y)\,, \\ \hat{\mathcal{K}}(x,Y) &= X^2(x)\, \hat{K}(Y)\,. \end{aligned} \tag{4.6}$$

The fields $b_u{}^a$ must satisfy

$$b_u{}^a b_v{}^b \eta_{ab} = \delta_{uv}\,, \tag{4.7}$$

and are identified up to $\mathrm{SO}(d-1)$ rotations acting on the u, v indices. Therefore, they parameterise the coset space

$$b_u{}^a \in \frac{\mathrm{SO}(d-1,N)}{\mathrm{SO}(d-1)\times \mathrm{SO}(N)}\,, \tag{4.8}$$

and together with $X \in \mathbb{R}^+$ they form the scalar manifold of half-maximal gauged SUGRA coupled to N vector multiplets

$$M_{\text{scalar}} = \frac{\mathrm{SO}(d-1,N)}{\mathrm{SO}(d-1)\times \mathrm{SO}(N)} \times \mathbb{R}+\,. \tag{4.9}$$

Using the formulae of section 5, in which we show how to construct the generalised metric from the half-maximal structure, we can then translate the above truncation Ansätze into the non-linear truncation Ansätze of the internal SUGRA fields.

For $D=7$, the remaining fields of the ExFT are expanded as

$$\begin{aligned} \mathcal{A}_\mu(x,Y) &= A_\mu{}^a(x)\, J_a(Y)\,, \\ \mathcal{B}_{\mu\nu}(x,Y) &= -B_{\mu\nu}(x)\, K(Y)\,, \\ \mathcal{G}_{\mu\nu}(x,Y) &= G_{\mu\nu}(x)\, \kappa^2(Y)\,, \end{aligned} \tag{4.10}$$

where $A_\mu{}^a$ are the $3+N$ vector fields, $B_{\mu\nu}$ are the two-form fields and $G_{\mu\nu}$ the metric of the seven-dimensional half-maximal gauged SUGRA.

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For $D=6$, the other fields of the ExFT are expanded as

$$\begin{aligned}\mathcal{A}_\mu(x,Y) &= A_\mu{}^a(x)\,J_a(Y)\,,\\ \mathcal{B}_{\mu\nu}(x,Y) &= B_{\mu\nu}(x)\,\hat{K}(Y) - \tilde{B}_{\mu\nu}(x)\,K(Y)\,,\\ \mathcal{C}_{\mu\nu\rho}(x,Y) &= C_{\mu\nu\rho}{}^a(x)\,\hat{J}_a(Y)\,,\\ \mathcal{G}_{\mu\nu}(x,Y) &= \sqrt{2}\,G_{\mu\nu}(x)\,\kappa^2(Y)\,.\end{aligned} \tag{4.11}$$

Here $G_{\mu\nu}$ is the metric, $A_\mu{}^a$ are the $4+N$ vector fields, $B_{\mu\nu}$ are the two-form fields and their duals $\tilde{B}_{\mu\nu}$ of the six-dimensional half-maximal gauged SUGRA. $C_{\mu\nu\rho}{}^a$ are the 3-form fields dual to the $A_\mu{}^a$, which appear via Stückelberg coupling in the field strength of $\tilde{B}_{\mu\nu}$. To see this, one can compute the ExFT field strengths (2.7). Using the truncation Ansatz (4.11) and the differential conditions (3.2) we find

$$\begin{aligned}\mathcal{F}_{\mu\nu}(x,Y) &= \tilde{F}_{\mu\nu}{}^a(x)\,J_a(Y)\,,\\ \mathcal{H}_{\mu\nu\rho}(x,Y) &= \tilde{F}_{\mu\nu\rho}(x)\,\hat{K}(Y) - \tilde{G}_{\mu\nu\rho}(x)\,K(Y)\,,\end{aligned} \tag{4.12}$$

where

$$\begin{aligned}\tilde{F}_{\mu\nu}{}^a &= 2\partial_{[\mu}A_{\nu]}{}^a + f^a{}_{bc}A_\mu{}^bA_\nu{}^c + \frac{2}{3}\Lambda^a B_{\mu\nu}\,,\\ \tilde{F}_{\mu\nu\rho} &= 3\partial_{[\mu}B_{\nu\rho]}\,,\\ \tilde{G}_{\mu\nu\rho} &= 3\partial_{[\mu}\tilde{B}_{\nu\rho]} + 3A_{[\mu}{}^a\partial_\nu A_{\rho]}{}^b\eta_{ab} + \Lambda_{uvw}\,A_\mu{}^uA_\nu{}^vA_\rho{}^w + \frac{2}{3}\Lambda_a\,C_{\mu\nu\rho}{}^a\,,\end{aligned} \tag{4.13}$$

where we defined

$$\Lambda^a = \left(\Lambda^u,\,\Lambda^{\bar{u}}\right) = \left(-\frac{1}{3!}\epsilon^{uvwx}\Lambda_{vwx},\,0\right)\,. \tag{4.14}$$

Clearly, $F_{\mu\nu}{}^a$ are the field strengths of the 6-dimensional half-maximal gauged SUGRA whose gauge group is determined by the structure constants f_{abc} and $F_{\mu\nu\rho}$ is the field strength of the two-form $B_{\mu\nu}$ of the gauged SUGRA. Using the ExFT/SUGRA dictionary of appendix B.3.1, we can use the above formulae to read off the consistent truncation Ansätze for the SUGRA fields.

5 Generalised metric from the half-maximal structure

To obtain expressions for the AdS vacua and their consistent truncations in terms of SUGRA fields, we need to know how the SUGRA fields are encoded in the ExFT objects used in the truncation Ansätze of sections 3.1 and 4.1. The SUGRA fields with at least one external leg are encoded in the ExFT tensor hierarchy fields $\mathcal{A}_\mu$, $\mathcal{B}_{\mu\nu}$, etc. and can be determined in the usual fashion via the SUGRA / ExFT dictionary, which we give for the $SO(5,5)$ case in appendix B.3.1. However, the purely internal SUGRA fields are encoded in the generalised metric of ExFT, via the dictionary we give in appendices A.2 and B.2, and therefore we must know how to obtain a generalised metric from the half-maximal structure.

Firstly, it is clear that one can construct a generalised metric from J_u and $\hat{K}$. Just like on a d-dimensional manifold, a Riemannian metric defines a $SO(d)\subset GL(d)$ structure, a generalised metric defines a (generalised) $H_d \subset E_{d(d)}$ structure, where H_d is the maximal compact subgroup of $E_{d(d)}$. On the other hand, J_u and $\hat{K}$ define a $\mathrm{G}_{\text{half}} = \mathrm{SO}(d-1) \subset H_d$ structure and, thus, J_u and $\hat{K}$ provide *more* information than the generalised metric. In ExFT, the generalised metric parameterises the coset space

$$\mathcal{M}_{MN} \in \frac{\mathrm{E}_{d(d)}}{H_d}\,. \tag{5.1}$$

Since J_u and $\hat{K}$ are by construction invariant under $\mathrm{G}_{\text{half}} = \mathrm{SO}(d-1) \subset H_d$, we must construct $\mathcal{M}_{MN}$ using an $\mathrm{SO}(d-1)_R$-invariant combination of J_u and $\hat{K}$. Therefore, the generalised metric must be given by

$$\mathcal{M}_{MN} = A\,\kappa^{6-2D}\hat{J}_{u\,M}\hat{J}^u{}_N + B\,\kappa^{4-D}\hat{K}_{MN} + C\,\epsilon^{u_1\ldots u_{d-1}}\left(J_{u_1}\ldots J_{u_{d-1}}\right)_{MN}\,. \tag{5.2}$$

The factors of κ are chosen so that $\mathcal{M}_{MN}$ has no weight under generalised diffeomorphisms and A, B and C are coefficients which are fixed by requiring $\mathcal{M}_{MN}$ to be an element of $\mathrm{E}_{d(d)}$. The final term schematically denotes an appropriate product of $(R_1)^{d-1} \longrightarrow R_1 \otimes R_1$. In the following subsections we will give the explicit expressions for the case of SL(5) ExFT and SO(5,5) ExFT.

5.1 Generalised metric in SL(5) ExFT

In SL(5) ExFT, the generalised metric is often used either in the $R_1 = \mathbf{10}$ representation or its dual representation, or in the fundamental representation, $R_2 = \mathbf{5}$, of SL(5). The two are related by [3]

$$\mathcal{M}_{ab,cd} = 2\mathcal{M}_{a[c}\mathcal{M}_{d]b}\,, \tag{5.3}$$

where $a, b = 1, \ldots, 5$ denote fundamental SL(5) indices. It will be useful to have explicit expressions for both representations.

The generalised metric in the $\mathbf{10}$ representation of SL(5) is given as in (5.2) which now explicitly becomes

$$\mathcal{M}_{ab,cd} = A\,\kappa^{-8}\hat{J}_{u\,ab}\hat{J}^u{}_{cd} + B\,\kappa^{-3}\epsilon_{abcde}\hat{K}^e + C\,\kappa^{-3}\epsilon^{uvw}\epsilon_{abefg}\epsilon_{cdhij}J_u{}^{ef}J_v{}^{hi}J_w{}^{gj}\,, \tag{5.4}$$

where $\hat{J}_{u\,ab}$ is defined as in (3.10), explicitly

$$\hat{J}_{u\,ab} = \frac{1}{4}\epsilon_{abcde}J_u{}^{cd}\hat{K}^e\,. \tag{5.5}$$

Requiring this to be an SL(5) element fixes $A = 8\sigma^2$, $B = -\sigma^2$ and $C = -\frac{\sigma^3}{6\sqrt{2}}$, up to a coefficient σ. Note that the minimal consistent truncation (3.11) corresponds precisely to a rescaling $\sigma \longrightarrow \sigma\,X$. σ is determined by the differential conditions (3.2) and can therefore be fixed by comparison of AdS vacua obtained from the half-maximal structures to known AdS_7 vacua, for example the maximally supersymmetric $\mathrm{AdS}_7 \times S^4$ vacua of 11-d SUGRA. This way we find $\sigma = 1$. Thus, the generalised metric and its inverse in the $\mathbf{10}$ and $\overline{\mathbf{10}}$ representations are given by

$$\begin{aligned}
\mathcal{M}_{ab,cd} &= 8\,\kappa^{-8}\hat{J}_{u\,ab}\hat{J}^u{}_{cd} - \kappa^{-3}\epsilon_{abcde}\hat{K}^e - \frac{1}{6\sqrt{2}}\kappa^{-3}\epsilon^{uvw}\epsilon_{abefg}\epsilon_{cdhij}J_u{}^{ef}J_v{}^{hi}J_w{}^{gj}\,,\\
\mathcal{M}^{ab,cd} &= 2\,\kappa^{-2}J_u{}^{ab}J^{u,cd} - \kappa^{-2}\epsilon^{abcde}K_e - \frac{2\sqrt{2}}{3}\kappa^{-12}\epsilon^{uvw}\epsilon^{abefg}\epsilon^{cdhij}\hat{J}_{u\,ef}\hat{J}_{v\,hi}\hat{J}_{w\,gj}\,.
\end{aligned} \tag{5.6}$$

Similarly, one can show that the generalised metric and its inverse in the **5** and $\overline{\mathbf{5}}$ representations of SL(5) are given by

$$
\begin{aligned}
\mathcal{M}_{ab} &= \kappa^{-4}\left(K_a K_b + \frac{4\sqrt{2}}{3}\kappa^{-5}\,\epsilon^{uvw}\hat{J}_{u,ac}\hat{J}_{v,bd}J_w{}^{cd}\right),\\
\mathcal{M}^{ab} &= \kappa^{-6}\left(\hat{K}^a\hat{K}^b + \frac{2\sqrt{2}}{3}\,\epsilon^{uvw}J_u{}^{ac}J_v{}^{bd}\hat{J}_{w,cd}\right).
\end{aligned} \tag{5.7}
$$

5.2 Generalised metric in SO(5,5) ExFT

In SO(5,5) ExFT, the generalised metric is often used either in the spinor or vector representation of SO(5,5). In the fundamental representation, the generalised metric $\mathcal{M}_{IJ}$ must satisfy

$$
\mathcal{M}_{IK}\mathcal{M}_{JL}\eta^{KL} = \eta_{IJ}\,, \tag{5.8}
$$

where $I = 1,\ldots,10$ labels the **10** representation of SO(5,5) and η_{IJ} is the constant SO(5,5)-invariant metric with which the I, J indices are raised/lowered. The generalised metric in the **16** is related to that in the **10** by

$$
\mathcal{M}_{MP}\mathcal{M}_{NQ}\left(\gamma_I\right)^{MN}\mathcal{M}^{IJ} = \left(\gamma^J\right)_{PQ}\,, \tag{5.9}
$$

where $M = 1,\ldots,16$ label the **16** representation and $(\gamma_I)^{MN}$ and $(\gamma_I)_{MN}$ are the SO(5,5) γ-matrices satisfying

$$
\left(\gamma_I\right)^{MP}\left(\gamma_J\right)_{NP} + \left(\gamma_J\right)^{MP}\left(\gamma_I\right)_{NP} = 2\,\eta_{IJ}\delta_P^M\,. \tag{5.10}
$$

We thus find the generalised metric and its inverse in the **16** are given by

$$
\begin{aligned}
\mathcal{M}_{MN} &= \frac{1}{\sqrt{2}}\Big(4\,\kappa^{-6}\,\hat{J}^u{}_M\hat{J}_{u\,N} - \kappa^{-2}\left(\gamma^I\right)_{MN}\hat{K}_I\\
&\quad - \frac{1}{4!}\,\kappa^{-6}\epsilon^{uvwx}\left(\gamma_I\right)_{MP}\left(\gamma_J\right)_{NQ}\left(\gamma^{IJ}\right)^S{}_R J_u{}^P J_v{}^Q J_w{}^R \hat{J}_{x,S}\Big),\\
\mathcal{M}^{MN} &= \frac{1}{\sqrt{2}}\Big(2\,\kappa^{-2}J_u{}^M J^{u\,N} - \kappa^{-2}\left(\gamma_I\right)^{MN}K^I\\
&\quad - \frac{2}{4!}\kappa^{-10}\epsilon_{uvwx}\left(\gamma_I\right)^{MP}\left(\gamma_J\right)^{NQ}\left(\gamma^{IJ}\right)^S{}_R\hat{J}^u{}_P\hat{J}^v{}_Q J_w{}^R\hat{J}_{x,S}\Big),
\end{aligned} \tag{5.11}
$$

where $\hat{J}_{u\,M}$ is defined in (3.10), and is given explicitly by

$$
\hat{J}^u{}_M = \frac{1}{2}\left(\gamma^I\right)_{MN}\hat{K}_I J^{u\,N}\,. \tag{5.12}
$$

Similarly, the generalised metric in the **10** is given by

$$
\mathcal{M}_{IJ} = \left(\frac{1}{4!}\epsilon^{uvwx}\left(\gamma_{IK}\right)_M{}^N\left(\gamma_J{}^K\right)_P{}^Q J_u{}^M\hat{J}_{v,N}J_w{}^P\hat{J}_{x,Q} + \kappa^{-4}K_IK_J + \kappa^{-4}\hat{K}_I\hat{K}_J\right). \tag{5.13}
$$

Just as in SL(5), there is a scaling degree of freedom which is generated by the minimal consistent truncation (3.11). Thus, the coefficients above can only be defined once those in the differential conditions (3.2) are fixed, or vice versa. Once one set of coefficients is fixed, the other can be obtained either by comparison with known AdS vacua, by a careful analysis of the ExFT BPS equations or by reducing the ExFT action to that of gauged SUGRA upon applying the consistent truncation.

We can now explicitly compute the relationship between $F_{\mu\nu\rho}$ and $\tilde{F}_{\mu\nu\rho}$ in (4.13). Using the expression for the generalised metric (5.13) and the scalar truncation Ansatz (4.6), we find

$$\begin{aligned}\langle \mathcal{M}^{IJ} \rangle \hat{K}_J &= X^{-4}(x)\, K^I(Y)\,, \\ \langle \mathcal{M}^{IJ} \rangle K_J &= X^{4}(x)\, \hat{K}^I(Y)\,,\end{aligned} \tag{5.14}$$

where $\langle \mathcal{M}_{IJ} \rangle$ denotes the generalised metric with the truncation Ansatz plugged in, i.e. that computed from $\mathcal{J}_u$, $\mathcal{K}$ and $\hat{\mathcal{K}}$ of (4.6). Therefore, the twisted self-duality equation (2.18) becomes

$$\tilde{G}_{(3)} = X^{-4} \star_6 \tilde{F}_{(3)}\,. \tag{5.15}$$

6 AdS$_7$ vacua from massive IIA supergravity

As shown in [30] and reviewed in section 3, supersymmetric AdS$_7$ vacua are characterised by three nowhere-vanishing generalised vector fields $J_u \in \Gamma(\mathcal{R}_1)$, transforming as a triplet of $\mathrm{SO}(3)_R$, and a nowhere-vanishing generalised tensor $\hat{K} \in \Gamma(\mathcal{R}_3)$, transforming as a singlet of $\mathrm{SO}(3)_R$. The differential conditions involve a constant totally antisymmetric 3-index tensor Λ_{uvw} which therefore takes the form

$$\Lambda_{uvw} = \sqrt{-c}\, \epsilon_{uvw}\,, \tag{6.1}$$

where the constant c is related to minus the seven-dimensional cosmological constant. The precise relation between c and the cosmological constant, or, equivalently, the AdS$_7$ radius, can be found from the ExFT BPS equations and using the formula for the generalised metric (5.6), or by comparison to known AdS$_7$ vacua of 10-/11-dimensional SUGRA. By comparing to the AdS$_7 \times S^4$ vacuum of 11-dimensional SUGRA, we find $\Lambda_{uvw} = 2\sqrt{2}\, R^{-1} \epsilon_{uvw}$, where R is the AdS$_7$ radius. Plugging this into the differential conditions (3.2), they become

$$\begin{aligned}\mathcal{L}_{J_u} J_v &= -\frac{2\sqrt{2}}{R}\, \epsilon_{uvw} J^w\,, \\ \mathcal{L}_{J_u} \hat{K} &= 0\,, \\ d\hat{K} &= \frac{2}{R} K\,,\end{aligned} \tag{6.2}$$

where K is defined via

$$J_u \wedge J_v = \delta_{uv}\, K\,. \tag{6.3}$$

6.1 Half-maximal structure

Here we are interested in studying AdS$_7$ vacua of massive IIA supergravity. As we discussed in section 3, the $\mathrm{SO}(d-1)_R = \mathrm{SO}(3)_R$ symmetry must be generated by spacetime Killing

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vectors. This suggests that the vacua are of the form $\text{AdS}_7 \times S^2 \times I$, with the Killing vectors on S^2 generating the $\text{SU}(2)_R$ symmetry. As we explain in appendix A.1, the generalised vector fields J_u and tensor $\hat{K}$ are formal sums of internal spacetime vector fields and differential forms as follows

$$\begin{aligned} J_u &= V_u + \lambda_u + \sigma_u + \phi_u \,, \\ \hat{K} &= \omega_{(0)} + \omega_{(2)} + \omega_{(3)} \,, \end{aligned} \tag{6.4}$$

where V_u, λ_u, σ_u and ϕ_u are the vector, 1-form, 2-form and scalar parts of J_u, while $\omega_{(p)}$ are the p-forms appearing in $\hat{K}$. Similarly, $K = \frac{1}{3} J_u \wedge J^u \in \Gamma(\mathcal{R}_2)$ is a formal sum of differential forms

$$K = \bar{\omega}_{(0)} + \bar{\omega}_{(1)} + \bar{\omega}_{(3)} \,, \tag{6.5}$$

where $\bar{\omega}_{(p)}$ are p-forms.

In terms of the above, the wedge products (2.10), (2.12) appearing in the algebraic conditions (3.1) become

$$\begin{aligned} J_u \wedge J_v &= 2\imath_{V_{(u}} \lambda_{v)} - 2\left(\lambda_{(u}\phi_{v)} + \imath_{V_{(u}}\sigma_{v)}\right) - 2\lambda_{(u} \wedge \sigma_{v)} \,, \\ \hat{K} \wedge K &= \omega_{(0)}\bar{\omega}_{(3)} + \omega_{(1)} \wedge \bar{\omega}_{(2)} + \omega_{(3)}\bar{\omega}_{(0)} \,, \end{aligned} \tag{6.6}$$

while the quadratic algebraic constraint on $\hat{K}$ is automatically fulfilled for $\text{SL}(5)$.

The differential operators appearing in the differential conditions (6.2) are modified as described in [19, 20] to capture the Romans mass, m, of massive IIA SUGRA. We explain in detail how to do this in appendix A.3. Including the Romans mass, and thus using equations (A.8), (A.10), (A.11), the differential operators appearing in the conditions (6.2) become

$$\begin{aligned} \mathcal{L}_{J_u} J_v &= L_{V_u} V_v + L_{V_u} \lambda_v + L_{V_u} \sigma_v + L_{V_u} \phi_v \\ &\quad + \imath_{V_v}\left(m\lambda_u - d\phi_u\right) - \imath_{V_v}\left(d\lambda_u\right) - \imath_{V_v}\left(d\sigma_u\right) + \phi_v\left(d\lambda_u\right) + \lambda_v \wedge \left(m\lambda_u - d\phi_u\right) \,, \\ \mathcal{L}_{J_u} \hat{K} &= L_{V_u}\omega_{(0)} + L_{V_u}\omega_{(2)} + L_{V_u}\omega_{(3)} \\ &\quad - \omega_{(0)}\left(d\lambda_u\right) - \omega_{(0)}\left(d\sigma_u\right) - \omega_{(2)} \wedge \left(m\lambda_u - d\phi_u\right) \,, \\ d\hat{K} &= -d\omega_{(0)} + d\omega_{(2)} \,. \end{aligned} \tag{6.7}$$

To describe supersymmetric AdS_7 vacua, we must therefore find the vector fields and differential forms satisfying the above algebraic and differential conditions. In doing so, we will use the differential equations

$$\mathcal{L}_{J_u} J_v = -\frac{2\sqrt{2}}{R}\epsilon_{uvw} J^w \,, \qquad \mathcal{L}_{J_u}\hat{K} = 0 \,, \tag{6.8}$$

as a guiding principle. These imply that the J_u's must transform as a triplet under $\text{SU}(2)_R$ and $\hat{K}$ as a singlet under $\text{SU}(2)_R$, which as we discussed before is generated by the Killing vector fields on S^2. Therefore, we will construct J_u out of spacetime tensors on $S^2 \times I$ that are triplets of $\text{SU}(2)_R$ as generated by the Killing vectors on S^2, and similarly $\hat{K}$ out of differential forms that are singlets of $\text{SU}(2)_R$.

In fact, the above decomposition (6.4), (6.5) of the generalised tensors in terms of vector fields and differential forms on the internal space is only true locally, because the generalised tangent bundles are twisted by the internal gauge potentials of the IIA supergravity, in this case the three-form potential C, two-form potential, B, and one-form potential, A. The gauge potentials mix the different components of the generalised tensors, for example, if $A = B = 0$ but $C \neq 0$, then

$$\sigma_u = \hat{\sigma}_u + \imath_{V_u} C \,, \qquad \omega_{(3)} = \omega_{(3)} + \omega_{(0)}\, C \,, \qquad \bar{\omega}_{(3)} = \hat{\bar{\omega}}_{(3)} - \bar{\omega}_{(0)}\, C \,, \tag{6.9}$$

where $\hat{\sigma}_u$, $\omega_{(3)}$, $\hat{\bar{\omega}}_{(3)}$ are the globally well-defined 2-forms and 3-forms, respectively, while σ_u, $\omega_{(3)}$ and $\bar{\omega}_{(3)}$ are only local 2-forms and 3-forms. Therefore, to construct the J_u and $\hat{K}$ we must understand what the possible form of the gauge potentials is. Since their field strengths must be invariant under the $\mathrm{SU}(2)_R$ symmetry, the gauge potentials must take the form

$$dB = R^3\, f(z)\, vol_{S^2} \wedge dz \,, \qquad dA = R^2\, l(z)\, vol_{S^2} \,, \tag{6.10}$$

for some functions $f(z)$ and $l(z)$, where z labels the coordinate on the interval I and vol_{S^2} is the volume form on S^2, see also appendix C for our S^2 conventions. C is always pure gauge since the internal space is three-dimensional. Moreover, we can choose a gauge such that

$$B = R^3\, F(z)\, vol_{S^2} \,, \tag{6.11}$$

with $\frac{dF(z)}{dz} = f(z)$, so that B is constructed from well-defined differential forms on S^2 and I. On the other hand, A cannot be written in terms of well-defined differential forms on S^2 since it necessarily breaks the R-symmetry. This implies that we can automatically cater for the twisting by the two-form potential by writing the most general J_u and $\hat{K}$ built of out spacetime tensors on S^2 and I. On the other hand, the twist by A will break the $\mathrm{SU}(2)_R$ symmetry and, therefore, we will keep track of it explicitly.

