Hyperkähler Metrics on the Regular Nilpotent Adjoint Orbit

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Referent: Prof. Dr. Roger Bielawski

Koreferent: Prof. Dr. Anna Maria Fino

Koreferent: Prof. Dr. Andrew Francis Swann

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GOTTFRIED WILHELM LEIBNIZ UNIVERSITÄT HANNOVER

Abstract

Fakultät für Mathematik und Physik
Institute of Differential Geometry

Doktor der Naturwissenschaften

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by Oliver Sonderegger

This thesis studies the Kronheimer hyperkähler metric on the adjoint orbit of the classical Lie group $SL_n(\mathbb{C})$ of a regular, nilpotent element in its Lie algebra $\mathfrak{sl}_n(\mathbb{C})$. We describe a Kähler potential of this hyperkähler metric in terms of the theta function on the Jacobian, consisting of invertible sheaves of degree $g-1$, of the nilpotent, spectral curve. By using an explicit description of matricial polynomials of degree two corresponding to invertible sheaves of degree $g-1$ without a non-trivial, global section on the nilpotent, spectral curve we construct some explicit solutions to Nahm's equations.

Kurzzusammenfassung

Diese Dissertation untersucht Kronheimers Hyperkählermetrik auf der adjungierten Bahn der klassischen Lie Gruppe $SL_n(\mathbb{C})$ eines regulären, nilpotenten Elements der Lie Algebra $\mathfrak{sl}_n(\mathbb{C})$. Wir beschreiben ein Kählerpotential dieser Hyperkählermetrik durch Ausdrücke der Thetafunktion, einer Funktion auf der Jacobischen der nilpotenten Spektralkurve. Wir benutzen eine explizite Beschreibung der zu den invertierbaren Garben vom Grad $g-1$ ohne nicht-trivialen, globalen Schnitt zugehörigen Matrixpolynomen um explizite Lösungen der Nahmgleichungen zu konstruieren.

Keywords:

Hyperkähler Geometry, Regular Nilpotent Adjoint Orbit, Nahm's equations, Moduli space, Kähler potential

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Contents

Chapter 1

Introduction

Kronheimer studied in [Kro90b] a hyperkähler structure on the complex nilpotent, adjoint orbits. These are the orbits of a nilpotent element of $\mathfrak{g}^{\mathbb{C}}$ of the adjoint action of $G^{\mathbb{C}}$ on $\mathfrak{g}^{\mathbb{C}}$, where G is a compact, connected, semisimple Lie group, $G^{\mathbb{C}}$ its complexification and $\mathfrak{g}^{\mathbb{C}}$ the corresponding Lie algebra of $G^{\mathbb{C}}$. He identified such orbits with a moduli space of $\mathfrak{g}$-valued solutions of Nahm's equations with certain boundary conditions. The hyperkähler structure on the orbit comes from the fact, that the moduli space can be seen as a infinite-dimensional version of a hyperkähler quotient.

It is remarkable, that Kronheimer showed the existence of a hyperkähler structure on complex, regular, semisimple, adjoint orbits too in [Kro90a] and later Biquard and Kovalev generalized the ideas of Kronheimer to arbitrary complex, adjoint orbits in [Biq96] and [Kov96] .

Kronheimer's hyperkähler metric on the moduli space is of the form

$$g_{B(t)}((b_0,b_1,b_2,b_3),(c_0,c_1,c_2,c_3)) = \int_{-\infty}^{0} \sum_{i=0}^{3} \langle b_i(t), c_i(t)\rangle dt,$$

where the b_i's and c_i's are tangent vectors of the tangent space at the point $B(t)$, which are solutions of the linearization of Nahm's equation at $B(t)$ with certain boundary conditions and $\langle\cdot,\cdot\rangle$ is an $Ad-$invariant inner product on $\mathfrak{g}$.

Since it relies on solving a system of ordinary differentianl equations, Kronheimer's hyperkähler metric on the nilpotent orbits is very difficult to write down explicitely. Hitchin proposed in [Hit98] an explicit description of a Kähler potential could help to make this metric more explicit. In [Hit+87] they described such a Kähler potential for a fixed complex strucure as a hamiltonian function of a hamiltonian circle action, which fixes one Kähler form and rotates the other two Kähler forms. If $G^{\mathbb{C}} = SL_n(\mathbb{C})$ this Kähler potential of the nilpotent, adjoint orbits is of the form

$$K(T_2(0) + iT_3(0)) = -\int_{-\infty}^{0} tr\left(T_2(t)^2 + T_3(t)^2\right) dt,$$

where the T_2 and T_3 arise in triples $(T_1(t), T_2(t), T_3(t))$ of solutions of Nahm's equations with certain boundary conditions.

By a result of Beauville in [Bea90] the integrand can be seen as a meromorphic function on the Jacobian of the nilpotent, spectral curve, which consists of invertible sheaves of degree $g-1$, where g is the arithmetic genus of the curve.

As long as the spectral curve is smooth Hitchin described this integrand in [Hit98] in terms of the theta function. We will call this formula *Hitchin's formula*. By allowing ordinary double points Bielawski generalized Hitchin's formula to reducible spectral curves in [Bie07] and so he was able to describe a Kähler potential on complex, regular, semisimple, adjoint orbits in terms of the theta function, what made Kronheimers hyperkähler metric more explicit.

In this thesis we will generalize Hitchin's formula to the case of a regular, nilpotent, adjoint $SL_n(\mathbb{C})$ - orbit, which allows us to describe a Kähler potential in terms of the theta function on the highly singular, nilpotent, spectral curve. Our approach is completely direct, using mainly tools of linear algebra. This leads to extensive computations, but also to explicit formulas. For example we will compute the theta function and regular, nilpotent, matricial polynomials corresponding to invertible sheaves of degree $g-1$ not lying in the Theta divisor. Using these expressions we are able to compute the integrand of the Kähler potential explicitly in terms of the theta function. Additionally we use these formulas with additional assumptions to extract explicit solutions of Nahm's equations.

Hitchin used in [Hit98] the fact, that the Jacobian variety of a smooth spectral curve is already compact - it is a torus. Then he compared both sides of Hitchin's formula, which are meromorphic functions on the Jacobian, see section 4.3, by computing the principal parts. These principal parts coincide and hence the difference of both sides defines a holomorphic function on the compact Jacobian. Thus it has to be constant. After computing this constant he got his formula. Bielawski in [Bie07] had to deal with a non-compact Jacobian and he used a certain compactification of the Jacobian, see [Ale96]. This compactification is based on invertible sheaves of partial normalisations of the considered reducible, spectral curve. This argumentation is reasonable, since all singularities are isolated. In our case it is unclear how to compactify the Jacobian in a usefull way. Since the singularities of the nilpotent, spectral curve are not isolated, we could not carry over the argument of partial normalisations to our case. Nevertheless during the computations of the principal parts we obtained some explicit results, which lead us in the end to a completely direct proof of Hitchin's formula in the case of the nilpotent, spectral curve and we could avoid the difficulties of the compactification of the Jacobian.

We want to mention some important results for nilpotent, adjoint orbits. Kronheimer obtained a hyperkähler structure by an infinite-dimensional hyperkähler quotient. In [KS93], [KS96], [KS01b], [KS01a] and [Vil05] they performed finite-dimensional hyperkähler quotients for adjoint orbits and studied hyperkähler potentials on orbits with low cohomogeneity. These potentials are simultaneously Kähler with respect to all complex structures. The regular, nilpotent, adjoint orbit in $\mathfrak{sl}_3(\mathbb{C})$ has cohomogeneity four and they gave explicit values for the hyperkähler potential, see[KS01b]. Since their hyperkähler potential and the Kähler potential above coincide in this case, they obtainted the formula of example 8 in chapter 5 already in a higher generality. Now we will describe the content of the chapters of this thesis.

Chapter 2 includes by no means any original content. We want to repeat and outline some basic concepts, conventions and notations for the later usage. In particular section 2.1 is dealing with complex analytic spaces, which arise in our case as the nilpotent, spectral curve. Moreover in section 2.2 and 2.3 we will outline the identification of the regular, nilpotent, adjoint $SL_n(\mathbb{C})$-orbit with the Kronheimer moduli space, to picture how the hyperkähler structure and a Kähler potential arise.

In Chapter 3 we start with a precise description of the nilpotent, spectral curve in section 3.1. In section 3.2 we characterize invertible sheaves of degree $g-1$ on the nilpotent, spectral curve and in section 3.3 we give an explicit formula of the theta function on the Jacobian as a determinantal function of a certain matrix M. This matrix M occur as a family of linear condition equations of a sheaf $\mathcal{F}$ in the theta divisor.

Chapter 4 contains the main result of this thesis. In section 4.1 we repeat the concept of the Beauville correspondence, an identification of isomorphism classes of invertible sheaves of degree $g-1$ without non-trivial, global sections on the nilpotent, spectral curve with conjugation classes of regular, nilpotent, matricial polynomials. In section 4.2 we show Hitchin's formula in the $SL_3(\mathbb{C})$-case. In section 4.3 we generalize the ideas and computations of section 4.2 to the $SL_n(\mathbb{C})$-case by proving the crucial *burning lemma*. This lemma is the main comparison tool to show the required equality of Hitchin's formula.

In Chapter 5 we study real sheaves and its theta function. Moreover in section 5.2 we restrict us to some special cases of real sheaves in order to establish explicit solutions of Nahm's equations. We will use and extend the ideas of chapter 4.

Chapter 2

Preliminaries

In this chapter we will repeat the definitions of complex analytic spaces from [GPR94] and [GR84]. Moreover we repeat some general Lie theory and concepts of [Kro90b], [Kro90a] and [Hit+87].

2.1 Complex Analytic Spaces

The focus in this section is the repetition of the definition of complex analytic spaces and invertible sheaves. We set the conventions and notations, which we will use later. In this thesis the invertible sheaves on a complex analytic curve play a crucial role.

2.1.1 Sheaves and Ringed Spaces

Let $(X, \mathcal{T})$ be a topological space. A presheaf of abelian groups (respectively of rings) on X, denoted by

$$\mathcal{F} := \left\{\mathcal{F}(U), res_U^V\right\}_{\substack{U,V \in \mathcal{T} \\ U \subseteq V}},$$

consists of a collection of abelian groups (respectively of rings) $(\mathcal{F}(U))_{U \in \mathcal{T}}$ and a collection of restriction homomorphisms of abelian groups (respectively of rings)

$$\left(res_U^V : \mathcal{F}(V) \to \mathcal{F}(U)\right)_{\substack{U,V \in \mathcal{T} \\ U \subseteq V}},$$

where the restriction morphisms have to satisfy for every inclusion of open sets $U \subseteq V \subseteq W \subseteq X$ the properties

$$res_U^U = Id_{\mathcal{F}(U)}, \qquad res_U^W = res_U^V \circ res_V^W.$$

The elements $s \in \mathcal{F}(U)$ are called local sections and the elements $s \in \mathcal{F}(X)$ are called global sections.

A presheaf $\mathcal{F}$ of abelian groups (respectively of rings) on a topological space $(X, \mathcal{T})$ is called a sheaf of abelian groups (respectively of rings) on $(X, \mathcal{T})$, if it satisfies the following two properties. Let $U \in \mathcal{T}$ be an arbitrary open set of X and $(U_i)_{i \in I}$ be an open cover of U. The first property is called locality and it means if $s, t \in \mathcal{F}(U)$ are two local sections with $res_{U_i}^U(s) = res_{U_i}^U(t)$ for all $i \in I$, then the two local sections coincide, $s = t$. The second property is called gluing property and it means if $(s_i)_{i \in I}$ is a collection of local sections of $\mathcal{F}(U_i)$ such that they satisfy $res_{U_i \cap U_j}^{U_i}(s_i) = res_{U_i \cap U_j}^{U_j}(s_j)$ for all $i, j \in I$, then there exists a local section $s \in \mathcal{F}(U)$ such that $res_{U_i}^U(s) = s_i$. We will drop from now on the letter $\mathcal{T}$ indicating the topology. If $\mathcal{A}_X$ is a sheaf of rings on a topological space X, then the pair $(X, \mathcal{A}_X)$ is called a ringed space with structure sheaf $\mathcal{A}_X$. Let $(X, \mathcal{A}_X)$ be a ringed space and $\mathcal{F}$ be a sheaf of abelian groups on X. If for every open set $U \subseteq X$ the abelian group $\mathcal{F}(U)$ has an $\mathcal{A}_X(U)$ - module structure, then $\mathcal{F}$ is called a sheaf of $\mathcal{A}_X$-modules. A sheaf of abelian groups $\mathcal{I}$ is called a sheaf of ideals, if all $\mathcal{I}(U)$ are $\mathcal{A}_X(U)$-ideals for all open sets $U \subset X$. Hence for every open set U we can consider the quotient ring $\mathcal{A}_X(U)/\mathcal{I}(U)$. These rings with the induced restriction morphisms define usually only a presheaf. The sheafification, see [Har77], of this presheaf is called quotient sheaf and is denoted by $\mathcal{A}_X/\mathcal{I}$. Let $\mathcal{F}$ be a sheaf of $\mathcal{A}_X$-modules, then the stalk at $x \in X$ is defined by the direct limit

$$\mathcal{F}_x := \varinjlim_{x \in U} \mathcal{F}(U).$$

The elements of $\mathcal{F}_x$ are denoted by $s_x = [V, s]$. It is in a natural way an $\mathcal{A}_{X,x}$-module. For the structure sheaf we denote the stalk by $\mathcal{A}_{X,x}$.

2.1.2 $\mathbb{C}$-ringed Spaces and Morphisms

Let $(X, \mathcal{A}_X)$ be a ringed space and let $\mathfrak{K}(U) := U \times \mathbb{C}$ be the constant sheaf of fields of complex numbers $\mathbb{C}$ with natural restriction morphisms. The stalk of $\mathfrak{K}$ is $\mathfrak{K}_x \cong \mathbb{C}$. If the sheaf of rings $\mathcal{A}_X$ is a sheaf of $\mathfrak{K}$-modules too, it is called a sheaf of $\mathbb{C}$-algebras. Futhermore suppose $\mathfrak{K}$ is a sheaf of submodules of the sheaf $\mathcal{A}_X$, such that the unit element $1_x \in \mathfrak{K}_x$ is also the unit element of $\mathcal{A}_{X,x}$. Hence we are able to identify the subalgebra $\mathbb{C} \cdot 1_x \subset \mathcal{A}_{X,x}$ with the field $\mathbb{C} \cong \mathfrak{K}_x$. Moreover let us assume that all $\mathcal{A}_{X,x}$ are local rings with unique maximal ideals $\mathfrak{m}(\mathcal{A}_{X,x})$, such that we have a decomposition of $\mathbb{C}$-vector spaces

$$\mathcal{A}_{X,x} \cong \overbrace{\mathbb{C} \cdot 1_x}^{\cong \mathbb{C} \cong \mathfrak{K}_x} \oplus \mathfrak{m}(\mathcal{A}_{X,x}).$$

If all these assumptions are satisfied we call the sheaf of $\mathbb{C}$-algebras $\mathcal{A}_X$ a sheaf of local $\mathbb{C}$-algebras. The pair $(X, \mathcal{A}_X)$ is then called a $\mathbb{C}$-ringed space. The most important example of a $\mathbb{C}$-ringed space comes from complex analysis.

Example 1. *Let D be a domain in $\mathbb{C}^n$ and let $U \subset D$ be an open set. Let*

$$\mathcal{O}_D := \left\{\mathcal{O}_D(U), res_U^V\right\}$$

be the sheaf of holomorphic functions on D, where $\mathcal{O}_D(U)$ is the ring of holomorphic functions on U and res_U^V are the natural restriction morphisms, where we sometimes just write $f|_U := res_U^V(f)$. It is obviously a $\mathfrak{K}$-module and hence a sheaf of $\mathbb{C}$-algebras.

The stalk $\mathcal{O}_{D,z_0}$ is isomorphic to the $\mathbb{C}$-algebra $\mathbb{C}\{z_0\}$ of power series with a non-zero radius of convergence around $z_0 \in D$. The isomorphism is given by

$$\begin{aligned} \psi : \mathcal{O}_{D,z_0} &\longrightarrow \mathbb{C}\{z_0\} \\ [U, f] &\longmapsto \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k. \end{aligned}$$

The stalk is a local ring with maximal ideal $\mathfrak{m}(\mathcal{O}_{D,z_0})$ of those convergent powerseries vanishing at z_0, i.e. the constant term is 0. Hence we have a decomposition $\mathcal{O}_{D,z_0} \cong \mathbb{C} \oplus \mathfrak{m}(\mathcal{O}_{D,z_0})$ and the pair $(D, \mathcal{O}_D)$ is a $\mathbb{C}$-ringed space. Because $\mathcal{O}_D$ is a sheaf of local $\mathbb{C}$-algebras we can consider the sheaf of abelian groups $\mathcal{O}_D^*$ consisting of the units of $\mathcal{O}_D$ with respect to the multiplication on $\mathcal{O}_D$. These are the nowhere vanishing holomorphic functions.

A morphism of $\mathbb{C}$-ringed spaces $(X, \mathcal{A}_X)$ and $(Y, \mathcal{A}_Y)$ is a pair

$$(\varphi, \varphi^{\#}) : (X, \mathcal{A}_X) \longrightarrow (Y, \mathcal{A}_Y),$$

where $\varphi : X \to Y$ is a continuous map and $\varphi^{\#} : \mathcal{A}_Y \to \varphi_*(\mathcal{A}_X)$ is a morphism of $\mathbb{C}$-algebras, i.e. a collection of $\mathbb{C}$-algebra homomorphisms

$$\varphi^{\#}(V) : \mathcal{A}_Y(V) \longrightarrow \mathcal{A}_X(\varphi^{-1}(V)) =: \varphi_*\mathcal{A}_X(V)$$

with the property $\varphi^{\#}(V) \circ res_U^V = res_{\varphi^{-1}(U)}^{\varphi^{-1}(V)} \circ \varphi^{\#}(V)$ for every inclusion of open sets $U \subset V \subset Y$.

Let $(X, \mathcal{A}_X)$ be a $\mathbb{C}$-ringed space and $U \subseteq X$ be an open set. Then the pair $(U, \mathcal{A}_U)$ is a $\mathbb{C}$-ringed space. Here $\mathcal{A}_U$ is the sheaf of local $\mathbb{C}$-algebras given by the following collection. If $V \subseteq U$ is an open subset, then $\mathcal{A}_U(V) := \mathcal{A}_X(U \cap V)$. The inclusion $\iota : U \to X$ induces a lifting homomorphism $\iota^{\#}$ via

$$\begin{aligned} \iota^{\#}(V) : \mathcal{A}_X(V) &\longrightarrow \mathcal{A}_U(V \cap U) \\ s &\longmapsto res_U^V(s). \end{aligned}$$

The pair $(\iota, \iota^{\#})$ is a morphism of $\mathbb{C}$-ringed spaces and $(U, \mathcal{A}_U)$ together with $(\iota, \iota^{\#})$ is called an open $\mathbb{C}$-ringed subspace of $(X, \mathcal{A}_X)$.

2.1.3 Model Spaces and Complex Analytic Spaces

We start with the definition of complex model spaces. Let $(D, \mathcal{O}_D)$ be the $\mathbb{C}$-ringed space of example 1 for a domain $D \subseteq \mathbb{C}^n$. A finite number of holomorphic functions on D, $f_1, \ldots, f_k \in \mathcal{O}_D(D)$, defines a sheaf of ideals by

$$\mathcal{I} := f_1\mathcal{O}_D + \cdots + f_k\mathcal{O}_D \subseteq \mathcal{O}_D.$$

Its zero-locus is a closed, topologcial subspace

$$Z := N(\mathcal{I}) := \{z \in D : f_1(z) = \cdots = f_k(z) = 0\} \subseteq D.$$

This allows us to consider the quotient sheaf $\mathcal{O}_D/\mathcal{I}$, which is the sheafification of the presheaf defined by $U \mapsto \mathcal{O}_{\mathcal{D}}(U)/\mathcal{I}(U)$, see for the definition [Har77]. This quotient sheaf is a sheaf of rings and the support of it is

$$Supp(\mathcal{O}_D/\mathcal{I}) := \{z \in D : \mathcal{O}_{D,z}/\mathcal{I}_z \neq 0\} = \{z \in D : \mathcal{I}_z \neq \mathcal{O}_{D,z}\},$$

which is exactly the zero-locus Z. For any open set $U \subseteq Z$ we define $\mathbb{C}$-algebras by

$$\mathcal{O}_Z(U) := (\mathcal{O}_D/\mathcal{I}|_Z)(U) := \varinjlim_{\substack{U \subset V \subseteq D \\ V \text{ open}}} ((\mathcal{O}_D/\mathcal{I})(V)).$$

With the natural restriction morphisms these $\mathbb{C}-$algebras define a sheaf of local $\mathbb{C}$-algebras. The pair

$$(Z, \mathcal{O}_Z)$$

is a $\mathbb{C}$-ringed space and called a complex model space. The dimension of $(Z, \mathcal{O}_Z)$ is the dimension of the topological space Z.

A $\mathbb{C}$-ringed space $(X, \mathcal{O}_X)$ with X Hausdorff is called a complex analytic space if for every $x \in X$ exists an open $\mathbb{C}$-ringed subspace $(U, \mathcal{O}_U)$, which is isomorphic as $\mathbb{C}$-ringed spaces to a complex model space. In other words a complex analytic space looks locally as complex model spaces or as a zero-locus of a finite number of holomorphic functions.

Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be two arbitrary complex analytic spaces. A morphism $(\varphi, \varphi^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of $\mathbb{C}$-ringed spaces is called a holomorphic map. If φ is a homeomorphism and $\varphi^\#$ is a sheaf isomorphism of $\mathbb{C}$-algebras, then the pair $\left(\varphi, \varphi^\#\right)$ is called a biholomorphism.

2.1.4 Gluing Property

In this subsection we will repeat the gluing device of [GR84] or [GPR94]. We will use this construction to describe in chapter 2 the nilpotent, spectral curve in details as a complex analytic space, which we obtain by gluing together two complex model spaces.

First let us recall some topological constructions. If $(X_i)_{i\in I}$ is a family of topological spaces, then $\left(\bigcup_{i\in I}\{i\} \times X_i\right)$ is a topological space with the disjoint union topology. Furthermore if there are open subsets $X_{ij} \subset X_j$ and homeomorphisms $\tau_{ij} : X_{ij} \to X_{ji}$ we get a topological quotient space

$$X := \left(\bigcup_{i\in I}\{i\} \times X_i\right) / \sim,$$

where $(i, x_i) \sim (j, x_j) \Leftrightarrow x_i \in X_{ij},\ x_j \in X_{ji}$ and $\tau_{ij}(x_i) = x_j$ with the induced quotient topology given by the natural projection $\pi : \left(\bigcup_{i\in I}\{i\} \times X_i\right) \to X,\ (i, x_i) \mapsto [i, x_i]$. If all topological spaces X_i are Hausdorff, then the disjoint union and the quotient space is Hausdorff too. Because of the definition of the disjoint union topology we have an open cover of $\left(\bigcup_{i\in I}\{i\} \times X_i\right)$ by the sets $\{i\} \times X_i$. The topological quotient space has an open cover given by the open sets $U_i := \pi\left(\{i\} \times X_i\right)$. Note that $\pi^{-1}\left(U_i \cap U_j\right) \cap \{i\} \times X_i = \{i\} \times X_{ij}$ and $\pi^{-1}\left(U_i \cap U_j\right) \cap \{j\} \times X_j = \{j\} \times X_{ji}$.

Let us assume now, that we have sheaves of local $\mathbb{C}$-algebras $\mathcal{O}_{X_i}$ on the topological spaces X_i, such that all pairs $(X_i, \mathcal{O}_{X_i})$ are complex analytic spaces and hence $\left(\{i\} \times X_i, \mathcal{O}_{\{i\}\times X_i}\right)$ are complex analytic spaces with $\mathcal{O}_{\{i\}\times X_i}(\{i\} \times V) := \mathcal{O}_{X_i}(V)$ for an open set $V \subseteq X_i$. If we have an open set $U \subset U_i$ we get sheaves of local $\mathbb{C}$-algebras on U_i defined by

$$\mathcal{O}_{U_i}(U) := \mathcal{O}_{\{i\}\times X_i}\left(\pi^{-1}(U) \cap \{i\} \times X_i\right).$$

We want to glue the sheaves $\mathcal{O}_{U_i}$ together to a sheaf on the space X with some additional gluing data. Before we do so we want to emphasise, that

$$\begin{aligned}\mathcal{O}_{U_i}|_{U_i\cap U_j}(U) &= \mathcal{O}_{\{i\}\times X_i}(\pi^{-1}(U \cap U_i \cap U_j) \cap \{i\} \times X_i)\\ &= \mathcal{O}_{\{i\}\times X_i}(\pi^{-1}(U) \cap \{i\} \times X_{ij})\\ &= \mathcal{O}_{\{i\}\times X_i}|_{\{i\}\times X_{ij}}(\pi^{-1}(U)).\end{aligned}$$

Suppose there are sheaf isomorphisms of $\mathbb{C}$-algebras

$$\tau_{ij}^\# : \mathcal{O}_{X_i}|_{X_{ij}} \longrightarrow \mathcal{O}_{X_j}|_{X_{ji}}$$

for all $i, j \in I$, which satisfy the gluing (cocycle) condition

$$\tau_{ij}^{\#} \circ \tau_{jk}^{\#} = \tau_{ik}^{\#}$$

on $\mathcal{O}_{X_i}|_{X_{ij} \cap X_{ik}}$ for all $i, j, k \in I$. With such isomorphisms we get immediatly new isomorphisms

$$\rho_{ij}^{\#} : \mathcal{O}_{U_i}|_{U_i \cap U_j} \longrightarrow \mathcal{O}_{U_j}|_{U_j \cap U_i}$$

for all $i, j \in I$, where $\rho_{ij}^{\#}(U) := \tau_{ij}^{\#}(\pi^{-1}(U))$, and they satisfy the cocycle condition too, i.e. $\rho_{ij}^{\#} \circ \rho_{jk}^{\#} = \rho_{ik}^{\#}$ on $\mathcal{O}_{U_i}|_{U_i \cap U_j \cap U_k}$ for all $i, j, k \in I$. With these isomorphisms we can write down the glued sheaf on X.

For an open set $U \subseteq X$ the $\mathbb{C}$-algebra $\mathcal{O}_X(U)$ is given by

$$\mathcal{O}_X(U) :=$$
$$\left\{ (s_i)_{i \in I} \in \prod_{i \in I} \mathcal{O}_{U_i}(U \cap U_i) : \rho_{ij}^{\#}\left(res_{U \cap U_i \cap U_j}^{U \cap U_i}(s_i) \right) = res_{U \cap U_j \cap U_i}^{U \cap U_j}(s_j), \quad \forall i, j \in I \right\}.$$

The restriction morphisms of $\mathcal{O}_X$ are coming from the restriction morphisms of $\mathcal{O}_{U_i}$ for all i. In other words let $\tilde{U} \subseteq U$ be an inclusion of open sets in X. The restriction morphisms are then given by

$$res_{\tilde{U}}^{U} : \mathcal{O}_X(U) \longrightarrow \mathcal{O}_X(\tilde{U}), \qquad (s_i)_{i \in I} \longmapsto (res_{\tilde{U}}^{U}(s_i))_{i \in I},$$

which is well defined, because the sheaf homomorphism property implies

$$g_{ij}^{\#}\left(res_{\tilde{U}}^{U}(s_i) \right) = res_{\tilde{U}}^{U}\left(g_{ij}^{\#}(s_i) \right) = res_{\tilde{U}}^{U}(s_j).$$

As a result we get a complex analytic space $(X, \mathcal{O}_X)$ and a natural holomorphic projection

$$\left(\pi, \pi^{\#}\right) : \left(\bigcup_{i \in I} \{i\} \times X_i, \prod_{i \in I} \mathcal{O}_{X_i} \right) \longrightarrow (X, \mathcal{O}_X),$$

where $\pi(i, x) \mapsto [i, x]$ and

$$\pi^{\#}(U) : \mathcal{O}_X(U) \longrightarrow \prod_{i \in I} \mathcal{O}_{X_i}(\pi^{-1}(U))$$
$$(s_i)_{i \in I} \longmapsto (s_i)_{i \in I}$$

for any open set $U \subset X$. Summarized in a proposition, see for more details [GPR94] and [GR84] we state the following proposition.

Proposition 1. *The sheaf $\mathcal{O}_X$ is a sheaf of local $\mathbb{C}$-algebras and $(X, \mathcal{O}_X)$ is a complex analytic space.*

2.1.5 Čech Cohomology

Later in this thesis we will study invertible sheaves on the nilpotent spectral curve. A usefull tool to study such sheaves is Čech cohomology theory. We want to recall some basic definitions and concepts of Čech cohomology. All definitions and constructions are from [GR84] or [GPR94].

Let $(X, \mathcal{O}_X)$ be a complex analytic space, let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules and let $\mathcal{U} := \{U_i : i \in I\}$ be an open cover of X. For any $q \in \mathbb{N}$ we define the $\mathcal{O}_X(X)$-modules

$$C^q(\mathcal{U}, \mathcal{F}) := \bigoplus_{i_0 < \cdots < i_q} \mathcal{F}\left(U_{i_0} \cap \cdots \cap U_{i_q}\right).$$

Its elements $\alpha \in C^q(\mathcal{U}, \mathcal{F})$ are called the q-cochains, which are given by a family $\alpha = (\alpha(i_0, \ldots, i_q))_{(i_0, \cdots, i_q) \in I^{q+1}}$, where the $\alpha(i_0, \ldots, i_q) \in \mathcal{F}\left(U_{i_0} \cap \cdots \cap U_{i_q}\right)$. The q-coboundary map, a $\mathcal{O}_X(X)$-module homomorphism, is defined by

$$\begin{aligned} \delta_q : C^q(\mathcal{U}, \mathcal{F}) &\longrightarrow C^{q+1}(\mathcal{U}, \mathcal{F}) \\ \alpha &\longmapsto \delta_q(\alpha), \end{aligned}$$

with

$$\delta_q(\alpha)(i_0, \cdots, i_{q+1}) := \sum_{\eta=0}^{q+1} (-1)^\eta res^{U_{i_0} \cap \cdots \cap U_{i_q}}_{U_{i_0} \cap \cdots \cap U_{i_q+1}} \left(\alpha(i_0, \cdots i_{\eta-1}, i_{\eta+1}, \cdots, i_{q+1})\right).$$

A direct computation shows $\delta_{q+1} \circ \delta_q = 0$ and hence $\left(C^q(\mathcal{U}, \mathcal{F}), \delta_q\right)_{q \in \mathbb{N}}$ is a complex, which is called the Čech-complex of $\mathcal{F}$ with respect to the open cover $\mathcal{U}$. The image $Im(\delta_{q-1})$, the so-called q-coboundaries, is an $\mathcal{O}_X(X)$-submodule of the kernel $Ker(\delta_q)$. The elements of the kernel are called q-cocycles. The q-th Čech cohomology module with respect to the open cover $\mathcal{U}$ is then defined by the quotient module

$$\check{\mathrm{H}}^q(\mathcal{U}, \mathcal{F}) := \frac{Z^q(\mathcal{U}, \mathcal{F})}{B^q(\mathcal{U}, \mathcal{F})} = \frac{Ker(\delta_q)}{Im(\delta_{q-1})}.$$

The definitions above depend on the choice of the open cover $\mathcal{U}$ and in general a short exact sequence of sheaves of abelian groups does not induce a long exact sequence of cohomology modules with respect to the open cover $\mathcal{U}$. The q-th Čech cohomology group or module of $\mathcal{F}$ is defined by a direct limit

$$\check{\mathrm{H}}^q(X, \mathcal{F}) := \varinjlim_{\mathcal{U}} \check{\mathrm{H}}^q(\mathcal{U}, \mathcal{F})$$

of the direct system given by refinements, see [GPR94]. We do not need to go deeper in details how this direct system is defined, because in our case we will have a Leray cover and then the Čech cohomology is already given by the Čech cohomology module with respect to this particular cover.

Let $\mathcal{U} := \{U_i : i \in I\}$ be an open cover of X and $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We call $\mathcal{U}$ a Leray cover, if $\check{\mathrm{H}}^p\left(U_{i_1} \cap \cdots \cap U_{i_q}, \mathcal{F}\right) = 0$ for all $p \geq 1$ and any non-empty finite set $\{i_1, \ldots, i_q\}$.

Proposition 2. *(Leray's Theorem) Let $(X, \mathcal{O}_X)$ be a compact, complex analytic space. Let us assume there is a Leray cover $\mathcal{U}$ of X, then $\check{\mathrm{H}}^q(\mathcal{U}, \mathcal{F}) \cong \check{\mathrm{H}}^q(X, \mathcal{F})$ for all $q \in \mathbb{N}$.*

For a proof of this proposition see for example [GPR94]. Leray's Theorem is crucial to compute Čech cohomology explicitly because it avoids the difficulty of the direct limit. Note if X is compact and $\mathcal{U}$ is a Leray cover, then we can find a locally finite Leray cover. Another observation is, if a compact, complex analytic space has a cover of Stein spaces, then we have by Cartan's theorem B, [GPR94], already a Leray cover. The spaces $\mathbb{C}$ and $\mathbb{C}^*$ are Stein. Additionally in this compact setting the ring $\mathcal{O}_X(X)$ consists only of the constant sections and hence is the field of complex numbers, which makes the Čech cohomology module in to a $\mathbb{C}$-vector space.

Two important results in cohomology theory are the following two properties.

Proposition 3. *Let $(X, \mathcal{O}_X)$ be a compact, complex analytic space. If we have a short exact sequence of $\mathcal{O}_X$-modules*

$$0 \to \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \to 0,$$

then we have a long exact sequence of $\mathbb{C}$-vector spaces

$$\cdots \to \check{\mathrm{H}}^q(X, \mathcal{F}) \longrightarrow \check{\mathrm{H}}^q(X, \mathcal{G}) \longrightarrow \check{\mathrm{H}}^q(X, \mathcal{H}) \longrightarrow \check{\mathrm{H}}^{q+1}(X, \mathcal{F}) \to \cdots.$$

Proposition 4. *Let $(X, \mathcal{O}_X)$ be a compact, complex analytic space of dimension n and $\mathcal{F}$ be a coherent, analytic sheaf on X, then all Čech Cohomology modules $\check{\mathrm{H}}^q(X, \mathcal{F})$ are finite dimensional $\mathbb{C}$-vector spaces. Furthermore we have $\dim_{\mathbb{C}} \check{\mathrm{H}}^q(X, \mathcal{F}) = 0$ for all $q > n$. In particular if X is a compact curve, then only the $\mathbb{C}$-vector spaces $\check{\mathrm{H}}^0(X, \mathcal{F})$ and $\check{\mathrm{H}}^1(X, \mathcal{F})$ are possibly non-trivial.*

See for the definition of coherence, the definition of the maps between the Čech cohomology modules and proofs [Har77], [GR84] and [GPR94]. The finite dimensionality of the Čech cohomology on a compact curve leads to some important invariants. For example the complex dimension of $\check{\mathrm{H}}^1(X, \mathcal{O}_X)$ is called the (arithmetic) genus of the curve and the Euler characteristic of an invertible sheaf $\mathcal{F}$ is defined by

$$\chi(X, \mathcal{F}) := \sum_{i=0}^{n} (-1)^i \dim_{\mathbb{C}}(\check{\mathrm{H}}^i(X, \mathcal{F})).$$

2.1.6 Invertible Sheaves

Let $(X, \mathcal{O}_X)$ be a complex analytic space. A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is called locally free of rank k, if for every $p \in X$ there is an open neighborhood U, a $k \in \mathbb{N}$ and an isomorphism of $\mathcal{O}_X(U)$-modules $\varphi^{\#}(U) : \mathcal{F}(U) \to \mathcal{O}_X(U)^{\oplus k}$. A locally free sheaf of rank 1 is called invertible. Let $\mathcal{F}$ be an invertible sheaf, then the dual sheaf $\mathcal{F}^v$ is defined as the sheaf hom $\mathcal{H}om(\mathcal{F}, \mathcal{O}_X)$, see [Har77] for a definition. The tensor product of two invertible sheaves $\mathcal{F}_1$ and $\mathcal{F}_2$, $\mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{F}_2$, is defined by the sheafification of the presheaf given by $U \mapsto \mathcal{F}_1(U) \otimes_{\mathcal{O}_X(U)} \mathcal{F}_2(U)$, which is again invertible [Har77]. We have $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{F}^v \cong \mathcal{O}_X$ as sheaves of $\mathcal{O}_X$-modules, which explains the name of invertible sheaves. We will often drop the subscript $\mathcal{O}_X$ in the tensor product. All together this implies, that the set of isomorphism classes of invertible sheaves together with the tensor product forms an abelian group with the structure sheaf as unit element. By Oka's theorem, [Oka50], the structure sheaf $\mathcal{O}_X$ of a complex analytic space is coherent and hence all locally free sheaves of rank n are coherent. In particular we can use proposition 4 on locally free sheaves.

Let $\mathcal{F}$ be an invertible sheaf and let $(U_i)_{i \in I}$ be an open cover of X, such that we have a collection of module isomorphisms, $g_i^{\#}(U_i) : \mathcal{F}(U_i) \cong \mathcal{O}_X(U_i)$. This gives us isomorphisms of $\mathcal{O}_X(U_i \cap U_j)$-modules, called transition functions,

$$g_{ij}^{\#}(U_i \cap U_j) := g_i^{\#}(U_i \cap U_j) \circ (g_j^{\#}(U_i \cap U_j))^{-1} : \mathcal{O}_X(U_i \cap U_j) \longrightarrow \mathcal{O}_X(U_i \cap U_j).$$

But because for any ring R we have $R \cong End_R(R)$, $r \mapsto (a \mapsto ra)$, there is an element $g_{ij} \in \mathcal{O}_X(U_i \cap U_j)$, such that $g_{ij}^{\#}(U_i \cap U_j)(s) = g_{ij}s$. But the map $g_{ij}^{\#}(U_i \cap U_j)$ is an isomorphism with inverse $g_{ij}^{\#}(U_i \cap U_j)^{-1} = g_{ji}^{\#}(U_i \cap U_j)$ and hence the element g_{ij} has to lie in $\mathcal{O}_X^*(U_i \cap U_j)$. Additionally we have $g_{ii}^{\#} = Id_{\mathcal{O}_X(U_i)}$ and $g_{ij}^{\#} \circ g_{jk}^{\#} =$

$(g_i^\#) \circ (g_j^\#)^{-1} \circ (g_j^\#) \circ (g_k^\#)^{-1} = g_i^\# \circ (g_k^\#)^{-1} = g_{ik}^\#$, which is just the cocylce-condition, i.e. it is an element in the kernel of the 1-coboundary map δ_1. This is equivalent to $1 = g_{ij}g_{jk}g_{ki}$. Note carefully, that the abelian group structure of $\mathcal{O}_X^*$ is given by the multiplication and hence the "sum" in the definition of the coboundary map is actually a product. To characterize isomorphism classes of invertible sheaves we have the next proposition from [GR84].

Proposition 5. *Let $(X, \mathcal{O}_X)$ be a compact, complex analytic space. Then there is an isomorphism of abelian groups between the abelian group of isomorphism classes of invertible sheaves and the Čech-cohomology group $\check{\mathrm{H}}^1(X, \mathcal{O}_X^*)$.*

Proof. We have already observed, that g_{ij} is invertible and hence an element of $C^1(U_0 \cap U_1, \mathcal{O}_X^*)$. The abelian group operation of $\mathcal{O}_X^*$ is given by multiplication, so if we compute the differential $\delta_1(g)_{ijk} = g_{ij}g_{ik}^{-1}g_{jk} = 1$ we see immediatly, that $g \in ker(\delta_1)$. In other words every invertible sheaf $\mathcal{F}$ induces a cohomology class given by $[(g_{ij})]$. If we choose now another trivialization $h_i : \mathcal{F}(U_i) \to \mathcal{O}_X(U_i)$, this induces another transition function h_{ij} and an isomorphism $L_i : \mathcal{O}_X(U_i) \to \mathcal{O}_X(U_i)$. The isomorphism L_i is just given by a multiplication of a $l_i \in \mathcal{O}_X^*(U_i)$. We compute $(\delta_0 l)_{ij} = l_i l_j^{-1}$ and get $h_i = l_i g_i$. Furthermore we have $h_{ij} = \delta_0(l)_{ij}g_{ij}$. And this means every invertible sheaf $\mathcal{G}$, isomorphic to $\mathcal{F}$, has a transition function h_{ij} of the form $h_{ij} = \delta_0(l)_{ij}g_{ij}$ for some $l \in C^0(U_i, \mathcal{O}_X^*)$. This shows every such invertible sheaf $\mathcal{G}$ defines the same cohomology class. On the other hand for a given cohomology class $[(g_{ij})] \in \check{\mathrm{H}}^1(X, \mathcal{O}_X^*)$ the gluing property proposition 1 says, there is an invertible sheaf $\mathcal{F}$ obtained by gluing the sheaves $\mathcal{O}_X(U_i)$ together via any representative of the cohomology class. Changing the representative gives just an invertible sheaf $\mathcal{G}$, which is isomorphic to $\mathcal{F}$. □

2.2 Lie groups and Hyperkähler Quotients

In this section we want to repeat some basic facts about Lie groups and hyperkähler manifolds. In particular we want to repeat a method how to get more hyperkähler manifolds by a quotient construction. We will follow [FH04], [Lee12] and [Hit+87].

2.2.1 Lie Group Actions and Quotient Manifolds

A smooth real (complex) manifold G is called a Lie group, if there is a group structure on G, such that the operation and the inversion are smooth (holomorphic) maps. A Lie algebra is a vector space $\mathfrak{g}$ over a field $\mathbb{K}$ with a Lie bracket, which is a bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying $[x, x] = 0$ and the Jacobi-identity $[x, [y, z]] + [y, [z, x]] + [y, [z, x]] = 0$ for all $x, y, z \in \mathfrak{g}$. An important Lie algebra is the Lie algebra associated to a Lie group denoted by $Lie(G)$. Since a Lie group G is a manifold, the infinite dimensional vector space of smooth vector fields on G forms an infinite dimensional Lie algebra via the Lie bracket given by $[X, Y](f) := (XY - YX)(f)$, where $X, Y \in Vec(G)$ and $f \in C^\infty(G)$. The Lie algebra $Lie(G)$ is now the $\dim(G)$-dimensional sub-Lie algebra given by the left-invariant smooth vector fields, which is Lie algebra-isomorphic to the tangent space T_eG with the commutator of derivations as Lie bracket.

Let M be a smooth (respectively complex) manifold and and $Diff(M)$ (respectively $Aut(M)$) its diffeomorphism group (respectively group of holomorphic automorphisms). A group homomorphism $\tilde{\sigma} : G \to Diff(M)$ (respectively $Aut(M)$) is called a Lie group action on the manifold M. It is called a smooth (holomorphic) Lie group action if the induced evaluation map

$$\begin{aligned} \sigma : G \times M &\longrightarrow M \\ (g,p) &\longmapsto \tilde{\sigma}(g)(p) \end{aligned}$$

is smooth (holomorphic). Moreover we have the maps $\sigma_g : M \to M$ given by $\sigma_g(p) := \sigma(g,p)$ and $\sigma_p : g \to M$ given by $\sigma_p(g) := \sigma(g,p)$. For a shorthand notation we will use sometimes $g.p := \sigma(g,p) \in M$. The set

$$G.p := \{g.p : g \in G\} \subset M$$

is called the orbit of G through p. The subgroup

$$G_p := \{g \in G : g.p = p\} < G$$

is called the stabilizer of $p \in M$. An action is called proper, if the map

$$\begin{aligned} \rho : G \times M &\to M \times M \\ (g,p) &\mapsto (\sigma(g)(p), p) \end{aligned}$$

is a proper map, i.e. if $K \subset M \times M$ is any compact subset then $\rho^{-1}(K)$ is compact too. An action is called free if all stabilizers G_p are trivial. An action is called transitive, if for two arbitrary $p, q \in M$ there exists always a $g \in G$ such that $p = \sigma(g,q)$. Let $\exp : \mathfrak{g} \to G$ denote the exponential map, see for a definition [FH04]. For an element $X \in \mathfrak{g}$ there is a vector field, called the fundamental vector field with respect to X, defined via the infinitesimally action by

$$X_p^{\#} := \frac{d}{ds}|_{s=0} \sigma(\exp(-sX), p) \in T_pM,$$

where $p \in M$.

Proposition 6. *Let $\tilde{\sigma} : G \to Diff(M)$ be a smooth, proper and free action on a smooth manifold M. Let $p \in M$ be a any point in M. Then the orbit $G.p$ is a closed, embedded submanifold of M and its tangent space at $e.p$ is*

$$T_{e.p}(G.p) = \left\{ X_p^{\#} : X \in Lie(G) \right\}.$$

Moreover M/G is a smooth manifold and the natural projection $\pi : M \to M/G$ is a submersion.

For a proof see for example [Lee12]. As a remark if the Lie group G is compact, then any action of G is proper. The next lemma shows, that the stabilizer itself is a closed Lie group.

Lemma 1. *If G is a Lie group acting smoothly on a manifold M. Then, for any element $p \in M$, the stabilizer G_p is a closed Lie subgroup of G. Moreover the corresponding Lie algebra is given by*

$$\mathfrak{g}_p := Lie(G_p) = \left\{ X \in \mathfrak{g} : X_p^{\#} = 0 \right\}.$$

If G is a compact Lie group, then G_p is compact too.

Proof. Because $e.p = p$ we see $e \in G_p$. Let $g, h \in G_p$ with $g.p = p$ and $h.p = p$. Then we have $g.(h.p) = p$ and therefore $(gh).p = p$. This means $gh \in G_p$, so G_p is closed under the group operation. Finally, for $g \in G_p$, we have $p = (g^{-1}g).p = g^{-1}.p$ and so $g^{-1} \in G_p$. Therefore G_p is a subgroup. Because the action of G is smooth it is also continuous. This gives us a continuous map $\sigma_p(g) := \sigma(g, p)$, and we can write $G_p = \sigma_p^{-1}(\{p\})$. Because M is Hausdorff the one-point set $\{p\}$ is closed and hence G_p is closed. By Cartan's theorem, which says that a closed subgroup of G is an embedded Lie subgroup, we get a smooth structure on G_p. For the first direction of the last assertion we fix an element $Y \in T_g G_p$. Its fundamental vector field is $X_p^\# = \frac{d}{ds}|_{s=0} \exp(-sY).p = \frac{d}{ds}|_{s=0} p = 0$. The converse follows by the observation, that the action of $\exp(-sX)$ on M is the flow of $X^\#$. This means $\varphi(0, p) = \exp(0).p = p$ and $\frac{d\varphi}{ds}(t, p) = X^\#_{\varphi(t,p)} = 0$. The last equation says, that the flow is locally constant and hence $\exp(-sX)$ fixes p for all $s \in (-\epsilon, \epsilon)$. □

2.2.2 The Orbit-Stabilizer Theorem

If we consider a smooth G-action on a manifold M and an orbit $G.p$ for a $p \in M$, then the action is automatically transitive and thus $G.p$ a homogeneous G-manifold. Because every homogeneous G-manifold can be written as a quotient of the Lie group G by a closed Lie subgroup, we want to write down the identification. A Lie group G acts in various natural ways smoothly on itself. The crucial actions are the left and right multiplications as well as their composition the conjugation, i.e.

$$\begin{aligned} L : G \times G &\to G & R : G \times G &\to G & c : G \times G &\to G \\ (g, h) &\mapsto gh, & (g, h) &\mapsto hg^{-1}, & (g, h) &\mapsto ghg^{-1}. \end{aligned}$$

These actions induce Lie group isomorphisms $G \to G$ given by $L_g(h) := L(g, h)$, $R_g(h) := R(g, h)$ and $c_g(h) := c(g, h)$. Note that the identity is a fixed point of the conjugation, i.e. $c_g(e) = e$. If $H < G$ is a Lie subgroup of G, then we denote the restriction of the right translation to H by $R|_H$. For any $g \in G$ the stabilizer of the action $R|_H$ is

$$H_g = \left\{ h \in H : g = R_g(h) = gh^{-1} \right\} = \{ e \in H \} .$$

Hence we have a free action.

Lemma 2. *If H is a closed subgroup of G, then the action of the right translation $R_h(g) = gh^{-1}$ is a proper action.*

Proof. We have to show, that $\rho : H \times G \to G \times G$ induced by R is a proper map. Let us define two auxilary maps by

$$\begin{aligned} l : G \times G &\to G \times G & \iota : H \times G &\to G \times G \\ (g_1, g_2) &\mapsto (g_2 g_1^{-1}, g_2), & (h, g) &\mapsto (h, g). \end{aligned}$$

We see immediatly, that l is continuous and bijective with inverse map $l^{-1}(k_1, k_2) := (k_1^{-1} k_2, k_2)$, which is also continuous. Hence l is a homeomorphism and therefore a proper map.

Since H is a closed subgroup of G the set $H \times G$ is closed in $G \times G$ too. If we pick a compact set $K \subset G \times G$ the intersection with the closed set $H \times G$, i.e. $K \cap H \times G$, is compact too. Because ι is the inclusion map we know $\iota^{-1}(K) = K \cap H \times G$ and so the inverse image of ι of an arbitrary compact set is compact. Hence ι is a proper map.

But since $\rho = l \circ \iota$, it is a composition of proper maps and therefore it is a proper map itself. □

As a consequence for a smooth action $\tilde{\sigma} : G \to Diff(M)$ we see, that the stabilizer G_p for any $p \in M$ acts freely and properly on G by the restriction of the right translation $R|_{G_p}$. By proposition 6 we get a smooth manifold G/G_p.

Proposition 7. *(Orbit-Stabilizer theorem) Let M be a smooth manifold and G be a Lie group acting on M smoothly and properly. Then the map*

$$\begin{aligned} \Phi : G/G_p &\longrightarrow G.p \\ [g] &\longmapsto g.p \end{aligned}$$

defines a diffeomorphism between the manifold G/G_p and the the orbit $G.p$ seen as a closed embedded manifold in M.

Proof. On the level of groups the orbit-stabilizer theorem says Φ is an G- equivariant isomorphism. Well-definition and bijectivity is easy to see. We see equivariance as follows. If $g_1 \in G$ and $[g_2] \in G/G_p$ then we compute

$$g_1.\Phi([g_2]) = g_1.(g_2.p) = (g_1.g_2).p = \Phi((g_1.g_2).G_p) = \Phi(g_1.[g_2]).$$

It remains to show, that we have a diffeomorphism. First we observe, that the tangent space at $[e]$ of the manifold G/G_p is $\mathfrak{g}/\mathfrak{g}_p$. Because we have $\sigma(\cdot, p) = \Phi \circ \pi$ we get $(d\sigma(\cdot,p))_e = (d\Phi)_{[e]} \circ (d\pi)_e$. But this means $(d\Phi)_{[e]}(\chi + \mathfrak{g}_p) = \chi_X(p)$. Thus the kernel of $(d\Phi)_{[e]}$ is trivial and hence $(d\Phi)_{[e]}$ is injecitve. Because of proposition 6 we know, that $T_p(G.p) = \left\{ X_p^{\#} : X \in Lie(G) \right\}$. This means $(d\Phi)_{[e]}$ is surjective and hence bijective. By using the left translation and equivariance of Φ see $d\Phi$ is everywhere invertible. The inverse function theorem says then, that Φ is everywhere a local diffeomorphism, hence Φ is a global diffeomorphism. □

2.2.3 Adjoint Orbits

Let G be a Lie group and $g \in G$. Recall from the last subsection we have a Lie group isomorphism given by the conjugation $c_g(h) = ghg^{-1}$, where $g, h \in G$. By differentiating this map we get isomorphisms of tangent spaces $(dc_g)_h : T_hG \to T_hG$. By taking $h = e$ and renaming $Ad(g) := (dc_g)_e$ we get a Lie algebra isomorphism

$$Ad(g) : \mathfrak{g} \longrightarrow \mathfrak{g}.$$

This isomorphism induces the Lie group homomorphism

$$\begin{aligned} Ad : G &\to GL(\mathfrak{g}) \\ g &\mapsto Ad(g), \end{aligned}$$

which is called the adjoint representation. The adjoint representation can be seen as a smooth action of the Lie group G on its associated Lie algebra $Lie(G) = \mathfrak{g}$ (seen as a smooth manifold), i.e.

$$\begin{aligned} Ad : G \times \mathfrak{g} &\longrightarrow \mathfrak{g} \\ (g, X) &\longmapsto Ad(g)X. \end{aligned}$$

If G is a matrix Lie group, say $SL_n(\mathbb{C})$ or $SU(n)$, then the adjoint representation is just given by the conjugation

$$Ad(g)(X) = \frac{d}{dt}|_{t=0} \left(g \exp(tX) g^{-1} \right) = gXg^{-1},$$

where $g \in G$ and $X \in \mathfrak{g}$.

For each element $Y \in \mathfrak{g}$ we get an adjoint orbit

$$\mathcal{O}(Y) = Ad(G)(Y) \subset \mathfrak{g},$$

which is a smooth (homogeneous) manifold and can be identified with

$$\mathcal{O}(Y) \cong G/G_Y, \qquad T_{Id}\mathcal{O}(Y) \cong T_eG/T_eG_Y = \mathfrak{g}/\mathfrak{g}_Y$$

by the orbit-stabilizer theorem.

In this thesis we are interested in the case when the Lie group G is either the compact Lie group $SU(n)$ of real dimension $n^2 - 1$ consisting of special, unitary matrices or its complexification $G^{\mathbb{C}}$, which is the complex Lie group $SL_n(\mathbb{C})$ of complex dimension $n^2 - 1$. Both are semisimple Lie groups , which means their asscociated Lie algebras do not contain any non-trivial abelian ideal. We call an element $Y \in \mathfrak{sl}_n(\mathbb{C})$ regular, nilpotent if $Y^n = 0$ and $Y^{n-1} \neq 0$. In other words Y is $GL_n(\mathbb{C})$-conjugated to the Jordan canonical form with exactly one Jordan block

$$N := \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

For a regular, nilpotent element $Y \in \mathfrak{sl}_n(\mathbb{C})$ we call the adjoint orbit $\mathcal{O}_{reg}(Y) := Ad\,(SL_n(\mathbb{C}))\,(Y)$ the complex, regular, nilpotent, adjoint orbit of $SL_n(\mathbb{C})$. Since $\mathcal{O}_{reg}(Y) \cong G^{\mathbb{C}}/G^{\mathbb{C}}_Y$ the orbit has a natural complex structure and it has complex dimension $n^2 - n$. Basically this dimension is obtained by the dimension of $SL_n(\mathbb{C})$ minus the dimension of the stabilizer of Y. But the dimension of the stabilizer is equal to the dimension of a Cartan subalgebra of $\mathfrak{sl}_n(\mathbb{C})$. Such a Cartan subalgebra is given by the complex, diagonal matrices with vanishing trace. Hence the complex dimension of $\mathcal{O}_{reg}(Y)$ is given by $(n^2 - 1) - (n - 1) = n^2 - n$. The regular, nilpotent, adjoint orbit is the unique maximal orbit in the nilpotent variety, see [Kos59] and [Kro90b]. Because of this uniqueness the regular, nilpotent orbit is independent of the choice of the regular, nilpotent element Y and we write sometimes only $\mathcal{O}_{reg}$ for the orbit.

The trace induces a conjugation invariant inner product on $\mathfrak{sl}_n(\mathbb{C})$ by

$$\kappa : \mathfrak{sl}_n(\mathbb{C}) \times \mathfrak{sl}_n(\mathbb{C}) \longrightarrow \mathbb{R}, \qquad (X, Y) \longmapsto tr(XY^*),$$

where $Y^* = \overline{Y}^T$.

2.2.4 Moment Maps

Let $\left(M^{2m}, \omega\right)$ be a symplectic manifold, i.e. a smooth manifold together with a smooth, closed, non-degenerated 2-form ω. Let G be a Lie group acting smoothly by symplectomorphisms on M. This means we have an action $\tilde{\sigma} : G \to Diff(M)$ such that $\tilde{\sigma}(g)^*\omega = \omega$ for all $g \in G$. As a consequence the Lie derivative along any fundamental vector field of the symplectic form vanishes, i.e.

$$\mathcal{L}_{X^\#}\omega = \frac{d}{ds}|_{s=0}\tilde{\sigma}(\exp(-sX))^*\omega = \frac{d}{ds}|_{s=0}\omega = 0,$$

where $X \in \mathfrak{g}$ and $X_p^{\#} := \frac{d}{ds}|_{s=0}\tilde{\sigma}(\exp(-sX))(p)$. With Cartan's magic formula and using closedness of the symplectic form we have

$$0 = \mathcal{L}_{X^{\#}}\omega = \iota(X^{\#})d\omega + d(\iota(X^{\#})\omega) = d(\iota(X^{\#})\omega)$$

and thus $\iota(X^{\#})\omega$ is a closed 1-form. In the case when the first de Rahm cohomology group $H^1_{dR}(M)$ vanishes every closed 1-form is exact and hence there exists for every $X \in \mathfrak{g}$ a smooth function

$$\mu^X : M \longrightarrow \mathbb{R},$$

which satisfies $d\mu^X = \iota(X^{\#})\omega$. This map is unique up to addition of a constant of integration and called a Hamiltonian function. Such a hamiltonian function induces a smooth map

$$\begin{aligned} \mu : M &\longrightarrow \mathfrak{g}^* \\ p &\longmapsto \mu(p)(X) = \mu^X(p). \end{aligned}$$

The co-adjoint representation acts on $\mathfrak{g}^*$ by $Ad(g^{-1})^*$. If the map μ equivariant, i.e. $\mu(\sigma(g,p)) = Ad(g^{-1})^*(\mu(p))$, then it is called a moment map. A symplectic group action is called Hamiltonian, if a moment map exists.

In the case $G = SU(n)$, as an application of Whitehead's lemmas, any symplectic action of G on a symplectic manifold (M,ω) is Hamiltonian [Wan15].

2.2.5 Hyperkähler Manifolds and Hyperkähler Quotients

Let us consider a triple $\left(M^{2m}, g, I\right)$, where M^{2m} is a smooth $2m$-dimensional manifold, g is a Riemannian metric and I is a complex structure. The triple is called a Kähler manifold if the metric g is compatible with the complex structure, i.e. $g(IX, IY) = g(X,Y)$ for all smooth vector fields X, Y, and if the so-called Kähler form $\omega_I(X,Y) := g(IX,Y)$ is closed, i.e. $d\omega_I = 0$. The Kähler form makes any Kähler manifold to a symplectic manifold.

The tuple $\left(M^{4m}, g, I, J, K\right)$ is called a hyperkähler manifold, if all triples $\left(M^{4m}, g, I\right)$, $\left(M^{4m}, g, J\right)$, $\left(M^{4m}, g, K\right)$ are Kähler manifolds and the three complex structures satisfy the quaternionic relations

$$I^2 = J^2 = K^2 = IJK = -Id.$$

The corresponding Kähler forms are denoted by ω_I, ω_J and ω_K.

We will recall the trivial example of a hyperkähler manifold, because at a later stage we need an infinite dimensional analogue of it.

Example 2. *Obviously $\mathbb{R}^4$ is a 4-dimensional, real vector space and a smooth manifold, where all tangent spaces are isomorphic to $\mathbb{R}^4$ too. For any point $p \in \mathbb{R}^4$ the standard inner product*

$$g_p\left((x_0,x_1,x_2,x_3),(y_0,y_1,y_2,y_3)\right) = \langle (x_0,x_1,x_2,x_3),(y_0,y_1,y_2,y_3)\rangle = \sum_{i=0}^{3} x_i y_i$$

defines a Riemannian metric on $\mathbb{R}^4$. With the matrices

$$I_p = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad J_p = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad K_p = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

we get three endomorphism on the tangent spaces $T_p\mathbb{R}^4$, which induce complex structures on $\mathbb{R}^4$. The metric satisfies

$$\langle I_pX, I_pX\rangle = \langle X, X\rangle, \qquad \langle J_pX, J_pX\rangle = \langle X, X\rangle, \qquad \langle K_pX, K_pX\rangle = \langle X, X\rangle$$

for all $X \in T_p\mathbb{R}^4 \cong \mathbb{R}^4$ and hence is compatible with all of the complex structures. For every such complex structure we consider the 2-forms

$$\begin{aligned} \omega_{I,p}(X_p, Y_p) &:= g_p(I_pX_p, Y_p) = \langle(-x_1, x_0, -x_3, x_2), (y_0, y_1, y_2, y_3)\rangle \\ &= -x_1y_0 + x_0y_1 - x_3y_2 + x_2y_3 \\ &= (dx^0 \wedge dx^1)_p(X_p, Y_p) + (dx^2 \wedge dx^3)_p(X_p, Y_p), \\ \omega_{J,p}(X_p, Y_p) &:= g_p(J_pX_p, Y_p) = \langle(-x_2, x_3, x_0, -x_1), (y_0, y_1, y_2, y_3)\rangle \\ &= -x_2y_0 + x_3y_1 + x_0y_2 - x_1y_3 \\ &= (dx^0 \wedge dx^2)_p(X_p, Y_p) + (dx^3 \wedge dx^1)_p(X_p, Y_p), \\ \omega_{K,p}(X_p, Y_p) &:= g_p(K_pX_p, Y_p) = \langle(-x_3, -x_2, x_1, x_0), (y_0, y_1, y_2, y_3)\rangle \\ &= -x_3y_0 - x_2y_1 + x_1y_2 + x_0y_3 \\ &= (dx^0 \wedge dx^3)_p(X_p, Y_p) + (dx^1 \wedge dx^2)_p(X_p, Y_p). \end{aligned}$$

These forms are obviously closed and non-degenerate and hence symplectic forms and therefore $\mathbb{R}^4$ is a hyperkähler manifold. If we identify $\mathbb{R}^4$ with the quaternions $\mathbb{H}$, then the endomorphisms I_p, J_p, K_p correspond to the multiplication by $i, j, k \in \mathbb{H}$.

Now let $\left(M^{4m}, g, I, J, K\right)$ be a hyperkähler manifold and G be a compact Lie group. Suppose that G acts smoothly, freely and by isometries on M, i.e. $\tilde{\sigma}(g)^*g = g$. Furthermore let us assume, that G is a Hamiltonian action with respect to all of the symplectic manifolds $\left(M^{4m}, \omega_I\right), \left(M^{4m}, \omega_J\right), \left(M^{4m}, \omega_K\right)$ with corresponding moment maps μ_I, μ_J and μ_K. Finally we suppose that $0 \in \mathfrak{g}^*$ is a regular value of all moment maps. Then the next theorem from [Hit+87] shows how to produce new hyperkähler manifolds as quotients.

