



# The staggered six-vertex model: Conformal invariance and corrections to scaling

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## Abstract

We study the emergence of non-compact degrees of freedom in the low energy effective theory for a class of  $\mathbb{Z}_2$ -staggered six-vertex models. In the finite size spectrum of the vertex model this shows up through the appearance of a continuum of critical exponents. To analyze this part of the spectrum we derive a set of coupled nonlinear integral equations from the Bethe ansatz solution of the vertex model which allow to compute the energies of the system for a range of anisotropies and of the staggering parameter. The critical theory is found to be independent of the staggering. Its spectrum and density of states coincide with the  $SL(2, \mathbb{R})/U(1)$  Euclidean black hole conformal field theory which has been identified previously in the continuum limit of the vertex model for a particular ‘self-dual’ choice of the staggering. We also study the asymptotic behavior of subleading corrections to the finite size scaling and discuss our findings in the context of the conformal field theory.

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## 1. Introduction

Studies of two-dimensional vertex models with local Boltzmann weights satisfying the Yang–Baxter equation or the related quantum spin chains have provided tremendous insights into the properties of quantum field theories in  $(1 + 1)$  dimension. The solution of the lattice model by

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means of Bethe ansatz methods allows a complete characterization of the spectrum in terms of elementary excitations and the analysis of the finite size spectrum allows for the identification of the model as a lattice regularization of the low energy effective field theory.

Based on a particular solution of the Yang–Baxter equation vertex models on lattices of arbitrary sizes can be defined using the co-multiplication property of the Yang–Baxter algebra. Introducing inhomogeneities – either by shifts in the spectral parameter and/or by using different representations of the underlying algebra for the internal local degrees of freedom [4,31] – allows to generalize these models further while keeping their integrability or to uncover relations to different systems: the equivalence of the  $q$ -state Potts model on the square lattice and a staggered six-vertex model [6,40] allowed for the solution of the Potts model by means of the Bethe ansatz [5]. Other choices for the staggering parameter in the six-vertex model lead to integrable quantum chains with longer ranged interactions [18,38] and have been found to appear in the zero charge sector of a vertex model based on alternating four-dimensional representations of the quantum group deformation of the Lie superalgebra  $sl(2|1)$  [17].

Recently, the critical properties of the staggered six-vertex model have been investigated for a particular choice of the staggering parameter corresponding to one of the integrable manifolds of the antiferromagnetic  $q$ -state Potts model [23,27]. For this ‘self-dual’ case the model has an additional discrete  $\mathbb{Z}_2$ -invariance leading to a conserved charge which can be used to classify the spectrum. Remarkably, it has been found that although the lattice model is defined in terms of (compact) spin-1/2 degrees of freedom the low energy effective theory has a continuous spectrum of critical exponents. From a finite size scaling analysis and the computation of the density of states in the continuum the latter has been identified with the non-compact  $SL(2, \mathbb{R})_k/U(1)$  sigma model, a conformal field theory (CFT) on the two-dimensional Euclidean black hole background [9,24]. The appearance of a non-compact continuum limit of a lattice model with finite number of states per site has also been observed in staggered vertex models with supergroup symmetries, see e.g. Refs. [8,15–17].

The spectrum for finite lattices is necessarily discrete. In the staggered six-vertex model the continuous spectrum emerges in the thermodynamic limit by closing the gaps between critical exponents as  $1/(\log L)^2$ . As a consequence the investigation of this scenario requires to compute energies for very large system sizes. For the integrable models mentioned above this is only possible after formulation of the spectral problem in terms of nonlinear integral equations (NLIEs) in which the system size enters as a parameter only. For the self-dual staggered six-vertex model such NLIEs have been derived and solved numerically to identify the low energy effective theory [9].

In this paper we derive a different set of NLIEs for the finite size spectrum of the spin chain from the Bethe ansatz solution of the staggered six-vertex model with anisotropy  $0 < \gamma < \pi/2$ . These equations hold for arbitrary values of the staggering parameter  $\gamma < \alpha < \pi - \gamma$ , in particular away from the self-dual line  $\alpha = \pi/2$ . After recalling what is known about the finite size spectrum of this model we propose a parametrization of its emerging continuous part in terms of the eigenvalues of the quasi-momentum operator. We solve the NLIEs numerically for systems with up to  $10^6$  lattice sites to verify this proposal. Both the finite size spectrum and the density of states are found to depend only on the anisotropy  $\gamma$  but not on the staggering parameter  $\alpha$ . This suggests that the effective theory for the staggered model is the black hole sigma model CFT  $SL(2, \mathbb{R})_k/U(1)$  at level  $k = \pi/\gamma > 2$ , independent of  $\alpha$ . We also study the subleading terms appearing in the finite size scaling of the energies of the system and propose a possible explanation for our findings in the context of this CFT.

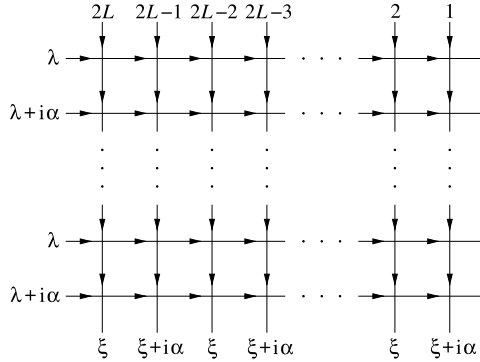


Fig. 1. The staggered six-vertex model on a square lattice. The local Boltzmann weights are elements of the  $R$ -matrix related to the six-vertex model.

**2. The staggered six-vertex model**

The six-vertex model on the square lattice, see Fig. 1, is defined through the  $R$ -matrix

$$R_{12}(\lambda, \mu) = \begin{pmatrix} a(\lambda, \mu) & 0 & 0 & 0 \\ 0 & b(\lambda, \mu) & c(\lambda, \mu) & 0 \\ 0 & c(\lambda, \mu) & b(\lambda, \mu) & 0 \\ 0 & 0 & 0 & a(\lambda, \mu) \end{pmatrix}, \tag{2.1}$$

containing the local Boltzmann weights  $a(\lambda, \mu) \equiv 1$ ,  $b(\lambda, \mu) = \frac{\text{sh}(\lambda - \mu)}{\text{sh}(\lambda - \mu + i\gamma)}$ ,  $c(\lambda, \mu) = \frac{\text{sh}(i\gamma)}{\text{sh}(\lambda - \mu + i\gamma)}$ . The matrix  $R_{12}(\lambda, \mu)$  should be read as an element of  $\text{End } V_1 \otimes \text{End } V_2$ ,  $V_i \simeq \mathbb{C}^2$ . The spin variables lying on the horizontal and vertical links of the lattice take values from  $V_1$  and  $V_2$ , respectively. Similarly, we associate the spectral parameters  $\lambda$  and  $\mu$  to horizontal and vertical lines while  $\gamma$  parametrizes the anisotropy of the model.

Taking the trace of an ordered product of  $R$ -matrices we construct a family of row-to-row transfer matrices on a lattice with  $2L$  horizontal sites with staggered spectral parameters

$$t(\lambda) = \text{tr}_0(R_{0,2L}(\lambda, \xi)R_{0,2L-1}(\lambda, \xi + i\alpha) \cdots R_{0,2}(\lambda, \xi)R_{0,1}(\lambda, \xi + i\alpha)). \tag{2.2}$$

The fact that the  $R$ -matrix (2.1) satisfies the Yang–Baxter equation implies that these transfer matrices commute for arbitrary values of  $\lambda$ , i.e.  $[t(\lambda), t(\mu)] = 0$ .

In the vertex model depicted in Fig. 1 we have introduced an additional staggering in the vertical direction. The corresponding double-row transfer matrix is the product of commuting operators

$$t^{(2)}(\lambda) = t(\lambda)t(\lambda + i\alpha). \tag{2.3}$$

On two lines in the space of coupling constants  $\gamma$  and  $\alpha$  this staggered six-vertex model exhibits additional quantum group symmetries: the first occurs for  $\alpha = \gamma$ . In this case the product of neighboring  $R$ -matrices in (2.2) degenerates and the underlying quantum group symmetry is that present in the integrable spin-1 XXZ chain [47]. A second special line follows from the spectral equivalence of the models with staggering parameter  $\alpha$  and  $\pi - \alpha$  [17]. This leads to the presence of a discrete  $\mathbb{Z}_2$ -invariance on the ‘self-dual’ line  $\alpha = \pi/2$  where the model is equivalent to one of the integrable manifolds of the antiferromagnetic  $q$ -state Potts model [5,23]. On this line the

model has a quantum group symmetry related to the twisted quantum algebra  $U_q[D_2^{(2)}]$  with  $q = e^{2i\gamma}$  [17].