In particular, we will write $\phi_u = \hat{\phi}_u + \imath_{V_u} A$ and $\sigma_u = \hat{\sigma}_u + \lambda_u \wedge A$ and $\omega_{(3)} = \hat{\omega}_{(3)} + \omega_{(2)} \wedge A$, where $\hat{\phi}_u$, $\hat{\sigma}_u$ and $\omega_{(3)}$ are spacetime tensors on $S^2 \times I$ that respect the $\mathrm{SU}(2)_R$ symmetry. In terms of the hatted objects, the differential operators appearing in the differential conditions become

$$\begin{aligned}
\mathcal{L}_{J_u} J_v &= L_{v_u} V_v + L_{V_u}\lambda_v + L_{V_u}\sigma_v + L_{V_u}\hat{\phi}_v \\
&\quad + \imath_{V_v}\left(m\lambda_u + \imath_{V_u} dA - d\hat{\phi}_u\right) - \imath_{V_v}\left(d\lambda_u\right) - \imath_{V_v}\left(d\hat{\sigma}_u - \lambda_u \wedge dA\right) + \phi_v\left(d\lambda_u\right) \\
&\quad + \lambda_v \wedge \left(m\lambda_u + \imath_{V_u} dA - d\phi_u\right) + \left(L_{V_u}\lambda_v - \imath_{V_v} d\lambda_u\right) \wedge A + \imath_{[V_u,V_v]} A \,, \\
\mathcal{L}_{J_u}\hat{K} &= L_{V_u}\omega_{(0)} + L_{V_u}\omega_{(2)} + L_{V_u}\hat{\omega}_{(3)} + L_{V_u}\omega_{(2)} \wedge A - \omega_{(0)} d\lambda_u \wedge A \\
&\quad - \omega_{(0)}\left(d\lambda_u\right) - \omega_{(0)}\left(d\hat{\sigma}_u - \lambda_u \wedge dA\right) - \omega_{(2)} \wedge \left(m\lambda_u + \imath_{V_u} dA - d\hat{\phi}_u\right) \,, \\
d\hat{K} &= -d\omega_{(0)} + d\omega_{(2)} \,.
\end{aligned} \tag{6.12}$$

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The most general J_u we can construct that is compatible with the $\mathrm{SU}(2)_R$ symmetry is

$$\begin{aligned} J_u = & \frac{2\sqrt{2}}{R} v_u + \frac{R}{4}\left(k(z)\, y_u\, dz + g(z)\, dy_u + r(z)\, \theta_u\right) - \frac{R}{2} p(z)\, y_u \\ & + \frac{R^3}{16\sqrt{2}}\left(n(z)\, y_u\, vol_{S^2} + h(z)\, \theta_u \wedge dz + v(z)\, dy_u \wedge dz\right) \\ & + \frac{2\sqrt{2}}{R}\imath_{v_u} A + \frac{R}{4}\left(k(z)\, y_u\, dz + g(z)\, dy_u + r(z)\, \theta_u\right) \wedge A\,, \end{aligned} \tag{6.13}$$

where $k(z)$, $g(z)$, $p(z)$, $n(z)$, $h(z)$, $v(z)$ and $r(z)$ are at this stage arbitrary functions of z, the coordinate on I, y_u are a triplet of functions on S^2, v_u are the Killing vector fields on S^2 and θ_u are 1-forms on S^2. The objects on S^2 are defined in appendix C. The algebraic conditions impose

$$\begin{aligned} J_u = & \frac{2\sqrt{2}}{R} v_u + \frac{R}{4}\left(-\frac{h(z)}{p(z)} y_u\, dz + g(z)\, dy_u\right) - \frac{R}{2} p(z)\, y_u \\ & + \frac{R^3}{16\sqrt{2}}\left(p(z)\, g(z)\, y_u\, vol_{S^2} + h(z)\, \theta_u \wedge dz + v(z)\, dy_u \wedge dz\right) \\ & + \frac{2\sqrt{2}}{R}\imath_{v_u} A + \frac{R}{4}\left(-\frac{h(z)}{p(z)} y_u\, dz + g(z)\, dy_u\right) \wedge A\,, \end{aligned} \tag{6.14}$$

such that K defined via

$$J_u \wedge J_v = \delta_{uv}\, K\,, \tag{6.15}$$

is given by

$$K = -\frac{R^2}{4} h(z)\, dz + \frac{R^4\, g(z)\, h(z)}{32\sqrt{2}} vol_{S^2} \wedge dz\,. \tag{6.16}$$

Furthermore, the most general $\hat{K}$ constructed from R-symmetry singlets is given by

$$\begin{aligned} \hat{K} = & \frac{R}{2}\, s(z) + \frac{R^3}{16\sqrt{2}}\left(g(z)\, s(z) - t(z)\right) vol_{S^2} + R^5\, u(z) vol_{S^2} \wedge dz \\ & + \frac{R^3}{16\sqrt{2}}\left(g(z)\, s(z) - t(z)\right) vol_{S^2} \wedge A\,. \end{aligned} \tag{6.17}$$

The algebraic condition $J_u \wedge J^u \wedge \hat{K} > 0$ then becomes

$$\frac{R^5}{64\sqrt{2}}\, h(z)\, t(z)\, vol_{S^2} \wedge dz > 0\,. \tag{6.18}$$

While this suggests that we must have $h(z)\, t(z) > 0$, this is not true at the endpoints of the interval parameterised by z. There we can in fact have $h(z)\, t(z) = 0$. Thus, we must impose

$$h(z)\, t(z) \geq 0\,, \tag{6.19}$$

with possible equality at the boundary. This ensures that the metric is non-singular everywhere. For holographic applications, we also want to impose that the internal space is compact by requiring that the S^2 shrinks at the endpoints of I, which will further refine (6.19). However, to determine the precise conditions, we must first construct the SUGRA fields of the AdS_7 solution.

With (6.14) and (6.17), the differential conditions (6.12) reduce to

$$\begin{aligned} m\lambda_u + \imath_{V_u} dA - d\hat{\phi}_u &= 0\,, \\ d\lambda_u &= 0\,, \\ d\hat{\sigma}_u - \lambda_u \wedge dA &= 0\,, \\ d\omega_{(0)} + \frac{2}{R}\bar{\omega}_{(1)} &= 0\,, \\ d\omega_{(2)} - \frac{2}{R}\bar{\omega}_{(3)} &= 0\,, \end{aligned} \tag{6.20}$$

where, as we discussed above, R-symmetry implies that

$$dA = R^2 l(z)\, vol_{S^2}\,. \tag{6.21}$$

Explicitly, the differential conditions imply the following set of ODEs

$$\dot{g} = -\frac{h}{p}\,, \qquad 2\,p\,\dot{p} = m\,h\,, \qquad \dot{s} = h\,, \qquad \dot{t} = -\frac{h\,s}{p}\,, \qquad l = -\frac{p}{4\sqrt{2}} - \frac{m\,g}{8\sqrt{2}}\,. \tag{6.22}$$

Note that the functions $u(z)$ and $v(z)$ do not appear in the differential conditions. This is due to the fact that they can be removed by gauge transformations of the gauge potentials A and C, as can be seen from (6.9) and (6.13). Thus, we can, and will, set $u = v = 0$ without loss of generality.

Using the ODEs (6.22) and having set $u = v = 0$ by gauge transformations, the half-maximal structures simplify to

$$\begin{aligned} J_u &= \frac{2\sqrt{2}}{R} v_u + \frac{R}{4} d\,(g\,y_u) - \frac{R}{2}\,p\,dy_u + \frac{R^3}{16\sqrt{2}} p\,(d\,(g\,\theta_u) - g\,y_u\,vol_{S^2}) \\ &\quad + \frac{2\sqrt{2}}{R}\imath_{v_u} A + \frac{R}{4} d\,(g\,y_u) \wedge A\,, \\ \hat{K} &= \frac{R}{2}\,s + \frac{R^3}{16\sqrt{2}}\,(g\,s - t)\,vol_{S^2} + \frac{R^3}{16\sqrt{2}}\,(g\,s - t)\,vol_{S^2} \wedge A\,, \end{aligned} \tag{6.23}$$

with

$$dA = -\frac{R^2}{4\sqrt{2}}\left(p + \frac{m}{2}\,g\right) vol_{S^2}\,. \tag{6.24}$$

Moreover, we can redefine the z coordinate to set $h(z)$ to anything we like. There are two convenient choices that help us solve the ODEs (6.22).

Choice 1. The first is to take $h(z) = p(z)$ so that we can integrate the equation $\dot{g} = -\frac{h}{p}$ to set $g = -z$, where we absorb the integration constant by shifting z. Then, the remaining ODEs are solved by

$$s = -\dot{t}\,, \qquad p = -\ddot{t}\,, \qquad l = \frac{\ddot{t}}{4\sqrt{2}} - \frac{m\,g}{8\sqrt{2}}\,, \tag{6.25}$$

where $t(z)$ must satisfy

$$\dddot{t} = -\frac{m}{2}\,. \tag{6.26}$$

Finally, with this gauge, the regularity condition (6.19) becomes

$$t(z)\,\ddot{t}(z) \leq 0\,. \tag{6.27}$$

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Choice 2. The second choice is to simply take $h(z) = 1$ and integrate the equation $\dot{s} = 1$ to set $s = z$ without loss of generality. The remaining ODEs now become

$$\dot{g} = -\frac{1}{p}\,, \qquad 2\,p\,\dot{p} = m\,, \qquad \dot{t} = -\frac{z}{p}\,, \qquad l = -\frac{p}{4\sqrt{2}} - \frac{m\,g}{8\sqrt{2}}\,. \tag{6.28}$$

The functions g and t are therefore determined in terms of p, its integral and its derivatives, and where p satisfies

$$\frac{\partial p^2}{\partial z} = m\,. \tag{6.29}$$

The regularity condition (6.19) becomes

$$t(z) \geq 0\,, \tag{6.30}$$

with equality only possible at ∂I.

To compare to the literature, especially the form of AdS_7 solutions given in [59], it is worthwhile to introduce

$$q = \frac{p}{\sqrt{2}}\,, \qquad \bar{y} = \frac{9}{4}z\,, \qquad \sqrt{\beta} = \frac{81}{4\sqrt{2}}t\,, \tag{6.31}$$

which now satisfy

$$\frac{\partial q^2}{\partial \bar{y}} = \frac{9}{2}m\,, \qquad q = -4\,\bar{y}\,\frac{\sqrt{\beta}}{\beta'}\,, \tag{6.32}$$

where $'$ is our shorthand notation for $\frac{\partial}{\partial \bar{y}}$. Moreover, we note the following identities

$$\frac{t}{p} = -\frac{1}{81}\frac{\beta'}{\bar{y}}\,, \qquad p\,t = -\frac{32}{81}\frac{\beta\,\bar{y}}{\beta'}\,, \qquad z^2 + 2\,p\,t = \frac{16}{81}\bar{y}\left(\bar{y} - \frac{4\beta}{\beta'}\right)\,, \tag{6.33}$$

which will allow us to recover precisely the description of AdS_7 vacua given in [59].

6.2 The supersymmetric AdS_7 vacua

It is now straightforward to compute the SUGRA fields of the supersymmetric AdS_7 vacua. We first plug J_u and $\hat{K}$, given in (6.23), into the generalised metric. We then use the ExFT / IIA dictionary worked out in [17] and summarised in appendix A.2 to read off the supergravity fields. In string frame, the warp factor of the AdS_7 part of the metric is given by [29, 30]

$$f_7 = |g_{\rm int}|^{-1/5}\kappa^2\, e^{4\psi/5}\,, \tag{6.34}$$

where $|g_{\rm int}|$ is the determinant of the internal metric, ψ is the dilaton and $\kappa^5 = \frac{1}{3}J_u \wedge J^u \wedge \hat{K}$.

Without fixing $h(z)$ we therefore find the following SUGRA fields in string frame

$$\begin{aligned}
ds^2_{10} &= \sqrt{\frac{t}{p}}ds^2_{AdS_7} + \frac{R^2}{8}\sqrt{\frac{t}{p}}\left(\frac{p\,t}{s^2+2\,p\,t}ds^2_{S^2} + \frac{h^2}{p\,t}dz^2\right)\,,\\
e^{\psi} &= \frac{2}{R}\left(\frac{t}{p}\right)^{3/4}\frac{1}{\sqrt{s^2+2\,p\,t}}\,,\\
B &= \frac{R^2}{8\sqrt{2}}\left(-g + \frac{s\,t}{s^2+2\,p\,t}\right) vol_{S^2}\,,
\end{aligned} \tag{6.35}$$

with 2-form field strength $F_2 = dA - m\,B_2$ and 3-form field strength $H = dB$ given by

$$
\begin{aligned}
F_2 &= -\frac{R^2}{8\sqrt{2}}\left(2\,p + \frac{m\,s\,t}{s^2+2\,p\,t}\right) vol_{S^2}\,,\\
H_3 &= \frac{R^2}{8\sqrt{2}}\frac{h\,t}{p}\left(\frac{3\,p}{s^2+2\,p\,t} - \frac{m\,s\,t}{(s^2+2\,p\,t)^2}\right) vol_{S^2}\wedge dz\\
&= \frac{2}{R}\left(3\left(\frac{t}{p}\right)^{-1/4} - \frac{m\,s}{s^2+2\,p\,t}\left(\frac{t}{p}\right)^{3/4}\right) vol_{M_3}\,,
\end{aligned}
\tag{6.36}
$$

where vol_{M_3} denotes the volume form on the internal manifold with the metric (6.35). Note that we have the opposite sign convention for our B field to [40] so that the Bianchi identity of F_2 is

$$
dF_2 = -mH_3\,. \tag{6.37}
$$

Choice 1. With the choice $h(z) = p(z)$, the expressions for the AdS$_7$ vacua (6.35) reduce to

$$
\begin{aligned}
ds_{10}^2 &= \sqrt{-\frac{t}{\ddot{t}}}\,ds^2_{AdS_7} + \frac{R^2}{8}\sqrt{-\frac{\ddot{t}}{t}}\left(\frac{t^2}{\dot{t}^2-2\,\ddot{t}\,t}ds^2_{S^2} + dz^2\right)\,,\\
e^{\psi} &= \frac{2}{R}\left(-\frac{t}{\ddot{t}}\right)^{3/4}\frac{1}{\sqrt{\dot{t}^2-2\,\ddot{t}\,t}}\,,\\
B &= \frac{R^2}{8\sqrt{2}}\left(z - \frac{\dot{t}\,t}{\dot{t}^2-2\,\ddot{t}\,t}\right) vol_{S^2}\,,
\end{aligned}
\tag{6.38}
$$

with field strengths

$$
\begin{aligned}
F_2 &= \frac{R^2}{8\sqrt{2}}\left(2\,\ddot{t} + \frac{m\,\dot{t}\,t}{\dot{t}^2-2\,\ddot{t}\,t}\right) vol_{S^2}\,,\\
H_3 &= \frac{R^2}{8\sqrt{2}}\left(\frac{m\,t^2\,\dot{t}}{(\dot{t}^2-2\,t\,\ddot{t})^2} - \frac{t\,\ddot{t}}{\dot{t}^2-2\,t\,\ddot{t}}\right) vol_{S^2}\wedge dz\\
&= \frac{2}{R}\left[3\left(-\frac{t}{\ddot{t}}\right)^{-1/4} + \frac{m\,\dot{t}}{\dot{t}^2-2\,t\,\ddot{t}}\left(-\frac{t}{\ddot{t}}\right)^{3/4}\right] vol_{M_3}\,,
\end{aligned}
\tag{6.39}
$$

where the function t satisfies

$$
\dddot{t} = -\frac{m}{2}\,, \qquad t \geq 0\,, \tag{6.40}
$$

with $t = 0$ at ∂I so that the internal space has no boundaries. For every such function t there is a supersymmetric AdS$_7$ vacuum of massive IIA supergravity. This matches the infinite family of supersymmetric AdS$_7$ vacua of [40] when we set the AdS radius to $R = 2$, and where our variables are related to those of [45] by a rescaling

$$
t = \frac{4\sqrt{2}}{81}\alpha\,, \qquad z = 2\sqrt{2}\,\pi\,\bar{z}\,, \tag{6.41}
$$

where we denote the "z" coordinate of [45] by $\bar{z}$ to distinguish it from our z coordinate.

Choice 2. With the choice $h(z)=1$ and using (6.33), the AdS_7 vacua (6.35) are given by

$$
\begin{aligned}
ds^2_{10} &= \frac{1}{9}\sqrt{-\frac{\beta'}{\bar y}}ds^2_{AdS_7} + \frac{1}{9}\sqrt{-\frac{\beta'}{\bar y}}R^2\left(\frac{\beta/4}{4\,\beta-\beta'\,\bar y}ds^2_{S^2} - \frac{1}{16}\frac{\beta' d\bar y^2}{\beta\,\bar y}\right),\\
e^{\psi} &= R^{-1}\frac{(-\beta'/\bar y)^{5/4}}{6\sqrt{4\,\beta-\beta'\bar y}}\,,\\
H_3 &= \frac{18}{R}\left(-\frac{\bar y}{\beta'}\right)^{1/4}\left(1-\frac{m}{108\,\bar y}\frac{(\beta')^2}{4\,\beta-\beta'\bar y}\right)vol_{M_3}\,,\\
F_2 &= \frac{R^2\,\bar y}{4}\frac{\sqrt{\beta}}{\beta'}\left(4+\frac{m}{18\,\bar y}\frac{(\beta')^2}{4\,\beta-\beta'\bar y}\right)vol_{S^2}\,,
\end{aligned}
\tag{6.42}
$$

Here vol_{M_3} is the internal volume form with respect to the full internal metric in (6.42), and β satisfies the ODE

$$
\frac{\partial q^2}{\partial \bar y} = \frac{9}{2}m\,, \quad \text{with } q = -4\,\bar y\,\frac{\sqrt{\beta}}{\beta'}\,. \tag{6.43}
$$

This matches the AdS vacua in the coordinates of [59] when the AdS radius is set to $R = 2$.

7 AdS_6 vacua from IIB supergravity

As we reviewed in section 3 and was shown in [30], supersymmetric AdS_6 vacua are described in ExFT by four nowhere-vanishing generalied vector fields $J_u \in \Gamma(\mathcal{R}_1)$, transforming as a **4** of SO(4), and a nowhere-vanishing generalised tensor $\hat K \in \Gamma(\mathcal{R}_3)$ which is invariant under SO(4).

Upon defining $\Lambda^u = -\frac{1}{3!}\epsilon^{uvwx}\Lambda_{vxw}$, the differential conditions (3.2) become

$$
\begin{aligned}
\mathcal{L}_{J_u}J_v &= -\epsilon_{uvwx}J^w\Lambda^x\,,\\
\mathcal{L}_{J_u}\hat K &= 0\,,\\
d\hat K &= \frac{2}{3}\Lambda^u J_u\,,
\end{aligned}
\tag{7.1}
$$

We can use a $SO(4)_R$ rotation to write, without loss of generality,

$$
\Lambda_u = \left(0,0,0,\frac{3}{\sqrt{2}\,R}\right)\,, \tag{7.2}
$$

with R the AdS_6 radius. The numerical coefficient in front of R have been fixed by comparing the solutions with known supersymmetric AdS_6 vacua of IIB [41].

Λ_{uvw} breaks the SO(4) symmetry to the $SO(3)_R$ R-symmetry of AdS_6 vacua. Let us therefore write $u = (A,4)$ with $A = 1,2,3$ labelling the vector representation of $SO(3)_R$. With respect to $(A,4)$ the differential conditions become

$$
\begin{aligned}
\mathcal{L}_{J_A}J_B &= -\frac{3}{\sqrt{2}\,R}\,\epsilon_{ABC}J^C\,,\\
\mathcal{L}_{J_A}J_4 &= 0\,,\\
\mathcal{L}_{J_A}\hat K &= 0\,,\\
d\hat K &= \frac{\sqrt{2}}{R}\,J_4\,.
\end{aligned}
\tag{7.3}
$$

Note that the conditions $\mathcal{L}_{J_4} J_u = 0$ and $\mathcal{L}_{J_4} \hat{K} = 0$ are automatically satisfied by $J_4 \propto d\hat{K}$ [30].

7.1 Half-maximal structure

We will now construct the half-maximal structures on the internal space that yields an AdS_6 vacuum. To do this, we will guide ourselves by the differential equations (7.3) determining the AdS vacuum. Recall from section 3 that these imply that the J_u's are generalised Killing vector fields and therefore either consist of a Killing vector field plus a compensating gauge transformation, or consist of a trivial gauge transformation. The latter can be written as dB for some $B \in \Gamma(\mathcal{R}_2)$ and will always generate a vanishing generalised Lie derivative on any vector field. We see from (7.3) that J_4 generates such a trivial gauge transformation, while J_A must generate the $\mathrm{SU}(2)_R$ symmetry of the AdS vacuum and therefore have non-vanishing vector components which generate this symmetry. The generalised tenors $\hat{K}$ and J_4 must be invariant under this R-symmetry.

To generate the $\mathrm{SU}(2)_R$ symmetry we take the internal space to contain an S^2 and on the remaining two-dimensional space, the Riemann surface Σ, we introduce coordinates x^α, $\alpha = 1, \ldots, 2$. We will raise/lower α in a Northwest/Southeast convention by the $\mathrm{SL}(2)$-invariants $\epsilon_{\alpha\beta} = \pm 1$ and $\epsilon^{\alpha\beta} = \pm 1$ with

$$\epsilon^{\alpha\gamma}\epsilon_{\beta\gamma} = \delta^\alpha_\beta \,. \tag{7.4}$$

Thus we write

$$x^\alpha = \epsilon^{\alpha\beta} x_\beta \,, \qquad x_\alpha = x^\beta \epsilon_{\beta\alpha} \,. \tag{7.5}$$

In IIB SUGRA with the conventions in appendix B.1, the J_u's and $\hat{K}$ become formal sums of spacetime vector fields and differential forms as follows

$$\begin{aligned} J_u &= V_u + \lambda_u^\alpha + \sigma_u \,, \\ \hat{K} &= \omega_{(0)}^\alpha + \omega_{(2)} + \omega_{(4)}^\alpha \,, \end{aligned} \tag{7.6}$$

where V_u, ${\lambda_u}^\alpha$ and σ_u denote the vector, 1-form and 3-form parts of J_u, while $\omega_{(p)}$ are p-forms appearing in $\hat{K}$. With our conventions B.1, the wedge products and tensor products appearing in the algebraic conditions (3.1) become

$$\begin{aligned} J_u \wedge J_v &= \sqrt{2}\left(\imath_{V_{(u}}\lambda^\alpha_{v)} + \lambda^\alpha_{(u} \wedge \sigma_{v)} + \left(-\imath_{V_{(u}}\sigma_{v)} - \frac{1}{2}\lambda_{u\,\beta} \wedge \lambda_v^\beta\right)\right) , \\ \hat{K} \otimes \hat{K}|_{R_c} &= \omega_{(2)} \wedge \omega_{(2)} + 2\,\omega_{(0)\alpha}\,\omega_{(4)}^\alpha \,, \\ \hat{K} \wedge K &= \omega_{(2)} \wedge \bar{\omega}_{(2)} + \omega_{(0)\alpha}\,\bar{\omega}_{(4)}^\alpha + \bar{\omega}_{(0)\alpha}\,\omega_{(4)}^\alpha \,, \end{aligned} \tag{7.7}$$

where we defined $K = \frac{1}{4} J_u \wedge J^u = \bar{\omega}_{(0)}^\alpha + \bar{\omega}_{(2)} + \bar{\omega}_{(4)}^\alpha$. Moreover, the differential operators appearing in the differential conditions (7.1) become

$$\begin{aligned} \mathcal{L}_{J_u} J_v &= L_{V_u} V_v + L_{V_u}\sigma_v + L_{V_u}\lambda_v^\alpha \\ &\quad - \imath_{V_v} d\lambda_u^\alpha - \imath_{V_v} d\sigma_u - \lambda_{v\,\beta} \wedge d\lambda_u^\beta \,, \\ \mathcal{L}_{J_u}\hat{K} &= L_{V_u}\omega_{(0)}^\alpha + L_{V_u}\omega_{(2)} + L_{V_u}\omega_{(4)}^\alpha \\ &\quad + \omega_{(0)\beta}\, d\lambda_u^\beta - \omega_{(0)}^\alpha\, d\sigma_u - \omega_{(2)} \wedge d\lambda_u^\alpha \,, \\ d\hat{K} &= -\sqrt{2}\, d\omega_{(2)} + \sqrt{2}\, d\omega_{(0)}^\alpha \,. \end{aligned} \tag{7.8}$$

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As discussed above, the J_A's will need to be formed out of vector fields and differential forms forming $\mathrm{SU}(2)_R$ triplets, while J_4 and $\hat{K}$ will need to be constructed from $\mathrm{SU}(2)_R$-invariant vector fields and differential forms. We will now construct the most general J_A and $\hat{K}$, up to gauge transformations, which transform as an SU(2)-triplet and singlet, and satisfy their algebraic conditions. We then calculate J_4 from $d\hat{K}$ and impose its algebraic condition $J_4 \wedge J_4 = \frac{1}{3} J_A \wedge J_A$ and $J_4 \wedge J_A = 0$ and finally solve the remaining differential conditions

$$\begin{aligned} \mathcal{L}_{J_A} J_B &= -\frac{3}{\sqrt{2}\,R} \epsilon_{ABC} J^C \,, \\ \mathcal{L}_{J_A} J_4 &= 0 \,, \\ \mathcal{L}_{J_A} \hat{K} &= 0 \,. \end{aligned} \tag{7.9}$$

Just like for AdS_7 vacua, we must first ascertain whether the gauge fields of the supergravity can be chosen in a way that respects the $\mathrm{SU}(2)_R$ symmetry and will thus naturally appear in the most general structures we write down, or whether the gauge fields necessarily break the $\mathrm{SU}(2)_R$ symmetry and need to be included by hand as a "twist" term. Since we are considering IIB SUGRA with an internal four-manifold, we will only have 3-form field strengths dC^α which must necessarily be $\mathrm{SU}(2)_R$ singlets. Therefore, they must be given by

$$dC^\alpha = b^\alpha \wedge vol_{S^2} \,, \tag{7.10}$$

for some 1-forms b^α on Σ. Therefore, we can choose a gauge such that locally

$$C^\alpha = c^\alpha vol_{S^2} \,, \tag{7.11}$$

for functions c^α on Σ which are $\mathrm{SU}(2)_R$ singlets. Hence, the gauge potentials can be chosen to be $\mathrm{SU}(2)_R$ invariant and will naturally appear in the most general structures we write down. This is in contrast with the mIIA AdS_7 vacua we studied in section 6, where the R-R 1-form potential had to be included via a "twist" term.