Theorem 1. *Let $N := \left(\mu_I^{-1}(0) \cap \mu_J^{-1}(0) \cap \mu_K^{-1}(0)\right) \subseteq M$. Then the $\dim(M) - 2\dim(G)$-dimensional manifold N/G has a hyperkähler structure.*

2.3 Adjoint Orbits and the Kronheimer Moduli Space

In this section we want to outline the constructions of Kronheimer and Biquard from [Kro90b], [Kro90a] and [Biq96]. Kronheimer found a hyperkähler structure on the nilpotent (and semisimple) adjoint orbits of a semisimple Lie algebra $\mathfrak{g}^{\mathbb{C}}$. Basically he identified such an orbit with a space of solutions of Nahm's equations with certain boundary conditions. Then he identified this solution space with an infinite-dimensional analogue of a hyperkähler quotient, which we will call the Kronheimer Moduli Space.

2.3.1 An infinite dimensional Hyperkähler Manifold

Let us fix now the compact, semisimple Lie group $G = SU(n)$ of unitary matrices with determinant equal to 1 and its Lie algebra $\mathfrak{su}(n)$ of traceless, skew-hermitian matrices. Its complexifications are the complex matrices with determinant equal to 1, i.e. $G^{\mathbb{C}} = SL_n(\mathbb{C})$, and its Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ of trace free complex matrices.

First we fix a nilpotent element $Y \in \mathfrak{g}^{\mathbb{C}}$. By the Jacobson-Morosov theorem [Kro90b] we know, there is a Lie algebra homomorphism

$$\rho^{\mathbb{C}} : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{g}^{\mathbb{C}},$$

such that the triple

$$Y = \rho^{\mathbb{C}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad X := \rho^{\mathbb{C}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad H := \rho^{\mathbb{C}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

forms an $\mathfrak{sl}_2(\mathbb{C})$-triple, i.e. they satisfy the relations

$$[H, X] = 2X, \qquad [H, Y] = -2Y, \qquad [X, Y] = H.$$

Because we fixed the Lie group we know $\mathfrak{sl}_n(\mathbb{C}) \subseteq \mathfrak{gl}(\mathbb{C}^n)$ and hence $\rho^{\mathbb{C}}$ is a complex-linear Lie algebra representation to the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. Now let

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

be the so-called Pauli-matrices. With $\tilde{\sigma}_1 := (-i)\sigma_3$, $\tilde{\sigma}_2 := (-i)\sigma_2$ and $\tilde{\sigma}_3 := (-i)\sigma_1$ we get a basis of $\mathfrak{su}(2)$, which satisfies

$$-2\tilde{\sigma}_1 = [\tilde{\sigma}_2, \tilde{\sigma}_3], \qquad -2\tilde{\sigma}_2 = [\tilde{\sigma}_3, \tilde{\sigma}_1], \qquad -2\tilde{\sigma}_3 = [\tilde{\sigma}_1, \tilde{\sigma}_2]$$

and

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \frac{1}{2}(\tilde{\sigma}_2 + i\tilde{\sigma}_1), \qquad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}(-\tilde{\sigma}_2 + i\tilde{\sigma}_1), \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\tilde{\sigma}_3.$$

Now we can define a homomorphism $\rho : \mathfrak{su}(2) \to \mathfrak{su}(n)$ by

$$\rho(\tilde{\sigma}_1) := \frac{1}{i}(Y + Y^H), \qquad \rho(\tilde{\sigma}_2) := (Y - Y^H), \qquad \rho(\tilde{\sigma}_3) := \frac{1}{i}[Y^H, Y].$$

If the $\mathfrak{sl}_2$-triple satisfies $X = Y^H$ and $H^H = H$, then this map is a Lie algebra homomorphism and its complexification coincides with $\rho^{\mathbb{C}}$.

Let us denote the triple $\sigma := (\tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3)$. Kronheimer studied in [Kro90b] gradient flow equations or in an equivalent manner by a substitution Nahm's equations, which are the three equations

$$\begin{aligned} \dot{T}_1(t) &= -[T_2(t), T_3(t)], \\ \dot{T}_2(t) &= -[T_3(t), T_1(t)], \\ \dot{T}_3(t) &= -[T_1(t), T_2(t)], \end{aligned} \tag{2.1}$$

where $T_i : (-\infty, 0] \to \mathfrak{g}$ are smooth maps for $i \in \{1, 2, 3\}$. Then Kronheimer considered the set of solutions of Nahm's equations with boundary conditions

$$\mathcal{M}(0,\sigma) :=$$

$$\Big\{ (T_1,T_2,T_3) \in C^\infty\left((-\infty,0],\mathfrak{g}^{\oplus 3}\right) : \text{ solution of (2.1)}, \lim_{t\to-\infty} T_i(t) = 0,$$
$$T_1(0) = Ad(g_0)\left(\rho(\tilde{\sigma}_1)\right), T_2(0) = Ad(g_0)\left(\rho(\tilde{\sigma}_2)\right), T_3(0) = Ad(g_0)\left(\rho(\tilde{\sigma}_3)\right)$$
$$\text{for some } g_0 \in G\Big\}.$$

By [Biq96] such solutions satisfy $|T_j(t) + \frac{Ad(g_0)(\rho(\tilde{\sigma}_j))}{2(t-1)}| \leq \frac{c}{(1+|t|)^{1+\delta}}$ for some positive constants $c,\delta > 0$ and all $t \in (-\infty,0]$. With such a constant δ he considered a space of tuples of smooth maps

$$\mathcal{A}_\delta := \Big\{ B := (B_0,B_1,B_2,B_3) \in C^\infty\left((-\infty,0],\mathfrak{g}^{\oplus 4}\right) : \|B_0(t)\|_{1,\delta} < \infty \text{ and}$$
$$\|B_j(t) - \frac{Ad(g_0)(\rho(\tilde{\sigma}_j))}{2(t-1)}\|_{1,\delta} < \infty \text{ for some } g_0 \in G \text{ and for all } j \in \{1,2,3\}\Big\},$$

where the norm is defined by

$$\|B_j(t)\|_{1,\delta} := \sup_{t\in(-\infty,0]} \left((1+|t|^{(1+\delta)})\,|B_j(t)| \right) + \sup_{t\in(-\infty,0]} \left((1+|t|^{(2+\delta)}) \left| \frac{dB_j}{dt}(t) \right| \right).$$

Here by an abuse of notation $|\cdot|$ denotes the norm induced by an Ad-invariant inner product of $\mathfrak{g}$ and the usual absolute value. The L^2-norm is well-defined and defines an inner product on $\mathcal{A}_\delta$ by

$$(B,C)_{L^2} := \int_{-\infty}^0 \sum_{i=0}^3 \langle B_i(t), C_i(t)\rangle dt,$$

where $\langle\cdot,\cdot\rangle$ is an Ad-invariant inner product of $\mathfrak{g}$. Its induced norm on $\mathcal{A}_\delta$ is complete by the Riesz-Fischer theorem. In other words $\mathcal{A}_\delta$ is a Banach space and hence a Banach manifold with tangent space $T_B\mathcal{A}_\delta$ at a point $B = (B_0,B_1,B_2,B_3)$, which is isomorphic to $\mathcal{A}_\delta$. We will denote the elements of $T_B\mathcal{A}_\delta$ by small letters (b_0,b_1,b_2,b_3). With the inner products $(b,c)_{L^2}$ for all $b,c \in T_B\mathcal{A}_\delta$ we get a Riemannian metric on $\mathcal{A}_\delta$.

As an infinite dimensional analogous of example 2 this Banach manifold carries a hyperkähler structure. Three almost complex structures are given by endomorphisms of $T_B\mathcal{A}_\delta$ defined by

$$I_B(b_0,b_1,b_2,b_3) := (-b_1,b_0,-b_3,b_2),$$
$$J_B(b_0,b_1,b_2,b_3) := (-b_2,b_3,b_0,-b_1),$$
$$K_B(b_0,b_1,b_2,b_3) := (-b_3,-b_2,b_1,b_0).$$

They clearly satisfy the quaternionic relations and they induce three 2-forms

$$\omega_{I,B}(b,c) := (I_Bb,c)_{L^2,B}, \quad \omega_{J,B}(b,c) := (J_Bb,c)_{L^2,B}, \quad \omega_{K,B}(b,c) := (K_Bb,c)_{L^2,B}.$$

The almost complex structures are integrable, the 2-forms are closed and non-degenerate and the metric is compatible with all of the three almost complex structures, see [Biq96].

2.3.2 Kronheimer Moduli Space

With the hyperkähler manifold $\mathcal{A}_\delta$ Biquard performed a hyperkähler quotient. We want to roughly outline his procedure. First he considered a gauge Banach Lie group

$$\mathcal{G}_{0,\delta} := \left\{ g \in C^\infty\left((-\infty, 0], G\right) : g(0) = Id, (\nabla_B g)g^{-1} \in \mathcal{A}_\delta \quad \forall B \in \mathcal{A}_\delta \right\}$$

with a smooth, in the sense of Banach manifolds, proper and free action on $\mathcal{A}_\delta$, a gauge transformation, defined by

$$\begin{aligned} \tau : \mathcal{G}_{0,\delta} \times \mathcal{A}_\delta &\longrightarrow \mathcal{A}_\delta \\ (g, B) &\longmapsto \left(Ad(g)B_0 - \frac{dg}{dt}g^{-1}, Ad(g)B_1, Ad(g)B_2, Ad(g)B_3 \right). \end{aligned}$$

The details to show smoothness, properness and freeness are computed in a slightly different situation in [Gal18]. The condition $(\nabla_B g)g^{-1} \in \mathcal{A}_\delta$ means that $\tau(g, B)$ lies in $\mathcal{A}_\delta$ too for all $g \in \mathcal{G}_{0,\delta}$ and $B \in \mathcal{A}_\delta$, see for definition and details [Kro90a] and [Biq96]. We have also the following claim.

Claim 1. *The group action τ is hamiltonian with respect to all of the complex structures I, J, K and the maps*

$$\begin{aligned} \mu_I^X : \mathcal{A}_\delta &\longrightarrow \mathbb{R} \\ B &\longmapsto \int_{-\infty}^0 \langle -\dot{B}_1(t) - [B_0, B_1] - [B_2, B_3], X_B \rangle dt, \\ \mu_J^X : \mathcal{A}_\delta &\longrightarrow \mathbb{R} \\ B &\longmapsto \int_{-\infty}^0 \langle -\dot{B}_2(t) - [B_0, B_2] - [B_3, B_1], X_B \rangle dt, \\ \mu_K^X : \mathcal{A}_\delta &\longrightarrow \mathbb{R} \\ B &\longmapsto \int_{-\infty}^0 \langle -\dot{B}_3(t) - [B_0, B_3] - [B_1, B_2], X_B \rangle dt, \end{aligned}$$

are the corresponding hamiltonian functions, where $X : \mathcal{A}_\delta \to T\mathcal{A}_\delta$ is a smooth vector field.

These maps are well defined by the definition of $\mathcal{A}_\delta$, for a proof and smoothness see [Gal18]. We say a tuple (B_0, B_1, B_2, B_3) satisfy the extended Nahm's equations if we have

$$\begin{aligned} \dot{B}_1(t) &= -[B_0, B_1] - [B_2, B_3] \\ \dot{B}_2(t) &= -[B_0, B_2] - [B_3, B_1] \\ \dot{B}_3(t) &= -[B_0, B_3] - [B_1, B_2], \end{aligned}$$

where the $B_i \in C^\infty((-\infty, 0], \mathfrak{g})$.

We denote $\mathcal{M}^{ext}(0, \sigma) := \mu^{-1}(0) := \mu_I^{-1}(0) \cap \mu_J^{-1}(0) \cap \mu_K^{-1}(0) \subset \mathcal{A}_\delta$ the solution space of the extended Nahm's equations in $\mathcal{A}_\delta$. Here the maps μ_I, μ_J, μ_K are the corresponding moment maps with respect to the hamiltonian functions of claim 1. Kronheimer and Biquard showed, that $\mathcal{M}^{ext}(0, \sigma)/\mathcal{G}_{0,\delta}$ is a smooth Banach manifold and that the hyperkähler structure of $\mathcal{A}_\delta$ descends to the quotient, i.e. it is a hyperkähler quotient. Furthermore we have an identification $\mathcal{M}(0, \sigma) \cong \mathcal{M}^{ext}(0, \sigma)/\mathcal{G}_{0,\delta}$ as smooth Banach manifolds, see [Biq96]. In other words $\mathcal{M}(0, \sigma)$ admits a hyperkähler structure. We call the hyperkähler manifold

$$\mathcal{M}^{ext}(0,\sigma)/\mathcal{G}_{0,\delta}$$

the Kronheimer moduli space. It could make sense to call this space the Biquard moduli space, but i wanted to emphasize Kronheimers beautiful ideas. Kronheimer showed then in [Kro90b] by studying equivalent complex trajectories, seen $\mathcal{O}_{reg}(Y)$ as a complex manifold and $\mathcal{M}^{ext}(0,\sigma)/\mathcal{G}_{0,\delta}$ seen as a complex manifold with complex structure I, that the map

$$\begin{aligned}(\mathcal{M}^{ext}(0,\sigma)/\mathcal{G}_{0,\delta}, I) &\longrightarrow (\mathcal{O}_{reg}(Y), i)\\ [T_0,T_1,T_2,T_3] &\longmapsto T_2(0)+iT_3(0)\end{aligned}$$

is a biholomorphism. Then this biholomorphism carries the hyperkähler structure of $\mathcal{M}^{ext}(0,\sigma)/\mathcal{G}_{0,\delta}$ to the regular, nilpotent orbit.

2.3.3 Circle Action and Kähler Potential

It is difficult to describe the hyperkähler metric on $\mathcal{O}_{reg}(Y)$ explicitely because of the behavior of the identification of the regular, nilpotent orbit with the Kronheimer moduli space, where we would have to solve a system of differential equations. Hitchin proposed to study a Kähler potential of the Kähler structure of a fixed complex structure. The next claim from [Hit+87] and [Hit98] describes how such a Kähler potential arise.

Proposition 8. *Let (M,g,I,J,K) be a hyperkähler manifold and let $\omega_I,\omega_J,\omega_K$ be the three Kähler forms. Moreover let X be a smooth vector field with the properties*

$$\mathcal{L}_X\omega_I=0,\qquad \mathcal{L}_X\omega_J=\omega_K,\qquad \mathcal{L}_X\omega_K=-\omega_J.$$

Furthermore let $\mu_I^X: M\to\mathbb{R}$ be a hamiltonian function with respect to the form ω_I. Then $2\mu_I^X$ is a Kähler potential for the Kähler manifold (M,g,J), i.e.

$$\omega_J=2i\partial_J\overline{\partial}_J\mu_I^X.$$

Proof. The function μ_I^X satisfies by assumption $d\mu_I^X=\iota_X\omega_I$, so we get for any smooth vector field Y

$$d\mu_I^X(JY)=(\partial_J+\overline{\partial_J})\mu_I^X(JY)=i(\partial_J-\overline{\partial_J})\mu_I^X(Y).$$

On the other hand we have by the hyperkähler structure and the compatibility of the metric

$$d\mu_I^X(JY)=\iota_X\omega_I(JY)=g(IX,JY)=g(KX,Y)=\omega_K(X,Y).$$

Together this means $i(\partial_J-\overline{\partial_J})\mu_I^X(Y)=\omega_K(X,Y)$ and hence we have

$$-2i\partial_J\overline{\partial_J}\mu_I^X=i(\overline{\partial_J}\partial_J-\partial_J\overline{\partial_J})\mu_I^X(Y)=di(\partial_J-\overline{\partial_J})\mu_I^X(Y)=d(\omega_K)=\mathcal{L}_X\omega_K=-\omega_J.$$

□

Following Hitchin in [Hit98], we have a natural $SO(3)$-action on $M^{ext}(0,\sigma)/G_{0,\delta}$ given by

$$\begin{aligned}\alpha : SO(3) \times M^{ext}(0,\sigma)/G_{0,\delta} &\longrightarrow M^{ext}(0,\sigma)/G_{0,\delta} \\ (P, [B_0(t), B_1(t), B_2(t), B_3(t)]) &\longmapsto \left[B_0(t), \sum_{j=1}^{3} P_{1j}B_j(t), \sum_{j=1}^{3} P_{2j}B_j(t), \sum_{j=1}^{3} P_{3j}B_j(t)\right].\end{aligned}$$

Let $SO(2) \subset SO(3)$ be a circle action which fixes B_1, i.e.

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{pmatrix}, \alpha \in [0, 2\pi).$$

With $b_i \in T_{B(t)}\mathcal{A}_\delta$ this circle action satisfies

$$\begin{aligned} I_B(\alpha(P,b)) &= I_B(b_0, b_1, \cos(\alpha)b_2 - \sin(\alpha)b_3, \sin(\alpha)b_2 + \cos(\alpha)b_3) \\ &= (-b_1, b_0, -\sin(\alpha)b_2 - \cos(\alpha)b_3, \cos(\alpha)b_2 - \sin(\alpha)b_3) = \alpha(P, I_B(b)) \end{aligned}$$

and

$$\begin{aligned} &g_B(\alpha(P,a), \alpha(P,b)) \\ &= \int_{-\infty}^{0} \langle a_0, b_0\rangle + \langle a_1, b_1\rangle + \langle \cos(\alpha)a_2 - \sin(\alpha)a_3, \cos(\alpha)b_2 - \sin(\alpha)b_3\rangle \\ &\quad + \langle \sin(\alpha)a_2 + \cos(\alpha)a_3, \sin(\alpha)b_2 + \cos(\alpha)b_3\rangle dt \\ &= \int_{-\infty}^{0} \langle a_0, b_0\rangle + \langle a_1, b_1\rangle + \cos(\alpha)^2\langle a_2, b_2\rangle - \cos(\alpha)\sin(\alpha)\langle a_2, b_3\rangle - \cos(\alpha)\sin(\alpha)\langle a_3, b_2\rangle \\ &\quad + \sin(\alpha)^2\langle a_3, b_3\rangle + \sin(\alpha)^2\langle a_2, b_2\rangle + \sin(\alpha)\cos(\alpha)\langle a_2, b_3\rangle \\ &\quad + \cos(\alpha)\sin(\alpha)\langle a_3, b_2\rangle + \cos(\alpha)^2\langle a_3, b_3\rangle dt \\ &= \int_{-\infty}^{0} \langle a_0, b_0\rangle + \langle a_1, b_1\rangle + \cos(\alpha)^2\langle a_2, b_2\rangle + \sin(\alpha)^2\langle a_3, b_3\rangle \\ &\quad + \sin(\alpha)^2\langle a_2, b_2\rangle + \cos(\alpha)^2\langle a_3, b_3\rangle dt \\ &= \int_{-\infty}^{0} \langle a_0, b_0\rangle + \langle a_1, b_1\rangle + \langle a_2, b_2\rangle + \langle a_3, b_3\rangle dt \\ &= g_B(a,b). \end{aligned}$$

Because $\omega_{I,B}(\alpha(P,a), \alpha(P,b)) = g_B(I_B\alpha(P,a), \alpha(P,b)) = g_B(\alpha(P, I_Ba), \alpha(P,b)) = g_B(I_Ba, b) = \omega_{I,B}(a,b)$ the circle action is a symplectic action. Furthermore we have

$$\begin{aligned} &\omega_{J,B}(\alpha(P,a), \alpha(P,b)) + i\omega_{K,B}(\alpha(P,a), \alpha(P,B)) \\ &\quad = (\cos(\alpha) + i\sin(\alpha))\left(\omega_{J,B}(a,b) + i\omega_{K,B}(a,b)\right). \end{aligned}$$

This means the action fixes the Kähler form ω_I and rotates the other two Kähler forms.

Claim 2. *Let* $X := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathfrak{so}(2)$ *and let* $X^\#$ *be its fundamental vector field with respect to the circle action which fixes* B_1. *Then the fundamental vector field is given* *by*

$$X^{\#}_{B(t)} = (0, 0, B_3(t), -B_2(t)) \,.$$

Moreover the following equations hold

$$\mathcal{L}_{X^{\#}}\omega_I = 0, \qquad \mathcal{L}_{X^{\#}}\omega_J = \omega_K, \qquad \mathcal{L}_{X^{\#}}\omega_K = -\omega_J.$$

Proof. Let $B(t) = (B_0(t), B_1(t), B_2(t), B_3(t)) \in \mathcal{A}_\delta$. Then the fundamental vector field is given by

$$\begin{aligned} X^{\#}_{B(t)} &= \frac{d}{d\theta}|_{\theta=0}\,(B_0(t), B_1(t), \cos(\theta)B_2(t) - \sin(\theta)B_3(t), \sin(\theta)B_2(t) + \cos(\theta)B_3(t)) \\ &= (0, 0, B_3(t), -B_2(t)) \end{aligned}$$

and therefore

$$\begin{aligned} I_{B(t)}X^{\#}_{B(t)} &= (0, 0, B_2(t), B_3(t)) \,, \\ J_{B(t)}X^{\#}_{B(t)} &= (-B_3(t), -B_2(t), 0, 0) \,, \\ K_{B(t)}X^{\#}_{B(t)} &= (B_2(t), -B_3(t), 0, 0) \,. \end{aligned}$$

Let Y and Z be two smooth vector fields. For a small enough $\epsilon > 0$ we choose curves $\gamma : (-\epsilon, \epsilon) \to \mathcal{A}_\delta$, $\delta : (-\epsilon, \epsilon) \to \mathcal{A}_\delta$, such that $\gamma(0) = B(t)$, $\frac{d\gamma}{d\kappa}(0) = Y_{B(t)}$ and $\delta(0) = B(t)$, $\frac{d\delta}{d\nu}(0) = Z_{B(t)}$. For any smooth function $f \in C^\infty(\mathcal{A}_\delta)$ we have $Y_{B(t)}(Z_\cdot(f)) = \frac{d}{d\kappa}|_{\kappa=0} Z_{\gamma(\kappa)}(f)$ and $Z_{B(t)}(Y_\cdot(f)) = \frac{d}{d\nu}|_{\nu=0} Y_{\delta(\nu)}(f)$. For the sake of clarity let us drop in the next computations the argument (t). We have

$$
\begin{aligned}
&(\mathcal{L}_X\omega_I)_B(Y_B, Z_B)\\
&= d(\iota_X\omega_I)_B(Y_B, Z_B) = Y_B(\iota_X\omega_I(Z)) - Z_B(\iota_X\omega_I(Y)) - \omega_{I,B}([Y,Z]_B)\\
&= \frac{d}{d\kappa}|_{\kappa=0}\left(\omega_I(X_{\gamma(\kappa)}, Z_{\gamma(\kappa)})\right) - \frac{d}{d\nu}|_{\nu=0}\left(\omega_I(X_{\delta(\nu)}, Y_{\delta(\nu)})\right) - \omega_I(X_B, [Y,Z]_B)\\
&= \frac{d}{d\kappa}|_{\kappa=0}\int_{-\infty}^0 \langle\gamma_2(\kappa), Z_{\gamma(\kappa),2}\rangle + \langle\gamma_3(\kappa), Z_{\gamma(\kappa),3}\rangle dt\\
&\quad - \frac{d}{d\nu}|_{\nu=0}\int_{-\infty}^0 \langle\delta_2(\nu), Y_{\delta(\nu),2}\rangle + \langle\delta_3(\nu), Y_{\delta(\nu),3}\rangle dt\\
&\quad - \int_{-\infty}^0 \langle B_2, [Y,Z]_{A,2}\rangle + \langle B_3, [Y,Z]_{A,3}\rangle dt\\
&= \int_{-\infty}^0 \langle\frac{d}{d\kappa}|_{\kappa=0}\gamma_2(\kappa), Z_{\gamma(0),2}\rangle + \langle\frac{d}{d\kappa}|_{\kappa=0}\gamma_3(\kappa), Z_{\gamma(0),3}\rangle dt\\
&\quad + \int_{-\infty}^0 \langle\gamma_2(0), \frac{d}{d\kappa}|_{\kappa=0}Z_{\gamma(\kappa),2}\rangle + \langle\gamma_3(0), \frac{d}{d\kappa}|_{\kappa=0}Z_{\gamma(\kappa),3}\rangle dt\\
&\quad - \int_{-\infty}^0 \langle\frac{d}{d\nu}|_{\nu=0}\delta_2(\nu), Y_{\delta(0),2}\rangle + \langle\frac{d}{d\nu}|_{\nu=0}\delta_3(\nu), Y_{\delta(0),3}\rangle dt\\
&\quad - \int_{-\infty}^0 \langle\delta_2(0), \frac{d}{d\nu}|_{\nu=0}Y_{\delta(\nu),2}\rangle + \langle\delta_3(0), \frac{d}{d\nu}|_{\nu=0}Y_{\delta(\nu),3}\rangle dt\\
&\quad - \int_{-\infty}^0 \langle B_2, [Y,Z]_{B,2}\rangle + \langle B_3, [Y,Z]_{B,3}\rangle dt\\
&= \int_{-\infty}^0 \langle Y_{B,2}, Z_{\gamma(0),2}\rangle + \langle Y_{B,3}, Z_{\gamma(0),3}\rangle dt - \int_{-\infty}^0 \langle Z_{B,2}, Y_{\delta(0),2}\rangle + \langle Z_{B,3}, Y_{\delta(0),3}\rangle dt\\
&\quad + \int_{-\infty}^0 \langle\gamma_2(0), Y_B(Z_{\cdot,2})\rangle + \langle\gamma_3(0), Y_B(Z_{\cdot,3})\rangle dt\\
&\quad - \int_{-\infty}^0 \langle\delta_2(0), Z_B(Y_{\cdot,2})\rangle + \langle\delta_3(0), Z_B(Y_{\cdot,3})\rangle dt\\
&\quad - \int_{-\infty}^0 \langle B_2, [Y,Z]_{B,2}\rangle + \langle B_3, [Y,Z]_{B,3}\rangle dt\\
&= \int_{-\infty}^0 \langle Y_{B,2}, Z_{B,2}\rangle + \langle Y_{B,3}, Z_{B,3}\rangle dt - \int_{-\infty}^0 \langle Z_{B,2}, Y_{B,2}\rangle + \langle Z_{B,3}, Y_{B,3}\rangle dt\\
&= 0.
\end{aligned}
$$

This shows the first equation. Moreover we have

$$
\omega_{J,B}(Y_B, Z_B) = \int_{-\infty}^0 \langle -Y_{B,2}, Z_{B,0}\rangle + \langle Y_{B,3}, Z_{B,1}\rangle + \langle Y_{B,0}, Z_{B,2}\rangle + \langle -Y_{B,1}, Z_{B,3}\rangle dt,
$$

$$
\omega_{K,B}(Y_B, Z_B) = \int_{-\infty}^0 \langle -Y_{B,3}, Z_{B,0}\rangle + \langle -Y_{B,2}, Z_{B,1}\rangle + \langle Y_{B,1}, Z_{B,2}\rangle + \langle Y_{B,0}, Z_{B,3}\rangle dt.
$$

With the last two formulas we compute

$$
\begin{aligned}
&(\mathcal{L}_X\omega_J)_B(Y_B,Z_B)\\
&= d(\iota_X\omega_J)_B(Y_B,Z_B)\\
&= Y_B(\iota_X\omega_J(Z)) - Z_B(\iota_X\omega_J(Y)) - \omega_{J,B}([Y,Z]_B)\\
&= \frac{d}{d\kappa}|_{\kappa=0}\left(\omega_J(X_{\gamma(\kappa)},Z_{\gamma(\kappa)})\right) - \frac{d}{d\nu}|_{\nu=0}\left(\omega_J(X_{\delta(\nu)},Y_{\delta(\nu)})\right) - \omega_J(X_B,[Y,Z]_B)\\
&= \frac{d}{d\kappa}|_{\kappa=0}\int_{-\infty}^{0}\langle-\gamma_3(\kappa),Z_{\gamma(\kappa),0}\rangle + \langle-\gamma_2(\kappa),Z_{\gamma(\kappa),1}\rangle dt\\
&\quad - \frac{d}{d\nu}|_{\nu=0}\int_{-\infty}^{0}\langle-\delta_3(\nu),Y_{\delta(\nu),0}\rangle + \langle-\delta_2(\nu),Y_{\delta(\nu),1}\rangle dt\\
&\quad - \int_{-\infty}^{0}\langle-B_3,[Y,Z]_{B,0}\rangle + \langle-B_2,[Y,Z]_{B,1}\rangle dt\\
&= \int_{-\infty}^{0}\langle-Y_{B,3},Z_{B,0}\rangle + \langle-Y_{B,2},Z_{B,1}\rangle dt + \int_{-\infty}^{0}\langle-B_3,Y_{B,0}(Z_{\cdot,0})\rangle + \langle-B_2,Y_{B,1}(Z_{\cdot,1})\rangle dt\\
&\quad - \int_{-\infty}^{0}\langle-Z_{B,3},Y_{B,0}\rangle + \langle-Z_{B,2},Y_{B,1}\rangle dt - \int_{-\infty}^{0}\langle-B_3,Z_{B,0}(Y_{\cdot,0})\rangle + \langle-B_2,Z_{B,1}(Y_{\cdot,1})\rangle dt\\
&\quad - \int_{-\infty}^{0}\langle-B_3,[Y,Z]_{B,0}\rangle + \langle-B_2,[Y,Z]_{B,1}\rangle dt\\
&= \int_{-\infty}^{0}\langle-Y_{B,3},Z_{B,0}\rangle + \langle-Y_{B,2},Z_{B,1}\rangle dt - \int_{-\infty}^{0}\langle-Z_{B,3},Y_{B,0}\rangle + \langle-Z_{B,2},Y_{B,1}\rangle dt\\
&= \omega_{K,N}(Y_B,Z_B).
\end{aligned}
$$

This shows the second equation. In the same way we see $(\mathcal{L}_X\omega_K)_B(Y_B,Z_B) = -\omega_{J,B}(Y_B,Z_B)$ and hence we have proven the claim. □

By proposition 8 we know, that a hamiltonian function $\mu_I^{X\#}$ defines a Kähler potential with respect to the complex structure J. The next claim from [Hit98] describes such a moment map.

Claim 3. *Let* $X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathfrak{so}(2)$ *and* $X^{\#}$ *its fundamental vector field with respect to the circle action fixing* B_1*. Then the map*

$$
\begin{aligned}
\mu_I^X : \mathcal{A}_\delta &\longrightarrow \mathbb{R}\\
B(t) &\longmapsto \frac{1}{2}\int_{-\infty}^{0}\langle B_2(t),B_2(t)\rangle + \langle B_3(t),B_3(t)\rangle dt
\end{aligned}
$$

is a hamiltonian function.

Proof. Let us fix a tangent vector $Y(t) := (y_0(t),y_1(t),y_2(t),y_3(t)) \in T_{B(t)}\mathcal{A}_\delta$. Let us consider the curve $\gamma : (-\epsilon,\epsilon) \to \mathcal{A}_\delta, s \mapsto B(t) + sY(t)$ with $\gamma(0)(t) = B(t)$ and

$\frac{d\gamma}{ds}(0)(t) = Y(t)$. Then we have

$$\begin{aligned}
&d(\mu_I^X)_{B(t)}(Y(t)_{B(t)}) \\
&= \frac{d}{ds}|_{s=0}\mu_I^X(\gamma(s)(t)) \\
&= \frac{1}{2}\frac{d}{ds}|_{s=0}\int_{-\infty}^{0} \langle B_2(t) + sy_2(t), B_2(t) + sy_2(t)\rangle + \langle B_3(t) + sy_3(t), B_3(t) + sy_3(t)\rangle dt \\
&= \int_{-\infty}^{0} \langle B_2(t), y_2(t)\rangle + \langle B_3(t), y_3(t)\rangle dt = g_{B(t)}(I_{B(t)}X^{\#}_{B(t)}, Y(t)) \\
&= \omega_{I,B(t)}(X^{\#}_{B(t)}, Y(t)) = \left(\iota(X^{\#})\omega_{I,B(t)}\right)(Y(t)).
\end{aligned}$$

□

In our case where $\mathfrak{g} = \mathfrak{su}(n)$ we have $\langle a, a\rangle = tr(aa^*) = -tr(a^2)$. So we write the Kähler potential in the form

$$K(B(t)) = 2\mu_I^X(B(t)) = -\int_{-\infty}^{0} tr\left(B_2(t)^2 + B_3(t)^2\right) dt. \tag{2.2}$$

The last formula is basically the start of this thesis, where we want to give a formula of $tr(B_2(t)^2 + B_3(t)^2)$ in terms of the theta function on the nilpotent, spectral curve.

Chapter 3

Nilpotent, Spectral Curve, its Jacobian Variety and the Generalized Theta Function

In this chapter we want to study the nilpotent, spectral curve, its Jacobian variety and its theta divisor. It turns out, that the theta divisor is given by the zero-set of a determinantal function. This polynomial is called the theta function and we will describe it in full details. The main tool to describe the Jacobian is, of course, the exponential sequence together with the characterization of invertible sheaves via Čech-cohomology.

3.1 Nilpotent, Spectral Curve

3.1.1 Ambient Space $|\mathcal{O}_{\mathbb{CP}^1}(2)|$

In this first subsection we will repeat well-known constructions of some complex analytic spaces. We begin with the construction the complex projective space $\mathbb{CP}^1$. Because this thesis is based on making explicit computations and working in local coordinates, it is important to write down carefully the used notations, even if the geometric object is widely well-known. Let us consider the sets

$$\begin{aligned} W_1 &:= \mathbb{C} \ni \tilde{\zeta}, & W_0 &:= \mathbb{C} \ni \zeta, \\ W_{01} &:= W_1 \setminus \{0\} = \mathbb{C}^*, & W_{10} &:= W_0 \setminus \{0\} = \mathbb{C}^*. \end{aligned}$$

Furthermore let us endow these sets with the sheaf of holomorphic functions $\mathcal{O}_{W_i}$ for $i \in \{0,1\}$. The two pairs $(W_1, \mathcal{O}_{W_1})$, $(W_0, \mathcal{O}_{W_0})$ are complex analytic model spaces. Furthermore the map

$$\left(\varphi_{10}, \varphi_{10}^{\#}\right) : \left(W_{10}, \mathcal{O}_{W_0|_{W_{10}}}\right) \longrightarrow \left(W_{01}, \mathcal{O}_{W_1}|_{W_{01}}\right),$$

given by $\varphi_{10}(\zeta) = \frac{1}{\zeta}$ for $\zeta \in W_{10}$ with lifting operator defined by the assignment

$$\begin{aligned} \varphi_{10}^{\#}(W) : \mathcal{O}_{W_1}|_{W_{01}}(W) &\longrightarrow \mathcal{O}_{W_0|_{W_{10}}}(\varphi_{10}^{-1}(W)) \\ s_1 &\longmapsto \varphi_{10}^{*} s_1 := s_1 \circ \varphi_{10} \end{aligned}$$

for any open subset $W \subset W_{01}$ is biholomorphic. Via this biholomorphism we are able to glue the two model spaces $(W_i, \mathcal{O}_{W_i})$, $i \in \{0,1\}$, together by proposition 1 to get the complex, projective space $\mathbb{CP}^1 := W_0 \cup W_1/_{\sim}$ with its structure sheaf $\mathcal{O}_{\mathbb{CP}^1}$ of holomorphic functions on $\mathbb{CP}^1$. If $W \subseteq \mathbb{CP}^1$ is an arbitrary open set, then an element $f \in \mathcal{O}_{\mathbb{CP}^1}(W)$ is given by a pair $f = (f_1, f_0) \in \mathcal{O}_{W_1}(W \cap W_1) \times \mathcal{O}_{W_0}(W \cap W_0)$ such that $\varphi_{10}^{\#}(W\cap W_{01})(s_1)(\zeta) = (\varphi_{10}^* s_1)(\zeta) = s_1(\frac{1}{\zeta}) = s_0(\zeta)$ holds for all $\zeta \in W_{10}\cap W$. The complex analytic space

$$\left(\mathbb{CP}^1, \mathcal{O}_{\mathbb{CP}^1}\right)$$

is one of the most beautiful complex analytic spaces. For example it is an Einstein-Kähler manifold with the Fubini-Study metric and admits an antipodal map, an antiholomorphic involution (see the lecture notes of Andrei Moroianu [Mor04]).

Let $\mathcal{O}_{\mathbb{CP}^1}(1)$ be Serre's twisting sheaf, i.e. the invertible sheaf on $\mathbb{CP}^1$ given by the transition function $h_{10}(\zeta) = \frac{1}{\zeta}$. Let us denote by $\mathcal{O}_{\mathbb{CP}^1}(-1) := \mathcal{O}_{\mathbb{CP}^1}(1)^v := \mathcal{H}om\left(\mathcal{O}_{\mathbb{CP}^1}(1), \mathcal{O}_{\mathbb{CP}^1}\right)$ the dual sheaf of Serre's twisting sheaf, which is the sheaf hom of $\mathcal{O}_{\mathbb{CP}^1}(1)$ and $\mathcal{O}_{\mathbb{CP}^1}$, see for definitions [Har77]. One can show, that the transition function of $\mathcal{O}_{\mathbb{CP}^1}(-1)$ is just given by $k_{10}(\zeta) = \frac{1}{h_{10}(\zeta)} = \zeta$. For an integer $k \in \mathbb{Z}$ we define

$$\mathcal{O}_{\mathbb{CP}^1}(k) := \mathcal{O}_{\mathbb{CP}^1}(1)^{\otimes k},$$

where we take the dual sheaf for negative k. These are all invertible sheaves and their transition functions are given by $l_{10}(\zeta) = \frac{1}{\zeta^k}$. Furthermore every other invertible sheaf on $\left(\mathbb{CP}^1, \mathcal{O}_{\mathbb{CP}^1}\right)$ is isomorphic to one of those $\mathcal{O}_{\mathbb{CP}^1}(k)$ by the famous Birkhoff - Grothendieck theorem or just in other symbols $\check{\mathrm{H}}^1(\mathbb{CP}^1, \mathcal{O}^*_{\mathbb{CP}^1}) \cong \mathbb{Z}$.

To every invertible sheaf $\mathcal{O}_{\mathbb{CP}^1}(k)$ we associate a 2-dimensional complex manifold $|\mathcal{O}_{\mathbb{CP}^1}(k)|$, called the total space, together with a natural projection $\pi_{\mathcal{O}_{\mathbb{CP}^1}(k)} : |\mathcal{O}_{\mathbb{CP}^1}(k)| \to \mathbb{CP}^1$. The next step is to describe the total space of the sheaf $\mathcal{O}_{\mathbb{CP}^1}(2)$ as a complex 2-dimensional manifold via gluing. First we take two complex analytic model spaces

$$(V_1, \mathcal{O}_{V_1}) \cong \left(\mathbb{C}^2, \mathcal{O}_{\mathbb{C}^2}\right), \qquad (V_0, \mathcal{O}_{V_0}) \cong \left(\mathbb{C}^2, \mathcal{O}_{\mathbb{C}^2}\right),$$

with coordinates $(\tilde{\zeta}, \tilde{\eta}) \in V_1$ and $(\zeta, \eta) \in V_0$. We define the spaces

$$V_{01} := \mathbb{C}^* \times \mathbb{C} = \left\{(\tilde{\zeta}, \tilde{\eta}) \in V_1 : \tilde{\zeta} \neq 0\right\}, \qquad V_{10} := \mathbb{C}^* \times \mathbb{C} = \{(\zeta, \eta) \in V_0 : \zeta \neq 0\}$$

and a biholomorphic map, by abusing the notation,

$$\left(\varphi_{10}, \varphi_{10}^{\#}\right) : (V_{10}, \mathcal{O}_{V_0}|_{V_{10}}) \longrightarrow (V_{01}, \mathcal{O}_{V_1}|_{V_{01}})$$

given by

$$\varphi_{10} : V_{10} \longrightarrow V_{01}, \qquad (\zeta, \eta) \longmapsto \left(\frac{1}{\zeta}, \frac{\eta}{\zeta^2}\right)$$

with lifting operator defined by

$$\begin{aligned} \varphi_{10}^{\#}(V) : \mathcal{O}_{V_1}|_{V_{01}}(V) &\longrightarrow \mathcal{O}_{V_0|_{V_{10}}}(\varphi_{10}^{-1}(V)) \\ s_1 &\longmapsto \varphi_{10}^* s_1 := s_1 \circ \varphi_{10} \end{aligned}$$

for any open subset $V \subset V_{01}$. By the gluing property, proposition 1, of complex analytic spaces we get a complex analytic space

$$(T, \mathcal{O}_T),$$

where $T := V_1 \cup V_0/_\sim$ and $(\zeta, \eta) \sim (\tilde{\zeta}, \tilde{\eta})$ if and only if $(\zeta, \eta) \in V_{10}$, $(\tilde{\zeta}, \tilde{\eta}) \in V_{01}$ and $(\tilde{\zeta}, \tilde{\eta}) \cong \varphi_{10}(\zeta, \eta) = (\frac{1}{\zeta}, \frac{\eta}{\zeta^2})$. For any open set $V \subseteq T$ the sections of $\mathcal{O}_T(V)$ are given by pairs $(f_1, f_0) \in \mathcal{O}_{V_1}(V \cap V_1) \times \mathcal{O}_{V_0}(V \cap V_0)$ such that $(\varphi_{10}^* f_1)(\zeta, \eta) = f_1(\frac{1}{\zeta}, \frac{\eta}{\zeta^2}) = f_0(\zeta, \eta)$ for all $(\zeta, \eta) \in V_{10} \cap V$. The space $(T, \mathcal{O}_T)$ has a natural holomorphic projection

$$\left(\pi_T, \pi_T^{\#}\right) : (T, \mathcal{O}_T) \to \left(\mathbb{CP}^1, \mathcal{O}_{\mathbb{CP}^1}\right),$$

which is given by $(\tilde{\zeta}, \tilde{\eta}) \mapsto \tilde{\zeta}$ and $(\zeta, \eta) \mapsto \zeta$ and with lifting operator

$$\begin{aligned} \pi_T^{\#}(W) : \mathcal{O}_{\mathbb{CP}^1}(W) &\longrightarrow \mathcal{O}_T(\pi_T^{-1}(W)) \\ s &\longmapsto \pi_T^* s := s \circ \pi_T \end{aligned}$$

for any open set $W \subset \mathbb{CP}^1$.

At this point we want to emphasise, that the pair $\left(|\mathcal{O}_{\mathbb{CP}^1}(2)|, \pi_{\mathcal{O}_{\mathbb{CP}^1}(2)}\right)$ defines a holomorphic line bundle on the complex, projective space. This holomorphic line bundle is the tangent bundle of $\mathbb{CP}^1$ and this is the reason, why we have choosen the letter T.

3.1.2 Nilpotent, Spectral Curve

A complex analytic space is locally the zero-set of a finite number of holomorphic functions. To describe the nilpotent, spectral curve as a complex analytic space we start with the two holomorphic functions $\eta^n \in \mathcal{O}_{V_0}(V_0)$ and $\tilde{\eta}^n \in \mathcal{O}_{V_1}(V_1)$. With these holomorphic functions we get two ideal sheaves

$$\mathcal{I}_{V_1}^n := \tilde{\eta}^n \mathcal{O}_{V_1} \subset \mathcal{O}_{V_1} \text{ and } \mathcal{I}_{V_0}^n := \eta^n \mathcal{O}_{V_0} \subset \mathcal{O}_{V_0},$$

which induce two zero-sets

$$\begin{aligned} U_1 &:= \left\{(\tilde{\zeta}, 0) \in V_1\right\} = \left\{(\tilde{\zeta}, \tilde{\eta}) \in V_1 : \tilde{\eta}^n = 0\right\} = zero(\mathcal{I}_{V_1}^n), \\ U_0 &:= \{(\zeta, 0) \in V_0\} = \{(\zeta, \eta) \in V_0 : \eta^n = 0\} = zero(\mathcal{I}_{V_0}^n). \end{aligned}$$

We want to define sheaves of local $\mathbb{C}$-algebras on the sets U_0 and U_1 to get two complex model spaces. The two ideal sheaves $\mathcal{I}_{V_1}^n$ and $\mathcal{I}_{V_0}^n$ produce two quotient presheaves on V_1 and V_0 defined by the assignment

$$V' \mapsto \mathcal{O}_{V_1}(V')/\mathcal{I}_{V_1}^n(V') \quad \text{and} \quad V \mapsto \mathcal{O}_{V_0}(V)/\mathcal{I}_{V_0}^n(V),$$

where V' is an open subset of V_1 and V an open subset of V_0. But before we continue to define our model spaces we will study the support of the two quotient presheaves, i.e. the set of those points, where the stalk is not trivial.

Claim 4. *Let $p_1 \in V_1 \setminus U_1$ and $p_0 \in V_0 \setminus U_0$ be two points. Then the stalks are trivial, i.e.*

$$\mathcal{O}_{V_1,p_1}/\mathcal{I}_{V_1,p_1}^n = 0 \quad \textit{and} \quad \mathcal{O}_{V_0,p_0}/\mathcal{I}_{V_0,p_0}^n = 0.$$

If $p_1 = (\tilde{\zeta}_1, 0) \in U_1$ and $p_0 = (\zeta_0, 0) \in U_0$, then we have

$$\begin{aligned} \mathcal{O}_{V_1,p_1}/\mathcal{I}_{V_1,p_1}^n &\cong \mathcal{O}_{W_1,\tilde{\zeta}_1}[\eta]/(\eta^n \mathcal{O}_{W_1,\tilde{\zeta}_1}), \\ \mathcal{O}_{V_0,p_0}/\mathcal{I}_{V_0,p_0}^n &\cong \mathcal{O}_{W_0,\zeta_0}[\eta]/(\eta^n \mathcal{O}_{W_0,\zeta_0}). \end{aligned}$$

In particular the quotient presheaves are supported on U_1 and U_0 respectively.

Proof. We will only prove the cases $p_0 \in V_0 \setminus U_0$ and $p_0 \in U_0$. The two other cases work similarly. So let $p_0 \in V_0 \setminus U_0$. Then there is an open, probably very small, neighborhood of p_0, say V', such that $V' \cap U_0 = \emptyset$. Thus on V' the holomorphic function $(\zeta, \eta) \mapsto \eta^n$ is a unit of the ring $\mathcal{O}_{V_0}(V')$. Since the stalk of the ideal sheaf is defined by $\mathcal{I}^n_{V_0,p_0} = \varinjlim_{V \ni p_0} (\eta^n \mathcal{O}_{V_0})(V)$ the pair (V', η^n) defines a representative of an element $[V', \eta^n]$ of $\mathcal{I}^n_{V_0,p_0}$. But the element $[V', \eta^n]$ is also a unit of the local ring $\mathcal{O}_{V_0,p_0}$, in other words we have $\mathcal{O}_{V_0,p_0} \cong \mathcal{I}^n_{V_0,p_0}$. This proves the first part. Now let $p_0 \in U_0$ and therefore it is of the form $p_0 = (\zeta_0, 0)$. By example 1 any element of $s_{p_0} \in \mathcal{O}_{V_0,p_0}$ is expandable into a convergent powerseries in $\mathbb{C}\{\zeta - \zeta_0, \eta\}$. If we only expand in the η-coordinate we can write $s_{p_0} = \sum_{l=0}^{\infty} s^l_{p_0} \eta^l$, where the $s^l_{p_0}$ are elements of $\mathcal{O}_{W_0,\zeta_0}$. An element of $\mathcal{I}^n_{V_0,p_0}$ expanded in the η-coordinate is of the form $\eta^n \sum_{l=0}^{\infty} k^l_{p_0} \eta^l$, where the $k^l_{p_0}$ are elements of $\mathcal{O}_{W_0,\zeta_0}$. Thus the quotient ring consists of elements of the form $s_{p_0} = \sum_{l=0}^{n-1} s^l_{p_0} \eta^l$ with $s^l_{p_0} \in \mathcal{O}_{W_0,\zeta_0}$, which proves the second part of the claim. □

The quotient sheaves with respect to the ideal sheaves $\mathcal{I}^n_{V_1}$ and $\mathcal{I}^n_{V_0}$ are given by the sheafification of the quotient presheaves above. If V is an open set of V_0, then the sheafification (in [Har77] it is called the sheaf associated to the presheaf) is defined as follows

$$\left(\mathcal{O}_{V_0}/\mathcal{I}^n_{V_0}\right)(V) := \Big\{ s = (s_p)_{p \in V} \in \prod_{p \in V} \left(\mathcal{O}_{V_0,p}/\mathcal{I}^n_{V_0,p}\right) : \forall p \in V \ \exists V' \subset V \text{ open} \\ \exists s' \in \mathcal{O}_{V_0}(V')/\mathcal{I}^n_{V_0}(V') \text{ s.t. } s_q = s'_q \quad \forall q \in V' \Big\}.$$

This definition delivers clearly a sheaf [Har77], where the stalks of the sheaf and the stalks of the presheaf coincide. We restrict these quotient sheaves to the sets U_1, U_0 to get sheaf of rings on U_1, U_0. So let $U \subseteq U_0$ and $U' \subseteq U_1$ be arbitrary open sets, then the restriction of the quotient sheaves is given by

$$\mathcal{O}_{U_1}(U') := \left(\mathcal{O}_{V_1}/\mathcal{I}^n_{V_1}\right)|_{U_1}(U') := \varinjlim_{U' \subset V'} \left(\mathcal{O}_{V_1}/\mathcal{I}^n_{V_1}\right)(V'),$$
$$\mathcal{O}_{U_0}(U) := \left(\mathcal{O}_{V_0}/\mathcal{I}^n_{V_0}\right)|_{U_0}(U) := \varinjlim_{U \subset V} \left(\mathcal{O}_{V_0}/\mathcal{I}^n_{V_0}\right)(V).$$

Since taking direct limits does not change the stalks by claim 4, these sheaves are not only sheaf of rings, but sheaves of local $\mathbb{C}$-algebras. We will describe the local sections in more details.

Claim 5. *Let $U \subseteq U_0$ and let $U' \subseteq U_1$ be open sets. Let $W \subset W_0$ and $W' \subset W_1$ be the open sets satisfying $U = W \times \{0\}$ and $U' = W' \times \{0\}$. Then a section $s_0 \in \mathcal{O}_{U_0}(U)$ and a section $s_1 \in \mathcal{O}_{U_1}(U')$ can be written as*

$$s_0(\zeta, \eta) = \sum_{l=0}^{n-1} s^l_0(\zeta)\,\eta^l, \qquad s_1(\tilde{\zeta}, \tilde{\eta}) = \sum_{l=0}^{n-1} s^l_1(\tilde{\zeta})\tilde{\eta}^l,$$

where $s^l_0 \in \mathcal{O}_{W_0}(W)$ and $s^l_1 \in \mathcal{O}_{W_1}(W')$.

Proof. We will prove only the case for U_0 and the case for the set U_1 works similarly. Let us consider an element $[V, s] \in \mathcal{O}_{U_0}(U)$, where V is an open set of V_0 containing the set U and let $s = (s_p)_{p \in V} \in \left(\mathcal{O}_{V_0}/\mathcal{I}^n_{V_0}\right)(V)$ be an element of the sheafification of the quotient presheaf. If $p \in V \setminus U$, then we have already seen in claim 4, that $s_p = 0$. Let us suppose $p \in U \cap V$. Then by definition of the sheafification there is an open neighborhood $V_p \subset V$ of p such that we can glue the s_q together to a section of the quotient presheaf for all $q \in V_p$. In other words, there is a section $s^p \in \mathcal{O}_{V_0}(V_p)/\mathcal{I}^n_{V_0}(V_p)$ with $s^p_q = s_q$ for all $q \in V_p$. We have a presheaf and so we have restriction morphisms too. Therefore we can shrink the open set $V_p \subset \mathbb{C}^2$ to an open set of the form $W_p \times K_p$, where $\zeta \in W_p, \eta \in K_p$ are open subsets of $\mathbb{C}$ and $p \in W_p \times K_p$. Because s^p is a holomorphic function on $W_p \times K_p$ we Taylor expand it in the η-coordinate around $\eta_0 = 0$ and we get

$$s^p(\zeta, \eta) = \sum_{l=0}^{n-1} s^{p,l}(\zeta)\eta^l,$$

with $s^{p,l} \in \mathcal{O}_{W_0}(W_p)$ and $(\zeta, \eta) \in W_p \times K_p$. But since s^p is a polynomial in the η-coordinate we can extend this holomorphic function to a holomorphic function on $W_p \times \mathbb{C}$. So for every $p \in U \cap V$ we get an open set $W_p \times \mathbb{C}$ and a holomorphic function s^p on $W^p \times \mathbb{C}$. The holomorphic functions s^{p_0} and s^{p_1} coincide on $(W_{p_0} \cap W_{p_1}) \times \mathbb{C}$ for two points $p_0, p_1 \in U \cap V$, because by the definition of the sheafification we have $s^{p_0}_q = s^{p_1}_q$ for all $q \in V_p$. Since $\mathcal{O}_{W_0}$ is a sheaf and $\cup_{p \in U \cap V} W_p = W$, we can glue the family of holomorphic functions $(s^{p,l})_{p \in U \cap V}$ together to a holomorphic function $s^l_0 \in \mathcal{O}_{W_0}(W)$. In other words we get a holomorphic function on $W \times \mathbb{C}$ defined by

$$s_0\left(\zeta, \eta\right) = \sum_{l=0}^{n-1} s^l_0\left(\zeta\right) \eta^l.$$

On the other hand such a holomorphic function defines clearly an element of

$$(\mathcal{O}_{V_0}/\mathcal{I}^n_{V_0})(W \times \mathbb{C})$$

and since $U \subset W \times \mathbb{C}$ we get an element $[W \times \mathbb{C}, s_0] \in \mathcal{O}_{U_0}(U)$. □

At this point we see immediatly, that the sheaves $\mathcal{O}_{U_1}$ and $\mathcal{O}_{U_0}$ are sheaves of local $\mathbb{C}$-algebras, where the multiplication of two holomorphic functions is given by the multiplication of the polynomials in claim 5 truncated by η^n respectively $\tilde{\eta}^n$. And finally we get two complex model spaces by the pairs $(U_1, \mathcal{O}_{U_1})$ and $(U_0, \mathcal{O}_{U_0})$. Let us define the sets

$$U_{10} := \left\{(\zeta, 0) \in U_0 : \zeta \neq 0\right\}, \quad U_{01} := \left\{(\tilde{\zeta}, 0) \in U_1 : \tilde{\zeta} \neq 0\right\}$$

and the biholomorphic map, again by abusing the notation,

$$\left(\varphi_{10}, \varphi^{\#}_{10}\right) : (U_{10}, \mathcal{O}_{U_0}|_{U_{10}}) \longrightarrow (U_{01}, \mathcal{O}_{U_1}|_{U_{01}})$$

given by

$$\varphi_{10} : U_{10} \longrightarrow U_{01}. \qquad (\zeta, \eta) \longmapsto \left(\frac{1}{\zeta}, \frac{\eta}{\zeta^2}\right)$$

with lifting operator

$$\begin{aligned}\varphi_{10}^{\#}(U): \mathcal{O}_{U_1}|_{U_{01}}(U) &\longrightarrow \mathcal{O}_{U_0|_{U_{10}}}(\varphi_{10}^{-1}(U))\\ s_1\left(\tilde{\zeta},\tilde{\eta}\right) &\longmapsto (\varphi_{10}^{*}s_1)(\zeta,\eta) := (s_1\circ\varphi_{10})(\zeta,\eta) = s_1\left(\frac{1}{\zeta},\frac{\eta}{\zeta^2}\right)\end{aligned}$$

for any open subset $U \subset U_{01}$. By the gluing property, proposition 1, we can glue them together to a complex analytic space, called the nilpotent, spectral curve, which we denote by

$$(C_n, \mathcal{O}_{C_n}),$$

where $C_n := U_1 \cup U_0/\sim$ with $(\tilde{\zeta},0) \sim (\zeta,0)$ if and only if $(\tilde{\zeta},0) \in U_{01}$, $(\zeta,0) \in U_{10}$ and $(\tilde{\zeta},0) = (\frac{1}{\zeta},0) = \varphi_{10}(\zeta,0)$. For any open set $U \subseteq C_n$ the local sections are given by pairs $(f_1,f_0) \in \mathcal{O}_{U_1}(U\cap U_1)\times\mathcal{O}_{U_0}(U\cap U_0)$ such that $(\varphi_{10}^{*}res_{U\cap U_{01}}^{U\cap U_1}(f_1))(\zeta,\eta) = res_{U\cap U_{01}}^{U\cap U_1}(f_1)(\frac{1}{\zeta},\frac{\eta}{\zeta^2}) = res_{U\cap U_{10}}^{U\cap U_0}(f_0)(\zeta,\eta)$. In the rest of the thesis we often do not write the restrictions of the local functions or sections, because it makes the most formulas unnecessarily confusing. Instead we will often write the pair (f_1,f_0) satisfies $f_1(\frac{1}{\zeta},\frac{\eta}{\zeta^2}) = f_0(\zeta,\eta)$ on U_{10}. Because $\{U_1,U_0\}$ is an open cover of the topological space C_n with $U_i = W_i\times\{0\}$, $i\in\{0,1\}$, the topological space C_n is homeomorphic to $\mathbb{CP}^1$ and is therefore compact and one dimensional. Moreover we have a holomorphic projection

$$\left(\pi_{C_n},\pi_{C_n}^{\#}\right): (C_n,\mathcal{O}_{C_n}) \longrightarrow \left(\mathbb{CP}^1,\mathcal{O}_{\mathbb{CP}^1}\right)$$

given by $(\tilde{\zeta},0)\mapsto\tilde{\zeta}$, $(\zeta,0)\mapsto\zeta$ and with lifting operator

$$\begin{aligned}\pi_{C_n}^{\#}(W): \mathcal{O}_{\mathbb{CP}^1}(W) &\longrightarrow \mathcal{O}_{C_n}(\pi_{C_n}^{-1}(W))\\ (s_1,s_0) &\longmapsto (s_1,s_0)\end{aligned}$$

for an open set $W\subseteq\mathbb{CP}^1$. Because of claim 5 the stalk of $\mathcal{O}_{C_n}$ at $p\in C_n$ is

$$\mathcal{O}_{C_n,p} \cong \mathcal{O}_{\mathbb{CP}^1,p}[\eta]/\langle\eta^n\mathcal{O}_{\mathbb{CP}^1,p}\rangle. \tag{3.1}$$

In other words the local rings $\mathcal{O}_{C_n,p}$ admit some nilpotent elements and hence the curve is non-reduced. Furthermore we see, that $\mathcal{O}_{C_n,p}$ is a finitely generated $\mathcal{O}_{\mathbb{CP}^1,p}$- module and hence the projection $\left(\pi_{C_n},\pi_{C_n}^{\#}\right)$ is a finite morphism of complex analytic spaces.

With the holomorphic projections $\left(\pi_T,\pi_T^{\#}\right)$ and $\left(\pi_{C_n},\pi_{C_n}^{\#}\right)$ we can consider the inverse image sheaf of an invertible sheaf $\mathcal{O}_{\mathbb{CP}^1}(k)$ on $\left(\mathbb{CP}^1,\mathcal{O}_{\mathbb{CP}^1}\right)$, $k\in\mathbb{Z}$. Let us denote the inverse image sheaves $\mathcal{O}_T(k) := \pi_T^{*}\mathcal{O}_{\mathbb{CP}^1}(k)$ and $\mathcal{O}_{C_n}(k) := \pi_{C_n}^{*}\mathcal{O}_{\mathbb{CP}^1}(k)$. These sheaves are also invertible sheaves and they have transition functions of the form $(\zeta,\eta)\mapsto\frac{1}{\zeta^k}$, seen as an element of $\mathcal{O}_T(V_0\cap V_1)$ or $\mathcal{O}_{C_n}(U_0\cap U_1)$ respectively. See for definitions and properties [GPR94].

At this point we want to investigate very roughly in an algebraic point of view. Because η^n and $\tilde{\eta}^n$ are polynomials, we can do all the above constructions in the algebraic category too to get a proper, algebraic curve $\left(C_n^{alg},\mathcal{O}_{C_n}^{alg}\right)$, which is non-reduced. It comes with a regular projection morphism to the complex projective space $\left(\mathbb{CP}^1,\mathcal{O}_{\mathbb{CP}^1}\right)$, such that the topological map is a homemorphism in the Zariski topology and hence this projection morphism makes the curve to a projective variety, see [Dre06]. We will use this observations later to carry over some algebraic properties to the analytic case.

3.1.3 Spectral Curve

A way more compact description of the nilpotent spectral curve is as follows, where all local considerations above play its role. Because of the gluing map $(\varphi_{10}, \varphi_{10}^{\#})$ we see $\varphi_{10}^{\#}(V_{01})(\tilde{\eta}) = \frac{1}{\zeta^2}\eta$. Hence the pair $(\tilde{\eta}, \eta)$ defines a global section of the invertible sheaf $\mathcal{O}_T(2) = \pi_T^*\mathcal{O}_{\mathbb{CP}^1}(2)$ on T. This section is called the tautological section [Hit83]. Furthermore the pair $(\tilde{\eta}^n, \eta^n)$ defines a global section of the invertible sheaf $\mathcal{O}_T(2n)$ and hence it induces a sheaf homomorphism

$$\begin{aligned}(\tilde{\eta}^n, \eta^n) \cdot (V) : \mathcal{O}_T(-2n)(V) &\longrightarrow \mathcal{O}_T(V)\\ (s_1, s_0) &\longmapsto (\tilde{\eta}^n s_1, \eta^n s_0)\end{aligned}$$

for every open set $V \subseteq T$. We get a sheaf of ideals defined by

$$\mathcal{I}^n := (\tilde{\eta}^n, \eta^n) \cdot \mathcal{O}_T(-2n) \subseteq \mathcal{O}_T.$$

The zero set of this sheaf of ideals gives us a topological space $C_n = zero(\mathcal{I}^n)$ and a sheaf of local $\mathbb{C}$-algebras on C_n defined by $\mathcal{O}_{C_n} := \mathcal{O}_T/\mathcal{I}^n\big|_{C_n}$. The pair $(C_n, \mathcal{O}_{C_n})$ is just the nilpotent, spectral curve.