The commuting operators generated by the double-row transfer matrix (2.3) can be written as sums over local (i.e. finite-range on the lattice) interactions, independent of the system size: as a consequence of  $R(\lambda, \lambda)$  being the permutation operator on  $V_1 \otimes V_2$ ,  $t^{(2)}(\xi)$  acts as a two-site translation operator. Therefore we can define the momentum operator as  $P = -i \log t^{(2)}(\xi) = -i \log(t(\xi)t(\xi + i\alpha))$ . The next term in the expansion of  $\log t^{(2)}(\xi)$  is the spin chain Hamiltonian

$$H \equiv -i\partial_\lambda \log(t(\lambda)t(\lambda + i\alpha))\Big|_{\lambda=\xi} \tag{2.4}$$

with nearest and next-nearest neighbor interaction. In terms of local Pauli matrices the Hamiltonian reads up to an overall factor

$$\begin{aligned} H \propto \sum_{j=1}^{2L} & \left[ -\frac{\sin^2 \alpha}{2} (\sigma_j \cdot \sigma_{j+2}) + \sin^2 \gamma \left\{ \frac{\cos \alpha}{\cos \gamma} (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y) + \sigma_j^z \sigma_{j+1}^z \right\} \right. \\ & + i(-1)^j \frac{\sin \gamma \sin \alpha}{2} (\sigma_j^x \sigma_{j+1}^y - \sigma_j^y \sigma_{j+1}^x) (\sigma_{j-1}^z - \sigma_{j+2}^z) \\ & \left. + i(-1)^j \frac{\sin \gamma \sin(2\alpha)}{4 \cos \gamma} (\sigma_{j-1}^x \sigma_{j+1}^y - \sigma_{j-1}^y \sigma_{j+1}^x) \sigma_j^z \right] \\ & + L(\cos(2\gamma) - \cos^2 \alpha). \end{aligned} \tag{2.5}$$

Note that this Hamiltonian differs from the one given by Ikhlef et al. [23] for  $\alpha = \pi/2$  by a unitary rotation of spins on one sublattice. Following Refs. [9,24] we also define a *quasi-shift* operator  $\tilde{\tau} \equiv t(\xi)[t(\xi + i\alpha)]^{-1}$  and the corresponding *quasi-momentum*

$$\tilde{P} \equiv \log(t(\xi)[t(\xi + i\alpha)]^{-1}). \tag{2.6}$$

Similar as in the self-dual case  $\tilde{\tau}$  acts as a diagonal-to-diagonal (light-cone) transfer matrix.

### 3. Bethe ansatz solution

In the six-vertex model the number of down arrows is conserved. This  $U(1)$  symmetry allows to diagonalize the transfer matrix starting from the reference state  $|0\rangle \equiv \binom{1}{0}^{\otimes 2L}$  by means of the algebraic Bethe ansatz [30]. The resulting eigenvalues in the sector with  $M \leq L$  down arrows are

$$\Lambda(\lambda) = \prod_{\ell=1}^M \frac{\text{sh}(\lambda - \lambda_\ell - i\gamma)}{\text{sh}(\lambda - \lambda_\ell)} + d(\lambda) \prod_{\ell=1}^M \frac{\text{sh}(\lambda - \lambda_\ell + i\gamma)}{\text{sh}(\lambda - \lambda_\ell)}, \tag{3.1}$$

with

$$d(\lambda) \equiv \left( \frac{\text{sh}(\lambda - \xi)}{\text{sh}(\lambda - \xi + i\gamma)} \frac{\text{sh}(\lambda - \xi - i\alpha)}{\text{sh}(\lambda - \xi - i\alpha + i\gamma)} \right)^L. \tag{3.2}$$

In Eq. (3.1) the rapidities  $\lambda_j$  are solutions to the Bethe equations

$$d(\lambda_j) \prod_{\substack{\ell=1 \\ \ell \neq j}}^M \frac{\text{sh}(\lambda_j - \lambda_\ell + i\gamma)}{\text{sh}(\lambda_j - \lambda_\ell - i\gamma)} = 1, \quad j = 1, \dots, M. \tag{3.3}$$

Note that these equations ensure the analyticity of the transfer matrix eigenvalues  $\Lambda(\lambda)$  at the points  $\lambda = \lambda_j, j = 1, \dots, M$ .

Since the expressions above depend only on the difference  $\lambda - \xi$  we are free to choose  $\xi \equiv i(\gamma - \alpha)/2$ . This results in the symmetric Bethe equations

$$\left( \frac{\text{sh}(\lambda_j + \frac{i\alpha}{2} + \frac{i\gamma}{2}) \text{sh}(\lambda_j - \frac{i\alpha}{2} + \frac{i\gamma}{2})}{\text{sh}(\lambda_j + \frac{i\alpha}{2} - \frac{i\gamma}{2}) \text{sh}(\lambda_j - \frac{i\alpha}{2} - \frac{i\gamma}{2})} \right)^L = \prod_{\substack{\ell=1 \\ \ell \neq j}}^M \frac{\text{sh}(\lambda_j - \lambda_\ell + i\gamma)}{\text{sh}(\lambda_j - \lambda_\ell - i\gamma)}, \quad j = 1, \dots, M. \quad (3.4)$$

The eigenvalues of the double-row transfer matrix (2.3) are the product of the individual ones from Eq. (3.1):

$$\Lambda^{(2)}(\lambda) = \Lambda(\lambda)\Lambda(\lambda + i\alpha). \quad (3.5)$$

Similarly, the energy eigenvalues  $E$  of the Hamiltonian (2.4) and the eigenvalue  $K$  of the quasi-momentum operator (2.6) are found to be:

$$E = -i\partial_\lambda \log(\Lambda(\lambda)\Lambda(\lambda + i\alpha))\Big|_{\lambda=\xi}, \quad K = \log(\Lambda(\xi)/\Lambda(\xi + i\alpha)). \quad (3.6)$$

Using the expression (3.1) for  $\Lambda(\lambda)$  in terms of the Bethe roots  $\{\lambda_j\}$  they are found to be sums over contributions from single rapidities, i.e.  $E = \sum_{j=1}^M \epsilon_0(\lambda_j)$  and  $K = \sum_{j=1}^M k_0(\lambda_j)$ , where

$$\begin{aligned} \epsilon_0(\lambda) &= \frac{4 \sin \gamma [\text{ch}(2\lambda) \cos \alpha - \cos \gamma]}{[\text{ch}(2\lambda) - \cos(\alpha - \gamma)][\text{ch}(2\lambda) - \cos(\alpha + \gamma)]}, \\ k_0(\lambda) &= \log \left( \frac{\text{ch}(2\lambda) - \cos(\alpha + \gamma)}{\text{ch}(2\lambda) - \cos(\alpha - \gamma)} \right). \end{aligned} \quad (3.7)$$

In the following we shall analyze the ground state and lowest excitations of the model with Hamiltonian (2.4) for staggering  $\gamma < \alpha < \pi - \gamma$  (denoted as phase B in Ref. [17]). In this regime the low energy states have been found to be described by configurations involving two types of Bethe roots for all anisotropies  $0 \leq \gamma \leq \pi/2$ , namely  $\{\lambda_j\}$  with  $\text{Im}(\lambda_j) = 0$  or  $\pi/2$ :

$$\{\lambda_\ell\}_{\ell=1}^M = \{v_j\}_{j=1}^{n_1} \cup \left\{ \frac{i\pi}{2} + \mu_j \right\}_{j=1}^{n_2} \quad \text{with } v_j, \mu_j \in \mathbb{R}. \quad (3.8)$$

Among these the ground state of the staggered spin chain is realized in the sector with  $M = n_1 + n_2 = L$  and roots  $v_j, \mu_j$  filling the entire real axis. Taking the thermodynamic limit  $L \rightarrow \infty$  with fixed  $n_1/L, n_2/L$  their distributions are described by densities  $\rho_1(v)$  and  $\rho_2(\mu)$  which are determined through coupled linear integral equations [14,44]. In the range of parameters considered here these equations have been solved to give [17]

$$\begin{aligned} \rho_1(v) &= \frac{\sin \frac{\pi(\alpha-\gamma)}{\pi-2\gamma}}{2(\pi-2\gamma)} \left( \cosh \frac{2\pi v}{\pi-2\gamma} - \cos \frac{\pi(\alpha-\gamma)}{\pi-2\gamma} \right)^{-1}, \\ \rho_2(\mu) &= \frac{\sin \frac{\pi(\alpha-\gamma)}{\pi-2\gamma}}{2(\pi-2\gamma)} \left( \cosh \frac{2\pi \mu}{\pi-2\gamma} + \cos \frac{\pi(\alpha-\gamma)}{\pi-2\gamma} \right)^{-1}. \end{aligned} \quad (3.9)$$

Integrating these expressions we find the total densities of the two types of roots in the ground state to be

$$\frac{n_1^{(0)}}{L} = \int_{-\infty}^{\infty} dv \rho_1(v) = \frac{\pi - \alpha - \gamma}{\pi - 2\gamma} = 1 - \frac{n_2^{(0)}}{L}. \tag{3.10}$$

Note that under the action of the duality transform  $\alpha \rightarrow \pi - \alpha$  the two types of roots are exchanged. This allows to restrict our analysis to staggering parameters  $\gamma < \alpha \leq \pi/2$  in the following.