The most general J_A we can construct as an $\mathrm{SU}(2)_R$-triplet is

$$\begin{aligned} J_A = \frac{1}{\sqrt{2}} \bigg(&\frac{3}{R} v_A + 4\, c_6\, R\, k^\alpha dy_A + 4\, c_6\, R\, y_A m^\alpha + n^\alpha \theta_A + \frac{16\, c_6^2\, R^3}{3} y_A h \wedge vol_{S^2} \\ &+ \frac{16\, c_6^2\, R^3}{3} l\, \theta_A \wedge vol_\Sigma + f\, dy_A \wedge vol_\Sigma \bigg) \,, \end{aligned} \tag{7.12}$$

where v_A are the Killing vectors, θ_A are 1-forms and vol_{S^2} is the volume form on S^2 as defined in appendix C and

$$vol_\Sigma = \frac{1}{2} \epsilon_{\alpha\beta} dx^\alpha \wedge dx^\beta \,. \tag{7.13}$$

l, k^α and n^α are at this stage arbitrary functions on Σ, while $h = h_\alpha dx^\alpha$ and $m^\alpha = m^\alpha{}_\beta\, dx^\beta$ are 1-forms on Σ. c_6 is a constant. It and the other numerical coefficients in front of R have been introduced for later convenience. We can further simplify (7.12) by using generalised diffeomorphisms, i.e. a combination of diffeomorphisms and gauge transformations: we can use the generalised vector field

$$V = \chi \wedge vol_{S^2} \,, \tag{7.14}$$

where χ is a one-form on Σ satisfying

$$d\chi = -\frac{R}{3} f\, vol_\Sigma\,, \tag{7.15}$$

to remove the term in J_A that depends on the function f by acting with $\mathcal{L}_V J_A$. In fact by working out the explicit twisting of the generalised tangent bundle by gauge potentials, e.g. using appendix E of [60], one sees that this generalised diffeomorphism corresponds to a gauge transformation of the R-R 4-form.

We now impose the algebraic conditions (3.1) such that the functions appearing in J_A are now no longer all independent. As a result, we find

$$J_A = \frac{1}{\sqrt{2}}\left(\frac{3}{R}v_A + 4\,c_6\,R\,(y_A m^\alpha + k^\alpha dy_A) + \frac{16\,c_6^2\,R^3}{3}\left(|m|\,\theta_A\wedge vol_\Sigma - y_A k^\beta m_\beta \wedge vol_{S^2}\right)\right), \tag{7.16}$$

where $|m| = \frac{1}{2} m_{\alpha\beta} m^{\alpha\beta}$.

Next, we construct $\hat{K}$ such that is an SU(2)$_R$-invariant and satisfies $\hat{K}\wedge\hat{K} = 0$ and $J_A \wedge J^A \wedge \hat{K} > 0$. We find the unique combination

$$\hat{K} = \frac{1}{\sqrt{2}}\left(4\,c_6\,p_\alpha + \frac{2\,c_6 R^2}{3} q_\alpha vol_{S^2}\wedge vol_\Sigma - \frac{16\,c_6^2 R^2}{3}\left(r + p_\beta k^\beta\right) vol_2 + \frac{p_\beta q^\beta}{r + p_\gamma k^\gamma} vol_\Sigma\right), \tag{7.17}$$

in terms of r, p_α and q_α which are so far arbitrary functions of x^α. However, just as for J_A we can use gauge transformations to further simplify this expression. A particular class of gauge transformations corresponds to shifts of $\hat{K}$ by d-exact terms,

$$\hat{K} \sim \hat{K} + dQ\,, \tag{7.18}$$

where $Q \in \Gamma(\mathcal{R}_3)$. Taking

$$Q = Q^{\alpha\beta} vol_{S^2}\wedge dx_\beta\,, \tag{7.19}$$

with $\partial_\beta Q^{\alpha\beta} \sim q^\alpha$ (with appropriate coefficients) we see that we can remove the functions q^α in (7.17). Thus, we are left with the general $\hat{K}$ up to gauge transformations given by

$$\hat{K} = \frac{1}{\sqrt{2}}\left(4\,c_6\,p_\alpha - \frac{16\,c_6^2 R^2}{3}\left(r + p_\beta k^\beta\right) vol_2\right). \tag{7.20}$$

The algebraic condition $J_u \wedge J^u \wedge \hat{K} > 0$ is equivalent to $J_A\wedge J^A\wedge\hat{K} > 0$ once we impose the remaining algebraic conditions. Therefore, we require

$$J_A \wedge J^A \wedge \hat{K} = 128\,c_6^4 R^4 r\,|m|\,vol_{S^2}\wedge vol_\Sigma > 0\,, \tag{7.21}$$

which implies that $r\,|m| \geq 0$ with equality at the points on Σ where the S^2 degenerates. From $\hat{K}$ we find

$$J_4 = \frac{R}{\sqrt{2}} d\hat{K} = \frac{1}{\sqrt{2}}\left(4\,c_6 R\,dp^\alpha - \frac{16\,c_6^2 R^3}{3} d\left(r + p_\beta k^\beta\right)\wedge vol_{S^2}\right). \tag{7.22}$$

The algebraic conditions

$$J_4 \wedge J_4 = \frac{1}{3} J_A \wedge J^A \,, \qquad J_4 \wedge J_A = 0 \,, \tag{7.23}$$

now impose

$$\begin{aligned} m_\alpha \wedge dp^\alpha &= 0 \,, \\ m^\alpha \wedge m^\beta &= dp^\alpha \wedge dp^\beta \,, \\ dr + p_\alpha dk^\alpha &= 0 \,. \end{aligned} \tag{7.24}$$

Note that the final condition can be used to simplify the expression of J_4

$$J_4 = \frac{1}{\sqrt{2}} \left(4\, c_6 R\, dp^\alpha - \frac{16\, c_6^2 R^3}{3} k_\beta\, dp^\beta \wedge vol_{S^2} \right) . \tag{7.25}$$

Finally, we are left to solve the differential conditions (7.9). Using the explicit expression of the generalised Lie derivative (7.8) and the fact that J_A, J_4 and $\hat{K}$ are SU(2) triplets, singlet and singlets, respectively, these equations reduce to

$$\begin{aligned} \imath_{V_A} d\lambda^\alpha_B &= 0 \,, \\ \imath_{V_A} d\sigma_B + \lambda_{A\,\alpha} \wedge d\lambda^\alpha_B &= 0 \,, \\ \lambda_{4\,\alpha} \wedge d\lambda^\alpha_A &= 0 \,, \\ \omega_{(0)\alpha} d\lambda^\alpha_A &= 0 \,, \\ \omega^\alpha_{(0)} d\sigma_A + \omega_{(2)} \wedge d\lambda^\alpha_A &= 0 \,. \end{aligned} \tag{7.26}$$

For our J_A's and $\hat{K}$ these further simplify to

$$d\lambda^\alpha_A = d\sigma_A = 0 \,, \tag{7.27}$$

which implies $m^\alpha = -dk^\alpha$.

Thus, we find that

$$\begin{aligned} J_A &= \frac{1}{\sqrt{2}} \left(\frac{3}{R} v_A + 4\, c_6 R\, d\left(k^\alpha\, y_A\right) + \frac{8\, c_6^2 R^3}{3} d\left(k^\alpha \theta_A \wedge dk_\alpha\right) \right) , \\ J_4 &= \frac{1}{\sqrt{2}} \left(4\, c_6 R\, dp^\alpha - \frac{16\, c_6^2 R^3}{3}\, k_\beta\, dp^\beta \wedge vol_{S^2} \right) , \\ \hat{K} &= \frac{1}{\sqrt{2}} \left(4\, c_6\, p_\alpha - \frac{16\, c_6^2 R^2}{3} \left(r + p_\beta k^\beta \right) vol_{S^2} \right) , \end{aligned} \tag{7.28}$$

where k^α and p^α are any SL(2)-doublets of functions on Σ subject to the differential conditions

$$dk^\alpha \wedge dk^\beta = dp^\alpha \wedge dp^\beta \,, \qquad dk^\alpha \wedge dp_\alpha = 0 \,, \tag{7.29}$$

and r is defined up to an integration constant by

$$dr = -p_\alpha\, dk^\alpha \,. \tag{7.30}$$

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The condition (7.21) implies

$$r|dk|vol_\Sigma \wedge vol_{S^2} > 0\,, \tag{7.31}$$

where $|dk| = \frac{1}{2}\partial_\alpha k_\beta \partial^\alpha k^\beta$. This seems to suggest that $r|dk| > 0$ but care needs to be taken at the boundaries of Σ. Instead, we must have

$$r|dk| \geq 0\,, \tag{7.32}$$

with equality only possible at the boundaries of Σ. In fact, as discussed in [41, 43], and as will become appart from the explicit SUGRA solution given in section 7.2 in order for the internal four-manifold not to have a boundary, we must have

$$r = 0\,, \qquad |dk| = 0\,, \tag{7.33}$$

on $\partial\Sigma$.

At this stage, one might wonder how the quadratic differential conditions (7.29) can underlie supersymmetric AdS vacua, which ought to be described by a first-order BPS equation. The answer is that we still have residual diffeomorphism symmetry on the Riemann surface Σ that can be used to turn (7.29) into first-order differential equations. We will show how to do this after calculating the supergravity fields from the structures.

We conclude this section by giving the explicit expressions for the objects $K = \frac{1}{4}J_u \wedge J^u$ and $\kappa^4 = K \wedge \hat{K}$, which appear in the truncation Ansatz (4.11). They are given by

$$\begin{aligned} K &= -8\sqrt{2}\, c_6^2 R^2\, |dk| \left(vol_\Sigma + \frac{4\, c_6 R^2}{3} vol_{S^2} \wedge vol_\Sigma \right)\,, \\ \kappa^4 &= \frac{128}{3} c_6^4 R^4\, r\, |dk|\, vol_{S^2} \wedge vol_\Sigma\,. \end{aligned} \tag{7.34}$$

7.2 The supersymmetric AdS$_6$ vacua

We will now compute the supergravity background corresponding to the half-maximal structures (7.28). The supergravity fields are encoded in the generalised metric (5.11), (5.13) as detailed in appendix B.2. Moreover, the AdS$_6$ part of the metric is warped by the factor [30]

$$f_6 = |g_{\text{int}}|^{-1/4} \kappa^2\,. \tag{7.35}$$

Thus, we find the following background

$$\begin{aligned} ds^2 &= \frac{4\, r^{5/4}\, \Delta^{1/4}\, c_6 R^2}{3^{3/4} |dk|^{1/2}} \left[\frac{3}{r} ds^2_{AdS_6} + \frac{|dk|^2}{\Delta} ds^2_{S^2} + \frac{1}{4\, r^2} dk^\alpha \otimes dp_\alpha \right]\,, \\ C_{(2)}{}^\alpha &= -\frac{4\, c_6 R^2}{3}\, vol_{S^2} \left(k^\alpha + \frac{r\, p_\gamma\, \partial_\beta k^\gamma\, \partial^\beta p^\alpha}{\Delta} |dk| \right)\,, \\ H_{\alpha\beta} &= \frac{1}{\sqrt{3}\,\Delta} \left(\frac{|dk|}{\sqrt{r}} p_\alpha p_\beta + 3\sqrt{r}\, \partial_\gamma k_\alpha \partial^\gamma p_\beta \right)\,, \end{aligned} \tag{7.36}$$

where

$$\Delta = 3\, r |dk|^2 + |dk| p_\gamma p_\delta \partial_\sigma k^\gamma \partial^\sigma p^\delta\,, \qquad |dk| = \frac{1}{2} \partial_\alpha k_\beta \partial^\alpha k^\beta\,. \tag{7.37}$$

The solutions are completely determined by the two pairs of functions p^α and k^α on Σ satisfying

$$dk^\alpha \wedge dk^\beta = dp^\alpha \wedge dp^\beta \,, \qquad dk^\alpha \wedge dp_\alpha = 0 \,. \tag{7.38}$$

r is defined in terms of these functions as

$$dr = -p_\alpha dk^\alpha \,. \tag{7.39}$$

In order to have a compact internal space, we must require that the S^2 shrinks on the boundary of Σ while the warp factor and the metric on Σ remain non-singular. From the explicit metric (7.36), one can easily see that this requires

$$r = |dk| = 0 \,, \tag{7.40}$$

on $\partial\Sigma$.

We will now show that the differential equations for k^α and p^α can be turned into first-order PDEs by coordinate choices. In particular, we can always use diffeomorphisms to make the metric on Σ conformally flat. From (7.36) we see that this requires

$$\partial_1 k^\alpha \partial_1 p_\alpha = \partial_2 k^\alpha \partial_2 p_\alpha \,, \qquad \partial_1 k^\alpha \partial_2 p_\alpha = 0 \,. \tag{7.41}$$

Together with (7.29), and imposing the condition (7.32), the differential conditions become the Cauchy-Riemann equations

$$dk^\alpha = I \cdot dp^\alpha \,, \tag{7.42}$$

where $I_\alpha^\beta = \delta_{\alpha\gamma}\epsilon^{\gamma\beta}$ is a complex structure on Σ. Therefore, p^α and k^α are the real and imaginary parts of two holomorphic functions on Σ

$$f^\alpha = -p^\alpha + i\, k^\alpha \,. \tag{7.43}$$

We now recover the description of supersymmetric AdS_6 vacua of [41] by identifying our holomorphic functions with the $\mathcal{A}_\pm$ of [41] via

$$\mathcal{A}_\pm = i\, f^1 \pm f^2 \,. \tag{7.44}$$

We present a dictionary between our objects and those of [41], as well as [50], in appendix D. As discussed in [42–44] these local solutions can be extended to globally regular solutions by including a boundary of the Riemann surface on which the holomorphic functions f^α have poles, and by introducing SL(2) monodromies.

8 Minimal consistent truncations

As shown in [30] and reviewed in section 3.1, given the half-supersymmetric structures describing an AdS vacua, one can automatically construct a consistent truncation around it containing a gravity multiplet and a scalar. This method was applied in [51] to construct the minimal consistent truncations around the supersymmetric AdS_7 and AdS_6 vacua, for the case where only the scalar fields of the lower-dimensional gauged SUGRA are turned on and are constant, agreeing with the consistent truncations found in [49] and [50] for the AdS_7 and AdS_6 vacua, respectively. Furthermore, as described in section 3.1, using the exceptional field theory tensor hierarchy and the dictionaries in appendix B.3.1, one can construct the uplift of all the fields of the minimal half-maximal gauged supergravity, including the p-forms. In the following, we summarise the results for the the minimal consistent truncations around AdS_7 and AdS_6. For the latter, we show explicitly how to construct the full ten-dimensional uplift, including all the fields of the 6-dimensional gauged SUGRA. This result will be generalised in section 10 to construct uplifts of half-maximal gauged supergravities around AdS_6 including matter multiplets.

8.1 AdS_7

We can now use (3.11), (4.10) to construct the consistent truncation Ansatz of IIA SUGRA around the supersymmetric AdS_7 vacua of section 6 to the pure 7-dimensional half-maximal SU(2) gauged SUGRA [47]. Here we will consider the truncation Ansatz where only the scalar fields of the 7-dimensional gauged SUGRA have been turned on and are constant. Thus, we compute the generalised metric of $\mathcal{J}_u(x,Y)$ and $\hat{\mathcal{K}}(x,Y)$ given in (3.11) and use the ExFT/IIA dictionary of appendix A.2 to find the supergravity expressions. This way, we obtain the truncation Ansatz in string frame

$$
\begin{aligned}
ds_{10}^2 &= X^{1/2}\sqrt{\frac{t}{p}}ds_7^2 + \frac{R^2}{8}\sqrt{\frac{t}{p}}\left[X^{5/2}\frac{p\,t}{X^5\,s^2+2\,p\,t}ds_{S^2}^2 + X^{-5/2}\frac{h^2}{p\,t}dz^2\right] ,\\
e^{\psi} &= \frac{2}{R}X^{5/4}\left(\frac{t}{p}\right)^{3/4}\frac{1}{\sqrt{X^5\,s^2+2\,p\,t}} ,\\
B &= \frac{R^2}{8\sqrt{2}}\left(-g+\frac{X^5\,s\,t}{X^5\,s^2+2\,p\,t}\right)vol_{S^2} ,
\end{aligned}
\tag{8.1}
$$

and field strengths

$$
\begin{aligned}
F_2 &= -\frac{R^2}{8\sqrt{2}}\left(2\,p+\frac{X^5\,m\,s\,t}{X^5\,s^2+2\,p\,t}\right)vol_{S^2} ,\\
H_3 &= \frac{2}{R}\left(\frac{t}{p}\right)^{-1/4}X^{-5/4}\left(3-\frac{t}{p}\frac{m\,s}{X^5\,s^2+2\,p\,t}\right)vol_{\tilde{M}_3}\\
&\quad+\frac{2}{R}\left(\frac{t}{p}\right)^{-1/4}X^{-5/4}\left(1-X^5\right)\left(1-\frac{4\,p\,t}{X^5\,s^2+2\,p\,t}+\frac{t}{p}\frac{m\,s}{X^5\,s^2+2\,p\,t}\right)vol_{\tilde{M}_3} ,
\end{aligned}
\tag{8.2}
$$

where $vol_{\tilde{M}_3}$ denotes the volume form of the internal 3-manifold with the metric (8.1).

Let us now evaluate the truncation Ansatz for our two gauge choices.

Choice 1. With $h(z) = p(z)$, the truncation Ansatz becomes

$$
\begin{aligned}
ds_{10}^2 &= X^{1/2}\sqrt{-\frac{t}{\ddot{t}}}ds_7^2 + \frac{R^2}{8}\sqrt{-\frac{\ddot{t}}{t}}\left[X^{5/2}\frac{t^2}{X^5\,\dot{t}^2-2\,t\ddot{t}}ds_{S^2}^2 + X^{-5/2}dz^2\right] ,\\
e^{\psi} &= \frac{2}{R}X^{5/4}\left(-\frac{t}{\ddot{t}}\right)^{3/4}\frac{1}{\sqrt{X^5\,\dot{t}^2-2\,t\,\ddot{t}}} ,\\
B &= \frac{R^2}{8\sqrt{2}}\left(z-\frac{X^5\,t\,\dot{t}}{X^5\,\dot{t}^2-2\,t\,\ddot{t}}\right)vol_{S^2} ,
\end{aligned}
\tag{8.3}
$$

and field strengths

$$
\begin{aligned}
F_2 &= \frac{R^2}{8\sqrt{2}}\left(2\,\ddot{t} + \frac{X^5\,m\,t\,\dot{t}}{X^5\,\dot{t}^2 - 2\,t\,\ddot{t}}\right) vol_{S^2}\,, \\
H_3 &= \frac{2}{R}\left(-\frac{\ddot{t}}{t}\right)^{1/4} X^{-5/4}\left(3 - \frac{t}{\ddot{t}}\frac{m\,\dot{t}}{X^5\,\dot{t}^2 - 2\,t\,\ddot{t}}\right) vol_{\tilde{M}_3} \\
&\quad + \frac{2}{R}\left(-\frac{\ddot{t}}{t}\right)^{1/4} X^{-5/4}\left(1 - X^5\right)\left(1 + \frac{4\,t\,\ddot{t}}{X^5\,\dot{t}^2 - 2\,t\,\ddot{t}} + \frac{t}{\ddot{t}}\frac{m\,\dot{t}}{X^5\,\dot{t}^2 - 2\,t\,\ddot{t}}\right) vol_{\tilde{M}_3}\,.
\end{aligned}
\tag{8.4}
$$

Choice 2. We now take $h(z) = 1$ and find

$$
\begin{aligned}
ds^2_{10} &= \frac{1}{9}\sqrt{-\frac{\beta'}{\bar{y}}}X^{1/2}ds^2_7 + \frac{R^2}{9}\sqrt{-\frac{\beta'}{\bar{y}}}\left[\frac{X^{5/2}\,\beta/4}{4\,\beta - X^5\,\bar{y}\,\beta'}ds^2_{S^2} - \frac{1}{16}X^{-5/2}\frac{\beta'\,d\bar{y}^2}{\beta\,\bar{y}}\right], \\
e^{\psi} &= R^{-1}X^{5/4}\frac{\left(-\beta'/\bar{y}\right)^{5/4}}{6\sqrt{4\,\beta - X^5\,\beta'\bar{y}}}\,, \\
F_2 &= \frac{R^2\,\bar{y}}{4}\frac{\sqrt{\beta}}{\beta'}\left(4 + \frac{X^5\,m}{18\,\bar{y}}\frac{(\beta')^2}{4\,\beta - X^5\,\beta'\bar{y}}\right) vol_{S^2}\,, \\
H_3 &= \frac{2}{R}\left(-\frac{\beta'}{\bar{y}}\right)^{-1/4} X^{-5/4}\left(9 - \frac{m}{12\bar{y}}\frac{(\beta')^2}{4\,\beta - X^5\,\beta'\bar{y}}\right) vol_{\tilde{M}_3} \\
&\quad + \frac{6}{R}\left(-\frac{\beta'}{\bar{y}}\right)^{-1/4} X^{-5/4}\left(1 - X^5\right)\left(1 - \frac{8\,\beta}{4\,\beta - X^5\,\beta'\,\bar{y}} + \frac{m}{36\bar{y}}\frac{(\beta')^2}{4\,\beta - X^5\,\beta'\bar{y}}\right) vol_{\tilde{M}_3}\,.
\end{aligned}
\tag{8.5}
$$

The truncation Ansatz is completely determined by the function $t(z)$ satisfying (6.40) for gauge choice 1 and $\beta(\bar{y})$ satisfying (6.32) for choice 2, and corresponds to the truncation Ansatz found in [49] in the coordinates of [45] and [59], respectively. Upon truncation, X becomes the scalar field of the minimal 7-dimensional gauged SUGRA [47] and all of the supersymmetric AdS vacua correspond to the same vacuum of the 7-dimensional theory.

8.2 AdS_6

We can similarly use (3.11) to find the minimal consistent truncation corresponding to the supersymmetric AdS_6 vacua of IIB SUGRA described in section 7. For example, the internal fields can be read off from the generalised metric (5.11), while the remaining fields can be determined using the truncation Ansatz (3.15). Recall that the AdS vacua are characterised in terms of two holomorphic functions f^α, with real/imaginary parts k^α, p^α, and a real function r defined through (7.39).

As before, we will denote by X the scalar field and A_A, A_4 the $\text{SU}(2)_R$ and U(1) gauge fields of the 6-dimensional gauged SUGRA, the so-called pure F(4) gauged SUGRA [48], obtained from the consistent truncation. In terms of these objects, we find that the metric

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in Einstein frame, the axio-dilaton and the 2-forms are given by

$$\begin{aligned} ds^2 &= \frac{4\,r^{5/4}\,\bar{\Delta}^{1/4}\,c_6 R^2}{3^{3/4}|dk|^{1/2}}\left[\frac{3}{R^2\,r}ds_6^2 + \frac{X^2\,|dk|^2}{\bar{\Delta}}ds^2_{\tilde{S}^2} + \frac{1}{X^2\,r^2}dk^\alpha\otimes dp_\alpha\right]\,,\\ H_{\alpha\beta} &= \frac{1}{\sqrt{3}\,\bar{\Delta}}\left(\frac{X^4\,|dk|}{\sqrt{r}}\,p_\alpha p_\beta + 3\sqrt{r}\,\partial_\gamma k_\alpha\partial^\gamma p_\beta\right)\,,\\ {C_{(2)}}^\alpha &= -\frac{4\,c_6 R^2}{3}\left(k^\alpha + \frac{X^4\,r\,p_\gamma\,\partial_\beta k^\gamma\,\partial^\beta p^\alpha}{\bar{\Delta}}|dk|\right) vol_{\tilde{S}^2} + 2\sqrt{2}\,c_6 R\,A^A\wedge\left(y_A dk^\alpha + k^\alpha\tilde{D}y_A\right)\\ &\quad + 2\sqrt{2}\,c_6 R\,A^4\wedge dp^\alpha + 4\,c_6 B\,p^\alpha - 3\,c_6 k^\alpha\epsilon_{ABC}y^A A^B\wedge A^C\,, \end{aligned} \tag{8.6}$$

where

$$\bar{\Delta} = 3\,r\,|dk|^2 + X^4\,|dk|\,p_\gamma p_\delta\partial_\sigma k^\gamma\partial^\sigma p^\delta\,. \tag{8.7}$$

Moreover,

$$\tilde{D}y^A = dy^A + \frac{3}{\sqrt{2}\,R}\epsilon^{ABC}A_B y_C\,, \tag{8.8}$$

is the $\mathrm{SU}(2)_R$ covariant derivative of y^A, in terms of which the $\mathrm{SU}(2)_R$ covariant S^2 metric and S^2 volume form are defined as

$$\begin{aligned} ds_{\tilde{S}^2} &= \delta_{AB}\tilde{D}y^A\otimes\tilde{D}y^B\,,\\ vol_{\tilde{S}^2} &= \frac{1}{2}\epsilon_{ABC}y^A\tilde{D}y^B\wedge\tilde{D}y^C\,. \end{aligned} \tag{8.9}$$

After applying a gauge transformation, the two-forms can equivalently be written as

$${C_{(2)}}^\alpha = -\frac{4\,c_6 R^2}{3}\left(k^\alpha + \frac{X^4\,r\,p_\gamma\,\partial_\beta k^\gamma\,\partial^\beta p^\alpha}{\bar{\Delta}}|dk|\right) vol_{\tilde{S}^2} + 2\sqrt{2}\,c_6 R\left(k^\alpha{\tilde{F}_{(2)}}{}^A y_A + {\tilde{F}_{(2)}}{}^4 p^\alpha\right)\,, \tag{8.10}$$

where

$$\begin{aligned} {\tilde{F}_{(2)}}{}^A &= dA^A + \frac{3}{2\sqrt{2}\,R}\epsilon^{ABC}A_B\wedge A_C\,,\\ {\tilde{F}_{(2)}}{}^4 &= dA^4 + \frac{\sqrt{2}}{R}B\,, \end{aligned} \tag{8.11}$$

are the 2-forms of the 6-dimensional gauged SUGRA as defined in (3.17) and using equations (7.1), (7.2).