There is a more general construction of such spectral curves, which explains the name. Let us consider a global section of

$$(\tilde{A}(\tilde{\zeta}), A(\zeta)) \in \check{\mathrm{H}}^0\left(\mathbb{CP}^1, \mathcal{O}_{\mathbb{CP}^1}(2) \otimes \mathfrak{gl}_n(\mathbb{C})\right).$$

Such a pair $(\tilde{A}(\tilde{\zeta}), A(\zeta))$ is given by polynomials of degree 2 with matrix coefficients

$$\tilde{A}(\tilde{\zeta}) = \tilde{A}_0 + \tilde{A}_1\tilde{\zeta} + \tilde{A}_2\tilde{\zeta}^2, \qquad A(\zeta) = A_0 + A_1\zeta + A_2\zeta^2,$$

where $A_i, \tilde{A}_i \in \mathfrak{gl}_n(\mathbb{C})$ and they satisfy $\tilde{A}\left(\frac{1}{\zeta}\right) = A(\zeta)$ for all $\zeta \in W_{10}$. This implies immediatly

$$\tilde{A}_0 = A_2, \qquad \tilde{A}_1 = A_1, \qquad \tilde{A}_2 = A_0$$

and we see, that a matricial polynomial $A(\zeta) = A_0 + A_1\zeta + A_2\zeta^2$ uniquely defines such a global section. In the rest of this thesis we will often consider only matricial polynomials $A(\zeta)$ on W_0, but we always have the global section $\left(\tilde{A}(\tilde{\zeta}), A(\zeta)\right)$ in mind.

The characteristic polynomials define two polynomial functions and hence holomorphic functions on V_0 and V_1 given by

$$P_1 := \det\left(\tilde{\eta} Id_n - \tilde{A}(\tilde{\zeta})\right) \in \mathcal{O}_{V_1}(V_1), \qquad P_0 := \det\left(\eta Id_n - A(\zeta)\right) \in \mathcal{O}_{V_0}(V_0).$$

They satisfy on V_{10}

$$\begin{aligned}(\varphi_{10}^* P_1)(\zeta, \eta) &= \det\left(\frac{\eta}{\zeta^2} Id_n - \left(A_2 + A_1\frac{1}{\zeta} + A_0\frac{1}{\zeta^2}\right)\right) = \frac{1}{\zeta^{2n}}\det\left(\eta Id_n - A(\zeta)\right)\\ &= \frac{1}{\zeta^{2n}} P_0(\zeta, \eta).\end{aligned}$$

In other words the pair (P_1, P_0) defines a global section of $\mathcal{O}_T(2n)$. We get again a sheaf of ideals by

$$\mathcal{I}^{(P_1,P_0)} := (P_1, P_0) \cdot \mathcal{O}_T(-2n),$$

which gives us a complex analytic space $\left(C_{(P_1,P_0)}, \mathcal{O}_{C_{(P_1,P_0)}}\right)$ called the spectral curve of the matricial polynomial $A(\zeta)$, where $\mathcal{O}_{C_{(P_1,P_0)}} := \mathcal{O}_T/\mathcal{I}^{(P_1,P_0)}|_{C_{(P_1,P_0)}}$. If we assume now, that the global section $(\tilde{A}\left(\tilde{\zeta}\right), A\left(\zeta\right))$ is nilpotent at all points $p \in \mathbb{CP}^1$, then the characteristic polynomials are just $P_1(\tilde{\zeta}, \tilde{\eta}) := \det\left(\tilde{\eta} Id_n - \tilde{A}\left(\tilde{\zeta}\right)\right) = \tilde{\eta}^n$ and $P_0(\zeta, \eta) := \det\left(\eta Id_n - A\left(\zeta\right)\right) = \eta^n$. In other words the corresponding spectral curve of a nilpotent, matricial polynomial is just the nilpotent, spectral curve we started with. We see immediatly, that any such nilpotent, global section induces the same nilpotent, spectral curve.

3.2 Jacobian Variety of Nilpotent, Spectral Curve

We want to describe the set of isomorphism classes of invertible sheaves of degree $g-1$ on $(C_n, \mathcal{O}_{C_n})$, which we call the Jacobian of degree $g-1$ of C_n and denoting it by $Jac^{g-1}(C_n)$. The set of isomorphism classes of degree k will be denoted by $Jac^k(C_n)$. The Picard group $Pic(C_n)$ consists of all isomorphism classes of invertible sheaves with the group structure defined by the tensor product of invertible sheaves, see [Har77]. The unit element is just the structure sheaf $\mathcal{O}_{C_n}$. We have already seen in proposition 5, that the isomorphism classes of invertible sheaves are characterized by $\check{\mathrm{H}}^1\left(C_n, \mathcal{O}^*_{C_n}\right)$. To study and computing a cohomology group it is often usefull to find a short exact sequence of sheaves of abelian groups, such that the corresponding long exact sequence, coming from the cohomology theory, inherits the cohomology group we want to compute. In the case of computing $\check{\mathrm{H}}^1(C_n, \mathcal{O}^*_{C_n})$ the helpful short exact sequence is the exponential sequence

$$0 \to \mathbb{Z} \xrightarrow{2\pi i \cdot} \mathcal{O}_{C_n} \xrightarrow{\exp} \mathcal{O}^*_{C_n} \to 1.$$

Note that the group structure of the sheaf of abelian groups $\mathcal{O}^*_{C_n}$ is given by multiplication and this explains the 1 at the far right. The sheaf of locally constant functions with values in the integers on C_n is denoted by $\mathbb{Z}$. The exponential map is defined by the standard exponential map truncated by η^n respectively $\tilde{\eta}^n$ in the sense of claim 5. Since the exponential map is surjective on the stalks, the exponential sequence is exact, see for more details [GPR94].

This short exact sequence leads to the long exact sequence of cohomology groups

$$\begin{aligned} 0 &\longrightarrow \check{\mathrm{H}}^0(C_n, \mathbb{Z}) \longrightarrow \check{\mathrm{H}}^0(C_n, \mathcal{O}_{C_n}) \longrightarrow \check{\mathrm{H}}^0(C_n, \mathcal{O}^*_{C_n}) \longrightarrow \check{\mathrm{H}}^1(C_n, \mathbb{Z}) \\ &\longrightarrow \check{\mathrm{H}}^1(C_n, \mathcal{O}_{C_n}) \longrightarrow \overbrace{\check{\mathrm{H}}^1(C_n, \mathcal{O}^*_{C_n})}^{\cong Pic(C_n)} \longrightarrow \check{\mathrm{H}}^2(C_n, \mathbb{Z}) \longrightarrow \check{\mathrm{H}}^2(C_n, \mathcal{O}_{C_n}) \longrightarrow \dots \end{aligned}$$

But because $\mathbb{Z}$ is a sheaf of abelian groups on the topological space C_n, which is homeomorphic to the simply-connected, topological space $\mathbb{CP}^1$, we can use Poincaré duality to see $\mathbb{Z} \cong H_0(\mathbb{CP}^1, \mathbb{Z}) \cong \check{\mathrm{H}}^2(C_n, \mathbb{Z})$. Moreover the sets U_0, U_1 are both homeomorphic to $\mathbb{C}$ and U_{10} is homeomorphic to $\mathbb{C}^*$ and hence they are all Stein spaces [GPR94]. For Stein manifolds X Cartan's theorem B holds, which says $\check{\mathrm{H}}^p(X, \mathcal{F}) = 0$ for all $p > 0$ and coherent $\mathcal{O}_X$-modules $\mathcal{F}$. Thus $\mathcal{U} := \{U_0, U_1\}$ forms a Leray cover with two open sets. This implies immediatly $\check{\mathrm{H}}^2(C_n, \mathcal{O}_{C_n}) = 0$, which is also clear because the topological dimension of C_n is 1. The curve C_n is simply connected and hence it has trivial first fundamental group. By the Hurewicz theorem $H_1(C_n, \mathbb{Z})$ vanishes and by the Poincaré duality again we have $\check{\mathrm{H}}^1(C_n, \mathbb{Z}) = 0$. Thus the long exact sequence of cohomology groups reduces to the short exact sequence

$$0 \to \check{\mathrm{H}}^1(C_n, \mathcal{O}_{C_n}) \xrightarrow{\exp} \overbrace{\check{\mathrm{H}}^1(C_n, \mathcal{O}^*_{C_n})}^{\cong Pic(C_n)} \xrightarrow{deg} \overbrace{\check{\mathrm{H}}^2(C_n, \mathbb{Z})}^{\cong \mathbb{Z}} \to 0. \tag{3.2}$$

We define the degree of an invertible sheaf on the nilpotent, spectral curve $(C_n, \mathcal{O}_{C_n})$ as the integer given by the image of the map *deg* in the long exact sequence above. The degree is additive with respect to the tensor product on the Picard group and therefore tensoring by an invertible sheaf of degree k gives an isomorphism between $Jac^0(C_n)$ and $Jac^k(C_n)$, [Har77]. We need to compute $Jac^0(C_n)$, which is given by the kernel of the degree map in the short exact sequence above. But the kernel is by exactness the image of the exponential map and hence we want to compute $\check{\mathrm{H}}^1(C_n, \mathcal{O}_{C_n})$ to describe $Im(\exp)$.

After the introduction of the degree of an invertible sheaf we can state some properties coming from the algebraic behavior of the nilpotent, spectral curve. For a proper scheme over $\mathbb{C}$ of dimension 1 there is a Riemann-Roch theorem for locally free sheaves, see for example [Sta19] or [BFM75]. But Serre made in [Ser56] a connection between the algebraic point of view and the analytic point of view. He shows, that there is a bijection between the set of algebraic, coherent sheaves and analytic, coherent sheaves. By a theorem of Oka [Oka50], we know that the structure sheaf of a complex analytic space is coherent and hence its invertible sheaves are coherent too. Moreover Serre showed, that the cohomology groups of both point of views are isomorphic as $\mathbb{C}$-vector spaces. In other words the Riemann-Roch theorem carries over to the analytic case of the nilpotent, spectral curve.

Proposition 9. *(Riemann-Roch) Let $\mathcal{F}$ be an invertible sheaf on the nilpotent, spectral curve $(C_n, \mathcal{O}_{C_n})$. Let $\chi(C_n, \mathcal{F}) = \dim_{\mathbb{C}}\left(\check{\mathrm{H}}^0(C_n, \mathcal{F})\right) - \dim_{\mathbb{C}}\left(\check{\mathrm{H}}^1(C_n, \mathcal{F})\right)$ be the complex Euler characteristic and let $g := \dim_{\mathbb{C}}\left(\check{\mathrm{H}}^1(C_n, \mathcal{O}_{C_n})\right)$ be the arithmetic genus of the nilpotent, spectral curve. Then we have*

$$\chi(C_n, \mathcal{F}) = \deg(\mathcal{F}) + \chi(\mathcal{O}_{C_n}) = \deg(\mathcal{F}) + 1 - g.$$

Another property of algebraic geometry uses the following. Via the holomorphic projection map $\left(\pi_{C_n}, \pi^{\#}_{C_n}\right)$ one can see the structure sheaf $\mathcal{O}_{C_n}$ as a finitely generated $\mathcal{O}_{\mathbb{CP}^1}$-module. In other words the projection $\left(\pi_{C_n}, \pi^{\#}_{C_n}\right)$ is a finite morphism. Since the projective space $\mathbb{CP}^1$ is a complex manifold we have the next proposition, see [Har77] and [GPR94]. Coherence of the direct image sheaf is Grauert's direct image theorem, [GPR94].

Proposition 10. *Let $\mathcal{F}$ be an invertible sheaf on the nilpotent, spectral curve $(C_n, \mathcal{O}_{C_n})$ and let $\pi_{C_n,*}\mathcal{F}$ be the direct image sheaf seen as a coherent $\mathcal{O}_{\mathbb{CP}^1}$- module. Then the sheaf $\pi_{C_n,*}\mathcal{F}$ is locally free of rank n and for all $q \geq 0$ the cohomology groups are isomorphic as $\mathbb{C}$-vector spaces, i.e.*

$$\check{\mathrm{H}}^q(C_n, \mathcal{F}) \cong \check{\mathrm{H}}^q(\mathbb{CP}^1, \pi_{C_n,*}\mathcal{F}).$$

An immediate consequence of proposition 10, proposition 9 and the Riemann-Roch theorem on $\mathbb{CP}^1$ is the following corollary.

Corollary 1. *We have the equation*

$$\deg(\mathcal{F}) + (1-g) = \deg(\pi_{C_n,*}\mathcal{F}) + n.$$

Proof. We compute with proposition 10 and proposition 9 and the Riemann-Roch theorem on $\mathbb{CP}^1$

$$\begin{aligned}\deg(\mathcal{F}) + 1 - g = \chi(C_n, \mathcal{F}) &= \dim_{\mathbb{C}}\left(\check{\mathrm{H}}^0(C_n, \mathcal{F})\right) - \dim_{\mathbb{C}}\left(\check{\mathrm{H}}^1(C_n, \mathcal{F})\right) \\ &= \dim_{\mathbb{C}}\left(\check{\mathrm{H}}^0(\mathbb{CP}^1, \pi_{C_n,*}\mathcal{F})\right) - \dim_{\mathbb{C}}\left(\check{\mathrm{H}}^1(\mathbb{CP}^1, \pi_{C_n,*}\mathcal{F})\right) \\ &= \chi(\mathbb{CP}^1, (\pi_{C_n,*}\mathcal{F})) = \deg(\pi_{C_n,*}\mathcal{F}) + rk(\pi_{C_n,*}\mathcal{F})\left(1 - g(\mathbb{CP}^1)\right) \\ &= \deg(\pi_{C_n,*}\mathcal{F}) + n.\end{aligned}$$

□

3.2.1 Generators of $\check{\mathrm{H}}^1(C_n, \mathcal{O}_{C_n})$ and the Arithmetic Genus

Next we want to prove the following proposition, which can be found in [AHH90] or [Hit98].

Proposition 11. *The cohomology group $\check{\mathrm{H}}^1(C_n, \mathcal{O}_{C_n})$ has a basis as a complex vector space by monomials of the form*

$$\zeta^{-k}\eta^l \in \mathcal{O}_{C_n}(U_{10}),$$

where $0 \leq l \leq n-1$ and $1 \leq k \leq 2l-1$. In particular we have

$$g := \dim_{\mathbb{C}}(\check{\mathrm{H}}^1(C_n, \mathcal{O}_{C_n})) = (n-1)^2.$$

Proof. Note that U_0, U_1 form a Leray cover of C_n and hence we only have to compute the Čech-cohomology group with respect to this open cover instead of computing the direct limit over refinements of open covers. In the definition of Čech-cohomology we have the maps $\delta_k : C_k(\mathcal{O}_{C_n}) \to C_{k+1}(\mathcal{O}_{C_n})$. Since $Ker(\delta_1) = \mathcal{O}_{C_n}(U_0 \cap U_1)$ and $Im(\delta_0) = \mathcal{O}_{C_n}(U_0) \oplus \mathcal{O}_{C_n}(U_1)$ the first Čech-cohomology group $\check{\mathrm{H}}^1(C_n, \mathcal{O}_{C_n})$ is given by holomorphic functions on $U_0 \cap U_1$ modulo holomorphic functions on U_0 and holomorphic functions on U_1, in symbols

$$\check{\mathrm{H}}^1(C_n, \mathcal{O}_{C_n}) = \mathcal{O}_{C_n}(U_0 \cap U_1)/\mathcal{O}_{C_n}(U_0) \oplus \mathcal{O}_{C_n}(U_1).$$

By claim 5 we can write a section on U_{10} in the form $s_0 = \sum_{l=0}^{n-1} s_0^l(\zeta)\eta^l$, where $s_0^l \in \mathcal{O}_{W_0}(W_{10})$. But we have $U_{10} \cong \mathbb{C}^*$ and so we can expand the s_0^l into Laurent series, hence we have

$$s_0(\zeta, \eta) = \sum_{l=0}^{n-1} \sum_{k=-\infty}^{\infty} a_{kl}\zeta^k\eta^l$$

with some complex coefficients $a_{kl} \in \mathbb{C}$. On U_0 and on U_1 the holomorphic functions expand into the form

$$\sum_{l=0}^{n-1} \sum_{k=-\infty}^{\infty} b_{kl}\zeta^k\eta^l \text{ on } U_0, \qquad \sum_{l=0}^{n-1} \sum_{k=0}^{\infty} c_{kl}\tilde{\zeta}^k\tilde{\eta}^l \text{ on } U_1.$$

We restrict the first function just to U_{10}. For the second function we have to restrict it to U_{01} first, i.e. to an element of $\mathcal{O}_{C_n}(U_{01})$ and then sending via $\varphi_{10}^{\#}(U_{01})$ to

$\mathcal{O}_{C_n}(U_{10}) = \mathcal{O}_{C_n}(\varphi_{10}^{-1}(U_{01}))$. Hence the second function, written as a holomorphic function on U_{10}, is given by

$$\sum_{l=0}^{n-1}\sum_{k=0}^{\infty} c_{kl}\tilde{\zeta}^k\tilde{\eta}^l = \sum_{l=0}^{n-1}\sum_{k=0}^{\infty} c_{kl}\frac{1}{\zeta^k}\left(\frac{\eta}{\zeta^2}\right)^l = \sum_{l=0}^{n-1}\sum_{k=-\infty}^{0} c_{kl}\zeta^{k-2l}\eta^l.$$

So an element of the quotient space $\mathcal{O}_{C_n}(U_0 \cap U_1)/\mathcal{O}_{C_n}(U_0) \oplus \mathcal{O}_{C_n}(U_1)$ has a representative of the form $\sum_{l=1}^{n-1}\sum_{k=-2l+1}^{-1} b_{kl}\zeta^k\eta^l$. We conclude that every element of $\check{\mathrm{H}}^1(C_n, \mathcal{O}_{C_n})$ is given by a representative of the form

$$b_{C_n}(\zeta,\eta) := \sum_{l=1}^{n-1}\sum_{k=1}^{2l-1} b_{kl}\zeta^{-k}\eta^l = \sum_{l=1}^{n-1}\sum_{k=1}^{2l-1} B_l\left(\frac{1}{\zeta}\right)\eta^l,$$

where $B_l(\frac{1}{\zeta}) := \sum_{k=1}^{2l-1} b_{kl}\zeta^{-k}$. The coefficients b_{kl} are complex numbers and every $B_l(\frac{1}{\zeta})$ is dependent of $2l-1$ of them. Therefore b_{C_n} is dependent of

$$\sum_{l=1}^{n-1}(2l-1) = 2\sum_{l=1}^{n-1} l - \sum_{l=1}^{n-1} 1 = 2\frac{n(n-1)}{2} - (n-1) = (n-1)^2$$

complex numbers. By the definition of the arithmetic genus we conclude

$$g := \dim_{\mathbb{C}} \check{\mathrm{H}}^1(C_n, \mathcal{O}_{C_n}) = (n-1)^2.$$

□

By using the projection formula, see [Har77], we have

$$\pi_{C_n,*}\pi_{C_n}^*\mathcal{O}_{\mathbb{CP}^1}(n-2) \cong (\pi_{C_n,*}\mathcal{O}_{C_n}) \otimes \mathcal{O}_{\mathbb{CP}^1}(n-2).$$

Moreover by using the isomorphism of claim 7 in section 4.1.4 we see immediatly $\deg(\pi_{C_n,*}\pi_{C_n}^*\mathcal{O}_{\mathbb{CP}^1}(n-2)) = -n$. By inserting the invertible sheaf $\pi_{C_n}^*\mathcal{O}_{\mathbb{CP}^1}(n-2)$ into corollary 1 we get

$$\deg(\pi_{C_n}^*\mathcal{O}_{\mathbb{CP}^1}(n-2)) + 1 - g = \deg(\pi_{C_n,*}\mathcal{F}) + n = 0$$

and thus we have

$$\deg(\mathcal{O}_{C_n}(n-2)) = \deg(\pi_{C_n}^*\mathcal{O}_{\mathbb{CP}^1}(n-2)) = n(n-2) = g-1. \tag{3.3}$$

3.2.2 Jacobian Variety

In this subsection we want to compute the Jacobian variety $Jac^{g-1}(C_n)$ and describe the transition functions of invertible sheaves of degree $g-1$ on the nilpotent, spectral curve.

Theorem 2. *Let $(C_n, \mathcal{O}_{C_n})$ be the nilpotent, spectral curve. Then the exponential map* exp *is an isomorphism*

$$\exp : \check{\mathrm{H}}^1(C_n, \mathcal{O}_{C_n}) \cong Jac^0(C_n).$$

In particular we have

$$Jac^{g-1}(C_n) \cong Jac^0(C_n) \cong \check{\mathrm{H}}^1(C_n, \mathcal{O}_{C_n}) \cong \mathbb{C}^g.$$

Furthermore the isomorphism classes of invertible sheaves of degree 0 *have transition functions of the form*

$$h_{10}(\zeta,\eta) = \left(d_{00} + \sum_{l=1}^{n-1}\sum_{k=1}^{2l-1} d_{kl}\frac{1}{\zeta^k}\eta^l \right),$$

where the coefficients $d_{kl} \in \mathbb{C}$ *and* d_{00} *is a non-zero complex number. Furthermore invertible sheaves of degree* $g-1$ *have transition functions of the form*

$$g_{10}(\zeta,\eta) = \frac{1}{\zeta^{n-2}}h_{10}(\zeta,\eta).$$

Proof. We have already seen in equation (3.2), that the long exact sequence reduces to

$$0 \to \check{\mathrm{H}}^1(C_n, \mathcal{O}_{C_n}) \xrightarrow{exp} \overbrace{\check{\mathrm{H}}^1(C_n, \mathcal{O}^*_{C_n})}^{\cong Pic(C_n)} \xrightarrow{deg} \overbrace{\check{\mathrm{H}}^2(C_n, \mathbb{Z})}^{\cong \mathbb{Z}} \to 0.$$

Hence by exactness we have injectivity of the exponential map and hence it is an isomorphism on its image, which is the group $Jac^0(C_n)$ of invertible sheaves of degree 0. To compute the transition functions we take a representative of a cohomology class of $\check{\mathrm{H}}^1(C_n, \mathcal{O}_{C_n})$

$$b_{C_n}(\zeta,\eta) := \sum_{l=1}^{n-1}\sum_{k=1}^{2l-1} B_l\left(\frac{1}{\zeta}\right)\eta^l$$

and compute its image under the exponential map truncated by η^n. Thus we get

$$\exp\left(b_{C_n}(\zeta,\eta)\right) = \prod_{k=1}^{n-1}\sum_{r=0}^{n-1}\frac{1}{r!}B_k\left(\frac{1}{\zeta}\right)^r \eta^{rk}.$$

Since we are on the nilpotent, spectral curve the term η^{rl} is at most η^{n-1}. If we fix a number q and if we consider two numbers r and l, such that $rl = q$, then we have the term $B_l(\frac{1}{\zeta})^r$, which has ζ-terms from $\left(\frac{1}{\zeta}\right)^r$ to $\left(\frac{1}{\zeta^{2l-1}}\right)^r$. The smallest exponentiation is $\frac{1}{\zeta^r}$ and the biggest possible exponentiation is $\left(\frac{1}{\zeta^{2l-1}}\right)^r = \frac{1}{\zeta^{2q-r}}$. Therefore we have the highest range of exponentiations if we set $r = 1$ and $q = l$. We can follow, that the transition functions of invertible sheaves of degree 0 on the nilpotent, spectral curve are of the form

$$1 + \sum_{l=1}^{n-1}\sum_{k=1}^{2l-1} c_{kl}\zeta^{-k}\eta^l.$$

If we multiply a transition function of an invertible sheaf $\mathcal{F}$ by a non-zero constant d_{00}, then the new invertible sheaf is isomorphic to $\mathcal{F}$. This follows basically by definition of $\check{\mathrm{H}}^1(C_n, \mathcal{O}^*_{C_n})$. We define $d_{kl} = c_{kl}d_{00}$ and the transition function gets

$$h_{10}(\zeta,\eta) = \left(d_{00} + \sum_{l=1}^{n-1}\sum_{k=1}^{2l-1} d_{kl}\zeta^{-k}\eta^l \right).$$

The last part of the proof follows from the fact, that the transition function of a tensor product of two invertible sheaves is just the multiplication of the two transition functions truncted by η^n. Because of equation (3.3) we know, that the degree of $\mathcal{O}_{C_n}(n-2) := \pi^*_{C_n}\mathcal{O}_{\mathbb{CP}^1}(n-2)$ is $n(n-2) = g-1$. The transition function of $\mathcal{O}_{C_n}(n-2)$

is just $\frac{1}{\zeta^{n-2}}$, which does not depend on η and so we do not have to truncate. In other words if we take an invertible sheaf $\mathcal{L}$ of degree 0 with transition function h_{10} and tensoring it with $\mathcal{O}_{C_n}(n-2)$ we get an invertible sheaf $\mathcal{F} = \mathcal{L} \otimes \mathcal{O}_{C_n}(n-2)$ of degree $g-1$ with transition function

$$g_{10}(\zeta, \eta) = \frac{1}{\zeta^{n-2}} h_{10}(\zeta, \eta).$$

□

3.3 Theta Divisor and Theta Function

In this section we want to describe the theta divisor, the set of invertible sheaves of degree $g-1$ on the nilpotent, spectral curve with a non-trivial, global section. It turns out, that it is an affine variety in $Jac^{g-1}(C_n)$, a zero locus of a polynomial function. This polynomial is given by a determinant of a certain matrix M and it is called the (generalized) theta function. The first step is to describe this matrix M and the second step is to compute its determinant.

3.3.1 Theta Divisor and Condition Equations

The theta divisor is defined by the set of invertible sheaves of degree $g-1$, which have a non-trivial, global section. As a formula this means

$$\Theta := \left\{ \mathcal{F} \in Jac^{g-1}(C_n) : \dim_{\mathbb{C}} \check{\mathrm{H}}^0(C_n, \mathcal{F}) \neq 0 \right\}.$$

We have seen in theorem 2, that the invertible sheaves of degree $g-1$ have transition functions of the form

$$g_{10}(\zeta, \eta) = \frac{1}{\zeta^{n-2}} \left(d_{00} + \sum_{l=1}^{n-1} \sum_{k=1}^{2l-1} d_{kl} \zeta^{-r} \eta^l s \right),$$

with complex coefficients $d_{kl} \in \mathbb{C}$ and a non-zero constant $d_{00} \neq 0$. An arbitrary global section of an invertible sheaf $\mathcal{F} \in Jac^{g-1}(C_n)$ is given by a pair (s_1, s_0) of holomorphic functions $s_0 \in \mathcal{O}_{U_0}(U_0)$ and $s_1 \in \mathcal{O}_{U_1}(U_1)$ such that they satisfy $s_1\left(\frac{1}{\zeta}, \frac{\eta}{\zeta^2}\right) = g_{10}(\zeta, \eta) s_0(\zeta, \eta)$ on U_{10}. By claim 5 we can write these two functions as power series, i.e.

$$s_1\left(\frac{1}{\zeta}, \frac{\eta}{\zeta^2}\right) = \sum_{q=0}^{n-1} \sum_{p=0}^{\infty} \tilde{a}_{pq} \frac{1}{\zeta^{p+2q}} \eta^q, \qquad s_0(\zeta, \eta) = \sum_{q=0}^{n-1} \sum_{p=0}^{\infty} a_{pq} \zeta^p \eta^q.$$

We are searching for all possible invertible sheaves, given by the coefficients $d_{kl} \in \mathbb{C}$, such that the global section is not the zero section. The pair (s_1, s_0) forms a non-trivial section if and only if at least one coefficient a_{pq} or $\tilde{a}_{pq}$ is non-zero. Hence we have to equate the coefficients of the following equation

$$\begin{aligned}\sum_{q=0}^{n-1}\sum_{p=0}^{\infty}\tilde{a}_{pq}\frac{1}{\zeta^{p+2q}}\eta^q &= \frac{1}{\zeta^{n-2}}\left(d_{00}+\sum_{l=1}^{n-1}\sum_{k=1}^{2l-1}d_{kl}\zeta^{-k}\eta^l\right)\sum_{q=0}^{n-1}\sum_{p=0}^{\infty}a_{pq}\zeta^p\eta^q \\ &= \sum_{q=0}^{n-1}\sum_{p=0}^{\infty}d_{00}a_{pq}\frac{1}{\zeta^{n-2-p}}\eta^q+\sum_{l=1}^{n-1}\sum_{k=1}^{2l-1}\sum_{q=0}^{n-1}\sum_{p=0}^{\infty}d_{kl}a_{pq}\frac{1}{\zeta^{n-2+k-p}}\eta^{l+q},\end{aligned} \tag{3.4}$$

which gives us conditions on the coefficients. We want to extract from the equation (3.4) the condition equations on the d_{kl} to be an invertible sheaf with a non-trivial global section. We start by considering the left hand side of (3.4). There are monomials of the form $\frac{1}{\zeta^{p+2q}}\eta^q$ and hence for a fixed integer $0\leq q'\leq n-1$ there is no integer p' with $p'\leq 2q'-1$ such that the monomial $\frac{1}{\zeta^{p'}}\eta^{q'}$ appears on the left hand side of (3.4). But such a monomial can appear on the right hand side. Thus for every pair (p',q') of integers with $0\leq q'\leq n-1$ and $p'\leq 2q'-1$ we get an equation in terms of a_{pq} and d_{kl}. We quickly compute

$$\begin{aligned}\frac{1}{\zeta^{n-2-p}}\eta^q &= \frac{1}{\zeta^{p'}}\eta^{q'} \Leftrightarrow p=n-2-p' \text{ and } q=q',\\ \frac{1}{\zeta^{n-2+k-p}}\eta^{q+l} &= \frac{1}{\zeta^{p'}}\eta^{q'} \Leftrightarrow p=n-2+k-p' \text{ and } q=q'-l.\end{aligned}$$

The sum on the right hand of (3.4) sums up every d_{kl}, thus for the coefficient $\frac{1}{\zeta^{p'}}\eta^{q'}$ we have the linear equation

$$\begin{aligned}0 &= d_{00}a_{n-2-p',q'}\\ &\quad + d_{11}a_{n-2-p'+1,q'-1}\\ &\quad + \begin{pmatrix} d_{12} & d_{22} & d_{32}\end{pmatrix}\begin{pmatrix} a_{n-2-p'+1,q'-2}\\ a_{n-2-p'+2,q'-2}\\ a_{n-2-p'+3,q'-2}\end{pmatrix}\\ &\quad + \begin{pmatrix} d_{13} & d_{23} & d_{33} & d_{43} & d_{53}\end{pmatrix}\begin{pmatrix} a_{n-2-p'+1,q'-3}\\ a_{n-2-p'+2,q'-3}\\ a_{n-2-p'+3,q'-3}\\ a_{n-2-p'+4,q'-3}\\ a_{n-2-p'+5,q'-3}\end{pmatrix}\\ &\quad + \dots\\ &\quad + \begin{pmatrix} d_{1q'} & d_{2q'} & \cdots & d_{2q'-1,q'}\end{pmatrix}\begin{pmatrix} a_{n-2-p'+1,0}\\ a_{n-2-p'+2,0}\\ \vdots\\ a_{n-2-p'+2q'-1,0}\end{pmatrix}.\end{aligned} \tag{3.5}$$

We call this linear equation the (p',q')-th condition equation.

Theorem 3. *Let $\mathcal{F}\in Jac^{g-1}(C_n)$ be an invertible sheaf characterized by the coefficients $d_{kl}\in\mathbb{C}$ of its transition function. Let (s_1,s_0) be a global, possibly trivial, section of $\mathcal{F}$ characterized by its coefficients a_{pq} and $\tilde{a}_{pq}$. Then we have*

$$0=a_{p0}=\tilde{a}_{p0},\quad \forall p\geq n-1.$$

For every integer $1 \leq q \leq n-1$ we have

$$a_{pq} = \tilde{a}_{pq} = 0, \quad \forall p \geq n-2.$$

In particular every global, possibly trivial, section of $\mathcal{F}$ is of polynomial type.

Proof. First let us consider the coefficient η^0. Equation (3.4) delivers the equation

$$\sum_{p=0}^{\infty} \tilde{a}_{p0} \frac{1}{\zeta^p} = \frac{1}{\zeta^{n-2}} \sum_{p=0}^{\infty} d_{00} a_{p0} \zeta^p = \sum_{p=0}^{\infty} a_{p0} \frac{1}{\zeta^{n-2-p}}.$$

Explicitly this gives us the equations

$$\begin{aligned}
&\vdots \\
0 &= d_{00} a_{(n-1)0}, \\
\tilde{a}_{00} &= d_{00} a_{n-2,0}, \\
\tilde{a}_{10} &= d_{00} a_{n-3,0}, \\
&\vdots \\
\tilde{a}_{p0} &= d_{00} a_{n-2-p,0}, \\
&\vdots \\
\tilde{a}_{n-2,0} &= d_{00} a_{00}, \\
\tilde{a}_{n-1,0} &= 0. \\
&\vdots
\end{aligned}$$

Since the coefficient d_{00} is non-zero, we get immediatly the first part of the theorem. We will split up the second part into two parts. First we will show $\tilde{a}_{pq} = 0$ for all $p \geq n-2$ and $1 \leq q \leq n-1$. Let us fix a $1 \leq q \leq n-1$ and let $p \geq n-2$. The monomial on the left hand side of (3.4) with coefficient $\tilde{a}_{pq}$ is $\frac{1}{\zeta^{p+2q}} \eta^q$. Note that $p + 2q \geq n - 2 + 2q$. We will show, that on the right hand side every monomial is of the form $\frac{1}{\zeta^r} \eta^q$ with $r \leq n-2+2q-1$ and hence there is no monomial on the right hand side of the form $\frac{1}{\zeta^{p+2q}} \eta^q$. Equating the coefficients says then $\tilde{a}_{pq} = 0$. In the first summand on the right hand side of (3.4) has monomials of the form $\frac{1}{\zeta^{n-2-p'}} \eta^q$, and hence $n-2-p' \leq n-2 < n-2+2q \leq p+2q$, what we wanted. We have to consider the second summand more carefully. Let us pick two integers (l, q') with $l + q' = q$, $1 \leq l \leq n-1$ and $0 \leq q' \leq n-1$. Thus the monomials on the right hand side of (3.4) are of the form $\frac{1}{\zeta^{n-2+k-p'}}$, where $1 \leq k \leq 2l-1 = 2(q-q')-1$. But this means

$$\begin{aligned}
n-2+k-p' &\leq n-2+2l-1-p' = n-2+2(q-q')-1-p' \leq n-2+2q-1 \\
&< n-2+2q \leq p+2q.
\end{aligned}$$

To finish the proof of the theorem we have to show now $a_{pq} = 0$ for all $p \geq n-2$ and $1 \leq q \leq n-1$. We do this via induction over the q's. If we consider $q = 1$ then equation (3.4) delivers

$$\begin{aligned}\sum_{p=0}^{\infty}\tilde{a}_{p1}\frac{1}{\zeta^{p+2}}\eta &= \sum_{p=0}^{\infty}d_{00}a_{p1}\frac{1}{\zeta^{n-2-p}}\eta^{1} + \sum_{k=1}^{1}\sum_{p=0}^{\infty}d_{k1}a_{p0}\frac{1}{\zeta^{n-2+k-p}}\eta^{1+0}\\ &= \sum_{p=0}^{\infty}d_{00}a_{p1}\frac{1}{\zeta^{n-2-p}}\eta^{1} + \sum_{p=0}^{\infty}d_{11}a_{p0}\frac{1}{\zeta^{n-2+1-p}}\eta^{1}.\end{aligned}$$

When we fix a $p \geq n-2$, then the monomial with coefficient a_{p1} is $\frac{1}{\zeta^{n-2-p}}\eta^{1}$. Note that $n-2-p \leq 0$, since $p \geq n-2$. Because $q = 1$ we have always $p' + 2q = p' + 2 > 0$ and hence on the left hand side of (3.4) appears no monomial of the form $\frac{1}{\zeta^{n-2-p}}\eta^{1}$. So the equation leads to

$$0 = d_{00}a_{p1}\frac{1}{\zeta^{n-2-p}}\eta^{1} + d_{11}a_{p+1,0}\frac{1}{\zeta^{n-2-p}}\eta^{1}.$$

But here we see, since $p \geq n-2$, $p+1 \geq n-2+1 = n-1$ and thus $a_{p+1,0} = 0$ by the first part of the theorem and the equation $0 = d_{00}a_{p1}$ remains. Again with $d_{00} \neq 0$ and the equation $0 = d_{00}a_{p1}$ we finish the base case $q = 1$ of the induction. Let us fix now an arbitrary $1 \leq q \leq n-1$ and let us suppose, that $a_{pq'} = 0$ for all $q' < q$ and $p \geq n-2$. This means we are considering an equation with monomial $\frac{1}{\zeta^{n-2-p}}\eta^{q}$. But since $p \geq n-2$, this means $n-2-p \leq 0$ and hence the left hand side does not have a monomial of this form. Thus the equation (3.5) gets

$$\begin{aligned}0 &= d_{00}a_{pq}\\ &\quad + d_{11}a_{(p+1)(q-1)}\\ &\quad + \begin{pmatrix} d_{12} & d_{22} & d_{32}\end{pmatrix}\begin{pmatrix} a_{(p+1)(q-2)}\\ a_{(p+2)(q-2)}\\ a_{(p+3)(q-2)}\end{pmatrix}\\ &\quad + \begin{pmatrix} d_{13} & d_{23} & d_{33} & d_{43} & d_{53}\end{pmatrix}\begin{pmatrix} a_{(p+1)(q-3)}\\ a_{(p+2)(q-3)}\\ a_{(p+3)(q-3)}\\ a_{(p+4)(q-3)}\\ a_{(p+5)(q-3)}\end{pmatrix}\\ &\quad + \cdots\\ &\quad + \begin{pmatrix} d_{1q} & d_{2q} & \dots & d_{(2q-1)q}\end{pmatrix}\begin{pmatrix} a_{(p+1)0}\\ a_{(p+2)0}\\ \vdots\\ a_{(p+2l'-1)0}\end{pmatrix}.\end{aligned}$$

But by induction hypothesis the most of the coefficients above vanish and the only remaining equation is

$$0 = d_{00}a_{pq},$$

which, with $d_{00} \neq 0$ again, gives us the last part of the theorem. □

3.3.2 The Matrix M

We have seen above, that for every pair (p', q') with $1 \leq q' \leq n-1$ and $p' \leq 2q' - 1$ we get the (p', q')-th condition equation

$$\begin{aligned}
0 = {} & d_{00} a_{n-2-p',q'} \\
& + d_{11} a_{n-2-p'+1,q'-1} \\
& + \begin{pmatrix} d_{12} & d_{22} & d_{32} \end{pmatrix} \begin{pmatrix} a_{n-2-p'+1,q'-2} \\ a_{n-2-p'+2,q'-2} \\ a_{n-2-p'+3,q'-2} \end{pmatrix} \\
& + \begin{pmatrix} d_{13} & d_{23} & d_{33} & d_{43} & d_{53} \end{pmatrix} \begin{pmatrix} a_{n-2-p'+1,q'-3} \\ a_{n-2-p'+2,q'-3} \\ a_{n-2-p'+3,q'-3} \\ a_{n-2-p'+4,q'-3} \\ a_{n-2-p'+5,q'-3} \end{pmatrix} \\
& + \dots \\
& + \begin{pmatrix} d_{1q'} & d_{2q'} & \dots & d_{2q'-1,q'} \end{pmatrix} \begin{pmatrix} a_{n-2-p'+1,0} \\ a_{n-2-p'+2,0} \\ \vdots \\ a_{n-2-p'+2q'-1,0} \end{pmatrix}.
\end{aligned}$$

But since $p' \in \mathbb{Z}$ there are infinitely many equations. But by the theorem 3 we know that for a $p' \leq 0$ every coefficient vanishes and hence the equation is trivial. Thus we get only for every $1 \leq q' \leq n-1$ and every $1 \leq p' \leq 2q' - 1$ a possibly non-trivial condition equation. In total we have g linear equations, since

$$\sum_{q'=1}^{n-1} \sum_{p'=1}^{2q'-1} 1 = \sum_{q'=1}^{n-1} (2q' - 1) = 2\frac{n(n-1)}{2} - (n-1) = (n-1)^2 = g.$$

By theorem 3 the number of possibly non-vanishing coefficients a_{pq} is

$$(n-1) + \sum_{q=1}^{n-1} (n-2) = (n-1) + (n-1)(n-2) = (n-1)^2 = g.$$

We put all these coefficients into a vector of $\mathbb{C}^g$ by defining

$$\vec{a} := \big(a_{00}, a_{10}, \dots, a_{n-2,0}, a_{01}, a_{11}, \dots, a_{n-3,1}, a_{02}, a_{12}, \dots, a_{n-3,2}, \dots, \\ a_{0,n-1}, a_{1,n-1}, \dots, a_{n-3,n-1}\big)^T.$$

By the g different linear condition equations of the form (3.5) we get a $g \times g$-matrix M, such that the system of linear equations

$$M\vec{a} = 0$$

is satisfied. Moreover every vector $\vec{a}$ satisfying $M\vec{a} = 0$ defines a global section of the invertible sheaf $\mathcal{F}$ characterized by the d_{kl}'s via the transition function. So an invertible sheaf has a non-trivial, global section if and only if the kernel of the matrix M is not trivial.

Proposition 12. *An invertible sheaf $\mathcal{F} \in Jac^{g-1}(C_n)$ lies in the theta divisor if and* *only if its corresponding matrix $M \in \mathbb{C}^{g\times g}$, arising from the condition equations (3.5), is not invertible.*

The determinant is our main tool to decide if this matrix M is invertible or not. Before we compute the determinant we have to study the structure of the matrix M in more details. An important observation of the condition equation is the following. If we have a product $d_{kl}a_{n-2-p'+k,q'-l}$ in the (p',q')-th condition equation, then the product $d_{kl}a_{n-2-(p'+1)+k,q'-l}$ is in the the $(p'+1,q')$-th condition equation. This motivates the next definition.

Definition 1. *Let us fix an invertible sheaf $\mathcal{F} \in Jac^{g-1}(C_n)$ characterized by the complex coefficients d_{kl} of its transition function. For $0 \le l \le n-1$ we call a matrix $A_l^{vw} \in \mathbb{C}^{v\times w}$ a $v\times w$-Hankel matrix of degree l with respect to the sheaf $\mathcal{F}$, if it is of the form*

$$A_l^{vw} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \\ \cdots & d_{l-2,l} & d_{l-1,l} & d_{ll} & \cdots \\ \cdots & d_{l-1,l} & d_{ll} & d_{l+1,l} & \cdots \\ \cdots & d_{ll} & d_{l+1,l} & d_{l+2,l} & \cdots \\ & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

For $l<0$ we set the convention $A_l^{vw} = 0 \in \mathbb{C}^{v\times w}$.

The Matrix M consists of blocks of such Hankel-matrices, i.e.

$$M = \begin{pmatrix} A_1^{1(n-1)} & A_0^{1(n-2)} & \cdots & A_{-(n-3)}^{1(n-2)} & A_{-(n-2)}^{1(n-2)} \\ A_2^{3(n-1)} & A_1^{3(n-2)} & \cdots & A_{-(n-4)}^{3(n-2)} & A_{-(n-3)}^{3(n-2)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{n-2}^{(2(n-2)-1)(n-1)} & A_{n-3}^{(2(n-2)-1)(n-2)} & \cdots & A_0^{(2(n-2)-1)(n-2)} & A_{-1}^{(2(n-2)-1)(n-2)} \\ A_{n-1}^{(2(n-1)-1)(n-1)} & A_{n-2}^{(2(n-1)-1)(n-2)} & \cdots & A_1^{(2(n-1)-1)(n-2)} & A_0^{(2(n-1)-1)(n-2)} \end{pmatrix},$$

where the A_l^{vw} are $v\times w$-Hankel matrices of degree l with respect to the sheaf $\mathcal{F}$.

We give the matrix M some matrix coordinates to determine the entries in the matrix. We define two index sets

$$P := \{(i,j) : 1 \le j \le n-1, 1 \le i \le 2j-1\} \subset \mathbb{N}\times\mathbb{N},$$
$$Q := \{(s,t) : 1 \le s \le n-2, 0 \le t \le n-1\} \cup \{(0,0)\} \subset \mathbb{N}\times\mathbb{N}.$$

The set P are the row-coordinates and the set Q are the column-coordinates of the matrix M. The pair of numbers (j,t) indicates a particular Hankel block A_l^{vw} and the pair of numbers (i,s) describes the coordinates of elements in this particular Hankel-block. We may write

$$((i,j),(s,t)) \in P\times Q$$

for the position of a matrix entry, i.e. first the row-coordinate and then the column-coordinate. We have now the following crucial observation.

Theorem 4. *Let $\mathcal{F} \in Jac^{g-1}(C_n)$ be an invertible sheaf and M its corresponding matrix M of the condition equations (3.5). Then the matrix entry $m_{(i,j),(s,t)}$ of the* *matrix M at the coordinate $((i,j),(s,t)) \in P \times Q$ is*

$$m_{(i,j),(s,t)} = d_{i-s,j-t}.$$

Moreover if either $1 \leq i-s \leq 2(j-t)-1$ and $1 \leq j-t \leq n-1$ or $i-s=0$ and $j-t=0$, then the entry is a possibly non-zero coefficient d_{kl}. Every other entry is zero.

Proof. As we defined the coordinates $((i,j),(s,t))$ we have all the information we need. The index (i,j) tells us, that we are in the (p',q')-th-conditon equation with $(p',q') = (i,j)$. The index (s,t) says, that we consider the coefficient $a_{n-2-s,t}$ in this particular condition equation. In a given (p',q')-condition equation we have products $d_{kl}a_{n-2-p'+k,q'-l}$. This means in the (p',q')-th-condition equation the coefficient $a_{n-2-s,t}$ appears exactly once in the product

$$d_{p'+(n-2-s)-(n-2),q'-t}a_{n-2-s,t} = d_{p'-s,q'-t}a_{n-2-s,t}.$$

With $(p',q') = (i,j)$ we have the desired product $d_{i-s,j-t}a_{n-2-s,t}$ and therefore we have

$$m_{(i,j),(s,t)} = d_{i-s,j-t}.$$

The last part of the claim follows by the fact, that the coefficients d_{kl} are possibly non-zero only if either $1 \leq l \leq n-1$ and $1 \leq k \leq 2l-1$ or $l=k=0$. □

In order to make the matrix coordinates $P \times Q$, the matrix M and theorem 4 more visible we refer to example 5

3.3.3 Theta Function

We want to compute the determinant of the matrix M. Recall we gave the matrix coordinates $((i,j),(s,t)) \in P \times Q$, where we defined

$$\begin{aligned} P &:= \{(i,j) : 1 \leq j \leq n-1, 1 \leq i \leq 2j-1\}, \\ Q &:= \{(s,t) : 1 \leq s \leq n-2, 0 \leq t \leq n-1\} \cup \{(0,0)\}. \end{aligned}$$

We want to use the Leibniz-formula

$$\det(M) = \sum_{\sigma \in S_g} \left(sign(\sigma) \prod_{i=1}^{g} m_{i,\sigma(i)} \right).$$

Here the symmetric group S_g consists of bijective maps $\sigma : \{1,\ldots,g\} \to \{1,\ldots,g\}$, where the set $\{1,\ldots,g\}$ play the role of matrix coordinates and g is the genus of the curve. We want to adjust this idea to matrix coordinates with index sets P, Q. We make the following definitions.

Definition 2. *A subset $\mathcal{D} \subseteq P \times Q$ is called regular, if it is of the following form*

$$\mathcal{D} := \{ ((i,j),(s,t)) \in P \times Q : \text{ each } (i,j) \text{ and each } (s,t) \text{ appears exactly one time and either } 1 \leq i-s \leq 2(j-t)-1 \text{ and } 1 \leq j-t \leq n-1 \text{ or } i-s=j-t=0\}.$$

The set of regular sets is denoted by $\mathcal{R}(P \times Q)$.

For each regular set we need a signum.

Definition 3. *Let P, Q be the index sets above and $\mathcal{D} \in \mathcal{R}(P \times Q)$ be a regular set. We define a map on P and a map on Q by*

$$-_P : P \times P \longrightarrow \mathbb{Z}$$
$$(i_2, j_2) -_P (i_1, j_1) := \begin{cases} j_2 - j_1, & j_2 \neq j_1 \\ i_2 - i_1, & j_2 = j_1 \end{cases},$$
$$-_Q : Q \times Q \longrightarrow \mathbb{Z}$$
$$(s_2, t_2) -_Q (s_1, t_1) := \begin{cases} t_2 - t_1, & t_2 \neq t_1 \\ s_1 - s_2, & t_2 = t_1 \end{cases}.$$

We define an order on P by $(i_1, j_1) < (i_2, j_2)$ if and only if $0 < (i_2, j_2) -_P (i_1, j_1)$. The signum of the regular set $\mathcal{D}$ is defined by

$$sign(\mathcal{D}) := \prod_{\substack{((i_1,j_1),(s_1,t_1)) \in \mathcal{D} \\ ((i_2,j_2),(s_2,t_2)) \in \mathcal{D} \\ (i_1,j_1)<(i_2,j_2)}} \frac{(s_2, t_2) -_Q (s_1, t_1)}{(i_2, j_2) -_P (i_1, j_1)}.$$

Now we can write down the determinant of the matrix M.

Theorem 5. *Let $(C_n, \mathcal{O}_{C_n})$ be the nilpotent, spectral curve. Let $\mathcal{F} \in Jac^{g-1}(C_n)$ be an invertible sheaf of degree $g - 1$ determined by $d_{kl} \in \mathbb{C}$, for $1 \leq l \leq n - 1$ and $1 \leq k \leq 2l - 1$. Then the invertible sheaf $\mathcal{F}$ belongs to the theta divisor, i.e.*

$$\mathcal{F} \in \Theta := \left\{ \mathcal{F} \in Jac^{g-1}(C_n) : \dim_{\mathbb{C}}(\check{\mathrm{H}}^0(C_n, \mathcal{F})) \neq 0 \right\}$$

if and only if the theta function, given by

$$\theta(\mathcal{F}) := \det(M) = \sum_{\mathcal{D} \in \mathcal{R}(P \times Q)} \left(sign(\mathcal{D}) \prod_{((i,j),(s,t)) \in \mathcal{D}} d_{i-s,j-t} \right),$$

vanishes.

The theorem provides us a big reduction of combinatorial computations. The symmetric group S_g is huge with $g!$ elements. But the elements of the symmetric group do not care about the zeros in the matrix M. The regular sets $\mathcal{D}$ play the role of those elements in the symmetric group, such that the product $\prod_{i=1}^{g} m_{i,\sigma(i)}$ is a product over the variables d_{kl} and hence is a priori non-zero. All other products are zero.

Proof. By theorem 4 we know, that the entry at $((i, j), (s, t))$ of M is $d_{i-s,j-t}$ if it exists and is zero otherwise. Therefore the determinant can be written as the sum over bijective maps $\sigma : P \to Q$, such that either $1 \leq i - pr_1(\sigma(i, j)) \leq 2(j - pr_2(\sigma(i, j))) - 1$ and $1 \leq i - pr_2(\sigma(i, j)) \leq n - 1$ or $i - pr_1(\sigma(i, j)) = i - pr_2(\sigma(i, j)) = 0$. Here pr_1, pr_2 are projections to the first resp. second variable. Every other element of the symmetric group induces a product in the Leibniz formula with a zero factor. Every such bijective map σ induces a subset

$$\mathcal{D}^\sigma := \{((i, j), \sigma(i, j)) \in P \times Q\} \subset P \times Q.$$

But by bijectivity of σ the set $\mathcal{D}^\sigma$ has to be regular. The signum of a regular set coincides with the signum of an element of the symmetric group. □

The next corollary is a statement about the indices, which helps to check, if the theta function is correctly computed.

Corollary 2. *Let $\mathcal{D} \in \mathcal{R}(P \times Q)$ be a regular set and let us consider the monomial*

$$Mon(\mathcal{D}) = \prod_{((i,j),(s,t))\in\mathcal{D}} d_{i-s,j-t}.$$

Then we have

$$\sum_{((i,j),(s,t))\in\mathcal{D}} (i-s) = \sum_{((i,j),(s,t))\in\mathcal{D}} (j-t) = \frac{1}{6}(n^3-n) = \frac{1}{6}(n+1)n(n-1).$$

Proof. Because $\mathcal{D}$ is regular each (i,j) and (s,t) appears exactly one time in $\mathcal{D}$. This means that the sum $\sum_{((i,j),(s,t))\in\mathcal{D}} i-s$ is over every possible i and s and therefore it is equal to $\sum_{(i,j)\in P} i - \sum_{(s,t)\in Q} s$. We have

$$\begin{aligned}
\sum_{(i,j)\in P} i &= \sum_{j=1}^{n-1}\sum_{i=1}^{2j-1} i = \sum_{j=1}^{n-1}\frac{(2j-1)2j}{2} = \sum_{j=1}^{n-1}(2j^2-j)\\
&= 2\frac{1}{6}(n-1)n(2(n-1)+1) - \frac{(n-1)n}{2},\\
\sum_{(s,t)\in Q} s &= \sum_{t=0}^{n-1}\sum_{s=0}^{n-2} s = n\frac{(n-2)(n-1)}{2}.
\end{aligned}$$

This means

$$\begin{aligned}
\sum_{((i,j),(s,t))\in\mathcal{D}} (i-s) &= 2\frac{1}{6}(n-1)n(2(n-1)+1) - \frac{(n-1)n}{2} - (n-2)\frac{(n-1)n}{2}\\
&= \frac{1}{6}(n-1)n\left(4n-2-3-3(n-2)\right)) = \frac{1}{6}(n-1)n(n+1).
\end{aligned}$$

For the second equation we calculate similarly

$$\begin{aligned}
\sum_{(i,j)\in P} j &= \sum_{j=1}^{n-1}\sum_{i=1}^{2j-1} i = 2\frac{1}{6}(n-1)n(2(n-1)+1) - \frac{(n-1)n}{2},\\
\sum_{(s,t)\in Q} t &= \sum_{s=1}^{n-2}\sum_{t=0}^{n-1} s = n-2\frac{(n-1)n}{2}.
\end{aligned}$$

Hence we get

$$\sum_{((i,j),(s,t))\in\mathcal{D}} (j-t) = \frac{1}{6}(n-1)n(n+1).$$

□

3.3.4 Examples

In this section we will compute the transition functions of invertible sheaves and the theta function for the nilpotent, spectral curves with $n \in \{2, 3, 4\}$.

Example 3. *Let us start with $n = 2$, the nilpotent, spectral curve $(C_2, \mathcal{O}_{C_2})$. We compute easily the arithmetic genus, which is $g = (n-1)^2 = 1$ and hence*

$$Jac^{g-1}(C_n) = Jac^0(C_n) \cong \mathbb{C}.$$

By proposition 11 the elements of $\check{\mathrm{H}}^1(C_2, \mathcal{O}_{C_2})$ are of the form

$$\sum_{k=1}^{n-1}\sum_{l=1}^{2k-1} a_{kl}\eta^k\zeta^{-l} = \sum_{k=1}^{1}\sum_{l=1}^{1} a_{kl}\eta^k\zeta^{-l} = a_{11}\frac{\eta}{\zeta},$$

where a_{11} is a complex number. In other words we have $\check{\mathrm{H}}^1(C_2, \mathcal{O}_{C_2}) = \langle \frac{\eta}{\zeta} \rangle \cong \mathbb{C}$. By the exponential map every element of $Jac^0(C_2)$ can be described as

$$\exp\left(a\frac{\eta}{\zeta}\right) = 1 + a\frac{\eta}{\zeta} + \frac{a}{2}\left(\frac{\eta}{\zeta}\right)^2 + \dots \ mod\ \eta^2 = 1 + a\frac{\eta}{\zeta}.$$

If we homogenize the transition function, every invertible sheaf has a transition function of the form

$$g_{10}(\zeta, \eta) = d_{00} + d_{11}\frac{\eta}{\zeta}.$$

The theta divisor is now given by the invertible sheaves of degree $g - 1 = 0$ with a non-trivial, global section. Hence if $\mathcal{F}$ is an invertible sheaf and the pair $(s_1, s_0) \in \mathcal{F}(C_2)$ is a global section, it has to satisfy $s_1(\frac{1}{\zeta}, \frac{\eta}{\zeta^2}) = g_{10}(\zeta, \eta)s_0(\zeta, \eta)$ on U_{10}. But because $s_1 \in \mathcal{O}_{U_1}(U_1)$ and $s_0 \in \mathcal{O}_{U_0}(U_0)$ we expand the two functions around the origin in U_i into powerseries, $i \in \{0, 1\}$. This gives us the equation

$$\begin{aligned}\left(\sum_{k=0}^{\infty}\tilde{a}_{k0}\frac{1}{\zeta^k} + \sum_{k=0}^{\infty}\tilde{a}_{k1}\frac{1}{\zeta^{k+2}}\eta\right) &= \left(d_{00} + d_{11}\frac{1}{\zeta}\eta\right)\left(\sum_{k=0}^{\infty}a_{k0}\zeta^k + \sum_{k=0}^{\infty}a_{k1}\zeta^k\eta\right)\\ &= \sum_{k=0}^{\infty}d_{00}a_{k0}\zeta^k + \sum_{k=0}^{\infty}\left(d_{11}a_{k0}\zeta^{k-1} + d_{00}a_{k1}\zeta^k\right)\eta.\end{aligned}$$

Equating coefficients gives us $\tilde{a}_{k0} = a_{k0} = 0$ for all $k > 0$. The only condition equation is

$$d_{11}a_{00} = 0.$$

Hence we have a non-trivial, global section, i.e. $a_{00} \neq 0$, if and only if $d_{11} = 0$. The matrix M is just $M = \left(d_{11}\right)$ and the theta function is

$$\theta(\mathcal{F}) = d_{11}.$$

The approach via regular sets is as follows. For $n = 2$ we have the index sets

$$P = \{(1, 1)\}, \qquad Q = \{(0, 0)\}$$

and the only possible regular set is then

$$\mathcal{D} = P \times Q = \{((1, 1), (0, 0))\}.$$

The signum is 1 *and we get the formula for the theta function* $\theta(\mathcal{F}) = d_{11}$.