The low energy excitations of the model have a linear dispersion and can be characterized by the deviations of the numbers  $n_{1,2}$  from their ground state values (3.10), i.e.

$$\Delta n_{1,2} = n_{1,2} - n_{1,2}^{(0)} \equiv \frac{1}{2}(m \pm \tilde{m}) \tag{3.11}$$

(note that  $m = n_1 + n_2 - L \in \mathbb{Z}$  by construction), and their momentum which can be parametrized by a single vorticity  $w \in \mathbb{Z}$ . Also within the root density approach the energy of these excitations has been found to be [17] (see also Refs. [23,27] for the self-dual case)

$$E(L) = L\varepsilon_\infty + \frac{2\pi v_F}{L} \left( -\frac{1}{6} + \frac{\gamma}{2\pi} m^2 + \frac{\pi}{2\gamma} w^2 + \kappa(L) \tilde{m}^2 + n_+ + n_- + R(L) \right). \tag{3.12}$$

Here  $\varepsilon_\infty$  is the bulk energy density of the state with root densities (3.9)

$$\varepsilon_\infty = -2 \int_{-\infty}^{\infty} dk \frac{\text{sh}(\frac{k\gamma}{2}) [\text{sh}(\frac{k\pi}{2} - \frac{k\gamma}{2}) \text{ch}(\frac{k\pi}{2} - k\alpha) - \text{sh}(\frac{k\gamma}{2})]}{\text{sh}(\frac{k\pi}{2}) \text{sh}(\frac{k\pi}{2} - k\gamma)}, \tag{3.13}$$

$v_F = 2\pi/(\pi - 2\gamma)$  is the Fermi velocity of the low energy modes,  $n_\pm$  are non-negative integers characterizing particle–hole type excitations and the corrections to scaling  $R(L)$  vanish as  $L \rightarrow \infty$ .

Similarly, the eigenvalue  $K$  of the quasi-momentum operator can be computed within the root density approach. For the ground state  $K$  is proportional to the system size, the corresponding ‘quasi-momentum density’ of the ground state in the thermodynamic limit is

$$\begin{aligned} k_\infty &= \frac{1}{L} K = \int_{-\infty}^{\infty} dv \rho_1(v) k_0(v) + \int_{-\infty}^{\infty} d\mu \rho_2(\mu) k_0(\mu + i\pi/2) \\ &= 2 \int_{-\infty}^{\infty} dk \frac{\text{sh}(\frac{k\gamma}{2}) \text{sh}(\frac{k\pi}{2} - k\alpha) \text{sh}(\frac{k\pi}{2} - \frac{k\gamma}{2})}{k \text{sh}(\frac{k\pi}{2}) \text{sh}(\frac{k\pi}{2} - k\gamma)}. \end{aligned} \tag{3.14}$$

Note that  $k_\infty = 0$  on the self-dual line  $\alpha = \pi/2$  as a consequence of  $k_0(\lambda) = -k_0(\lambda + i\pi/2)$  and  $\rho_1(\lambda) \equiv \rho_2(\lambda)$ .

To analyze the finite size spectrum (3.12) further and to identify the low energy effective theory for the staggered six-vertex model one has to study the system size dependence of the coupling constant  $\kappa(L)$ . Based on numerical solutions of the Bethe equations (3.4) for systems with several thousand lattice sites it has been established that  $\kappa(L) \propto 1/(\log L)^2$  for large  $L$  [23, 27] indicating the emergence of a continuous component in the spectrum of critical exponents of the model in the thermodynamic limit. Such a finite size behavior can be traced back to a singularity in the linear Bethe ansatz integral equations, first observed in a staggered  $sl(2|1)$  super

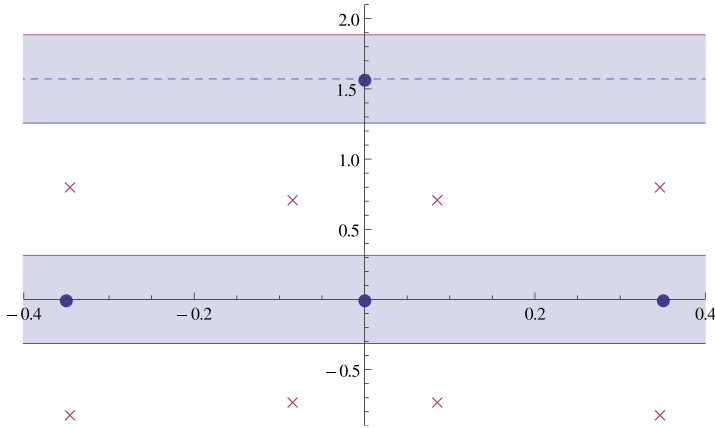


Fig. 2. Solution to the Bethe equations (3.4) for  $L = 4$ , anisotropy  $\gamma = \frac{\pi}{5}$  and staggering  $\alpha = \frac{2\pi}{5}$ : displayed is a configuration (3.8) containing  $n_1 = 3$  and  $n_2 = 1$  Bethe roots (●) with  $\text{Im}(\lambda_j) = 0$  and  $\pi/2$ , respectively. Also shown are the  $2L = 8$  additional ‘hole-type’ solutions (×) for this configuration, see discussion below Eq. (4.2). Note that the latter are outside the shaded strips  $|\text{Im} z| \leq \frac{\gamma}{2}$  and  $|\text{Im} z - \frac{\alpha}{2}| \leq \frac{\gamma}{2}$ .

spin chain [15]. Due to this logarithmic dependence on the system size a quantitative analysis of the spectrum requires more sophisticated methods. In a first step Ikhlef et al. have refined the root density approach using Wiener–Hopf methods which allowed them to derive an analytical expression for the asymptotic behavior of  $\kappa(L)$  in the self-dual case [24]

$$\kappa(L) = \frac{\pi^2 \gamma}{8(\pi - 2\gamma)} \frac{1}{\log L^2}, \tag{3.15}$$

valid for anisotropies  $0 \leq \gamma < \pi/2$ .<sup>1</sup> Numerical data for general staggering suggested that the finite-size spectrum is in fact independent of  $\alpha \in (\gamma, \pi/2]$  and consistent with this expression [17].

To go further, Candu and Ikhlef [9] have reformulated the spectral problem for the self-dual model in terms of nonlinear integral equations. In the next section we shall derive such integral equations which are valid for arbitrary staggering  $\gamma < \alpha \leq \pi/2$  and which are then used to analyze the continuous part of the spectrum and the corrections to scaling  $R(L)$  in (3.12).

#### 4. Nonlinear integral equations

To be specific we consider solutions to the Bethe equations corresponding to low energy excitations (3.12) with  $(m, w) = (0, 0)$ . These configurations contain a total of  $M = n_1 + n_2 \equiv L$  roots (3.8) distributed symmetrically around the imaginary axis

$$\{v_j\}_{j=1}^{n_1} = \{-v_j\}_{j=1}^{n_1}, \quad \{\mu_j\}_{j=1}^{n_2} = \{-\mu_j\}_{j=1}^{n_2}, \tag{4.1}$$

see Fig. 2 for an example. Following Refs. [28,29] we introduce auxiliary functions

<sup>1</sup> For  $\gamma = \pi/2$  the model has an  $OSP(2|2)$  symmetry [20]. The coupling constant can be obtained from RG arguments which determine its system size dependence at intermediate scales, giving  $\kappa(L) \simeq 1/\log L$  [23].