The five-form field strength can easily be computed from (3.15) and using its self-duality. We find

$$F_{(5)} = F_{(2,3)} + F_{(3,2)} + F_{(4,1)}\,, \tag{8.12}$$

where $F_{(p,q)}$ are the parts of the 5-form with p external and q internal legs, appropriately $\mathrm{SU}(2)_R$-covariantised. Explicitly, they are given by

$$\begin{aligned}
F_{(2,3)} &= \frac{8\sqrt{2}\,c_6^2 R^3\,|dk|}{3}\left(\tilde{F}_{(2)}{}^A\wedge\tilde{\theta}_A\wedge vol_\Sigma+\frac{X^4\,r|dk|}{\bar{\Delta}}p_\alpha\left(y_A\,\tilde{F}_{(2)}{}^A\wedge dp^\alpha-\tilde{F}_{(2)}{}^4\wedge dk^\alpha\right)\wedge vol_{\tilde{S}^2}\right),\\
F_{(3,2)} &= 16\,c_6^2 R^2|dk|\left(\frac{r^2\,|dk|}{\bar{\Delta}}\tilde{F}_{(3)}\wedge vol_{\tilde{S}^2}+X^{-4}\star_6\tilde{F}_{(3)}\wedge vol_\Sigma\right),\\
F_{(4,1)} &= 8\sqrt{2}\,c_6^2 R\,X^2\left(-r\star_6\tilde{F}_{(2)}{}^A\wedge\tilde{D}y_A+p_\alpha\left(\star_6\tilde{F}_{(2)}{}^4\wedge dp^\alpha+y_A\star_6\tilde{F}_{(2)}{}^A\wedge dk^\alpha\right)\right),
\end{aligned} \tag{8.13}$$

where $F_{(2,3)}$ and $F_{(3,2)}$ can be read off directly from (3.17) and $F_{(4,1)}$ can be obtained from $F_{(2,3)}$ by self-duality of the 5-form field strength. Above $\star_6$ refers to the Hodge dual of the metric of the six-dimensional gauged SUGRA, and $\tilde{F}_{(3)}$ is, as defined in (3.17), the field strength of the 2-form potential

$$\tilde{F}_{(3)} = dB_{(2)}\,. \tag{8.14}$$

Moreover, we have used (5.15) to replace $\tilde{G}_{(3)}$ by $X^{-4}\star_6\tilde{F}_{(3)}$. A non-trivial check of the truncation Ansatz is that the component $F_{(3,2)}$ is self-dual.

In deriving these relations, we used the fact that the 10-dimensional Hodge dual is related to the 6-dimensional Hodge dual and the Hodge dual on S^2 and Σ as

$$\begin{aligned}
\star_{10}F_{(2)}\wedge\Theta_A\wedge vol_\Sigma &= \frac{f_6}{f_\Sigma}\star_{S^2}\Theta_A\wedge\star_6 F_{(2)}\,,\\
\star_{10}F_{(2)}\wedge\omega\wedge vol_{S^2} &= \frac{f_6}{f_{S^2}}\star_\Sigma\,\omega\wedge\star_6 F_{(2)}\,,
\end{aligned} \tag{8.15}$$

for any 1-form $\omega\in\Omega^{(1)}(\Sigma)$ and where

$$f_6=\frac{3}{R^2\,r}\,,\qquad f_\Sigma=\frac{|dk|}{X^2\,r^2}\,,\qquad f_{S^2}=\frac{X^2|dk|^2}{\bar{\Delta}}\,, \tag{8.16}$$

denote the relative factors of the 6-dimensional, S^2 and Riemann surface metric in (8.6). Also,

$$\star_\Sigma\,dk^\alpha=-|dk|\,dp^\alpha\,,\qquad \star_\Sigma dp^\alpha=|dk|\,dk^\alpha\,. \tag{8.17}$$

After the consistent truncation, all the 10-dimensional AdS vacua correspond to the same vacuum of the 6-dimensional gauged SUGRA. Our truncation Ansatz includes the previously-found consistent truncation of a particular AdS_6 vacuum in this family [61] as a particular example. This arises by using the form of the holomorphic function given in [50].

9 Consistent truncations with vector multiplets for AdS_7

Here we will now search for consistent truncation with vector multiplets around the supersymmetric AdS_7 vacua of massive IIA SUGRA. There are in fact many 7-dimensional half-maximal gauged SUGRAs that contain supersymmetric AdS_7 vacua [58] and could, in principle, arise as a consistent truncation of 10-dimensional SUGRA. We will see that in fact only the pure SU(2) gauged SUGRA [47] and coupled to one vector multiplet can be uplifted, where in the latter case the Romans mass must vanish.

As we discussed above, we can only have $N \leq 3$ vector multiplets in a consistent truncation and the corresponding generalised vector fields must form representations of the $\mathrm{SU}(2)_R$ symmetry group generated by the J_u of the AdS_7 vacua. Therefore, we must consider generalised vector fields that are singlets or triplets under $\mathrm{SU}(2)_R$, and satisfy the algebraic conditions (4.1) as well as the differential conditions (4.4). Doublets under $\mathrm{SU}(2)_R$ do not lead to f_{abc} of the form required in (4.4). Moreover, plugging in the form of the J_u for the AdS_7 vacua, we have

$$\mathcal{L}_{J_u} J_{\bar{v}} = 2\sqrt{2} R^{-1} L_{v_u} J_{\bar{v}} \,, \tag{9.1}$$

where on the right-hand side we have the usual three-dimensional Lie derivative generated by v_u acting on the vector, scalar, 1-form and 2-form parts of $J_{\bar{u}}$ separately. This implies that the $J_{\bar{u}}$ must form a representation of $\mathrm{SU}(2)_R$ under the Lie derivative generated by the $\mathrm{SU}(2)_R$ Killing vector fields on S^2.

In the following, we choose the gauge $h(z) = p(z)$ so that the AdS vacua are described by a cubic function $t(z)$.

9.1 Singlets under $\mathrm{SU}(2)_R$

For the $J_{\bar{u}}$ to form singlets under $\mathrm{SU}(2)_R$, they must take the general form

$$J_{\bar{u}} = f_{\bar{u}}(z)\, \partial_z + g_{\bar{u}}(z) + l_{\bar{u}}(z) \iota_{\partial_z} A + h_{\bar{u}}(z)\, dz + k_{\bar{u}}(z)\, vol_{S^2} + r_{\bar{u}}(z)\, dz \wedge A \,. \tag{9.2}$$

Plugging the above parametrisation into the algebraic conditions (4.3), we find they can be solved by only one generalised vector field which is unique (up to an overall sign which just amounts to a redefinition of the scalar field in the truncation Ansatz)

$$J_{\bar{1}} = \frac{R}{2} \ddot{t} + \frac{R}{4} dz + \frac{R^3}{16\sqrt{2}} \ddot{t}\, z\, vol_2 + \frac{R}{4} dz \wedge A \,. \tag{9.3}$$

Therefore, the algebraic conditions already restrict us to having at most 1 vector multiplet that transforms as a singlet under $\mathrm{SU}(2)_R$. However, we must now also check the differential conditions (4.4) but find that

$$\mathcal{L}_{J_{\bar{1}}} \hat{K} = -m \frac{R^4\, t}{32\sqrt{2}} vol_2 \wedge dz \neq 0 \text{ unless } m = 0 \,. \tag{9.4}$$

Therefore, if the Romans mass is non-vanishing, it is impossible to have a consistent truncation with singlet vector multiplets.

For vanishing Romans mass the existence of this consistent truncation is not surprising. In this case the vacuum lifts to a $\mathrm{AdS}_7 \times S^4$ solution of 11-dimensional SUGRA, where the S^4 is written as a S^3 fibred over an interval. It is known that there is a maximally supersymmetric consistent truncation of 11-dimensional SUGRA around this vacuum with gauge group SO(5). This truncation of 11-dimensional SUGRA can be further consistently truncated to a consistent truncation with gauge group $\mathrm{SU}(2) \times \mathrm{U}(1) \subset \mathrm{SO}(5)$ by keeping only the singlets under a Cartan $\mathrm{U}(1) \subset \mathrm{SU}(2)_L \subset \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \subset \mathrm{SO}(5)$. Moreover, the generators of $\mathrm{SU}(2) \times \mathrm{U}(1)$ are independent of one of the four internal coordinates, which can be identified with the Hopf fibre of S^3 when writing S^4 as a S^3 fibred over an interval, see for example [14] for an explicit realisation of the SO(4) generators on S^3, albeit in $\mathrm{O}(3,3)$ generalised geometry. Thus we find a consistent truncation of IIA SUGRA giving rise to $\mathrm{SU}(2) \times \mathrm{U}(1)$ gauge group, which corresponds precisely to the above setup.

9.2 Triplets under $\mathrm{SU}(2)_R$

We repeat the above analysis but consider $J_{\bar{u}}$ with $\bar{u} = 1, \ldots, 3$ forming a triplet under $\mathrm{SU}(2)_R$, which implies they must take the general form (6.13). The algebraic conditions (4.3) then lead to (up to an overall sign)

$$\begin{aligned} J_{\bar{u}} = \frac{2\sqrt{2}}{R} v_{\bar{u}} - \epsilon \frac{R}{2} \ddot{t}\, y_{\bar{u}} + \frac{2\sqrt{2}}{R} \iota_{v_{\bar{u}}} A - \frac{R}{4} (z\, dy_{\bar{u}} + \epsilon\, y_{\bar{u}} dz) \\ + \frac{R^3}{16\sqrt{2}} \ddot{t}\, (\theta_{\bar{u}} \wedge dz - \epsilon\, z\, y_{\bar{u}} Vol_{S^2}) - \frac{R}{4} (z\, dy_{\bar{u}} + \epsilon\, y_{\bar{u}} dz) \wedge A\,, \end{aligned} \tag{9.5}$$

where $\epsilon = \pm 1$. Finally, one needs to check the differential conditions (4.4). However, we find

$$\begin{aligned} \mathcal{L}_{J_{\bar{u}}} \hat{K} = \frac{R^2}{8} (1-\epsilon)\, \dot{t}\, dy_{\bar{u}} \wedge dz - \frac{R^4 \left(m\, \epsilon\, t + 2\, \dot{t}\, \ddot{t}\right)}{32\sqrt{2}} y_{\bar{u}}\, vol_{S^2} \wedge dz \\ + \frac{R^2}{8} (1-\epsilon)\, \dot{t}\, dy_{\bar{u}} \wedge dz \wedge A\,. \end{aligned} \tag{9.6}$$

Looking at the two form part of (9.6) we observe that it can only vanish when $\epsilon = 1$, since $\dot{t}$ cannot vanish for non-zero Romans mass. In this case, (9.6) vanishes if the condition

$$(m\, t + 2\, \dot{t}\, \ddot{t}) = 0\,, \tag{9.7}$$

is satisfied. However, by taking a z-derivative of this condition we find that it implies

$$\dddot{t} = 0\,, \tag{9.8}$$

which can never be satisfied for $m \neq 0$ due to the condition (6.26). We therefore conclude that, if the Romans mass is non-vanishing, consistent truncations with a $\mathrm{SU}(()2)_R$ triplet of vector multiplets do not exist.

Moreover, even when $m = 0$, the truncation is only consistent if $\epsilon = 1$ and $\dot{t}\,\ddot{t} = 0$ and hence requires $\ddot{t} = 0$, or $\epsilon = -1$ and $\dot{t} = \ddot{t} = 0$. However, from (6.27) we see that for the AdS_7 solution to be non-singular requires $t\,\ddot{t} \leq 0$ with equality only allowed at ∂I. Therefore, if $\ddot{t} = 0$, the AdS_7 solutions would be badly singular, as is also apparent by direct inspection of (6.38). Therefore, there are no consistent truncations around AdS_7 vacua of IIA with a triplet of vector multiplets.

10 Consistent truncations with vector multiplets for AdS_6

We now turn to consistent truncations with vector multiplets around AdS_6 vacua of IIB. In principle there are a large number of 6-dimensional half-maximal gauged SUGRAs (containing vector multiplets) that contain supersymmetric AdS_6 vacua [52], and which could thus arise from a consistent truncation of AdS_6 vacua of IIB. Here we will now address the question of which of these 6-dimensional gauged SUGRAs can be uplifted to IIB.

Since we can only keep $N \leq 4$ vector multiplets in a consistent truncation and the generalised vector fields corresponding to the vector multiplets must transform as representations under the $\mathrm{SU}(2)_R$ we have the following possibilities:

- up to 4 singlets,
- a triplet,
- a triplet plus singlet.

Once again, doublets under $\mathrm{SU}(2)_R$ are forbidden by (4.4).

Just as for AdS_7 vacua, the form of the generalised Lie derivative simplifies when plugging in the form of the J_u for the AdS_6 vacua. We find

$$\begin{aligned} \mathcal{L}_{J_A} J_{\bar{u}} &= \frac{3}{\sqrt{2}\,R} L_{v_A} J_{\bar{u}}\,, \\ \mathcal{L}_{J_4} J_{\bar{u}} &= 0\,, \end{aligned} \tag{10.1}$$

where in the first equation on the right-hand side we have the usual four-dimensional Lie derivative generated by v_u acting on the vector, 1-form and 3-form parts of $J_{\bar{u}}$ separately. This implies that the $J_{\bar{u}}$ must form a representation of $\mathrm{SU}(2)_R$ under the Lie derivative generated by the $\mathrm{SU}(2)_R$ Killing vector fields on S^2.

10.1 One singlet under $\mathrm{SU}(2)_R$

We first consider a single vector multiplet whose corresponding generalised vector field satisfies the differential conditions

$$\begin{aligned} \mathcal{L}_{J_A} J_{\bar{1}} &= 0\,, \\ \mathcal{L}_{J_{\bar{1}}} \hat{K} &= 0\,. \end{aligned} \tag{10.2}$$

Note the algebraic conditions (4.1) together with the above immediately imply that

$$\mathcal{L}_{J_{\bar{1}}} J_a = 0\,, \tag{10.3}$$

while $J_4 \propto d\hat{K}$ implies

$$\mathcal{L}_{J_4} J_{\bar{1}} = 0\,. \tag{10.4}$$

The corresponding consistent truncation will lead to a half-maximal gauged SUGRA with one vector multiplet and gauge group $\mathrm{SU}(2) \times \mathrm{U}(1)$.

The most general Ansatz we can write for a generalised vector field that transforms as a singlet under $\mathrm{SU}(2)_R$ is

$$J_{\bar{1}} = \frac{1}{\sqrt{2}} \left(w(z) + 4\, R\, c_6\, n^\alpha(z) + \frac{16\, R^3\, c_6^2}{3}\, l(z) \wedge vol_{S^2} \right) , \tag{10.5}$$

where $w(z)$ is a vector field on Σ and $n^\alpha(z)$ is an SL(2)-doublet of 1-forms on Σ and $l(z)$ is a 1-forms on Σ. The algebraic conditions (4.3) now impose that

$$w(z) = 0 , \qquad l(z) = k_\alpha(z) n^\alpha(z) , \tag{10.6}$$

and further imposes on n_α that

$$\begin{aligned} n_\alpha \wedge dk^\alpha &= n_\alpha \wedge dp^\alpha = 0 , \\ n_\alpha \wedge n^\alpha &= -dk_\alpha \wedge dk^\alpha . \end{aligned} \tag{10.7}$$

Thus, the generalised vector field simplifies to

$$J_{\bar{1}} = \frac{1}{\sqrt{2}} \left(4\, R\, c_6\, n^\alpha + \frac{16\, R^3\, c_6^2}{3}\, k_\alpha n^\alpha \wedge vol_{S^2} \right) . \tag{10.8}$$

The conditions (10.7) fix n^α up to one degree of freedom. The explicit form of n^α depends on the precise relation between dk^α and dp^α. For example, if we impose the Cauchy-Riemann equations (7.42), then n^α can be nicely expressed in terms of the holomorphic function $f^\alpha = -p^\alpha + i\, k^\alpha$ and complex coordinate $z = x_1 + i\, x_2$ on Σ

$$n^\alpha = \frac{1}{2} g\, \partial f^\alpha\, d\bar{z} + \frac{1}{2} \bar{g}\, \bar{\partial} \bar{f}^\alpha dz . \tag{10.9}$$

Here $g \in \mathrm{U}(1)$ is the single degree of freedom left in n^α.

The differential condition

$$\mathcal{L}_{J_{\bar{1}}} \hat{K} = 0 , \tag{10.10}$$

imposes that we must have

$$dn^\alpha = 0 . \tag{10.11}$$

If we impose the Cauchy-Riemann equations then together with (10.9) this becomes

$$\partial \left(g \partial f^\alpha \right) - c.c = 0 , \tag{10.12}$$

where *c.c.* stands for complex conjugate. Eq. (10.12), with g valued in U(1), is a sufficient and necessary condition for having a consistent truncation with a single vector multiplet.

10.1.1 Uplift formulae

By computing the generalised metric using (5.11), (5.13) and using the ExFT / IIB SUGRA dictionary (B.13), (B.16), we can read off the consistent truncation Ansatz for the purely internal components of the metric, 2-form, 4-form and axio-dilaton. The components with some external legs can be read off from the ExFT fields of the tensor hierarchy, $\mathcal{A}_\mu$ and $\mathcal{B}_{\mu\nu}$, and using their IIB parameterisation given in section B.3.1. Moreover, we can also compute the field strengths of IIB supergravity from the ExFT field strengths (2.7), which become (4.13) upon plugging in the truncation Ansatz.

It is now straightforward to read off the uplift formulae for the consistent truncation including a vector multiplet by using the ExFT/IIB dictionary B.3. The result is best expressed in terms of the scalar fields

$$m_a = (m_A,\, m_4,\, m_5)\ , \tag{10.13}$$

which satisfy

$$m_a \eta^{ab} m_b = -1\,. \tag{10.14}$$

Therefore, they parameterise the coset space

$$m_a \in \frac{\mathrm{SO}(4,1)}{\mathrm{SO}(4)}\,, \tag{10.15}$$

and are the scalar fields of the half-maximal gauged SUGRA. They are related to the ${b_u}^a$ of the truncation Ansatz (4.6) up to SO(3) transformations. In particular, they satisfy

$$m^a m^b = \delta^{uv} {b_u}^a {b_v}^b - \eta^{ab}\,, \tag{10.16}$$

so that m_a and ${b_u}^a$ parameterise the same coset space $\frac{\mathrm{SO}(4,1)}{\mathrm{SO}(4)}$. Moreover, we define

$$m\cdot y \equiv m_A\, y^A\,, \tag{10.17}$$

and the SU(2)-covariant derivative in the **3** representation of SU(2)

$$\begin{aligned} \tilde{D}y^A &= dy^A + \frac{3}{\sqrt{2}\,R}\epsilon^{ABC} A_B\, y_C\,, \\ \tilde{D}m^A &= dm^A + \frac{3}{\sqrt{2}\,R}\epsilon^{ABC} A_B\, m_C\,. \end{aligned} \tag{10.18}$$

Similarly, we define the SU(2)-covariantised 1-forms

$$\tilde{\theta}_A = \epsilon_{ABC}\, y^B\, \tilde{D}y^C\,, \tag{10.19}$$

and the SU(2)-covariantised volume on S^2

$$vol_{\tilde{S}^2} = \frac{1}{2}\epsilon_{ABC} y^A\, \tilde{D}y^B \wedge \tilde{D}y^C\,. \tag{10.20}$$

In all our uplift formulae, we will throughout impose the Cauchy-Riemann equations (7.42) on k^α and p^α, so that n^α is given by (10.9), although one can use the above method to derive the uplift formulae in a different gauge as well. Then, with the above conventions, the metric is given by

$$\begin{aligned} ds^2 = \frac{4\,c_6\,R^2\,r^{5/4}|dk|^{3/2}}{3^{3/4}\bar{\Delta}^{3/4}} \Bigg[&\frac{3\,\bar{\Delta}}{R^2\,r\,|dk|^2} ds_6^2 + X^2\left(\delta_{AB}\tilde{D}y^A\otimes\tilde{D}y^B + w\otimes w - \frac{1}{r^2}p_\alpha p_\beta n^\alpha\otimes n^\beta\right) \\ &+ \frac{\tilde{\Delta}}{X^2\,r^2\,|dk|^2} dk^\alpha\otimes dp_\alpha - \frac{3}{X^2\,r} n_\alpha \otimes \left(m_4\, dk^\alpha - m\cdot y\, dp^\alpha\right) \Bigg]\,, \end{aligned} \tag{10.21}$$

where

$$\begin{aligned}\bar{\Delta} &= X^4|dk|\,p_\alpha\,p_\beta\left(m_5\partial_\gamma k^\alpha\partial^\gamma p^\beta + n^{\alpha\gamma}\left((m\cdot y)\,\partial_\gamma p^\beta - m_4\partial_\gamma k^\beta\right)\right)\\ &\quad + 3\,r\,|dk|^2\left(m_5^2 - m_4^2 - (m\cdot y)^2\right)\,,\\ \tilde{\Delta} &= 3\,r\,m_5|dk|^2 + X^4|dk|p_\alpha p_\beta\partial_\gamma k^\alpha\partial^\gamma p^\beta\,,\\ w &= m_A\,\tilde{D}y^A + \frac{1}{3\,r^2}p_\alpha\,\sigma^\alpha\,,\\ \sigma^\alpha &= 3\,r\,(m_5\,n^\alpha - m_4\,dp^\alpha - m\cdot y\,dk^\alpha) - X^4p^\alpha\,p_\beta\,\star_2 n^\beta\,.\end{aligned} \tag{10.22}$$

Here $\star_2 n^\alpha$ denotes the Hodge dual of n^α with respect to the flat metric on the Riemann surface.

The axio-dilaton is given by

$$H^{\alpha\beta} = \frac{X^4\,p^\alpha\,p^\beta\,|dk|}{\sqrt{3\,r\,\bar{\Delta}}} + \sqrt{\frac{3\,r}{\bar{\Delta}}}\left(m_5\,\partial_\gamma k^\alpha\partial^\gamma p^\beta + n^{\alpha\gamma}\left(m\cdot y\,\partial_\gamma p^\beta - m_4\partial_\gamma k^\beta\right)\right)\,, \tag{10.23}$$

and the 2-form by

$$\begin{aligned}C_{(2)}{}^\alpha &= -\frac{4\,c_6R^2}{3}vol_{\tilde{S}^2}\,(k^\alpha + L^\alpha) - \frac{4\,c_6R^2}{3}\frac{|dk|^2}{\bar{\Delta}}\sigma^\alpha\wedge\tilde{\theta}_A\,m^A\\ &\quad + 2\sqrt{2}\,c_6R\left(k^\alpha\,\tilde{F}_{(2)}{}^A y_A + p^\alpha\,\tilde{F}_{(2)}{}^4 + A^{\bar{1}}\wedge n^\alpha\right)\,,\end{aligned} \tag{10.24}$$

where we have defined the SL(2)-doublet function

$$L^\alpha = \frac{X^4\,r\,|dk|}{\bar{\Delta}}p_\beta\left[m_5\partial_\gamma k^\beta\partial^\gamma p^\alpha + n^{\beta\gamma}\left(m\cdot y\,\partial_\gamma p^\alpha - m_4\partial_\gamma k^\alpha\right)\right]\,. \tag{10.25}$$

Moreover,

$$\begin{aligned}\tilde{F}_{(2)}{}^A &= dA^A + \frac{3}{2\sqrt{2}\,R}\epsilon^{ABC}\,A_B\wedge A_C\,,\\ \tilde{F}_{(2)}{}^4 &= dA^4 + \frac{\sqrt{2}}{R}B\,,\\ \tilde{F}_{(2)}{}^{\bar{1}} &= dA^{\bar{1}}\,,\end{aligned} \tag{10.26}$$

are the 6-dimensional two-form field strengths as defined in (4.13), using (7.1) and (7.2). In constructing $C_{(2)}{}^\alpha$ from the truncation Ansatz (4.6) and (4.11), we have performed a gauge transformation to write the 2-form in terms of the field strengths of the 6-dimensional gauged SUGRA, just like we did in the minimal case in going from (8.6) to (8.10). When n^α is exact (it must always be closed), i.e. $n^\alpha = d\chi^\alpha$, e.g. if $H^{(1)}(\Sigma) = 0$, we can perform a further gauge transformation to write the 2-form as

$$\begin{aligned}C_{(2)}{}^\alpha &= -\frac{4\,c_6R^2}{3}vol_{\tilde{S}^2}\,(k^\alpha + L^\alpha) - \frac{4\,c_6R^2}{3}\frac{|dk|^2}{\bar{\Delta}}\sigma^\alpha\wedge\tilde{\theta}_A\,m^A\\ &\quad + 2\sqrt{2}\,c_6R\left(k^\alpha\,\tilde{F}_{(2)}{}^A y_A + p^\alpha\,\tilde{F}_{(2)}{}^4 + \chi^\alpha\,\tilde{F}_{(2)}{}^{\bar{1}}\right)\,.\end{aligned} \tag{10.27}$$

The self-dual 5-form of IIB supergravity is given by

$$F_{(5)} = F_{(1,4)} + F_{(2,3)} + F_{(3,2)} + F_{(4,1)} + F_{(5,0)}\,, \tag{10.28}$$