Example 4. *Next we will consider the nilpotent spectral curve with* $n = 3$. *We compute* $g = (n-1)^2 = 4$ *and for the transition functions we get*

$$\begin{aligned}\frac{1}{\zeta^{3-2}} g_{10}(\zeta,\eta) &= \frac{1}{\zeta^{3-2}}\left(d_{00} + \sum_{l=1}^{3-1}\sum_{k=1}^{2l-1} d_{kl}\zeta^{-k}\eta^l\right)\\ &= \frac{1}{\zeta}\left(d_{00} + d_{11}\frac{1}{\zeta}\eta + \left(d_{12}\frac{1}{\zeta} + d_{22}\frac{1}{\zeta^2} + d_{32}\frac{1}{\zeta^2}\right)\eta^2\right).\end{aligned}$$

We see immeditaly

$$Jac^3(C_3) \cong \mathbb{C}^4.$$

To compute the theta divisor we pick again an arbitrary invertible sheaf and assume, that the pair (s_1, s_0) *is a global section. We get the equation*

$$\begin{aligned}&\sum_{l=0}^{3-1}\sum_{k=0}^{\infty} \tilde{a}_{kl}\frac{1}{\zeta^{k+2l}}\eta^l\\ &= \sum_{k=0}^{\infty}\tilde{a}_{k0}\frac{1}{\zeta^k} + \sum_{k=0}^{\infty}\tilde{a}_{k1}\frac{1}{\zeta^{k+2}}\eta + \sum_{k=0}^{\infty}\tilde{a}_{k2}\frac{1}{\zeta^{k+4}}\eta^2\\ &= \left(d_{00}\frac{1}{\zeta^1} + d_{11}\frac{1}{\zeta^2}\eta + \left(d_{12}\frac{1}{\zeta^2} + d_{22}\frac{1}{\zeta^3} + d_{32}\frac{1}{\zeta^4}\right)\eta^2\right)\sum_{l=0}^{3-1}\sum_{k=0}^{\infty} a_{kl}\zeta^k\eta^l\\ &= \sum_{k=0}^{\infty} d_{00}a_{k0}\frac{1}{\zeta^{1-k}} + \left(\sum_{k=0}^{\infty} d_{00}\frac{1}{\zeta^{1-k}}a_{k1} + \sum_{k=0}^{\infty} d_{11}\frac{1}{\zeta^{2-k}}a_{k0}\right)\eta\\ &\quad + \Big(\sum_{k=0}^{\infty}\frac{1}{\zeta^{1-k}}d_{00}a_{k2} + \sum_{k=0}^{\infty} d_{11}\frac{1}{\zeta^{2-k}}a_{k1} + \sum_{k=0}^{\infty} d_{12}\frac{1}{\zeta^{2-k}}a_{k0}\\ &\qquad + \sum_{k=0}^{\infty} d_{22}\frac{1}{\zeta^{3-k}}a_{k0} + \sum_{k=0}^{\infty} d_{32}\frac{1}{\zeta^{4-k}}a_{k0}\Big)\eta^2.\end{aligned}$$

By equating the coefficients we get the condition equations

$$\begin{aligned}0 &= d_{00}a_{01} + d_{11}a_{10},\\ 0 &= d_{00}a_{02} + d_{12}a_{10},\\ 0 &= d_{11}a_{01} + d_{12}a_{00} + d_{22}a_{10},\\ 0 &= d_{22}a_{00} + d_{32}a_{10},\end{aligned}$$

which induce the matrix M

$$M = \begin{pmatrix} 0 & d_{11} & d_{00} & 0\\ 0 & d_{12} & 0 & d_{00}\\ d_{12} & d_{22} & d_{11} & 0\\ d_{22} & d_{32} & 0 & 0\end{pmatrix}.$$

The determinant and thus the theta function is then

$$\theta(\mathcal{F}) := \det(M) = d_{00}^2 d_{12}d_{32} + d_{00}d_{11}^2 d_{22} - d_{00}^2 d_{22}^2.$$

On the other hand the regular sets are

$$\begin{aligned}
\mathcal{D}_1 &:= \{((1,2)(1,2)),((1,1)(1,1)),((2,2)(1,0)),((3,2)(0,0))\},\\
\mathcal{D}_2 &:= \{((1,2)(1,2)),((1,1)(1,1)),((3,2)(1,0)),((2,2)(0,0))\},\\
\mathcal{D}_3 &:= \{((1,2)(1,2)),((2,2)(1,1)),((3,2)(1,0)),((1,1)(0,0))\}.
\end{aligned}$$

The signs of these regular sets are

$$sign(\mathcal{D}_1) = +1, \qquad sign(\mathcal{D}_2) = -1, \qquad sign(\mathcal{D}_3) = +1.$$

This gives us the same theta function as above

$$\theta(\mathcal{F}) := d_{00}^2 d_{12} d_{32} + d_{00} d_{11}^2 d_{22} - d_{00}^2 d_{22}^2.$$

Example 5. *Now we deal with the case $n = 4$. First note that $g = (n-1)^2 = 3^2 = 9$. The transition functions of invertible sheaves of degree $g - 1 = 8$ are given by*

$$\begin{aligned}
g_{10}(\zeta,\eta) = \frac{1}{\zeta^2}\Big(& d_{00} + d_{11}\frac{1}{\zeta}\eta + \left(d_{12}\frac{1}{\zeta} + d_{22}\frac{1}{\zeta^2} + d_{32}\frac{1}{\zeta^3}\right)\eta^2 \\
&+ \left(d_{13}\frac{1}{\zeta} + d_{23}\frac{1}{\zeta^2} + d_{33}\frac{1}{\zeta^3} + d_{43}\frac{1}{\zeta^4} + d_{53}\frac{1}{\zeta^5}\right)\eta^3\Big),
\end{aligned}$$

where we see

$$Jac^8(C_4) \cong \mathbb{C}^9.$$

The matrix M associated to the invertible sheaf $\mathcal{F}$ is given by

$$M(\mathcal{F}) = \begin{pmatrix}
0 & 0 & d_{11} & 0 & d_{00} & 0 & 0 & 0 & 0\\
0 & 0 & d_{12} & d_{00} & 0 & 0 & d_{00} & 0 & 0\\
0 & d_{12} & d_{22} & 0 & d_{11} & d_{00} & 0 & 0 & 0\\
d_{12} & d_{22} & d_{32} & d_{11} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & d_{13} & 0 & 0 & 0 & 0 & 0 & d_{00}\\
0 & d_{13} & d_{23} & 0 & d_{12} & 0 & d_{11} & d_{00} & 0\\
d_{13} & d_{23} & d_{33} & d_{12} & d_{22} & d_{11} & 0 & 0 & 0\\
d_{23} & d_{33} & d_{43} & d_{22} & d_{32} & 0 & 0 & 0 & 0\\
d_{33} & d_{43} & d_{53} & d_{32} & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

Here we see the Hankel-block structure very well. The theta function is

$$\begin{aligned}
&\theta(\mathcal{F}) = \\
&d_{00}^3\left(d_{00}d_{11}d_{22}(d_{12}d_{33}d_{32} + d_{23}d_{43}d_{11} + d_{33}d_{22}d_{22} - d_{11}d_{33}d_{33} - d_{22}d_{23}d_{32} - d_{12}d_{22}d_{43})\right)\\
&- d_{00}^3\left(d_{00}d_{11}d_{32}(d_{12}d_{23}d_{32} + d_{13}d_{43}d_{11} + d_{33}d_{22}d_{12} - d_{11}d_{23}d_{33} - d_{12}d_{43}d_{12} - d_{32}d_{22}d_{13})\right)\\
&+ d_{00}^3\left(d_{00}^2d_{12}(d_{23}d_{43}d_{32} + d_{33}d_{53}d_{12} + d_{43}d_{33}d_{22} - d_{12}d_{43}d_{43} - d_{22}d_{53}d_{23} - d_{32}d_{33}d_{33})\right)\\
&- d_{00}^3\left(d_{00}^2d_{22}(d_{13}d_{43}d_{32} + d_{23}d_{53}d_{12} + d_{33}d_{33}d_{22} - d_{12}d_{43}d_{33} - d_{22}d_{53}d_{13} - d_{32}d_{33}d_{23})\right)\\
&+ d_{00}^3\left(d_{00}^2d_{32}(d_{13}d_{33}d_{32} + d_{23}d_{43}d_{12} + d_{33}d_{23}d_{22} - d_{12}d_{33}d_{33} - d_{22}d_{43}d_{13} - d_{32}d_{23}d_{23})\right)\\
&- d_{00}^3\left(d_{00}^2d_{11}(d_{13}d_{33}d_{53} + d_{23}d_{43}d_{33} + d_{33}d_{23}d_{43} - d_{33}d_{33}d_{33} - d_{13}d_{43}d_{43} - d_{53}d_{23}d_{23})\right)\\
&- d_{00}^3\left(d_{11}^2d_{12}(d_{33}d_{11}d_{32} - d_{12}d_{32}d_{32})\right)\\
&- d_{00}^3\left(d_{11}^3(d_{12}d_{33}d_{32} + d_{23}d_{43}d_{11} + d_{33}d_{22}d_{22} - d_{11}d_{33}d_{33} - d_{22}d_{43}d_{12} - d_{32}d_{22}d_{23})\right)\\
&- d_{00}^3\left(d_{00}d_{11}d_{12}(d_{12}d_{43}d_{32} + d_{23}d_{53}d_{11} + d_{33}d_{32}d_{22} - d_{11}d_{43}d_{33} - d_{22}d_{53}d_{12} - d_{32}d_{32}d_{23})\right)\\
&+ d_{00}^3\left(d_{00}d_{11}d_{22}(d_{12}d_{33}d_{32} + d_{23}d_{43}d_{11} + d_{33}d_{22}d_{22} - d_{11}d_{33}d_{33} - d_{22}d_{43}d_{12} - d_{32}d_{22}d_{23})\right).
\end{aligned}$$

Note at this point, that the sum over all left indices in a monomial is equal to the sum of all right indices, which is just $\frac{1}{6}(n-1)n(n+1) = 10$*, see corollary 2.*

Chapter 4

Hitchin's Formula on the Regular, Nilpotent, Adjoint Orbit

We have seen in chapter 2 by Kronheimer, that the regular, nilpotent, adjoint orbit $\mathcal{O}_{reg}(\mathfrak{sl}_n(\mathbb{C}))$ admits a hyperkähler structure induced by the L^2-norm on the Kronheimer moduli space. Nevertheless an explicit description is still not known. A possibility to express a hyperkähler metric is to find a Kähler potential by fixing a complex structure. Because a hyperkähler structure induces an S^2 of Kähler structures, there is a natural $SO(3)$-action on the space of Kähler forms. By fixing the Kähler form ω_I and rotating the other two Kähler forms we get an $SO(2)$-action. This action is Hamiltonian and hence it induces a momentum map. Its hamiltonian function is $SU(n)$-conjugation invariant and it induces a Kähler potential with respect to the complex structure J, i.e. $g = i\partial_J\overline{\partial}_J(2\mu_I^X)$ with $X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathfrak{so}(2)$ where $K(T_1,T_2,T_3) = 2\mu_I^X(T_1,T_2,T_3) = -\int_{-\infty}^0 tr\left(T_2(t)^2 + T_3(t)^2\right)dt$. The goal of this section is to rephrase this expression in terms of invertible sheaves on the nilpotent, spectral curve of degree $g-1$. The Kähler potential gets a bit more explicit, because one does not have to solve a system of differential equations. If (T_1,T_2,T_3) is an element of $\mathcal{M}(0,\sigma)$, then

$$A(\zeta)(t) := \overbrace{(T_2(t) + iT_3(t))}^{=A_0(t)} + \overbrace{2iT_1(t)}^{=A_1(t)}\zeta + \overbrace{(T_2(t) - iT_3(t))}^{=A_2(t)}\zeta^2$$

defines a regular, nilpotent, matricial polynomial. With $A_+(\zeta)(t) := \frac{1}{2}A_1(t) + A_2(t)\zeta$, solutions of Nahm's equations imply a Lax equation

$$\frac{d}{dt}A(\zeta)(t) = [A(\zeta)(t), A_+(\zeta)(t)]\,.$$

But this implies

$$\frac{d}{dt}tr(A^n) = tr((n-1)A^{n-1}A') = (n-1)tr(A^{n-1}[A,A_+]) = (n-1)tr([A^n,A_+]) = 0$$

for all $n \in \mathbb{N}$ and hence the characteristic polynomial of $A(\zeta)(t)$ is independent of the variable t. The zero-locus of the characteristic polynomial is then the nilpotent, spectral curve. Beauville showed in [Bea90] that there is a correspondence between isomorphism classes of invertible sheaves of degree $g-1$ not lying in the theta divisor and $GL_n(\mathbb{C})$-conjugation classes of regular, nilpotent, matricial polynomials satisfying the characteristic equation. So for every $t_0 \in (-\infty, 0]$ the matrix polynomial $A(\zeta)(t_0)$ gives us an isomorphism class of invertible sheaves of degree $g-1$ not lying in the theta divisor. For smooth, spectral curves Hitchin described in [Hit98] the expression $tr\left(T_2^2 + T_3^2\right)$ in terms of invertible sheaves. Bielawski generalized in [Bie07] these ideas to reducible, spectral curves,the case of a semi-simple adjoint orbit, by allowing ordinary double points. This chapter wants to transform the ideas of Hitchin and Bielawski to the nilpotent, spectral curve and prove an analogous expression for $tr\left(T_1^2+T_2^2+T_3^2\right)=tr\left(A_0A_2-\frac{1}{4}A_1^2\right)$ and $tr\left(T_2^2+T_3^2\right)=tr\left(A_0A_2\right)$ in terms of the theta function. The computations are completely direct.

4.1 Global Sections, Evaluation Map and Flows

In order to compute explicit regular, nilpotent, matricial polynomials we need a precise description of the cohomology module $\check{\mathrm{H}}^0\left(C_n,\mathcal{F}(1)\right)$.

4.1.1 Inverting the Matrix M

Let $\mathcal{F}\in Jac^{g-1}(C_n)$ be an invertible sheaf of degree $g-1$ and let us consider again the corresponding matrix $M\in\mathbb{C}^{g\times g}$ from section 3.3.2. Suppose $\theta(\mathcal{F})\neq 0$, i.e. M is invertible or equivantly $\mathcal{F}$ does not lie in the theta divisor. We want to invert the matrix M for a later usage and to do so, we want to use Cramer's rule, [Fis05]. Thus we have to compute all the possible cofactors of the matrix M. Recall in section 3.3.2 we defined the index sets

$$\begin{aligned}P&:=\left\{(i,j):1\leq j\leq n-1,1\leq i\leq 2j-1\right\},\\ Q&:=\left\{(s,t):1\leq s\leq n-2,0\leq t\leq n-1\right\}\cup\left\{(0,0)\right\}.\end{aligned}$$

Let us fix an index $((a,b),(u,v))\in P\times Q$. We denote the matrix arising from the matrix M without the (a,b)-row and (u,v)-column by $\widehat{M}_{((a,b),(u,v))}$. We do the same strategy as in section 3.3.3 and so we define two index sets

$$P_{(a,b)}:=P\setminus\left\{(a,b)\right\},\qquad Q_{(u,v)}:=Q\setminus\left\{(u,v)\right\}.$$

Verbally the same as in the case of $P\times Q$ we have the following definition.

Definition 4. *A subset $\mathcal{D}\subseteq P_{(a,b)}\times Q_{(u,v)}$ is called regular, if it is of the following form $\mathcal{D}:=$*

$$\left\{\begin{matrix}((i,j),(s,t))\in P_{(a,b)}\times Q_{(u,v)}: & \textit{each } (i,j) \textit{ and each } (s,t) \textit{ appears exactly once}\\ \textit{and either } 1\leq i-s\leq 2(j-t)-1 & \textit{and } 1\leq j-t\leq n-1 \textit{ or } i-s=j-t=0\end{matrix}\right\}.$$

The set of regular sets is denoted by $\mathcal{R}\left(P_{(a,b)}\times Q_{(u,v)}\right)$.

Because we have to be carefull with the sign of a cofactor we need additionally the next definition.

Definition 5. *Let us define the functions*

$$\begin{aligned}\iota_{row}:P&\longrightarrow\mathbb{N} & \iota_{column}:Q&\longrightarrow\mathbb{N}\\ (a,b)&\longmapsto a+(b-1)^2, & (u,v)&\longmapsto\begin{cases}v(n-2)+n-u & ,v\geq 1\\ (n-1)-u & ,v=0\end{cases}\end{aligned}$$

and

$$\begin{aligned}\nu:P\times Q&\longrightarrow\mathbb{N}\\ ((a,b),(u,v))&\longmapsto\iota_{row}(a,b)+\iota_{column}(u,v).\end{aligned}$$

Claim 6. *If we number the rows and the columns of the matrix M with $\{1, \dots, g\}$, then $1 \leq \iota_{row}(a,b) \leq g$, $1 \leq \iota_{column}(u,v) \leq g$ and for a fixed index $((a,b),(u,v)) \in P \times Q$ the (a,b)-th row is the $\iota_{row}(a,b)$-th row and the (u,v)-th column is the $\iota_{column}(u,v)$-th column.*

Proof. We compute the number of the row with index $(a,b) \in P$. By the definition this is

$$\begin{aligned} a + \sum_{\alpha=1}^{b-1} \sum_{\beta=1}^{2\alpha-1} 1 &= a + \sum_{\alpha=1}^{b-1} (2\alpha - 1) = a + 2\sum_{\alpha=1}^{b-1} \alpha - \sum_{\alpha=1}^{b-1} 1 \\ &= a + 2\frac{(b-1)b}{2} - (b-1) = a + (b-1)(b-1) \\ &= \iota_{row}(a,b). \end{aligned}$$

Furthermore by the definition of the column-index we have $(n-2) - (u-1) = (n-1) - u = \iota_{column}(u,0)$ if $v = 0$ and $(n-1) + (v-1)(n-2) + (n-2) - (u-1) = v(n-2) + n - u = \iota_{column}(u,v)$ if $v \neq 0$. This shows the claim. □

Finally we can state the next theorem, which gives us the possibility to invert the matrix M.

Theorem 6. *Let $((a,b),(u,v)) \in P \times Q$ be a fixed index. The cofactor $C_{((a,b),(u,v))}$ of M, i.e. the determinant of the matrix $\widehat{M}_{((a,b),(u,v))}$ and multiplied by $(-1)^{\iota_{row}(a,b)+\iota_{column}(u,v)}$, is given by*

$$C_{((a,b)(u,v))} = (-1)^{\nu(a,b,u,v)} \sum_{\mathcal{D} \in \mathcal{R}(P_{(a,b)} \times Q_{(u,v)})} \left(sign(\mathcal{D}) \prod_{((i,j),(s,t)) \in \mathcal{D}} d_{i-s,j-t} \right).$$

The inverse of the matrix M is given by

$$M^{-1} = \frac{1}{\theta} \left(C_{((a,b),(u,v))} \right)^T_{((a,b)(u,v)) \in P \times Q}.$$

Proof. The formula $m_{(i,j),(s,t)} = d_{i-s,j-t}$ of theorem 4 still holds for the matrix $\widehat{M}_{((a,b),(u,v))}$ with index set $P_{(a,b)} \times Q_{(u,v)}$. This means the arguments in the proof of theorem 5 transforms in the exact way to this case to compute the determinant of $\widehat{M}_{((a,b),(u,v))}$. The inverted matrix is then given by Cramer's rule $M^{-1} = \frac{\operatorname{adj}(M)}{\det(M)}$. □

4.1.2 Basis of $\check{\mathrm{H}}^0\big(C_n, \mathcal{F}(1)\big)$

Recall by theorem 3 that a global section of an invertible sheaf $\mathcal{F}(1) \in Jac^{g-1+n}(C_n)$ with transition function $\frac{1}{\zeta^{n-1}} g_{10}(\zeta,\eta)$ is a pair $(s_1, s_0) \in \mathcal{O}_{U_1}(U_1) \times \mathcal{O}_{U_0}(U_0)$ of local sections satisfying on U_{10} the equation

$$(\varphi_{10}^* s_1)(\zeta,\eta) = s_1\left(\frac{1}{\zeta}, \frac{\eta}{\zeta^2}\right) = \frac{1}{\zeta^{n-1}} g_{10}(\zeta,\eta) s_0(\zeta,\eta)$$

and we can write these local holomorphic functions in the form

$$s_1(\tilde{\zeta},\tilde{\eta}) = \sum_{l=0}^{n-1} s_1^l(\tilde{\zeta})\tilde{\eta}^l, \qquad s_0(\zeta,\eta) = \sum_{l=0}^{n-1} s_0^l(\zeta)\eta^l,$$

where s_0^0 is a polynomial of degree $n-1$ and all other s_0^l , $l \neq 0$, are polynomials of degree $n-2$. Let us write

$$s_0^0(\zeta) = \sum_{i=0}^{n-1} a_{i0}\zeta^i = a_{00} + a_{10}\zeta + \cdots + a_{n-1,0}\zeta^{n-1},$$
$$s_0^l(\zeta) = \sum_{i=0}^{n-2} a_{il}\zeta^i = a_{0l} + a_{1l}\zeta + \cdots + a_{n-2,l}\zeta^{n-2}$$

for $\zeta \in W_0$ and let us define the vector $\vec{a} \in \mathbb{C}^{g+n}$

$$\vec{a} := \big(a_{00}, a_{10}, \ldots, a_{n-1,0}, a_{01}, a_{11}, \ldots, a_{n-2,1}, a_{02}, a_{12}, \ldots, a_{n-2,2}, \ldots, a_{0,n-1}, a_{1,n-1}, \ldots, a_{n-2,n-1}\big)^T.$$

By the same strategy as in section 3.3.2, i.e. equating coefficients of a global section of $\mathcal{F}(1)(C_n)$, we obtain similar condition equations. In other words the vector $\vec{a} \in \mathbb{C}^{g+n}$ defines a global section (s_1, s_0) of $\mathcal{F}(1)$ if and only if

$$A\vec{a} = 0 \tag{4.1}$$

for a matrix $A \in \mathbb{C}^{g \times g+n}$. The matrix A has the same structure of Hankel-blocks as the matrix M. We extend the matrix coordinates as follows. We define

$$P^{ext} := P = \{(i,j) : 1 \leq j \leq n-1, 1 \leq i \leq 2j-1\},$$
$$Q^{ext} := \{(s,t) : 1 \leq s \leq n-1, 0 \leq t \leq n-1\} \cup \{(0,0)\}.$$

Proposition 13. *The entry $\kappa_{(i,j),(s,t)} \in \mathbb{C}$ of the matrix A at the coordinate $((i,j),(s,t)) \in P^{ext} \times Q^{ext}$ is*

$$\kappa_{(i,j),(s,t)} = d_{i-s,j-t}.$$

In particular if either $1 \leq i-s \leq 2(j-t)-1$ and $1 \leq j-t \leq n-1$ or $i-s=0$ and $j-t=0$, then the entry $\kappa_{(i,j),(s,t)}$ is possibly non-zero. Every other entry is zero.

Note that A is essentially obtained by the matrix M with n new columns with column-index $(n-1,i)$, $i \in \{0,...,n-1\}$, since we only extended the index set Q to Q^{ext} and by proposition 13.

Lemma 3. *If the invertible sheaf $\mathcal{F} \in Jac^{g-1}(C_n)$ does not lie in the theta divisor, then the matrix A has full rank, i.e. $rk(A) = g$. Moreover we have*

$$\check{\mathrm{H}}^0(C_n, \mathcal{F}(1)) \cong Ker(A) \cong \mathbb{C}^n.$$

Proof. The matrix A has the $g \times g$ matrix M as a submatrix. With our condition of a non-vanishing theta function this submatrix is invertible and therefore we can find g linear independent rows in the matrix A. We conclude that A is surjective and the image of A is g-dimensional. By the rank-nulity theorem [Fis05] we know, that the kernel is $(g+n)-g = n$-dimensional as a $\mathbb{C}$-vector space. □

Now we want to describe a basis of the vector space $\check{\mathrm{H}}^0(C_n, \mathcal{F}(1))$ by using M^{-1}. First we split up the equation $A\vec{a} = 0$. Let C be the matrix obtained by replacing all elements in the $(n-1,i)$-column of A with a 0. Let B be the matrix obtained by replacing all elements in the other columns of A with a 0. We have $A = C + B \in$

$\mathbb{C}^{g\times g+n}$ and equation (4.1) gets

$$A\vec{a} = C\vec{a} + B\vec{a} = 0.$$

Let us define the vector $\vec{\tau} \in \mathbb{C}^g$ by
$\vec{\tau} :=$

$$\left(a_{10}, \dots, a_{(n-1)0}, a_{11}, \dots, a_{n-2,1}, a_{12}, \dots, a_{n-2,2}, \dots, a_{1,n-1}, \dots, a_{n-2,n-1}\right)^T.$$

Because the zeroes in the matrix C we have then

$$M\vec{\tau} = C\vec{a} = -B\vec{a}.$$

But the matrix M is invertible and so we get

$$\vec{\tau} = -M^{-1}B\vec{a}. \tag{4.2}$$

The matrix B is non-zero at the column-coordinate $(n-1, i)$ and everywhere else 0. This means the vector $\vec{b} = \left(b_{(i,j)}\right)_{(i,j)\in P} := B\vec{a} \in \mathbb{C}^g$ is given by

$$b_{(i,j)} = \sum_{\mu=0}^{n-1} \kappa_{(i,j),(n-1,\mu)} a_{0\mu}.$$

Equation (4.2) describes the dependency of $\vec{\tau}$ in terms of the variables a_{0l} for $l \in \{0, \dots, n-1\}$ and the sheaf $\mathcal{F}$. We see, that the complex vector space of global sections $\check{\mathrm{H}}^0(C_n, \mathcal{F}(1))$ is n-dimensional by considering the a_{0l} as our free variables of the global sections (s_1, s_0). Because $\kappa_{(i,j),(n-1,\mu)} = d_{i-(n-1),j-\mu}$ the index i has to be bigger or equal then $n-1$. Hence we have $n-1 \le i \le 2j-1$ and it follows $\lfloor \frac{n}{2} \rfloor \le j$. For every j smaller then $\lfloor \frac{n}{2} \rfloor$ the element $b_{(i,j)}$ is zero and therefore we have

$$b_{(i,j)} = \begin{cases} \sum_{\mu=0}^{n-1} d_{i-(n-1),j-\mu} a_{0\mu}, & \lfloor \frac{n}{2} \rfloor \le j, \\ 0, & \lfloor \frac{n}{2} \rfloor > j. \end{cases}$$

By theorem 6 with $l \in \{0, \dots, n-1\}$ and $k \in \{1, \dots, 2l-1\}$ we have

$$a_{kl} = -\frac{1}{\theta} \sum_{\substack{(i,j)\in P \\ \lfloor \frac{n}{2} \rfloor \le j}} C_{((i,j),(n-1-k,l))} b_{(i,j)}.$$

This formula describes every coefficient of a global section in terms of the free variables a_{0l}. Now we describe a basis of $\check{\mathrm{H}}^0(C_n, \mathcal{F}(1))$.

Theorem 7. *For a $l_0 \in \{1, ..., n\}$ we fix the free varibales*

$$a^{l_0}_{0,l_0-1} = 1 \textit{ and } a^{l_0}_{0l} = 0 \textit{ for all } l \neq l_0 - 1.$$

Then we have

$$b^{l_0}_{(i,j)} = \kappa_{(i,j),(n-1,l_0-1)} a^{l_0}_{0,l_0-1} = d_{i-(n-1),j-(l_0-1)}.$$

Furthermore let $(r^{l_0}_1, r^{l_0}_0)$ be the global section induced by the coefficients

$$a^{l_0}_{kl} := -\frac{1}{\theta} \sum_{\substack{(i,j)\in P \\ \lfloor \frac{n}{2} \rfloor \le j}} C_{((i,j),(n-1-k,l))} d_{i-(n-1),j-(l_0-1)}, \qquad k \neq 0,$$

then the n global sections $(r_1^{l_0}, r_0^{l_0})$ form a basis of the $\mathbb{C}$-vector space $\check{\mathrm{H}}^0(C_n, \mathcal{F}(1))$.

Proof. This follows by the previous calculations. □

Remark 1. *By construction of the basis of theorem 7 we have the following property. If we evaluate the section $r_0^{l_0}$ at the point $\zeta_0 = 0$, then we get*

$$ev(0)(r_0^{l_0}) = \eta^{l_0 - 1}.$$

Formally speaking it is non-zero in the $(l_0 - 1)$-th formal neighborhood and zero in every other formal neighborhood over the point $\{0\} \subset U_0$.

4.1.3 Evaluation Map

In the last section we saw that $\check{\mathrm{H}}^0\left(C_n, \mathcal{F}(1)\right)$ is n-dimensional as is the $\mathbb{C}$-vector space $\mathbb{C}[\eta]/\langle \eta^n \rangle$. We want to make a connection between these two spaces, such that we can represent the multiplication by η by an endomorphism.

Let $p \in C_n$ be an arbitrary point in our nilpotent spectral curve and $\pi_{C_n}(p) \in \mathbb{CP}^1$ its corresponding point in the complex projective space. We want to evaluate a global section $s \in \check{\mathrm{H}}^0\left(C_n, \mathcal{F}(1)\right)$ of an invertible sheaf $\mathcal{F} \in Jac^{g-1}(C_n) \setminus \Theta$ at $\pi_{C_n}(p)$.

By equation (3.1) the stalk of any invertible sheaf is given by

$$\mathcal{O}_{C_n,p} \cong \mathcal{O}_{\mathbb{CP}^1,p}[\eta]/_{\eta^n \mathcal{O}_{\mathbb{CP}^1,p}} \cong (\mathcal{O}_{\mathbb{CP}^1,p} \otimes \mathbb{C}[\eta]/\langle \eta^n \rangle).$$

Let $\mathfrak{m}\left(\mathcal{O}_{\mathbb{CP}^1,p}\right)^e$ be the extended maximal ideal of the stalk $\mathcal{O}_{\mathbb{CP}^1,p}$ into $\mathcal{O}_{C_n,p} \cong \mathcal{O}_{\mathbb{CP}^1,p} \otimes \mathbb{C}[\eta]/_{\langle \eta^n \rangle}$. With $D_p := \{p\}$ and the $\mathbb{C}$-algebra $\mathcal{O}_{D_p}(D_p) := \mathcal{O}_{C_n,p}/\mathfrak{m}\left(\mathcal{O}_{\mathbb{CP}^1,p}\right)^e$ we get a 0-dimensional, complex analytic space

$$\left(D_p, \mathcal{O}_{D_p}\right).$$

We set here $\mathcal{O}_{D_p}(\emptyset) = 0$. By natural isomorphisms of $\mathbb{C}$-algebras, see [GW10], we get

$$\mathcal{O}_{D_p}(D_p) = \mathcal{O}_{C_n,p}/\mathfrak{m}\left(\mathcal{O}_{\mathbb{CP}^1,p}\right)^e \cong \overbrace{\left(\mathcal{O}_{\mathbb{CP}^1,p}/\mathfrak{m}(\mathcal{O}_{\mathbb{CP}^1,p})\right)}^{\cong \mathbb{C}} \otimes (\mathbb{C}[\eta]/\langle \eta^n \rangle) \cong \mathbb{C}[\eta]/\langle \eta^n \rangle) \cong \mathbb{C}^n.$$

If we take now an invertible sheaf $\mathcal{F} \in Jac^{g-1}(C_n)$, then the stalks $\mathcal{F}(1)_p \cong \mathcal{O}_{C_n,p}$ are isomorphic and we get maps

$$\begin{aligned} \check{\mathrm{H}}^0\left(C_n, \mathcal{F}(1)\right) &\longrightarrow \mathcal{F}(1)_p && \longrightarrow \mathcal{O}_{C_n,p}/\mathfrak{m}\left(\mathcal{O}_{\mathbb{CP}^1,p}\right)^e \cong \mathbb{C}[\eta]/_{\langle \eta^n \rangle} \\ s &\longmapsto s_p = [U_p, res^{C_n}_{U_p}(s)] && \longmapsto [s_p] := [U_p, res^{C_n}_{U_p}(s)] + \mathfrak{m}(\mathcal{O}_{\mathbb{CP}^1,p})^e. \end{aligned}$$

We denote the composition of these two maps by $ev(p)$, i.e. $ev(p)(s) = [s_p]$ and call it the evaluation map at $p \in C_n$. If we take a point $p \in U_0$ and a global section s of $\mathcal{F}(1)$ then it induces an element of the stalk $\mathcal{F}(1)_p$ via $[U_p, res^{C_n}_{U_p}(s)]$, where U_p is an open neighborhood of p in C_n. By choosing U_p small enough such that $U_p \subseteq U_0$ and writing the global section as a pair $s = (s_1, s_0)$ we see that $res^{U_1}_{U_p \cap U_1}(s_1)$ is uniquely determined by $res^{U_0}_{U_p \cap U_0}(s_0)$ via $s_1(\frac{1}{\zeta}, \frac{\eta}{\zeta^2}) := \frac{1}{\zeta^{n-1}} g_{10}(\zeta, \eta) s_0(\zeta, \eta)$. In particular with $p = (\zeta_0, 0) \in U_0$ the evaluation map $ev(p)(s)$ is uniquely determined by $s_0(\zeta_0, \eta) \in$

$\mathbb{C}[\eta]/\langle\eta^n\rangle$. In this case we can write

$$ev(p)(s) = [s_p] = s_0(\zeta_0, \eta) = \sum_{l=0}^{n-1} s_0^l(\zeta_0)\eta^l.$$

If $p = (0,0) \in U_1$, then the last equation works if we replace s_0 with s_1.

We call a point $p \in C_n$ a common zero of a global section s if $ev(p)(s) = 0$. A point $p = (\zeta_0, 0) \in U_0$ is a common zero if all the $s_0^l(\zeta_0)$ vanish simultaneously for all $l \in \{0, \ldots, n-1\}$.

Theorem 8. *Let $\mathcal{F}$ be an invertible sheaf in $Jac^{g-1}(C_n)$. Then the sheaf $\mathcal{F}(1)$ has a non-zero, global section $s \in \check{\mathrm{H}}^0(C_n, \mathcal{F}(1))$ with a common zero $p \in C_n$ if and only if the invertible sheaf $\mathcal{F}$ lies in the theta divisor, i.e. $\mathcal{F} \in \Theta$.*

Proof. " $\Rightarrow$ " : Let $s = (s_1, s_0)$ be such a section with a common zero $p \in C_n$. If $p = (0,0) \in U_0$, then $ev(p)(s) = 0$ implies $s_0(\zeta, \eta) = \zeta t_0(\zeta, \eta)$ for a $t_0 \in \mathcal{O}_{U_0}(U_0)$. Thus for $(\zeta, \eta) \in U_0 \cap U_1$ we have $s_1(\frac{1}{\zeta}, \frac{\eta}{\zeta^2}) = \frac{1}{\zeta^{n-1}} g_{10}(\zeta, \eta) s_0(\zeta, \eta) = \frac{1}{\zeta^{n-2}} g_{10}(\zeta, \eta) t_0(\zeta, \eta)$. In other words the pair (s_1, t_0) defines a global section of $\mathcal{F}$. Since the section s in non-zero, the section (s_1, t_0) is non-zero too. In the same way the claim follows for $p = (0,0) \in U_1$. Now let us assume $p \in U_1 \cap U_0$ and let us write $p = (\zeta_0, 0) \in U_0$ and $p = (\frac{1}{\zeta_0}, 0) \in U_1$. Because the s_0^l and s_1^l are polynomials with a common zero p we have for all $l \in \{0, \ldots, n-1\}$

$$s_0^l(\zeta) = (\zeta_0 - \zeta) t_0^l(\zeta), \qquad s_1^l\left(\frac{1}{\zeta}\right) = \left(\frac{1}{\zeta_0} - \frac{1}{\zeta}\right) t_1^l\left(\frac{1}{\zeta}\right),$$

where the t_0^l and t_1^l are polynomials with $\deg(t_i^l) = \deg(s_i^l) - 1$. Summing up over the monomials η^l we get

$$s_0(\zeta, \eta) = (\zeta_0 - \zeta) t_0(\zeta, \eta), \qquad s_1\left(\frac{1}{\zeta}, \frac{\eta}{\zeta^2}\right) = \left(\frac{1}{\zeta_0} - \frac{1}{\zeta}\right) t_1\left(\frac{1}{\zeta}, \frac{\eta}{\zeta^2}\right),$$

such that

$$\left(\frac{1}{\zeta_0} - \frac{1}{\zeta}\right) t_1\left(\frac{1}{\zeta}, \frac{\eta}{\zeta^2}\right) = \frac{1}{\zeta^{n-1}} g_{10}(\zeta, \eta)(\zeta_0 - \zeta) t_0(\zeta, \eta),$$

which is clearly satisfied for $\zeta = \zeta_0$. For $\zeta \neq \zeta_0$ we have the equivalent equation

$$t_1\left(\frac{1}{\zeta}, \frac{\eta}{\zeta^2}\right) = \frac{1}{\zeta^{n-1}} \frac{(\zeta_0 - \zeta)}{\left(\frac{1}{\zeta_0} - \frac{1}{\zeta}\right)} g_{10}(\zeta, \eta) t_0(\zeta, \eta).$$

But since $\frac{(\zeta_0 - \zeta)}{\left(\frac{1}{\zeta_0} - \frac{1}{\zeta}\right)} = -\zeta\zeta_0$ for all $\zeta \in U_0 \cap U_1 \setminus \{\zeta_0\}$ the last equality is

$$t_1\left(\frac{1}{\zeta}, \frac{\eta}{\zeta^2}\right) = \frac{1}{\zeta^{n-1}} (-\zeta\zeta_0) g_{10}(\zeta, \eta) t_0(\zeta, \eta) = \frac{1}{\zeta^{n-2}} \left(-\zeta_0 g_{10}(\zeta, \eta)\right) t_0(\zeta, \eta).$$

The pair (t_1, t_0) is indeed a global section of the invertible sheaf given by the transition function $\frac{1}{\zeta^{n-2}} \left((-\zeta_0) g_{10}(\zeta, \eta)\right)$. Since a multiplication by a non-zero constant does not change the isomorphism class of an invertible sheaf, $(t_1, t_0) \in \check{\mathrm{H}}^0(C_n, \mathcal{F})$. Since the global section (s_1, s_0) is non-trivial, the section (t_1, t_0) is a global, non-trivial section too and therefore $\mathcal{F} \in \Theta$.

" $\Leftarrow$ " : Let us consider now an invertible sheaf in the theta divisor $\mathcal{F} \in \Theta$. Then there is a non-zero, global section $(t_1, t_0) \in \check{\mathrm{H}}^0(C_n, \mathcal{F})$ satisfying

$$t_1\left(\frac{1}{\zeta}, \frac{\eta}{\zeta^2}\right) = \frac{1}{\zeta^{n-2}} g_{10}(\zeta, \eta) t_0(\zeta, \eta)$$

on U_{10}. By multiplying the transition function with a non-zero constant $(-\zeta_0)$ and using the computations in the first part of the proof, we see

$$s_0(\zeta, \eta) := (\zeta_0 - \zeta) t_0(\zeta, \eta), \qquad s_1(\tilde{\zeta}, \tilde{\zeta}) := (\tilde{\zeta}_0 - \tilde{\zeta}) t_1(\tilde{\zeta}, \tilde{\zeta})$$

is a non-zero global section of $\mathcal{F}(1)$ with a common zero $p = (\zeta_0, 0) \in U_0$. This proves the theorem. □

The theorem says if an invertible sheaf $\mathcal{F} \in Jac^{g-1}(C_n)$ does not lie in the theta divisor, then a non-trivial, global section $(s_1, s_0) \in \check{\mathrm{H}}^0(C_n, \mathcal{F}(1))$ cannot have a common zero.

Corollary 3. *Let $\mathcal{F} \in Jac^{g-1}(C_n) \setminus \Theta$ and $p \in C_n$. Then the evaluation map*

$$ev(p) : \check{\mathrm{H}}^0(C_n, \mathcal{F}(1)) \to \check{\mathrm{H}}^0\left(D_p, \mathcal{O}_{D_p}\right)$$

is an isomorphism of n-dimensional $\mathbb{C}$-vector spaces.

Proof. We have already seen, that $\check{\mathrm{H}}^0(C_n, \mathcal{F}(1))$ and $\check{\mathrm{H}}^0\left(D_p, \mathcal{O}_{D_p}\right)$ are both n- dimensional $\mathbb{C}$-vector spaces. Theorem 8 says if the image of a section s of the evaluation map $ev(p)$ is zero, i.e. p is a common zero of the section s, then it is the zero section. In other words the kernel of the evaluation map is trivial and hence the homomorphism is injective. The dimensions of the two $\mathbb{C}$-vector spaces coincide and hence $ev(p)$ is an isomorphism. □

4.1.4 Beauville Correspondence

In this section we follow [Bea90] and [AHH90]. With the holomorphic projection map $(\pi_{C_n}, \pi^{\#}_{C_n})$ of the nilpotent, spectral curve $(C_n, \mathcal{O}_{C_n})$ to the complex projective space $(\mathbb{CP}^1, \mathcal{O}_{\mathbb{CP}^1})$ we are able to consider direct image sheaves. We have already seen in proposition 10, that the direct image $\pi_{C_n,*}\mathcal{O}_{C_n}$ is a locally free sheaf of rank n on $\mathbb{CP}^1$. By the famous Birkhoff-Grothendieck-theorem, see [Hit98], this locally free sheaf is decomposible into a direct sum of invertible sheaves.

Claim 7. *The direct image sheaf $\pi_{C_n,*}\mathcal{O}_{C_n}$ of the structure sheaf $\mathcal{O}_{C_n}$, seen as an $\mathcal{O}_{\mathbb{CP}^1}$-module, is isomorphic to*

$$\mathcal{O}_{\mathbb{CP}^1} \oplus \mathcal{O}_{\mathbb{CP}^1}(-2) \oplus \cdots \oplus \mathcal{O}_{\mathbb{CP}^1}(-2(n-1)).$$

Proof. Let $W \subset \mathbb{CP}^1$ and $U := W \times \{0\} \subset C_n$. Then the $\mathcal{O}_{\mathbb{CP}^1}(W)$-module of the direct image is $\pi_{C_n,*}\mathcal{O}_{C_n}(W) = \mathcal{O}_{C_n}(\pi^{-1}_{C_n}(W)) = \mathcal{O}_{C_n}(U)$. Hence an element of $\pi_{C_n,*}\mathcal{O}_{C_n}(W)$ is given by a pair of holomorphic functions $(s_1, s_0) \in \mathcal{O}_{U_1}(U_1 \cap U) \times \mathcal{O}_{U_0}(U_0 \cap U)$ such that $(\varphi^*_{10}s_1)(\zeta, \eta) = s_1(\frac{1}{\zeta}, \frac{\eta}{\zeta^2}) = s_0(\zeta, \eta)$. We write these holomorphic functions in the form

$$s_1(\tilde{\zeta}, \tilde{\eta}) = \sum_{l=0}^{n-1} s_1^l(\tilde{\zeta})\tilde{\eta}^l, \qquad s_0(\zeta, \eta) = \sum_{l=0}^{n-1} s_0^l(\zeta)\eta^l.$$

Here the $s_1^l \in \mathcal{O}_{\mathbb{CP}^1}(W_1 \cap W)$ and $s_0^l \in \mathcal{O}_{\mathbb{CP}^1}(W_0 \cap W)$ are local functions for all $l \in \{0, \ldots, n-1\}$. By equating the coefficients and using $\varphi_{10}(\zeta, \eta) = (\frac{1}{\zeta}, \frac{\eta}{\zeta^2})$ we have

$$s_1^l\left(\frac{1}{\zeta}\right)\frac{1}{\zeta^{2l}} = s_0^l(\zeta).$$

Thus the pair (s_1^l, s_0^l) is a local section of $\mathcal{O}_{\mathbb{CP}^1}(-2l)(W)$. Hence the map

$$\begin{aligned}\chi(W) : \pi_{C_n,*}\mathcal{O}_{C_n}(W) &\longrightarrow \mathcal{O}_{\mathbb{CP}^1}(W) \oplus \mathcal{O}_{\mathbb{CP}^1}(-2)(W) \oplus \cdots \oplus \mathcal{O}_{\mathbb{CP}^1}(-2(n-1))(W)\\ (s_1, s_0) &\longmapsto \left((s_1^0, s_0^0), (s_1^1, s_0^1), \ldots, (s_1^{n-1}, s_0^{n-1})\right)\end{aligned}$$

defines is an isomorphism of $\mathcal{O}_{\mathbb{CP}^1}(W)$-modules with inverse map given by

$$\left((s_1^0, s_0^0), (s_1^1, s_0^1), \ldots, (s_1^{n-1}, s_0^{n-1})\right) \longmapsto \left(\sum_{l=0}^{n-1} s_1^l(\tilde{\zeta})\tilde{\eta}^l, \sum_{l=0}^{n-1} s_0^l(\zeta)\eta^l\right).$$

Because we have choosen an arbitrary open set W these isomorphisms define an isomorphism of $\mathcal{O}_{\mathbb{CP}^1}$-modules. □

Let $\mathcal{F} \in Jac^{g-1}(C_n)\backslash\Theta$ be an invertible sheaf of degree $g-1$ without a non-trivial, global section. The theta divisor condition says $\check{\mathrm{H}}^0(C_n, \mathcal{F}) = 0$. By the Riemann-Roch-theorem, proposition 9, we have $\check{\mathrm{H}}^1(C_n, \mathcal{F}) = 0$, since

$$\begin{aligned}-\dim_{\mathbb{C}}\left(\check{\mathrm{H}}^1(C_n, \mathcal{F})\right) &= \dim_{\mathbb{C}}\left(\check{\mathrm{H}}^0(C_n, \mathcal{F})\right) - \dim_{\mathbb{C}}\left(\check{\mathrm{H}}^1(C_n, \mathcal{F})\right)\\ &= \deg(\mathcal{F}) + 1 - g = 0.\end{aligned}$$

With proposition 10 we have

$$\check{\mathrm{H}}^0(C_n, \mathcal{F}_n) = \check{\mathrm{H}}^0(\mathbb{CP}^1, \pi_{C_n,*}\mathcal{F}) = 0, \quad \check{\mathrm{H}}^1(C_n, \mathcal{F}) = \check{\mathrm{H}}^1(\mathbb{CP}^1, \pi_{C_n,*}\mathcal{F}) = 0$$

and by corollary 1 we get

$$-deg(\pi_{C_n,*}\mathcal{F}) = rk(\pi_{C_n,*}\mathcal{F}) = n.$$

But the only locally free sheaves on $(\mathbb{CP}^1, \mathcal{O}_{\mathbb{CP}^1})$ of rank n of degree $-n$ with trivial cohomology are isomorphic to $\mathcal{O}_{\mathbb{CP}^1}(-1)^{\oplus n}$ and so we get an isomorphism of $\mathcal{O}_{\mathbb{CP}^1}$-modules

$$\xi : \pi_{C_n,*}\mathcal{F} \cong \overbrace{\mathcal{O}_{\mathbb{CP}^1}(-1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{CP}^1}(-1)}^{n-times}. \tag{4.3}$$

This means that all direct image sheaves $\pi_{C_n,*}\mathcal{F}$ of invertible sheaves $\mathcal{F}$ of degree $g-1$ not lying in the theta divisor look equivalent as $\mathcal{O}_{\mathbb{CP}^1}$-modules. But since $\mathcal{F}$ is an $\mathcal{O}_{C_n}$-module the direct image $\pi_{C_n,*}\mathcal{F}$ has an additional structure given by a $\pi_{C_n,*}\mathcal{O}_{C_n}$-module structure, i.e. a sheaf homomorphism of $\mathbb{C}$-algebras

$$\pi_{C_n,*}m : \pi_{C_n,*}\mathcal{O}_{C_n} \longrightarrow \mathcal{E}nd(\pi_{C_n,*}\mathcal{F}).$$

Here $\mathcal{E}nd$ is the sheaf hom. With the isomorphism of claim 7 the morphism $\pi_{C_n,*}m$ induces a morphism of $\mathcal{O}_{\mathbb{CP}^1}$-modules

$$\pi_{C_n,*}\tilde{m} : \mathcal{O}_{\mathbb{CP}^1} \oplus \mathcal{O}_{\mathbb{CP}^1}(-2) \oplus \cdots \oplus \mathcal{O}_{\mathbb{CP}^1}(-2(n-1)) \longrightarrow \mathcal{E}nd(\pi_{C_n,*}\mathcal{F})$$

and thus we get a morphism of $\mathcal{O}_{\mathbb{CP}^1}$-modules

$$\pi_{C_n,*}\tilde{m}|_{\mathcal{O}_{\mathbb{CP}^1}(-2)} : \mathcal{O}_{\mathbb{CP}^1}(-2) \longrightarrow \mathcal{E}nd(\pi_{C_n,*}\mathcal{F}) \cong \mathcal{E}nd(\pi_{C_n,*}\mathcal{F}(1)).$$

With the isomorphism ξ we see $\pi_{C_n,*}\mathcal{F}(1) \cong \mathcal{O}_{\mathbb{CP}^1}^{\oplus n}$ and so we have in a non-canonical way

$$\begin{aligned}\pi_{C_n,*}\tilde{m}|_{\mathcal{O}_{\mathbb{CP}^1}(-2)} \in\ & Hom\left(\mathcal{O}_{\mathbb{CP}^1}(-2), \mathcal{E}nd(\pi_{C_n,*}\mathcal{F})\right)\\ &= \check{\mathrm{H}}^0(\mathbb{CP}^1, \mathcal{H}om\left(\mathcal{O}_{\mathbb{CP}^1}(-2), \mathcal{E}nd(\pi_{C_n,*}\mathcal{F})\right))\\ &\cong \check{\mathrm{H}}^0(\mathbb{CP}^1, \mathcal{H}om\left(\pi_{C_n,*}\mathcal{F}(1), \pi_{C_n,*}\mathcal{F}(1) \otimes \mathcal{O}_{\mathbb{CP}^1}(2)\right))\\ &\cong \check{\mathrm{H}}^0(\mathbb{CP}^1, \mathcal{H}om(\mathcal{O}_{\mathbb{CP}^1}^{\oplus n}, \mathcal{O}_{\mathbb{CP}^1}^{\oplus n} \otimes \mathcal{O}_{\mathbb{CP}^1}(2)))\\ &\cong \check{\mathrm{H}}^0(\mathbb{CP}^1, \mathcal{E}nd(\mathcal{O}_{\mathbb{CP}^1}^{\oplus n})) \otimes \check{\mathrm{H}}^0(\mathbb{CP}^1, \mathcal{O}_{\mathbb{CP}^1}(2))\\ &\cong End(\mathcal{O}_{\mathbb{CP}^1}^{\oplus n}) \otimes \check{\mathrm{H}}^0(\mathbb{CP}^1, \mathcal{O}_{\mathbb{CP}^1}(2))\\ &\cong \mathfrak{gl}_n(\mathbb{C}) \otimes \check{\mathrm{H}}^0(\mathbb{CP}^1, \mathcal{O}_{\mathbb{CP}^1}(2))\\ &\cong \check{\mathrm{H}}^0(\mathbb{CP}^1, \mathfrak{gl}_n(\mathbb{C}) \otimes \mathcal{O}_{\mathbb{CP}^1}(2)).\end{aligned}$$

Hence the morphism $\pi_{C_n,*}\tilde{m}|_{\mathcal{O}_{\mathbb{CP}^1}(-2)}$ can be described as a pair of matricial polynomials of degree 2

$$\left(\tilde{A}(\tilde{\zeta}), A(\zeta)\right) \in \check{\mathrm{H}}^0\left(\mathbb{CP}^1, \mathfrak{gl}_n(\mathbb{C}) \otimes \mathcal{O}_{\mathbb{CP}^1}(2)\right).$$

The map $\pi_{C_n,*}\tilde{m}|_{\mathcal{O}_{\mathbb{CP}^1}(-2)}(\mathbb{CP}^1)$, seen as an element of

$$Hom\left(\pi_{C_n,*}\mathcal{F}(1)(\mathbb{CP}^1), \pi_{C_n,*}\mathcal{F}(3)(\mathbb{CP}^1)\right)$$

and using proposition 10, is given by its contruction by the multiplication of the global section $(\tilde{\eta}, \eta) \in \mathcal{O}_{C_n}(2)$

$$\begin{aligned}m_{(\tilde{\eta},\eta)}(C_n) : \check{\mathrm{H}}^0(C_n, \mathcal{F}(1)) &\longrightarrow \check{\mathrm{H}}^0(C_n, \mathcal{F}(3))\\ (s_1, s_0) &\longmapsto (\tilde{\eta}s_1, \eta s_0).\end{aligned}$$

Therefore the morphism $\pi_{C_n,*}\tilde{m}|_{\mathcal{O}_{\mathbb{CP}^1}(-2)}(\mathbb{CP}^1)$ satisfies

$$\left(\pi_{C_n,*}\tilde{m}|_{\mathcal{O}_{\mathbb{CP}^1}(-2)}(\mathbb{CP}^1)\right)^n = 0 \text{ and } \left(\pi_{C_n,*}\tilde{m}|_{\mathcal{O}_{\mathbb{CP}^1}(-2)}(\mathbb{CP}^1)\right)^{n-1} \neq 0$$

and so the matricial polynomial $A(\zeta)$ is nilpotent and regular for all $\zeta \in W_0$.

In other words every invertible sheaf $\mathcal{F} \in Jac^{g-1}(C_n) \setminus \Theta$ induces a global section $(\tilde{A}(\tilde{\zeta}), A(\zeta)) \in \check{\mathrm{H}}^0\left(\mathbb{CP}^1, \mathfrak{gl}_n(\mathbb{C}) \otimes \mathcal{O}_{\mathbb{CP}^1}(2)\right)$, such that $A(\zeta) = A_0 + A_1\zeta + A_2\zeta^2$ is a regular, nilpotent, matricial polynomial of degree 2. Because W_0 is dense in $\mathbb{CP}^1$ the matricial polynomial $A(\zeta)$ already defines the global section $(\tilde{A}(\tilde{\zeta}), A(\zeta))$ completely. The matricial polynomial depends on the choice of the isomorphism ξ and thus the regular, nilpotent, matricial polynomial $A(\zeta)$ corresponding to a sheaf $\mathcal{F} \in Jac^{g-1}(C_n) \setminus \Theta$ is only unique up to $GL_n(\mathbb{C})$-conjugations, i.e. conjugation by automorphisms of $\mathcal{O}_{\mathbb{CP}^1}(-1)^{\oplus n}$.

Beauville showed in [Bea90] the following central theorem.

Theorem 9. *(Beauville Correspondence) There is a bijection between* $Jac^{g-1}(C_n)\setminus\Theta$ *and the set of* $GL_n(\mathbb{C})$*-conjugation classes of regular, nilpotent, matricial polynomials*

$$\left(\tilde{A}(\tilde{\zeta}), A(\zeta)\right) \in \check{\mathrm{H}}^0\left(\mathbb{CP}^1, \mathfrak{gl}_n(\mathbb{C}) \otimes \mathcal{O}_{\mathbb{CP}^1}(2)\right).$$

We have seen above how we get a conjugation class of regular, nilpotent, matricial polynomials from an invertible sheaf of degree $g-1$ not lying in the theta divisor. The inverse map is given by mapping a regular, nilpotent, matricial polynomial to the invertible sheaf $\mathcal{G}(-1)|_{C_n}$. The sheaf $\mathcal{G}$ is a sheaf on T with support on C_n given by the cokernel sheaf of the short exact sequence

$$0 \to \mathcal{O}_T(-2)^{\oplus n} \overset{\eta \cdot Id_n - A(\zeta)}{\longrightarrow} \mathcal{O}_T^{\oplus n} \longrightarrow \mathcal{G} \to 0.$$

For more details and a proof of invertibility of $\mathcal{G}(-1)|_{C_n}$ as $\mathcal{O}_{C_n}$-module see [Bea90] or [AHH90].

Now we want to make use of the evaluation map. The global section $(\tilde{\eta}, \eta) \in \mathcal{O}_{C_n}(2)(C_n)$ induces a morphism by multiplication

$$m_{(\tilde{\eta},\eta)}(U) : \mathcal{F}(U) \longrightarrow \mathcal{F}(2)(U), \qquad (s_1, s_0) \longmapsto (\tilde{\eta}s_1, \eta s_0).$$

This morphism defines more morphisms given by

$$m_{(\tilde{\eta},\eta),p} : \mathcal{O}_{C_n,p} \cong \mathcal{F}_p \longrightarrow \mathcal{F}_p \cong \mathcal{O}_{C_n,p}, \qquad s_p = [U_p, s] \longmapsto [U_p, m_{(\tilde{\eta},\eta)}s]$$

and

$$[m_{(\tilde{\eta},\eta),p}] : \mathcal{O}_{D_p}(D_p) \longrightarrow \mathcal{O}_{D_p}(D_p)$$
$$[s_p] = [U_p, s] + \mathfrak{m}(\mathcal{O}_{\mathbb{CP}^1})^e \longmapsto [U_p, m_{(\tilde{\eta},\eta)}s] + \mathfrak{m}(\mathcal{O}_{\mathbb{CP}^1})^e.$$

Then, just by definition, we have

$$[(m_{(\tilde{\eta},\eta)}s)_p] = [m_{(\tilde{\eta},\eta),p}][s_p]$$

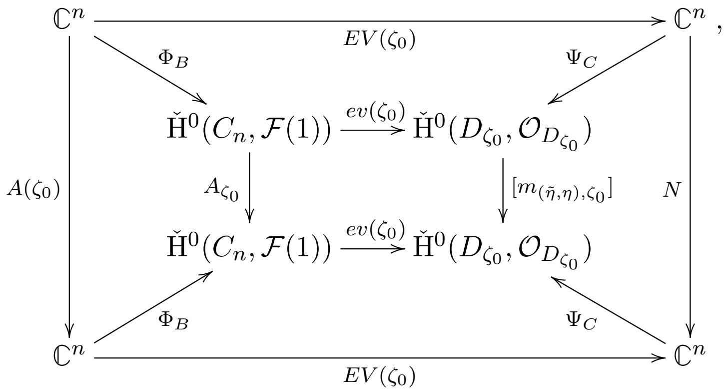
and so we have a commutative diagram

$$\begin{array}{ccc} \check{\mathrm{H}}^0(C_n, \mathcal{F}(1)) & \xrightarrow{ev(p)} & \check{\mathrm{H}}^0(D_p, \mathcal{O}_{D_p}) \\ {\scriptstyle m_{(\tilde{\eta},\eta)}}\downarrow & & \downarrow{\scriptstyle [m_{(\tilde{\eta},\eta),p}]} \\ \check{\mathrm{H}}^0(C_n, \mathcal{F}(3)) & \xrightarrow{ev(p)} & \check{\mathrm{H}}^0(D_p, \mathcal{O}_{D_p}). \end{array}$$

The evaluation map in the bottom row is not an isomorphism. A choosen isomorphism ξ 4.3 induces an isomorphism $\check{\mathrm{H}}^0(C_n, \mathcal{F}(3)) \cong \check{\mathrm{H}}^0(C_n, \mathcal{F}(1)) \otimes \check{\mathrm{H}}^0(\mathbb{CP}^1, \mathcal{O}_{\mathbb{CP}^1}(2))$. Thus the morphism $m_{(\tilde{\eta},\eta)}$ induces a map $(\tilde{A}^{end}, A^{end}) \in End(\check{\mathrm{H}}^0(C_n, \mathcal{F}(1))) \otimes \check{\mathrm{H}}^0(\mathbb{CP}^1, \mathcal{O}_{\mathbb{CP}^1}(2))$. By writing $(\tilde{A}^{end}, A^{end})$ in the form

$$\tilde{A}^{end}(\tilde{\zeta}) = \tilde{A}_0^{end} + \tilde{A}_1^{end}\tilde{\zeta} + \tilde{A}_2^{end}\tilde{\zeta}^2, \qquad A^{end}(\zeta) = A_0^{end} + A_1^{end}\zeta + A_2^{end}\zeta^2$$

we get for every $p = (\zeta_0, 0) \in U_0$ an endomorphism $(\tilde{A}_{\tilde{\zeta}_0}, A_{\zeta_0}) \in End(\check{\mathrm{H}}^0(C_n, \mathcal{F}(1)))$. By choosing now a basis of $\check{\mathrm{H}}^0(C_n, \mathcal{F}(1))$ then the matrix corresponding to $(\tilde{A}_{\tilde{\zeta}_0}, A_{\zeta_0})$ is just the regular, nilpotent, matrical polynomial at the point ζ_0. In other words we have a commutative diagram



where Ψ_C is the coordinate function coming from the standard basis $\{1, \ldots, \eta^{n-1}\}$ of $\check{\mathrm{H}}^0(D_p, \mathcal{O}_{D_p})$ and Φ_B is the coordinate function of any choice of a basis of $\check{\mathrm{H}}^0(C_n, \mathcal{F}(1))$. The map N is just the Jordan canonical form of a nilpotent matrix with only one Jordan block as a lower-triangular matrix.

We will use this diagram and the basis of theorem 7 in the next sections to compute $A(\zeta_0)$ for every $\zeta_0 \in W_0$ explicitely. Moreover the commutativity of this diagram reflects the fact, that multiplication by η evaluated at ζ_0 can be seen as a multiplication of an Eigenvalue of $A(\zeta_0)$.

4.1.5 Flows

Elements of the Kronheimer moduli space induce linear flows on the jacobian in the direction of the invertible sheaf $\mathcal{L}^t \in Jac^0(C_n)$ with transition function $\exp(t\frac{\eta}{\zeta})$ but with varying starting points $\mathcal{F} \in Jac^{g-1}(C_n)$ at $t = 0$, see [Hit83], [SC97] and [HM89]. Thus we are interested in invertible sheaves of the form $\mathcal{F} \otimes \mathcal{L}^t \in Jac^{g-1}(C_n)$ for $t \in \mathbb{C}$. In the next theorem and the rest of this thesis we have taken $-t$, but since $t \in \mathbb{C}$ it has no influence. This choice of the sign will give us solutions of Nahm's equations on $[0, \infty)$ instead of $(-\infty, 0]$ in chapter 5.

Theorem 10. *Let $\mathcal{F} \in Jac^{g-1}(C_n)$ be an invertible sheaf characterized by its transition function $\frac{1}{\zeta^{n-2}} g_{10}(\zeta, \eta)$ and its coefficients $d_{kl} \in \mathbb{C}$. Furthermore let $\mathcal{L}^t \in Jac^0(C_n)$ be the invertible sheaf with transition function $\exp(-t\frac{\eta}{\zeta})$ for every $t \in \mathbb{C}$ and let $\mathcal{F}^t := \mathcal{F} \otimes \mathcal{L}^t \in Jac^{g-1}(C_n)$. Then the coefficients of the transition function of the invertible sheaf $\mathcal{F}^t$ are given by*

$$d_{kl}(t) := \sum_{j=0}^{k} (-1)^j d_{k-j,l-j} \frac{t^j}{j!} \in \mathbb{C}.$$

Proof. If we consider the truncated powerseries of $\exp\left(-t\frac{\eta}{\zeta}\right)$, i.e.

$$1 - t\frac{\eta}{\zeta} + \frac{t^2}{2}\frac{\eta^2}{\zeta^2} - \frac{t^3}{3!}\frac{\eta^3}{\zeta^3} + \cdots \pm \frac{t^{n-1}}{(n-1)!}\frac{\eta^{n-1}}{\zeta^{n-1}}.$$

and multiply it with the transition function of $\mathcal{F}(1)$ we get the result. More precisely , if we consider the formula of the multiplication of polynomials $((a_i) * (b_j))_l = \sum_{i+j=l} a_i b_j$, for a fixed η^l we get $\sum_{i+j=l}\left(\sum_{k=1}^{2i-1} d_{ki} \frac{(-t)^j}{j!} \frac{1}{\zeta^{k+j}}\right)$. Now we fix a $\frac{1}{\zeta^m}$, i.e. we consider $\frac{\eta^l}{\zeta^m}$. The summation index j goes from 0 to l, i goes from l to 0 and

$k = m - j$ goes from m to 0. In other words we get

$$d_{ml}(t) = d_{ml} + d_{m-1,l-1}\frac{(-t)^1}{1!} + d_{m-2,l-2}\frac{(-t)^2}{2!} + \cdots + d_{0,l-m}\frac{(-t)^m}{m!}.$$

□

4.2 Hitchin's Formula in the case $n = 3$

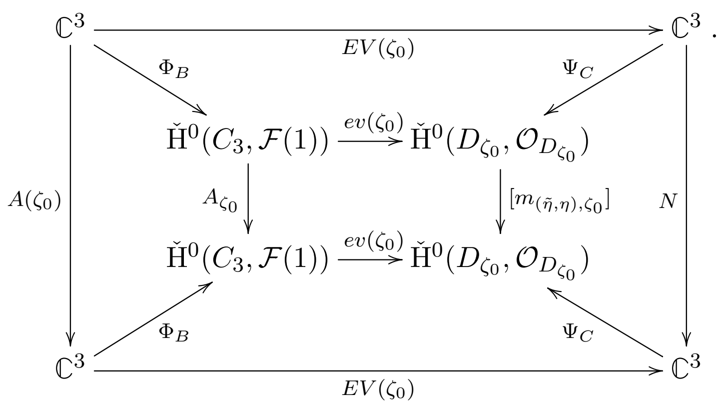
Hitchin studied in [Hit98] the integrand of the Kähler potential for a description of the (Hyper-) Kähler metric in terms of invertible sheaves in the Jacobian of its corresponding smooth spectral curve. We will call the analogous formula for the nilpotent, spectral curve *Hitchin's* formula.

Theorem 11. *(Hitchin's formula, n=3) Let $\mathcal{F} \in Jac^3(C_3) \setminus \Theta$ an invertible sheaf of degree 3 $(= g-1)$ on the nilpotent, spectral curve C_3. Let $t \in \mathbb{C}$ and let $\mathcal{L}^t \in Jac^0(C_3)$ be the invertible sheaf of degree 0 with transition function $\exp\left(-t\frac{\eta}{\zeta}\right)$. Let $\mathcal{F}^t := \mathcal{F} \otimes \mathcal{L}^t \in Jac^3(C_3)$ and let $A(\zeta, t) = A_0(t) + A_1(t)\zeta + A_2(t)\zeta^2$ be a representative of the $GL_3(\mathbb{C})$-conjugation class of regular, nilpotent, matricial polynomials corresponding to the invertible sheaf $\mathcal{F}^t$. Then we have the equation*

$$tr\left(A_0(t)A_2(t) - \frac{1}{4}A_1(t)^2\right) = \frac{3}{2}\frac{\theta''(\mathcal{F}^t)\theta(\mathcal{F}^t) - \theta'(\mathcal{F}^t)\theta'(\mathcal{F}^t)}{\theta(\mathcal{F}^t)^2} = \frac{3}{2}\frac{d^2}{dt^2}\log\left(\theta(\mathcal{F}^t)\right),$$

for all $t \in \mathbb{C}$ whereever $\mathcal{F}^t \notin \Theta$.