$$a_1(\lambda) = a(\lambda) \equiv d(\lambda) \prod_{\ell=1}^L \frac{\text{sh}(\lambda - \lambda_\ell + i\gamma)}{\text{sh}(\lambda - \lambda_\ell - i\gamma)} \tag{4.2}$$

and  $a_2(\lambda) \equiv a_1(\lambda + i\pi/2)$ , thereby encoding the Bethe roots (3.8) in the zeroes of  $(1 + a_1)(\lambda)$  and  $(1 + a_2)(\lambda)$ , respectively. The additional zeroes of these expressions are called hole-type solutions, cf. Fig. 2. This allows to rewrite the Bethe equations (3.4) in the sector of  $n_1 + n_2 = M = L$  roots and parameter ranges  $0 < \gamma < \alpha \leq \pi/2$  in terms of coupled nonlinear integral equations (NLIEs)

$$\begin{aligned} \log a_1(\lambda) = & 2i\gamma L + L \log \left( \frac{\text{sh}(\lambda + \frac{i\alpha}{2} - \frac{i\gamma}{2}) \text{sh}(\lambda - \frac{i\alpha}{2} - \frac{i\gamma}{2})}{\text{sh}(\lambda + \frac{i\alpha}{2} + \frac{i\gamma}{2}) \text{sh}(\lambda - \frac{i\alpha}{2} + \frac{i\gamma}{2})} \right) \\ & - \int_{\mathcal{C}} \frac{d\omega}{2\pi} K_{i\gamma}(\lambda - \omega) \log(1 + a_1)(\omega) \\ & + \int_{\mathcal{C}} \frac{d\omega}{2\pi} K_{i\gamma'}(\lambda - \omega) \log(1 + a_2)(\omega), \end{aligned} \tag{4.3}$$

$$\begin{aligned} \log a_2(\lambda) = & 2i\gamma L + L \log \left( \frac{\text{ch}(\lambda + \frac{i\alpha}{2} - \frac{i\gamma}{2}) \text{ch}(\lambda - \frac{i\alpha}{2} - \frac{i\gamma}{2})}{\text{ch}(\lambda + \frac{i\alpha}{2} + \frac{i\gamma}{2}) \text{ch}(\lambda - \frac{i\alpha}{2} + \frac{i\gamma}{2})} \right) \\ & - \int_{\mathcal{C}} \frac{d\omega}{2\pi} K_{i\gamma}(\lambda - \omega) \log(1 + a_2)(\omega) \\ & + \int_{\mathcal{C}} \frac{d\omega}{2\pi} K_{i\gamma'}(\lambda - \omega) \log(1 + a_1)(\omega). \end{aligned} \tag{4.4}$$

Here we have chosen the branch cut of the logarithm along the negative real axis. Note that unlike in the NLIEs derived earlier for the self-dual model [9] the convolution integrals in (4.3) and (4.4) are computed along a fixed non-intersecting closed contour  $\mathcal{C}$  (cf. Fig. 3). As a consequence, the system size  $L$  enters in the NLIEs only as a parameter in the driving terms. A solution of the NLIEs with the analytical properties of the auxiliary functions (4.2) corresponds to the physical state (4.1). The kernels

$$K_{i\gamma}(x) = \frac{1}{i} \frac{\text{sh}(2i\gamma)}{\text{sh}(x - i\gamma) \text{sh}(x + i\gamma)}, \quad \gamma' \equiv \frac{\pi}{2} - \gamma, \tag{4.5}$$

have no poles which need to be considered in the contour integration. Furthermore, their Fourier transformations on the contour are regular which is particularly useful for the numerical evaluation of the convolution integrals.

The Bethe numbers and thus the auxiliary functions  $a_1$  and  $a_2$  fix the eigenvalue of the transfer matrix. For  $\lambda$  inside the closed contour  $\mathcal{C}$  this yields

$$\begin{aligned} \log \Lambda(\lambda) = & -i\gamma L + \int_{\mathcal{C}} \frac{d\omega}{2\pi i} \frac{\text{sh}(i\gamma) \log(1 + a_1)(\omega)}{\text{sh}(\lambda - \omega) \text{sh}(\lambda - \omega - i\gamma)} \\ & - \int_{\mathcal{C}} \frac{d\omega}{2\pi i} \frac{\text{sh}(i\gamma) \log(1 + a_2)(\omega)}{\text{ch}(\lambda - \omega) \text{ch}(\lambda - \omega - i\gamma)}. \end{aligned} \tag{4.6}$$



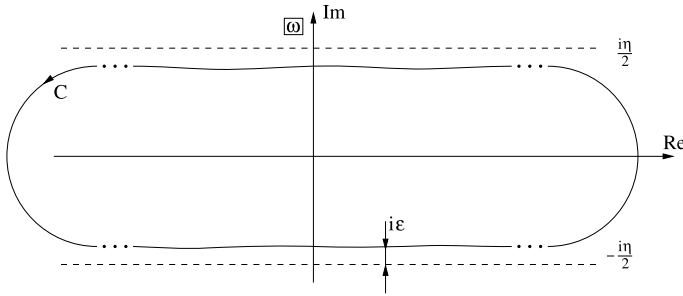


Fig. 3. The closed contour  $C$  in the complex plane: the parameter  $\eta \equiv \frac{1}{2} \min\{\gamma, \alpha - \gamma\}$  determines the height of both shaded strips in Fig. 2 separating the Bethe roots from the hole-type solutions and  $\alpha$ -dependent singularities of the auxiliary functions. Evaluating  $a_1$  for  $\lambda \in C$  by enclosing the real Bethe numbers  $\{v_j\}_{j=1}^{n_1}$  (similarly  $\{\mu_j\}_{j=1}^{n_2}$  for  $a_2$ ) one has to introduce a distance parameter  $\varepsilon > 0$  to avoid singularities at the border  $\text{Im } \lambda = \pm \eta/2$ , e.g. from the kernels. Extending  $C$  to  $\pm\infty$  covers all system sizes  $L$ .

The proof is straightforward and only involves Cauchy’s theorem. Considering the logarithmic derivative of  $a(\lambda)$ ,

$$\begin{aligned} \partial \log \frac{a(\lambda)}{d(\lambda)} &= - \sum_{\ell=1}^{n_1} \frac{\text{sh}(2i\gamma)}{\text{sh}(\lambda - v_\ell + i\gamma) \text{sh}(\lambda - v_\ell - i\gamma)} \\ &\quad + \sum_{\ell=1}^{n_2} \frac{\text{sh}(2i\gamma)}{\text{ch}(\lambda - \mu_\ell + i\gamma) \text{ch}(\lambda - \mu_\ell - i\gamma)} \end{aligned}$$

the summation part can be cast into an integral representation involving the auxiliary functions: for any analytic  $f(\lambda)$  the relations

$$\sum_{\ell=1}^{n_1} f(v_\ell) = \int_C \frac{d\omega}{2\pi i} \frac{f(\omega) \partial a_1(\omega)}{1 + a_1(\omega)}, \quad \sum_{\ell=1}^{n_2} f(\mu_\ell) = \int_C \frac{d\omega}{2\pi i} \frac{f(\omega) \partial a_2(\omega)}{1 + a_2(\omega)} \tag{4.7}$$

hold if the contour encloses all Bethe roots but none of the hole-type solutions and  $\alpha$ -dependent singularities of  $1 + a_{1,2}(\lambda) = 0$ . For the example shown in Fig. 2 this is guaranteed for the contour  $C$  according to Fig. 3. As the expression  $d(\lambda)$  is analytic within  $C$  Eq. (4.3) immediately follows, where the integration constant can be fixed by considering (4.2) for e.g.  $\lambda \rightarrow \infty$ . Eq. (4.4) is just a shift in the argument by  $i\pi/2$  formally exchanging  $a_1 \leftrightarrow a_2$ . The eigenvalue  $\Lambda(\lambda)$  can be treated in a similar way: by factorizing the eigenvalue (3.1) in favor of the auxiliary function  $1 + a_1(\lambda)$  and taking the logarithmic derivative,

$$\begin{aligned} \partial \log \Lambda(\lambda) &= \sum_{\ell=1}^{n_1} \frac{\text{sh}(i\gamma)}{\text{sh}(\lambda - v_\ell) \text{sh}(\lambda - v_\ell - i\gamma)} - \sum_{\ell=1}^{n_2} \frac{\text{sh}(i\gamma)}{\text{ch}(\lambda - \mu_\ell) \text{ch}(\lambda - \mu_\ell - i\gamma)} \\ &\quad + \partial \log[1 + a_1(\lambda)], \end{aligned}$$

the summation part can be cast into an integral representation according to (4.7) as long as  $\lambda$  and  $\lambda - i\gamma$  remain outside the closed contour. For  $\lambda$  inside the closed contour  $C$  the additional term  $\partial \log[1 + a_1(\lambda)]$  can be absorbed into the contour integral involving  $a_1$  by Cauchy’s theorem. Lifting the derivatives immediately yield (4.6).