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with

$$
\begin{aligned}
F_{(1,4)} &= \frac{16\, c_6^2 R^4}{3} \frac{|dk|^3\, r}{\bar{\Delta}} \epsilon^{ABC} y_A\, m_B\, \tilde{D} m_C \wedge vol_{\tilde{S}^2} \wedge vol_\Sigma\,, \\
F_{(2,3)} &= \frac{8\sqrt{2}\, c_6^2 R^3\, |dk|}{3} \Big[\tilde{F}_{(2)}{}^A \wedge \tilde{\theta}_A \wedge vol_\Sigma \\
&\quad + \frac{|dk|}{\bar{\Delta}} \left(6\, r\, |dk| \left(\frac{1}{2} \left((m\cdot y)\, y_A + m_A \right) \tilde{F}_{(2)}{}^A + m_4\, \tilde{F}_{(2)}{}^4 + m_5\, \tilde{F}_{(2)}{}^{\bar{1}} \right) \right. \\
&\quad \left. + X^4\, p_\alpha\, p_\beta \left(n^{\alpha\gamma} \left(\partial_\gamma k^\beta \tilde{F}_{(2)}{}^4 - \partial_\gamma p^\beta\, y_A\, \tilde{F}_{(2)}{}^A \right) + \partial_\gamma k^\alpha \partial^\gamma p^\beta\, \tilde{F}_{(2)}{}^{\bar{1}} \right) \right) \wedge vol_\Sigma \wedge \tilde{\theta}_B\, m^B \\
&\quad + \frac{X^4\, r\, |dk|}{\bar{\Delta}} p_\alpha \left(\tilde{F}_{(2)}{}^A \wedge (y_A \lambda^\alpha + m_A \star_2 n^\alpha) - \tilde{F}_{(2)}{}^4 \wedge \rho^\alpha + \tilde{F}_{(2)}{}^{\bar{1}} \wedge \star_2 \sigma^\alpha \right) \wedge vol_{\tilde{S}^2}\,, \\
F_{(3,2)} &= 16\, c_6^2 R^2 \frac{r^2 |dk|^2}{\bar{\Delta}} \tilde{F}_{(3)} \wedge \left(\left(m_5^2 - m_4^2 - (m\cdot y)^2 \right) vol_{\tilde{S}^2} - \omega \wedge \tilde{\theta}_B m^B \right) \\
&\quad + 16\, c_6^2 R^2 |dk| X^{-4} \left(\star_6 \tilde{F}_{(3)} \right) \wedge vol_\Sigma\,, \\
F_{(4,1)} &= \star_{10} F_{(2,3)}\,, \\
F_{(5,0)} &= 48\, c_6^2\, r\, \epsilon^{ABC} y_A\, m_B\, \star_6\, \tilde{D} m_A\,,
\end{aligned}
\tag{10.29}
$$

where

$$
\begin{aligned}
\lambda^\alpha &= m_5\, dp^\alpha - m_4\, n^\alpha\,, \\
\rho^\alpha &= m_5\, dk^\alpha - (m\cdot y)\, n^\alpha - m_4 \star_2 n^\alpha\,,
\end{aligned}
\tag{10.30}
$$

and

$$
\star_2 \sigma^\alpha = m_5 \star_2 n^\alpha + (m\cdot y)\, dp^\alpha - m_4\, dk^\alpha\,, \tag{10.31}
$$

is the Hodge dual of σ^α with respect to the flat metric on Σ. $F_{(p,q)}$ are the $\mathrm{SU}(2)_R$ covariantised components of the 5-form field strength with p external and q internal legs. $\star_{10}$ refers to the Hodge dual operator with respect to the full 10-dimensional metric (10.21), while $\star_6$ refers to the Hodge dual operator of the metric of the six-dimensional gauged SUGRA whose line element is ds_6^2. $\tilde{F}_{(3)}$ is as defined in (3.17) the field strength of the two-form

$$
\tilde{F}_{(3)} = dB_{(2)}\,. \tag{10.32}
$$

In the above, we have used (5.15) to replace $\tilde{G}_{(3)}$ by $X^{-4} \star_6 \tilde{F}_{(3)}$. The self-duality of the five-form relates the components $F_{(p,q)}$ to $F_{(6-p,4-q)}$.In particular, it implies that $F_{(3,2)}$ should be self-dual, which can easily be checked using (10.21). This provides a non-trivial check of the truncation Ansatz. Moreover, we have used the self-duality of the 5-form to compute $F_{(5,0)}$ and $F_{(4,1)}$ from $F_{(1,4)}$ and $F_{(2,3)}$ rather than using the truncation Ansatz of section 4.1.

10.2 Multiple singlets under $\mathrm{SU}(2)_R$

We next consider the situation where we have $N \leq 4$ vector multiplets transforming as singlets under $\mathrm{SU}(2)_R$. The corresponding consistent truncation will lead to a half-maximal gauged SUGRA with gauge group $\mathrm{SU}(2) \times G$, where as we will see we can only have $G =$

U(1) or $G = \mathrm{U}(1)^2$. Following the same logic as in the case for one single vector multiplet, the most general solution to the algebraic conditions (4.1) and differential conditions

$$\mathcal{L}_{J_A} J_{\bar{u}} = 0\,, \qquad \mathcal{L}_{J_{\bar{u}}} \hat{K} = 0\,, \qquad \mathcal{L}_{J_{\bar{u}}} J_{\bar{v}} = -f_{\bar{u}\bar{v}}{}^{\bar{w}} J_{\bar{w}}\,, \tag{10.33}$$

is

$$J_{\bar{u}} = \frac{1}{\sqrt{2}} \left(4\, R\, c_6\, n_{\bar{u}}{}^{\alpha} + \frac{16\, R^3\, c_6^2}{3}\, k_\alpha n_{\bar{u}}{}^{\alpha} \wedge vol_{S^2} \right)\,, \qquad \text{with } \bar{u} = 1, \ldots, N\,, \tag{10.34}$$

where the $n_{\bar{u}}{}^{\alpha}$ have to satisfy

$$\begin{aligned} n_{\bar{u}\,\alpha} \wedge dk^{\alpha} &= n_{\bar{u}\,\alpha} \wedge dp^{\alpha} = 0\,, \\ n_{\bar{u}\,\alpha} \wedge n_{\bar{v}}{}^{\alpha} &= -\delta_{\bar{u}\bar{v}}\, dk_\alpha \wedge dk^{\alpha}\,, \end{aligned} \tag{10.35}$$

as well as

$$d\, n_{\bar{u}\,\alpha} = 0\,. \tag{10.36}$$

As in the one vector multiplet case, we can solve the algebraic conditions (10.35) by

$$n_{\bar{u}}{}^{\alpha} = \frac{1}{2} g_{\bar{u}}\, \partial f^{\alpha}\, d\bar{z} + \frac{1}{2} \bar{g}_{\bar{u}}\, \bar{\partial} \bar{f}^{\alpha} dz\,, \tag{10.37}$$

with $g_{\bar{u}} \in \mathrm{U}(1)$. But the second of the conditions (10.35) now imposes that

$$g_{\bar{u}} \bar{g}_{\bar{v}} + \bar{g}_{\bar{u}} g_{\bar{v}} = 2\, \delta_{\bar{u}\bar{v}}\,. \tag{10.38}$$

It is easy to check that these conditions can only be solved when $N \leq 2$, which implies that consistent truncations with $N = 3, 4$ vector multiplets which are singlets under $\mathrm{SU}(2)_R$ cannot exist. For the case $N = 2$, the condition is solved by

$$g_{\bar{2}} = \pm i\, g_{\bar{1}}\,, \qquad g_{\bar{1}} \in \mathrm{U}(1)\,. \tag{10.39}$$

Without loss of generality, we can take $g_{\bar{2}} = i\, g_{\bar{1}}$, by suitably redefining the scalar fields of the truncation Ansatz. In this case, the differential conditions (10.36) implies

$$\partial \left(g_{\bar{1}} \partial f^{\alpha} \right) = 0\,, \tag{10.40}$$

which admits non-trivial solutions only in the cases where

$$\partial f^2 = \lambda\, \partial f^1\,, \tag{10.41}$$

with λ a constant. In this case,

$$g_{\bar{1}} = e^{i\,c}\, \frac{\bar{\partial} \bar{f}^1}{\partial f^1} = e^{i\,c}\, \frac{\bar{\partial} \bar{f}^2}{\partial f^2}\,, \tag{10.42}$$

where c is a real integration constant.

Recall that $J_4 \propto d\hat{K}$ immediately implies

$$\mathcal{L}_{J_4} J_{\bar{u}} = 0\,, \tag{10.43}$$

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while one can also easily check that

$$\mathcal{L}_{J_{\bar{u}}} J_{\bar{v}} = 0\,. \tag{10.44}$$

Therefore, the consistent truncation leads to a $\mathrm{SU}(2) \times \mathrm{U}(1)^2$ gauged SUGRA.

One can then wonder whether AdS_6 vacua described by two holomorphic functions satisfying the relation $\partial f^2 = \lambda\, \partial f^1$ exist. Firstly, we see that this rules out having SL(2) monodromies. Moreover, we observe that, in this situation,

$$|dk| = \frac{1}{2} i\, \partial f^\alpha\, \bar{\partial} \bar{f}_\alpha = \frac{1}{2} i(\lambda - \bar{\lambda})\, |\partial f^1|^2\,. \tag{10.45}$$

However, as explained in section 7.2 and [42–44], any globally regular vacuum must be described by functions satisfying the condition $r \geq 0$ and $|dk| \geq 0$, with equality on the boundary of the Riemann surface Σ. The latter ensures that the total space has no boundary. For (10.45) this condition implies that $\lambda \neq \bar{\lambda}$ and that $\partial f^1 = \partial f^2 = 0$ on the boundary of Σ. However, since Σ is compact, we must have $\partial f^1 = \partial f^2 = 0$ everywhere. Therefore, although the differential and algebraic conditions for consistent truncations with two vector multiplets can be locally solved, there are no half-supersymmetric compactifications to AdS_6 vacua with an internal space without boundaries that allow such a consistent truncation.

10.2.1 Uplift formulae for two singlets under $\mathrm{SU}(2)_R$

As we discussed above, a consistent truncation with two vector multiplets and gauge group SU(2) around an AdS_6 vacua of IIB SUGRA necessarily requires the internal space to have a boundary. Although this is not particularly interesting from a holographic perspective, we can nonetheless use the formalism described in [30] to derive the consistent truncation. For simplicity, we will only give the truncation Ansatz which preserves the full $\mathrm{SO}(5,2)$ symmetry of the AdS vacuum since this is sufficient for a wide variety of applications. Therefore, we will consider the case where only the scalar fields of the six-dimensional gauged SUGRA are non-zero and depend only on the internal four coordinates. Moreover, as in the case of only one $\mathrm{SU}(2)_R$ singlet, we will impose the Cauchy-Riemann equations (7.42) on k^α and p^α throughout. However, it is straightforward to obtain the uplift formulae in a different gauge.

The scalar manifold of the six-dimensional SUGRA obtained from the consistent truncation is

$$M_{\text{scalar}} = \frac{\mathrm{SO}(4,2)}{\mathrm{SO}(4) \times \mathrm{SO}(2)}\,, \tag{10.46}$$

and can be parameterised by ${m_i}^a$ with $a = 1, \ldots, 6$ labelling the vector representation of $\mathrm{SO}(4,2)$ and $i = 1, 2$ the doublet of SO(2). The ${m_i}^a$ must satisfy

$${m_i}^a {m_j}^b \eta_{ab} = -\delta_{ij}\,, \tag{10.47}$$

and are related to the ${b_u}^a$ of (4.6) by

$$\delta^{ij} {m_i}^a {m_j}^b = \delta^{uv} {b_u}^a {b_v}^b - \eta^{ab}\,. \tag{10.48}$$

Moreover, we can decompose SO(4, 2) $\longrightarrow$ SO(3) $\times$ SO(2) such that

$$\mathbf{6} \longrightarrow (\mathbf{3},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2})\,. \tag{10.49}$$

We accordingly write

$$m_i{}^a = \left(m_i{}^A,\, m_i,\, \lambda_i{}^{\bar{u}}\right)\,, \tag{10.50}$$

where $\lambda_i{}^{\bar{u}}$ are constrained by (10.47), i.e.

$$\lambda_i{}^{\bar{u}}\lambda_j{}^{\bar{v}}\delta_{\bar{u}\bar{v}} = m_i{}^A m_j{}^B \delta_{AB} + m_i m_j + \delta_{ij}\,. \tag{10.51}$$

The uplift formulae can be conveniently formulated in terms of

$$\begin{aligned}
n_i{}^\alpha &= \lambda_i{}^{\bar{u}}\, n_{\bar{u}}{}^\alpha\,,\\
\omega_i{}^\alpha &= (m_i\cdot y)\, dk^\alpha + m_i\, dp^\alpha\,,\\
w_i &= m_i{}^A\, dy_A - \frac{1}{r} p_\alpha\, \omega_i{}^\alpha - \frac{1}{r} p_\alpha\, n_i{}^\alpha\,,\\
\sigma &= |\lambda| - \epsilon^{ij} m_i\, (m_j\cdot y)\,,\\
\bar{\Delta} &= X^4\, |dk|\, p_\alpha p_\beta \left(\sigma\, \partial_\gamma k^\alpha \partial^\gamma p^\beta - \epsilon^{ij} n_i{}^{\alpha\gamma}\, \omega_j{}^\beta{}_\gamma\right) + 3\, r\, |dk|^2 \Big[|\lambda|^2\\
&\quad + \epsilon^{ik}\epsilon^{jl}\left(-\lambda_k{}^{\bar{u}}\lambda_l{}^{\bar{v}}\delta_{\bar{u}\bar{v}}\, (m_i m_j + (m_i\cdot y)\,(m_j\cdot y)) + m_i\, m_j\, (m_k\cdot y)\,(m_l\cdot y)\right)\Big]\,,
\end{aligned} \tag{10.52}$$

where $|\lambda|$ denotes the determinant of the 2×2 matrix $\lambda_i{}^{\bar{u}}$. Just as in the singlet vector multiplet case, we use the shorthand

$$m_i\cdot y = m_i{}^A\, y_A\,. \tag{10.53}$$

The metric is given by

$$\begin{aligned}
ds^2 = \frac{4\, c_6\, R^2\, r^{5/4} |dk|^{3/2}}{3^{3/4} \bar{\Delta}^{3/4}} &\left[\frac{3\, \bar{\Delta}}{R^2 r\, |dk|^2} ds_6^2 + X^2 \left(\delta_{AB} dy^A \otimes dy^B + \delta^{ij} w_i \otimes w_j\right)\right.\\
&\left. + \frac{3}{X^2\, r} \left(\left(\sigma + 2\, \epsilon^{ij} m_i m_j{}^u y_u\right) dk^\alpha \otimes dp_\alpha + \epsilon^{ij} n_{i\,\alpha} \otimes \omega_j{}^\alpha\right)\right]\,,
\end{aligned} \tag{10.54}$$

the axio-dilaton by

$$H^{\alpha\beta} = \frac{X^4\, p^\alpha\, p^\beta\, |dk|}{\sqrt{3\, r\, \bar{\Delta}}} + \sqrt{\frac{3\, r}{\bar{\Delta}}} \left(\sigma\, \partial_\gamma k^\alpha \partial^\gamma p^\beta - \epsilon^{ij} n_i{}^{\alpha\gamma} \omega_j{}^\beta{}_\gamma\right)\,, \tag{10.55}$$

and the 2-form by

$$\begin{aligned}
C_{(2)}{}^\alpha = &-\frac{4\, c_6 R^2}{3} vol_{S^2} \left(k^\alpha + \frac{X^4\, r\, |dk|}{\bar{\Delta}} p_\beta \left[\sigma\, \partial_\gamma k^\beta \partial^\gamma p^\alpha - \epsilon^{ij} n_i{}^{\beta\gamma}\, \omega_j{}^\alpha{}_\gamma\right]\right)\\
&+ \frac{4 c_6 R^2 |dk|^2}{3\bar{\Delta}} \left(X^4\, p^\alpha\, p_\beta \epsilon^{ij} + 3\, r\, \delta^\alpha_\beta\, \delta^{ij}\right) \left(\omega_j{}^\beta + n_j{}^\beta\right) \wedge \Theta_A m_i{}^A\\
&- \frac{2 c_6 R^2 |dk|^2}{3\bar{\Delta}} r \left(\epsilon_{ABC} \epsilon^{kl} m_k{}^A m_l{}^B y^C\right) \epsilon^{ij} \left(\omega_i{}^\alpha + n_i{}^\alpha\right) \wedge w_j\,.
\end{aligned} \tag{10.56}$$

Since we are considering the subsector of the truncation where only the scalar fields are turned on and are constant, the IIB five-form field strength vanishes

$$F_{(5)} = 0\,. \tag{10.57}$$

10.3 Triplet under $\mathrm{SU}(2)_R$

We next move to the case where we have $N = 3$ vector multiplets transforming as a triplet of $\mathrm{SU}(()2)_R$, i.e.

$$\mathcal{L}_{J_A} J_{\bar{B}} = -\frac{3}{\sqrt{2}\,R}\epsilon_{A\bar{B}}{}^{\bar{C}} J_{\bar{C}}\,, \tag{10.58}$$

where $\bar{A} = 1, 2, 3$. As we will see shortly, this leads to a ISO(3) gauged SUGRA. Equation (10.58) implies that the most general ansatz for the fields $J_{\bar{A}}$ must be of the form given in (7.12). The algebraic conditions (4.3) then fix $J_{\bar{A}}$ to be

$$\begin{aligned} J_{\bar{A}} = \frac{1}{\sqrt{2}}\bigg(&\frac{3}{R}v_{\bar{A}} + 4\,c_6\,R\,y_{\bar{A}}\,\pi^\alpha + 4\,c_6\,R\,k^\alpha dy_{\bar{A}} \\ &+ \frac{16\,c_6^2\,R^3}{3}\,y_{\bar{A}}\,k_\alpha\pi^\alpha\wedge vol_{S^2} - \frac{16\,c_6^2 R^3}{3}|\pi|\theta_{\bar{A}}\wedge vol_\Sigma\bigg), \end{aligned} \tag{10.59}$$

where $|\pi| = \frac{1}{2}\pi^\alpha{}_\beta\pi_\alpha{}^\beta$ and π^α is a SL(2)-doublet of one-forms on Σ satisfying the algebraic conditions

$$\begin{aligned} \pi_\alpha\wedge dk^\alpha &= \pi_\alpha\wedge dp^\alpha = 0\,, \\ \pi_\alpha\wedge n^\alpha &= -dk_\alpha\wedge dk^\alpha\,. \end{aligned} \tag{10.60}$$

Furthermore, in order to satisfy the differential condition

$$\mathcal{L}_{J_{\bar{A}}}\hat{K} = 0\,, \tag{10.61}$$

one needs to impose the conditions

$$\begin{aligned} p_\alpha\pi^\alpha &= p_\alpha dk^\alpha \\ d\pi^\alpha &= \frac{1}{r}\,p^\alpha\,\pi_\beta\wedge\pi^\beta\,. \end{aligned} \tag{10.62}$$

As in the case of the singlet vector multiplet, conditions (10.60) can are solved by

$$\pi^\alpha = \frac{1}{2}g_\pi\,\partial f^\alpha\,d\bar{z} + \frac{1}{2}\bar{g}_\pi\,\bar{\partial}\bar{f}^\alpha dz\,, \tag{10.63}$$

where again $g_\pi \in \mathrm{U}(1)$. In this case, however, the first condition of (10.62) fixes the phase g_π to

$$g_\pi = i\frac{p_\alpha\bar{\partial}\bar{f}^\alpha}{p_\beta\partial f^\beta}\,, \tag{10.64}$$

thereby fixing the one-forms π^α completely. The second equation in (10.62) then give an extra differential condition on f^α that has to be satisfied for the vacua to allow consistent truncations with a $\mathrm{SU}(()2)_R$ triplet of vector multiplets.

Using (10.59) and the above differential conditions, we find

$$\mathcal{L}_{J_{\bar{A}}} J_{\bar{B}} = -\frac{3\sqrt{2}}{R}\epsilon_{\bar{A}\bar{B}}{}^{\bar{C}} J_{\bar{C}} + \frac{3}{\sqrt{2}\,R}\epsilon_{\bar{A}\bar{B}}{}^{C} J_C\,, \qquad \mathcal{L}_{J_4} J_{\bar{A}} = 0\,, \tag{10.65}$$

where $\bar{A}$ and A are raised/lowered with $\delta_{\bar{A}\bar{B}}$ and δ_{AB}, respectively. Together with the relations

$$\begin{aligned}
\mathcal{L}_{J_A} J_B &= -\frac{3}{\sqrt{2}\,R}\epsilon_{AB}{}^{C} J_C\,, \\
\mathcal{L}_{J_A} J_{\bar{B}} &= -\frac{3}{\sqrt{2}\,R}\epsilon_{A\bar{B}}{}^{\bar{C}} J_{\bar{C}}\,, \\
\mathcal{L}_{J_4} J_a &= \mathcal{L}_{J_a} J_4 = 0\,,
\end{aligned} \tag{10.66}$$

this implies that the gauge group of the six-dimensional half-maximal gauged SUGRA is ISO(3).

10.3.1 Uplift formulae

Just as for the case of two vector multiplets forming SU(2) singlets, we will here only give the consistent truncation Ansatz preserving the SO(5,2) symmetry of the AdS_6 vacuum, i.e. where the scalar fields are the only non-zero fields of the six-dimensional half-maximal gauged SUGRA and are constant. The full consistent truncation Ansatz including general values for all gauge fields of the six-dimensional gauged SUGRA can be obtained as discussed above 4.1 and demonstrated explicitly for the case of a single vector multiplet in section 10.1.1.

The scalar manifold of the six-dimensional half-maximal gauged SUGRA is

$$M_{\text{scalar}} = \frac{\mathrm{SO}(4,3)}{\mathrm{SO}(4)\times\mathrm{SO}(3)} \times \mathbb{R}^+\,. \tag{10.67}$$

We will parameterise the coset space $\frac{\mathrm{SO}(4,3)}{\mathrm{SO}(4)\times\mathrm{SO}(3)}$ by

$$m_I{}^a = \left(m_I{}^A - \lambda_I{}^A,\, m_I,\, \lambda_I{}^A\right)\,, \tag{10.68}$$

where $I = 1,2,3$ and which satisfies

$$m_I{}^a m_J{}^b \eta_{ab} = -\delta_{IJ}\,. \tag{10.69}$$

The $m_I{}^a$ are related to the $b_u{}^a$ of (4.6) by

$$m_I{}^a m_J{}^b \delta^{IJ} = b_u{}^a b_v{}^b \delta^{uv} - \eta^{ab}\,. \tag{10.70}$$

The uplift formulae can be conveniently expressed in terms of

$$\begin{aligned}
\omega_I &= (m_I\cdot y)\, p_\alpha\, dk^\alpha + m_I\, p_\alpha\, dp^\alpha\,, \\
\sigma_I^\pm &= (\lambda_I\cdot y)\, p_\alpha\, dp^\alpha \pm m_I\, p_\alpha\, dk^\alpha\,, \\
\Lambda &= p_\alpha\, p_\beta\, \partial_\gamma k^\alpha \partial^\gamma p^\beta\,, \\
\bar{\Delta} &= X^4 \Lambda\, |m_I{}^A||dk| \\
&\quad - 3\, r\, |dk|^2 \left(|m_I{}^A|\, |m_I{}^A - 2\,(\lambda_I\cdot y)\, y^A| + \frac{1}{4}\left(\epsilon_{ABC}\epsilon^{IJK} m_I y^A m_J{}^B m_K{}^C\right)^2 \right),
\end{aligned} \tag{10.71}$$

with

$$(m_I\cdot y) = m_I{}^A y_A\,, \qquad (\lambda_I\cdot y) = \lambda_I{}^A y_A\,. \tag{10.72}$$

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The metric, axio-dilaton and 2-form are given by

$$
\begin{aligned}
ds^2 =& \frac{4\, c_6 R^2 r^{5/4} |dk|^{3/2}}{3^{3/4}\bar{\Delta}^{3/4}} \Bigg[\frac{3\,\bar{\Delta}}{R^2 r\, |dk|^2} ds_6^2 \\
&+X^2 \left(m_I{}^A dy_A - \frac{1}{r}\omega_I \right) \otimes \left(m^{I\,B} dy_B - \frac{1}{r}\omega^I \right) + \frac{X^2}{r^2} p_\alpha p_\beta dp^\alpha \otimes dp^\beta \\
&+ \frac{3\,|dk|}{X^2 \Lambda\, r} \Big(|m_I{}^A| p_\alpha p_\beta \left(dk^\alpha \otimes dk^\beta - dp^\alpha \otimes dp^\beta \right) \\
&+ \frac{1}{2} \epsilon_{ABC} \epsilon^{IJK} y^A m_J{}^B m_K{}^C p_\alpha \left(\sigma_I^+ \otimes dp^\alpha + dp^\alpha \otimes \sigma_I^+ \right) \Big) \Bigg] \,, \\
H^{\alpha\beta} =& \frac{1}{\sqrt{\bar{\Delta}}} \Bigg[\left(\frac{X^4\, |dk|}{\sqrt{3\, r}} - \frac{2\,\sqrt{3\, r}}{\Lambda} |m_I{}^A| |dk|^2 \right) p^\alpha p^\beta + \sqrt{3\, r}\, |m_I{}^A| \partial_\gamma k^\alpha \partial^\gamma p^\beta \\
&+ \frac{\sqrt{3\, r}}{2\,\Lambda} |dk|\, \epsilon^{IJK} \epsilon_{ABC} y^A m_I{}^B m_J{}^C\, \sigma_K^{-\,\gamma} \left(p^\alpha \partial_\gamma p^\beta + p^\beta \partial_\gamma p^\alpha \right) \Bigg] \,, \\
\sqrt{\bar{\Delta}} H^{\alpha\beta} C_{(2)\,\beta} =& \frac{\sqrt{3}\, |dk|}{2\sqrt{r}} \Big[- \epsilon^{IJK} m_J{}^A \lambda_{K\,A} \omega_I{}^\gamma \partial_\gamma p^\alpha \\
&+ 2\, p^\alpha\, |dk| \epsilon^{IJK} m_I m_J{}^A \lambda_K{}^B \left(\delta_{AB} - y_A y_B \right) \Big] vol_\Sigma \\
&+ \sqrt{3\, r}\, |dk| \epsilon^{IJK}\, m_I{}^A\, dy_A \wedge \Bigg[\frac{2\, p^\alpha}{\Lambda} p_\beta\, |dk| \Big(m_J{}^B \lambda_K{}^C \left(\delta_{BC} - y_B\, y_C \right) dk^\beta \\
&+ 2\, |dk|\, m_J \left(m_K{}^B - \lambda_K{}^B \right) y_B\, dp^\beta \Big) + m_K{}^B \left(m_J\, y_B\, dk^\alpha + \lambda_{JB}\, dp^\alpha \right) \Bigg] \\
&+ \frac{\sqrt{3\, r}}{2} \Big(- \frac{2\, X^4 |dk|\, p^\alpha}{3\, r} \left(r + k^\beta p_\beta \right) - |m_i{}^u| k_\beta \partial_\gamma p^\beta \partial^\gamma k^\alpha \\
&+ \frac{4\, p^\alpha\, |dk|^2}{\Lambda} p_\beta k^\beta |m_i{}^u| \Big) vol_{S^2} \\
&- \frac{\sqrt{3\, r}\, |dk|}{2} \left(k^\alpha\, m_I + \frac{2}{\Lambda} p^\alpha\, \sigma^-_{I\,\beta} k_\gamma \partial^\beta p^\gamma \right) \epsilon^{IJK}\, m_J{}^A\, m_K{}^B\, dy_A \wedge dy_B \,,
\end{aligned}
\tag{10.73}
$$

and the five-form vanishes by our assumption that the scalar fields are the only non-vanishing fields and they are constant.