In order to prove this theorem directly we want to compute the left hand side in terms of the coefficients of the transition functions. Since the trace is conjugation invariant, the term $tr\left(A_0(t)A_2(t) - \frac{1}{4}A_1(t)^2\right)$ is independent of the choice of representative of the conjugation class. We will describe such a representative, a regular, nilpotent, matricial polynomial $A(\zeta)$ coming from a global section $\left(\tilde{A}(\tilde{\zeta}), A(\zeta)\right) \in \check{\mathrm{H}}^0\left(\mathbb{CP}^1, \mathfrak{gl}_n(\mathbb{C}) \otimes \mathcal{O}_{\mathbb{CP}^1}(2)\right)$, in terms of coefficients of the transition function of an invertible sheaf $\mathcal{F} \in Jac^3(C_3)\setminus\Theta$. To compute the matricial polynomial $A(\zeta)$, $\zeta \in W_0$, we will fix a $\zeta_0 \in W_0$ and consider the following diagram of the Beauville correspondence



Because we consider $n = 3$, we have $\dim_{\mathbb{C}}\left(\check{\mathrm{H}}^0\left(C_3, \mathcal{F}(1)\right)\right) = 3$. If we denote a basis of $\check{\mathrm{H}}^0\left(C_3, \mathcal{F}(1)\right)$ by $B = \left\{r^1, r^2, r^3\right\}$, then the map Φ_B is the coordinate function of the $\mathbb{C}$-vector space $\check{\mathrm{H}}^0(C_3, \mathcal{F}(1))$ with respect to the basis B, i.e.

$$\Phi_B : \mathbb{C}^3 \longrightarrow \check{\mathrm{H}}^0\left(C_3, \mathcal{F}(1)\right), \quad (x_1, x_2, x_3) \longmapsto x_1 r^1 + x_2 r^2 + x_3 r^3$$

and Ψ_C is the coordinate function of the $\mathbb{C}$-vector space $\frac{\mathbb{C}[\eta]}{\langle \eta^3 \rangle}$ with respect to the basis $C := \left\{1, \eta, \eta^2\right\}$. The multiplication with η, the map $[m_{(\tilde{\eta},\eta),\zeta_0}]$, is of course a linear map and induces a matrix $N := \Psi_C^{-1} \circ [m_{(\tilde{\eta},\eta),\zeta_0}] \circ \Psi_C$, which is just the Jordan canonical form with exactly one Jordan block

$$N := \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Let us denote the transformation matrix of the evaluation map $ev(\zeta_0)$ with $Ev(\zeta_0)$ with respect to the bases B and C. The regular, nilpotent, matricial polynomial at $\zeta_0 \in W_0$ is then given by

$$\begin{aligned} A(\zeta_0) &= \Phi_B^{-1} \circ A_{\zeta_0} \circ \Phi_B \\ &= \Phi_B^{-1} \circ ev(\zeta_0)^{-1} \circ [m_{(\tilde{\eta},\eta),\zeta_0}] \circ ev(\zeta_0) \circ \Phi_B \\ &= \Phi_B^{-1} \circ ev(\zeta_0)^{-1} \circ \Psi_C \circ N \circ \Psi_C^{-1} \circ ev(\zeta_0) \circ \Phi_B \\ &= EV(\zeta_0)^{-1} N EV(\zeta_0). \end{aligned}$$

Thus we want to compute the matrix $EV(\zeta_0)$ and its inverse.

4.2.1 A Basis of $\check{\mathrm{H}}^0\left(C_3, \mathcal{F}(1)\right)$

To compute $EV(\zeta_0)$ we use the basis of theorem 7 to get the coordinate function Φ_B. Since the basis is given by coefficients a^i_{kl}, $i \in \{1, 2, 3\}$, consisting of cofactors of the matrix M of section 3.3.2, we will compute in a first step these cofactors. The index sets of the matrix M in $n = 3$ are

$$P = \{(1,1), (1,2), (2,2), (3,2)\}, \quad Q = \{(1,0), (0,0), (1,1), (1,2)\}$$

and we have the following

Lemma 4. *The cofactors of the matrix M corresponding to an invertible sheaf $\mathcal{F} \in Jac^3(C_3) \setminus \Theta$ are given by*

$$\begin{aligned} C_{((2,2),(1,0))} &= d_{32} d_{00}^2, & C_{((3,2),(1,0))} &= -(d_{00}^2 d_{22} - d_{00} d_{11}^2), \\ C_{((2,2),(0,0))} &= -d_{00}^2 d_{22}, & C_{((3,2),(0,0))} &= d_{00}^2 d_{12}, \\ C_{((2,2),(1,1))} &= d_{22} d_{11} d_{00}, & C_{((3,2),(1,1))} &= -d_{12} d_{11} d_{00}, \\ C_{((2,2),(1,2))} &= d_{00} d_{12} d_{22}, & C_{((3,2),(1,2))} &= -d_{00} d_{12}^2. \end{aligned}$$

Proof. This follows immediatly by direct computations. Let us recall the matrix M is given by

$$M(\mathcal{F}) = \begin{pmatrix} 0 & d_{11} & d_{00} & 0 \\ 0 & d_{12} & 0 & d_{00} \\ d_{12} & d_{22} & d_{11} & 0 \\ d_{22} & d_{32} & 0 & 0 \end{pmatrix}.$$

We just have to be careful with the correct signs. □

With $\lfloor \frac{n}{2} \rfloor = \lfloor \frac{3}{2} \rfloor = 2$, by theorem 7 and lemma 4 we can write down the coefficients of the basis vectors. The first basis vector is given by $\left(a^1_{00}, a^1_{01}, a^1_{02}\right) = (1, 0, 0)$, $\vec{b}^1 = (0, 0, 0, d_{12})$ and

$$\begin{aligned}
a^1_{10} &= -\frac{1}{\theta} C_{((3,2),(1,0))} d_{12} = -\frac{(d_{00}d_{11}^2 - d_{00}^2 d_{22})d_{12}}{\theta}, \\
a^1_{20} &= -\frac{1}{\theta} C_{((3,2),(0,0))} d_{12} = -\frac{d_{00}^2 d_{12}^2}{\theta}, \\
a^1_{11} &= -\frac{1}{\theta} C_{((3,2),(1,1))} d_{12} = \frac{d_{00}d_{11}d_{12}^2}{\theta}, \\
a^1_{12} &= -\frac{1}{\theta} C_{((3,2),(1,2))} d_{12} = \frac{d_{00}d_{12}^3}{\theta}.
\end{aligned}$$

The second basis vector is given by $\left(a^2_{00}, a^2_{01}, a^2_{02}\right) = (0, 1, 0)$, $\vec{b}^2 = (0, 0, 0, d_{11})$ and

$$\begin{aligned}
a^2_{10} &= -\frac{1}{\theta} C_{((3,2),(1,0))} d_{11} = -\frac{(d_{00}d_{11}^2 - d_{00}^2 d_{22})d_{11}}{\theta}, \\
a^2_{20} &= -\frac{1}{\theta} C_{((3,2),(0,0))} d_{11} = -\frac{d_{00}^2 d_{11} d_{12}}{\theta}, \\
a^2_{11} &= -\frac{1}{\theta} C_{((3,2),(1,1))} d_{11} = \frac{d_{00}d_{11}^2 d_{12}}{\theta}, \\
a^2_{12} &= -\frac{1}{\theta} C_{((3,2),(1,2))} d_{11} = \frac{d_{00}d_{11}d_{12}^2}{\theta}.
\end{aligned}$$

The third basis vector is given by $\left(a^3_{00}, a^3_{01}, a^3_{02}\right) = (0, 0, 1)$, $\vec{b}^3 = (0, 0, d_{00}, 0)$ and

$$\begin{aligned}
a^3_{10} &= -\frac{1}{\theta} C_{((2,2),(1,0))} d_{00} = -\frac{d_{00}^3 d_{32}}{\theta}, \\
a^3_{20} &= -\frac{1}{\theta} C_{((2,2),(0,0))} d_{00} = \frac{d_{00}^3 d_{22}}{\theta}, \\
a^3_{11} &= -\frac{1}{\theta} C_{((2,2),(1,1))} d_{00} = -\frac{d_{00}^2 d_{11} d_{22}}{\theta}, \\
a^3_{12} &= -\frac{1}{\theta} C_{((2,2),(1,2))} d_{00} = -\frac{d_{00}^2 d_{12} d_{22}}{\theta}.
\end{aligned}$$

With these computations we have a basis of $\check{\mathrm{H}}^0\left(C_3, \mathcal{F}(1)\right)$ given by the polynomials

$$\begin{aligned}
r_0^1(\zeta,\eta) &:= a_{00}^1 + a_{10}^1\zeta + a_{20}^1\zeta^2 + a_{01}^1\eta + a_{11}^1\zeta\eta + a_{02}^1\eta^2 + a_{12}^1\zeta\eta^2 \\
&= 1 - \frac{1}{\theta}C_{((3,2),(1,0))}d_{12}\zeta - \frac{1}{\theta}C_{((3,2),(0,0))}d_{12}\zeta^2 \\
&\quad - \frac{1}{\theta}C_{((3,2),(1,1))}d_{12}\zeta\eta - \frac{1}{\theta}C_{((3,2),(1,2))}d_{12}\zeta\eta^2, \\
&= 1 - \frac{(d_{00}d_{11}^2 - d_{00}^2 d_{22})d_{12}}{\theta}\zeta - \frac{d_{00}^2 d_{12}^2}{\theta}\zeta^2 \\
&\quad + \frac{d_{00}d_{11}d_{12}^2}{\theta}\zeta\eta + \frac{d_{00}d_{12}^3}{\theta}\zeta\eta^2 \\
r_0^2(\zeta,\eta) &:= a_{00}^2 + a_{10}^2\zeta + a_{20}^2\zeta^2 + a_{01}^2\eta + a_{11}^2\zeta\eta + a_{02}^2\eta^2 + a_{12}^2\zeta\eta^2 \\
&= -\frac{1}{\theta}C_{((3,2),(1,0))}d_{11}\zeta - \frac{1}{\theta}C_{((3,2),(0,0))}d_{11}\zeta^2 \\
&\quad + \eta - \frac{1}{\theta}C_{((3,2),(1,1))}d_{11}\zeta\eta - \frac{1}{\theta}C_{((3,2),(1,2))}d_{11}\zeta\eta^2, \\
&= -\frac{(d_{00}d_{11}^2 - d_{00}^2 d_{22})d_{11}}{\theta}\zeta - \frac{d_{00}^2 d_{11}d_{12}}{\theta}\zeta^2 \\
&\quad + \eta + \frac{d_{00}d_{11}^2 d_{12}}{\theta}\zeta\eta + \frac{d_{00}d_{11}d_{12}^2}{\theta}\zeta\eta^2 \\
r_0^3(\zeta,\eta) &:= a_{00}^3 + a_{10}^3\zeta + a_{20}^3\zeta^2 + a_{01}^3\eta + a_{11}^3\zeta\eta + a_{02}^3\eta^2 + a_{12}^3\zeta\eta^2 \\
&= -\frac{1}{\theta}C_{((2,2),(1,0))}d_{00}\zeta - \frac{1}{\theta}C_{((2,2),(0,0))}d_{00}\zeta^2 \\
&\quad - \frac{1}{\theta}C_{((2,2),(1,1))}d_{00}\zeta\eta + \eta^2 - \frac{1}{\theta}C_{((2,2),(1,2))}d_{00}\zeta\eta^2 \\
&= -\frac{d_{00}^3 d_{32}}{\theta}\zeta + \frac{d_{00}^3 d_{22}}{\theta}\zeta^2 \\
&\quad - \frac{d_{00}^2 d_{11}d_{22}}{\theta}\zeta\eta + \eta^2 - \frac{d_{00}^2 d_{12}d_{22}}{\theta}\zeta\eta^2.
\end{aligned}$$

Here $r_0^i \in \mathcal{O}_{U_0}(U_0)$ and $r_1^i \in \mathcal{O}_{U_1}(U_1)$ is uniquely determined by r_0^i to get a global section $(r_1^i, r_0^i) \in \check{\mathrm{H}}^0(C_n, \mathcal{F}(1))$, $i \in \{1,2,3\}$.

4.2.2 Inverting the Evaluation Map

In this subsection we want to compute $EV(\zeta_0)$ and its inverse for a fixed $(\zeta_0, 0) \in U_0$. Because U_0 is dense in C_n and a global section $(s_1, s_0) \in \check{\mathrm{H}}^0\left(C_3, \mathcal{F}(1)\right)$ is completely determined by s_0 we will often call s_0 already a global section, but we always think in terms of pairs (s_1, s_0). Let $s_0 = \Phi_B\left(x_1, x_2, x_3\right) = x_1 r^1 + x_2 r^2 + x_3 r^3$ be a global section of $\mathcal{F}(1)$ for some $(x_1, x_2, x_3) \in \mathbb{C}^3$. The transformation matrix $EV(\zeta_0)$ with respect of the bases B, C of the evaluation map $ev(\zeta_0)$ is $EV(\zeta_0) = \Psi_C^{-1} \circ ev(\zeta_0) \circ \Psi_B$. By definition of the evaluation map and the considerations in section 4.1.3 we get

$$
\begin{aligned}
&ev(\zeta_0) \circ \Psi_B\,(x_1, x_2, x_3) \\
&\quad = ev(\zeta_0)\,\big(x_1 r_0^1(\zeta,\eta) + x_2 r_0^2(\zeta,\eta) + x_3 r_0^3(\zeta,\eta)\big) \\
&\quad = x_1 ev(\zeta_0)(r_0^1) + x_2 ev(\zeta_0)(r_0^2(\zeta,\eta)) + x_3 ev(\zeta_0)(r_0^3(\zeta,\eta)) \\
&\quad = x_1\Big(1 - \frac{1}{\theta}C_{((3,2),(1,0))}d_{12}\zeta_0 - \frac{1}{\theta}C_{((3,2),(0,0))}d_{12}\zeta_0^2 - \frac{1}{\theta}C_{((3,2),(1,1))}d_{12}\zeta_0\eta \\
&\qquad - \frac{1}{\theta}C_{((3,2),(1,2))}d_{12}\zeta_0\eta^2\Big) \\
&\qquad + x_2\Big(- \frac{1}{\theta}C_{((3,2),(1,0))}d_{11}\zeta_0 - \frac{1}{\theta}C_{((3,2),(0,0))}d_{11}\zeta_0^2 + \eta - \frac{1}{\theta}C_{((3,2),(1,1))}d_{11}\zeta_0\eta \\
&\qquad - \frac{1}{\theta}C_{((3,2),(1,2))}d_{11}\zeta_0\eta^2\Big) \\
&\qquad + x_3\Big(- \frac{1}{\theta}C_{((2,2),(1,0))}d_{00}\zeta_0 - \frac{1}{\theta}C_{((2,2),(0,0))}d_{00}\zeta_0^2 - \frac{1}{\theta}C_{((2,2),(1,1))}d_{00}\zeta_0\eta \\
&\qquad + \eta^2 - \frac{1}{\theta}C_{((2,2),(1,2))}d_{00}\zeta_0\eta^2\Big).
\end{aligned}
$$

Thus the transformation matrix of $ev(\zeta_0)$ is

$$
\begin{aligned}
EV(\zeta_0) = \mathbb{1} &- \frac{\zeta_0}{\theta}\begin{pmatrix} C_{((3,2),(1,0))}d_{12} & C_{((3,2),(1,0))}d_{11} & C_{((2,2),(1,0))}d_{00} \\ C_{((3,2),(1,1))}d_{12} & C_{((3,2),(1,1))}d_{11} & C_{((2,2),(1,1))}d_{00} \\ C_{((3,2),(1,2))}d_{12} & C_{((3,2),(1,2))}d_{11} & C_{((2,2),(1,2))}d_{00} \end{pmatrix} \\
&- \frac{\zeta_0^2}{\theta}\begin{pmatrix} C_{((3,2),(0,0))}d_{12} & C_{((3,2),(0,0))}d_{11} & C_{((2,2),(0,0))}d_{00} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\end{aligned}
$$

For $\zeta_0 \in W_0$ let us define

$$
\begin{aligned}
G &:= \frac{1}{\theta}\begin{pmatrix} C_{((3,2),(1,0))}d_{12} & C_{((3,2),(1,0))}d_{11} & C_{((2,2),(1,0))}d_{00} \\ C_{((3,2),(1,1))}d_{12} & C_{((3,2),(1,1))}d_{11} & C_{((2,2),(1,1))}d_{00} \\ C_{((3,2),(1,2))}d_{12} & C_{((3,2),(1,2))}d_{11} & C_{((2,2),(1,2))}d_{00} \end{pmatrix}, \\
H &:= \frac{1}{\theta}\begin{pmatrix} C_{((3,2),(0,0))}d_{12} & C_{((3,2),(0,0))}d_{11} & C_{((2,2),(0,0))}d_{00} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
F(\zeta_0) &:= G + \zeta_0 H.
\end{aligned}
$$

In this notation we see $EV(\zeta_0) := \mathbb{1} - \zeta_0 F(\zeta_0)$. If the operator norm is $\|\zeta_0 F(\zeta_0)\|_\infty < 1$, then the Neumann series $\sum_{k=0}^{\infty}(\zeta_0 N(\zeta_0))^k$ converges and the inverse of $EV(\zeta_0) = \mathbb{1} - \zeta_0 F(\zeta_0)$ is

$$
EV(\zeta_0)^{-1} = \mathbb{1} + \zeta_0 F(\zeta_0) + \zeta_0^2 F(\zeta_0)^2 + \dots .
$$

For properties of the Neumann series see for example [Wer07]. But a priori the operator norm $\|\zeta_0 F(\zeta_0)\|_\infty$ can be huge. Let us take an $\epsilon > 0$ small enough, such that every $\zeta_0 \in B_\epsilon(0) \subset W_0$ satisfies $\|\zeta_0 F(\zeta_0)\|_\infty < 1$. Hence in a probably very small neighborhood of $0 \in W_0$ we are able to invert the matrix $EV(\zeta_0)$ via the Neumann series.

4.2.3 Matricial Polynomials

The regular, nilpotent, matricial polynomial at $\zeta_0 \in W_0$ with respect to an invertible sheaf $\mathcal{F} \in Jac^3(C_3) \setminus \Theta$ is given by $A(\zeta_0) = EV(\zeta_0)^{-1} N EV(\zeta_0)$. In a small neighborhood of $0 \in W_0$ we can write

$$A(\zeta_0) = \left(\mathbb{1} + \zeta_0 F(\zeta_0) + \zeta_0^2 F(\zeta_0)^2 + \cdots\right) N \left(\mathbb{1} - \zeta_0 F(\zeta_0)\right).$$

By the Beauville correspondence, theorem 9, or see [AHH90], we know the matricial polynomial $A(\zeta)$ is of degree 2. This means we are only interested in the terms $1, \zeta_0, \zeta_0^2$ and all other higher terms will cancel out. The truncated series (by ζ_0^3) of $EV(\zeta_0)^{-1}$ is

$$\mathbb{1} + \zeta_0 F(\zeta_0) + \zeta_0^2 G^2 = \mathbb{1} + \zeta_0 G + \zeta_0^2 (H + G^2)$$

and we get

$$\begin{aligned} A(\zeta_0) &= \left(\mathbb{1} + \zeta_0 F(\zeta_0) + \zeta_0^2 G^2\right) N \left(\mathbb{1} - \zeta_0 F(\zeta_0)\right) \qquad \text{modulo } \zeta_0^3 \\ &= N\left(\mathbb{1} - \zeta_0 F(\zeta_0)\right) + \zeta_0 G N + \zeta_0^2 \left(-GNG + HN\right) + \zeta_0^2 G^2 N \\ &= N + \zeta_0 \left(GN - NG\right) + \zeta_0^2 \left(-GNG + HN - NH + G^2 N\right). \end{aligned}$$

Now we compute the matrices $GN - NG$ and $-GNG + HN - NH + G^2N$.

Lemma 5. *The entries of the matrix $\theta^2 G^2$ are*

$$\begin{aligned}
(G^2)_{11} &= C_{((3,2),(1,0))} d_{12} C_{((3,2),(1,0))} d_{12} + C_{((3,2),(1,1))} d_{12} C_{((3,2),(1,0))} d_{11} \\ &\quad + C_{((3,2),(1,2))} d_{12} C_{((2,2),(1,0))} d_{00}, \\
(G^2)_{21} &= C_{((3,2),(1,0))} d_{12} C_{((3,2),(1,1))} d_{12} + C_{((3,2),(1,1))} d_{12} C_{((3,2),(1,1))} d_{11} \\ &\quad + C_{((3,2),(1,2))} d_{12} C_{((2,2),(1,1))} d_{00}, \\
(G^2)_{31} &= C_{((3,2),(1,0))} d_{12} C_{((3,2),(1,2))} d_{12} + C_{((3,2),(1,1))} d_{12} C_{((3,2),(1,2))} d_{11} \\ &\quad + C_{((3,2),(1,2))} d_{12} C_{((2,2),(1,2))} d_{00}, \\
(G^2)_{12} &= C_{((3,2),(1,0))} d_{12} C_{((3,2),(1,0))} d_{11} + C_{((3,2),(1,0))} d_{11} C_{((3,2),(1,1))} d_{11} \\ &\quad + C_{((2,2),(1,0))} d_{00} C_{((3,2),(1,2))} d_{11}, \\
(G^2)_{22} &= C_{((3,2),(1,0))} d_{11} C_{((3,2),(1,1))} d_{12} + C_{((3,2),(1,1))} d_{11} C_{((3,2),(1,1))} d_{11} \\ &\quad + C_{((3,2),(1,2))} d_{11} C_{((2,2),(1,1))} d_{00}, \\
(G^2)_{32} &= C_{((3,2),(1,0))} d_{11} C_{((3,2),(1,2))} d_{12} + C_{((3,2),(1,1))} d_{11} C_{((3,2),(1,2))} d_{11} \\ &\quad + C_{((3,2),(1,2))} d_{11} C_{((2,2),(1,2))} d_{00}, \\
(G^2)_{13} &= C_{((3,2),(1,0))} d_{12} C_{((2,2),(1,0))} d_{00} + C_{((3,2),(1,0))} d_{11} C_{((2,2),(1,1))} d_{00} \\ &\quad + C_{((2,2),(1,0))} d_{00} C_{((2,2),(1,2))} d_{00}, \\
(G^2)_{23} &= C_{((2,2),(1,0))} d_{00} C_{((3,2),(1,1))} d_{12} + C_{((2,2),(1,1))} d_{00} C_{((3,2),(1,1))} d_{11} \\ &\quad + C_{((2,2),(1,2))} d_{00} C_{((2,2),(1,1))} d_{00}, \\
(G^2)_{33} &= C_{((2,2),(1,0))} d_{00} C_{((3,2),(1,2))} d_{12} + C_{((2,2),(1,1))} d_{00} C_{((3,2),(1,2))} d_{11} \\ &\quad + C_{((2,2),(1,2))} d_{00} C_{((2,2),(1,2))} d_{00}.
\end{aligned}$$

Proof. This follows by matrix multiplication. □

Lemma 6. *We have*

$$NG = \frac{1}{\theta}\begin{pmatrix} 0 & 0 & 0 \\ C_{((3,2),(1,0))}d_{12} & C_{((3,2),(1,0))}d_{11} & C_{((2,2),(1,0))}d_{00} \\ C_{((3,2),(1,1))}d_{12} & C_{((3,2),(1,1))}d_{11} & C_{((2,2),(1,1))}d_{00} \end{pmatrix},$$

$$GN = \frac{1}{\theta}\begin{pmatrix} C_{((3,2),(1,0))}d_{11} & C_{((2,2),(1,0))}d_{00} & 0 \\ C_{((3,2),(1,1))}d_{11} & C_{((2,2),(1,1))}d_{00} & 0 \\ C_{((3,2),(1,2))}d_{11} & C_{((2,2),(1,2))}d_{00} & 0 \end{pmatrix},$$

$$NH = \frac{1}{\theta}\begin{pmatrix} 0 & 0 & 0 \\ C_{((3,2),(0,0))}d_{12} & C_{((3,2),(0,0))}d_{11} & C_{((2,2),(0,0))}d_{00} \\ 0 & 0 & 0 \end{pmatrix},$$

$$HN = \frac{1}{\theta}\begin{pmatrix} C_{((3,2),(0,0))}d_{11} & C_{((2,2),(0,0))}d_{00} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$GNG = \frac{1}{\theta^2}\begin{pmatrix} C_{((3,2),(1,0))}d_{11}C_{((3,2),(1,0))}d_{12} + C_{((2,2),(1,0))}d_{00}C_{((3,2),(1,1))}d_{12} \\ C_{((3,2),(1,0))}d_{11}C_{((3,2),(1,0))}d_{11} + C_{((2,2),(1,0))}d_{00}C_{((3,2),(1,1))}d_{11} \\ C_{((3,2),(1,0))}d_{11}C_{((2,2),(1,0))}d_{00} + C_{((2,2),(1,0))}d_{00}C_{((2,2),(1,1))}d_{00} \\ C_{((3,2),(1,1))}d_{11}C_{((3,2),(1,0))}d_{12} + C_{((2,2),(1,1))}d_{00}C_{((3,2),(1,1))}d_{12} \\ C_{((3,2),(1,1))}d_{11}C_{((3,2),(1,0))}d_{11} + C_{((2,2),(1,1))}d_{00}C_{((3,2),(1,1))}d_{11} \\ C_{((3,2),(1,1))}d_{11}C_{((2,2),(1,0))}d_{00} + C_{((2,2),(1,1))}d_{00}C_{((2,2),(1,1))}d_{00} \\ C_{((3,2),(1,2))}d_{11}C_{((3,2),(1,0))}d_{12} + C_{((2,2),(1,2))}d_{00}C_{((3,2),(1,1))}d_{12} \\ C_{((3,2),(1,2))}d_{11}C_{((3,2),(1,0))}d_{11} + C_{((2,2),(1,2))}d_{00}C_{((3,2),(1,1))}d_{11} \\ C_{((3,2),(1,2))}d_{11}C_{((2,2),(1,0))}d_{00} + C_{((2,2),(1,1))}d_{00}C_{((2,2),(1,1))}d_{00} \end{pmatrix},$$

$$G^2N = \frac{1}{\theta^2}\begin{pmatrix} (G^2)_{12} & (G^2)_{13} & 0 \\ (G^2)_{22} & (G^2)_{23} & 0 \\ (G^2)_{32} & (G^2)_{33} & 0 \end{pmatrix}.$$

In particular we have

$$GN - NG = \frac{1}{\theta}\begin{pmatrix} -d_{00}^2d_{11}d_{22} + d_{00}d_{11}^3 & d_{32}d_{00}^3 & 0 \\ d_{00}^2d_{12}d_{22} - 2d_{00}d_{11}^2d_{12} & 2d_{00}^2d_{11}d_{22} - d_{00}d_{11}^3 & -d_{32}d_{00}^3 \\ 0 & d_{00}d_{11}^2d_{12} + d_{00}^2d_{12}d_{22} & -d_{00}^2d_{11}d_{22} \end{pmatrix}$$

and

$$HN - NH = \frac{1}{\theta^2}\begin{pmatrix} d_{00}^2d_{11}d_{12}\theta & -d_{00}^3d_{22}\theta & 0 \\ -d_{00}^2d_{12}^2\theta & -d_{00}^2d_{11}d_{12}\theta & d_{00}^3d_{22}\theta \\ 0 & 0 & 0 \end{pmatrix}.$$

Moreover we have

$$-GNG = -\frac{1}{\theta^2}\begin{pmatrix} (d_{22}-d_{11}^2)d_{11}(d_{22}-d_{11}^2)d_{12} + (-d_{32})d_{11}d_{12}^2 \\ (d_{22}-d_{11}^2)d_{11}(d_{22}-d_{11}^2)d_{11} + (d_{32})d_{11}^2d_{12} \\ (d_{22}-d_{11}^2)d_{11}(-d_{32}) + (-d_{32})(-d_{11}d_{22}) \\ d_{11}^2d_{12}(d_{22}-d_{11}^2)d_{12} + (-d_{11}d_{22})d_{11}d_{12}^2 \\ d_{11}^2d_{12}(d_{22}-d_{11}^2)d_{11} + (-d_{11}d_{22})d_{11}^2d_{12} \\ d_{11}^2d_{12}(-d_{32}) + (-d_{11}d_{22})(-d_{11}d_{22}) \\ d_{11}d_{12}^2(d_{22}-d_{11}^2)d_{12} + (-d_{12}d_{22})d_{11}d_{12}^2 \\ d_{11}d_{12}^2(d_{22}-d_{11}^2)d_{11} + (-d_{12}d_{22})d_{11}^2d_{12} \\ d_{11}d_{12}^2(-d_{32}) + (-d_{12}d_{22})(-d_{11}d_{22}) \end{pmatrix}$$

and

$$G^2N = \frac{1}{\theta^2}\begin{pmatrix} (d_{22}-d_{11}^2)d_{12}(d_{22}-d_{11}^2)d_{11} + (d_{22}-d_{11}^2)d_{11}d_{11}^2d_{12} + (-d_{32})d_{11}d_{12}^2 \\ (d_{22}-d_{11}^2)d_{12}(-d_{32}) + (d_{22}-d_{11}^2)d_{11}(-d_{11}d_{22}) + (-d_{32})(-d_{12}d_{22}) \\ 0 \\ d_{11}d_{12}^2(d_{22}-d_{11}^2)d_{11} + d_{11}^2d_{12}d_{11}^2d_{12} + (-d_{11}d_{22})d_{11}d_{12}^2 \\ d_{11}d_{12}^2(-d_{32}) + d_{11}^2d_{12}(-d_{11}d_{22}) + (-d_{11}d_{22})(-d_{12}d_{22}) \\ 0 \\ d_{12}^3(d_{22}-d_{11}^2)d_{11} + d_{11}d_{12}^2d_{11}^2d_{12} + (-d_{12}d_{22})d_{11}d_{12}^2 \\ d_{12}^3(-d_{32}) + d_{11}d_{12}^2(-d_{11}d_{22}) + (-d_{12}d_{22})(-d_{12}d_{22}) \\ 0 \end{pmatrix},$$

where we set $d_{00} = 1$ in the last two equations just to write it down more compactly.

Proof. This follows by standard matrix multiplication and lemma 4. □

Lemma 6 contains everything we need to describe one direction of the Beauville correspondence explicitly.

Theorem 12. *Let $(C_3, \mathcal{O}_{C_3})$ be the nilpotent, spectral curve and let $\mathcal{F} \in Jac^3(C_3) \setminus \Theta$ be an invertible sheaf of degree 3 without a non-trivial global section. We set $d_{00} = 1$. Then the corresponding $GL_3(\mathbb{C})$-conjugation class of regular, nilpotent, matricial polynomials has a representative of the form*

$$A_0 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$A_1 = GN - NG$$

$$= \frac{1}{\theta}\begin{pmatrix} -d_{11}d_{22} + d_{11}^3 & d_{32} & 0 \\ d_{12}d_{22} - 2d_{11}^2d_{12} & 2d_{11}d_{22} - d_{11}^3 & -d_{32} \\ 0 & d_{11}^2d_{12} + d_{12}d_{22} & -d_{11}d_{22} \end{pmatrix}$$

$$A_2 = -GNG + HN - NH + G^2N$$

$$= \frac{1}{\theta^2}\begin{pmatrix} d_{12}d_{11}(\theta - d_{11}^4 + d_{11}^2d_{22}) & (d_{22}-d_{11}^2)^3 - d_{12}d_{32}d_{22} & d_{11}d_{32}(-d_{11}^2) \\ -d_{12}^2(\theta - d_{11}^4) & -d_{11}d_{12}(2\theta - d_{11}^4) & d_{12}d_{32}(d_{22}+d_{11}^2) - d_{22}^3 \\ d_{12}^3d_{11}^3 & -d_{12}^2(\theta - d_{11}^4) & d_{11}d_{12}(\theta - d_{11}^2d_{22}) \end{pmatrix}.$$

Proof. This follows by the previous three lemmas and the formula $A(\zeta_0) = N + \zeta_0(GN - NG) + \zeta_0^2(-GNG + HN - NH + G^2N)$. □

Remark 2. *Note that $-C_{((3,2),(1,1))}d_{12} + C_{((3,2),(1,2))}d_{11} = 0$. This is the crucial observation to formulate in a later stage the burning lemma.*

4.2.4 Trace $tr\left(A_0A_2 - \frac{1}{4}A_1^2\right)$

After the description of nilpotent, regular, matricial polynomials in the previous section we are able to state a theorem, which is basically the theorem 11 without the ingredient of flows.

Theorem 13. *Let $(C_3, \mathcal{O}_{C_3})$ be the nilpotent, spectral curve and let $A(\zeta) = A_0 + A_1\zeta + A_2\zeta^2$ be the corresponding matricial polynomial of theorem 12 corresponding to an invertible sheaf $\mathcal{F} \in Jac^3(C_3) \setminus \Theta$. Then we have the following equation*

$$tr\left(A_0A_2 - \frac{1}{4}A_1^2\right) = \frac{3}{2}\frac{3d_{00}^2d_{11}^2\theta - (d_{00}d_{11}^3)^2}{\theta^2}.$$

To prove this directly we need some lemmas. The first lemma is

Lemma 7. *The trace of A_1 vanishes. Furthermore we have*

$$tr(A_1^2) = \frac{1}{\theta^2}\Big(2C^2_{((3,2),(1,0))}d^2_{11} - 2C_{((2,2),(1,0))}d_{00}C_{((3,2),(1,0))}d_{12} - 2C_{((3,2),(1,0))}d_{11}C_{((2,2),(1,1))}d_{00} - 2C_{((2,2),(1,0))}d_{00}C_{((2,2),(1,2))}d_{00} + 4C_{((2,2),(1,0))}d_{00}C_{((3,2),(1,1))}d_{11} + 2C^2_{((2,2),(1,1))}d^2_{00}\Big).$$

Proof. From theorem 12 we see immediatly the vanishing trace of A_1. Now we compute

$$\begin{aligned}
tr(A_1^2) = \frac{1}{\theta^2}\Big(&C^2_{((3,2),(1,0))}d^2_{11} + C_{((2,2),(1,0))}d_{00}\big(-C_{((3,2),(1,0))}d_{12} + C_{((3,2),(1,1))}d_{11}\big)\\
&+ C_{((2,2),(1,0))}d_{00}\big(-C_{((3,2),(1,0))}d_{12} + C_{((3,2),(1,1))}d_{11}\big)\\
&+ \big(-C_{((3,2),(1,0))}d_{11} + C_{((2,2),(1,1))}d_{00}\big)^2\\
&- C_{((2,2),(1,0))}d_{00}\big(-C_{((3,2),(1,1))}d_{11} + C_{((2,2),(1,2))}d_{00}\big)\\
&- C_{((2,2),(1,0))}d_{00}\big(-C_{((3,2),(1,1))}d_{11} + C_{((2,2),(1,2))}d_{00}\big) + \big(-C_{((2,2),(1,1))}d_{00}\big)^2\Big)\\
= \frac{1}{\theta^2}\Big(&C^2_{((3,2),(1,0))}d^2_{11} - C_{((2,2),(1,0))}d_{00}C_{((3,2),(1,0))}d_{12}\\
&+ C_{((2,2),(1,0))}d_{00}C_{((3,2),(1,1))}d_{11} - C_{((2,2),(1,0))}d_{00}C_{((3,2),(1,0))}d_{12}\\
&+ C_{((2,2),(1,0))}d_{00}C_{((3,2),(1,1))}d_{11} + C^2_{((3,2),(1,0))}d^2_{11}\\
&- 2C_{((3,2),(1,0))}d_{11}C_{((2,2),(1,1))}d_{00} + C^2_{((2,2),(1,1))}d^2_{00}\\
&+ C_{((2,2),(1,0))}d_{00}C_{((3,2),(1,1))}d_{11} - C_{((2,2),(1,0))}d_{00}C_{((2,2),(1,2))}d_{00}\\
&+ C_{((2,2),(1,0))}d_{00}C_{((3,2),(1,1))}d_{11} - C_{((2,2),(1,0))}d_{00}C_{((2,2),(1,2))}d_{00}\\
&+ C^2_{((2,2),(1,1))}d^2_{00}\Big)\\
= \frac{1}{\theta^2}\Big(&2C^2_{((3,2),(1,0))}d^2_{11} - 2C_{((2,2),(1,0))}d_{00}C_{((3,2),(1,0))}d_{12}\\
&- 2C_{((3,2),(1,0))}d_{11}C_{((2,2),(1,1))}d_{00} - 2C_{((2,2),(1,0))}d_{00}C_{((2,2),(1,2))}d_{00}\\
&+ 4C_{((2,2),(1,0))}d_{00}C_{((3,2),(1,1))}d_{11} + 2C^2_{((2,2),(1,1))}d^2_{00}\Big).
\end{aligned}$$

□

Lemma 8. *The trace of A_0A_2 is given by $tr\,(A_0A_2) = Z_1 + Z_2$, where*

$$\begin{aligned}
Z_1 = &\frac{1}{\theta}C_{((2,2),(0,0))}d_{00} - \frac{1}{\theta^2}C_{((3,2),(1,0))}d_{11}C_{((3,2),(1,0))}d_{11}\\
&- \frac{1}{\theta^2}C_{((2,2),(1,0))}d_{00}C_{((3,2),(1,1))}d_{11} + \frac{1}{\theta^2}C_{((3,2),(1,0))}d_{12}C_{((2,2),(1,0))}d_{00}\\
&+ \frac{1}{\theta^2}C_{((3,2),(1,0))}d_{11}C_{((2,2),(1,1))}d_{00} + \frac{1}{\theta^2}C_{((2,2),(1,0))}d_{00}C_{((2,2),(1,2))}d_{00},\\
Z_2 = &-\frac{1}{\theta}C_{((2,2),(0,0))}d_{00} - \frac{1}{\theta^2}C_{((3,2),(1,1))}d_{11}C_{((2,2),(1,0))}d_{00}\\
&- \frac{1}{\theta^2}C_{((2,2),(1,1))}d_{00}C_{((2,2),(1,1))}d_{00}.
\end{aligned}$$

Proof. The matrix A_0 is the Jordan canonical form with exactly one Jordan block seen as a lower-triangular matrix. So by doing the matrix multiplication A_0A_2 we just need to use the computations of lemma 6 and read out the correct terms. □

Now we are able to prove theorem 13.

Proof. We have $tr(A_0A_2 - \frac{1}{4}A_1^2) = -\frac{1}{4}tr(A_1^2) + Z_1 + Z_2$. With lemma 4 , lemma 7 and lemma 8 we have

$$
\begin{aligned}
&tr(A_0A_2 - \tfrac{1}{4}A_1^2) = \\
&-\frac{1}{4}\frac{1}{\theta^2}\Big(2C^2_{((3,2),(1,0))}d_{11}^2 - 2C_{((2,2),(1,0))}d_{00}C_{((3,2),(1,0))}d_{12} \\
&\quad - 2C_{((3,2),(1,0))}d_{11}C_{((2,2),(1,1))}d_{00} - 2C_{((2,2),(1,0))}d_{00}C_{((2,2),(1,2))}d_{00} \\
&\quad + 4C_{((2,2),(1,0))}d_{00}C_{((3,2),(1,1))}d_{11} + 2C^2_{((2,2),(1,1))}d_{00}^2\Big) \\
&\quad + \frac{1}{\theta}C_{((2,2),(0,0))}d_{00} - \frac{1}{\theta^2}C_{((3,2),(1,0))}d_{11}C_{((3,2),(1,0))}d_{11} \\
&\quad - \frac{1}{\theta^2}C_{((2,2),(1,0))}d_{00}C_{((3,2),(1,1))}d_{11} + \frac{1}{\theta^2}C_{((3,2),(1,0))}d_{12}C_{((2,2),(1,0))}d_{00} \\
&\quad + \frac{1}{\theta^2}C_{((3,2),(1,0))}d_{11}C_{((2,2),(1,1))}d_{00} + \frac{1}{\theta^2}C_{((2,2),(1,0))}d_{00}C_{((2,2),(1,2))}d_{00} \\
&\quad - \frac{1}{\theta}C_{((2,2),(0,0))}d_{00} - \frac{1}{\theta^2}C_{((3,2),(1,1))}d_{11}C_{((2,2),(1,0))}d_{00} \\
&\quad - \frac{1}{\theta^2}C_{((2,2),(1,1))}d_{00}C_{((2,2),(1,1))}d_{00} \\
&= \frac{1}{\theta^2}\frac{3}{2}\big(- C^2_{((3,2),(1,0))}d_{11}^2 + C_{((2,2),(1,0))}d_{00}C_{((3,2),(1,0))}d_{12} \\
&\quad + C_{((3,2),(1,0))}d_{11}C_{((2,2),(1,1))}d_{00} + C_{((2,2),(1,0))}d_{00}C_{((2,2),(1,2))}d_{00} \\
&\quad - 2C_{((2,2),(1,0))}d_{00}C_{((3,2),(1,1))}d_{11} - C^2_{((2,2),(1,1))}d_{00}^2\big) \\
&= \frac{1}{\theta^2}\frac{3}{2}\big(- (d_{00}^2d_{22} - d_{00}d_{11}^2)^2d_{11}^2 - d_{32}d_{00}^2d_{00}(d_{00}^2d_{22} - d_{00}d_{11}^2)d_{12} \\
&\quad - (d_{00}^2d_{22} - d_{00}d_{11}^2)d_{11}d_{22}d_{11}d_{00}d_{00} + d_{32}d_{00}^2d_{00}d_{00}d_{12}d_{22}d_{00} \\
&\quad + 2d_{32}d_{00}^2d_{00}d_{12}d_{11}d_{00}d_{11} - (-d_{22}d_{11}d_{00})^2d_{00}^2\big) \\
&= \frac{1}{\theta^2}\frac{3}{2}\big(- (d_{00}^4d_{22}^2 - 2d_{00}^2d_{22}d_{00}d_{11}^2 + d_{00}^2d_{11}^4)d_{11}^2 - d_{12}d_{32}d_{00}^2d_{00}d_{00}^2d_{22} \\
&\quad + d_{00}d_{12}d_{32}d_{00}^2d_{00}d_{11}^2 - d_{00}^2d_{22}d_{11}d_{22}d_{11}d_{00}d_{00} + d_{00}d_{11}^2d_{11}d_{22}d_{11}d_{00}d_{00} \\
&\quad + d_{32}d_{00}^2d_{00}d_{00}d_{12}d_{22}d_{00} + 2d_{32}d_{00}^2d_{00}d_{12}d_{11}d_{00}d_{11} - d_{22}^2d_{11}^2d_{00}^2d_{00}^2\big) \\
&= \frac{1}{\theta^2}\frac{3}{2}\big(- 3d_{00}^4d_{11}^2d_{22}^2 + 3d_{00}^3d_{22}d_{11}^4 - d_{00}^2d_{11}^6 + 3d_{00}^2d_{12}d_{32}d_{00}^2d_{11}^2\big) \\
&= \frac{1}{\theta^2}\frac{3}{2}\big(3d_{00}^2d_{11}^2(d_{00}^2d_{12}d_{32} + d_{00}d_{22}d_{11}^2 - d_{00}^2d_{22}^2) - d_{00}^2d_{11}^6\big) \\
&= \frac{3}{2}\frac{3d_{00}^2d_{11}^2\theta - (d_{00}d_{11}^3)^2}{\theta^2}.
\end{aligned}
$$

□

4.2.5 Flows and Hitchin's formula

Let $\mathcal{F} \in Jac^3(C_3)$ and let $\mathcal{L}^t \in Pic^0(C_3)$, $t \in \mathbb{C}$, be a family of invertible sheaves given by transition functions $\exp\left(-t\frac{\eta}{\zeta}\right)$. We get a family of invertible sheaves in the Jacobian via $\mathcal{F}^t = \mathcal{F} \otimes \mathcal{L}^t \in Jac^3(C_3)$, which we can describe in terms of transition functions depending on the variable t.

Lemma 9. *Let* $\frac{1}{\zeta}\left(\sum_{l=0}^{2}\sum_{k=1}^{2l-1} d_{kl}\frac{\eta^l}{\zeta^k}\right)$ *be the transition function of an invertible sheaf* $\mathcal{F} \in Jac^3(C_3)$*. The transition function of the invertible sheaf* $\mathcal{F} \otimes \mathcal{L}^t$ *is given by*

$\frac{1}{\zeta}g_{10}(\zeta,\eta)(t) =$

$$\frac{1}{\zeta}\left(d_{00} + (d_{11} - d_{00}t)\frac{\eta}{\zeta} + \left(d_{12}\frac{1}{\zeta} + \left(d_{22} - d_{11}t + \frac{1}{2}d_{00}t^2\right)\frac{1}{\zeta^2} + d_{32}\frac{1}{\zeta^3}\right)\eta^2\right).$$

Proof. This is basically the statement of theorem 10 with $n = 3$. □

The theta function of an invertible sheaf $\mathcal{F}$ is $\theta(\mathcal{F}) = d_{00}^2 d_{12}d_{32} + d_{00}d_{11}^2 d_{22} - d_{00}^2 d_{22}^2$. The sheaf $\mathcal{F}^t$ defines a theta function too, which is holomorphic in the parameter t. The first two derivatives are important in order to prove Hitchin's formula.

Lemma 10. *We have the following formulas,*

$$\begin{aligned}
\theta(\mathcal{F}^t) &= \theta(\mathcal{F}) - d_{00}d_{11}^3 t + \frac{3}{2}d_{00}^2 d_{11}^2 t^2 - d_{00}^3 d_{11}t^3 + \frac{1}{4}d_{00}^4 t^4\\
&= \left(\theta(\mathcal{F}) - \frac{1}{4}d_{11}^4\right) + \frac{1}{4}\left(d_{11} - d_{00}t\right)^4,\\
\theta(\mathcal{F}^t)' &= -d_{00}(d_{11} - d_{00}t)^3,\\
\theta(\mathcal{F}^t)'' &= 3d_{00}^2(d_{11} - d_{00}t)^2.
\end{aligned}$$

Moreover if $\mathcal{F}^t \in Jac^3(C_3)\setminus\Theta$, then we have

$$\begin{aligned}
\frac{d^2}{dt^2}\log\left(\theta(\mathcal{F}^t)\right) &= \frac{\theta(\mathcal{F}^t)(\theta(\mathcal{F}^t))'' - (\theta(\mathcal{F}^t))'(\theta(\mathcal{F}^t))'}{\theta(\mathcal{F}^t)^2}\\
&= \frac{\theta(\mathcal{F}^t)3d_{00}^2(d_{11} - d_{00}t)^2 - (d_{00}(d_{11} - d_{00}t)^3)^2}{\theta(\mathcal{F}^t)^2}.
\end{aligned}$$

Proof. We just compute $\theta(\mathcal{F}^t) =$

$$\begin{aligned}
&d_{00}^2 d_{12}d_{32} + d_{00}(d_{11} - d_{00}t)^2(d_{22} - d_{11}t + \frac{1}{2}d_{00}t^2) - d_{00}^2(d_{22} - d_{11}t + \frac{1}{2}d_{00}t^2)^2\\
&= d_{00}^2 d_{12}d_{32} + d_{00}(d_{11}^2 - 2d_{11}d_{00}t + d_{00}^2 t^2)(d_{22} - d_{11}t + \frac{1}{2}d_{00}t^2)\\
&\quad - d_{00}^2(d_{22}^2 - 2d_{11}d_{22}t + d_{00}d_{22}t^2 + d_{11}^2 t^2 - d_{00}d_{11}t^3 + \frac{1}{4}d_{00}^2 t^4)\\
&= d_{00}^2 d_{12}d_{32} + d_{00}d_{22}d_{11}^2 - d_{00}d_{11}^3 t + \frac{1}{2}d_{00}^2 d_{11}^2 t^2\\
&\quad - 2d_{00}^2 d_{11}d_{22}t + 2d_{00}^2 d_{11}^2 t^2 - d_{00}^3 d_{11}t^3 + d_{00}^3 d_{22}t^2 - d_{00}^3 d_{11}t^3 + \frac{1}{2}d_{00}^4 t^4\\
&\quad - d_{00}^2 d_{22}^2 + 2d_{00}^2 d_{11}d_{22}t - d_{00}^3 d_{22}t^2 - d_{00}^2 d_{11}^2 t^2 + d_{00}^3 d_{11}t^3 - \frac{1}{4}d_{00}^4 t^4\\
&= d_{00}^2 d_{12}d_{32} + d_{00}d_{22}d_{11}^2 - d_{00}^2 d_{22}^2\\
&\quad + (-d_{00}d_{11}^3 - 2d_{00}^2 d_{11}d_{22} + 2d_{00}^2 d_{11}d_{22})t\\
&\quad + (\frac{1}{2}d_{00}^2 d_{11}^2 + 2d_{00}^2 d_{11}^2 + d_{00}^3 d_{22} - d_{00}^3 d_{22} - d_{00}^2 d_{11}^2)t^2\\
&\quad + (-d_{00}^3 d_{11} - d_{00}^3 d_{11} + d_{00}^3 d_{11})t^3\\
&\quad + (\frac{1}{2}d_{00}^4 - \frac{1}{4}d_{00}^4)t^4\\
&= \theta(\mathcal{F}) - d_{00}d_{11}^3 t + \frac{3}{2}d_{00}^2 d_{11}^2 t^2 - d_{00}^3 d_{11}t^3 + \frac{1}{4}d_{00}^4 t^4\\
&= \theta(\mathcal{F}) - \frac{1}{4}d_{11}^4 + \frac{1}{4}\left(d_{11} - d_{00}t\right)^4.
\end{aligned}$$

Then the derivatives are

$$\begin{aligned}\theta(\mathcal{F}^t)' &= -d_{00}d_{11}^3 + 3d_{00}^2 d_{11}^2 t - 3d_{00}^3 d_{11} t^2 + d_{00}^4 t^3 = -d_{00}(d_{11} - d_{00}t)^3,\\ \theta(\mathcal{F}^t)'' &= 3d_{00}^2 d_{11}^2 - 6d_{00}^3 d_{11} t + 3d_{00}^4 t^2 = 3d_{00}^2(d_{11} - d_{00}t)^2.\end{aligned}$$

With

$$\frac{d^2}{dt^2}\log\left(\theta(\mathcal{F}^t)\right) = \frac{\theta(\mathcal{F}^t)\theta(\mathcal{F}^t)'' - \theta(\mathcal{F}^t)'\theta(\mathcal{F}^t)'}{\theta(\mathcal{F}^t)^2}$$

and by inserting the derivatives of the theta function we get the last formula. □

At this stage we can prove Hitchin's formula, theorem 11, in the case $n = 3$.

Proof. If we replace the invertible sheaf $\mathcal{F}$ with the invertible sheaf $\mathcal{F}^t$ we know by lemma 9, that we have to replace θ by $\theta(\mathcal{F}^t)$, d_{11} by $(d_{11} - d_{00}t)$ and d_{22} by $(d_{22} - d_{11}t + \frac{1}{2}d_{00}t^2)$. Theorem 13 gets

$$tr\left(A_0(t)A_2(t) - \frac{1}{4}A_1(t)^2\right) = \frac{3}{2}\frac{3d_{00}^2(d_{11} - d_{00}t)^2\theta(\mathcal{F}^t) - (d_{00}(d_{11} - d_{00}t)^3)^2}{\theta(\mathcal{F}^t)^2}.$$

By lemma 10 this is just equal to

$$\frac{3}{2}\frac{d^2}{dt^2}\log(\theta(\mathcal{F}^t)) = \frac{3}{2}\frac{\theta(\mathcal{F}^t)(\theta(\mathcal{F}^t))'' - (\theta(\mathcal{F}^t))'(\theta(\mathcal{F}^t))'}{\theta(\mathcal{F}^t)^2}.$$

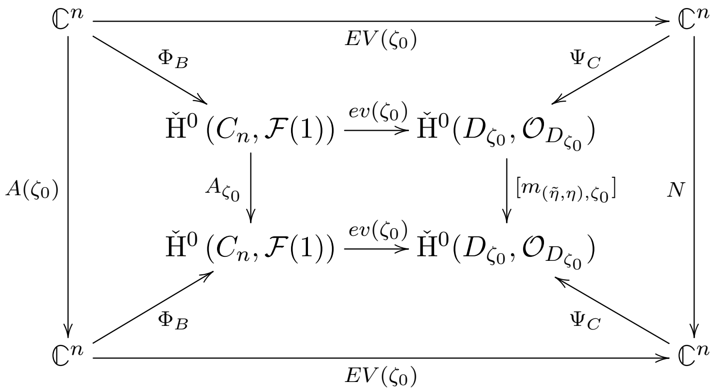
This finishes the proof of Hitchin's formula in the case $n = 3$. □

4.3 Hitchin's Formula

In this section we want to adjust the computations and ideas of the case $n = 3$ to the general case. We start again by describing a representative of the $GL_n(\mathbb{C})$-conjugacy class of regular, nilpotent, matricial polynomials corresponding to an invertible sheaf $\mathcal{F} \in Jac^{g-1}(C_n) \setminus \Theta$. With this explicit description we are able to compute the trace $tr\left(A_0A_2 - \frac{1}{4}A_1^2\right)$. Then we will compute the term $\frac{3}{2}\frac{d^2}{dt^2}\log\left(\theta(\mathcal{F}^t)\right)$ and compare it with $tr\left(A_0A_2 - \frac{1}{4}A_1^2\right)$ by using the crucial *burning lemma.*

4.3.1 Beauville Correspondence

To describe a representative of the $GL_n(\mathbb{C})$-conjugacy class of regular, nilpotent, matricial polynomials $A(\zeta) = A_0 + A_1\zeta + A_2\zeta^2$ corresponding to an invertible sheaf $\mathcal{F} \in Jac^{g-1}(C_n) \setminus \Theta$ we will use again the following commutative diagramm.



The first step is the computation of the transformation matrix of the evaluation map and its inverse. Recall that a global section $s \in \check{\mathrm{H}}^0(C_n, \mathcal{F}(1))$ is a pair $(s_1, s_0) \in \mathcal{O}_{U_1}(U_1) \times \mathcal{O}_{U_0}(U_0)$, which we can write $s_0(\zeta,\eta) = \sum_{l=0}^{n-1} s_0^l(\zeta)\eta^l$, where s_0^0 is a polynomial of degree $n-1$ and the s_0^l, $l \neq 0$, are polynomials of degree $n-2$. Let us denote the choosen basis of $\check{\mathrm{H}}^0(C_n, \mathcal{F}(1))$ of theorem 7 by $B = \{r^1, \dots, r^n\}$. Since $r^i = (r_1^i, r_0^i)$ and r_1^i is uniquely determined by r_0^i we do the computations for r_0^i. Let a_{kl}^i be the coefficients of the basis vector r_0^i. We define the coefficient vector without the free variables a_{0l}^i

$$\tau^i := \left(a_{10}^i, \dots, a_{n-1,0}^i, a_{11}^i, \dots, a_{n-2,1}^i, \dots, a_{1,n-1}^i, \dots, a_{n-2,n-1}^i\right)^T \in \mathbb{C}^{n+g}$$

and the matrix of all such coefficient vectors

$$T := \begin{pmatrix} \tau^1, & \dots, & \tau^n \end{pmatrix} \in \mathbb{C}^{(g+n)\times n}.$$

Furthermore, for a $\zeta_0 \in W_0 \subset \mathbb{CP}^1$, let us define a matrix $L(\zeta_0) \in \mathbb{C}^{n\times(g+n)}$ by

$$L(\zeta_0) := \begin{pmatrix} \zeta_0 & \zeta_0^2 & \zeta_0^3 & \cdots & \zeta_0^{n-1} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & \zeta_0 & \zeta_0^2 & \cdots & \zeta_0^{n-2} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \zeta_0 & \zeta_0^2 & \cdots & \zeta_0^{n-2} \end{pmatrix}.$$

Then by definition of Ψ_C, if the standard basis vectors of $\mathbb{C}^n$ are denoted by e_i, we can write

$$\Psi_C^{-1}\left(ev(\zeta_0)(r^i)\right) = e_i + L(\zeta_0)\tau_i \in \mathbb{C}^n.$$

For an arbitrary global section $s = (s_1, s_0) \in \check{\mathrm{H}}^0(C_n, \mathcal{F}(1))$, written as a linear combination of the basis vectors $s = \sum_{i=1}^n x_i r^i = \Phi_B(x_1, \dots, x_n)$, we get

$$\begin{aligned} EV(\zeta_0)\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} &= \Psi_C^{-1} \circ ev(\zeta_0) \circ \Phi_B(x_1, \dots, x_n) = \sum_{i=1}^n x_i \Psi_C^{-1}\left(ev(\zeta_0)(r_0^i)\right) \\ &= \left(\Psi_C^{-1}\left(ev(\zeta_0)(r_0^1)\right), \cdots, \quad \Psi_C^{-1}\left(ev(\zeta_0)(r_0^n)\right)\right)\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}. \end{aligned}$$

Thus the transformation matrix of $ev(\zeta_0)$ with respect to the bases B and C is given by

$$\begin{aligned} EV(\zeta_0) &= \left(\Psi_C^{-1}\left(ev(\zeta_0)(r_0^1)\right), \quad \dots, \quad \Psi_C^{-1}\left(ev(\zeta_0)(r_0^n)\right)\right) \\ &= \mathbb{1}_n + L(\zeta_0)T. \end{aligned}$$

By considering a probably very small neighborhood of $0 \in W_0$ we can always make the operator norm $\|L(\zeta_0)T\| < 1$ for all ζ_0 in this neighborhood. Hence the Neumann series of $L(\zeta_0)T$ converges and the inversion of $EV(\zeta_0) = \mathbb{1} - (-L(\zeta_0)T)$ is given by

$$EV(\zeta_0)^{-1} = \sum_{k=0}^{\infty} \left(-L(\zeta_0)T\right)^k.$$

The regular, nilpotent, matricial polynomial in a small neighbhorhood of $0 \in W_0$ is

$$\begin{aligned} A(\zeta_0) = EV(\zeta_0)^{-1}NEV(\zeta_0) &= \left(\sum_{k=0}^{\infty}(-L(\zeta_0)T)^k\right) N\left(\mathbb{1}_n + L(\zeta_0)T\right) \\ &= N + NL(\zeta_0)T - L(\zeta_0)TN - L(\zeta_0)TNL(\zeta_0)T + (L(\zeta_0)T)^2N \\ &\quad + \text{ higher terms in } \zeta_0. \end{aligned}$$

By [AHH90] or the Beauville correspondence we know, that the matricial polynomial $A(\zeta_0)$ has to have degree 2 and hence we are only interested in $1, \zeta_0$ and ζ_0^2. All terms of higher order than 2 cancel out.

Lemma 11. *The only ζ_0^1 terms in $A(\zeta_0)$ are in the term $NL(\zeta_0)T - L(\zeta_0)TN$ and they are given by*

$$\begin{pmatrix} -a_{10}^2 & -a_{10}^3 & \cdots & -a_{10}^n & 0 \\ a_{10}^1 - a_{11}^2 & a_{10}^2 - a_{11}^3 & \cdots & a_{10}^{n-1} - a_{11}^n & a_{10}^n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{1n-2}^1 - a_{1n-1}^2 & a_{1n-2}^2 - a_{1n-1}^3 & \cdots & a_{1n-2}^{n-1} - a_{1n-1}^n & a_{1n-2}^n \end{pmatrix}.$$

Proof. Because the ζ_0^1-term appears only in n different columns in $L(\zeta_0)$ we get

$$Q := L(\zeta_0)T \text{ modulo } \zeta_0^2 = \begin{pmatrix} a_{10}^1 & a_{10}^2 & \cdots & a_{10}^n \\ a_{11}^1 & a_{11}^2 & \cdots & a_{11}^n \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n-1}^1 & a_{1n-1}^2 & \cdots & a_{1n-1}^n \end{pmatrix} \in \mathbb{C}^{n\times n}.$$

Moreover we have

$$-QN = -\begin{pmatrix} a_{10}^2 & a_{10}^3 & \cdots & a_{10}^n & 0 \\ a_{11}^2 & a_{11}^3 & \cdots & a_{11}^n & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 \\ a_{1n-1}^2 & a_{1n-1}^3 & \cdots & a_{1n-1}^n & 0 \end{pmatrix},$$

$$NQ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a_{10}^1 & a_{10}^2 & \cdots & a_{10}^n \\ a_{11}^1 & a_{11}^2 & \cdots & a_{11}^n \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n-2}^1 & a_{1n-2}^2 & \cdots & a_{1n-2}^n \end{pmatrix}$$

and therefore

$$NQ - QN = \begin{pmatrix} -a_{10}^2 & -a_{10}^3 & \cdots & -a_{10}^n & 0 \\ a_{10}^1 - a_{11}^2 & a_{10}^2 - a_{11}^3 & \cdots & a_{10}^{n-1} - a_{11}^n & a_{10}^n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{1n-2}^1 - a_{1n-1}^2 & a_{1n-2}^2 - a_{1n-1}^3 & \cdots & a_{1n-2}^{n-1} - a_{1n-1}^n & a_{1n-2}^n \end{pmatrix}.$$

□

Lemma 12. *The ζ_0^2-term of $NL(\zeta_0)T - L(\zeta_0)TN$ is*

$$\begin{pmatrix} -a_{20}^2 & -a_{20}^3 & \cdots & -a_{20}^n & 0 \\ a_{20}^1 - a_{21}^2 & a_{20}^2 - a_{21}^3 & \cdots & a_{20}^{n-1} - a_{21}^n & a_{20}^n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{2n-2}^1 - a_{2n-1}^2 & a_{2n-2}^2 - a_{2n-1}^3 & \cdots & a_{2n-2}^{n-1} - a_{2n-1}^n & a_{2n-2}^n \end{pmatrix}.$$

Proof. This follows by the same computations as in lemma 11. □

Lemma 13. *The ζ_0^2-term of $(L(\zeta_0)T)^2N$ is (v_{st}) with $s,t \in \{1,\cdots,n\}$, where*

$$v_{st} := \sum_{j=0}^{n-1} a_{1s-1}^{j+1} a_{1j}^{t+1}$$

for $t \neq n$ and $v_{sn} = 0$.