### 4.1. Mixed eigenvalues

The energy  $E$  and quasi-momentum  $K$  of the eigenstates are related to the logarithmically combined expressions (3.6) evaluated at the points  $\xi$  and  $\xi + i\alpha$  being zeroes of  $a(\lambda)$ . The bulk parts can be split off by considering the mixed eigenvalues

$$\Lambda_{\text{mix}}(\lambda) \equiv \Lambda(\lambda)\Lambda\left(\lambda + \frac{i\pi}{2}\right), \quad \Omega_{\text{mix}}(\lambda) \equiv \frac{\Lambda(\lambda)}{\Lambda(\lambda + \frac{i\pi}{2})}. \tag{4.8}$$

Due to  $i\pi$ -periodicity of (3.1) the mixed eigenvalues are periodic,  $\log \Lambda_{\text{mix}}(\lambda + i\pi/2) = \log \Lambda_{\text{mix}}(\lambda)$ , and antiperiodic,  $\log \Omega_{\text{mix}}(\lambda + i\pi/2) = -\log \Omega_{\text{mix}}(\lambda)$  with respect to  $i\pi/2$  satisfying the functional equations

$$\begin{aligned} &\Lambda_{\text{mix}}\left(x - \frac{i\gamma}{2}\right)\Lambda_{\text{mix}}\left(x + \frac{i\gamma}{2}\right) \\ &= d\left(x - \frac{i\gamma}{2}\right)d\left(x - \frac{i\gamma}{2} + \frac{i\pi}{2}\right)\left[1 + a^{-1}\left(x - \frac{i\gamma}{2}\right)\right]\left[1 + a^{-1}\left(x + \frac{i\pi}{2} - \frac{i\gamma}{2}\right)\right] \\ &\quad \times \left[1 + a\left(x - \frac{i\pi}{2} + \frac{i\gamma}{2}\right)\right]\left[1 + a\left(x + \frac{i\gamma}{2}\right)\right] \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} &\Omega_{\text{mix}}\left(x - \frac{i\gamma}{2}\right)\Omega_{\text{mix}}\left(x + \frac{i\gamma}{2}\right) \\ &= \frac{d\left(x - \frac{i\gamma}{2}\right)}{d\left(x - \frac{i\gamma}{2} + \frac{i\pi}{2}\right)} \frac{[1 + a^{-1}\left(x - \frac{i\gamma}{2}\right)][1 + a\left(x + \frac{i\gamma}{2}\right)]}{[1 + a^{-1}\left(x + \frac{i\pi}{2} - \frac{i\gamma}{2}\right)][1 + a\left(x - \frac{i\pi}{2} + \frac{i\gamma}{2}\right)]}, \end{aligned} \tag{4.10}$$

respectively. To finally evaluate (3.6) for general staggering Eqs. (4.9) and (4.10) can be solved in Fourier space for  $\Lambda_{\text{mix}}$  and  $\Omega_{\text{mix}}$  using the transformation pair

$$\hat{f}(k) = \int_{-\infty}^{\infty} dx e^{-ikx} f(x), \quad f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \hat{f}(k). \tag{4.11}$$

As  $\Lambda_{\text{mix}}(\lambda)$  and  $\Omega_{\text{mix}}(\lambda)$  are analytic<sup>2</sup> in  $\{\lambda \in \mathbb{C} \mid \frac{\pi+2\gamma}{4} < \text{Im } \lambda < \frac{\pi-\gamma}{2}\}$  and the region enclosed by the contour  $\mathcal{C}$  a standard manipulation in Fourier space yields with the  $i\pi/2$ -(anti)periodicity

$$\begin{aligned} \partial \log \Lambda_{\text{mix}}\left(x - \frac{i\pi}{4} + \frac{i\gamma}{2}\right) &= -iL \int_{-\infty}^{\infty} \frac{dk e^{ikx} \text{sh}\left(\frac{k\gamma}{2}\right) \text{ch}\left(\frac{k\pi}{4} - \frac{k\alpha}{2}\right)}{\text{sh}\left(\frac{k\pi}{4}\right) \text{ch}\left(\frac{k\pi}{4} - \frac{k\gamma}{2}\right)} \\ &\quad - \sum_{j=1}^2 \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{ikx} e^{-\frac{k\gamma}{2} + k\varepsilon}}{2 \text{ch}\left(\frac{k\pi}{4} - \frac{k\gamma}{2}\right)} \int_{-\infty}^{\infty} dy e^{-iky} \partial \log(1 + a_j)\left(y - \frac{i\gamma}{2} + i\varepsilon\right) \\ &\quad + \sum_{j=1}^2 \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{ikx} e^{\frac{k\gamma}{2} - k\varepsilon}}{2 \text{ch}\left(\frac{k\pi}{4} - \frac{k\gamma}{2}\right)} \int_{-\infty}^{\infty} dy e^{-iky} \partial \log(1 + a_j^{-1})\left(y + \frac{i\gamma}{2} - i\varepsilon\right), \end{aligned} \tag{4.12}$$

<sup>2</sup> Valid for the range  $0 < \gamma \leq \frac{\alpha}{2}$ ; similar results can be obtained for  $\frac{\alpha}{2} < \gamma < \alpha \leq \frac{\pi}{2}$ .

$$\begin{aligned}
 \partial \log \Omega_{\text{mix}} \left( x - \frac{i\pi}{4} + \frac{i\gamma}{2} \right) &= iL \int_{-\infty}^{\infty} \frac{dk e^{ikx} \operatorname{sh}(\frac{k\gamma}{2}) \operatorname{sh}(\frac{k\pi}{4} - \frac{k\alpha}{2})}{\operatorname{ch}(\frac{k\pi}{4}) \operatorname{sh}(\frac{k\pi}{4} - \frac{k\gamma}{2})} \\
 &+ \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{ikx} e^{-\frac{k\gamma}{2} + k\varepsilon}}{2 \operatorname{sh}(\frac{k\pi}{4} - \frac{k\gamma}{2})} \int_{-\infty}^{\infty} dy e^{-iky} \partial \log \frac{1 + a_1(y - \frac{i\gamma}{2} + i\varepsilon)}{1 + a_2(y - \frac{i\gamma}{2} + i\varepsilon)} \\
 &- \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{ikx} e^{\frac{k\gamma}{2} - k\varepsilon}}{2 \operatorname{sh}(\frac{k\pi}{4} - \frac{k\gamma}{2})} \int_{-\infty}^{\infty} dy e^{-iky} \partial \log \frac{1 + a_1^{-1}(y + \frac{i\gamma}{2} - i\varepsilon)}{1 + a_2^{-1}(y + \frac{i\gamma}{2} - i\varepsilon)},
 \end{aligned} \tag{4.13}$$

where we used the Fourier transform of  $\log d(\lambda)$ ,

$$\widehat{\partial \log d}(k) = -\frac{4\pi iL \operatorname{sh}(\frac{k\gamma}{2})}{\operatorname{sh}(\frac{k\pi}{2})} \operatorname{ch}\left(\frac{k\pi}{2} - \frac{k\alpha}{2}\right). \tag{4.14}$$

Note that this system (4.12) and (4.13) already describes the energy  $E$  and quasi-momentum  $K$  according to (3.6) in the self-dual case  $\alpha = \pi/2$ . However, as  $\Lambda_{\text{mix}}(\lambda)$  and  $\Omega_{\text{mix}}(\lambda)$  are composed from simple eigenvalues  $\Lambda(\lambda)$ , cf. (4.8), one can solve the system for  $\Lambda(x - \frac{i\pi}{4} + \frac{i\gamma}{2})$  and  $\Lambda(x + \frac{i\pi}{4} + \frac{i\gamma}{2})$ . Recombining after suitably shifting the arguments (3.6) reads in Fourier representation

$$\begin{aligned}
 &\partial \log \Lambda(x + \xi) + \partial \log \Lambda(x + \xi + i\alpha) \\
 &= -iL \int_{-\infty}^{\infty} \frac{dk e^{ikx} \operatorname{sh}(\frac{k\gamma}{2}) \operatorname{ch}^2(\frac{k\pi}{4} - \frac{k\alpha}{2})}{\operatorname{sh}(\frac{k\pi}{4}) \operatorname{ch}(\frac{k\pi}{4} - \frac{k\gamma}{2})} - iL \int_{-\infty}^{\infty} \frac{dk e^{ikx} \operatorname{sh}(\frac{k\gamma}{2}) \operatorname{sh}^2(\frac{k\pi}{4} - \frac{k\alpha}{2})}{\operatorname{ch}(\frac{k\pi}{4}) \operatorname{sh}(\frac{k\pi}{4} - \frac{k\gamma}{2})} \\
 &- \sum_{j=1}^2 \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{\operatorname{ch}(\frac{k\pi}{4} - \frac{k\alpha}{2})}{2 \operatorname{ch}(\frac{k\pi}{4} - \frac{k\gamma}{2})} e^{ikx} e^{-\frac{k\gamma}{2} + k\varepsilon} \int_{-\infty}^{\infty} dy e^{-iky} \partial \log(1 + a_j) \left( y - \frac{i\gamma}{2} + i\varepsilon \right) \\
 &+ \sum_{j=1}^2 \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{\operatorname{ch}(\frac{k\pi}{4} - \frac{k\alpha}{2})}{2 \operatorname{ch}(\frac{k\pi}{4} - \frac{k\gamma}{2})} e^{ikx} e^{\frac{k\gamma}{2} - k\varepsilon} \int_{-\infty}^{\infty} dy e^{-iky} \partial \log(1 + a_j^{-1}) \left( y + \frac{i\gamma}{2} - i\varepsilon \right) \\
 &- \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{\operatorname{sh}(\frac{k\pi}{4} - \frac{k\alpha}{2})}{2 \operatorname{sh}(\frac{k\pi}{4} - \frac{k\gamma}{2})} e^{ikx} e^{-\frac{k\gamma}{2} + k\varepsilon} \int_{-\infty}^{\infty} dy e^{-iky} \partial \log \left( \frac{1 + a_1}{1 + a_2} \right) \left( y - \frac{i\gamma}{2} + i\varepsilon \right) \\
 &+ \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{\operatorname{sh}(\frac{k\pi}{4} - \frac{k\alpha}{2})}{2 \operatorname{sh}(\frac{k\pi}{4} - \frac{k\gamma}{2})} e^{ikx} e^{+\frac{k\gamma}{2} - k\varepsilon} \int_{-\infty}^{\infty} dy e^{-iky} \partial \log \left( \frac{1 + a_1^{-1}}{1 + a_2^{-1}} \right) \left( y + \frac{i\gamma}{2} - i\varepsilon \right),
 \end{aligned} \tag{4.15}$$