10.4 Triplet plus singlet under $\mathbf{SU(2)}_R$

We finally consider the possibility of having consistent truncations with four vector multiplets forming a triplet and a singlet of SU(()2)$_R$, i.e.

$$
\mathcal{L}_{J_A} J_{\bar{B}} = - \frac{3}{\sqrt{2}\, R} \epsilon_{A\bar{B}}{}^{\bar{C}} J_{\bar{C}} \,, \qquad \mathcal{L}_{J_A} J_{\bar{4}} = 0 \,. \tag{10.74}
$$

Since $J_4 \propto d\hat{K}$ we automatically have

$$
\mathcal{L}_{J_4} J_a = 0 \,. \tag{10.75}
$$

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For a vacuum to allow such consistent truncations around it, it must allow both a truncation with a single vector multiplet, characterised by (10.9), and a truncation with a triplet of vector multiplets, characterised by (10.63). The resulting gauge group will clearly be ISO(3) × U(1). Futhermore, in order to have both simultaneously, we need to satisfy the condition

$$J_{\bar{4}} \wedge J_{\bar{A}} = 0\,, \tag{10.76}$$

where $\bar{A} = 1, 2, 3$ labels the triplet and $\bar{4}$ the extra singlet. Similar to the case of two singlets, the above condition fixes the phase g that characterises the singlet to be (as before, up to a sign which can be absorbed by a field redefinition of the scalar fields in the truncation)

$$g = -i\, g_\pi = \frac{p_\alpha \bar{\partial} \bar{f}^\alpha}{p_\beta \partial f^\beta}\,. \tag{10.77}$$

Therefore, a vacuum allows a consistent truncation with four vector multiplets only in the case where it allows a consistent truncation with a $\mathrm{SU}(()2)_R$ triplet of vector multiplets and a consistent truncation with a single vector multiplet characterised precisely by the phase (10.77).

11 Conclusions

In this paper, we showed how to use ExFT to easily recover the infinite families of supersymmetric AdS_7 and AdS_6 solutions of massive IIA and IIB SUGRA, respectively, known in the literature [40, 41, 43, 44]. The ExFT description of these vacua allowed us to immediately construct the "minimal" consistent truncation of 10-dimensional SUGRA around these solutions [49–51, 61], in which we keep only the gravitational supermultiplet of the lower-dimensional gauged SUGRA. We then analysed whether it is possible to construct consistent truncations around the supersymmetric AdS vacua keeping more modes, which would result in lower-dimensional gauged SUGRAs coupled to vector multiplets. Assuming the method developed in [29, 30] is the most general one for constructing consistent truncations with vector multiplets, we found that

- there are no consistent truncations with vector multiplets around AdS_7 vacua of massive IIA, unless the Romans mass vanishes. For vanishing Romans mass, there is a consistent truncation that is itself a truncation (and dimensional reduction) of the maximally supersymmetric consistent truncation of 11-dimensional SUGRA on S^4,
- there are consistent truncations with vector multiplets of IIB SUGRA around its supersymmetric AdS_6 solutions. In this case, the holomorphic functions describing the AdS_6 solutions must satisfy further differential constraints.

In particular, we found that the only consistent truncations with vector multiplets of IIB SUGRA around the supersymmetric AdS_6 vacua yield $N \leq 4$ vector multiplets with gauge group $\mathrm{SU}(2) \times \mathrm{U}(1)$, $\mathrm{SU}(2) \times \mathrm{U}(1)^2$, ISO(3) and $\mathrm{ISO}(3) \times \mathrm{U}(1)$, when the holomorphic functions f^α satisfy the following differential conditions.

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Consistent truncation with one vector multiplet. The differential condition (10.12) is

$$\partial\left(g\,\partial f^\alpha\right) - c.c. = 0\,, \tag{11.1}$$

for some function $g \in \mathrm{U}(1)$, where *c.c.* denotes the complex conjugate. While we will not attempt to find general solutions of (11.1) that are holomorphic and satisfy (7.33) and (7.32), it is easy to show that if one of the holomorphic functions is linear in the complex coordinate z, i.e. $f^1 = A_0 + A_1\,z$, then the other function must be quadratic, i.e. $f^2 = B_0 + B_1\,z + B_2\,z^2$, where A_0, A_1, B_0, B_1 and B_2 are constant complex numbers. This implies that the Abelian T-dual to the Brandhuber-Oz solution [62], which is described by a linear and quadratic holomorphic function [41, 63], admits a consistent truncation with a single vector multiplet, while the non-Abelian T-dual to the Brandhuber-Oz solution [37], which is described by a linear and cubic holomorphic function [50, 63], does not. The consistent truncation Ansatz is given in section 10.1.1 and leads to F(4) gauged SUGRA coupled to one vector multiplet.

Consistent truncation with two vector multiplets. The differential condition that the holomorphic functions must satisfy is now

$$\partial f^2 = \lambda\,\partial f^1\,, \tag{11.2}$$

for some constant λ. As we discussed in section 10.2, this necessarily implies that the internal space of the AdS_6 solutions has a boundary. While such solutions are not interesting from a holographic perspective, we can nonetheless compute the consistent truncation Ansatz, which we have given in 10.2.1, and which leads to F(4) gauged SUGRA coupled to two Abelian vector multiplets.

Consistent truncation with three vector multiplets. To allow for a consistent truncation with three vector multiplets, the following differential condition must be satisfied:

$$d\pi^\alpha = \frac{1}{r}\,p^\alpha\,\pi_\beta \wedge \pi^\beta\,, \tag{11.3}$$

where

$$\pi^\alpha = \frac{1}{2}i\frac{p_\alpha \bar\partial \bar f^\alpha}{p_\beta \partial f^\beta}\,\partial f^\alpha\,d\bar z - \frac{1}{2}i\frac{p_\alpha \partial f^\alpha}{p_\beta \bar\partial \bar f^\beta}\,\bar\partial \bar f^\alpha dz\,, \tag{11.4}$$

and $dr = -k_\alpha\,dp^\alpha$ with p^α, k^α the real/imaginary parts of the holomorphic functions $f^\alpha = -p^\alpha + i\,k^\alpha$. For any pair of holomorphic functions f^α satisfying the above condition, there is a consistent truncation of IIB SUGRA around that AdS_6 solution to 6-dimensional half-maximal ISO(3) gauged SUGRA. The uplift formulae for the scalar fields is given in section 10.3.1. It is unclear whether there are globally regular supersymmetric AdS_6 solutions satisfying the differential conditions (11.3).

Consistent truncation with four vector multiplets. To admit a consistent truncation with four vector multiplets, the AdS_6 vacua must satisfy the differential condition for the triplet, i.e. (11.3) with π^α as in (11.4), as well as

$$\partial\left(\frac{p_\beta \bar\partial \bar f^\beta}{p_\gamma \partial f^\gamma}\partial f^\alpha\right) - c.c. = 0\,. \tag{11.5}$$

For any pair of holomorphic functions f^α satisfying the above, the corresponding AdS_6 solution admits a consistent truncation to 6-dimensional half-maximal $\mathrm{ISO}(3)\times\mathrm{U}(1)$ gauged SUGRA. Once again, it is unclear whether there are such globally regular supersymmetric AdS_6 solutions of IIB SUGRA.

It would be interesting to classify for which Riemann surfaces Σ these consistent truncations exist, i.e. for which Riemann surfaces one can have holomorphic functions f^α which satisfy the above differential conditions and lead to closed internal manifolds, thus also satisfying (7.32) and (7.33), or even to find a complete list of such holomorphic functions. For example, it would be interesting to see whether some of the examples studied in [64, 65] for the case where the Riemann surface is a disc allow for consistent truncations with vector multiplets. The dual SCFTs have large global symmetry groups and one might hope that a subset of these symmetries could be captured via a consistent truncation.

For now, we are able to say that the Abelian T-dual of the Brandhuber-Oz solution admits a consistent truncation with one vector multiplet, the non-Abelian T-dual does not, and there are no globally regular solutions that admit a consistent truncation with two vector multiplets. Moreover, the only possible gauge groups in six dimensions are $\mathrm{SU}(2)\times\mathrm{U}(1)$, $\mathrm{SU}(2)\times\mathrm{U}(1)^2$, $\mathrm{ISO}(3)$ and $\mathrm{ISO}(3)\times\mathrm{U}(1)$. This is only a subset of all possible 6-dimensional half-maximal gauged SUGRAs that admit supersymmetric AdS vacua [52]. The other six-dimensional gauged SUGRAs do not have uplifts to IIB SUGRA.

Our results can be used to uplift the 6-dimensional solutions found in [66–69][3] and to complete their holographic study, while they also suggest that there are no IIB uplifts of the 6-dimensional solutions [70] which requires the six-dimensional gauge group $\mathrm{SU}(2)\times\mathrm{SU}(2)\,(\times\mathrm{U}(1))$. Similarly, we found that of all the 7-dimensional half-maximal gauged SUGRAs that admit a supersymmetric AdS_7 vacuum [58], only the pure SU(2) gauged SUGRA [47] and it coupled to an Abelian vector multiplet can be uplifted to IIA SUGRA, where in the latter case the Romans mass is necessarily zero. This suggests that the other 7-dimensional gauged SUGRAs with supersymmetric AdS_7 vacua are lower-dimensional artifacts without a clear relation to 10-dimensional SUGRA. However, as suggested in [71], one may also wonder whether it would be possible to construct consistent truncations of 10-/11-dimensional SUGRA coupled to DBI actions to obtain lower-dimensional gauged SUGRAs with higher-derivative terms. We leave this open challenge for the future.

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[3]While [69] studied 6-dimensional half-maximal $\mathrm{SU}(2)\times\mathrm{SU}(2)$ gSUGRA, the solutions constructed therein are also solutions of F(4) gSUGRA coupled to a single vector multiplet and therefore can be uplifted using our Ansatz.

A SL(5) ExFT conventions and ExFT/IIA dictionary

A.1 Embedding IIA into SL(5) ExFT

To embed IIA SUGRA in SL(5) ExFT we decompose $SL(5) \longrightarrow GL(4)^+ \longrightarrow GL(3)^+ \times \mathbb{R}^+$, where $GL(n)^+ = SL(n) \times \mathbb{R}^+$. The GL(4) is the geometric group realised by the internal manifold of a 11-dimensional compactification, which is broken to $GL(3) \times \mathbb{R}^+$ by reducing to IIA SUGRA. Accordingly, we decompose an object in the fundamental SL(5) representation as

$$F^a = \left(F^i,\, F^4,\, F^5\right) , \tag{A.1}$$

where $a = 1, \ldots, 5$ is the SL(5) fundamental index and $i = 1, 2, 3$ labels the fundamental of GL(3).

We will need to decompose the generalised tensors of the half-maximal structure, i.e. generalised vector fields and generalised tensors in the $\overline{\mathbf{5}}$ and $\mathbf{5}$ representation. A generalised vector field, $\mathcal{A}^{ab}$, decomposes as

$$\mathcal{A}^{i5} = V^i \,, \qquad \mathcal{A}^{ij} = -\epsilon^{ijk}\,\omega_{(1)k} \,, \qquad \mathcal{A}^{i4} = \frac{1}{2}\epsilon^{ijk}\,\omega_{(2)jk} \,, \qquad \mathcal{A}^{45} = \omega_{(0)} \,, \tag{A.2}$$

a generalised tensor field $\mathcal{B}_a$ in the $\overline{\mathbf{5}}$ as

$$\mathcal{B}^i = \frac{1}{2}\epsilon^{ijk}\,\omega_{(2)jk} \,, \qquad \mathcal{B}^4 = \frac{1}{3!}\epsilon^{ijk}\,\omega_{(3)ijk} \,, \qquad \mathcal{B}^5 = \omega_{(0)} \,, \tag{A.3}$$

and a generalised tensor field $\mathcal{C}^a$ in the $\mathbf{5}$ as

$$\mathcal{C}_i = \omega_{(1)i} \,, \qquad \mathcal{C}_4 = \omega_{(0)} \,, \qquad \mathcal{C}_5 = \frac{1}{3!}\epsilon^{ijk}\,\omega_{(3)ijk} \,, \tag{A.4}$$

where V are spacetime vector fields, $\omega_{(p)}$ are spacetime p-forms and $\epsilon^{ijk} = \pm 1$ denotes the three-dimensional alternating symbol, i.e. the tensor *density*.

Just as in the above, we also decompose the SL(5) "extended derivatives" as

$$\partial_{ab} = \left(\partial_{i5},\, \partial_{ij},\, \partial_{i4},\, \partial_{45}\right) , \tag{A.5}$$

These derivatives $\partial_{i5} \neq 0$ are the usual IIA internal spacetime derivatives, and solve the SL(5) ExFT section conditions

$$\partial_{[ab}\partial_{cd]} = 0 \,. \tag{A.6}$$

A.2 IIA parameterisation of the generalised metric

The IIA parameterisation of the SL(5) generalised metric is given in [17]. Here we translate the parameterisation given there to the string-frame metric which we use in section 6 when describing the supersymmetric AdS_7 vacua. The components of the generalised metric

$\mathcal{M}^{ab}$ are parameterised as

$$\begin{aligned}
\mathcal{M}^{ij} &= |g|^{2/5} e^{2\psi/5} \left(g^{ij} + |g|^{-1} B^i B^j \right) , \\
\mathcal{M}^{i4} &= |g|^{2/5} e^{2\psi/5} \left(-A^i + |g|^{-1} B^i C \right) , \\
\mathcal{M}^{i5} &= -|g|^{-3/5} e^{2\psi/5} B^i , \\
\mathcal{M}^{44} &= |g|^{2/5} e^{2\psi/5} \left(e^{-2\psi} + g_{ij} A^i A^j + |g|^{-1} C^2 \right) , \\
\mathcal{M}^{45} &= -|g|^{-3/5} e^{2\psi/5} C , \\
\mathcal{M}^{55} &= |g|^{-3/5} e^{2\psi/5} ,
\end{aligned} \tag{A.7}$$

where g_{ij} is the internal 3-dimensional IIA string frame metric, A_i is the 1-form, B_{ij} is the 2-form and C_{ijk} is the 3-form. The 2- and 3-form appear as $B^i = \frac{1}{2}\epsilon^{ijk} B_{jk}$ and $C = \frac{1}{3!}\epsilon^{ijk} C_{ijk}$ where $\epsilon^{ijk} = \pm 1$ is the alternating symbol, i.e. a tensor *density*.

A.3 Including the Romans mass

As discussed in [19, 20], the Romans mass of IIA SUGRA appears like a deformation of the differential structure of ExFT and EGG, similar to a gauging of lower-dimensional gauged SUGRAs. In particular, the generalised Lie derivative (2.1) now takes the form

$$\mathcal{L}_\xi V^{ab} = \mathcal{L}_\xi^{(0)} V^{ab} + \frac{1}{2} Z^{cd,[a} V^{b]e} \xi^{fg} \epsilon_{cdefg} , \tag{A.8}$$

where $\mathcal{L}^{(0)}$ is the undeformed generalised Lie derivative (2.1) and $Z^{ab,c}$ satisfies $Z^{[ab,c]} = 0$ and encodes the deformation of the generalised Lie derivative. For the Romans mass m, the only non-vanishing component of $Z^{ab,c}$ is

$$Z^{45,4} = m . \tag{A.9}$$

The deformation $Z^{ab,c}$ generates an SL(5) transformation and thus can easily be worked out for the generalised Lie derivative acting in another representation. In particular, to describe AdS_7 vacua, we require the massive generalised Lie derivative acting in the **5** which is

$$\mathcal{L}_\xi \mathcal{B}^a = \mathcal{L}_\xi^{(0)} \mathcal{B}^a - \frac{1}{4} Z^{bc,a} \mathcal{B}^d \xi^{ef} \epsilon_{bcdef} . \tag{A.10}$$

The differential operators d of (2.11) are also modified. Their deformations by $Z^{ab,c}$ can be determined by requiring them to be covariant under the deformed generalised Lie derivative (A.8). In fact, the d operator appearing in the differential conditions remains unmodified

$$d\mathcal{C}_a = d^{(0)} \mathcal{C}_a , \tag{A.11}$$

where $d^{(0)}$ is the unmodified $d : \Gamma(\mathcal{R}_3) \longrightarrow \Gamma(\mathcal{R}_2)$ given in (2.11).

B SO(5,5) ExFT conventions and ExFT/IIB dictionary

B.1 Embedding IIB into SO(5,5) ExFT

To connect the SO(5,5) ExFT with IIB SUGRA we decompose $SO(5,5) \longrightarrow SL(4) \times SL(2)_S \times SL(2)_A$, where $SL(2)_S$ corresponds to S-duality while $SL(2)_A$ is an accidental symmetry in the decomposition relevant to six dimensions and which will be broken by the IIB solution to the section condition [53, 72]. For our purposes, we will need the decomposition of the **16** and **10** representations of SO(5,5) which is

$$\begin{aligned} \mathbf{16} &\longrightarrow (\mathbf{4},\mathbf{2},\mathbf{1}) \oplus \left(\bar{\mathbf{4}},\mathbf{1},\mathbf{2}\right) , \\ \mathbf{10} &\longrightarrow (\mathbf{1},\mathbf{2},\mathbf{2}) \oplus (\mathbf{6},\mathbf{1},\mathbf{1}) . \end{aligned} \tag{B.1}$$

Thus, a generalised vector field becomes

$$\mathcal{A}^M = \left(\mathcal{A}^{U,i},\, \mathcal{A}^{\alpha}{}_i\right) , \tag{B.2}$$

where we use $i = 1,\ldots,4$ for the SL(4) spatial indices, $\alpha = 1,2$ as $\mathrm{SL}(2)_S$ indices and $U,V = +,-$ for the $\mathrm{SL}(2)_A$ indices. We identify these components with spacetime tensors as follows

$$\mathcal{A}^{+,i} = V^i\,, \qquad \mathcal{A}^{-,i} = \frac{1}{3!}\epsilon^{ijkl}\,\omega_{(3)jkl}\,, \qquad \mathcal{A}^{\alpha}{}_i = \omega_{(1)}{}^{\alpha}{}_i\,, \tag{B.3}$$

where V is a spacetime tensor, $\omega_{(p)}$ are spacetime p-forms, α is as before a fundamental $\mathrm{SL}(2)_S$ index and ϵ^{ijkl} is the 4-dimensional alternating symbol, i.e. tensor *density*.

Similarly, a tensor in the **10** decomposes as $\mathcal{B}^I = \left(\mathcal{B}^{U,\alpha},\, \mathcal{B}^{ij}\right)$ which contain the spacetime tensors

$$\mathcal{B}^{+,\alpha} = \omega_{(0)}{}^{\alpha}\,, \qquad \mathcal{B}^{-,\alpha} = \frac{1}{4!}\epsilon^{ijkl}\,\omega_{(4)ijkl}\,, \qquad \mathcal{B}^{ij} = \frac{1}{2}\epsilon^{ijkl}\,\omega_{(2)kl}\,, \tag{B.4}$$

where $\omega_{(p)}$ are p-forms and $\alpha = 1,2$ is an $\mathrm{SL}(2)_S$ index.

Furthermore, with these conventions the SO(5,5) invariant metric is given by

$$\eta_{IJ} = \begin{pmatrix} \epsilon_{\alpha\beta}\epsilon_{UV} & 0 \\ 0 & \epsilon_{ijkl} \end{pmatrix} , \tag{B.5}$$

with inverse

$$\eta^{IJ} = \begin{pmatrix} \epsilon^{\alpha\beta}\epsilon^{UV} & 0 \\ 0 & \epsilon^{ijkl} \end{pmatrix} . \tag{B.6}$$

We employ the following summation convention over the **10** indices

$$\mathcal{B}_1{}^I\mathcal{B}_2{}^J\eta_{IJ} = \mathcal{B}_1{}^I\mathcal{B}_{2\,I} = \mathcal{B}_1{}^{U,\alpha}\mathcal{B}_{2\,U,\alpha} + \frac{1}{2}\mathcal{B}_1{}^{ij}\mathcal{B}_{2\,ij}\,. \tag{B.7}$$

The identity matrix in the **10** takes the following form due to the summation convention (B.7)

$$\delta_I^J = \begin{pmatrix} \delta_\alpha^\beta\delta_U^V & 0 \\ 0 & 2\delta_{ij}^{kl} \end{pmatrix} , \tag{B.8}$$

where $\delta_{ij}^{kl} = \frac{1}{2}\left(\delta_i^k\delta_j^l - \delta_i^l\delta_j^k\right)$.

Finally, the $(\gamma_I)^{MN}$-matrices are given by

$$\begin{aligned} (\gamma_{\alpha\,U})^{V\,i\,\beta}{}_j &= \sqrt{2}\,\delta_\alpha^\beta\,\delta_U^V\,\delta_j^i\,, \\ (\gamma_{ij})^{V\,k\,W\,l} &= 2\sqrt{2}\epsilon^{VW}\,\delta_{ij}^{kl}\,, \\ (\gamma_{ij})^{\beta}{}_k{}^{\gamma}{}_l &= -\sqrt{2}\epsilon_{ijkl}\epsilon^{\beta\gamma}\,, \end{aligned} \tag{B.9}$$

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and the $(\gamma_I)_{MN}$-matrices are

$$\begin{aligned}(\gamma_{\alpha U})_{V\,i\,\beta}{}^{j} &= \sqrt{2}\epsilon_{\alpha\beta}\epsilon_{UV}\delta_i{}^j\,,\\ (\gamma_{ij})_{V\,k\,W\,l} &= \sqrt{2}\epsilon_{VW}\epsilon_{ijkl}\,,\\ (\gamma_{ij})_{\beta}{}^{k}{}_{\gamma}{}^{l} &= -2\sqrt{2}\delta^{kl}_{ij}\epsilon_{\beta\gamma}\,.\end{aligned} \tag{B.10}$$

With the above decomposition, the "extended derivatives" are given by

$$\partial_M = \left(\partial_{U,i},\, \partial_\alpha{}^i\right)\,, \tag{B.11}$$

with only $\partial_{+,i} \neq 0$. This corresponds to the IIB solution of the section condition (2.2)

$$(\gamma_I)^{MN}\,\partial_M \otimes \partial_N = 0\,, \tag{B.12}$$

which we use.