Proof. We have

$$Q(QN) = \begin{pmatrix} a_{10}^1 & a_{10}^2 & \cdots & a_{10}^n \\ a_{11}^1 & a_{11}^2 & \cdots & a_{11}^n \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n-1}^1 & a_{1n-1}^2 & \cdots & a_{1n-1}^n \end{pmatrix} \begin{pmatrix} a_{10}^2 & a_{10}^3 & \cdots & a_{10}^n & 0 \\ a_{11}^2 & a_{11}^3 & \cdots & a_{11}^n & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{1n-1}^2 & a_{1n-1}^3 & \cdots & a_{1n-1}^n & 0 \end{pmatrix}$$

and therefore we have

$$v_{st} := \sum_{j=0}^{n-1} a_{1s-1}^{j+1} a_{1j}^{t+1}.$$

□

Lemma 14. *The ζ_0^2-term of $L(\zeta_0)TNL(\zeta_0)T$ is (w_{st}) with $s,t \in \{1,\cdots,n\}$, where*

$$w_{st} := \sum_{j=0}^{n-2} a_{1s-1}^{j+2} a_{1j}^{t}.$$

Proof. We have

$$QNQ = \begin{pmatrix} a_{10}^1 & a_{10}^2 & \cdots & a_{10}^n \\ a_{11}^1 & a_{11}^2 & \cdots & a_{11}^n \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n-1}^1 & a_{1n-1}^2 & \cdots & a_{1n-1}^n \end{pmatrix} \begin{pmatrix} 0 & 0 & \cdots & 0 \\ a_{10}^1 & a_{10}^2 & \cdots & a_{10}^n \\ a_{11}^1 & a_{11}^2 & \cdots & a_{11}^n \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n-2}^1 & a_{1n-2}^2 & \cdots & a_{1n-2}^n \end{pmatrix}$$

and therefore we have

$$w_{st} := \sum_{j=0}^{n-2} a_{1s-1}^{j+2} a_{1j}^{t}.$$

□

Theorem 14. *Let $\mathcal{F} \in Jac^{g-1}(C_n) \setminus \Theta$ be an invertible sheaf of degree $g-1$ not lying in the theta divisor. Then a representative of its corresponding $GL_n(\mathbb{C})$-conjugacy* *class of nilpotent, regular, matricial polynomials is given by* $A(\zeta) = A_0 + A_1\zeta + A_2\zeta^2$, *where*

$$A_0 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} -a_{10}^2 & -a_{10}^3 & \cdots & -a_{10}^n & 0 \\ a_{10}^1 - a_{11}^2 & a_{10}^2 - a_{11}^3 & \cdots & a_{10}^{n-1} - a_{11}^n & a_{10}^n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{1n-2}^1 - a_{1n-1}^2 & a_{1n-2}^2 - a_{1n-1}^3 & \cdots & a_{1n-2}^{n-1} - a_{1n-1}^n & a_{1n-2}^n \end{pmatrix},$$

$$A_2 = \begin{pmatrix} -a_{20}^2 & -a_{20}^3 & \cdots & -a_{20}^n & 0 \\ a_{20}^1 - a_{21}^2 & a_{20}^2 - a_{21}^3 & \cdots & a_{20}^{n-1} - a_{21}^n & a_{20}^n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{2n-2}^1 - a_{2n-1}^2 & a_{2n-2}^2 - a_{2n-1}^3 & \cdots & a_{2n-2}^{n-1} - a_{2n-1}^n & a_{2n-2}^n \end{pmatrix}$$
$$- \begin{pmatrix} w_{11} & w_{12} & \cdots & w_{1n} \\ w_{21} & w_{22} & \cdots & w_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{n1} & w_{n2} & \cdots & w_{nn} \end{pmatrix} + \begin{pmatrix} v_{11} & v_{12} & \cdots & v_{1n-1} & 0 \\ v_{21} & v_{22} & \cdots & v_{2n-1} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ v_{n1} & v_{n2} & \cdots & v_{nn-1} & 0 \end{pmatrix}.$$

Proof. The matrix A_1 is given by lemma 11. The matrix A_2 is given by lemma 12, lemma 13 and lemma 14. □

We have an immediate observation.

Corollary 4. *The matrices A_0, A_1, A_2 of theorem 14 are trace-free. In particular the matricial polynomial $A(\zeta) = A_0 + A_1\zeta + A_2\zeta^2$ induces an element of* $\check{\mathrm{H}}^0(\mathbb{CP}^1, \mathfrak{sl}_n(\mathbb{C}) \otimes \mathcal{O}_{\mathbb{CP}^1}(2))$.

Proof. Obviously A_0 is trace-free. The trace of A_1 vanishes, because it is the commutator of the two matrices Q and N, which is

$$-a_{10}^2 + \left(a_{10}^2 - a_{11}^3\right) + \cdots + a_{1n-2}^n = 0.$$

With the same argument the first matrix in the expression of A_2 has vanishing trace too. It remains to show, that the matrix $(v_{st})_{st} - (w_{st})_{st}$ is trace-free. Let us denote $v_{st}^j := a_{1s-1}^{j+1} a_{1j}^{t+1}$ and $w_{st}^j := a_{1s-1}^{j+2} a_{1j}^t$. So we have $w_{ss} = \sum_{j=0}^{n-2} w_{ss}^j$ and $v_{ss} = \sum_{j=0}^{n-1} v_{ss}^j$. The trace of the matrix $(v_{st})_{st} - (w_{st})_{st}$ is

$$\begin{aligned}\sum_{s=1}^n (v_{ss} - w_{ss}) &= \sum_{s=1}^n \left(\sum_{j=0}^{n-1} v_{ss}^j - \sum_{j=0}^{n-2} w_{ss}^j \right) = \sum_{s=1}^n \sum_{j=0}^{n-1} v_{ss}^j - \sum_{s=1}^n \sum_{j=0}^{n-2} w_{ss}^j \\ &= \sum_{j_0=0}^{n-1} \left(\sum_{s=1}^n v_{ss}^{j_0} - \sum_{j=0}^{n-2} w_{j_0+1,j_0+1}^j \right) = \sum_{j_0=0}^{n-1} \left(\sum_{s=1}^{n-1} v_{ss}^{j_0} - \sum_{j=0}^{n-2} w_{j_0+1,j_0+1}^j \right) \\ &= \sum_{j_0=0}^{n-1} \sum_{s=1}^{n-1} \left(v_{ss}^{j_0} - w_{j_0+1,j_0+1}^{s-1} \right),\end{aligned}$$

where we used $v_{nn} = 0$. But we have

$$v_{ss}^{j_0} = a_{1s-1}^{j_0+1} a_{1j_0}^{s+1} = w_{j_0+1,j_0+1}^{s-1}$$

and hence the trace vanishes. □

4.3.2 Trace $tr\left(A_0A_2 - \frac{1}{4}A_1^2\right)$

In this subsection we will use theorem 14 to compute the expression $tr\left(A_0A_2 - \frac{1}{4}A_1^2\right)$. An immediate corollary of the theorem is the following corollary.

Corollary 5. *With $q \in \{2, \dots, n\}$ and $r \in \{1, \dots, n-1\}$ we have*

$$(A_1)_{1r} = -a_{10}^{r+1}, \qquad (A_1)_{1n} = 0,$$
$$(A_1)_{qr} = a_{1q-2}^{r} - a_{1q-1}^{r+1}, \qquad (A_1)_{qn} = a_{1q-2}^{n}.$$

Furthermore we have

$$\begin{aligned}
(A_2)_{1r} &= -a_{20}^{r+1} - w_{1r} + v_{1r} \\
&= -a_{20}^{r+1} - \sum_{j=0}^{n-2} a_{10}^{j+2} a_{1j}^{r} + \sum_{j=0}^{n-1} a_{10}^{j+1} a_{1j}^{r+1}, \\
(A_2)_{1n} &= -w_{1n} = -\sum_{j=0}^{n-2} a_{10}^{j+2} a_{1j}^{n}, \\
(A_2)_{qr} &= a_{2q-2}^{r} - a_{2q-1}^{r+1} - w_{qr} + v_{qr} \\
&= a_{2q-2}^{r} - a_{2q-1}^{r+1} - \sum_{j=0}^{n-2} a_{1q-1}^{j+2} a_{1j}^{r} + \sum_{j=0}^{n-1} a_{1q-1}^{j+1} a_{1j}^{r+1}, \\
(A_2)_{qn} &= a_{2q-2}^{n} - w_{qn} = -a_{2q-2}^{n} - \sum_{j=0}^{n-2} a_{1q-1}^{j+2} a_{1j}^{n}.
\end{aligned}$$

In particular we have

$$(A_2)_{qq+1} = a_{2q-2}^{q+1} - a_{2q-1}^{q+2} - \sum_{j=0}^{n-2} a_{1q-1}^{j+2} a_{1j}^{q+1} + \sum_{j=0}^{n-1} a_{1q-1}^{j+1} a_{1j}^{q+2}.$$

In chapter 5 we will denote the term $(A_2)_{qq+1}$ by α_q.

Lemma 15. *We have*

$$\begin{aligned}
tr\,(A_0A_2) &= \sum_{q=1}^{n-1} \left(-\sum_{j=0}^{n-2} a_{1q-1}^{j+2} a_{1j}^{q+1} + \sum_{j=0}^{n-1} a_{1q-1}^{j+1} a_{1j}^{q+2} \right), \\
tr\,(A_1^2) &= \sum_{s=1}^{n} \sum_{q=1}^{n} \left(a_{1s-2}^{q} - a_{1s-1}^{q+1} \right) \left(a_{1q-1}^{s+1} - a_{1q-2}^{s} \right).
\end{aligned}$$

Proof. Because A_0 is the regular, nilpotent matrix with only one Jordan block and by corollary 5 we have

$$tr\,(A_0A_2) = \sum_{q=1}^{n-1}(A_2)_{qq+1} = \sum_{q=1}^{n-1}\left(a_{2q-1}^{q+2} - a_{2q-2}^{q+1} - \sum_{j=0}^{n-2} a_{1q-1}^{j+2}a_{1j}^{q+1} + \sum_{j=0}^{n-1} a_{1q-1}^{j+1}a_{1j}^{q+2}\right)$$
$$= \sum_{q=1}^{n-1}\left(-\sum_{j=0}^{n-2} a_{1q-1}^{j+2}a_{1j}^{q+1} + \sum_{j=0}^{n-1} a_{1q-1}^{j+1}a_{1j}^{q+2}\right).$$

Moreover, just by matrix multiplication, we have $(A_1^2)_{st} = \sum_{q=1}^{n}(A_1)_{sq}(A_1)_{qt}$ and so the diagonal elements of A_1^2 are

$$(A_1^2)_{ss} = \sum_{q=1}^{n}(A_1)_{sq}(A_1)_{qs} = \sum_{q=1}^{n}\left(a_{1s-1}^{q+1} - a_{1s-2}^{q}\right)\left(a_{1q-1}^{s+1} - a_{1q-2}^{s}\right).$$

Thus the trace of A_1^2 is

$$tr(A_1^2) = \sum_{s=1}^{n}(A_1^2)_{ss} = \sum_{s=1}^{n}\sum_{q=1}^{n}\left(a_{1s-1}^{q+1} - a_{1s-2}^{q}\right)\left(a_{1q-1}^{s+1} - a_{1q-2}^{s}\right).$$

□

Now we can state the main lemma in this subsection.

Lemma 16. *We have*

$$tr\left(A_0A_2 - \frac{1}{4}A_1^2\right) = \frac{3}{2}\left(\sum_{s=1}^{n}\sum_{q=1}^{n-2} a_{1q-1}^{s}a_{1s-1}^{q+2} - \sum_{s=1}^{n-1}\sum_{q=1}^{n-1} a_{1q-1}^{s+1}a_{1s-1}^{q+1}\right).$$

Proof. We combine the two expressions of lemma (15) and compute

$$tr\left(A_0A_2 - \frac{1}{4}A_1^2\right)$$
$$= \sum_{q=1}^{n-1}\left(-\sum_{j=0}^{n-2} a_{1q-1}^{j+2}a_{1j}^{q+1} + \sum_{j=0}^{n-1} a_{1q-1}^{j+1}a_{1j}^{q+2}\right) - \frac{1}{4}\sum_{s=1}^{n}\sum_{q=1}^{n}(a_{1s-1}^{q+1} - a_{1s-2}^{q})(a_{1q-1}^{s+1} - a_{1q-2}^{s})$$
$$= \sum_{q=1}^{n-1}\left(-\sum_{s=1}^{n-1} a_{1q-1}^{s+1}a_{1s-1}^{q+1} + \sum_{j=s}^{n} a_{1q-1}^{s}a_{1s-1}^{q+2}\right) - \frac{1}{4}\sum_{s=1}^{n-1}\sum_{q=1}^{n-1} a_{1q-1}^{s+1}a_{1s-1}^{q+1}$$
$$- \frac{1}{4}\sum_{s=1}^{n-1}\sum_{q=1}^{n-1} a_{1s-1}^{q+1}a_{1q-1}^{s+1} + \frac{1}{4}\sum_{s=1}^{n}\sum_{q=1}^{n-2} a_{1q-1}^{s}a_{1s-1}^{q+2} + \frac{1}{4}\sum_{s=1}^{n}\sum_{q=1}^{n-2} a_{1q-1}^{s}a_{1s-1}^{q+2}$$
$$= \frac{3}{2}\left(\sum_{s=1}^{n}\sum_{q=1}^{n-2} a_{1q-1}^{s}a_{1s-1}^{q+2} - \sum_{s=1}^{n-1}\sum_{q=1}^{n-1} a_{1q-1}^{s+1}a_{1s-1}^{q+1}\right).$$

□

4.3.3 Burning Lemma

This section is about a lemma, which we will use later to compare the trace

$$tr\left(A_0(t)A_2(t) - \frac{1}{4}A_1(t)^2\right)$$

and $\frac{3}{2}\frac{d^2}{dt^2}\log(\theta(\mathcal{F}^t))$. The invertible sheaf $\mathcal{F}^t$ is from section 4.3.4. The burning lemma is the heart of the direct proof of Hitchin's formula and it is called *burning lemma* because it *burns* out all unnecessary terms. We will always use the notation $C_{((i,j),(u,v))}$ for the cofactor of the matrix M by canceling out the (i,j)-row and the (u,v)-column. Similarly we use the notation $M_{((i,j),(u,v))}$ for the minor and a multiple subscripts if we cancel out several rows and columns.

Theorem 15 (Burning Lemma)**.** *Let $\mathcal{F} \in Jac^{g-1}(C_n) \setminus \Theta$ be an invertible sheaf not lying in the theta divisor and M the corresponding matrix.*

i) Let $(s,t) \in Q$ such that $s \leq n-4$ and $t \leq n-3$. Then we have

$$\sum_{(i,j)\in P} C_{((i,j),(s,t))} d_{i-s-2,j-t-2} = 0.$$

ii) Let $(s,t),(u,v) \in Q$ such that $(s,t) \neq (u+1,v+1)$ and $(u+1,v+1) \in Q$. Then we have

$$\sum_{(\alpha,\beta)\in P} C_{((\alpha,\beta),(s,t))} d_{\alpha-u-1,\beta-v-1} = 0.$$

iii) Let us fix a $1 \leq q \leq n$. With the indices $(s,t) = (n-2, q-1)$ and $(u,v) = (n-3, q-2)$ we get $(s,t) = (u+1, v+1)$. Then we have

$$\sum_{(\alpha,\beta)\in P}\sum_{(a,b)\in P} C_{((\alpha,\beta),(s,t))}(-d_{\alpha-u-1,\beta-v-1}) C_{((a,b),(u,v))}(-d_{a-s-1,b-t-1})$$
$$= \theta\left(\sum_{(a,b)\in P} C_{((a,b),(u,v))} d_{a-u-2,b-v-2}\right).$$

Furthermore with the elements $(u,v) = (n-2, q-1)$ and $(s,t) = (n-3, q-2)$ we have $(u,v) = (s+1, t+1)$. Then we have

$$\sum_{(\alpha,\beta)\in P}\sum_{(a,b)\in P} C_{((\alpha,\beta),(s,t))}(-d_{\alpha-u-1,\beta-v-1}) C_{((a,b),(u,v))}(-d_{a-s-1,b-t-1})$$
$$= \left(\sum_{(\alpha,\beta)\in P} C_{((\alpha,\beta),(s,t))} d_{\alpha-s-2,\beta-t-2}\right)\theta.$$

iv) Let us consider indices $(s,t),(u,v) \in Q$ with $s \neq n-2$ and $u \neq n-2$ satisfying $(s,t) = (u+1, v+1)$. Then we have

$$\sum_{(\alpha,\beta)\in P}\sum_{(a,b)\in P} C_{((\alpha,\beta),(s,t))}(-d_{\alpha-u-1,\beta-v-1}) C_{((a,b),(u,v))}(-d_{a-s-1,b-t-1}) = 0.$$

Furthermore if we have elements $(s,t),(u,v) \in Q$ with $s \neq n-2$ and $u \neq n-2$ satisfying $(u,v) = (s+1,t+1)$. Then we have

$$\sum_{(\alpha,\beta)\in P}\sum_{(a,b)\in P} C_{((\alpha,\beta),(s,t))}(-d_{\alpha-u-1,\beta-v-1})C_{((a,b),(u,v))}(-d_{a-s-1,b-t-1}) = 0.$$

v) We have the equation

$$\sum_{(s,t)\in Q}\sum_{(i,j)\in P}\frac{C_{((i,j),(s,t))}}{\theta}d_{i-s-2,j-t-2} = \sum_{\substack{(s,t)\in Q\\ n-3\leq s\leq n-2}}\sum_{(i,j)\in P}\frac{C_{((i,j),(s,t))}}{\theta}d_{i-s-2,j-t-2}.$$

vi) We have the equation

$$\begin{aligned}&\sum_{(s,t)\in Q}\sum_{(u,v)\in Q}\sum_{(\alpha,\beta)\in P}\sum_{(a,b)\in P}\frac{C_{((\alpha,\beta),(s,t))}}{\theta}(-d_{\alpha-u-1,\beta-v-1})\frac{C_{((a,b),(u,v))}}{\theta}(-d_{a-s-1,b-t-1})\\ &=\sum_{\substack{(s,t)\in Q\\ s=n-2}}\sum_{\substack{(u,v)\in Q\\ u=n-2}}\sum_{(\alpha,\beta)\in P}\sum_{(a,b)\in P}\frac{C_{((\alpha,\beta),(s,t))}}{\theta}(-d_{\alpha-u-1,\beta-v-1})\frac{C_{((a,b),(u,v))}}{\theta}(-d_{a-s-1,b-t-1})\\ &\quad+2\frac{\theta}{\theta}\sum_{\substack{(u,v)\in Q\\ u=n-3}}\sum_{(a,b)\in P}\frac{C_{((a,b),(u,v))}}{\theta}d_{a-u-2,b-v-2}).\end{aligned}$$

vii) Let $(a,b),(\alpha,\beta) \in P$. Then we have

$$\sum_{t=1}^{n} C_{((a,b),(n-2,t-1))}d_{\alpha-n+1,\beta-t+1} = \theta\delta_{(a,b),(\alpha-1,\beta)} - \sum_{\substack{(s,t)\in Q\\ s\neq n-2}} C_{((a,b),(s,t))}d_{\alpha-s-1,\beta-t}.$$

Since there will be a lot of indices, we want to compute a little archetyplical example. It describes the idea behind the burning lemma.

Example 6. *Let us consider the matrix*

$$D := \begin{pmatrix} d_{00} & d_{11}\\ d_{11} & d_{22}\end{pmatrix}.$$

The determinant is $\det(D) = d_{00}d_{22} - d_{11}^2$, which we write as Laplace expansion $\det(D) = C_{11}d_{00} + C_{12}d_{11} = \overbrace{M_{11}}^{=d_{22}}d_{00} - \overbrace{M_{12}}^{=d_{11}}d_{11}$. With C indicating the cofactors and M the minors. But we also have

$$C_{11}d_{11} + C_{12}d_{22} = \overbrace{M_{11}}^{=d_{22}}d_{11} - \overbrace{M_{12}}^{=d_{11}}d_{22} = 0.$$

The summation of cofactors multiplied with its elements in a neighbor column vanishes.

For the rest of this section we use the abbreviation ι_r for the function ι_{row} and ι_c for the function ι_{column} of definition 5 to indicate the number of the row and the column and it allows us to indicate the signs of the cofactors precisely.

Proof. i) First we observe that if $(s,t) \in Q$ with $s \le n-4$ and $t \le n-3$, then $(s+2,t+2) \in Q$ too. We Laplace expand the minor $M_{((i,j),(s,t))}$ along the $(s+2,t+2)$-th column. This means in a formula

$$\begin{aligned} M_{((i,j),(s,t))} &= \sum_{\substack{(a,b)\in P\\(a,b)\neq(i,j)}} C_{\substack{((i,j),(s,t))\\((a,b),(s+2,t+2))}} d_{a-s-2,b-t-2} \\ &= \sum_{\substack{(a,b)\in P\\(a,b)\neq(i,j)}} (-1)^{\iota_r(a,b)-\delta_{(a,b)>(i,j)}}(-1)^{\iota_c(s+2,t+2)-\delta_{(s+2,t+2)>(s,t)}} M_{\substack{((i,j),(s,t))\\((a,b),(s+2,t+2))}} d_{a-s-2,b-t-2}. \end{aligned}$$

We sum up and get

$$\begin{aligned} &\sum_{(i,j)\in P} C_{((i,j),(s,t))} d_{i-s-2,j-t-2} = \sum_{(i,j)\in P} (-1)^{\nu(i,j,s,t)} M_{((i,j),(s,t))} d_{i-s-2,j-t-2} \\ &= \sum_{(i,j)\in P} \sum_{\substack{(a,b)\in P\\(a,b)\neq(i,j)}} (-1)^{\nu(i,j,s,t)} (-1)^{\iota_r(a,b)-\delta_{(a,b)>(i,j)}}(-1)^{\iota_c(s+2,t+2)-\delta_{(s+2,t+2)>(s,t)}} \\ &\qquad \cdot M_{\substack{((i,j),(s,t))\\((a,b),(s+2,t+2))}} d_{a-s-2,b-t-2} d_{i-s-2,j-t-2}. \end{aligned}$$

Now we consider an index of the double sum $((i_0,j_0),(a_0,b_0)) \in P \times P$ with $(i_0,j_0) \neq (a_0,b_0)$. Since $(i_0,j_0) \neq (a_0,b_0)$ there appears the double index $((a_0,b_0),(i_0,j_0)) \in P \times P$ in the double sum too. The summand of the index $((i_0,j_0),(a_0,b_0))$ is

$$\begin{aligned} &(-1)^{\nu(i_0,j_0,s,t)}(-1)^{\iota_r(a_0,b_0)-\delta_{(a_0,b_0)>(i_0,j_0)}}(-1)^{\iota_c(s+2,t+2)-\delta_{(s+2,t+2)>(s,t)}} \\ &\quad \cdot M_{\substack{((i_0,j_0),(s,t))\\((a_0,b_0),(s+2,t+2))}} d_{a_0-s-2,b_0-t-2} d_{i_0-s-2,j_0-t-2} \end{aligned}$$

and the summand of the index $((a_0,b_0),(i_0,j_0))$ is

$$\begin{aligned} &(-1)^{\nu(a_0,b_0,s,t)}(-1)^{\iota_r(i_0,j_0)-\delta_{(i_0,j_0)>(a_0,b_0)}}(-1)^{\iota_c(s+2,t+2)-\delta_{(s+2,t+2)>(s,t)}} \\ &\quad \cdot M_{\substack{((a_0,b_0),(s,t))\\(i_0,j_0),(s+2,t+2)}} d_{i_0-s-2,j_0-t-2} d_{a_0-s-2,b_0-t-2}. \end{aligned}$$

The minors are equal and so these two summands differ only by the sign. Note that $\nu(a,b,s+2,t+2) = \nu(a,b,s,t)$ and thus we have

$$\begin{aligned} &(-1)^{\nu(i_0,j_0,s,t)}(-1)^{\iota_r(a_0,b_0)-\delta_{(a_0,b_0)>(i_0,j_0)}}(-1)^{\iota_c(s+2,t+2)-\delta_{(s+2,t+2)>(s,t)}} \\ &= (-1)^{\nu(i_0,j_0,s,t)}(-1)^{\nu(a_0,b_0,s+2,t+2)}(-1)^{-\delta_{(a_0,b_0)>(i_0,j_0)}-\delta_{(s+2,t+2)>(s,t)}} \\ &= (-1)^{\nu(a_0,b_0,s,t)}(-1)^{\nu(i_0,j_0,s+2,t+2)}(-1)(-1)^{\delta_{(i_0,j_0)>(a_0,b_0)}+\delta_{(s+2,t+2)>(s,t)}} \\ &= (-1)(-1)^{\nu(a_0,b_0,s,t)}(-1)^{\iota_r(i_0,j_0)-\delta_{(i_0,j_0)>(a_0,b_0)}}(-1)^{-\iota_c(s+2,t+2)-\delta_{(s+2,t+2)>(s,t)}}. \end{aligned}$$

In other words the summands are equal with opposite sign. Therefore in the double sum for each double index we find exactly one other double index such that the summands cancel each other out. Hence we have proved the claim.

ii) We follow the same strategy as above. We expand the $M_{(\alpha,\beta)(s,t)}$ minor along the $(u+1, v+1)$-th column and we get

$$\sum_{(\alpha,\beta)\in P} C_{((\alpha,\beta),(s,t)}d_{\alpha-u-1,\beta-v-1} = \sum_{(\alpha,\beta)\in P} (-1)^{\nu(\alpha,\beta,s,t)} M_{((\alpha,\beta),(s,t)}d_{\alpha-u-1,\beta-v-1}$$
$$= \sum_{(\alpha,\beta)\in P} (-1)^{\nu(\alpha,\beta,s,t)} \sum_{\substack{(a,b)\in P\\(\alpha,\beta)\neq(a,b)}} (-1)^{\iota_r(a,b)-\delta_{(a,b)>(\alpha,\beta)}}(-1)^{\iota_c(u+1,v+1)-\delta_{(u+1,v+1)>(s,t)}}$$
$$\cdot M_{\substack{((\alpha,\beta),(s,t))\\((a,b),(u+1,v+1))}} d_{a-u-1,b-v-1}d_{\alpha-u-1,\beta-v-1}.$$

Again by comparing $((\alpha_0,\beta_0),(a_0,b_0)) \in P\times P$ and $((a_0,b_0),(\alpha_0,\beta_0)) \in P\times P$ we see, the summands only differ by the sign, which is exactly the opposite. And therefore they canceling each other out.

iii) Note that $s = n-2$ and $(s,t) = (u+1, v+1)$ implies $u = n-3$. Just by the Laplace expansion we have

$$\sum_{(\alpha,\beta)\in P}\sum_{(a,b)\in P} C_{((\alpha,\beta),(s,t))}(-d_{\alpha-u-1,\beta-v-1})C_{((a,b),(u,v))}(-d_{a-s-1,b-t-1})$$
$$= \left(\sum_{(\alpha,\beta)\in P} C_{((\alpha,\beta),(u+1,v+1))}d_{\alpha-u-1,\beta-v-1}\right)\left(\sum_{(a,b)\in P} C_{((a,b),(u,v))}d_{a-(u+1)-1,b-(v+1)-1}\right)$$
$$= \theta\left(\sum_{(a,b)\in P} C_{((a,b),(u,v))}d_{a-u-2,b-v-2}\right).$$

The case $u = n-2$ and $(u,v) = (s+1, t+1)$ follows analogous.

iv) Note that $s \leq n-3$ and $(s,t) = (u+1,v+1)$ implies $u \leq n-4$. We have by the Laplace expansion

$$\sum_{(\alpha,\beta)\in P}\sum_{(a,b)\in P} C_{((\alpha,\beta),(s,t))}(-d_{\alpha-u-1,\beta-v-1})C_{((a,b),(u,v))}(-d_{a-s-1,b-t-1})$$
$$= \left(\sum_{(\alpha,\beta)\in P} C_{((\alpha,\beta),(u+1,v+1))}d_{\alpha-u-1,\beta-v-1}\right)\left(\sum_{(a,b)\in P} C_{((a,b),(u,v))}d_{a-(u+1)-1,b-(v+1)-1}\right)$$
$$= \theta\left(\sum_{(a,b)\in P} C_{((a,b),(u,v))}d_{a-u-2,b-v-2}\right).$$

But the sum in the right parenthesis, because $u \leq n-4$, is of the form of *i*) and therefore vanishes.

The case $(u,v) = (s+1,t+1)$ follows analogous.

v) This follows by *i*).

vi) This follows by *ii*) and *iii*).

vii) The case $(a,b) = (\alpha-1,\beta)$ is just the usual Laplace expansion in a complicated way written down. For the other case we have to show

$$\sum_{(s,t)\in Q} C_{((a,b),(s,t))}d_{\alpha-s-1,\beta-t} = 0.$$

We follow the same strategy as in i) and expand the minor $M_{((a,b),(s,t))}$ along the $(\alpha-1,\beta)$-row. This means we have

$$M_{((a,b),(s,t))} = \sum_{\substack{(u,v)\in Q\\(u,v)\neq(s,t)}} (-1)^{\iota_r(\alpha-1,\beta)-\delta_{(\alpha-1,\beta)>(a,b)}} (-1)^{\iota_c(u,v)-\delta_{(u,v)>(s,t)}} \cdot M_{\substack{((a,b),(s,t))\\((\alpha-1,\beta),(u,v))}} d_{\alpha-1-u,\beta-v}.$$

Hence we have

$$\begin{aligned}&\sum_{(s,t)\in Q} C_{((a,b),(s,t))} d_{\alpha-s-1,\beta-t}\\ &= \sum_{(s,t)\in Q} (-1)^{\nu(a,b,s,t)} \sum_{\substack{(u,v)\in Q\\(u,v)\neq(s,t)}} (-1)^{\iota_r(\alpha-1,\beta)-\delta_{(\alpha-1,\beta)>(a,b)}} (-1)^{\iota_c(u,v)-\delta_{(u,v)>(s,t)}}\\ &\quad \cdot M_{\substack{((a,b),(s,t))\\((\alpha-1,\beta),(u,v))}} d_{\alpha-1-u,\beta-v} d_{\alpha-s-1,\beta-t}.\end{aligned}$$

We fix again two double indices $((s_0,t_0),(u_0,v_0)) \in Q$ and $((u_0,v_0),(s_0,t_0)) \in Q$ with $(s_0,t_0) \neq (u_0,v_0)$. The summand of the index $((s_0,t_0),(u_0,v_0))$ is

$$\begin{aligned}&(-1)^{\nu(a,b,s_0,t_0)} (-1)^{\iota_r(\alpha-1,\beta)-\delta_{(\alpha-1,\beta)>(a,b)}} (-1)^{\iota_c(u_0,v_0)-\delta_{(u_0,v_0)>(s_0,t_0)}}\\ &\quad \cdot M_{\substack{((a,b),(s_0,t_0))\\((\alpha-1,\beta),(u_0,v_0))}} d_{\alpha-1-u_0,\beta-v_0} d_{\alpha-s_0-1,\beta-t_0}\end{aligned}$$

and the summand of the index $((s_0,t_0),(u_0,v_0))$ is

$$\begin{aligned}&(-1)^{\nu(a,b,u_0,v_0)} (-1)^{\iota_r(\alpha-1,\beta)-\delta_{(\alpha-1,\beta)>(a,b)}} (-1)^{\iota_c(s_0,t_0)-\delta_{(s_0,t_0)>(u_0,v_0)}}\\ &\quad \cdot M_{\substack{((a,b),(u_0,v_0))\\((\alpha-1,\beta),(s_0,t_0))}} d_{\alpha-1-s_0,\beta-t_0} d_{\alpha-u_0-1,\beta-v_0}.\end{aligned}$$

But the signs are opposite since it differs by the terms $(-1)^{\delta_{(u_0,v_0)>(s_0,t_0)}}$ and $(-1)^{\delta_{(s_0,t_0)>(u_0,v_0)}}$. Hence the two summands are equal with opposite sign and therefore in the summation they cancel each other out. □

4.3.4 Derivatives of the Theta function

In this subsection we want to compute $\frac{d^2}{dt}\log\left(\theta(\mathcal{F}^t)\right)$ and reduce the number of terms in its expression via the burning lemma.

Let $\mathcal{F} \in Jac^{g-1}(C_n) \setminus \Theta$ and $\mathcal{L}^t \in Jac^0(C_n)$ be a family of invertible sheaves corresponding to the transition function $\exp\left(-t\frac{\eta}{\zeta}\right)$ for $t \in \mathbb{C}$. Then we define $\mathcal{F}^t := \mathcal{F} \otimes \mathcal{L}^t \in Jac^{g-1}(C_n)$. If M is the matrix corresponding to $\mathcal{F}$ with $\theta(\mathcal{F}) = \det(M)$, then we denote the corresponding matrix of $\mathcal{F}^t$ with $\theta\left(\mathcal{F}^t\right) = \det\left(M(t)\right)$ by $M(t)$. We know already by theorem 10, that the matrix $M(t)$ is obtained by replacing the elements d_{kl} (even if they are 0) in the matrix M by the $d_{kl}(t)$ below. The $d_{kl}(t)$ are polynomials in the variable t and we have

$$\begin{aligned} d_{kl}(t) &:= \sum_{j=0}^{k}(-1)^j d_{k-j,l-j}\frac{t^j}{j!}, \\ d'_{kl}(t) &= -\sum_{j=0}^{k}(-1)^j d_{k-j-1,l-j-1}\frac{t^j}{j!} = -d_{k-1l-1}(t), \\ d''_{kl}(t) &= \sum_{j=0}^{k}(-1)^j d_{k-j-2,l-j-2}\frac{t^j}{j!} = d_{k-2l-2}(t). \end{aligned}$$

The matrices $M(t), M'(t), M''(t)$ are then just given by the entries at $(i,j) \in P$ and $(s,t) \in Q$,

$$\begin{aligned} (M(t))_{(i,j)(s,t)} &= \quad d_{i-s,j-t}(t), \\ (M'(t))_{(i,j)(s,t)} &= -d_{i-s-1,j-t-1}(t), \\ (M''(t))_{(i,j)(s,t)} &= \quad d_{i-s-2,j-t-2}(t). \end{aligned}$$

In the rest of the section we drop the arguments "(t)" and "$(\mathcal{F}^t)$" in the formulas.

Lemma 17. *Let us assume $\mathcal{F}^t$ does not lie in the theta divisor for all t in a neighborhood of $t_0 \in \mathbb{C}$. Then, in this neighborhood of t_0, we have*

$$\frac{3}{2}\frac{d^2}{dt^2}\log(\theta) = \frac{3}{2}\frac{\theta''\theta - \theta'\theta'}{\theta^2} = \frac{3}{2}\left(tr\left(M^{-1}M''\right) - tr\left(\left(M^{-1}M'\right)^2\right)\right).$$

Proof. With the differential rule of the determinant we have

$$\theta' = tr\left(M^{-1}M'\right)\det(M) = tr\left(M^{-1}M'\right)\theta.$$

Furthermore we compute

$$\begin{aligned} \theta'' &= \frac{d^2}{dt^2}\det(M) = \frac{d}{dt}tr(M^{-1}M')\det(M) \\ &= tr\left(\frac{d}{dt}(M^{-1}M')\right)\det(M) + tr\left(M^{-1}M'\right)tr\left(M^{-1}M'\right)\det(M) \\ &= tr\left(-M^{-1}M'M^{-1}M' + M^{-1}M''\right)\theta + tr\left(M^{-1}M'\right)tr\left(M^{-1}M'\right)\theta. \end{aligned}$$

Hence we finish the proof with the following computation

$$\begin{aligned} \frac{\theta''\theta - \theta'\theta'}{\theta^2} &= tr(-(M^{-1}M')^2 + M^{-1}M'') + tr(M^{-1}M')^2 - tr(M^{-1}M')^2 \\ &= tr(-(M^{-1}M')^2) + tr(M^{-1}M''). \end{aligned}$$

□

Now we compute the terms $tr\left((M^{-1}M')^2\right)$ and $tr\left(M^{-1}M''\right)$. We need these two computations to prove Hitchin's formula. We will use the burning lemma very frequently to reduce the number of summands. It is indicated by $\stackrel{bl}{=}$.

Theorem 16. *We have*

i)

$$tr\left(M^{-1}M''\right) = \frac{1}{\theta}\sum_{\substack{(s,t)\in Q\\ n-2\leq s\leq n-3}}\left(\sum_{(i,j)\in P} C_{((i,j),(s,t))}d_{i-s-2,j-t-2}\right),$$

ii) $tr((M^{-1}M')^2)$

$$= \sum_{\substack{(s,t)\in Q\\ s=n-2}}\sum_{\substack{(u,v)\in Q\\ u=n-2}}\sum_{(\alpha,\beta)\in P}\sum_{(a,b)\in P}\frac{C_{((\alpha,\beta),(s,t))}}{\theta}d_{\alpha-u-1,\beta-v-1}\frac{C_{((a,b),(u,v))}}{\theta}d_{a-s-1,b-t-1}$$
$$-2\frac{\theta}{\theta}\sum_{\substack{(u,v)\in Q\\ u=n-3}}\sum_{(a,b)\in P}\frac{C_{((a,b),(u,v))}}{\theta}d_{a-u-2,b-v-2}$$

and

iii) $tr(M^{-1}M'') - tr\left((M^{-1}M')^2\right)$

$$= \sum_{\substack{(s,t)\in Q\\ s=n-2}}\sum_{(i,j)\in P}\frac{C_{((i,j),(s,t))}}{\theta}d_{i-s-2,j-t-2} - \sum_{\substack{(s,t)\in Q\\ s=n-3}}\sum_{(i,j)\in P}\frac{C_{((i,j),(s,t))}}{\theta}d_{i-s-2,j-t-2}$$
$$-\sum_{\substack{(s,t)\in Q\\ s=n-2}}\sum_{\substack{(u,v)\in Q\\ u=n-2}}\sum_{(\alpha,\beta)\in P}\sum_{(a,b)\in P}\frac{C_{((\alpha,\beta),(s,t))}}{\theta}d_{\alpha-u-1,\beta-v-1})\frac{C_{((a,b),(u,v))}}{\theta}d_{a-s-1,b-t-1}.$$

Proof. Recall that the inverse is given by $M^{-1} = \frac{1}{\theta}(C_{((i,j)(s,t))})^T_{(i,j)(s,t)}$. Therefore we have

$$(M^{-1}M')_{(s,t)(u,v)} = \sum_{(i,j)\in P}\frac{C_{((i,j),(s,t))}}{\theta}m'_{(i,j)(u,v)} = \sum_{(i,j)\in P}\frac{C_{((i,j),(s,t))}}{\theta}(-d_{i-u-1,j-v-1}),$$
$$(M^{-1}M'')_{(s,t)(u,v)} = \sum_{(i,j)\in P}\frac{C_{((i,j),(s,t))}}{\theta}m''_{(i,j)(u,v)} = \sum_{(i,j)\in P}\frac{C_{((i,j),(s,t))}}{\theta}d_{i-u-2,j-v-2}.$$

Furthermore we have

$$(M^{-1}M')^2_{(s,t)(s,t)} = \sum_{(u,v)\in Q}(M^{-1}M')_{(s,t)(u,v)}(M^{-1}M')_{(u,v)(s,t)}$$
$$= \sum_{(u,v)\in Q}\left(\sum_{(i,j)\in P}\frac{C_{((i,j),(s,t))}}{\theta}(-d_{i-u-1,j-v-1})\right)\left(\sum_{(i,j)\in P}\frac{C_{((i,j),(u,v))}}{\theta}(-d_{i-s-1,j-t-1})\right).$$

With the *burning lemma*, theorem 15, we can compute the traces. We have

$$tr(M^{-1}M'') = \sum_{(s,t)\in Q}(M^{-1}M'')_{(s,t)(s,t)} = \frac{1}{\theta}\sum_{(s,t)\in Q}\left(\sum_{(i,j)\in P}C_{((i,j),(s,t))}d_{i-s-2,j-t-2}\right)$$
$$\stackrel{bl}{=}\frac{1}{\theta}\sum_{\substack{(s,t)\in Q\\ n-2\leq s\leq n-3}}\left(\sum_{(i,j)\in P}C_{((i,j),(s,t))}d_{i-s-2,j-t-2}\right)$$

and

$$
\begin{aligned}
tr((M^{-1}M')^2) &= \sum_{(s,t)\in Q} (M^{-1}M')^2_{(s,t)(s,t)} \\
&= \sum_{(s,t)\in Q} \sum_{(u,v)\in Q} \left(\sum_{(i,j)\in P} \frac{C_{((i,j),(s,t))}}{\theta} (-d_{i-u-1,j-v-1}) \right) \\
&\quad \cdot \left(\sum_{(i,j)\in P} \frac{C_{((i,j),(u,v))}}{\theta} (-d_{i-s-1,j-t-1}) \right) \\
&= \sum_{(s,t)\in Q} \sum_{(u,v)\in Q} \sum_{(\alpha,\beta)\in P} \sum_{(a,b)\in P} \frac{C_{((\alpha,\beta),(s,t))}}{\theta} d_{\alpha-u-1,\beta-v-1} \frac{C_{((a,b),(u,v))}}{\theta} d_{a-s-1,b-t-1} \\
&\overset{bl}{=} \sum_{\substack{(s,t)\in Q\\ s=n-2}} \sum_{\substack{(u,v)\in Q\\ u=n-2}} \sum_{(\alpha,\beta)\in P} \sum_{(a,b)\in P} \frac{C_{((\alpha,\beta),(s,t))}}{\theta} d_{\alpha-u-1,\beta-v-1} \frac{C_{((a,b),(u,v))}}{\theta} d_{a-s-1,b-t-1} \\
&\quad - 2\frac{\theta}{\theta} \sum_{\substack{(u,v)\in Q\\ u=n-3}} \sum_{(a,b)\in P} \frac{C_{((a,b),(u,v))}}{\theta} d_{a-u-2,b-v-2}.
\end{aligned}
$$

This shows i) and ii). iii) follows by combining i) and ii). □

4.3.5 Statement and Proof of Hitchin's Formula

Finally we have all necessary lemmas to state and prove Hitchin's Formula on the nilpotent, spectral curve $(C_n, \mathcal{O}_{C_n})$.

Theorem 17. *Let $\mathcal{F}$ be any invertible sheaf in the Jacobian $Jac^{g-1}(C_n) \setminus \Theta$. Let $\mathcal{L}^t$ be the family of invertible sheaves corresponding to the transition function* $\exp\left(-t\frac{\eta}{\zeta}\right)$ *with $t \in \mathbb{C}$ and let $\mathcal{F}^t := \mathcal{F} \otimes \mathcal{L}^t \in Jac^{g-1}(C_n)$. Let $A(\zeta)(t) = A_0(t) + A_1(t)\zeta + A_2(t)\zeta^2$ be the corresponding regular, nilpotent, matricial polynomial to the invertible sheaf $\mathcal{F}^t$ as long as $\mathcal{F}^t \notin \Theta$. Then for each t such that $\mathcal{F}^t$ does not lie in the theta divisor holds the equation*

$$tr\left(A_0(t)A_2(t) - \frac{1}{4}A_1(t)^2\right) = \frac{3}{2}\frac{d^2}{dt^2}\log\left(\theta(\mathcal{F}^t)\right).$$

In order to prove this theorem we will do some smaller computations and at the end we patch everything together. When we write $\overset{sv}{=}$ this means "swapping the variable" and it indicates the commutativitiy $A \cdot b \cdot C \cdot d = A \cdot d \cdot C \cdot b$.

Lemma 18. *Let us fix a $q \in \{1, \dots, n-2\}$. Then we have*

$$
\begin{aligned}
\sum_{r=1}^{n} a^r_{1q-1} a^{q+2}_{1r-1} &= \frac{1}{\theta} \sum_{(\alpha,\beta)\in P} C_{((\alpha,\beta),(n-2,q-1))} d_{\alpha-(n-2)-2,\beta-(q-1)-2} \\
&\quad - \frac{1}{\theta} \sum_{(a,b)\in P} C_{((a,b),(n-3,q-2))} d_{a-(n-3)-2,b-(q-2)-2}.
\end{aligned}
$$

Proof. We compute

$$\sum_{r=1}^{n} a^r_{1q-1} a^{q+2}_{1r-1}$$

$$= \sum_{r=1}^{n} \left(- \sum_{(\alpha,\beta)\in P} \frac{C_{((\alpha,\beta),(n-2,q-1))}}{\theta} d_{\alpha-n+1,\beta-r+1} \right)$$

$$\cdot \left(- \sum_{(a,b)\in P} \frac{C_{((a,b),(n-2,r-1))}}{\theta} d_{a-n+1,b-(q+2)+1} \right)$$

$$= \sum_{r=1}^{n} \sum_{(\alpha,\beta)\in P} \sum_{(a,b)\in P} \frac{C_{((\alpha,\beta),(n-2,q-1))}}{\theta} d_{\alpha-n+1,\beta-r+1} \frac{C_{((a,b),(n-2,r-1))}}{\theta} d_{a-n+1,b-q-1}$$

$$\overset{sv}{=} \sum_{r=1}^{n} \sum_{(\alpha,\beta)\in P} \sum_{(a,b)\in P} \frac{C_{((\alpha,\beta),(n-2,q-1))}}{\theta} d_{a-n+1,b-q-1} \frac{C_{((a,b),(n-2,r-1))}}{\theta} d_{\alpha-n+1,\beta-r+1}$$

$$= \sum_{(\alpha,\beta)\in P} \sum_{(a,b)\in P} \frac{C_{((\alpha,\beta),(n-2,q-1))}}{\theta} d_{a-n+1,b-q-1} \left(\sum_{r=1}^{n} \frac{C_{((a,b),(n-2,r-1))}}{\theta} d_{\alpha-n+1,\beta-r+1} \right)$$

$$\overset{bl}{=} \sum_{(\alpha,\beta)\in P} \sum_{(a,b)\in P} \frac{C_{((\alpha,\beta),(n-2,q-1))}}{\theta} d_{a-n+1,b-q-1} \Big(\frac{\theta}{\theta} \delta_{\substack{a=\alpha-1,\\ b=\beta}} - \sum_{\substack{(s,t)\in Q\\ s\neq n-2}} \frac{C_{((a,b),(s,t))}}{\theta} d_{\alpha-s-1,\beta-t} \Big)$$

$$= \sum_{(\alpha,\beta)\in P} \sum_{(a,b)\in P} \frac{C_{((\alpha,\beta),(n-2,q-1))}}{\theta} d_{a-n+1,b-q-1} \frac{\theta}{\theta} \delta_{\substack{a=\alpha-1,\\ b=\beta}}$$

$$- \sum_{(\alpha,\beta)\in P} \sum_{(a,b)\in P} \left(\frac{C_{((\alpha,\beta),(n-2,q-1))}}{\theta} d_{a-n+1,b-q-1} \right) \left(\sum_{\substack{(s,t)\in Q\\ s\neq n-2}} \frac{C_{((a,b),(s,t))}}{\theta} d_{\alpha-s-1,\beta-t} \right)$$

$$\overset{sv}{=} \sum_{(\alpha,\beta)\in P} \frac{C_{((\alpha,\beta),(n-2,q-1))}}{\theta} d_{(\alpha-1)-n+1,\beta-(q-1)-2} \frac{\theta}{\theta}$$

$$- \sum_{\substack{(s,t)\in Q\\ s\neq n-2}} \sum_{(a,b)\in P} \left(\frac{C_{((a,b),(s,t))}}{\theta} d_{a-n+1,b-(q-1)-2} \right) \left(\sum_{(\alpha,\beta)\in P} \frac{C_{((\alpha,\beta),(n-2,q-1))}}{\theta} d_{\alpha-s-1,\beta-t} \right)$$

$$\overset{bl,ii)}{=} \sum_{(\alpha,\beta)\in P} \frac{C_{((\alpha,\beta),(n-2,q-1))}}{\theta} d_{\alpha-(n-2)-2,\beta-(q-1)-2} \frac{\theta}{\theta}$$

$$- \sum_{(a,b)\in P} \sum_{(\alpha,\beta)\in P} \frac{C_{((a,b),(n-3,q-2))}}{\theta} d_{a-n+1,b-(q-1)-2} \frac{C_{((\alpha,\beta),(n-2,q-1))}}{\theta} d_{\alpha-(n-3)-1,\beta-(q-2)}$$

$$\overset{bl,iii)}{=} \sum_{(\alpha,\beta)\in P} \frac{C_{((\alpha,\beta),(n-2,q-1))} 1}{\theta} d_{\alpha-(n-2)-2,\beta-(q-1)-2} \frac{\theta}{\theta}$$

$$- \sum_{(a,b)\in P} \frac{C_{((a,b),(n-3,q-2))}}{\theta} d_{a-(n-3)-2,b-(q-2)-2} \frac{\theta}{\theta}.$$

□

Lemma 19. *For all $r, q \in \{1, ..., n-1\}$ we have*

$$a^{r+1}_{1q-1}a^{q+1}_{1r-1} = \sum_{\substack{(s,t)\in Q\\ s=n-2\\ t=q-1}}\sum_{\substack{(u,v)\in Q\\ u=n-2\\ v=r-1}}\sum_{(\alpha,\beta)\in P}\sum_{(a,b)\in P}\frac{C_{((\alpha,\beta),(s,t))}}{\theta}d_{\alpha-u-1,\beta-v-1}\frac{C_{((a,b),(u,v))}}{\theta}d_{a-s-1,b-s-1}.$$

Proof. We compute

$$\begin{aligned}
&a^{r+1}_{1q-1}a^{q+1}_{1r-1}\\
&=\Big(\sum_{(\alpha,\beta)\in P}\frac{C_{((\alpha,\beta),(n-2,q-1))}}{\theta}d_{\alpha-n+1,\beta-(r+1)+1}\Big)\Big(\sum_{(a,b)\in P}\frac{C_{((a,b),(n-2,r-1))}}{\theta}d_{a-n+1,b-(q+1)+1}\Big)\\
&=\sum_{(\alpha,\beta)\in P}\sum_{(a,b)\in P}\frac{C_{((\alpha,\beta),(n-2,q-1))}}{\theta}d_{\alpha-n+1,\beta-(r+1)+1}\frac{C_{((a,b),(n-2,r-1))}}{\theta}d_{a-n+1,b-(q+1)+1}\\
&=\sum_{\substack{(s,t)\in Q\\ s=n-2\\ t=q-1}}\sum_{\substack{(u,v)\in Q\\ u=n-2\\ v=r-1}}\sum_{(\alpha,\beta)\in P}\sum_{(a,b)\in P}\frac{C_{((\alpha,\beta),(s,t))}}{\theta}d_{\alpha-u-1,\beta-v-1}\frac{C_{((a,b),(u,v))}}{\theta}d_{a-s-1,b-t-1}.
\end{aligned}$$

□

Now we have everything necessary to prove theorem 17.

Proof. From lemma 16 we know

$$tr\left(A_0A_2-\frac{1}{4}A_1^2\right)=\frac{3}{2}\left(\sum_{t=1}^{n}\sum_{q=1}^{n-2}a^t_{1q-1}a^{q+2}_{1t-1}-\sum_{r=1}^{n-1}\sum_{q=1}^{n-1}a^{r+1}_{1q-1}a^{q+1}_{1r-1}\right).$$

By the previous two lemmas 18 and 19 we have

$$\begin{aligned}
&\sum_{r=1}^{n}\sum_{q=1}^{n-2}a^r_{1q-1}a^{q+2}_{1r-1}-\sum_{r=1}^{n-1}\sum_{q=1}^{n-1}a^{r+1}_{1q-1}a^{q+1}_{1r-1}\\
&=\sum_{q=1}^{n-2}\Big(\frac{1}{\theta}\sum_{(\alpha,\beta)\in P}C_{((\alpha,\beta),(n-2,q-1))}d_{\alpha-(n-2)-2,\beta-(q-1)-2}\\
&\quad-\frac{1}{\theta}\sum_{(a,b)\in P}C_{((a,b),(n-3,q-2))}d_{a-(n-3)-2,b-(q-2)-2}\Big)\\
&\quad-\sum_{r=1}^{n-1}\sum_{q=1}^{n-1}\sum_{(\alpha,\beta)\in P}\sum_{(a,b)\in P}\frac{C_{((\alpha,\beta),(n-2,q-1))}}{\theta}d_{\alpha-(n-2)-1,\beta-(r-1)-1}\\
&\qquad\cdot\frac{C_{((a,b),(n-2,r-1))}}{\theta}d_{a-(n-2)-1,b-(q-1)-1}\\
&=\sum_{\substack{(s,t)\in Q\\ s=n-2}}\sum_{(i,j)\in P}\frac{C_{((i,j),(s,t))}}{\theta}d_{i-s-2,j-t-2}-\sum_{\substack{(s,t)\in Q\\ s=n-3}}\sum_{(i,j)\in P}\frac{C_{((i,j),(s,t))}}{\theta}d_{i-s-2,j-t-2}\\
&\quad-\sum_{\substack{(s,t)\in Q\\ s=n-2}}\sum_{\substack{(u,v)\in Q\\ u=n-2}}\sum_{(\alpha,\beta)\in P}\sum_{(a,b)\in P}\frac{C_{((\alpha,\beta),(s,t))}}{\theta}d_{\alpha-u-1,\beta-v-1}\frac{C_{((a,b),(u,v))}}{\theta}d_{a-s-1,b-t-1}.
\end{aligned}$$

Finally by theorem 16 and lemma 17 this is equal to

$$tr(M^{-1}M'') - tr\left((M^{-1}M')^2\right) = \frac{d^2}{dt^2}\log\left(\theta(\mathcal{F}^t)\right).$$

This proves the theorem 17 with *Hitchin*'s formula. □

Two corollaries of the proof of theorem 17 and lemma 15 are the following.

Corollary 6. *As long as the invertible sheaf $\mathcal{F}^t$ does not lie in the theta divisor we have the formula*

$$tr\,(A_0(t)A_2(t)) = \frac{d^2}{dt^2}\log(\theta(\mathcal{F}^t)).$$

Proof. Via lemma 15 and lemma 16 we see, that the difference of the two equations, one with the term $tr\left(A_0A_2 - \frac{1}{4}A_1^2\right)$ and the other with the term $tr\,(A_0A_2)$, is only given by the factor $\frac{3}{2}$. Then we use the proof of Hitchin's formula to get equality with the right-hand side. □

Corollary 7. *Let $X \in \mathcal{O}_{reg}\left(\mathfrak{sl}_n(\mathbb{C})\right)$ and let $(T_1, T_2, T_3) \in \mathcal{M}(0, \sigma)$ be the element corresponding to X via Kronheimer's identification. Let*

$$A(\zeta, t) = (T_2(-t) + iT_3(-t)) + (2iT_1(-t))\,\zeta + (T_2(-t) - iT_3(-t))\,\zeta^2$$

and let $\mathcal{F}^{-t} \in Jac^{g-1}(C_n) \setminus \Theta$ be the corresponding invertible sheaf to the regular, nilpotent, matricial polynomial $A(\zeta, -t)$. Then the value of the Kähler potential at X is

$$\begin{aligned} K(X) &= -\int_{-\infty}^{0} tr(T_2(t)^2 + T_3(t)^2)dt = -\int_{-\infty}^{0} tr(A_0(-t)A_2(-t))dt \\ &= -\int_{-\infty}^{0} \frac{d^2}{dt^2}\log(\theta(\mathcal{F}^{-t}))dt = -[\frac{d}{dt}\log(\theta(\mathcal{F}^{-t}))]_{-\infty}^{0} dt \\ &= \frac{\theta'(\mathcal{F}^0)}{\theta(\mathcal{F}^0)}. \end{aligned}$$

Proof. The first equality is equation (2.2). Then we just integrate the equality of corollary 6. Note that the improper integral is well-defined because the theta function is of polynomial type in the variable t □

Chapter 5

Real Sheaves and Special Solutions of Nahm's Equations

5.1 Real Sheaves

Elements of the Kronheimer moduli space, triples (T_1, T_2, T_3) with

$$T_i \in C^\infty\left((-\infty, 0], \mathfrak{su}(n)\right)$$

solving Nahm's equations, induce regular, nilpotent, matricial polynomials of the form

$$A(\zeta) := \overbrace{(T_2(0) + iT_3(0))}^{A_0:=} + \overbrace{2iT_1(0)}^{A_1:=}\zeta + \overbrace{(T_2(0) - iT_3(0))}^{A_2:=}\zeta^2,$$

which satisfy $A_0^T = -\overline{A_2}, A_1^T = \overline{A_1}$. This property is called the reality condition. The corresponding invertible sheaf $\mathcal{F} \in Jac^{g-1}(C_n) \setminus \Theta$ satisfies such a reality condition too, see for example [Hit83] or [Bie07]. In this section we want to characterize such real sheaves on the nilpotent, spectral curve and prove their theta function is real-valued.

5.1.1 Reality Condition

In this subsection we recall the definitions of a real sheaf, see [Hit83] or [Bie07]. At the end we apply these definitions to the nilpotent, spectral curve. The antipodal map on $\mathbb{CP}^1$ induces on the total space of $\mathcal{O}_{\mathbb{CP}^1}(2)$ an anti-holomorphic involution, called a real structure, given by the map

$$\begin{aligned} \tau : |\mathcal{O}_{\mathbb{CP}^1}(2)| &\longrightarrow |\mathcal{O}_{\mathbb{CP}^1}(2)| \\ \left[(\tilde{\zeta}, \tilde{\eta}), (\zeta, \eta)\right] &\longmapsto \left[\left(-\frac{1}{\overline{\tilde{\zeta}}}, -\frac{\overline{\tilde{\eta}}}{\overline{\tilde{\zeta}}^2}\right), \left(-\frac{1}{\overline{\zeta}}, -\frac{\overline{\eta}}{\overline{\zeta}^2}\right)\right]. \end{aligned}$$

Clearly it satisfies the involutive property $\tau^2 = id$ and it is anti-holomorphic. The nilpotent, spectral curve C_n is invariant under this real structure. Note that $\tau|_{U_0}(U_0) = U_1$ and $\tau|_{U_1}(U_1) = U_0$, where U_0 and U_1 are the open sets given by the standard open cover of C_n and hence $\tau|_{U_1 \cap U_0}(U_1 \cap U_0) = U_0 \cap U_1$, see subsection 3.1.2.

This real structure τ induces a real structure σ on the Jacobian as follows. Let $g_{10}(\zeta, \eta)$ be a transition function of an invertible sheaf $\mathcal{E}$ of an arbitrary degree. A local section $(s_1, s_0) \in \mathcal{E}(U)$ on an open set U has to satisfy the equation $s_1(\frac{1}{\zeta}, \frac{\eta}{\zeta^2}) = g_{10}(\zeta, \eta)s_0(\zeta, \eta)$ on U_{10}. We get a new transition function defined by

$$\tilde{g}_{01}\left(\frac{1}{\zeta}, \frac{\eta}{\zeta^2}\right) := \overline{g_{10}\left(\tau(\zeta, \eta)\right)},$$

where the local sections $(\tilde{s}_1, \tilde{s}_0)$ satisfy $\tilde{g}_{01}(\frac{1}{\zeta}, \frac{\eta}{\zeta^2})\tilde{s}_1(\frac{1}{\zeta}, \frac{\eta}{\zeta^2}) = \tilde{s}_0(\zeta, \eta)$ on U_{10}. We denote the invertible sheaf given by the new transition function $\tilde{g}_{01}$ by $\sigma(\mathcal{E})(= \overline{\tau^*\mathcal{E}})$ and we get a map

$$\sigma : Pic(C_n) \longrightarrow Pic(C_n)$$
$$\mathcal{E} \longmapsto \sigma(\mathcal{E}).$$

In particular, for a fixed $k \in \mathbb{Z}$, the invertible sheaf $\mathcal{O}_{C_n}(k)$ with transition function $g_{10}(\zeta, \eta) = \frac{1}{\zeta^k}$ induces the transition function $\tilde{g}_{01}\left(\frac{1}{\zeta}, \frac{\eta}{\zeta^2}\right) = \zeta^k$. In other words we have $\sigma(\mathcal{O}_{C_n}(k)) \cong \mathcal{O}_{C_n}(k)$ and hence the real structure preserves the sheaves $\mathcal{O}_{C_n}(k)$. Therefore it preserves the degree of the sheaves.

Let $\mathcal{E}$ be an arbitrary invertible sheaf in the Jacobian and let $(s_1, s_0) \in \mathcal{E}(U)$ be a local section on an open set $U \subseteq C_n$. Then the real structure induces a local section of $\sigma(\mathcal{E})$ on $\tau(U)$ given by

$$\left(\overline{\tau^* s_0}, \overline{\tau^* s_1}\right) \in \sigma(\mathcal{E})(\tau(U)).$$

So if the sheaf $\mathcal{E}$ has no non-trivial, global sections, then a $\sigma(\mathcal{E})$ has no non-trivial, global sections too and therefore the real structure preserves the Theta divisor in $Jac^{g-1}(C_n)$. The restriction of σ gives us a map

$$\sigma|_{Jac^{g-1}(C_n)\setminus\Theta} : Jac^{g-1}(C_n) \setminus \Theta \longrightarrow Jac^{g-1}(C_n) \setminus \Theta.$$

The next definition is from [Bie07].

Definition 6. *We call an invertible sheaf $\mathcal{E}$ of degree kn real, if we have as $\mathcal{O}_{C_n}$-modules*

$$\mathcal{E} \cong \sigma(\mathcal{E})^v \otimes_{\mathcal{O}_{C_n}} \mathcal{O}_{C_n}(2k).$$

We characterize now the real sheaves of degree 0 on the nilpotent, spectral curve.

Lemma 20. *An invertible sheaf $\mathcal{L}$ of degree 0 on the nilpotent, spectral curve, given by the transition function $g_{10}(\zeta, \eta) = \sum_{l=0}^{n-1}\sum_{k=1}^{2l-1} d_{kl}\frac{\eta^l}{\zeta^k}$, is real if and only if the coefficients d_{kl} satisfy the equation*

$$d_{kl} = (-1)^{k+l}\overline{d_{2l-k,l}}.$$

Proof. Invertible sheaves of degree 0 are real if and only if $\mathcal{L} \cong \sigma(\mathcal{L})^v$. This is satisfied if and only if their transition functions coincide (up to a multiplication by a non-zero constant). The transition function of $\sigma(\mathcal{L})$ is given by

$$\tilde{g}_{01}\left(\frac{1}{\zeta}, \frac{\eta}{\zeta^2}\right) = \overline{g_{10}(\tau(\zeta, \eta))} = \overline{\sum_{l=0}^{n-1}\sum_{k=1}^{2l-1} d_{kl}\frac{(-\frac{\overline{\eta}}{\overline{\zeta}^2})^l}{(-\frac{1}{\overline{\zeta}})^k}} = \sum_{l=0}^{n-1}\sum_{k=1}^{2l-1}(-1)^{l+k}\overline{d_{kl}}\frac{1}{\zeta^{2l-k}}\eta^l.$$

The local sections of $\sigma(\mathcal{L})$ satisfy $\tilde{g}_{01}(\frac{1}{\zeta}, \frac{\eta}{\zeta^2})s_1(\frac{1}{\zeta}, \frac{\eta}{\zeta^2}) = s_0(\zeta, \eta)$ on U_{10} and hence the local sections $(\tilde{s}_1, \tilde{s}_0)$ of its dual satisfy $\tilde{s}_1(\frac{1}{\zeta}, \frac{\eta}{\zeta^2}) = \tilde{g}_{01}(\zeta, \eta)\tilde{s}_0(\zeta, \eta)$ U_{10}. Hence we can equate the coefficients of the transition function g_{10} and $\tilde{g}_{01}$. This gives us our desired result. □

Because solutions of Nahm's equations of the Kronheimer moduli space induce real sheaves for all $t \in (-\infty, 0]$ we are interested in flows of real sheaves.