$$\begin{aligned}
 &\partial \log \Lambda(x + \xi) - \partial \log \Lambda(x + \xi + i\alpha) \\
 &= iL \int_{-\infty}^{\infty} \frac{dk e^{ikx} \operatorname{sh}(\frac{k\gamma}{2}) \operatorname{sh}(\frac{k\pi}{2} - k\alpha)}{2 \operatorname{ch}(\frac{k\pi}{4}) \operatorname{sh}(\frac{k\pi}{4} - \frac{k\gamma}{2})} + iL \int_{-\infty}^{\infty} \frac{dk e^{ikx} \operatorname{sh}(\frac{k\gamma}{2}) \operatorname{sh}(\frac{k\pi}{2} - k\alpha)}{2 \operatorname{sh}(\frac{k\pi}{4}) \operatorname{ch}(\frac{k\pi}{4} - \frac{k\gamma}{2})} \\
 &+ \sum_{j=1}^2 \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{\operatorname{sh}(\frac{k\pi}{4} - \frac{k\alpha}{2})}{2 \operatorname{ch}(\frac{k\pi}{4} - \frac{k\gamma}{2})} e^{ikx} e^{-\frac{k\gamma}{2} + k\varepsilon} \int_{-\infty}^{\infty} dy e^{-iky} \partial \log(1 + a_j) \left( y - \frac{i\gamma}{2} + i\varepsilon \right)
 \end{aligned}$$

$$\begin{aligned}
 & - \sum_{j=1}^2 \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{\text{sh}\left(\frac{k\pi}{4} - \frac{k\alpha}{2}\right)}{2 \text{ch}\left(\frac{k\pi}{4} - \frac{k\gamma}{2}\right)} e^{ikx} e^{\frac{k\gamma}{2} - k\varepsilon} \int_{-\infty}^{\infty} dy e^{-iky} \partial \log(1 + a_j^{-1}) \left(y + \frac{i\gamma}{2} - i\varepsilon\right) \\
 & + \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{\text{ch}\left(\frac{k\pi}{4} - \frac{k\alpha}{2}\right)}{2 \text{sh}\left(\frac{k\pi}{4} - \frac{k\gamma}{2}\right)} e^{ikx} e^{-\frac{k\gamma}{2} + k\varepsilon} \int_{-\infty}^{\infty} dy e^{-iky} \partial \log\left(\frac{1 + a_1}{1 + a_2}\right) \left(y - \frac{i\gamma}{2} + i\varepsilon\right) \\
 & - \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{\text{ch}\left(\frac{k\pi}{4} - \frac{k\alpha}{2}\right)}{2 \text{sh}\left(\frac{k\pi}{4} - \frac{k\gamma}{2}\right)} e^{ikx} e^{+\frac{k\gamma}{2} - k\varepsilon} \int_{-\infty}^{\infty} dy e^{-iky} \partial \log\left(\frac{1 + a_1^{-1}}{1 + a_2^{-1}}\right) \left(y + \frac{i\gamma}{2} - i\varepsilon\right)
 \end{aligned} \tag{4.16}$$

for staggering  $\gamma \leq \alpha/2$  (and a similar expression for  $\alpha/2 < \gamma < \alpha \leq \pi/2$ ). From the bulk parts of these expressions we can read off the energy density  $\varepsilon_\infty$  (3.13) and quasi-momentum densities  $k_\infty$  (3.14) already obtained within the root density approach above.

Using the relation  $a^{-1}(-\lambda) = a(\lambda)$  provided by the Bethe root's symmetry (4.1) energy and momentum reduce for all  $0 < \gamma < \alpha \leq \pi/2$  to

$$\begin{aligned}
 E - L\varepsilon_\infty &= i \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{\text{ch}\left(\frac{k\pi}{4} - \frac{k\alpha}{2}\right)}{\text{ch}\left(\frac{k\pi}{4} - \frac{k\gamma}{2}\right)} e^{-\frac{k\eta}{2} + k\varepsilon} \sum_{j=1}^2 \int_{-\infty}^{\infty} dy e^{-iky} \partial \log(1 + a_j) \left(y - \frac{i\eta}{2} + i\varepsilon\right) \\
 & + i \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{\text{sh}\left(\frac{k\pi}{4} - \frac{k\alpha}{2}\right)}{\text{sh}\left(\frac{k\pi}{4} - \frac{k\gamma}{2}\right)} e^{-\frac{k\eta}{2} + k\varepsilon} \int_{-\infty}^{\infty} dy e^{-iky} \partial \log\left(\frac{1 + a_1}{1 + a_2}\right) \left(y - \frac{i\eta}{2} + i\varepsilon\right),
 \end{aligned} \tag{4.17}$$

$$\begin{aligned}
 K - Lk_\infty &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{\text{sh}\left(\frac{k\pi}{4} - \frac{k\alpha}{2}\right)}{\text{ch}\left(\frac{k\pi}{4} - \frac{k\gamma}{2}\right)} e^{-\frac{k\eta}{2} + k\varepsilon} \sum_{j=1}^2 \int_{-\infty}^{\infty} dy e^{-iky} \log(1 + a_j) \left(y - \frac{i\eta}{2} + i\varepsilon\right) \\
 & + \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{\text{ch}\left(\frac{k\pi}{4} - \frac{k\alpha}{2}\right)}{\text{sh}\left(\frac{k\pi}{4} - \frac{k\gamma}{2}\right)} e^{-\frac{k\eta}{2} + k\varepsilon} \int_{-\infty}^{\infty} dy e^{-iky} \log\left(\frac{1 + a_1}{1 + a_2}\right) \left(y - \frac{i\eta}{2} + i\varepsilon\right).
 \end{aligned} \tag{4.18}$$

Note that the singularity from the kernel in the last line of (4.18) is compensated by the zero of the auxiliary functions

$$\int_{-\infty}^{\infty} dy \log\left(\frac{1 + a_1}{1 + a_2}\right) \left(y - \frac{i\eta}{2} + i\varepsilon\right) = 0. \tag{4.19}$$

### 5. Analysis of the finite size spectrum

Based on the formulation of the eigenvalues of the Hamiltonian and the quasi-momentum operator in terms of the solutions to the NLIEs (4.3) and (4.4) we can now proceed with our analysis of the low energy spectrum (3.12) of the staggered six-vertex model. For this it will be particularly important to vary the parameter  $\tilde{m}$  (3.11) in a controlled way as the system size is increased. As discussed in Section 3 the density of the two types of Bethe roots (3.8) depends on

the staggering parameter  $\alpha$  in the model, see Eq. (3.10). This implies that the ground state can be realized only for certain commensurate system sizes  $L = L_c$ : for staggering

$$\alpha = \alpha(p, q) \equiv \frac{p\gamma + q(\pi - \gamma)}{p + q}, \tag{5.1}$$

with  $p, q \in \mathbb{N}$  prime to each other ( $p = q = 1$  on the self-dual line) we have to choose  $L_c$  being an integer multiple of  $(p + q)$ . For this choice of parameters the ground state corresponding to  $(m, w) = (0, 0)$  and  $\tilde{m} = 0$  is described by integer numbers  $n_1^{(0)} = Lp/(p + q)$  and  $n_2^{(0)} = Lq/(p + q)$  of Bethe roots on the real line and with  $\text{Im}(\lambda_j) = i\pi/2$ , respectively. Excitations can be constructed by shifting  $t$  roots between these two sets, i.e. with  $n_{1,2} = n_{1,2}^{(0)} \pm t$ , resulting in  $\tilde{m} = 2t$ . Similarly, we can consider the spectrum in sectors where the commensurability condition is not satisfied: let  $L = \ell_0(p + q) + r$  with  $\ell_0 \in \mathbb{N}, r = 1, 2, \dots, (p + q - 1)$ . The Bethe states are described by  $n_1 = \ell_0 p + t + r$  and  $n_2 = \ell_0 q - t$  roots of the two types. From (3.11) we obtain  $\tilde{m} = 2t + 2rq/(p + q)$  for these states. Together this allows to vary  $\tilde{m}$  in steps of  $2q/(p + q)$  for the staggering parameter (5.1).