B.2 IIB parameterisation of the generalised metric

Here we give the IIB parameterisation of the SO(5, 5) generalised metric in the **16** and **10** representations. The generalised metric in the **16** is given by

$$\begin{aligned}\mathcal{M}_{+i\,+j} &= e^{1/2}g_{ij} + e^{-3/2}\left(C_{(4)}^2\,g_{ij} + \frac{1}{4}C_{ik\,\alpha}\,\beta^{kr\,\alpha}C_{js\,\gamma}\beta^{st\,\gamma}\,g_{rt}\right)\\ &\quad - \frac{1}{2}e^{-3/2}C_{(4)}\left(g_{ir}\,C_{jk\,\alpha}\beta^{kr\,\alpha} + (i\leftrightarrow j)\right) + e^{1/2}\,C_{ik\,\alpha}\,C_{jl\,\gamma}\,g^{kl}\,H^{\alpha\gamma}\,,\\ \mathcal{M}_{+i\,-j} &= e^{-3/2}\left(C_{(4)}\,g_{ij} - \frac{1}{2}C_{ik\,\alpha}\beta^{kl\,\alpha}g_{lj}\right)\,,\\ \mathcal{M}_{-i\,-j} &= e^{-3/2}g_{ij}\,,\\ \mathcal{M}_{+i\;\alpha}{}^{j} &= e^{-3/2}\left(C_{(4)}\,g_{ik}\,\beta^{jk}{}_{\alpha} - \frac{1}{2}C_{ik\,\gamma}\,\beta^{kl\,\gamma}\,g_{lm}\,\beta^{jm}{}_{\alpha}\right) - e^{1/2}C_{ik\,\gamma}g^{kj}H^{\gamma}{}_{\alpha}\,,\\ \mathcal{M}_{-i\;\alpha}{}^{j} &= e^{-3/2}g_{ik}\,\beta^{jk}{}_{\alpha}\,,\\ \mathcal{M}_{\alpha}{}^{i}{}_{\beta}{}^{j} &= e^{1/2}g^{ij}H_{\alpha\beta} + e^{-3/2}\beta^{ik}{}_{\alpha}\,\beta^{jl}{}_{\beta}\,g_{kl}\,.\end{aligned} \tag{B.13}$$

Here g_{ij} is the internal 4-d Einstein-frame metric, $C_{(4)} = \frac{1}{4!}\epsilon^{ijkl}C_{ijkl}$ is the dual of the fully internal 4-form, $C_{ij\,\alpha}$ denotes the SL(2)-dual of R-R 2-forms and $\beta^{ij}{}_{\alpha} = \frac{1}{2}\epsilon^{ijkl}C_{kl\,\alpha}$ is its dual. Throughout we dualise with $\epsilon^{ijkl} = \pm 1$, the four-dimensional alternating symbol, i.e. the tensor *density*. $H_{\alpha\beta}$ is the SL(2) matrix parameterised by the axio-dilaton $\tau = e^{\psi} + i\,C_0$,

$$H_{\alpha\beta} = \frac{1}{\operatorname{Im}\tau}\begin{pmatrix}|\tau|^2 & \operatorname{Re}\tau\\ \operatorname{Re}\tau & 1\end{pmatrix}\,. \tag{B.14}$$

All our $SL(2)_S$ indices are raised/lowered by the SL(2) invariant $\epsilon_{\alpha\beta} = \epsilon^{\alpha\beta} = \pm 1$ in a Northwest/Southeast convention. The $\epsilon_{\alpha\beta}$'s are normalised as

$$\epsilon_{\alpha\gamma}\epsilon^{\beta\gamma} = \delta^{\beta}_{\alpha}\,. \tag{B.15}$$

The generalised metric in the **10** is given by

$$\begin{aligned}
\mathcal{M}_{+\alpha+\beta} &= \frac{1}{e}\left(e^2+C_{(4)}^2\right)H_{\alpha\beta}+\frac{1}{4e}\star\left(C_{(2)\alpha}\wedge C_{(2)\gamma}\right)\star\left(C_{(2)\beta}\wedge C_{(2)\delta}\right)H^{\gamma\delta}\\
&\quad+\frac{C_{(4)}}{2e}\left(H_{\alpha_i\gamma}\star\left(C_{(2)\beta}\wedge C_{(2)\delta}\right)\epsilon^{\gamma\delta}+(\alpha\leftrightarrow\beta)\right)+\frac{e}{2}\,C_{(2)ij\,\alpha_i}\,C_{(2)kl\,\beta}\,g^{ik}g^{jl}\,,\\
\mathcal{M}_{+\alpha-\beta} &= \frac{C_{(4)}}{e}H_{\alpha\beta}+\frac{1}{2e}\star\left(C_{(2)\alpha}\wedge C_{(2)}{}^{\gamma}\right)H_{\gamma\beta}\,,\\
\mathcal{M}_{-\alpha-\beta} &= \frac{1}{e}H_{\alpha\beta}\,,\\
\mathcal{M}_{+\alpha}{}^{ij} &= \frac{1}{e}\left(C_{(4)}H_{\alpha\beta}+\frac{1}{2}\star\left(C_{(2)\alpha}\wedge C_{(2)}{}^{\gamma}\right)H_{\gamma\beta}\right)\beta^{j_1j_2\,\beta}+e\,g^{ik}\,g^{jl}\,C_{(2)\alpha\,kl}\,,\\
\mathcal{M}_{\alpha-}{}^{ij} &= \frac{1}{e}H_{\alpha\beta}\,\beta^{ij\,\beta}\,,\\
\mathcal{M}^{ij\,kl} &= e\left(g^{ik}g^{jl}-g^{il}g^{jk}\right)+\frac{1}{e}\beta^{ij}{}_{\alpha}\,\beta^{kl}{}_{\beta}\,H^{\alpha\beta}\,.
\end{aligned} \tag{B.16}$$

B.3 IIB parameterisation of the ExFT tensor hierarchy

To complete the embedding of type IIB supergravity into exceptional field theory, one needs to embed the supergravity fields with legs along both the internal and external directions. These are encoded into the the ExFT tensor hierarchy fields $\mathcal{A}_\mu$, $\mathcal{B}_{\mu\nu}$, …The map between supergravity and ExFT fields can be obtained by comparing how both transform under gauge transformations or by comparing their corresponding field strengths. We summarise the findings in the next section B.3.1 and give details of the derivations in sections B.3.2–B.3.4.

B.3.1 Summary of IIB parameterisation

The 10-dimensional IIB metric is given by

$$ds_{10}^2 = g_{ij}\tilde{D}y^i\tilde{D}y^j+g_{\mu\nu}dx^\mu dx^\nu\,, \tag{B.17}$$

where g_{ij} is the internal four-dimensional metric as computed from the generalised metric (B.13), (B.16) and

$$\tilde{D}y^i = dy^i+(\imath_{A_\mu}dy^i)dx^\mu\,, \tag{B.18}$$

are the Kaluza-Klein covariantised derivatives of the internal coordinates with $A_\mu{}^i$ the Kaluza-Klein vector field. The "external" metric $g_{\mu\nu}$ is related to the ExFT metric $\mathcal{G}_{\mu\nu}$ by

$$\mathcal{G}_{\mu\nu} = g_{\mu\nu}|g_{\rm int}|^{-1/4}\,, \tag{B.19}$$

where $|g_{\rm int}|$ denotes the determinant of the internal metric g_{ij}.

For the remainder of this appendix, we will follow the conventions of [73] and denote the 10-dimensional type IIB supergravity gauge fields by a hat, i.e. $\hat{C}_{(\hat{2})}{}^{\alpha}$ and $\hat{C}_{(\hat{4})}$, unlike in the main part of this paper. We will reserve the unhatted objects for later purposes in this appendix. Under the splitting of the 10 dimensions into six external and four internal directions, we write them as

$$\hat{C}_{(\hat{2})}{}^{\alpha} = \frac{1}{2}\bar{C}_{\mu\nu}{}^{\alpha} dx^{\mu} \wedge dx^{\nu} + \bar{C}_{\mu n}{}^{\alpha} dx^{\mu} \wedge \tilde{D}y^{n} + \frac{1}{2}\bar{C}_{mn}{}^{\alpha}\tilde{D}y^{m} \wedge \tilde{D}y^{n}\,, \tag{B.20}$$

and, analogously, for $\hat{C}_{(\hat{4})}$. The fields $\bar{C}_{\mu\nu}{}^{\alpha}$, $\bar{C}_{\mu n}{}^{\alpha}$, ... are the components of the KK-redefined form-fields $\bar{C}_{\mu\nu\,(0)}{}^{\alpha}$, $\bar{C}_{\mu\,(1)}{}^{\alpha}$, ... defined in (B.39). The barred fields that are totally internal, i.e. $\bar{C}_{(2)}{}^{\alpha}$ and $\bar{C}_{(4)}$, are embedded into ExFT through the generalised metric (B.13), (B.16). The rest can be read off from the ExFT tensor hierarchy fields as (see (B.58))

$$\begin{aligned}
A_{\mu} &= (\mathcal{A}_{\mu})_{(v)}\,,\\
\bar{C}_{\mu\,(1)}{}^{\alpha} &= (\mathcal{A}_{\mu})_{(1)}{}^{\alpha}\,,\\
\bar{C}_{\mu\nu\,(0)}{}^{\alpha} &= \sqrt{2}\,(\mathcal{B}_{\mu\nu})_{(0)}{}^{\alpha} + \iota_{(\mathcal{A}_{[\mu})_{(v)}}(\mathcal{A}_{\nu]})_{(1)}{}^{\alpha}\\
\bar{C}_{\mu\,(3)} &= (\mathcal{A}_{\mu})_{(3)} + \frac{1}{2}\epsilon_{\alpha\beta}\bar{C}_{\mu\,(1)}{}^{\alpha} \wedge \bar{C}_{(2)}{}^{\beta}\,,\\
\bar{C}_{\mu\nu\,(2)} &= -\sqrt{2}\,(\mathcal{B}_{\mu\nu})_{(2)} + \iota_{(\mathcal{A}_{[\mu})_{(v)}}(\mathcal{A}_{\nu]})_{(3)} + \frac{1}{2}\epsilon_{\alpha\beta}\bar{C}_{\mu\nu\,(0)}{}^{\alpha}\bar{C}_{(2)}{}^{\beta}\,,
\end{aligned} \tag{B.21}$$

where $(\mathcal{A}_{\mu})_{(v)}$, $(\mathcal{A}_{\mu})_{(1)}{}^{\alpha}$, ..., $(\mathcal{B}_{\mu\nu})_{(0)}{}^{\alpha}$, ... are components of the tensor hierarchy fields $\mathcal{A}_{\mu}$ and $\mathcal{B}_{\mu\nu}$. The fields $\bar{C}_{\mu\nu\rho\,(1)}$ and $\bar{C}_{\mu\nu\rho\sigma\,(0)}$ involve further fields of the tensor hierarchy. However, they can also be determined (up to gauge transformations) from $\bar{C}_{(4)}$, $\bar{C}_{\mu\,(3)}$ and $\bar{C}_{\mu\nu\,(2)}$ through the self-duality condition of the 10-dimensional four-form.

In addition to the dictionaries between tensor hierarchy and supergravity gauge fields, one can also embed the supergravity field strengths $\hat{F}_{(\hat{3})}{}^{\alpha}$ and $\hat{F}_{(\hat{5})}$ into the ExFT field strengths. As in the case of gauge fields, we write the 10-dimensional field strengths as

$$\begin{aligned}
\hat{F}_{(\hat{3})}{}^{\alpha} = \frac{1}{3!}\bar{F}_{\mu\nu\rho}{}^{\alpha} dx^{\mu} \wedge dx^{\nu} \wedge dx^{\rho} + \frac{1}{2}\bar{F}_{\mu\nu m}{}^{\alpha} dx^{\mu} \wedge dx^{\nu} \wedge \tilde{D}y^{m}\\
+ \frac{1}{2}\bar{F}_{\mu mn}{}^{\alpha} dx^{\mu} \wedge \tilde{D}y^{m} \wedge \tilde{D}y^{n} + \frac{1}{3!}\bar{F}_{mnp}{}^{\alpha}\tilde{D}y^{m} \wedge \tilde{D}y^{n} \wedge \tilde{D}y^{p}\,,
\end{aligned} \tag{B.22}$$

and analogously for $\hat{F}_{(\hat{5})}$. The barred F fields are the components of the form-fields (B.45). Since the internal space is four-dimensional, the only 10-dimensional field strength with a totally internal part is $\hat{F}_{(\hat{3})}{}^{\alpha}$, given by

$$\bar{F}_{(3)}{}^{\alpha} = d\bar{C}_{(2)}{}^{\alpha}\,, \tag{B.23}$$

with $\bar{C}_{(2)}{}^{\alpha}$ the internal part of the 10-dimensional two-form, which is embedded into the ExFT through the generalised metric. Since all ExFT field strength have at least two external indices, the components of the 10-dimensional field strengths with one external leg can only be obtained directly from the gauge fields. These are (see (B.50) and (B.54))

$$\begin{aligned}
\bar{F}_{\mu\,(2)}{}^{\alpha} &= D_{\mu}^{KK}\bar{C}_{(2)}{}^{\alpha} - d\bar{C}_{\mu\,(1)}{}^{\alpha}\,,\\
\bar{F}_{\mu\,(4)} &= D_{\mu}^{KK}\bar{C}_{(4)} - d\bar{C}_{\mu\,(3)} - \frac{1}{2}\epsilon_{\alpha\beta}\bar{C}_{(2)}{}^{\alpha} \wedge \bar{F}_{\mu\,(2)}{}^{\beta} - \frac{1}{2}\epsilon_{\alpha\beta}\bar{C}_{\mu\,(1)}{}^{\alpha} \wedge \bar{F}_{(3)}{}^{\beta}\,.
\end{aligned} \tag{B.24}$$

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The rest of the components can be read off from the field strengths of the ExFT tensor hierarchy fields as (see (B.60))

$$
\begin{aligned}
F_{\mu\nu} &= (\mathcal{F}_{\mu\nu})_{(v)}\,, \\
\bar{F}_{\mu\nu\,(1)}{}^{\alpha} &= (\mathcal{F}_{\mu\nu})_{(1)}{}^{\alpha} + \iota_{F_{\mu\nu}}\bar{C}_{(2)}{}^{\alpha}\,, \\
\bar{F}_{\mu\nu\rho\,(0)}{}^{\alpha} &= \sqrt{2}\,(\mathcal{H}_{\mu\nu\rho})_{(0)}{}^{\alpha}\,, \\
\bar{F}_{\mu\nu\,(3)} &= (\mathcal{F}_{\mu\nu})_{(3)} + \epsilon_{\alpha\beta}\bar{F}_{\mu\nu\,(1)}{}^{\alpha}\wedge\bar{C}_{(2)}{}^{\beta} + \iota_{F_{\mu\nu}}\bar{C}_{(4)} + \frac{1}{2}\epsilon_{\alpha\beta}\bar{C}_{(2)}{}^{\alpha}\wedge\iota_{F_{\mu\nu}}\bar{C}_{(2)}{}^{\beta}\,, \\
\bar{F}_{\mu\nu\rho\,(2)} &= -\sqrt{2}\,(\mathcal{H}_{\mu\nu\rho})_{(2)} + \epsilon_{\alpha\beta}\bar{F}_{\mu\nu\rho\,(2)}{}^{\alpha}\bar{C}_{(2)}{}^{\beta}\,,
\end{aligned}
\tag{B.25}
$$

where $F_{\mu\nu}$ is the field strength of the KK gauge field A_μ (see (B.40)) and $(\mathcal{F}_{\mu\nu})_{(v)}$, ..., $(\mathcal{H}_{\mu\nu\rho})_{(0)}{}^{\alpha}$, ... are the components of the ExFT field strengths $\mathcal{F}_{\mu\nu}$ and $\mathcal{H}_{\mu\nu\rho}$ defined in (2.5) and (2.7). As for the gauge fields, the components $\bar{F}_{\mu\nu\rho\sigma,(1)}$ and $\bar{F}_{\mu\nu\rho\sigma\delta,(0)}$ can be obtained from the 10-dimensional self-duality condition for $\hat{F}_{(\hat{5})}$.

B.3.2 Tensor hierarchy of SO(5,5) exceptional field theory

The tensor hierarchy of SO(5,5) ExFT containts the fields $\mathcal{A}_\mu$, $\mathcal{B}_{\mu\nu}$, $\mathcal{C}_{\mu\nu\rho}$, ... as listed in equation (2.14). As discussed in section B.1, taking the type IIB solution to the section constraint [53, 72] these decompose into

$$
\begin{aligned}
\mathcal{A}_\mu &= A_{\mu,(v)} + A_{\mu\,(1)}{}^{\alpha} + A_{\mu\,(3)}\,, \\
\mathcal{B}_{\mu\nu} &= B_{\mu\nu\,(0)}{}^{\alpha} + B_{\mu\nu\,(2)} + B_{\mu\nu\,(4)}{}^{\alpha}\,, \\
\mathcal{C}_{\mu\nu\rho} &= C_{\mu\nu\rho\,(1)} + \bar{C}_{\mu\nu\rho\,(1)} + C_{\mu\nu\rho\,(3)\,\alpha}\,.
\end{aligned}
\tag{B.26}
$$

The gauge variations of $\mathcal{A}_\mu$ and $\mathcal{B}_{\mu\nu}$ are given by

$$
\begin{aligned}
\delta\mathcal{A}_\mu &= \mathcal{D}_\mu\Lambda - d\,\Xi_\mu\,, \\
\delta\mathcal{B}_{\mu\nu} &= 2\mathcal{D}_{[\mu}\Xi_{\nu]} + \Lambda\wedge\mathcal{F}_{\mu\nu} - \mathcal{A}_{[\mu}\wedge\delta\mathcal{A}_{\nu]} - d\,\Theta_{\mu\nu}\,,
\end{aligned}
\tag{B.27}
$$

where $\Lambda\in\mathbf{16}$, $\Xi_\mu\in\mathbf{10}$ and $\Theta_{\mu\nu}\in\overline{\mathbf{16}}$ are generalised gauge parameters, the derivative $\mathcal{D}_\mu$ is defined as

$$
\mathcal{D}_\mu = \partial_\mu - \mathcal{L}_{\mathcal{A}_\mu}\,,
\tag{B.28}
$$

and $\mathcal{F}_{\mu\nu}$ is the field strength of $\mathcal{A}_\mu$ defined in (2.5).

Gauge variations and field strength of $\mathcal{A}_\mu$. In the type IIB solution of the section constraint, the variation $\delta\mathcal{A}_\mu$ decomposes as

$$
\begin{aligned}
\delta A_{\mu\,(v)} &= D^{KK}_\mu\Lambda_{(v)}\,, \\
\delta A_{\mu\,(1)}{}^{\alpha} &= D^{KK}_\mu\Lambda_{(1)}{}^{\alpha} + L_{\Lambda_{(v)}}A_{\mu\,(1)}{}^{\alpha} - \,d\tilde{\Xi}_{\mu\,(0)}{}^{\alpha}\,, \\
\delta A_{\mu\,(3)} &= D^{KK}_\mu\Lambda_{(3)} + L_{\Lambda_{(v)}}A_{\mu\,(3)} - \epsilon_{\alpha\beta}dA_{\mu\,(1)}{}^{\alpha}\wedge\Lambda_{(1)}{}^{\beta} + \,d\tilde{\Xi}_{\mu\,(2)}\,,
\end{aligned}
\tag{B.29}
$$

where now L is the usual Lie derivatives, the derivative D^{KK}_μ is defined as

$$
D^{KK}_\mu = \partial_\mu - L_{A_{\mu\,(v)}}\,,
\tag{B.30}
$$

and the fields $\tilde{\Xi}_{\mu\,(0)}{}^\alpha$ and $\tilde{\Xi}_{\mu\,(2)}$ are

$$\begin{aligned}\tilde{\Xi}_{\mu\,(0)}{}^\alpha &= \sqrt{2}\,\Xi_{\mu\,(0)}{}^\alpha + \iota_{\Lambda_{(v)}} A_{\mu\,(1)}\,,\\ \tilde{\Xi}_{\mu\,(2)} &= \sqrt{2}\,\Xi_{\mu\,(2)} - \iota_{\Lambda_{(v)}} A_{\mu\,(3)}\,,\end{aligned} \tag{B.31}$$

with $\Xi_{\mu\,(0)}{}^\alpha$ and $\Xi_{\mu\,(2)}$ being the zero- and two- form components of the gauge parameter Ξ_μ. The field strength $\mathcal{F}_{\mu\nu}$, defined in (2.5), decomposes as

$$\begin{aligned}(\mathcal{F}_{\mu\nu})_{(v)} &= 2\partial_{[\mu}A_{\nu]\,(v)} - [A_{\mu\,(v)}, A_{\nu\,(v)}]\,,\\ (\mathcal{F}_{\mu\nu\,(1)})_{(1)}{}^\alpha &= 2D^{KK}_{[\mu}A_{\nu]\,(1)}{}^\alpha + d\tilde{B}_{\mu\nu\,(0)}{}^\alpha\,,\\ (\mathcal{F}_{\mu\nu})_{(3)} &= 2D^{KK}_{[\mu}A_{\nu]\,(3)} - d\tilde{B}_{\mu\nu\,(2)} - \epsilon_{\alpha\beta}A_{[\mu\,(1)}{}^\alpha \wedge dA_{\nu]\,(1)}{}^\beta\,,\end{aligned} \tag{B.32}$$

where, analogously to (B.31), the fields $\tilde{B}_{\mu\nu\,(0)}{}^\alpha$ and $\tilde{B}_{\mu\nu\,(2)}$ are defined as

$$\begin{aligned}\tilde{B}_{\mu\nu\,(0)}{}^\alpha &= \sqrt{2}B_{\mu\nu\,(0)}{}^\alpha + \iota_{A_{[\mu\,(v)}}A_{\nu]\,(1)}{}^\alpha\,,\\ \tilde{B}_{\mu\nu\,(2)} &= \sqrt{2}B_{\mu\nu\,(2)} - \iota_{A_{[\mu\,(v)}}A_{\nu]\,(3)}\,,\end{aligned} \tag{B.33}$$

with $B_{\mu\nu\,(0)}{}^\alpha$ and $B_{\mu\nu\,(2)}$ components of $\mathcal{B}_{\mu\nu}$.

Gauge variations and field strength of $\boldsymbol{\mathcal{B}_{\mu\nu}}$. The fields $\tilde{B}_{\mu\nu\,(0)}{}^\alpha$ and $\tilde{B}_{\mu\nu\,(2)}$ transform under gauge variations as

$$\begin{aligned}\delta\tilde{B}_{\mu\nu\,(0)}{}^\alpha &= \sqrt{2}\,(\delta\mathcal{B}_{\mu\nu})_{(0)}{}^\alpha + \iota_{\delta A_{[\mu\,(v)}}A_{\nu]\,(1)}{}^\alpha + \iota_{A_{[\mu\,(v)}}\delta A_{\nu]\,(1)}{}^\alpha\\ &= 2D^{KK}_{[\mu}\tilde{\Xi}_{\nu]\,(0)}{}^\alpha + L_{\Lambda_{(v)}}\tilde{B}_{\mu\nu\,(0)}{}^\alpha + \iota_{(\mathcal{F}_{\mu\nu})_{(v)}}\Lambda_{(1)}{}^\alpha\,,\\ \delta\tilde{B}_{\mu\nu\,(2)} &= \sqrt{2}\,(\delta\mathcal{B}_{\mu\nu})_{(0)} - \iota_{\delta A_{[\mu\,(v)}}A_{\nu]\,(3)} - \iota_{A_{[\mu\,(v)}}\delta A_{\nu]\,(3)}\\ &= 2D^{KK}_{[\mu}\left(\tilde{\Xi}_{\nu]\,(2)} - \frac{1}{2}\epsilon_{\alpha\beta}A_{\nu]\,(1)}{}^\alpha \wedge \Lambda^\beta_{(1)}\right) + \epsilon_{\alpha\beta}dA_{[\mu\,(1)}{}^\alpha\tilde{\Xi}_{\nu]\,(0)}{}^\beta + L_{\Lambda_{(v)}}\tilde{B}_{\mu\nu\,(2)}\\ &\quad - \frac{1}{2}\epsilon_{\alpha\beta}\Lambda_{(1)}{}^\alpha \wedge d\tilde{B}_{\mu\nu\,(0)}{}^\beta - \iota_{(\mathcal{F}_{\mu\nu})_{(v)}}\Lambda_{(3)} - \frac{1}{2}\epsilon_{\alpha\beta}(\mathcal{F}_{\mu\nu})_{(1)}{}^\alpha \wedge \Lambda_{(1)}{}^\beta + d\tilde{\Theta}_{\mu\nu\,(1)}\,,\end{aligned} \tag{B.34}$$

where the field $\tilde{\Theta}_{\mu\nu\,(1)}$ is a redefinition of the one-form part of $\Theta_{\mu\nu}$. Finally, the field strengths of the fields $\tilde{B}_{\mu\nu\,(0)}{}^\alpha$ and $\tilde{B}_{\mu\nu\,(2)}$ can be obtained from the field strength $\mathcal{H}_{\mu\nu\rho}$ (without tilde), defined in (2.7), as

$$\begin{aligned}\tilde{H}_{\mu\nu\rho\,(0)}{}^\alpha &\equiv \sqrt{2}\,(\mathcal{H}_{\mu\nu\rho})_{(0)}{}^\alpha = 3D^{KK}_{[\mu}\tilde{B}_{\nu\rho]\,(0)}{}^\alpha - 3\,\iota_{(\mathcal{F}_{[\mu\nu})_{(v)}}A_{\rho]\,(1)}{}^\alpha\,,\\ \tilde{H}_{\mu\nu\rho\,(2)} &\equiv \sqrt{2}\,(\mathcal{H}_{\mu\nu\rho})_{(2)}\\ &= 3D^{KK}_{[\mu}\tilde{B}_{\nu\rho]\,(2)} + 3\,\iota_{(\mathcal{F}_{[\mu\nu})_{(v)}}A_{\rho]\,(3)} + 3\epsilon_{\alpha\beta}A_{[\mu\,(1)}{}^\alpha \wedge D^{KK}_{\nu}A_{\rho]\,(1)}{}^\beta\\ &\quad + 3\epsilon_{\alpha\beta}\,dA_{[\mu\,(1)}{}^\alpha\tilde{B}_{\nu\rho]\,(0)}{}^\beta + d\tilde{C}_{\mu\nu\rho\,(1)}\,,\end{aligned} \tag{B.35}$$

where $\tilde{C}_{\mu\nu\rho\,(1)}$ is again some redefinition of the one-form part of $\tilde{C}_{\mu\nu\rho}$.