Lemma 21. *Let $\mathcal{F} \in Jac^{g-1}(C_n)$ and $\mathcal{L}^t$ be the invertible sheaf of degree 0 with transition function $\exp\left(-t\frac{\eta}{\zeta}\right)$ for a $t \in \mathbb{C}$. Let $\mathcal{F}^t := \mathcal{F} \otimes \mathcal{L}^t$. If $\mathcal{F}$ is a real sheaf, then $\mathcal{F}^t$ is real for all $t \in \mathbb{R}$.*

Proof. The coefficients of the transition functions of $\mathcal{F}^t$ are given by

$$d_{kl}(t) = \sum_{i=0}^{k} (-1)^i d_{k-i,l-i} \frac{t^i}{i!}.$$

By using Lemma 20 we get

$$\begin{aligned} d_{kl}(t) &= \sum_{i=0}^{k} (-1)^i d_{k-i,l-i} \frac{t^i}{i!} = \sum_{i=0}^{k} (-1)^{k-i+l-i} \overline{d_{2(l-i)-(k-i),l-i} \frac{t^i}{i!}} \\ &= (-1)^{k+l} \sum_{i=0}^{k} \overline{d_{2l-k-i,l-i} \frac{t^i}{i!}} = (-1)^{k+l} \overline{d_{2l-k,l}(t)}. \end{aligned}$$

We used here $\bar{t} = t$, since $t \in \mathbb{R}$. □

5.1.2 Theta Function of Real Sheaves

In this subsection we study the theta function of real sheaves. We have already seen, that a Kähler potential on the regular, nilpotent, adjoint orbit is given by $K(X) = \frac{\theta'(\mathcal{F}^0)}{\theta(\mathcal{F}^0)}$, where $\mathcal{F}^t$ is the flow of invertible sheaves induced by elements of the Kronheimer moduli space. A Kähler potential is a real function and hence $\frac{\theta'(\mathcal{F}^0)}{\theta(\mathcal{F}^0)} \in \mathbb{R}$. We will show, that the theta function of any real sheaf is real-valued.

Before we state the result of the theta function of arbitrary real sheaves, we start with an auxillary lemma. The theta function is a determinant of a matrix M with a certain structure. If we Laplace expand the matrix M we can write

$$\theta = \det(M) = d_{00}^b \det\left(\tilde{M}\right)$$

for some $b \in \mathbb{N}$ and $\tilde{M}$ some submatrix of M. The remaining matrix $\tilde{M}$ is kind of persymmetric, which we will use later and which is the reason why we wanted to exclude the factor d_{00}^b. Because the theta function is a summation of monomials over regular index sets, every regular set has to have a subset B corresponding to the factor d_{00}^b. First we define now this set B.

Definition 7. *Let $u \in \mathbb{N}$, $n \geq 2$. We define*

$$\begin{aligned} B_{-1} &:= \{((i,j),(s,t)) \in P \times Q : j = t = n-1, i = s, 1 \leq s \leq n-2\}, \\ B_u &:= \{((i,j),(s,t)) \in P \times Q : i = s, j = t = n-2-u, 1 \leq s \leq n-3-2u\}. \end{aligned}$$

The index set B is defined by

$$\begin{aligned}B :=& B_{-1} \cup \bigcup_{u=0}^{\lfloor \frac{n}{2} \rfloor -2} B_u \\ =& \Big\{((i,j),(s,t)) \in P \times Q : j = t = n-1, i = s, 1 \leq s \leq n-2\Big\} \\ &\cup \bigcup_{u=0}^{\lfloor \frac{n}{2} \rfloor -2} \Big\{((i,j),(s,t)) \in P \times Q : i = s, j = t = n-2-u, 1 \leq s \leq n-3-2u\Big\}.\end{aligned}$$

Lemma 22. *For every regular set $\mathcal{D} \in \mathcal{R}(P \times Q)$ we have $B \subset \mathcal{D}$. In particular we have a decomposition of the regular set*

$$\mathcal{D} = B \cup \mathcal{D}^B.$$

Proof. Let us take an arbitrary regular set $\mathcal{D}$ and its elements are of the form

$$((i,j),(s,t)) \in P \times Q.$$

First let $t = n-1$. The columns $(s, n-1)$ of M contain the elements $d_{i-s,j-(n-1)}$. But, because $1 \leq j \leq n-1$, this is only possible at $j = n-1$. Because $t - j = 0$ we need to have $i - s = 0$ too, i.e. we have a d_{00} at this position. This is exactly B_{-1}. In other words every regular set contains this set. If $n \leq 3$ we are already done.

Now we assume $n \geq 4$ and we use induction over u. The base case is $u = 0$, i.e. we will show $B_0 \subset \mathcal{D}$ for all regular sets $\mathcal{D}$. If $\mathcal{D}$ is an arbitrary regular set, we are considering the indices with $t = n-2$. There are two possibilities. The first $j = n-2$ and the second $j = n-1$. In the case $j = n-1$ we have $j - t = n-1-(n-2) = 1$ and therefore $i - s = 1$ too. If $1 \leq s \leq n-3$ then $2 \leq i \leq n-2$. But the index $(i, n-1)$ is already in B_{-1}. Because every index appears exactly one time in a regular set, this possibility cancels. It only remains $j = t = n-2$ with $j - t = 0$ and therefore $i - s = 0$. This shows $B_0 \subset \mathcal{D}$.

The induction assumption is $\bigcup_{u=0}^{l-1} B_u \subset \mathcal{D}$ for all regular sets $\mathcal{D}$. In the induction step we have to show, that B_l is a subset too. Observe that if $\lfloor \frac{n}{2} \rfloor - 2 \leq l$, then B_l is empty (there is no s). This means we are considering indices with $t = n-2-l$ and $1 \leq s \leq n-3-2l$.

If $j = n-2-l$, then $j - t = 0$ and we have $1 \leq i = s \leq n-3-2l$. This is the index in B_l. Therefore we have to show, that for all other j the possible indices (i,j) are already in a B_u for $-1 \leq u \leq l-1$.

Now we consider the case $j = n-1$ and we fix an s. Because $j - t = n-1-(n-2-l) = l+1$ we have $1 \leq i - s \leq 2(l+1) - 1 = 2l+1$. Hence $i \leq 2l+1+s \leq 2l+1+n-3-2l = n-2$. But these indices $(i, n-1)$ are already in B_{-1}.

The last case is to consider if $n-2-l+1 \leq j \leq n-2$. Hence $1 \leq i-s \leq 2(j-t)-1$ and hence $i \leq 2j-2(n-2-l)-1+s \leq 2j-2n+3+2l+n-3-2l = 2j-n$. Now we take B_u with $u = n-2-j$. Observe that if $j = n-2$, then $u = 0$ and if $j = n-2-(l-1)$ then $u = l-1$. By induction assumption we know such B_u are subsets of all regular sets $\mathcal{D}$. Because $i = s$ in B_u we have $1 \leq i \leq n-3-2u = n-3-2(n-2-j) = 2j-(n-1)$. This means all the above calculated indices (i,j) are allready in B_u, with $u = n-2-j$.

Hence the only indices remaining are given by B_l and therefore $B_l \subset \mathcal{D}$ for all regular sets $\mathcal{D}$. □

Theorem 18. *If an invertible sheaf $\mathcal{F} \in Jac^{g-1}(C_n)$ is real, then the theta function is real-valued, i.e.*

$$\theta(\mathcal{F}) \in \mathbb{R}.$$

Proof. Recall that the theta function is given by a sum of monomials comming from some regular sets. Let us denote $Mon(\mathcal{D}) := sign(\mathcal{D}) \prod_{((i,j),(s,t))\in P\times Q} d_{i-s,j-t}$ the monomial of the regular set $\mathcal{D} \in \mathcal{R}(P \times Q)$. We cannot expect, that all monomials are already real. We will show, that for every regular set $\mathcal{D}$ we can find another, possibly the same, regular set $\overline{\mathcal{D}}$ such that $\overline{Mon(\mathcal{D})} = Mon(\overline{\mathcal{D}})$. If $\overline{\mathcal{D}} = \mathcal{D}$ the monomial is already real. If $\overline{\mathcal{D}} \neq \mathcal{D}$ then we have

$$Mon(\mathcal{D}) + Mon(\overline{\mathcal{D}}) = Mon(\mathcal{D}) + \overline{Mon(\mathcal{D})} \in \mathbb{R}.$$

By lemma 22 we have a decomposition $\mathcal{D} = B \cup \mathcal{D}^B$. Hence for a regular set $\mathcal{D}$ we define the following map:

$$\beta : \mathcal{D}^B \to P \times Q$$
$$((i,j),(s,t)) \mapsto ((\tilde{i},\tilde{j}),(\tilde{s},\tilde{t})),$$

where

$$\tilde{i} = n-1-2t+s, \qquad \tilde{j} = n-1-t,$$
$$\tilde{s} = n-1-2j+i, \qquad \tilde{t} = n-1-j.$$

First we have to check, that this map is well-defined, i.e. $\beta(\mathcal{D}^B) \subset P \times Q$.

$1 \le \tilde{i} \le 2\tilde{j}-1$: This inequality is equivalent to $1 \le n-1-2t+s$ and $n-1-2t+s \le 2(n-1-t)-1 = 2(n-1)-2t-1$. And this again is equivalent to $2t-s \le n-2$ and $s \le n-2$. The second inequality holds obviously. To show the first, we note that if $((i,j),(s,t)) \in \mathcal{D}^B$, then $t \neq n-1$ and if $t = n-2-u$ we have $n-3-2u+1 \le s$. Therefore with a $t = n-2-u$ we compute $2t-s \le 2(n-2-u)-(n-3-2u+1) = n-2$. This is exactly the first inequality.

$1 \le \tilde{j} \le n-1$: This means $1 \le n-1-t \le n-1$. The second inequalitiy is immediate. The first follows from the fact, that $t = n-1$ holds if and only if the element lies in B.

$1 \le \tilde{s} \le n-2,$ or $0 = \tilde{s} = \tilde{t}$: First we want to show $1 \le n-1-2j+i \le n-2$, what is equivalent to $1 \le 2j-i \le n-2$. The inequality $1 \le 2j-i$ is immediate. Because of the definition of B (the index is in $\mathcal{D}^B = \mathcal{D}\setminus B$) we see, if we have the case $j = n-1$, then we have $n-1 \le i$ and by the definition of P we have $i \le 2(n-1)-1$. The inequality $2(n-1)-i \le n-2$ holds for all $n \le i$ and the only remaining index is $i = n-1$. In this case we have $i = j = n-1$ and it is clear by definition, that we have $0 = \tilde{s} = \tilde{t}$. In the case $j = n-2-u$, by the definition of B, we have $n-3-2u+1 \le i \le 2(n-2-u)-1$. But then we have $2j-i \le 2(n-2-u)-(n-3-2u+1) = n-2$. Therefore we are done.

$0 \le \tilde{t} \le n-1$: This is equivalent to $0 \le n-1-j \le n-1$ and this is again equivalent to $0 \le j \le n-1$. This is always satisfied.

Because we have

$$\begin{aligned}\beta(\beta((i,j),(s,t))) &= \beta((n-1-2t+s, n-1-t),(n-1-2j+i, n-1-j))\\ &= ((n-1-2(n-1-j)+(n-1-2j+i), n-1-(n-1-j)),\\ &\qquad (n-1-2(n-1-t)+(n-1-2t+s), n-1-(n-1-t))\\ &= ((i,j),(s,t)),\end{aligned}$$

the map β is bijective on its image. Now we define

$$\overline{\mathcal{D}^B} := \beta(\mathcal{D}^B), \qquad \overline{\mathcal{D}} := B \cup \overline{\mathcal{D}^B}.$$

Because β is bijective, the set $\overline{\mathcal{D}}$ is regular again.

We show now, if $\mathcal{D} = \overline{\mathcal{D}}$ then the monomial $Mon(\mathcal{D})$ is already real. The equality $\mathcal{D} = \overline{\mathcal{D}}$ implies

$$\beta((i,j),(s,t)) = ((n-1-2t+s, n-1-t),(n-1-2j+i, n-1-j)) \in \mathcal{D}.$$

If $((i,j),(s,t)) = \beta((i,j),(s,t))$ we get $i-s = n-1-2t$ and $i-s = -(n-1)+2j$. The sum of these equations gives $k = i-s = j-t = l$ and by the reality condition, lemma 20, d_{ll} is real. If $((i,j),(s,t)) \neq \beta((i,j),(s,t))$ in the monomial $Mon(\mathcal{D})$ we have both factors $d_{i-s,j-t}$ and $d_{n-1-2t+s-(n-1-2j+i),n-1-t-(n-1-j)} = d_{2(j-t)-(i-s),j-t}$. But by the reality condition we have

$$d_{i-s,j-t} d_{2(j-t)-(i-s),j-t} = (-1)^{i-s+j-t} \overline{d_{2(j-t)-(i-s),j-t}} d_{2(j-t)-(i-s),j-t} \in \mathbb{R}.$$

Therefore if $\mathcal{D} = \overline{\mathcal{D}}$ the monomial $Mon(\mathcal{D})$ consists only of factors of real numbers or pairs of complex numbers and its complex conjugate as above and hence it is a real-valued monomial.

We show now $sign(\mathcal{D}) = sign(\overline{\mathcal{D}})$. Recall $sign(\mathcal{D}) = \prod_{(i_1,j_1)<(i_2,j_2)} \frac{(s_2,t_2)-(s_1,t_1)}{(i_2,j_2)-(i_1,j_1)}$. We compute

$$\begin{aligned}
(\tilde{i}_2,\tilde{j}_2) -_P (\tilde{i}_1,\tilde{j}_1) &= (n-1-2t_2+s_2, n-1-t_2) -_P (n-1-2t_1+s_1, n-1-t_1) \\
&= \begin{cases} -2t_2+2t_1+s_2-s_1, & t_1 = t_2 \\ t_1 - t_2, & t_1 \neq t_2 \end{cases} \\
&= \begin{cases} s_2 - s_1, & t_1 = t_2 \\ t_1 - t_2, & t_1 \neq t_2 \end{cases} \\
&= (-1)\left((s_2,t_2) -_Q (s_1,t_1)\right).
\end{aligned}$$

and

$$\begin{aligned}
(\tilde{s}_2,\tilde{t}_2) -_Q (\tilde{s}_1,\tilde{t}_1) &= (n-1-2j_2+i_2, n-1-j_2) -_Q (n-1-2j_1+i_1, n-1-j_1) \\
&= \begin{cases} 2j_2-2j_1+i_1-i_2, & j_1 = j_2 \\ n-1-j_2-(n-1-j_1), & j_1 \neq j_2 \end{cases} \\
&= \begin{cases} i_1 - i_2, & j_1 = j_2 \\ j_1 - j_2, & j_1 \neq j_2 \end{cases} \\
&= (-1)\left((i_2,j_2) -_P (i_1,j_1)\right).
\end{aligned}$$

Because

$$\frac{(s_2,t_2) -_Q (s_1,t_1)}{(i_2,j_2) -_P (i_1,j_1)} = \frac{(-1)((\tilde{i}_2,\tilde{j}_2) -_P (\tilde{i}_1,\tilde{j}_1))}{(-1)((\tilde{s}_2,\tilde{t}_2) -_Q (\tilde{s}_1,\tilde{t}_1))} = \frac{(\tilde{i}_2,\tilde{j}_2) -_P (\tilde{i}_1,\tilde{j}_1)}{(\tilde{s}_2,\tilde{t}_2) -_Q (\tilde{s}_1,\tilde{t}_1))}$$

we see

$$sign(\mathcal{D}) = sign(B) sign(\mathcal{D}^B) = sign(B)\frac{1}{sign(\overline{\mathcal{D}^B})} = sign(B) sign(\overline{\mathcal{D}^B}) = sign(\overline{\mathcal{D}}).$$

Finally if we take a regular set $\mathcal{D}$ and its monomial $Mon(\mathcal{D})$ we get, with lemma 20 and lemma 22,

$$\begin{aligned}
Mon(\mathcal{D}) &= \prod_{((i,j),(s,t))\in\mathcal{D}} d_{i-s,j-t} = \prod_{((i,j),(s,t))\in B} d_{i-s,j-t} \prod_{((i,j),(s,t))\in\mathcal{D}^B} d_{i-s,j-t} \\
&= \prod_{((i,j),(s,t))\in B} d_{i-s,j-t} \prod_{((i,j),(s,t))\in\mathcal{D}^B} (-1)^{i-s+j-t}\overline{d_{2(j-t)-(i-s),j-t}} \\
&= \prod_{((i,j),(s,t))\in B} d_{i-s,j-t} \prod_{((i,j),(s,t))\in\mathcal{D}^B} (-1)^{2(i-s)} \prod_{((i,j),(s,t))\in\mathcal{D}^B} \overline{d_{2(j-t)-(i-s),j-t}} \\
&= \overline{\prod_{((i,j),(s,t))\in B} d_{i-s,j-t} \prod_{((i,j),(s,t))\in\mathcal{D}^B} d_{2(j-t)-(i-s),j-t}} \\
&= \overline{\prod_{((i,j),(s,t))\in B} d_{i-s,j-t} \prod_{((\tilde{i},\tilde{j}),(\tilde{s},\tilde{t}))\in\overline{\mathcal{D}^B}} d_{\tilde{i}-\tilde{s},\tilde{j}-\tilde{t}}} \\
&= \overline{Mon(\overline{\mathcal{D}})}.
\end{aligned}$$

The equality

$$\prod_{((i,j),(s,t))\in\mathcal{D}^B} (-1)^{i-s+j-t} = \prod_{((i,j),(s,t))\in\mathcal{D}^B} (-1)^{2(i-s)}$$

follows by regularity of the set $\mathcal{D}^B$. This shows the theorem. □

5.2 Special Solutions of Nahm's Equations

In order to produce explicit solutions of Nahm's equations for a flow $\mathcal{F}^t \in Jac^{g-1}(C_n)\setminus \Theta$ Hitchin defined in [Hit83] a covariant derivate on the rank n bundle

$$E_t := \check{\mathrm{H}}^0\left(C_n, \mathcal{F}^t(1)\right)$$

over the interval $(-\infty, 0]$ by

$$\nabla_t := \frac{d}{dt} + \left(\frac{1}{2}A_1(t) + A_2(t)\zeta\right),$$

where $A(t)(\zeta)$ is the corresponding matricial polynomial to the invertible sheaf $\mathcal{F}^t$. Then by taking a covariantly constant basis of E_t with respect to ∇_t on $(-\infty, 0]$ he trivialized the vector bundle E_t. He showed, if $Q(t)$ is the corresponding basis tranformation matrix, then the triple

$$\begin{aligned}
&(T_1(t), T_2(t), T_3(t)) := \\
&\left(\frac{i}{2}Q(t)^{-1}A_1(t)Q(t), \frac{1}{2}Q(t)^{-1}(A_0(t)+A_2(t))Q(t), \frac{-i}{2}Q(t)^{-1}(A_0(t)-A_2(t))Q(t)\right)
\end{aligned}$$

satisfies Nahm's equations. Moreover he showed, if $\mathcal{F}$ is a real sheaf such that Hitchin's Hermitian form on $\check{\mathrm{H}}^0\left(C_n, \mathcal{F}^t(1)\right)$ is positive definite, see [Hit83], then we have $\overline{T_i(t)}^T = -T_i(t)$ and hence the solutions of Nahm's equations are $\mathfrak{su}(n)$-valued.

To avoid the difficulties of finding a covariantly constant basis and computing Hitchin's Hermitian form to characterize those sheaves with definite Hermitian form we will restrict ourselves to a very special case.

5.2.1 Matricial Polynomials in "Jordan"-form

Theorem 19. *Let $\mathcal{F} \in Jac^{g-1}(C_n)\setminus\Theta$ be an invertible sheaf on the nilpotent, spectral curve of degree $g-1$ not lying in the Theta divisor. Let $d_{kl} \in \mathbb{C}$ be the coefficients of the transition function of the sheaf $\mathcal{F}$ and let $A(\zeta)$ be the corresponding matricial polynomial of theorem 14. Let us suppose $d_{kl} = 0$ for all $k \neq l$. Then A_1 is a diagonal matrix of the form*

$$A_1 = \begin{pmatrix} -a_{10}^2 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & a_{10}^2 - a_{11}^3 & 0 & \ddots & & \vdots \\ 0 & 0 & a_{11}^3 - a_{12}^4 & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & a_{1n-3}^{n-1} - a_{1n-2}^{n} & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & a_{1n-2}^{n} \end{pmatrix}$$

and A_2 is of "Jordan"-form, i.e.

$$A_2 = \begin{pmatrix} 0 & \alpha_0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \alpha_1 & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \alpha_{n-3} & 0 \\ \vdots & & & & 0 & \alpha_{n-2} \\ 0 & \cdots & \cdots & \cdots & 0 & 0 \end{pmatrix},$$

where $\alpha_s := a_{2s-1}^{s+2} - a_{2s}^{s+3} - w_{s+1,s+2} + v_{s+1,s+2}$, with $s \in \{0,\dots,n-2\}$.

We suspect, that the assumption $d_{kl} = 0$ for all $k \neq l$ ensures for real sheaves, that the basis of theorem 7 is already orthogonal with respect to Hitchin's inner product. First we need to study cofactors of the matrix M.

Lemma 23. *Let $\mathcal{F}$ be an invertible sheaf as in theorem 19. Let $(i,j) \in P$, $l \in \{0,\dots,n-1\}$ and $0 \leq k \leq 2l-1$. If $i-(n-1-k) \neq j-l$ holds, then the cofactor $C_{((i,j),(n-1-k,l))} = 0$ vanishes.*

Proof. Let us fix an index $(i_0,j_0) \in P$ and let us define

$$\begin{aligned} K^{(i_0,j_0)} &:= \{(k,l) \in Q : i_0 - k = j_0 - l\} \\ &= \{(i_0-\tau, j_0-\tau) \in Q : \tau \in \{0,\dots,n-1\}\}. \end{aligned}$$

The indices in $K^{(i_0,j_0)}$ indicate, if at $((i_0,j_0),(k,l))$ is a variable of the form $d_{\tau\tau}$, but it can be zero if the variable is for example $d_{-1,-1}$. For each such $(i_0-\tau, j_0-\tau)$-column we define

$$\begin{aligned} I^{(i_0-\tau,j_0-\tau)} &:= \{(i,j) \in P : i-(i_0-\tau) = j-(j_0-\tau)\} \\ &= \{(i_0-\eta, j_0-\eta) \in P : \eta \in \{0,\dots,n-1\}\}. \end{aligned}$$

These indices indicate, if at $((i,j),(i_0-\tau,j_0-\tau))$ is a variable of the form $d_{\eta\eta}$. Since the elements of $K^{(i_0,j_0)}$ are of the form $(k,l) = (i_0-\tau, j_0-\tau)$, $\tau \in \{0,\dots,n-1\}$ and the elements of $I^{(i_0-\tau,j_0-\tau)}$ are of the form $(i,j) = ((i_0-\tau)+\eta, (j_0-\tau)+\eta) = (i_0-\tilde{\tau}, j_0-\tilde{\tau})$, $\tilde{\tau} \in \{0,\dots,n-1\}$, the number of indices in $K^{(i_0,j_0)}$ is the same as the number of indices in $I^{(i_0-\tau,j_0-\tau)}$ for all τ . Let us take now the matrix M and cancel the (i_0,j_0)-row and the $(n-1-k_0,l_0)$-column, where (k_0,l_0) satisfies $i_0-(n-1-k_0)\neq j_0-l_0$. Canceling means, we remove an index of $I^{(i_0-\tau,j_0-\tau)}$ for each $\tau\in\{0,\dots,n-1\}$. But because we have $i_0-(n-1-k_0)\neq j_0-l_0$, we do not remove an index of $K^{(i_0,j_0)}$. In other words in the matrix without the (i_0,j_0)-row and the $(n-1-k_0,l_0)$-column we can find $|K^{(i_0,j_0)}|$-vectors with $\big(|K^{(i_0,j_0)}|-1\big)$-rows with possibly non-zero entry. This implies linear dependence of these vectors and hence the minor as well as the cofactor $C_{((i_0,j_0),(n-1-k_0,l_0))}$ is zero. $|K^{(i_0,j_0)}|$ means the number of elements in the finite set $K^{(i_0,j_0)}$. □

Now we prove theorem 19.

Proof. Let us consider the coefficients of the basis of theorem 7

$$a^{\tau}_{kl}=-\frac{1}{\theta}\sum_{\substack{(i,j)\in P\\ \lfloor\frac{n}{2}\rfloor\leq j}}C_{((i,j),(n-1-k,l))}d_{i-(n-1),j-(\tau-1)}.$$

A summand vanishes if and only if $C_{((i,j),(n-1-k,l))}=0$ or $d_{i-(n-1),j-(\tau-1)}=0$. If $i-(n-1)\neq j-(\tau-1)$ the second equation is satisfied and we get a first condition, that a summand does not vanish, namely

$$i-(n-1)=j-(\tau-1)$$

needs to be satisfied. By lemma 23 we get a second condition,

$$i-(n-1-k)=j-l.$$

Because of theorem 14 we are only interested in $k=1$ and $k=2$. With these two conditions the coefficient a^{τ}_{1l} is possibly non-zero if $l=\tau-2$ and the coefficient a^{τ}_{2l} is possibly non-zero if $l=\tau-3$. Hence the only possibly non-zero coefficients are of the form

$$a^{s+2}_{1s},a^{s+3}_{2s},$$

where $s\in\{0,\dots,n-2\}$. This implies that the expressions $w_{st}=\sum_{j=0}^{n-2}a^{j+2}_{1s-1}a^{t}_{1j}$ and $v_{st}=\sum_{j=0}^{n-1}a^{j+1}_{1s-1}a^{t+1}_{1j}$ are only possibly non-zero if $t=s+1$. Moreover we have

$$w_{ss+1}=a^{s+1}_{1s-1}a^{s+1}_{1s-1},\quad v_{ss+1}=a^{s+1}_{1s-1}a^{s+2}_{1s}.$$

In other words the matrices $V = (v_{st})_{st}$ and $W = (w_{st})_{st}$ are of "$Jordan$"-form. If we consider A_1, by canceling out every a^{τ}_{1l} without $l = \tau - 2$, we get

$$A_1 = \begin{pmatrix} -a^2_{10} & -a^3_{10} & \cdots & -a^n_{10} & 0 \\ a^1_{10} - a^2_{11} & a^2_{10} - a^3_{11} & \cdots & a^{n-1}_{10} - a^n_{11} & a^n_{10} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a^1_{1n-2} - a^2_{1n-1} & a^2_{1n-2} - a^3_{1n-1} & \cdots & a^{n-1}_{1n-2} - a^n_{1n-1} & a^n_{1n-2} \end{pmatrix}$$

$$= \begin{pmatrix} -a^2_{10} & 0 & 0 & \cdots & \cdots & 0 \\ 0 & a^2_{10} - a^3_{11} & 0 & & \ddots & \vdots \\ 0 & 0 & a^3_{11} - a^4_{12} & \cdots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & a^{n-1}_{1n-3} - a^n_{1n-2} & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & a^n_{1n-2} \end{pmatrix},$$

which is of course a diagonal matrix. If we consider the part of A_2 with the coefficients a^{τ}_{2l}, , by canceling out every a^{τ}_{2l} without $l = \tau - 3$, we get

$$\begin{pmatrix} -a^2_{20} & -a^3_{20} & \cdots & -a^n_{20} & 0 \\ a^1_{20} - a^2_{21} & a^2_{20} - a^3_{21} & \cdots & a^{n-1}_{20} - a^n_{21} & a^n_{20} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a^1_{2n-2} - a^2_{2n-1} & a^2_{2n-2} - a^3_{2n-1} & \cdots & a^{n-1}_{2n-2} - a^n_{2n-1} & a^n_{2n-2} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -a^3_{20} & 0 & \cdots & \cdots & 0 \\ 0 & 0 & a^3_{20} - a^4_{21} & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & a^{n-1}_{2n-4} - a^n_{2n-3} & 0 \\ \vdots & & & & 0 & a^n_{2n-3} \\ 0 & \cdots & \cdots & \cdots & 0 & 0 \end{pmatrix}.$$

Putting these and the matrices V, W together, we get

$$A_2 = \begin{pmatrix} 0 & -a^3_{20} - w_{12} + v_{12} & 0 & \cdots & \cdots & 0 \\ 0 & 0 & a^3_{20} - a^4_{21} - w_{23} + v_{23} & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & a^{n-1}_{2n-4} - a^n_{2n-3} - w_{n-2n-1} + v_{n-2n-1} & 0 \\ \vdots & & & & 0 & a^n_{2n-3} - w_{n-1n} \\ 0 & \cdots & \cdots & \cdots & 0 & 0 \end{pmatrix},$$

which is of "$Jordan$"-form. □

5.2.2 Explicit Description of Special Solutions of Nahm's Equations

In this subsection we are still considering only sheaves $\mathcal{F} \in Jac^{g-1}(C_n) \setminus \Theta$, where the coefficients of its transition function satisfy $d_{kl} = 0$ for all $k \neq l$. Because we want to describe explicit solutions of Nahm's equations we need a nicely choosen basis of $\check{\mathrm{H}}^0\left(C_n, \mathcal{F}^t(1)\right)$. Because of theorem 19 the condition $d_{kl} = 0$ for all $k \neq l$ says, that the basis of theorem 7 is already in such a nice form. The "$Jordan$"-form of the regular, nilpotent, matricial polynomial corresponding to $\mathcal{F}$ simplifies the problem significantly and we do not need to compute Hitchin's Hermitian form of [Hit83].

We will consider only real sheaves, such that the obtained solutions of Nahm's equations are $\mathfrak{su}(n)$-valued. By lemma 20 the coefficients d_{ll} of the transition function of $\mathcal{F}$ are all real numbers and hence all coefficients of the basis a^{τ}_{kl} and all α_s are real too.

If we fix an open interval $(t_0, t_1) \subseteq \mathbb{R}$ such that $\mathcal{F}^t = \mathcal{F} \otimes \mathcal{L}^t \in Jac^{g-1}(C_n) \setminus \Theta$ for all $t \in (t_0, t_1)$ we know by lemma 21, that all sheaves $\mathcal{F}^t$ are real and of the special case $d_{kl}(t) = 0$ for all $k \neq l$ and $t \in (t_0, t_1)$. For such an open interval the coefficients $a^{\tau}_{kl}(t)$ and $\alpha_s(t)$ are polynomials in t and so smooth real-valued functions on (t_0, t_1). Since the matricial polynomial is regular we have $\alpha_s(t) \neq 0$. Let us suppose, that the $\alpha_s(t) = a^{s+2}_{2s-1}(t) - a^{s+3}_{2s}(t) - w_{s+1,s+2}(t) + v_{s+1,s+2}(t)$ are negative for all $t \in (t_0, t_1)$ and $s \in \{0, \ldots, n-2\}$, i.e. $\alpha_s(t) < 0$. We write $\alpha_s(t) = i^2|\alpha_s(t)|$ and we define

$$P_\mu(t) := \sqrt{\prod_{s=\mu-1}^{n-2} (i^2\, |\alpha_s(t)|)}, \qquad P_n(t) := 1$$

for $\mu \in \{1, \ldots, n-1\}$. All $P_\mu(t)$ are non-zero too and we define an invertible matrix

$$P(t) := \begin{pmatrix} P_1(t) & 0 & \cdots & 0 \\ 0 & P_2(t) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & P_n(t) \end{pmatrix}.$$

We have

$$\frac{P_\mu(t)}{P_{\mu+1}(t)} = \frac{\sqrt{i^{2(n-2-(\mu-1))} \left(\prod_{s=\mu-1}^{n-2} |\alpha_s(t)|\right)}}{\sqrt{i^{2(n-2-\mu)} \left(\prod_{s=\mu}^{n-2} |\alpha_s(t)|\right)}} = i\sqrt{|\alpha_{\mu-1}(t)|}$$

for all $\mu \in \{1, \ldots, n-1\}$. We conjugate the regular, nilpotent, matrical polynomial $A(t, \zeta) = A_0(t) + A_1(t)\zeta + A_2(t)\zeta^2$ of theorem 19 by the matrix $P(t)$,

$$\hat{A}_0(t) := P(t)^{-1} A_0(t) P(t), \quad \hat{A}_1(t) := P(t)^{-1} A_1(t) P(t), \quad \hat{A}_2(t) := P(t)^{-1} A_2(t) P(t),$$

and we get the matrices

$$\hat{A}_0(t) = \begin{pmatrix} 0 & 0 & 0 & \cdots & \cdots & 0 \\ i\sqrt{|\alpha_0(t)|} & 0 & 0 & \ddots & & \vdots \\ 0 & i\sqrt{|\alpha_1(t)|} & 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & i\sqrt{|\alpha_{n-3}(t)|} & 0 & 0 \\ 0 & \cdots & \cdots & 0 & i\sqrt{|\alpha_{n-2}(t)|} & 0 \end{pmatrix},$$

$$\hat{A}_1(t) = \begin{pmatrix} -a_{10}^2(t) & 0 & 0 & \cdots & \cdots & 0 \\ 0 & a_{10}^2(t) - a_{11}^3(t) & 0 & & & \vdots \\ 0 & 0 & a_{11}^3(t) - a_{12}^4(t) & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & a_{1n-3}^{n-1}(t) - a_{1n-2}^{n}(t) & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & a_{1n-2}^{n}(t) \end{pmatrix},$$

$$\hat{A}_2(t) = \begin{pmatrix} 0 & i\sqrt{|\alpha_0(t)|} & 0 & \cdots & \cdots & 0 \\ 0 & 0 & i\sqrt{|\alpha_1(t)|} & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & i\sqrt{|\alpha_{n-3}(t)|} & 0 \\ \vdots & & & & 0 & i\sqrt{|\alpha_{n-2}(t)|} \\ 0 & \cdots & \cdots & \cdots & 0 & 0 \end{pmatrix}.$$

We see immediatly the properties $\overline{\hat{A}_0(t)}^T = -\hat{A}_2(t)$ and $\overline{\hat{A}_1(t)}^T = \hat{A}_1(t)$ as we wished to have. We want to remark, that the basis transformation on $\check{\mathrm{H}}^0\left(C_n, \mathcal{F}^t(1)\right)$ given by the matrix $P(t)$ is not a normalization with respect to Hitchin's Hermitian form of the choosen (orthogonal) basis of theorem 7 because $P_n = 1$ and hence it is not covariantly constant with respect to ∇_t. Moreover we suspect if Hitchin's Hermitian form is definite, that the assumption $\alpha_s(t) < 0$ is satisfied. We define now candidates of solutions of Nahm's equations

$$
\begin{aligned}
T_1(t) &:= \frac{i}{2}\hat{A}_1(t)\\
&= \frac{i}{2}\begin{pmatrix}
-a_{10}^2(t) & 0 & 0 & \cdots & \cdots & 0\\
0 & a_{10}^2(t)-a_{11}^3(t) & 0 & \ddots & & \vdots\\
0 & 0 & a_{11}^3(t)-a_{12}^4(t) & \ddots & & \vdots\\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots\\
\vdots & & & \ddots & a_{1n-3}^{n-1}(t)-a_{1n-2}^{n}(t) & 0\\
0 & \cdots & \cdots & \cdots & 0 & a_{1n-2}^{n}(t)
\end{pmatrix},\\
T_2(t) &:= \frac{1}{2}\left(\hat{A}_0(t)+\hat{A}_2(t)\right)\\
&= \frac{i}{2}\begin{pmatrix}
0 & \sqrt{|\alpha_0(t)|} & 0 & \cdots & \cdots & 0\\
\sqrt{|\alpha_0(t)|} & 0 & \sqrt{|\alpha_1(t)|} & \ddots & & \vdots\\
0 & \sqrt{|\alpha_1(t)|} & 0 & \ddots & & \vdots\\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots\\
\vdots & & \ddots & \ddots & 0 & \sqrt{|\alpha_{n-2}(t)|}\\
0 & \cdots & \cdots & \cdots & \sqrt{|\alpha_{n-2}(t)|} & 0
\end{pmatrix},\\
T_3(t) &:= -\frac{i}{2}\left(\hat{A}_0(t)-\hat{A}_2(t)\right)\\
&= \frac{1}{2}\begin{pmatrix}
0 & -\sqrt{|\alpha_0(t)|} & 0 & \cdots & \cdots & 0\\
\sqrt{|\alpha_0(t)|} & 0 & -\sqrt{|\alpha_1(t)|} & & & \vdots\\
0 & \sqrt{|\alpha_1(t)|} & 0 & \ddots & & \vdots\\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots\\
\vdots & & \ddots & \ddots & 0 & -\sqrt{|\alpha_{n-2}(t)|}\\
0 & \cdots & \cdots & \cdots & \sqrt{|\alpha_{n-2}(t)|} & 0
\end{pmatrix}.
\end{aligned}
$$

These matrices are trace-free and they satisfy the condition $T_i(t)^T = -\overline{T_i(t)}$. In other words the matrices $T_i(t) \in \mathfrak{su}(n)$. The commutators of these matrices are

$[T_2, T_3] =$

$$
\frac{i}{2}\begin{pmatrix}
|\alpha_0| & 0 & \cdots & 0\\
0 & |\alpha_1|-|\alpha_0| & \cdots & \vdots\\
\vdots & & \ddots & \vdots\\
0 & \cdots & \cdots & -|\alpha_{n-2}|
\end{pmatrix} = -\frac{i}{2}\begin{pmatrix}
\alpha_0 & 0 & \cdots & 0\\
0 & \alpha_1-\alpha_0 & \cdots & \vdots\\
\vdots & & \ddots & \vdots\\
0 & \cdots & \cdots & -\alpha_{n-2}
\end{pmatrix},
$$

$[T_3, T_1] =$

$$\frac{i}{4}\begin{pmatrix} 0 & (-2a^2_{10} + a^3_{11})\sqrt{|\alpha_0|} & 0 & \cdots & \cdots & \cdots & 0 \\ (-2a^2_{10} + a^3_{11})\sqrt{|\alpha_0|} & 0 & (a^2_{10} - 2a^3_{11} + a^4_{12})\sqrt{|\alpha_1|} & \ddots & \ddots & & \vdots \\ 0 & (a^2_{10} - 2a^3_{11} + a^4_{12})\sqrt{|\alpha_1|} & 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & (a^{n-2}_{1n-4} - 2a^{n-1}_{1n-3} + a^n_{1n-2})\sqrt{|\alpha_{n-3}|} & 0 & (a^{n-1}_{1n-3} - 2a^n_{1n-2})\sqrt{|\alpha_{n-2}|} \\ 0 & \cdots & \cdots & \cdots & 0 & (a^{n-1}_{1n-3} - 2a^n_{1n-2})\sqrt{|\alpha_{n-2}|} & 0 \end{pmatrix},$$

$[T_1, T_2] =$

$$\frac{i}{4}\begin{pmatrix} 0 & (-2a^2_{10} + a^3_{11})\sqrt{|\alpha_0|} & 0 & \cdots & \cdots & \cdots & 0 \\ -(-2a^2_{10} + a^3_{11})\sqrt{|\alpha_0|} & 0 & (a^2_{10} - 2a^3_{11} + a^4_{12})\sqrt{|\alpha_1|} & \ddots & \ddots & & \vdots \\ 0 & -(a^2_{10} - 2a^3_{11} + a^4_{12})\sqrt{|\alpha_1|} & 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & -(a^{n-2}_{1n-4} - 2a^{n-1}_{1n-3} + a^n_{1n-2})\sqrt{|\alpha_{n-3}|} & 0 & (a^{n-1}_{1n-3} - 2a^n_{1n-2})\sqrt{|\alpha_{n-2}|} \\ 0 & \cdots & \cdots & \cdots & 0 & -(a^{n-1}_{1n-3} - 2a^n_{1n-2})\sqrt{|\alpha_{n-2}|} & 0 \end{pmatrix}.$$

Now we state the main result of this chapter.

Theorem 20. *Let $\mathcal{F} \in Jac^{g-1}(C_n) \setminus \Theta$. Let us suppose, that the coefficients d_{kl} of the transition function of $\mathcal{F}$ satisfy $d_{kl} = 0$ for all $k \neq l$. Additionally let us assume $\mathcal{F}$ is a real sheaf, i.e. $d_{ll} \in \mathbb{R}$. Let $(t_0, t_1) \subseteq \mathbb{R}$ be an open interval such that $\mathcal{F}^t \in Jac^{g-1}(C_n) \setminus \Theta$ for all $t \in (t_0, t_1)$. Then for $s \in \{0, \ldots, n-2\}$ we have the equations*

i) $\frac{d}{dt} a^{s+2}_{1s}(t) = \alpha_s(t) = a^{s+2}_{2s-1}(t) - a^{s+3}_{2s}(t) - w_{s+1,s+2}(t) + v_{s+1,s+2}(t),$

ii) $\frac{d}{dt} a^{s+3}_{2s}(t) = -a^{s+2}_{1s}(t)\alpha_{s+1}(t)$

$$= -a^{s+2}_{1s}(t)\left(a^{s+3}_{2s}(t) - a^{s+4}_{2s+1}(t) - a^{s+3}_{1s+1}(t)a^{s+3}_{1s+1}(t) + a^{s+3}_{1s+1}(t)a^{s+4}_{1s+2}(t)\right),$$

iii) $\frac{i}{2}\frac{d}{dt}\left(\sqrt{|\alpha_s(t)|}\right) = \frac{i}{4}\left(a^{s+1}_{1s-1}(t) - 2a^{s+2}_{1s}(t) + a^{s+3}_{1s+1}(t)\right)\sqrt{|\alpha_s(t)|}.$

In particular the triple of $\mathfrak{su}(n)$-valued functions $(T_1(t), T_2(t), T_3(t))$ satisfies Nahm's equations on the interval (t_0, t_1), i.e.

$$\frac{d}{dt}T_1(t) = [T_2(t), T_3(t)],$$
$$\frac{d}{dt}T_2(t) = [T_3(t), T_1(t)],$$
$$\frac{d}{dt}T_3(t) = [T_1(t), T_2(t)].$$

Remark 3. *The Nahm's equations in theorem 20 are different from those in the Kronheimer moduli space by the sign. Since we considered the invertible sheaf $\mathcal{L}^t$ with transition function $\exp(-t\frac{\eta}{\zeta})$, where we have $\lim_{t\to\infty}\exp(-t\frac{\eta}{\zeta}) = 0$, we want $t \in [0,\infty)$ instead of $(-\infty, 0]$ as in the Kronheimer moduli space. To get elements of the Kronheimer moduli we just need to take the triple $(T_1(-t), T_2(-t), T_3(-t))$.*

Before we start the proof of theorem 20, we are in need of a lot technical lemmas. The ideas of the lemmas are similar to the burning lemma. A lot of the lemmas are vanishing results, which need a precise study of the signs of cofactors. For the rest of this section we use the abbreviation ι_r for the function ι_{row} and ι_c for the function ι_{column} of definition 5 to indicate the number of the row and the column and it allows us to indicate the signs of the cofactors precisely. Moreover we will always use the notation $C_{((i,j),(u,v))}$ for the cofactor of the matrix M by canceling out the (i,j)-row and the (u,v)-column. Similarly we use the notation $M_{((i,j),(u,v))}$ for the minor and a multiple subscript if we cancel out several rows and columns. Additionally we consider in this section only invertible sheaves satisfying the assumptions in theorem 20. The first lemma is a swapping index porperty.

Lemma 24. *Let $(i,j) \in P$ and $(p,q), (a,b) \in P \setminus \{(i,j)\}$. Let $s \in \{0,\dots,n-2\}$ and let $v \in \{0,\dots,n-2\} \setminus \{s\}$. Then we have*

i) $C_{\substack{((i,j),(n-2,s))\\((p,q),(n-2,v))}} C_{((a,b),(n-2,v))} = C_{\substack{((i,j),(n-2,s))\\((a,b),(n-2,v))}} C_{((p,q),(n-2,v))}$,

ii) $C_{\substack{((i,j),(n-3,s))\\((p,q),(n-2,v))}} C_{((a,b),(n-2,v))} = C_{\substack{((i,j),(n-3,s))\\((a,b),(n-2,v))}} C_{((p,q),(n-2,v))}$.

iii) For $v \in \{0,\dots,n-2\} \setminus \{s-1, s\}$ we have

$$C_{\substack{((i,j),(n-3,s))\\((p,q),(n-4,v))}} C_{((a,b),(n-4,v))} = C_{\substack{((i,j),(n-3,s))\\((a,b),(n-4,v))}} C_{((p,q),(n-4,v))}.$$

Proof. i) The idea of the proof is just to Laplace-expand both sides carefully and compare then these expressions. The main difficulty is to deal with the signs of the cofactors, which makes the proof very unwieldy and lengthy. First we write

$$C_{\substack{((i,j),(n-2,s))\\((p,q),(n-2,v))}} (-1)^{\nu(a,b,n-2,v)} M_{((a,b),(n-2,v))} = C_{\substack{((i,j),(n-2,s))\\((a,b),(n-2,v))}} (-1)^{\nu(p,q,n-2,v)} M_{((p,q),(n-2,v))}.$$

We first Laplace-expand both sides and we get

$$C_{\substack{((i,j),(n-2,s))\\((p,q),(n-2,v))}} M_{((a,b),(n-2,v))} = C_{\substack{((i,j),(n-2,s))\\((p,q),(n-2,v))}} \sum_{\substack{(\alpha_1,\beta_1)\in P\\(\alpha_1,\beta_1)\neq(a,b)}} C_{\substack{((a,b),(n-2,v))\\((\alpha_1,\beta_1),(n-2,s))}} d_{\alpha_1-(n-2),\beta_1-s},$$

$$C_{\substack{((i,j),(n-2,s))\\((a,b),(n-2,v))}} M_{((p,q),(n-2,v))} = C_{\substack{((i,j),(n-2,s))\\((a,b),(n-2,v))}} \sum_{\substack{(\alpha_2,\beta_2)\in P\\(\alpha_2,\beta_2)\neq(p,q)}} C_{\substack{((p,q),(n-2,v))\\((\alpha_2,\beta_2),(n-2,s))}} d_{\alpha_2-(n-2),\beta_2-s}.$$

The cofactors are only possibly non-zero if $p-q=a-b=n-2-v$ and $\alpha_1-\beta_1=\alpha_2-\beta_2=n-2-s$ by lemma 23. But since $v\neq s$ we have $(\alpha_1,\beta_1)\neq(p,q)$ and $(\alpha_2,\beta_2)\neq(a,b)$ anyway. This means, we only have to show

$$(-1)^{\nu(a,b,n-2,v)}C_{\substack{((i,j),(n-2,s))\\((p,q),(n-2,v))}}C_{\substack{((a,b),(n-2,v))\\((\alpha,\beta),(n-2,s))}}=(-1)^{\nu(p,q,n-2,v)}C_{\substack{((i,j),(n-2,s))\\((a,b),(n-2,v))}}C_{\substack{((p,q),(n-2,v))\\((\alpha,\beta),(n-2,s))}}$$

for $(\alpha,\beta)\neq(p,q)$ and $(\alpha,\beta)\neq(a,b)$. Just for notation we define $Q^*:=Q\setminus\{(n-2,v),(n-2,s)\}$. We expand now all terms above and get

$$\begin{aligned}
C_{\substack{((a,b),(n-2,v))\\((\alpha,\beta),(n-2,s))}} &= (-1)^{\iota_r(\alpha,\beta)-\delta_{(\alpha,\beta)>(a,b)}}(-1)^{\iota_c(n-2,s)-\delta_{s>v}}M_{\substack{((a,b),(n-2,v))\\((\alpha,\beta),(n-2,s))}}\\
&= (-1)^{\iota_r(\alpha,\beta)-\delta_{(\alpha,\beta)>(a,b)}}(-1)^{\iota_c(n-2,s)-\delta_{s>v}}\\
&\quad\cdot\sum_{(w,x)\in Q^*}C_{\substack{((a,b),(n-2,v))\\((\alpha,\beta),(n-2,s))\\((p,q),(w,x))}}d_{p-w,q-x}\\
&= (-1)^{\iota_r(\alpha,\beta)-\delta_{(a,b)<(\alpha,\beta)}}(-1)^{\iota_c(n-2,s)-\delta_{s>v}}\\
&\quad\cdot(-1)^{\iota_r(p,q)-\delta_{(p,q)>(a,b)}-\delta_{(p,q)>(\alpha,\beta)}}(-1)^{\iota_c(w,x)-\delta_{(w,x)>(n-2,v)}-\delta_{(w,x)>(n-2,s)}}\\
&\quad\cdot\sum_{(w,x)\in Q^*}M_{\substack{((a,b),(n-2,v))\\((\alpha,\beta),(n-2,s))\\((p,q),(w,x))}}d_{p-w,q-x}\\
&= (-1)^{\iota_r(\alpha,\beta)-\delta_{(a,b)<(\alpha,\beta)}}(-1)^{\iota_c(n-2,s)-\delta_{s>v}}\\
&\quad\cdot(-1)^{\iota_r(p,q)-\delta_{(p,q)>(a,b)}-\delta_{(p,q)>(\alpha,\beta)}}(-1)^{\iota_c(w,x)-\delta_{(w,x)>(n-2,v)}-\delta_{(w,x)>(n-2,s)}}\\
&\quad\cdot\sum_{(y,z)\in Q^*\setminus\{(w,x)\}}\sum_{(w,x)\in Q^*}C_{\substack{((a,b),(n-2,v))\\((\alpha,\beta),(n-2,s))\\((p,q),(w,x))\\((i,j),(y,z))}}d_{p-w,q-x}d_{i-y,j-z},\\
C_{\substack{((i,j),(n-2,s))\\((p,q),(n-2,v))}} &= (-1)^{\iota_r(p,q)-\delta_{(p,q)>(i,j)}}(-1)^{\iota_c(n-2,v)-\delta_{v>s}}\\
&\quad\cdot\sum_{(\epsilon,\phi)\in Q^*}C_{\substack{((i,j),(n-2,s))\\((p,q),(n-2,v))\\((a,b),(\epsilon,\phi))}}d_{a-\epsilon,b-\phi}\\
&= (-1)^{\iota_r(p,q)-\delta_{(p,q)>(i,j)}}(-1)^{\iota_c(n-2,v)-\delta_{v>s}}\\
&\quad\cdot(-1)^{(a,b)-\delta_{(a,b)>(i,j)}-\delta_{(a,b)>(p,q)}}(-1)^{(\epsilon,\phi)-\delta_{(\epsilon,\phi)>(n-2,v)}-\delta_{(\epsilon,\phi)>(n-2,s)}}\\
&\quad\cdot\sum_{(\eta,\xi)\in Q^*\setminus\{(\epsilon,\phi)\}}\sum_{(\epsilon,\phi)\in Q^*}C_{\substack{((i,j),(n-2,s))\\((p,q),(n-2,v))\\((a,b),(\epsilon,\phi))\\((\alpha,\beta),(\eta,\xi))}}d_{a-\epsilon,b-\phi}d_{\alpha-\eta,\beta-\xi},
\end{aligned}$$

$$
\begin{aligned}
C_{\substack{((p,q),(n-2,v))\\((\alpha,\beta),(n-2,s))}} &= (-1)^{\iota_r(\alpha,\beta)-\delta_{(\alpha,\beta)>(p,q)}}(-1)^{\iota_c(n-2,s)-\delta_{s>v}}\\
&\quad\cdot \sum_{(w,x)\in Q^*} C_{\substack{((p,q),(n-2,v))\\((\alpha,\beta),(n-2,s))\\((a,b),(w,x))}} d_{a-w,b-x}\\
&= (-1)^{\iota_r(\alpha,\beta)-\delta_{(\alpha,\beta)>(p,q)}}(-1)^{\iota_c(n-2,s)-\delta_{s>v}}\\
&\quad\cdot (-1)^{\iota_r(a,b)-\delta_{(a,b)>(\alpha,\beta)}-\delta_{(a,b)>(p,q)}}(-1)^{\iota_c(w,x)-\delta_{(w,x)>(n-2,v)}-\delta_{(w,x)>(n-2,s)}}\\
&\quad\cdot \sum_{(y,z)\in Q^*\setminus\{(w,x)\}}\ \sum_{(w,x)\in Q^*} C_{\substack{((p,q),(n-2,v))\\((\alpha,\beta),(n-2,s))\\((a,b),(w,x))\\((i,j),(y,z))}} d_{a-w,b-x}d_{i-y,j-z},\\
C_{\substack{((i,j),(n-2,s))\\((a,b),(n-2,v))}} &= \sum_{(\epsilon,\phi)\in Q^*} C_{\substack{((i,j),(n-2,s))\\((a,b),(n-2,v))\\((p,q),(\epsilon,\phi))}} d_{p-\epsilon,q-\phi}\\
&= (-1)^{\iota_r(a,b)-\delta_{(a,b)>(i,j)}}(-1)^{\iota_c(n-2,v)-\delta_{v>s}}\\
&\quad\cdot (-1)^{\iota_r(p,q)-\delta_{(p,q)>(i,j)}-\delta_{(p,q)>(a,b)}}(-1)^{\iota_c(\epsilon,\phi)-\delta_{(\epsilon,\phi)>(n-2,v)}-\delta_{(\epsilon,\phi)>(n-2,s)}}\\
&\quad\cdot \sum_{(\eta,\xi)\in Q^*\setminus\{(\epsilon,\phi)\}}\ \sum_{(\epsilon,\phi)\in Q^*} C_{\substack{((i,j),(n-2,s))\\((a,b),(n-2,v))\\((p,q),(\epsilon,\phi))\\((\alpha,\beta),(\eta,\xi))}} d_{a-\epsilon,b-\phi}d_{\alpha-\eta,\beta-\xi}.
\end{aligned}
$$

If we fix now all indices and multiply the cofactors, we get

$$
\begin{aligned}
&(-1)^{\iota_r(\alpha,\beta)-\delta_{(a,b)<(\alpha,\beta)}}(-1)^{\iota_c(n-2,s)-\delta_{s>v}}\\
&\cdot (-1)^{\iota_r(p,q)-\delta_{(p,q)>(a,b)}-\delta_{(p,q)>(\alpha,\beta)}}(-1)^{\iota_c(w,x)-\delta_{(w,x)>(n-2,v)}-\delta_{(w,x)>(n-2,s)}}\\
&\cdot C_{\substack{((a,b),(n-2,v))\\((\alpha,\beta),(n-2,s))\\((p,q),(w,x))\\((i,j),(y,z))}} d_{i-y,j-z}d_{p-w,q-x}\\
&\cdot (-1)^{\iota_r(p,q)-\delta_{(p,q)>(i,j)}}(-1)^{\iota_c(n-2,v)-\delta_{v>s}}\\
&\cdot (-1)^{\iota_r(a,b)-\delta_{(a,b)>(i,j)}\delta_{(a,b)>(p,q)}}(-1)^{\iota_c(\epsilon,\phi)-\delta_{(\epsilon,\phi)>(n-2,v)}-\delta_{(\epsilon,\phi)>(n-2,s)}}\\
&\cdot C_{\substack{((i,j),(n-2,s))\\((p,q),(n-2,v))\\((a,b),(\epsilon,\phi))\\((\alpha,\beta),(\eta,\xi))}} d_{\alpha-\eta,\beta-\xi}d_{a-\epsilon,b-\phi}\\
&= (-1)^{\iota_r(\alpha,\beta)-\delta_{(\alpha,\beta)>(p,q)}}(-1)^{\iota_c(n-2,s)-\delta_{s>v}}\\
&\quad\cdot (-1)^{\iota_r(a,b)-\delta_{(a,b)>(\alpha,\beta)}-\delta_{(a,b)>(p,q)}}(-1)^{\iota_c(w,x)-\delta_{(w,x)>(n-2,v)}-\delta_{(w,x)>(n-2,s)}}\\
&\quad C_{\substack{((p,q),(n-2,v))\\((\alpha,\beta),(n-2,s))\\((a,b),(w,x))\\((i,j),(y,z))}} d_{i-y,j-z}d_{a-w,b-x}\\
&\quad\cdot (-1)^{\iota_r(a,b)-\delta_{(a,b)>(i,j)}}(-1)^{\iota_c(n-2,v)-\delta_{v>s}}\\
&\quad\cdot (-1)^{\iota_r(p,q)-\delta_{(p,q)>(i,j)}-\delta_{(p,q)>(a,b)}}(-1)^{\iota_c(\epsilon,\phi)-\delta_{(\epsilon,\phi)>(n-2,v)}-\delta_{(\epsilon,\phi)>(n-2,s)}}\\
&\quad\cdot C_{\substack{((i,j),(n-2,s))\\((a,b),(n-2,v))\\((p,q),(\epsilon,\phi))\\((\alpha,\beta),(\eta,\xi))}} d_{\alpha-\eta,\beta-\xi}d_{p-\epsilon,q-\phi}.
\end{aligned}
$$

By checking every term on each side, we see they are equal and this proves i). The parts ii) and iii) work identically the same. □

We need a certain extension of the burning lemma.

Lemma 25. *Let $(p,q),(i,j) \in P$ be two fixed indices. Then we have*

$$\sum_{(u,v)\in Q} C_{((p,q),(u,v))} d_{i-u-2,j-v-1} = \theta \delta_{(p,q)=(i-2,j-1)}.$$

Proof. The case $(p,q) = (i-2, j-1)$ is just the usual Laplace expansion. For the case $(p,q) \neq (i-2, j-1)$ we expand the cofactor $C_{((p,q),(u,v))}$ along the $(i-2, j-1)$-th row and we get

$$\begin{aligned} C_{((p,q),(u,v))} &= (-1)^{\nu(p,q,u,v)} M_{((p,q),(u,v))} \\ &= (-1)^{\nu(p,q,u,v)} \sum_{(\rho,\delta)\in Q\setminus\{(u,v)\}} C^{((p,q),(u,v))}_{((i-2,j-1),(\rho,\delta))} d_{i-2-\rho,j-1-\delta} d_{i-u-2,j-v-1}. \end{aligned}$$

So we only have to show $C^{((p,q),(u,v))}_{((i-2,j-1),(\rho,\delta))} = -C^{((p,q),(\rho,\delta))}_{((i-2,j-1),(u,v))}$. By considering the signs of the expansions

$$\begin{aligned} C_{((p,q),(u,v))} &= (-1)^{\nu(p,q,u,v)} M_{((p,q),(u,v))}, \\ C^{((p,q),(u,v))}_{((i-2,j-1),(\rho,\delta))} &= (-1)^{\nu(p,q,u,v)} (-1)^{\iota_r(i-2,j-1)} (-1)^{\iota_c(\rho,\delta)} (-1)^{\delta_{(i-2,j-1)>(p,q)} + \delta_{(\rho,\delta)>(u,v)}} \\ &\quad \cdot M^{((p,q),(u,v))}_{((i-2,j-1),(\rho,\delta))}, \\ C_{((p,q),(\rho,\delta))} &= (-1)^{\nu(p,q,\rho,\delta)} M_{((p,q),(\rho,\delta))}, \\ C^{((p,q),(\rho,\delta))}_{((i-2,j-1),(u,v))} &= (-1)^{\nu(p,q,\rho,\delta)} (-1)^{\iota_r(i-2,j-1)} (-1)^{\iota_c(u,v)} (-1)^{\delta_{(i-2,j-1)>(p,q)} + \delta_{(u,v)>(\rho,\delta)}} \\ &\quad \cdot M^{((p,q),(\rho,\delta))}_{((i-2,j-1),(u,v))}, \end{aligned}$$

we see the difference lies in the term $(-1)^{\delta_{(\rho,\delta)>(u,v)}}$. In one term this expression is positive and in the other it has to be negative. □

The next lemma is helpful to cancel out needless terms.