The observed presence of a continuous component of the spectrum implies that there should be a corresponding continuous quantum number in the thermodynamic limit  $L \rightarrow \infty$ . For the self-dual model, i.e.  $\alpha = \pi/2$ , it has been shown that this quantum number is in fact related to the conserved quasi-momentum  $K$  of the corresponding excitation [9,24]: for large  $L$  the number  $\tilde{m}$  characterizing the Bethe configuration (3.11) is related to the rescaled quasi-momentum  $s \equiv \frac{\pi - 2\gamma}{4\pi\gamma} K$  as

$$\tilde{m} \simeq \frac{4s}{\pi} \left( \log \frac{L}{L_0} + B(s) \right). \tag{5.2}$$

Here  $L_0$  is a non-universal length scale which only depends on the anisotropy  $\gamma$  while the function  $B(s)$  determines the (finite part) of the density of states in the continuum part of the spectrum.

This line of arguments can be implemented in a straightforward way for staggering away from the self-dual line: here the quasi-momentum has a non-zero value (3.14) in the ground state ( $(m, w) = (0, 0)$  and  $\tilde{m} = 0$ ). Since the expression (3.15) has been found to be consistent with numerical results [17] for the finite size spectrum for  $\alpha \in (\gamma, \pi/2]$  we propose that the quasi-momentum should be replaced by its deviation from the ground state value, i.e.  $(K - Lk_\infty)$  in the relations above. In particular, the quantum number for the continuous component of the finite size spectrum appearing in (5.2) should read

$$s \equiv \frac{\pi - 2\gamma}{4\pi\gamma} (K - Lk_\infty). \tag{5.3}$$

To support this conjecture we have computed the quasi-momentum (4.18) numerically: initializing the auxiliary functions (4.2) with a degenerate configuration (3.8)  $v_j = \mu_j = 0$  the NLIEs (4.3), (4.4) are solved by iteration. The convolution integrals are dealt with in Fourier space using a fast Fourier transform (FFT) with  $2^{18}$  sample points. The procedure converges to a solution within  $\approx 50$  (500) iterations for systems sizes  $L$  of order  $10^2$  ( $10^6$ ). In Fig. 4 we present data for the ratio  $\tilde{m}/s$  obtained using this procedure as a function of the system size for various values of the anisotropy  $\gamma$  and the staggering  $\alpha$ . Apart from the corrections for small  $L$  the numerical values show the predicted linear dependence (5.2) on  $\log L$  with an offset independent of the staggering.

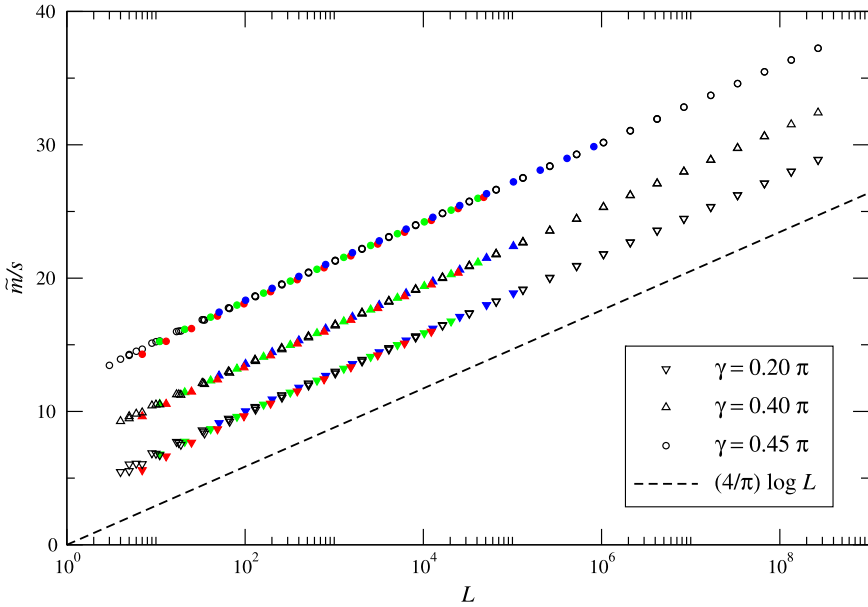


Fig. 4. Ratio of the number  $\tilde{m}$  characterizing the configuration of Bethe roots according to (3.11) and the quantum number  $s$  (5.3) related to the quasi-momentum of that state as a function of the system size. The plot shows data of various states for anisotropies  $\gamma/\pi = 0.2, 0.4$  and  $0.45$  and different values of the staggering  $\alpha(2, 1) = \frac{1}{3}(\pi + \gamma)$  (red symbols in the web version),  $\alpha(3, 2) = \frac{1}{3}(2\pi + \gamma)$  (green symbols in the web version),  $\alpha(13, 12) = \frac{1}{25}(12\pi + \gamma)$  (blue symbols in the web version) and the self-dual case  $\alpha(1, 1) = \frac{\pi}{2}$  (black open symbols). The linear dependence on  $\log L$  (dashed line) with an offset depending only on  $\gamma$  is clearly seen.

Rewriting the expression (3.12) for the finite size spectrum in terms of the quantum number (5.3) the effective field theory describing the low energy excitations of the lattice model can now be identified using the predictions of conformal field theory (CFT) [1,7]

$$\Delta E = \frac{2\pi v_F}{L} \left( -\frac{c}{12} + h + \bar{h} \right), \quad P = \frac{2\pi}{L} (h - \bar{h}). \tag{5.4}$$

The finite size energy of the ground state  $((m, w) = (0, 0)$  and  $\tilde{m} = 0$ ) implies that the effective central charge of the vertex model is  $c_{\text{eff}} = 2$ , as expected for a model with two species of excitations (holes in the distributions of Bethe roots with  $\text{Im}(\lambda) = 0$  or  $\pi/2$ , see Eq. (3.8)). On the other hand, because of the presence of the continuous quantum number (5.3), the continuum limit of the spin chain has to correspond to a non-rational CFT with non-normalizable Virasoro vacuum and therefore a state with  $h = \bar{h} = 0$  should not be part of the spectrum of the lattice model. This has led Ikhlef et al. [23] to identify the continuum limit of the staggered six-vertex model for  $\alpha = \pi/2$  with the  $SL(2, \mathbb{R})/U(1)$  sigma model at level  $k = \pi/\gamma \in (2, \infty)$  describing a two-dimensional Euclidean black hole [43]. This CFT has central charge  $c = 2\frac{k+1}{k-2}$  and primary fields with dimensions

$$h = \frac{(m - kw)^2}{4k} + \frac{s^2 + 1/4}{k - 2}, \quad \bar{h} = \frac{(m + kw)^2}{4k} + \frac{s^2 + 1/4}{k - 2}. \tag{5.5}$$

Here,  $j = (-1/2 + is)$  with real  $s$  is the spin of the  $SL(2, \mathbb{R})$  affine primaries from the principal continuous representations. The ground state of the lattice model corresponds to the state with

lowest conformal weight  $h_0 = \bar{h}_0 = 1/(4(k - 2))$  which results in the observed effective central charge. Further evidence for this proposal has been provided in Refs. [9,24] where the density of states in the continuum has been computed from the finite part  $\propto B(s)$  in the relation (5.2) and shown to agree with the known result for the  $SL(2, \mathbb{R})/U(1)$  sigma model [22,34]

$$\begin{aligned} \rho_{BH}(s) &= \frac{1}{\pi} (\log \epsilon + \partial_s (s B_{BH}(s))), \\ B_{BH}(s) &= \frac{1}{2s} \operatorname{Im} \log \left[ \Gamma \left( \frac{1}{2} (1 - m + wk) - is \right) \Gamma \left( \frac{1}{2} (1 - m - wk) - is \right) \right] \end{aligned} \tag{5.6}$$

(the term  $\log \epsilon$  arises from the regularization needed to handle divergencies arising in the string theory).