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B.3.3 KK decompositon of type IIB supergravity

The bosonic field content of type IIB supergravity is given by a metric field $\hat{g}$, an axio-dilaton field H, a SL(2)-doublet of two-form fields $\hat{C}_{(\hat{2})}{}^{\alpha}$ and a four-form field $\hat{C}_{(\hat{4})}$. The sub-index $(\hat{p})$ indicates that the object is a p-form from the 10-dimensional point of view. The gauge variations of $\hat{C}_{(\hat{2})}{}^{\alpha}$ and a $\hat{C}_{(\hat{4})}$ are

$$\begin{aligned}\delta\hat{C}_{\hat{\mu}\hat{\nu}}{}^{\alpha} &= d\hat{\lambda}_{(\hat{1})}{}^{\alpha}\,,\\ \delta\hat{C}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} &= d\hat{\lambda}_{(\hat{3})} + \frac{1}{2}\epsilon_{\alpha\beta}\hat{\lambda}_{(\hat{1})}{}^{\alpha}\wedge\hat{F}_{(\hat{3})}{}^{\beta}\,,\end{aligned} \tag{B.36}$$

and their field strengths

$$\begin{aligned}\hat{F}_{(\hat{3})}{}^{\alpha} &= d\hat{C}_{(\hat{2})}{}^{\alpha}\,,\\ \hat{F}_{(\hat{5})} &= d\hat{C}_{(\hat{4})} - \frac{1}{2}\epsilon_{\alpha\beta}\hat{C}_{(\hat{2})}{}^{\alpha}\wedge d\hat{C}_{(\hat{2})}{}^{\beta}\,.\end{aligned} \tag{B.37}$$

Next we split the 10-dimensional space into a six-dimensional external and a four-dimensional internal spaces. Throughout the rest of this section, we will use the field redefinitions of [73]. Our conventions for the coordinates are: $x^{\hat{\mu}}$ are the ten dimensional coordinates, x^{μ} are the external ones and y^n the internal, with $\hat{\mu} = 1,\ldots,10$, $\mu = 1,\ldots,6$ and $n = 1,\ldots,4$. The two-forms fields $\hat{C}_{(\hat{2})}{}^{\alpha}$ decomposes under this splitting as

$$\begin{aligned}\hat{C}_{(\hat{2})}{}^{\alpha} &= \frac{1}{2}\hat{C}_{\hat{\mu}\hat{\nu}}{}^{\alpha}dx^{\hat{\mu}}\wedge dx^{\hat{\nu}}\\ &= \frac{1}{2}\hat{C}_{\mu\nu}{}^{\alpha}dx^{\mu}\wedge dx^{\nu} + \hat{C}_{\mu n}{}^{\alpha}dx^{\mu}\wedge dy^{n} + \frac{1}{2}\hat{C}_{mn}{}^{\alpha}dy^{m}\wedge dy^{n}\\ &\equiv \frac{1}{2}\hat{C}_{\mu\nu\,(0)}{}^{\alpha}dx^{\mu}\wedge dx^{\nu} + dx^{\mu}\wedge\hat{C}_{\mu\,(1)}{}^{\alpha} + \hat{C}_{(2)}{}^{\alpha}\,,\end{aligned} \tag{B.38}$$

where now the subscript (p) indicates that the object is a p-form from the point of view of the internal space. The four-form $\hat{C}_{(\hat{4})}$ decomposes in an analogous way. Next, in a standard Kaluza-Klein manner, we redefine these form-fields by projecting the 10-dimensional curved indices into six-dimensional ones using the projector $P_{\mu}{}^{\hat{\mu}} = e_{\mu}{}^{a}e_{a}{}^{\hat{\mu}}$, where a are the external flat indices and $e_{\hat{\mu}}{}^{\hat{a}}$ is the 10-dimensional metric vielbein in a frame where it is upper-triangular. We obtain

$$\begin{aligned}\bar{C}_{(2)}{}^{\alpha} &= \hat{C}_{(2)}{}^{\alpha}\,,\\ \bar{C}_{\mu\,(1)}{}^{\alpha} &= \hat{C}_{\mu\,(1)}{}^{\alpha} - \iota_{A_\mu}\hat{C}_{(2)}{}^{\alpha}\,,\\ \bar{C}_{\mu\nu\,(0)}{}^{\alpha} &= \hat{C}_{\mu\nu\,(0)}{}^{\alpha} + 2\iota_{A_{[\mu}}\hat{C}_{\nu]\,(1)}{}^{\alpha} - \iota_{A_\mu}\iota_{A_\nu}\hat{C}_{(2)}{}^{\alpha}\,,\\ \bar{C}_{(4)} &= \hat{C}_{(4)}\,,\\ \bar{C}_{\mu\,(3)} &= \hat{C}_{\mu\,(3)} - \iota_{A_\mu}\hat{C}_{(4)}\,,\\ \bar{C}_{\mu\nu\,(2)} &= \hat{C}_{\mu\nu\,(2)} + 2\iota_{A_{[\mu}}\hat{C}_{\nu]\,(3)} - \iota_{A_\mu}\iota_{A_\nu}\hat{C}_{(4)}\,,\\ &\;\;\vdots\end{aligned} \tag{B.39}$$

where A_{μ} is the KK gauge field, with field strength

$$F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]} - [A_{\mu}, A_{\nu}]\,. \tag{B.40}$$

For computational purposes it is worth noticing that the 10-dimensional two-forms (B.38) can now be written as

$$\hat{C}_{(\hat{2})}{}^{\alpha} = \frac{1}{2}\bar{C}_{\mu\nu}{}^{\alpha} dx^{\mu} \wedge dx^{\nu} + \bar{C}_{\mu n}{}^{\alpha} dx^{\mu} \wedge \tilde{D}y^{n} + \frac{1}{2}\bar{C}_{mn}{}^{\alpha}\tilde{D}y^{m} \wedge \tilde{D}y^{n} \,, \tag{B.41}$$

with

$$\tilde{D}y^{n} = dy^{n} + (\iota_{A_\mu} dy^{n})\, dx^{\mu} \,, \tag{B.42}$$

and analogously for any other 10-dimensional form. Furthemore, because of the Chern-Simons term in the five-form field strength, the four-form needs some extra redefinitions (see for instance [73, 74]), namely

$$\begin{aligned} C_{(4)} = \bar{C}_{(4)} \,, \qquad & C_{\mu\,(3)} = \bar{C}_{\mu\,(3)} - \frac{1}{2}\epsilon_{\alpha\beta}\bar{C}_{\mu\,(1)}{}^{\alpha} \wedge \bar{C}_{(2)}{}^{\beta} \,, \\ C_{\mu\nu\,(2)} = \bar{C}_{\mu\nu\,(2)} - \frac{1}{2}\epsilon_{\alpha\beta}\bar{C}_{\mu\nu\,(0)}{}^{\alpha}\bar{C}_{(2)}{}^{\beta} \,, \end{aligned} \tag{B.43}$$

For completeness, we also define

$$C_{(2)}{}^{\alpha} = \bar{C}_{(2)}{}^{\alpha} \,, \qquad C_{\mu\,(1)}{}^{\alpha} = \bar{C}_{\mu\,(1)}{}^{\alpha} \,, \qquad C_{\mu\nu\,(0)}{}^{\alpha} = \bar{C}_{\mu\nu\,(0)}{}^{\alpha} \,. \tag{B.44}$$

Analogous definitions apply also for the field strengths. In particular, now,

$$\begin{aligned} \bar{F}_{(3)}{}^{\alpha} &= \hat{F}_{(3)}{}^{\alpha} \,, \\ \bar{F}_{\mu\,(2)}{}^{\alpha} &= \hat{F}_{\mu\,(2)}{}^{\alpha} - \iota_{A_\mu}\hat{F}_{(3)}{}^{\alpha} \,, \\ &\vdots \\ \bar{F}_{\mu\,(4)} &= \hat{F}_{\mu\,(4)} \,, \\ \bar{F}_{\mu\nu\,(3)} &= \hat{F}_{\mu\nu\,(3)} + 2\iota_{A_{[\mu}}\hat{F}_{\nu]\,(4)} \,, \\ &\vdots \end{aligned} \tag{B.45}$$

where we recall that there is no internal five-form because the internal space is four-dimensional. Furthermore,

$$\begin{aligned} F_{(3)}{}^{\alpha} = \bar{F}_{(3)}{}^{\alpha} \,, \qquad & F_{\mu\,(2)}{}^{\alpha} = \bar{F}_{\mu\,(2)}{}^{\alpha} \,, \qquad F_{\mu\nu\,(1)}{}^{\alpha} = \bar{F}_{\mu\nu\,(1)}{}^{\alpha} - \iota_{F_{\mu\nu}} C_{(2)}{}^{\alpha} \,, \\ F_{\mu\nu\rho\,(0)}{}^{\alpha} = \bar{F}_{\mu\nu\rho\,(0)}{}^{\alpha} \,, \qquad & F_{\mu\,(4)} = \bar{F}_{\mu\,(4)} \,, \\ F_{\mu\nu\,(3)} = \bar{F}_{\mu\nu\,(3)} - \epsilon_{\alpha\beta}F_{\mu\nu\,(1)}{}^{\alpha} \wedge C_{(2)}{}^{\beta} &- \iota_{F_{\mu\nu}} C_{(4)} + \frac{1}{2}\epsilon_{\alpha\beta} C_{(2)}{}^{\alpha} \wedge \iota_{F_{\mu\nu}} C_{(2)}{}^{\beta} \,, \\ F_{\mu\nu\rho\,(2)} = \bar{F}_{\mu\nu\rho\,(2)} - \epsilon_{\alpha\beta}F_{\mu\nu\rho\,(0)}{}^{\alpha} C_{(2)}{}^{\beta} \,, \end{aligned} \tag{B.46}$$

where $F_{\mu\nu}$ is the KK field strenght (B.40).

Gauge variations and field strength of $\hat{C}_{(\hat{2})}{}^{\alpha}$. Following the above redefinitions, the fields coming from the decomposition of $\hat{C}_{(\hat{2})}{}^{\alpha}$ transform under the gauge transformations (B.36) as

$$\begin{aligned} \delta C_{(2)}{}^{\alpha} &= d\lambda_{(1)}{}^{\alpha} \,, \\ \delta C_{\mu(1)}{}^{\alpha} &= D_{\mu}^{KK}\lambda_{(1)}{}^{\alpha} - d\lambda_{\mu\,(0)}{}^{\alpha} \,, \\ \delta C_{\mu\nu\,(0)}{}^{\alpha} &= 2D_{[\mu}^{KK}\lambda_{\nu](0)}{}^{\alpha} + \iota_{F_{\mu\nu}}\lambda_{(1)}{}^{\alpha} \,, \end{aligned} \tag{B.47}$$

where $F_{\mu\nu}$ is the KK field strength (B.40) and the derivative D_μ^{KK} is, as above, defined as

$$D_\mu^{KK} = \partial_\mu - L_{A_\mu}\,. \tag{B.48}$$

Analogous to the gauge fields, the λ-parameters are defined as

$$\begin{aligned}\lambda_{(1)}{}^\alpha &= \bar\lambda_{(1)}{}^\alpha = \hat\lambda_{(1)}{}^\alpha\,,\\ \lambda_{\mu\,(0)}{}^\alpha &= \bar\lambda_{\mu\,(0)}{}^\alpha = \hat\lambda_{\mu\,(0)}{}^\alpha - \iota_{A_\mu}\hat\lambda_{(1)}{}^\alpha\,.\end{aligned} \tag{B.49}$$

After redefinitions, the field strengths coming from $\hat F_{(\hat 3)}{}^\alpha$ become

$$\begin{aligned}F_{(3)}{}^\alpha &= dC_{(2)}{}^\alpha\,,\\ F_{\mu\,(2)}{}^\alpha &= D_\mu^{KK}C_{(2)}{}^\alpha - dC_{\mu\,(1)}{}^\alpha\,,\\ F_{\mu\nu\,(1)}{}^\alpha &= 2D_{[\mu}^{KK}C_{\nu]\,(1)}{}^\alpha + dC_{\mu\nu\,(0)}{}^\alpha\,,\\ F_{\mu\nu\rho\,(0)}{}^\alpha &= 3D_{[\mu}^{KK}C_{\nu\rho]\,(0)}{}^\alpha - 3\iota_{F_{[\mu\nu}}C_{\rho]\,(1)}{}^\alpha\,.\end{aligned} \tag{B.50}$$

Gauge variations and field strength of $\hat C_{(\hat 4)}$. The redefined fields coming from $\hat C_{(\hat 4)}$ transform under gauge transformations as

$$\begin{aligned}\delta C_{(4)} &= d\lambda_{(3)} + \frac{1}{2}\epsilon_{\alpha\beta}\lambda_{(1)}{}^\alpha\wedge F_{(3)}{}^\beta\,,\\ \delta C_{\mu\,(3)} &= D_\mu^{KK}\lambda_{(3)} - d\lambda_{\mu\,(2)} + \epsilon_{\alpha\beta}\lambda_{(1)}{}^\alpha\wedge dC_{\mu\,(1)}{}^\beta\,,\\ \delta C_{\mu\nu\,(2)} &= 2D_{[\mu}^{KK}\left(\lambda_{\nu]\,(2)} + \frac{1}{2}\epsilon_{\alpha\beta}\lambda_{(1)}{}^\alpha\wedge C_{\nu]\,(1)}{}^\beta\right) + d\lambda_{\mu\nu\,(1)} + \iota_{F_{\mu\nu}}\lambda_{(3)}\\ &\quad + \frac{1}{2}\epsilon_{\alpha\beta}(\lambda_{(1)}{}^\alpha\wedge F_{\mu\nu\,(1)}{}^\beta - 2\lambda_{[\mu\,(0)}{}^\alpha dC_{\nu]\,(1)}{}^\beta + dC_{\mu\nu\,(0)}{}^\alpha\wedge\lambda_{(1)}{}^\beta)\,,\end{aligned} \tag{B.51}$$

where the new λ-fields are defined, analogous to the gauge fields, as

$$\begin{aligned}\lambda_{(3)} &= \bar\lambda_{(3)} - \frac{1}{2}\epsilon_{\alpha\beta}\bar\lambda_{(1)}{}^\alpha\wedge C_{(2)}{}^\beta\,,\\ \lambda_{\mu\,(2)} &= \bar\lambda_{\mu\,(2)} - \frac{1}{2}\epsilon_{\alpha\beta}\left(\bar\lambda_{(1)}{}^\alpha\wedge C_{\mu\,(1)}{}^\beta + \bar\lambda_{\mu\,(0)}C_{(2)}{}^\beta\right)\,,\\ \lambda_{\mu\nu\,(1)} &= \bar\lambda_{\mu\nu\,(1)} - \frac{1}{2}\epsilon_{\alpha\beta}\bar\lambda_{(1)}{}^\alpha C_{\mu\nu\,(0)}{}^\beta\,,\end{aligned} \tag{B.52}$$

together with

$$\begin{aligned}\bar\lambda_{(3)} &= \hat\lambda_{(3)}\,,\\ \bar\lambda_{\mu\,(2)} &= \hat\lambda_{\mu\,(2)} - \iota_{A_\mu}\hat\lambda_{(3)}\,,\\ \bar\lambda_{\mu\nu\,(1)} &= \hat\lambda_{\mu\nu\,(1)} + 2\iota_{A_{[\mu}}\hat\lambda_{\nu]\,(2)} - \iota_{A_\mu}\iota_{A_\nu}\hat\lambda_{(3)}\,.\end{aligned} \tag{B.53}$$

After redefinitions, the field strengths coming from $\hat F_{(\hat 5)}$ become

$$\begin{aligned}F_{\mu\,(4)} &= D_\mu^{KK}C_{(4)} - dC_{\mu\,(3)} - \frac{1}{2}\epsilon_{\alpha\beta}C_{(2)}{}^\alpha\wedge F_{\mu\,(2)}{}^\beta + \frac{1}{2}\epsilon_{\alpha\beta}C_{(2)}{}^\alpha\wedge dC_{\mu\,(1)}{}^\beta\,,\\ F_{\mu\nu\,(3)} &= 2D_{[\mu}^{KK}C_{\nu]\,(3)} + dC_{\mu\nu\,(2)} - \epsilon_{\alpha\beta}C_{[\mu\,(1)}{}^\alpha\wedge dC_{\nu]\,(1)}{}^\beta\,,\\ F_{\mu\nu\rho\,(2)} &= 3D_{[\mu}^{KK}C_{\nu\rho]\,(2)} - dC_{\mu\nu\rho\,(1)} - 3\iota_{F_{[\mu\nu}}C_{\rho]\,(3)} - 3\epsilon_{\alpha\beta}dC_{[\mu\nu\,(0)}{}^\alpha\wedge C_{\rho]\,(1)}{}^\beta\\ &\quad - 3\epsilon_{\alpha\beta}C_{[\mu\,(1)}{}^\alpha\wedge D_\nu^{KK}C_{\rho]\,(1)}{}^\beta\,.\end{aligned} \tag{B.54}$$

Summary of variations. Combining the results above together with diffeomorphism variations along a vector χ in the internal space one finally obtains

$$\begin{aligned}\delta C_{(2)}{}^{\alpha} &= d\lambda_{(1)}{}^{\alpha} + L_{\chi} C_{(2)}{}^{\alpha}\,,\\ \delta C_{\mu(1)}{}^{\alpha} &= D^{KK}_{\mu}\lambda_{(1)}{}^{\alpha} - d\lambda_{\mu\,(0)}{}^{\alpha} + L_{\chi} C_{\mu(1)}{}^{\alpha}\,,\\ \delta C_{\mu\nu\,(0)}{}^{\alpha} &= 2D^{KK}_{[\mu}\lambda_{\nu](0)}{}^{\alpha} + \iota_{F_{\mu\nu}}\lambda_{(1)}{}^{\alpha} + L_{\chi} C_{\mu\nu\,(0)}{}^{\alpha}\,,\end{aligned} \tag{B.55}$$

and

$$\begin{aligned}\delta C_{(4)} &= d\lambda_{(3)} + \frac{1}{2}\epsilon_{\alpha\beta}\lambda_{(1)}{}^{\alpha}\wedge F_{(3)}{}^{\beta} + L_{\chi} C_{(4)}\,,\\ \delta C_{\mu\,(3)} &= D^{KK}_{\mu}\lambda_{(3)} - d\lambda_{\mu\,(2)} + \epsilon_{\alpha\beta}\lambda_{(1)}{}^{\alpha}\wedge dC_{\mu\,(1)}{}^{\beta} + L_{\chi} C_{\mu\,(3)}\,,\\ \delta C_{\mu\nu\,(2)} &= 2D^{KK}_{[\mu}\left(\lambda_{\nu]\,(2)} + \frac{1}{2}\epsilon_{\alpha\beta}\lambda_{(1)}{}^{\alpha}\wedge C_{\nu]\,(1)}{}^{\beta}\right) + d\lambda_{\mu\nu\,(1)} + \iota_{F_{\mu\nu}}\lambda_{(3)}\\ &\quad + \frac{1}{2}\epsilon_{\alpha\beta}(\lambda_{(1)}{}^{\alpha}\wedge F_{\mu\nu\,(1)}{}^{\beta} - 2\lambda_{[\mu\,(0)}{}^{\alpha} dC_{\nu]\,(1)}{}^{\beta} + dC_{\mu\nu\,(0)}{}^{\alpha}\wedge\lambda_{(1)}{}^{\beta}) + L_{\chi} C_{\mu\nu\,(2)}\,,\end{aligned} \tag{B.56}$$

The KK gauge field A_μ transforms as

$$\delta A_{\mu} = D^{KK}_{\mu}\chi\,. \tag{B.57}$$

B.3.4 Dictionaries SO(5,5) ExFT — IIB supergravity

By comparing (B.29) and (B.34) with (B.55) and (B.56) we can identify

$$\begin{aligned}A_{\mu} &= (\mathcal{A}_{\mu})_{(v)}\,, &\quad C_{\mu\,(1)}{}^{\alpha} &= (\mathcal{A}_{\mu})_{(1)}{}^{\alpha}\,, &\quad C_{\mu\nu\,(0)}{}^{\alpha} &= \tilde{B}_{\mu\nu\,(0)}{}^{\alpha}\\ C_{\mu\,(3)} &= (\mathcal{A}_{\mu})_{(3)}\,, & C_{\mu\nu\,(2)} &= -\tilde{B}_{\mu\nu\,(2)}\,,\end{aligned} \tag{B.58}$$

and analogously for the gauge parameters

$$\begin{aligned}\chi &= \Lambda_{(v)}\,, &\quad \lambda_{(1)}{}^{\alpha} &= \Lambda_{(1)}{}^{\alpha}\,, &\quad \lambda_{\mu\,(0)}{}^{\alpha} &= \tilde{\Xi}_{\mu\,(0)}{}^{\alpha}\\ \lambda_{(3)} &= \Lambda_{(3)}\,, & \lambda_{\mu\,(2)} &= -\tilde{\Xi}_{\mu\,(2)}\,.\end{aligned} \tag{B.59}$$

We can also establish dictionaries between field strenghts. Comparing (B.32), (B.35), (B.50) and (B.54) we obtain

$$\begin{aligned}F_{\mu\nu} &= (\mathcal{F}_{\mu\nu})_{(v)}\,, &\quad F_{\mu\nu\,(1)}{}^{\alpha} &= (\mathcal{F}_{\mu\nu})_{(1)}{}^{\alpha}\,, &\quad F_{\mu\nu\rho\,(0)}{}^{\alpha} &= \tilde{H}_{\mu\nu\rho\,(0)}{}^{\alpha}\\ F_{\mu\nu\,(3)} &= (\mathcal{F}_{\mu\nu})_{(3)}\,, & F_{\mu\nu\rho\,(2)} &= -\tilde{H}_{\mu\nu\rho\,(2)}\,.\end{aligned} \tag{B.60}$$

C S^2 conventions

We describe the S^2 by three functions y_u, $u = 1, \ldots, 3$ satisfying

$$y_u y^u = 1\,, \tag{C.1}$$

where we raise/lower $u, v = 1, \ldots, 3$ indices with δ_{uv}. In terms of these functions, the round metric on S^2 and its volume form are given by

$$ds^2_{S^2} = dy_u dy^u\,, \qquad vol_{S^2} = \frac{1}{2}\epsilon_{uvw} y^u\, dy^v\wedge dy^w\,. \tag{C.2}$$

The Killing vectors of the round S^2 are given by

$$v_u^i = g^{ij}\epsilon_{uvw}y^v\partial_j y^w\,, \tag{C.3}$$

where $i,j=1,2$ denote a local coordinate basis and g^{ij} is the inverse metric of the round S^2. Alternatively, the Killing vectors can be defined as in [14].

We also make repeated use of the 1-forms that are Hodge dual to dy_u with respect to the round metric (C.2)

$$\theta_u = \star dy_u = \epsilon_{uvw}y^v dy^w\,. \tag{C.4}$$

These form a "dual span" of the $T^*(S^2)$ to the Killing vectors, i.e.

$$\imath_{v_u}\theta_v = \delta_{uv} - y_u\,y_v\,. \tag{C.5}$$

Note that the 1-forms dy_u, θ_u and Killing vectors v_u satisfy

$$y_u dy^u = y_u\theta^u = y_u v^u = 0\,. \tag{C.6}$$

All the objects we introduced above transform naturally under the $\mathrm{SU}(2)_R$ symmetry generated by the Killing vector fields.

$$\begin{aligned}
L_{v_u}v_v &= -\epsilon_{uvw}\,v^w\,,\\
L_{v_u}y_v &= -\epsilon_{uvw}\,y^w\,,\\
L_{v_u}dy_v &= -\epsilon_{uvw}\,dy^w\,,\\
L_{v_u}\theta_v &= -\epsilon_{uvw}\theta^w\,.
\end{aligned} \tag{C.7}$$

D Dictionary between $\mathbf{AdS_6}$ conventions

Upon imposing the Cauchy-Riemann equations (7.42) and identifying the holomorphic functions as in (7.44), we find the following match between our objects and those of [41].

$$r = \frac{1}{8}\mathcal{G}\,,\qquad |dk| = \frac{1}{4}\underline{\kappa}^2\,,\qquad \Delta = \frac{3\underline{\kappa}^4\mathcal{G}}{128}\tilde{\mathcal{D}}\,, \tag{D.1}$$

where, as in [50],

$$\tilde{\mathcal{D}} = 1 + \frac{2\,|\partial\mathcal{G}|^2}{3\,\underline{\kappa}^2\,\mathcal{G}}\,. \tag{D.2}$$

Here, to differentiate our κ from the objects denoted by the same symbols in [41] we denoted theirs by an underline: $\underline{\kappa}$.

Our SL(2) doublet of 2-forms, ${C_{(2)}}^\alpha$, are related to the complex 2-form, $\mathcal{C}_{(2)}$, of [41] by

$$\mathcal{C}_{(2)} = -{C_{(2)}}^1 + i\,{C_{(2)}}^2\,. \tag{D.3}$$

Similarly, our axio-dilaton, $H^{\alpha\beta}$ is mapped to the complex scalar $\mathcal{B}$ of [41] via

$$\mathcal{B} = -2\,\frac{1+(H_{12})^2+(H_{22})^2}{1+(H_{12}+i\,H_{22})^2}\,. \tag{D.4}$$

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We can similarly match our minimal consistent truncation with that found in [50]. To differentiate between our scalar field X, gauge fields A^A, A^4, two-form fields $B_{(2)}$ and those of [50], we will denote the objects of [50] by an underline, i.e. $\underline{X}$, $\underline{A}^A$, $\underline{A}^4$, $\underline{B}$. We use the same notation for the field strengths, i.e. our objects are $\tilde{F}_{(2)}{}^A$, $\tilde{F}_{(2)}{}^4$ and $\tilde{F}_{(3)}$ and those of [50] are $\underline{\tilde{F}}_{(2)}{}^A$, $\underline{\tilde{F}}_{(2)}{}^4$ and $\underline{\tilde{F}}_{(3)}$. The map is now given by

$$\begin{aligned} X &= \underline{X}^{-1}\,, \qquad & A^A &= \frac{\sqrt{2}}{3}\underline{A}^A\,, \qquad & A^4 &= \underline{A}^4\,, \qquad B = \underline{B}\,, \\ \tilde{F}_{(2)}{}^A &= \frac{\sqrt{2}}{3}\underline{\tilde{F}}_{(2)}{}^A\,, \qquad & \tilde{F}_{(2)}{}^4 &= \underline{\tilde{F}}_{(2)}{}^4\,, \qquad & \tilde{F}_{(3)} &= \underline{\tilde{F}}_{(3)}\,, \end{aligned} \tag{D.5}$$

and our function $\bar{\Delta}$ is related to $\mathcal{D}$ of [50] by

$$\bar{\Delta} = \frac{3\underline{\kappa}^4\mathcal{G}}{128}\underline{X}^{-4}\mathcal{D}\,. \tag{D.6}$$

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