Lemma 26. *Let $(i,j) \in P$.*

i) Let $(u,v) \in Q$ be an index such that $u \neq n-2$ and $u \neq n-3$ and $s \in \{0,\ldots,n-2\}$. Then we have

$$\sum_{(p,q)\in P\setminus\{(i,j)\}} C^{((i,j),(n-2,s))}_{((p,q),(u,v))} d_{p-u-1,q-v-1} = 0.$$

ii) Let $s \in \{0,\ldots,n-2\}$ and let $(u,v) \in Q$ such that $(u,v) \neq (n-2, s+1)$ and $(u,v) \neq (n-3, s)$. Then we have

$$\sum_{(p,q)\in P} C_{((p,q),(u,v))} d_{p-u-1,q-v} = 0.$$

iii) Let $s \in \{0,\ldots,n-2\}$ and let $v \in \{0,\ldots,n-2\} \setminus \{s-1\}$. Then we we have

$$\sum_{(p,q)\in P\setminus\{(i,j)\}} \frac{C^{((i,j),(n-2,s))}_{((p,q),(n-3,v))}}{\theta} (-d_{p-(n-3)-1,q-v-1}) d_{i-(n-1),j-s-1} = 0$$

Proof. i) First we consider the case $v = n-1$. But then we have $d_{p-u-1,q-v-1} = d_{p-u-1,q-n} = 0$ and we are done. In the case $v \neq n-1$ we know $(u+1, v+1) \in Q$. We expand the cofactors along the $(u+1, v+1)$-th column and we get

$$\begin{aligned}
&\sum_{(p,q)\in P\setminus\{(i,j)\}} C_{\substack{((i,j),(n-2,s))\\((p,q),(u,v))}} d_{p-u-1,q-v-1} \\
&= \sum_{(p,q)\in P\setminus\{(i,j)\}} (-1)^{\iota_r(i,j)-\delta_{(p,q)>(i,j)}}(-1)^{\iota_c(u,v)-\delta_{(u,v)>(n-2,s)}} \\
&\quad \cdot M_{\substack{((i,j),(n-2,s))\\((p,q),(u,v))}} d_{p-u-1,q-v-1} \\
&= \sum_{(p,q)\in P\setminus\{(i,j)\}} (-1)^{\iota_r(i,j)-\delta_{(p,q)>(i,j)}}(-1)^{\iota_c(u,v)-\delta_{(u,v)>(n-2,s)}} \\
&\quad \left(\sum_{(a,b)\in P\setminus\{(i,j),(p,q)\}} C_{\substack{((i,j),(n-2,s))\\((p,q),(u,v))\\((a,b),(u+1,v+1))}} d_{a-(u+1),b-(v+1)} \right) d_{p-u-1,q-v-1}.
\end{aligned}$$

It remains to show

$$\begin{aligned}
&C_{\substack{((i,j),(n-2,s))\\((p,q),(u,v))\\((a,b),(u+1,v+1))}} d_{a-(u+1),b-(v+1)} d_{p-u-1,q-v-1} \\
&\quad = -C_{\substack{((i,j),(n-2,s))\\((a,b),(u,v))\\((p,q),(u+1,v+1))}} d_{p-(u+1),q-(v+1)} d_{a-u-1,b-v-1}.
\end{aligned}$$

But their signs are

$$\begin{aligned}
&(-1)^{\iota_r(p,q)-\delta_{(p,q)>(i,j)}}(-1)^{\iota_c(u,v)-\delta_{(u,v)>(n-2,s)}}(-1)^{\iota_r(a,b)-\delta_{(a,b)>(i,j)}-\delta_{(a,b)>(p,q)}} \\
&(-1)^{\iota_c(u+1,v+1)-\delta_{(u+1,v+1)>(n-2,s)}-\delta_{(u+1,v+1)>(u,v)}}
\end{aligned}$$

and

$$\begin{aligned}
&(-1)^{\iota_r(a,b)-\delta_{(a,b)>(i,j)}}(-1)^{\iota_c(u,v)-\delta_{(u,v)>(n-2,s)}}(-1)^{\iota_r(p,q)-\delta_{(p,q)>(i,j)}-\delta_{(p,q)>(a,b)}} \\
&(-1)^{\iota_c(u+1,v+1)-\delta_{(u+1,v+1)>(n-2,s)}-\delta_{(u+1,v+1)>(u,v)}},
\end{aligned}$$

where the only difference is in terms $(-1)^{\delta_{(p,q)>(a,b)}}$ and $(-1)^{\delta_{(a,b)>(p,q)}}$. In other words one cofactor has positive sign and the other negative sign and so, if we sum up, they cancel each other out.

ii) Because $u \leq n-4$ we know $(u+1, v) \in Q$ and $(u+1, v) \neq (u, v)$. We expand the cofactors along the $(u+1, v)$-th column and we get

$$\begin{aligned}
&\sum_{(p,q)\in P} C_{((p,q),(u,v))} d_{p-u-1,q-v} \\
&= \sum_{(p,q)\in P} (-1)^{\nu(p,q,u,v)} \left(\sum_{(a,b)\in P\setminus\{(p,q)\}} C_{\substack{((p,q),(u,v))\\((a,b),(u+1,v))}} d_{a-u-1,b-v} \right) d_{p-u-1-q-v}.
\end{aligned}$$

We have

$$
\begin{aligned}
&(-1)^{\iota_r(p,q)}(-1)^{\iota_c(u,v)}C_{\substack{((p,q),(u,v))\\((a,b),(u+1,v))}}\\
&=(-1)^{\iota_r(p,q)}(-1)^{\iota_c(u,v)}(-1)^{\iota_r(a,b)-\delta_{(a,b)>(p,q)}}(-1)^{\iota_c(u+1,v)-\delta_{(u+1,v)>(u,v)}}M_{\substack{((p,q),(u,v))\\((a,b),(u+1,v))}},\\
&(-1)^{\iota_r(a,b)}(-1)^{\iota_c(u,v)}C_{\substack{((a,b),(u,v))\\((p,q),(u+1,v))}}\\
&=(-1)^{\iota_r(p,q)}(-1)^{\iota_c(u,v)}(-1)^{\iota_r(p,q)-\delta_{(p,q)>(a,b)}}(-1)^{\iota_c(u+1,v)-\delta_{(u+1,v)>(u,v)}}M_{\substack{((a,b),(u,v))\\((p,q),(u+1,v))}}.
\end{aligned}
$$

We see the sign changes. Because we sum over $(p,q)\in P$ and $(a,b)\in P\setminus\{(p,q)\}$ we can always find such a pair with opposite sign and so they cancel out.

iii) If we expand the theta function along the $(n-3,v)$-column we get with lemma 24

$$
\begin{aligned}
&C_{\substack{((i,j),(n-2,s))\\((p,q),(n-3,v))}}(-d_{p-(n-3)-1,q-v-1})d_{i-(n-1),j-s-1}\theta\\
&=\sum_{(a,b)\in P}C_{\substack{((i,j),(n-2,s))\\((p,q),(n-3,v))}}(-d_{p-(n-3)-1,q-v-1})d_{i-(n-1),j-s-1}C_{((a,b),(n-3,v))}d_{a-(n-3),b-v}\\
&=\sum_{(a,b)\in P}C_{\substack{((i,j),(n-2,s))\\((a,b),(n-3,v))}}d_{a-(n-3),b-v}d_{i-(n-1),j-s-1}C_{((p,q),(n-3,v))}(-d_{p-(n-3)-1,q-v-1})\\
&=M_{((i,j),(n-2,s))}d_{i-(n-1),j-s-1}C_{((p,q),(n-3,v))}(-d_{p-(n-3)-1,q-v-1}).
\end{aligned}
$$

Summing over the $(p,q)\in P$ the last term vanishes by ii) of the burning lemma. □

The next lemma is the crucial lemma to prove the equation i) of theorem 20.

Lemma 27. *We have*

i) $\frac{\frac{d}{dt}\theta(\mathcal{F}^t)}{\theta(\mathcal{F}^t)}=\sum_{m=0}^{n-2}a_{1m}^{m+2}(t)$,

ii) $-\sum_{\substack{(i,j)\in P\\ \lfloor\frac{n}{2}\rfloor\le j}}\frac{\frac{d}{dt}\left(C_{((i,j),(n-2,s))}(t)\right)}{\theta(\mathcal{F}^t)}d_{i-(n-1),j-s-1}(t)=a_{2s-1}^{s+2}(t)+\left(\sum_{\substack{m=0\\m\neq s}}^{n-2}a_{1m}^{m+2}(t)\right)a_{1s}^{s+2}(t)$,

iii) $-\sum_{\substack{(i,j)\in P\\ \lfloor\frac{n}{2}\rfloor\le j}}\frac{C_{((i,j),(n-2,s))}(t)}{\theta(\mathcal{F}^t)}\frac{d}{dt}\left(d_{i-(n-1),j-s-1}(t)\right)=-a_{2s}^{s+3}(t)+v_{s+1,s+2}(t)$.

Proof. In the proof we will drop the arguments (t) and $(\mathcal{F}^t)$ to get a little bit more clearness. i) By theorem 16 and its proof we have

$$\frac{\theta'}{\theta}=tr(M^{-1}M')=\sum_{(u,v)\in Q}\left(\sum_{(i,j)\in P}\frac{C_{((i,j),(u,v))}}{\theta}(-d_{i-u-1,j-v-1})\right).$$

Now we apply ii) of the burning lemma and every term with $u\neq n-2$ cancels out. It remains

$$tr(M^{-1}M')=\sum_{v=0}^{n-2}\left(-\sum_{(i,j)\in P}\frac{C_{((i,j),(n-2,v))}}{\theta}d_{i-(n-2)-1,j-v-1}\right)=\sum_{v=0}^{n-2}a_{1v}^{v+2}.$$

ii) Let us fix an $(i,j) \in P$ and consider $\frac{1}{\theta}\frac{d}{dt}\left(C_{((i,j),(n-2,s))}\right) d_{i-(n-1),j-s-1}$. By the formula of the derivative of a determinantal function we have

$$
\begin{aligned}
&\frac{1}{\theta}\frac{d}{dt}\left(C_{((i,j),(n-2,s))}\right) d_{i-(n-1),j-s-1} \\
&= (-1)^{\nu(i,j,n-2,s)} \sum_{\substack{(u,v)\in Q \\ (u,v)\neq(n-2,s)}} \sum_{(p,q)\in P\setminus\{(i,j)\}} \frac{C_{((i,j),(n-2,s))}^{\ ((p,q),(u,v))}}{\theta}(-d_{p-u-1,q-v-1})d_{i-(n-1),j-s-1} \\
&= (-1)^{\nu(i,j,n-2,s)} \sum_{\substack{v=0 \\ v\neq s}}^{n-2} \sum_{(p,q)\in P\setminus\{(i,j)\}} \frac{C_{((i,j),(n-2,s))}^{\ ((p,q),(n-2,v))}}{\theta}(-d_{p-(n-2)-1,q-v-1})d_{i-(n-1),j-s-1} \\
&\quad + (-1)^{\nu(i,j,n-2,s)} \sum_{v=0}^{n-2} \sum_{(p,q)\in P\setminus\{(i,j)\}} \frac{C_{((i,j),(n-2,s))}^{\ ((p,q),(n-3,v))}}{\theta}(-d_{p-(n-3)-1,q-v-1})d_{i-(n-1),j-s-1} \\
&\quad + (-1)^{\nu(i,j,n-2,s)} \sum_{\substack{(u,v)\in Q \\ u\neq n-2 \\ u\neq n-3}} \sum_{(p,q)\in P\setminus\{(i,j)\}} \frac{C_{((i,j),(n-2,s))}^{\ ((p,q),(u,v))}}{\theta}(-d_{p-u-1,q-v-1})d_{i-(n-1),j-s-1}.
\end{aligned}
$$

The third term vanishes because of lemma 26. For the first term let us fix a $v \neq s$. Then we write $1 = \frac{\theta}{\theta} = \sum_{(a,b)\in P} \frac{C_{((a,b),(n-2,v))}}{\theta} d_{a-(n-2),b-v}$ and with lemma 24 we get

$$
\begin{aligned}
&(-1)^{\nu(i,j,n-2,s)} \frac{C_{((i,j),(n-2,s))}^{\ ((p,q),(n-2,v))}}{\theta}(-d_{p-(n-2)-1,q-v-1})d_{i-(n-1),j-s-1} \\
&= (-1)^{\nu(i,j,n-2,s)} \frac{C_{((i,j),(n-2,s))}^{\ ((p,q),(n-2,v))}}{\theta}(-d_{p-(n-2)-1,q-v-1})d_{i-(n-1),j-s-1} \\
&\quad \left(\sum_{(a,b)\in P} \frac{C_{((a,b),(n-2,v))}}{\theta} d_{a-(n-2),b-v}\right) \\
&= (-1)^{\nu(i,j,n-2,s)} \sum_{(a,b)\in P} \frac{C_{((i,j),(n-2,s))}^{\ ((p,q),(n-2,v))}}{\theta} \frac{C_{((a,b),(n-2,v))}}{\theta}(-d_{p-(n-2)-1,q-v-1}) \\
&\quad d_{i-(n-1),j-s-1} d_{a-(n-2),b-v} \\
&= (-1)^{\nu(i,j,n-2,s)} \sum_{(a,b)\in P} \frac{C_{((i,j),(n-2,s))}^{\ ((a,b),(n-2,v))}}{\theta} \frac{C_{((p,q),(n-2,v))}}{\theta}(-d_{p-(n-2)-1,q-v-1}) \\
&\quad d_{i-(n-1),j-s-1} d_{a-(n-2),b-v} \\
&= (-1)^{\nu(i,j,n-2,s)} \left(\sum_{(a,b)\in P} \frac{C_{((i,j),(n-2,s))}^{\ ((a,b),(n-2,v))}}{\theta} d_{a-(n-2),b-v}\right) \frac{C_{((p,q),(n-2,v))}}{\theta} \\
&\quad (-d_{p-(n-2)-1,q-v-1})d_{i-(n-1),j-s-1} \\
&= (-1)^{\nu(i,j,n-2,s)} \frac{M_{((i,j),(n-2,s))}}{\theta} \frac{C_{((p,q),(n-2,v))}}{\theta}(-d_{p-(n-2)-1,q-v-1})d_{i-(n-1),j-s-1} \\
&= \frac{C_{((i,j),(n-2,s))}}{\theta} \frac{C_{((p,q),(n-2,v))}}{\theta} \left(-d_{p-(n-2)-1,q-v-1}\right) d_{i-(n-1),j-s-1}.
\end{aligned}
$$

This means, we get

$$- \sum_{(i,j)\in P} \sum_{(p,q)\in P\setminus\{(i,j)\}} (-1)^{\nu(i,j,n-2,s)} \frac{C^{((i,j),(n-2,s))}_{((p,q),(n-2,v))}}{\theta} (-d_{p-(n-2)-1,q-v-1}) d_{i-(n-1),j-s-1}$$
$$= \sum_{(i,j)\in P} \sum_{(p,q)\in P} \frac{C_{((i,j),(n-2,s))}}{\theta} \frac{C_{((p,q),(n-2,v))}}{\theta} d_{p-(n-2)-1,q-v-1} d_{i-(n-1),j-s-1}$$
$$= \Big(- \sum_{(i,j)\in P} \frac{C_{((i,j),(n-2,s))}}{\theta} d_{i-(n-1),j-s-1} \Big) \Big(- \sum_{(p,q)\in P} \frac{C_{((p,q),(n-2,v))}}{\theta} (-d_{p-(n-1),q-v-1}) \Big)$$
$$= a^{s+2}_{1s} a^{v+2}_{1v}.$$

For the second term we fix an arbitrary v with $v \neq s-1$. Then by lemma 26 we have

$$(-1)^{\nu(i,j,n-2,s)} \sum_{(p,q)\in P\setminus\{(i,j)\}} \frac{C^{((i,j),(n-2,s))}_{((p,q),(n-3,v))}}{\theta} (-d_{p-(n-3)-1,q-v-1}) d_{i-(n-1),j-s-1} = 0.$$

With $v = s-1$ we have

$$(-1)^{\nu(i,j,n-2,s)} \sum_{(p,q)\in P} (-1) \frac{C^{\ ((i,j),(n-2,s))}_{((p,q),(n-3,s-1))}}{\theta} (-d_{p-(n-3)-1,q-(s-1)-1}) d_{i-(n-1),j-s-1}$$
$$= (-1)^{\nu(i,j,n-2,s)} \sum_{(p,q)\in P} (-1)^{\iota_r(p,q)-\delta_{(p,q)>(i,j)}} (-1)^{\iota_c(n-3,s-1)-\overbrace{\delta_{(n-3,s-1)>(n-2,s)}}^{=0}}$$
$$\cdot \frac{M^{\ ((i,j),(n-2,s))}_{((p,q),(n-3,s-1))}}{\theta} d_{p-(n-3)-1,q-(s-1)-1} d_{i-(n-1),j-s-1}$$
$$= (-1)^{\iota_r(i,j)} (-1)^{\iota_c(n-2,s)} \sum_{(p,q)\in P} (-1)^{\iota_r(p,q)-\delta_{(p,q)>(i,j)}} (-1)^{\iota_c(n-3,s-1)} (-1)$$
$$\cdot (-1)^{\delta_{(n-2,s)>(n-3,s-1)}} \frac{M^{((i,j),(n-3,s-1))}_{((p,q),(n-2,s))}}{\theta} d_{p-(n-3)-1,q-(s-1)-1} d_{i-(n-1),j-s-1}$$
$$= (-1)(-1)^{\iota_r(i,j)} (-1)^{\iota_c(n-3,s-1)} \sum_{(p,q)\in P} (-1)^{\iota_r(p,q)-\delta_{(p,q)>(i,j)}} (-1)^{\iota_c(n-2,s)-\delta_{(n-2,s)>(n-3,s-1)}}$$
$$\cot \frac{M^{((i,j),(n-3,s-1))}_{((p,q),(n-2,s))}}{\theta} d_{p-(n-3)-1,q-(s-1)-1} d_{i-(n-1),j-s-1}$$
$$= (-1)(-1)^{\nu(i,j,n-3,s-1)} \sum_{(p,q)\in P} \frac{C^{((i,j),(n-3,s-1))}_{((p,q),(n-2,s))}}{\theta} d_{p-(n-3)-1,q-(s-1)-1} d_{i-(n-1),j-s-1}$$
$$= (-1)(-1)^{\nu(i,j,n-3,s-1)} \frac{M_{((i,j),(n-3,s-1))}}{\theta} d_{i-(n-1),j-s-1}$$
$$= -\frac{C_{((i,j),(n-3,s-1))}}{\theta} d_{i-(n-1),j-s-1}.$$

Summing up over $(i,j) \in P$ gives a^{s+2}_{2s-1} and this proves ii).

iii) First we use the trick to expand θ along the $(n-2,s)$-column, i.e. $1 = \frac{\theta}{\theta} = \sum_{(p,q)\in P} \frac{C_{((p,q),(n-2,s))}}{\theta} d_{p-(n-2),q-s}$. Now we consider $-a^{s+3}_{2s} + v_{s+1,s+2}$ and fix the indices $(i,j) \in P$ and $(p,q) \in P$. We get with the burning lemma vii)

$$\begin{aligned}
&\frac{-C_{((p,q),(n-2,s))}}{\theta} d_{p-(n-2),q-s} \frac{-C_{((i,j),(n-3,s))}}{\theta} d_{i-(n-1),j-s-2} \\
&\qquad + \frac{-C_{((p,q),(n-2,s))}}{\theta} d_{p-(n-1),q-s-1} \frac{-C_{((i,j),(n-2,s+1))}}{\theta} d_{i-(n-1),j-s-2} \\
&\qquad = \frac{C_{((p,q),(n-2,s))}}{\theta} d_{i-(n-1),j-s-2} \\
&\qquad \left(\frac{C_{((i,j),(n-2,s+1))}}{\theta} d_{p-(n-1),q-s-1} + \frac{C_{((i,j),(n-3,s))}}{\theta} d_{p-(n-2),q-s} \right) \\
&\qquad = \frac{C_{((p,q),(n-2,s))}}{\theta} d_{i-(n-1),j-s-2} \\
&\qquad \left(\frac{\theta}{\theta} \delta_{(i,j)=(p-1,q)} - \sum_{\substack{(u,v)\in Q \\ (u,v)\neq(n-2,s+1) \\ (u,v)\neq(n-3,s)}} \frac{C_{((i,j),(u,v))}}{\theta} d_{p-u-1,q-v} \right) \\
&\qquad = \frac{C_{((p,q),(n-2,s))}}{\theta} d_{(p-1)-(n-1),q-s-2} \\
&\qquad - \sum_{\substack{(u,v)\in Q \\ (u,v)\neq(n-2,s+1) \\ (u,v)\neq(n-3,s)}} \frac{C_{((p,q),(n-2,s))}}{\theta} d_{p-u-1,q-v} \frac{C_{((i,j),(u,v))}}{\theta} d_{i-(n-1),j-s-2}
\end{aligned}$$

If we sum up over $(p,q) \in P$, because of lemma 26, the last term vanishes and the only remaining non-vanishing term is

$$\sum_{(p,q)\in P} \frac{C_{((p,q),(n-2,s))}}{\theta} d_{(p-1)-(n-1),q-s-2} = - \sum_{(p,q)\in P} \frac{C_{((p,q),(n-2,s))}}{\theta} \frac{d}{dt} \left(d_{p-(n-1),q-s-1} \right).$$

The last minus sign appears, because $\frac{d}{dt} d_{p-(n-1),q-s-1}(t) = -d_{(p-1)-(n-1),q-s-2}(t)$. □

To prove ii) of theorem 20 we need again some vanishing formulas.

Lemma 28. *i) For $s \in \{0,\dots,n-2\}$ we have*

$$\sum_{(i,j)\in P} \sum_{(p,q)\in P\setminus\{(i,j)\}} \left(C_{\substack{((i,j),(n-3,s)) \\ ((p,q),(n-2,s+1))}} d_{p-(n-2)-1,q-(s+1)-1} \right) d_{i-(n-1),j-s-2} = 0.$$

ii) We have

$$\sum_{(i,j)\in P} \sum_{\substack{v=0 \\ v\neq s}}^{n-2} \sum_{(p,q)\in P\setminus\{(i,j)\}} \left(C_{\substack{((i,j),(n-3,s)) \\ ((p,q),(n-3,v))}} d_{p-(n-3)-1,q-v-1} \right) d_{i-(n-1),j-s-2} = 0.$$

iii) For all $v \in \{0,\dots,n-2\} \setminus \{s-1\}$ we have

$$\sum_{(i,j)\in P} \sum_{v\neq s} \sum_{(p,q)\in P\setminus\{(i,j)\}} \left(C_{\substack{((i,j),(n-3,s)) \\ ((p,q),(n-4,v))}} d_{p-(n-4)-1,q-v-1} \right) d_{i-(n-1),j-s-2} = 0.$$

Proof. i) Observe, that for each pair of indices $((i,j),(p,q)) \in P \times P$ with $(i,j) \neq (p,q)$ there exists exactly one other pair $((p,q),(i,j)) \in P \times P$. This means, it suffices to show for fixed indices (i,j) and (p,q) we have

$$C_{\substack{((i,j),(n-3,s))\\((p,q),(n-2,s+1))}} = -C_{\substack{((p,q),(n-3,s))\\((i,j),(n-2,s+1))}}.$$

We have

$$C_{\substack{((i,j),(n-3,s))\\((p,q),(n-2,s+1))}} = (-1)^{\nu(p,q,n-2,s+1)-\delta_{(p,q)>(i,j)}-\delta_{(n-2,s+1)>(n-3,s)}} M_{\substack{((i,j),(n-3,s))\\((p,q),(n-2,s+1))}},$$
$$C_{\substack{((p,q),(n-3,s))\\((i,j),(n-2,s+1))}} = (-1)^{\nu(i,j,n-2,s+1)-\delta_{(i,j)>(p,q)}-\delta_{(n-2,s+1)>(n-3,s)}} M_{\substack{((p,q),(n-3,s))\\((i,j),(n-2,s+1))}}.$$

The only difference of these two expressions is in the sign $(-1)^{\delta_{(i,j)>(p,q)}}$ and hence we have the desired claim.

ii) Now we consider the formula of the burning lemma ii). It says if $(s,t),(u,v) \in Q$ such that $(u,v) \neq (s+1,t+1)$ and $(s+1,t+1) \in Q$, then we have

$$\sum_{(\alpha,\beta)\in P} C_{((\alpha,\beta),(s,t))} d_{\alpha-u-1,\beta-v-1} = 0.$$

In particular for $(s,t) = (u,v) = (n-3,v)$ we can write

$$\sum_{v\neq s} \sum_{(\alpha,\beta)\in P} C_{((\alpha,\beta),(n-3,v))} d_{\alpha-(n-3)-1,\beta-v-1} = 0.$$

Let us fix $(i,j) \in P$ such that $i-j = n-3-s$. Since $v \neq s$ and Lemma 23 we get

$$\sum_{(\alpha,\beta)\in P\setminus\{(i,j)\}} C_{((\alpha,\beta),(n-3,v))} d_{\alpha-(n-3)-1,\beta-v-1} = -C_{((i,j),(n-3,v))} d_{i-(n-3)-1,j-v-1} = 0.$$

We expand now all cofactors $C_{((\alpha,\beta),(n-3,v))}$ on the left-hand side along the $(n-3,s)$-column and we get

$$\begin{aligned}
0 &= \sum_{(\alpha,\beta)\in P} C_{((\alpha,\beta),(n-3,v))} d_{\alpha-(n-3)-1,\beta-v-1} \\
&= \sum_{(\alpha,\beta)\in P} (-1)^{\nu(\alpha,\beta,n-3,v)} \sum_{(i,j)\in P\setminus\{(\alpha,\beta)\}} C_{\substack{((\alpha,\beta),(n-3,v))\\((i,j),(n-3,s))}} d_{i-(n-3)-1,j-v-1} d_{\alpha-(n-3)-1,\beta-v-1}.
\end{aligned}$$

iii) We expand the theta function along the $(n-4,v)$th-column and we get

$$\theta = \sum_{(a,b)\in P} C_{((a,b),(n-4,v))} d_{a-(n-4),b-v}.$$

With lemma 24 we have

$$\begin{aligned}
&C_{\substack{((i,j),(n-3,s))\\((p,q),(n-4,v))}} d_{p-(n-4)-1,q-v-1} d_{i-(n-1),j-s-2} C_{((a,b),(n-4,v))} d_{a-(n-4),b-v} \\
&= C_{\substack{((i,j),(n-3,s))\\((a,b),(n-4,v))}} d_{a-(n-4),b-v} d_{i-(n-1),j-s-2} C_{((p,q),(n-4,v))} d_{p-(n-4)-1,q-v-1}.
\end{aligned}$$

Summing up over the $(a,b) \in P$ we get

$$M_{((i,j),(n-3,s))}d_{i-(n-1),j-s-2}C_{((p,q),(n-4,v))}d_{p-(n-4)-1,q-v-1}.$$

Summing up over the $(p,q) \in P \setminus \{(i,j)\}$ and with ii) of the burning lemma we get zero. □

The next lemma is the crucial lemma to prove ii) of theorem 20.

Lemma 29. *Let $s \in \{0,\dots,n-2\}$. Then we have the formulas*

i) $-\sum_{\substack{(i,j)\in P\\ \lfloor\frac{n}{2}\rfloor\le j}} \frac{C_{((i,j),(n-3,s))}(t)}{\theta(\mathcal{F}^t)}\frac{d}{dt}d_{i-(n-1),j-s-2}(t)$

$$= \sum_{\substack{(i,j)\in P\\ \lfloor\frac{n}{2}\rfloor\le j}} \frac{C_{((i,j),(n-3,s))}(t)}{\theta(\mathcal{F}^t)}d_{i-(n-1)-1,j-s-3}(t)$$

$$= -a_{1s}^{s+2}(t)\Big(a_{2s+1}^{s+4}(t)-a_{1s+1}^{s+3}(t)a_{1s+2}^{s+4}(t)\Big)+\sum_{(i,j)\in P}\frac{C_{((i,j),(n-2,s))}(t)}{\theta(\mathcal{F}^t)}d_{i-n-1,j-(s+3)}(t),$$

ii) $a_{1s+1}^{s+3}(t)\left(-\sum_{(i,j)\in P}\frac{C_{((i,j),(n-2,s))}(t)}{\theta(\mathcal{F}^t)}d_{i-n,j-s-2}(t)\right)$

$$= -a_{1s}^{s+2}(t)a_{2s}^{s+3}(t)-\sum_{(p,q)\in P}\left(\frac{C_{((p,q)),(n-4,s-1))}(t)}{\theta(\mathcal{F}^t)}d_{p-(n-1),q-(s+2)}(t)\right)$$

$$+\sum_{(i,j)\in P}\left(\frac{C_{((i,j),(n-2,s))}(t)}{\theta(\mathcal{F}^t)}d_{(i-2)-(n-1),(j-1)-(s+2)}(t)\right),$$

iii) $-\sum_{\substack{(i,j)\in P\\ \lfloor\frac{n}{2}\rfloor\le j}} \frac{\frac{d}{dt}C_{((i,j),(n-3,s))}(t)}{\theta(\mathcal{F}^t)}d_{i-(n-1),j-s-2}(t)-\frac{\frac{d}{dt}\theta(\mathcal{F}^t)}{\theta(\mathcal{F}^t)}a_{2s}^{s+3}(t)$

$$= -a_{1s+1}^{s+3}(t)a_{2s}^{s+3}(t)-\sum_{(p,q)\in P}\frac{C_{((p,q)),(n-4,s-1))}(t)}{\theta(\mathcal{F}^t)}d_{p-(n-1),q-(s+2)}(t),$$

iv)

$$-a_{1s+1}^{s+3}(t)a_{2s}^{s+3}(t) = -a_{1s}^{s+2}(t)a_{1s+1}^{s+3}(t)a_{1s+1}^{s+3}(t)$$

$$+a_{1s+1}^{s+3}(t)\left(\sum_{(i,j)\in P}\frac{C_{((i,j),(n-2,s))}(t)}{\theta(\mathcal{F}^t)}d_{i-n,j-s-2}(t)\right),$$

v) $-a_{1s+1}^{s+3}(t)a_{2s}^{s+3}(t) = -a_{1s}^{s+2}(t)a_{1s+1}^{s+3}(t)a_{1s+1}^{s+3}(t)+a_{1s}^{s+2}(t)a_{2s}^{s+3}(t)$

$$-\sum_{(i,j)\in P}\frac{C_{((i,j),(n-2,s))}(t)}{\theta(\mathcal{F}^t)}d_{i-n-1,j-(s+3)}(t)$$

$$+\sum_{(p,q)\in P}\frac{C_{((p,q)),(n-4,s-1))}(t)}{\theta(\mathcal{F}^t)}d_{p-(n-1),q-(s+2)}(t),$$

vi) $-\sum_{\substack{(i,j)\in P\\ \lfloor\frac{n}{2}\rfloor\leq j}} \frac{\frac{d}{dt}C_{((i,j),(n-3,s))}(t)}{\theta(\mathcal{F}^t)} d_{i-(n-1),j-s-2}(t) - \frac{\frac{d}{dt}\theta(\mathcal{F}^t)}{\theta(\mathcal{F}^t)} a_{2s}^{s+3}(t)$

$$= a_{1s}^{s+2}(t)\left(a_{2s}^{s+3}(t) - w_{s+1s+2}(t)\right) - \sum_{(i,j)\in P} \frac{C_{((i,j),(n-2,s))}(t)}{\theta(\mathcal{F}^t)} d_{i-n-1,j-(s+3)}(t).$$

Proof. In the proof we will drop the arguments (t) and $(\mathcal{F}^t)$ to get a little bit more clearness. i) By lemma 27, the burning lemma vii) and lemma 26 we get

$$\begin{aligned}
&a_{1s}^{s+2}\left(-a_{2s+1}^{s+4} + a_{1s+1}^{s+3}a_{1s+2}^{s+4}\right)\\
&= a_{1s}^{s+2}\Big(\sum_{(p,q)\in P} \frac{C_{((p,q),(n-2s+1))}}{\theta} d_{p-1-(n-1),q-(s+3)}\Big)\\
&= \Big(-\sum_{(i,j)\in P} \frac{C_{((i,j),(n-2,s))}}{\theta} d_{i-(n-1),j-(s+1)}\Big)\Big(\sum_{(p,q)\in P} \frac{C_{((p,q),(n-2,s+1))}}{\theta} d_{p-1-(n-1),q-(s+3)}\Big)\\
&= \sum_{(i,j)\in P}\sum_{(p,q)\in P}\left(-\frac{C_{((i,j),(n-2,s))}}{\theta} d_{p-1-(n-1),q-(s+3)}\right)\\
&\quad\Big(\frac{\theta}{\theta}\delta_{(p,q)=(i-1,j)} - \frac{C_{((p,q),(n-3,s))}}{\theta} d_{i-(n-2),j-s} - \sum_{\substack{(u,v)\in Q\\ (u,v)\neq(n-2,s+1)\\ (u,v)\neq(n-3,s)}} \frac{C_{((p,q),(u,v))}}{\theta} d_{i-u-1,j-v}\Big)\\
&= \left(-\sum_{(i,j)\in P} \frac{C_{((i,j),(n-2,s))}}{\theta} d_{i-1-1-(n-1),j-(s+3)}\right)\frac{\theta}{\theta}\\
&\quad + \left(\sum_{(i,j)\in P} \frac{C_{((i,j),(n-2,s))}}{\theta} d_{i-(n-2),j-s}\right)\left(\sum_{(p,q)\in P} \frac{C_{((p,q),(n-3,s))}}{\theta} d_{p-1-(n-1),q-(s+3)}\right)\\
&= -\sum_{(i,j)\in P} \frac{C_{((i,j),(n-2,s))}}{\theta} d_{i-1-1-(n-1),j-(s+3)} + \frac{\theta}{\theta}\sum_{(p,q)\in P} \frac{C_{((p,q),(n-3,s))}}{\theta} d_{p-1-(n-1),q-(s+3)}\\
&= -\sum_{(i,j)\in P} \frac{C_{((i,j),(n-2,s))}}{\theta} d_{i-n-1,j-(s+3)} - \sum_{(i,j)\in P} \frac{C_{((i,j),(n-3,s))}}{\theta}\frac{d}{dt}\left(d_{i-(n-1),j-(s+2)}\right).
\end{aligned}$$

ii) With lemma 25 we get

$$
\begin{aligned}
&a_{1s+1}^{s+3}\left(-\sum_{(i,j)\in P}\frac{C_{((i,j),(n-2,s))}}{\theta}d_{i-n,j-s-2}\right)\\
&=\left(-\sum_{(p,q)\in P}\frac{C_{((p,q),(n-2,s+1))}}{\theta}d_{p-(n-1),q-(s+2)}\right)\left(-\sum_{(i,j)\in P}\frac{C_{((i,j),(n-2,s))}}{\theta}d_{i-n,j-s-2}\right)\\
&=\sum_{(i,j)\in P}\sum_{(p,q)\in P}\left(\frac{C_{((i,j),(n-2,s))}}{\theta}d_{p-(n-1),q-(s+2)}\frac{C_{((p,q),(n-2,s+1))}}{\theta}d_{i-n,j-s-2}\right)\\
&=\sum_{(i,j)\in P}\sum_{(p,q)\in P}\left(\frac{C_{((i,j),(n-2,s))}}{\theta}d_{p-(n-1),q-(s+2)}\right)\left(\frac{\theta}{\theta}\delta_{(p,q)=(i-2,j-1)}\right.\\
&\quad-\frac{C_{((p,q),(n-3,s))}}{\theta}d_{i-(n-1),j-(s+1)}-\frac{C_{((p,q)),(n-4,s-1))}}{\theta}d_{i-(n-4)-2,j-(s-1)-1}\\
&\quad\left.-\sum_{\substack{(u,v)\in Q\\(u,v)\neq(n-2,s+1)\\(u,v)\neq(n-3,s)\\(u,v)\neq(n-4,s-1)}}\frac{C_{((p,q)),(u,v))}}{\theta}d_{i-u-2,j-v-1}\right)\\
&=-\sum_{(i,j)\in P}\sum_{(p,q)\in P}\frac{C_{((i,j),(n-2,s))}}{\theta}d_{i-(n-1),j-(s+1)}\frac{C_{((p,q),(n-3,s))}}{\theta}d_{p-(n-1),q-(s+2)}\\
&\quad-\sum_{(i,j)\in P}\sum_{(p,q)\in P}\frac{C_{((i,j),(n-2,s))}}{\theta}d_{i-(n-4)-2,j-(s-1)-1}\frac{C_{((p,q)),(n-4,s-1))}}{\theta}d_{p-(n-1),q-(s+2)}\\
&\quad-\sum_{(i,j)\in P}\sum_{(p,q)\in P}\sum_{\substack{(u,v)\in Q\\(u,v)\neq(n-2,s+1)\\(u,v)\neq(n-3,s)\\(u,v)\neq(n-4,s-1)}}\frac{C_{((i,j),(n-2,s))}}{\theta}d_{i-u-2,j-v-1}\frac{C_{((p,q),(u,v))}}{\theta}d_{p-(n-1),q-(s+2)}\\
&\quad+\sum_{(i,j)\in P}\sum_{(p,q)\in P}\left(\frac{C_{((i,j),(n-2,s))}}{\theta}d_{p-(n-1),q-(s+2)}\right)\left(\frac{\theta}{\theta}\delta_{(p,q)=(i-2,j-1)}\right).
\end{aligned}
$$

If $u-v\neq(n-2)-(s+1)$ lemma 23 ensures, that the term

$$\sum_{(p,q)\in P}\frac{C_{((p,q),(u,v))}}{\theta}d_{p-(n-1),q-(s+2)}=0$$

vanishes. In the case $u\leq n-5$ and $u-v=(n-2)-(s+1)$ we have $(u+1,v+1)\in Q$ and so by the burning lemma ii)

$$\sum_{(i,j)\in P}\frac{C_{((i,j),(n-2,s))}}{\theta}d_{i-u-2,j-v-1}=0.$$

Thus the third term vanishes and by using the Laplace expansion for the theta function it remains

$$a_{1s+1}^{s+3}\left(-\sum_{(i,j)\in P}\frac{C_{((i,j),(n-2,s))}}{\theta}d_{i-n,j-s-2}\right)$$
$$=-a_{1s}^{s+2}a_{2s}^{s+3}-\sum_{(p,q)\in P}\frac{C_{((p,q)),(n-4,s-1))}}{\theta}d_{p-(n-1),q-(s+2)}$$
$$+\sum_{(i,j)\in P}\frac{C_{((i,j),(n-2,s))}}{\theta}d_{(i-2)-(n-1),(j-1)-(s+2)}.$$

iii) We will write $\nu := \nu(i,j,n-3,s)$ to shorten the formulas. Let us compute

$$-\sum_{\substack{(i,j)\in P\\ \lfloor\frac{n}{2}\rfloor\le j}}\frac{\frac{d}{dt}C_{((i,j),(n-3,s))}}{\theta}d_{i-(n-1),j-s-2}$$
$$=-\sum_{\substack{(i,j)\in P\\ \lfloor\frac{n}{2}\rfloor\le j}}\frac{(-1)^{\nu(i,j,n-3,s)}\frac{d}{dt}M_{((i,j),(n-3,s))}}{\theta}d_{i-(n-1),j-s-2}$$
$$=-\sum_{\substack{(i,j)\in P\\ \lfloor\frac{n}{2}\rfloor\le j}}(-1)^{\nu}\sum_{\substack{(u,v)\in Q\\ (u,v)\neq(n-3,s)}}\sum_{(p,q)\in P\setminus\{(i,j)\}}\frac{C_{((i,j),(n-3,s))}^{\;((p,q),(u,v))}}{\theta}(-d_{p-u-1,q-v-1})d_{i-(n-1),j-s-2}$$
$$=\sum_{\substack{(i,j)\in P\\ \lfloor\frac{n}{2}\rfloor\le j}}(-1)^{\nu}\sum_{\substack{(n-3,v)\in Q\\ v\neq s}}\sum_{(p,q)\in P\setminus\{(i,j)\}}\frac{C_{((i,j),(n-3,s))}^{\;((p,q),(n-3,v))}}{\theta}d_{p-(n-3)-1,q-v-1}d_{i-(n-1),j-s-2}$$
$$+\sum_{\substack{(i,j)\in P\\ \lfloor\frac{n}{2}\rfloor\le j}}(-1)^{\nu}\sum_{(n-2,v)\in Q}\sum_{(p,q)\in P\setminus\{(i,j)\}}\frac{C_{((i,j),(n-3,s))}^{\;((p,q),(n-2,v))}}{\theta}d_{p-(n-2)-1,q-v-1}d_{i-(n-1),j-s-2}$$
$$+\sum_{\substack{(i,j)\in P\\ \lfloor\frac{n}{2}\rfloor\le j}}(-1)^{\nu}\sum_{(n-4,v)\in Q}\sum_{(p,q)\in P\setminus\{(i,j)\}}\frac{C_{((i,j),(n-3,s))}^{\;((p,q),(n-4,v))}}{\theta}d_{p-(n-4)-1,q-v-1}d_{i-(n-1),j-s-2}$$
$$+\sum_{\substack{(i,j)\in P\\ \lfloor\frac{n}{2}\rfloor\le j}}(-1)^{\nu}\sum_{\substack{(u,v)\in Q\\ u\neq n-2\\ u\neq n-3\\ u\neq n-4}}\sum_{(p,q)\in P\setminus\{(i,j)\}}\frac{C_{((i,j),(n-3,s))}^{\;((p,q),(u,v))}}{\theta}d_{p-u-1,q-v-1}d_{i-(n-1),j-s-2}.$$

By lemma 28 a lot of terms vanish and it remains

$$
\begin{aligned}
&- \sum_{\substack{(i,j)\in P\\ \lfloor \frac{n}{2}\rfloor \le j}} \frac{\frac{d}{dt} C_{((i,j),(n-3,s))}}{\theta} d_{i-(n-1),j-s-2} \\
&= \sum_{\substack{(i,j)\in P\\ \lfloor \frac{n}{2}\rfloor \le j}} (-1)^{\nu(i,j,n-3,s)} \sum_{\substack{v=0\\ v\neq s+1}}^{n-2} \sum_{(p,q)\in P\setminus\{(i,j)\}} \frac{C_{\substack{((i,j),(n-3,s))\\ ((p,q),(n-2,v))}}}{\theta} d_{p-(n-2)-1,q-v-1} d_{i-(n-1),j-s-2} \\
&\quad + \sum_{\substack{(i,j)\in P\\ \lfloor \frac{n}{2}\rfloor \le j}} (-1)^{\nu(i,j,n-3,s)} \sum_{(p,q)\in P\setminus\{(i,j)\}} \frac{C_{\substack{((i,j),(n-3,s))\\ ((p,q),(n-4,s-1))}}}{\theta} d_{p-(n-4)-1,q-(s-1)-1} d_{i-(n-1),j-s-2}.
\end{aligned}
$$

Now we expand the theta function along the $(n-2,v)$-column and we have $1 = \frac{\theta}{\theta} = \frac{1}{\theta}\sum_{(a,b)\in P} C_{((a,b),(n-2,v))} d_{a-(n-2),b-v}$. We fix the indices $(i,j),(p,q) \in P$ and $v \in \{0,\dots,n-2\} \setminus \{s+1\}$. With lemma 24 we have

$$
\begin{aligned}
&- \sum_{(a,b)\in P} (-1)^{\nu(i,j,n-3,s)} \frac{C_{\substack{((i,j),(n-3,s))\\ ((p,q),(n-2,v))}}}{\theta} (-d_{p-(n-2)-1,q-v-1}) d_{i-(n-1),j-s-2} \\
&\quad \cdot \frac{C_{((a,b),(n-2,v))}}{\theta} d_{a-(n-2),b-v} \\
&= (-1)^{\nu(i,j,n-3,s)} \left(- \sum_{(a,b)\in P} \frac{C_{\substack{((i,j),(n-3,s))\\ ((a,b),(n-2,v))}}}{\theta} d_{a-(n-2),b-v} \right) d_{i-(n-1),j-s-2} \\
&\quad \frac{C_{((p,q),(n-2,v))}}{\theta} (-d_{p-(n-2)-1,q-v-1}) \\
&= (-1)^{\nu(i,j,n-3,s)} \frac{-M_{((i,j),(n-3,s))}}{\theta} d_{i-(n-1),j-s-2} \frac{-C_{((p,q),(n-2,v))}}{\theta} d_{p-(n-2)-1,q-v-1} \\
&= \frac{-C_{((i,j),(n-3,s))}}{\theta} d_{i-(n-1),j-s-2} \frac{-C_{((p,q),(n-2,v))}}{\theta} d_{p-(n-2)-1,q-v-1}.
\end{aligned}
$$

Summing up over the indices $(i,j),(p,q) \in P$ and v we get

$$
\begin{aligned}
&\sum_{\substack{v=0\\ v\neq s+1}}^{n-2} \left(\sum_{(i,j)\in P} \frac{-C_{((i,j),(n-3,s))}}{\theta} d_{i-(n-1),j-s-2} \right) \left(\sum_{(p,q)\in P} \frac{-C_{((p,q),(n-2,v))}}{\theta} d_{p-(n-2)-1,q-v-1} \right) \\
&= a_{2s}^{s+3} \left(\sum_{v\neq s+1} a_{1v}^{v+2} \right).
\end{aligned}
$$

Note that $v \neq s+1$ implies always $(i,j) \neq (p,q)$. For the remaining term we compute

$$\sum_{\substack{(i,j)\in P\\ \lfloor\frac{n}{2}\rfloor\leq j}} (-1)^{\nu(i,j,n-3,s)} \sum_{(p,q)\in P\setminus\{(i,j)\}} \frac{C^{\ ((i,j),(n-3,s))}_{((p,q),(n-4,s-1))}}{\theta} d_{p-(n-4)-1,q-(s-1)-1} d_{i-(n-1),j-s-2}$$
$$= \sum_{\substack{(i,j)\in P\\ \lfloor\frac{n}{2}\rfloor\leq j}} (-1)^{\iota_r(i,j)}(-1)^{\iota_c(n-3,s)} \sum_{(p,q)\in P\setminus\{(i,j)\}} (-1)^{\iota_r(p,q)-\delta_{(p,q)>(i,j)}}$$
$$\cdot (-1)^{\iota_c(n-4,s-1)-\delta_{(n-4,s-1)>(n-3,s)}} \frac{M^{\ ((i,j),(n-3,s))}_{((p,q),(n-4,s-1))}}{\theta} d_{p-(n-4)-1,q-(s-1)-1} d_{i-(n-1),j-s-2}$$
$$= \sum_{\substack{(i,j)\in P\\ \lfloor\frac{n}{2}\rfloor\leq j}} (-1)^{\iota_r(i,j)}(-1)^{\iota_c(n-4,s-1)} \sum_{(p,q)\in P\setminus\{(i,j)\}} (-1)^{\iota_r(p,q)-\delta_{(p,q)>(i,j)}}(-1)^{\iota_c(n-3,s)}$$
$$\cdot (-1)(-1)^{\delta_{(n-3,s)>(n-4,s-1)}} \frac{M^{((i,j),(n-4,s-1))}_{((p,q),(n-3,s))}}{\theta} d_{p-(n-4)-1,q-(s-1)-1} d_{i-(n-1),j-s-2}$$
$$= -\sum_{\substack{(i,j)\in P\\ \lfloor\frac{n}{2}\rfloor\leq j}} (-1)^{\nu(i,j,n-4,s-1)} \sum_{(p,q)\in P\setminus\{(i,j)\}} \frac{C^{((i,j),(n-4,s-1))}_{((p,q),(n-3,s))}}{\theta} d_{p-(n-4)-1,q-(s-1)-1} d_{i-(n-1),j-s-2}$$
$$= -\sum_{\substack{(i,j)\in P\\ \lfloor\frac{n}{2}\rfloor\leq j}} (-1)^{\nu(i,j,n-4,s-1)} \frac{M_{((i,j),(n-4,s-1))}}{\theta} d_{i-(n-1),j-s-2}$$
$$= -\sum_{\substack{(i,j)\in P\\ \lfloor\frac{n}{2}\rfloor\leq j}} \frac{C_{((i,j),(n-4,s-1))}}{\theta} d_{i-(n-1),j-s-2}.$$

Putting the last two computations together we get the formula of iii). The formula of iv) is just the multiplication of a^{s+3}_{1s+1} with the formula in iii) of lemma 27. The formula in v) follows immediatly by iv) and ii) and the formula of vi) follows by v) and iii). □

Now we prove theorem 20.

Proof. We will drop the arguments (t) and $(\mathcal{F}^t)$. Let $s \in \{0,\ldots,n-2\}$.

i) We compute the derivative of a^{s+2}_{1s} and we get

$$\frac{d}{dt}a^{s+2}_{1s} = -\sum_{\substack{(i,j)\in P\\ \lfloor\frac{n}{2}\rfloor\leq j}} \frac{\frac{d}{dt}\left(C_{((i,j),(n-2,s))}d_{i-(n-1),j-s-1}\right)\theta - \theta' C_{((i,j),(n-2,s))}d_{i-(n-1),j-s-1}}{\theta^2}$$
$$= -\sum_{\substack{(i,j)\in P\\ \lfloor\frac{n}{2}\rfloor\leq j}} \frac{\frac{d}{dt}C_{((i,j),(n-2,s))}}{\theta} d_{i-(n-1),j-s-1} - \sum_{\substack{(i,j)\in P\\ \lfloor\frac{n}{2}\rfloor\leq j}} \frac{C_{((i,j),(n-2,s))}}{\theta}\frac{d}{dt}\left(d_{i-(n-1),j-s-1}\right)$$
$$-\frac{\theta'}{\theta}(-a^{s+2}_{1s}).$$

By applying lemma 27 we get the result directly.

ii) Let us derive the coefficient a_{2s}^{s+3}. We get

$$\frac{d}{dt}a_{2s}^{s+3} = -\sum_{\substack{(i,j)\in P\\ \lfloor\frac{n}{2}\rfloor\le j}} \frac{\frac{d}{dt}C_{((i,j),(n-3,s))}}{\theta} d_{i-(n-1),j-s-2}$$
$$-\sum_{\substack{(i,j)\in P\\ \lfloor\frac{n}{2}\rfloor\le j}} \frac{C_{((i,j),(n-3,s))}}{\theta}\frac{d}{dt}\left(d_{i-(n-1),j-s-2}\right) - \frac{\theta'}{\theta}(-a_{2s}^{s+3})$$

Now we apply lemma 29 and we are done.

iii) We have

$$\frac{i}{2}\frac{d}{dt}\sqrt{|\alpha_s(t)|} = \frac{i}{2}\sqrt{i^2}\frac{d}{dt}\sqrt{\alpha_s(t)} = \frac{i}{2}\sqrt{i^2}\frac{1}{2}\frac{1}{\sqrt{\alpha_s(t)}}\frac{d}{dt}\alpha_s(t) = -\frac{i}{2}\frac{1}{2}\frac{1}{\sqrt{|\alpha_s(t)|}}\frac{d}{dt}\alpha_s(t)$$

and so the equation in iii) gets

$$-\frac{i}{2}\frac{1}{2}\frac{1}{\sqrt{|\alpha_s(t)|}}\frac{d}{dt}\alpha_s(t) = \frac{i}{4}\left(a_{1s-1}^{s+1}(t) - 2a_{1s}^{s+2}(t) + a_{1s+1}^{s+3}(t)\right)\sqrt{|\alpha_s|},$$

or equivalently

$$\frac{d}{dt}\alpha_s(t) = \left(a_{1s-1}^{s+1}(t) - 2a_{1s}^{s+2}(t) + a_{1s+1}^{s+3}(t)\right)\alpha_s.$$

With i) and ii) we compute

$$\begin{aligned}
\frac{d}{dt}\alpha_s &= \frac{d}{dt}\left(a_{2s-1}^{s+2} - a_{2s}^{s+3} - w_{s+1,s+2} + v_{s+1,s+2}\right)\\
&= \frac{d}{dt}a_{2s-1}^{s+2} - \frac{d}{dt}a_{2s}^{s+3} - 2a_{1s}^{s+2}\frac{d}{dt}a_{1s}^{s+2} + \left(\frac{d}{dt}a_{1s}^{s+2}\right)a_{1s+1}^{s+3} + a_{1s}^{s+2}\left(\frac{d}{dt}a_{1s+1}^{s+3}\right)\\
&= a_{1s-1}^{s+1}\alpha_s - a_{1s}^{s+2}\alpha_{s+1} - 2a_{1s}^{s+2}\alpha_s + \alpha_s a_{1s+1}^{s+3} + a_{1s}^{s+2}\alpha_{s+1}\\
&= \left(a_{1s-1}^{s+1} - 2a_{1s}^{s+2} + a_{1s+1}^{s+3}\right)\alpha_s\\
&= \left(a_{1s-1}^{s+1} - 2a_{1s}^{s+2} + a_{1s+1}^{s+3}\right)\left(a_{2s-1}^{s+2} - a_{2s}^{s+3} - a_{1s}^{s+2}a_{1s}^{s+2} + a_{1s}^{s+2}a_{1s+1}^{s+3}\right).
\end{aligned}$$

Finally the matrices $(T_1(t), T_2(t), T_3(t))$ of theorem 20 satisfy Nahm's equations by i) and iii). This proves the theorem. □

5.2.3 Examples

Example 7. *We will consider the case $n = 2$ and hence $g = (n-1)^2 = 1$. The transition function of any invertible sheaf $\mathcal{F} \in Jac^0(C_2)$ is of the form $g_{10}(\zeta,\eta) = d_{00} + d_{11}\frac{\eta}{\zeta}$ and the theta function is $\theta(\mathcal{F}) = d_{11}$. This means all invertible sheaves are of the special case above and we have $a_{10}^2 = -\frac{d_{00}}{d_{11}}$. An invertible sheaf $\mathcal{F}$ is real, if $d_{11} \in \mathbb{R}$. In terms of flows we have $d_{11}(t) = d_{11} - td_{00}$. A representative of the $GL_2(\mathbb{C})$-conjugation class of regular, nilpotent, matricial polynomials of a sheaf $\mathcal{F}^t \in Jac^0(C_2)\setminus\Theta$ is given by*

$$A(\zeta,t) = \begin{pmatrix}0&0\\1&0\end{pmatrix} + \begin{pmatrix}\frac{d_{00}}{d_{11}(t)}&0\\0&-\frac{d_{00}}{d_{11}(t)}\end{pmatrix}\zeta + \begin{pmatrix}0&-\frac{d_{00}^2}{d_{11}(t)^2}\\0&0\end{pmatrix}\zeta^2.$$

This leads to solutions of Nahm's equations

$$T_1(t) = \frac{i}{2}\begin{pmatrix} \frac{d_{00}}{d_{11}(t)} & 0 \\ 0 & -\frac{d_{00}}{d_{11}(t)} \end{pmatrix}, \ T_2(t) = \frac{i}{2}\begin{pmatrix} 0 & \frac{d_{00}}{d_{11}(t)} \\ \frac{d_{00}}{d_{11}(t)} & 0 \end{pmatrix}, \ T_3(t) = \frac{1}{2}\begin{pmatrix} 0 & -\frac{d_{00}}{d_{11}(t)} \\ \frac{d_{00}}{d_{11}(t)} & 0 \end{pmatrix}.$$

Any element of $\mathcal{O}_{reg}(\mathfrak{sl}_2(\mathbb{C}))$ is $SU(2)$-conjugated to a matrix of the form

$$X := \hat{A}_0(0) = i\begin{pmatrix} 0 & 0 \\ \frac{d_{00}}{d_{11}} & 0 \end{pmatrix}$$

for some negative real number d_{11}. For such an element, via the solutions of Nahm's equations above, the Kähler potential is given by $K(X) = \frac{\theta'(\mathcal{F}^t)}{\theta(\mathcal{F}^t)} = -\frac{d_{00}}{d_{11}} > 0$. Hence a Kähler potential is given by

$$K(X) = \sqrt{tr\left(X\overline{X}^T\right)}$$

up to a multiplication of a constant. This is not surprising, because the regular, nilpotent orbit coincides with the minimal, nilpotent orbit in the case $n = 2$ and it is well-known, that $K(X) = \sqrt{tr\left(X\overline{X}^T\right)}$ (up to a multiplication by a constant) is a Kähler potential on the minimal, nilpotent orbit in any complex, simple Lie algebra, see [KS01c].

Example 8. *In this last example we consider the case $n = 3$. The genus is $g = 4$. Let $\mathcal{F} \in Jac^3(C_3) \setminus \Theta$ be an invertible sheaf of degree 3. Let us suppose its transition function is of the form $g_{10}(\zeta, \eta) := \frac{1}{\zeta}\left(d_{00} + d_{11}\frac{\eta}{\zeta} + d_{22}\frac{\eta^2}{\zeta^2}\right)$ with $d_{11}, d_{22} \in \mathbb{R}$. Moreover let us assume the inequalities*

$$d_{11}^2 > d_{22} > 0,$$

which ensures α_0 and α_1 are negative terms. The theta function is $\theta(\mathcal{F}) = d_{00}d_{11}^2 d_{22} - d_{00}^2 d_{22}^2 > 0$. For simplicity we set $d_{00} = 1$. We get matrices by conjugating the matrices A_i by the coordinate transformation matrix P,

$$\hat{A}_0 = i\begin{pmatrix} 0 & 0 & 0 \\ \frac{(d_{11}^2 - d_{22})^{\frac{3}{2}}}{\theta} & 0 & 0 \\ 0 & \frac{(d_{22})^{\frac{3}{2}}}{\theta} & 0 \end{pmatrix},$$

$$\hat{A}_1 = \begin{pmatrix} \frac{(d_{11}^2 - d_{22})d_{11}}{\theta} & 0 & 0 \\ 0 & \frac{(2d_{22} - d_{11}^2)d_{11}}{\theta} & 0 \\ 0 & 0 & \frac{-d_{11}d_{22}}{\theta} \end{pmatrix},$$

$$\hat{A}_2 = i\begin{pmatrix} 0 & \frac{(d_{11}^2 - d_{22})^{\frac{3}{2}}}{\theta} & 0 \\ 0 & 0 & \frac{(d_{22})^{\frac{3}{2}}}{\theta} \\ 0 & 0 & 0 \end{pmatrix}.$$

By theorem 20 we get $\mathfrak{su}(n)$*-valued solutions of Nahm's equations*

$$T_1(t) = \frac{i}{2}\begin{pmatrix} \frac{\left(d_{11}(t)^2 - d_{22}(t)\right)d_{11}(t)}{\theta(\mathcal{F}^t)} & 0 & 0 \\ 0 & \frac{\left(2d_{22}(t) - d_{11}(t)^2\right)d_{11}(t)}{\theta(\mathcal{F}^t)} & 0 \\ 0 & 0 & \frac{-d_{11}(t)d_{22}(t)}{\theta(\mathcal{F}^t)} \end{pmatrix},$$

$$T_2(t) = \frac{i}{2}\begin{pmatrix} 0 & \frac{(d_{11}(t)^2 - d_{22}(t))^{\frac{3}{2}}}{\theta(\mathcal{F}^t)} & 0 \\ \frac{(d_{11}(t)^2 - d_{22}(t))^{\frac{3}{2}}}{\theta(\mathcal{F}^t)} & 0 & \frac{(d_{22}(t))^{\frac{3}{2}}}{\theta(\mathcal{F}^t)} \\ 0 & \frac{(d_{22}(t))^{\frac{3}{2}}}{\theta(\mathcal{F}^t)} & 0 \end{pmatrix},$$

$$T_3(t) = \frac{1}{2}\begin{pmatrix} 0 & -\frac{(d_{11}(t)^2 - d_{22}(t))^{\frac{3}{2}}}{\theta(\mathcal{F}^t)} & 0 \\ \frac{(d_{11}(t)^2 - d_{22}(t))^{\frac{3}{2}}}{\theta(\mathcal{F}^t)} & 0 & -\frac{(d_{22}(t))^{\frac{3}{2}}}{\theta(\mathcal{F}^t)} \\ 0 & \frac{(d_{22}(t))^{\frac{3}{2}}}{\theta(\mathcal{F}^t)} & 0 \end{pmatrix}.$$

These matrices already appeared in [KS93] up to a reparametrization, where they are called flow lines. Let us consider the matrix

$$R = \begin{pmatrix} -\frac{\sqrt{3}}{2} + \frac{1}{2}i & 0 & 0 \\ 0 & -\frac{1}{2} - \frac{\sqrt{3}}{2}i & 0 \\ 0 & 0 & \frac{\sqrt{3}}{2} - \frac{1}{2}i \end{pmatrix} \in SU(3).$$

Now we conjugate the matrix

$$\hat{A}_0 = i\begin{pmatrix} 0 & 0 & 0 \\ \frac{(d_{11}^2 - d_{22})^{\frac{3}{2}}}{\theta} & 0 & 0 \\ 0 & \frac{(d_{22})^{\frac{3}{2}}}{\theta} & 0 \end{pmatrix}$$

by the matrix R *and we get*

$$\hat{A}_0^{conj} = R\hat{A}_0R^{-1} = R\hat{A}_0\overline{R}^T = \begin{pmatrix} 0 & 0 & 0 \\ -\frac{(d_{11}^2 - d_{22})^{\frac{3}{2}}}{\theta} & 0 & 0 \\ 0 & -\frac{(d_{22})^{\frac{3}{2}}}{\theta} & 0 \end{pmatrix}.$$

With $a = -\frac{(d_{11}^2 - d_{22})^{\frac{3}{2}}}{\theta}$ *and* $c = -\frac{(d_{22})^{\frac{3}{2}}}{\theta}$ *(it is the notation of [KS93]) we get*

$$a^{\frac{2}{3}} + c^{\frac{2}{3}} = (-1)^{\frac{2}{3}}\frac{d_{11}^2}{\theta^{\frac{2}{3}}}.$$

Thus the value of the Kähler potential is

$$\sqrt{\left(a^{\frac{2}{3}} + c^{\frac{2}{3}}\right)^3} = \frac{-d_{11}^3}{\theta} = \frac{\theta'(\mathcal{F}^0)}{\theta(\mathcal{F}^0)} = K(\hat{A}_0^{conj}) = K(\hat{A}_0)$$

This formula coincides with the formula in higher generality of [KS93], [KS01b] with $b = 0$ *in their notation.*

With fixed numbers $d_{11} = -\sqrt{2}$ *and* $d_{00} = d_{22} = 1$ *we have the property* $d_{11}^2 > d_{22} > 0$*. We have* $d_{22}(t) = 1 + \sqrt{2}t + \frac{t^2}{2} = \left(\frac{t}{\sqrt{2}} + 1\right)^2$, $d_{11}(t) = -\sqrt{2} - t = -\sqrt{2}(\frac{t}{\sqrt{2}} + 1)$,

$d_{11}(t)^2 = 2d_{22}(t)$ *and* $\theta(\mathcal{F}^t) = \left(\frac{t}{\sqrt{2}} + 1\right)^4$. *This leads to solutions of Nahm's equations*

$$T_1(t) = \frac{i}{2}\begin{pmatrix} \frac{-\sqrt{2}}{\left(\frac{t}{\sqrt{2}}+1\right)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{\left(\frac{t}{\sqrt{2}}+1\right)} \end{pmatrix} = \begin{pmatrix} \frac{-i}{(t+\sqrt{2})} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{i}{(t+\sqrt{2})} \end{pmatrix},$$

$$T_2(t) = \frac{i}{2}\begin{pmatrix} 0 & \frac{1}{\left(\frac{t}{\sqrt{2}}+1\right)} & 0 \\ \frac{1}{\left(\frac{t}{\sqrt{2}}+1\right)} & 0 & \frac{1}{\left(\frac{t}{\sqrt{2}}+1\right)} \\ 0 & \frac{1}{\left(\frac{t}{\sqrt{2}}+1\right)} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{i}{\sqrt{2}(t+\sqrt{2})} & 0 \\ \frac{i}{\sqrt{2}(t+\sqrt{2})} & 0 & \frac{i}{\sqrt{2}(t+\sqrt{2})} \\ 0 & \frac{i}{\sqrt{2}(t+\sqrt{2})} & 0 \end{pmatrix},$$

$$T_3(t) = \frac{1}{2}\begin{pmatrix} 0 & \frac{-1}{\left(\frac{t}{\sqrt{2}}+1\right)} & 0 \\ \frac{1}{\left(\frac{t}{\sqrt{2}}+1\right)} & 0 & \frac{-1}{\left(\frac{t}{\sqrt{2}}+1\right)} \\ 0 & \frac{1}{\left(\frac{t}{\sqrt{2}}+1\right)} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{-1}{\sqrt{2}(t+\sqrt{2})} & 0 \\ \frac{1}{\sqrt{2}(t+\sqrt{2})} & 0 & \frac{-1}{\sqrt{2}(t+\sqrt{2})} \\ 0 & \frac{1}{\sqrt{2}(t+\sqrt{2})} & 0 \end{pmatrix}.$$

We see easily, that these matrices are well-defined on the interval $[0, \infty)$ *and they solve Nahm's equations*

$$\begin{aligned} \frac{d}{dt}T_1(t) &= [T_2(t), T_3(t)], \\ \frac{d}{dt}T_2(t) &= [T_3(t), T_1(t)], \\ \frac{d}{dt}T_3(t) &= [T_1(t), T_2(t)]. \end{aligned}$$

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Curriculum Vitae

Personal Details

Oliver Sonderegger
Limmerstr. 81
30451 Hannover

E-Mail: sonderegger@math.uni-hannover.de

Date of birth: 15. 09. 1988 in Frauenfeld
Nationality: Switzerland

Education

1995–2003	Primary School in Wigoltingen
2003–2007	Matura at Kantonsschule Frauenfeld (emphasis mathematics and physics)

Academic Education

2009–2013	B.Sc. ETH in mathematics at ETH Zurich
2013–2015	M.Sc. ETH in mathematics at ETH Zurich
2016–2019	Doctorate at Leibniz Universität Hannover at Prof. Dr. Bielawski
2016–2017	Member of the Graduiertenkolleg 1463: Analysis, Geometry and Stringtheory
2017–2019	Research assistant at the institute of differential geometry
2009–2014	Teaching Assistant at ETH Zurich

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