From its effect on the finite size spectrum a staggering  $\alpha \neq \pi/2$  is an irrelevant perturbation of the low energy effective theory. This is consistent with the assumption that the critical theory for the entire phase  $0 \leq \gamma < \alpha < \pi - \gamma$  is the Euclidean black hole sigma model CFT. Due to the presence of a continuous spectrum, however, it is not sufficient for this identification to rely on the finite size spectrum alone. In addition the density of states in the continuum has to be computed. According to the considerations at the beginning of this section, the allowed values of  $\tilde{m}$  for given staggering  $\alpha$  and system size  $L$  differ by multiples of 2. Therefore, the density of states in the continuum follows from (5.2) to be (see also Refs. [9,24])

$$\rho(s) = \frac{1}{2} \partial_s \tilde{m} = \frac{2}{\pi} \left( \log \frac{L}{L_0} + \partial_s (s B(s)) \right). \tag{5.7}$$

We have computed the function  $B(s)$  for staggering  $\alpha \neq \pi/2$  by numerical solution of the Bethe equations (3.4) for systems sizes up to  $L \lesssim 2400$  and from the NLIEs (4.3), (4.4) for larger  $L$ . By comparing the data for the largest system size available with the corresponding quantity for the black hole sigma model (5.6) we have determined the non-universal length scale  $L_0(\gamma)$ . As seen in Fig. 5 the convergence of the data obtained for the lattice model towards  $B_{BH}(s)$  is excellent, just as in the previous studies for the self-dual case [9,24].

### 6. Corrections to scaling

For the identification of the CFT describing the continuum limit of the staggered six-vertex model in the previous section we have made use of the consequences of conformal invariance on the finite size spectrum (5.4) which hold asymptotically for  $L \rightarrow \infty$ . Having revealed the origin of the logarithmic terms in the spectrum to be the presence of a continuous spectrum of critical exponents we now turn to analyze the dominant corrections to finite size scaling (3.12) of an energy eigenvalue  $E_a(L)$  corresponding to the operator  $\Phi_a$  in the CFT with conformal weights  $(h_a, \bar{h}_a)$ , i.e.

$$R_a(L) = \frac{L}{2\pi v_F} (E_a(L) - L\varepsilon_\infty) + \frac{c}{12} - (h_a + \bar{h}_a). \tag{6.1}$$

These corrections arise from deviations of the lattice Hamiltonian from the fixed-point Hamiltonian  $H^*$  of the CFT by terms involving irrelevant operators [11] and therefore should provide additional insights into the particular lattice regularization of the CFT considered. If the deviations are small these terms can be written as

$$H_{\text{lattice}} = H^* + \sum_b g_b \int dx \Phi_b(x), \tag{6.2}$$

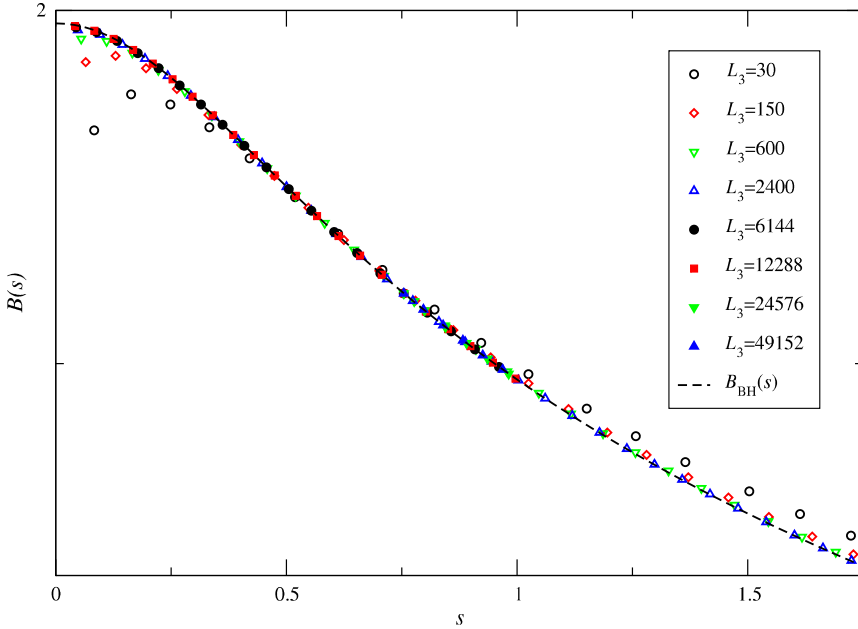


Fig. 5. Numerical values for the finite part  $B(s)$  of the density of states in the continuous part of the spectrum of the staggered six-vertex model. Data are presented for system sizes  $L = L_c + r$ ,  $r = 0, 1, 2$ , at anisotropy  $\gamma = \frac{\pi}{5}$  and staggering  $\alpha(p = 2, q = 1) = \frac{1}{3}(\pi + \gamma)$ . The dashed line is the result for the  $SL(2, \mathbb{R})/U(1)$  sigma model, see Eq. (5.6).

where  $\Phi_b$  are conformal fields with scaling dimension  $X_b = h_b + \bar{h}_b > 2$  and conformal spin  $s_b = h_b - \bar{h}_b$ . The coupling constants  $g_b$  are in general unknown.

The effect of these terms on the finite size spectrum can be studied within perturbation theory [2,11]: to second order one finds

$$\begin{aligned}
 R_a(L) \simeq & 2\pi \sum_b g_b C_{a,a,b} \left(\frac{2\pi}{L}\right)^{X_b-2} \\
 & + 4\pi^2 \sum_{\substack{b,b',a' \\ a' \neq a}} g_b g_{b'} \frac{C_{a,a',b} C_{a',a,b'}}{X_a - X_{a'}} \left(\frac{2\pi}{L}\right)^{X_b + X_{b'} - 4} + \dots
 \end{aligned}
 \tag{6.3}$$

Here conformal invariance has been used to compute the matrix elements of the perturbation (6.2) from the three point functions  $\langle \Phi_a(z_1) \Phi_b(z_2) \Phi_c(z_3) \rangle$  in the complex plane

$$\langle a | \Phi_b(x) | c \rangle = C_{a,b,c} \left(\frac{2\pi}{L}\right)^{X_b} e^{\frac{2\pi i}{L}(s_a - s_c)x},
 \tag{6.4}$$

$C_{a,b,c}$  are the universal coefficients appearing in the operator product expansion (OPE) of primary fields in the CFT. The first order term in (6.3) is absent for the ground state, an exception being contributions due to descendants with dimension  $X = 4$  of the identity operator which are expected to be present in any theory [10,11,39]. Based on these insights the irrelevant operators present in lattice formulations of various unitary models have been identified: the ‘analytic’ corrections to scaling resulting from operators in the conformal block of the identity have been



studied to even higher orders as in (6.3), cf. [25,39,42]. For the six-vertex model including its higher spin variants subject to various boundary conditions some of the deviations from the respective fixed point have been identified based on numerical studies of the finite size spectrum [2, 3,37]. Very recently, it has been shown that similar arguments as above can also be applied to non-unitary models where the spectrum may contain zero norm states and therefore the scalar product used in conformal perturbation theory has to be adapted to properly deal with Jordan cells, see e.g. Ref. [13]. With such a modification the perturbative analytical corrections to scaling in a logarithmic minimal CFT with central charge  $c = -2$  describing critical dense polymers have been found to coincide with the exactly known spectrum for a lattice model [26].

Corrections to scaling similar to (6.3) are expected to arise in systems with a continuous spectrum [46]. In fact, our numerical results for the corrections to the finite size scaling of the ground state of the staggered six-vertex model, i.e.  $(m, w) = (0, 0)$  and  $\tilde{m} = 0$ , indicate that  $R_0(L)$  vanishes asymptotically as a power of the system size, i.e.  $R_0(L) \sim L^{-\delta}$ , see Figs. 6 and 7. For  $\gamma \lesssim \pi/3$  the dominant term vanishes as  $L^{-2}$  with an amplitude which changes its sign for an anisotropy in  $0.1\pi \lesssim \gamma \lesssim 0.15\pi$ . For  $\gamma > \pi/3$  we observe a slow crossover from the  $L^{-2}$  behavior to a different power law which takes place over several orders of magnitude in the system size. Based on numerical results for  $L$  up to  $10^6$  obtained by solving the NLIEs we conjecture that the exponent  $\delta$  governing the asymptotic behavior of  $R_0(L)$  (i.e. the threshold to the continuous spectrum above the lowest state with  $(m, w) = (0, 0)$ ) is given by

$$\delta = \begin{cases} 2 & \text{for } 0 < \gamma < \frac{\pi}{3}, \\ \frac{2\pi}{\gamma} - 4 & \text{for } \frac{\pi}{3} \leq \gamma < \frac{\pi}{2}. \end{cases} \quad (6.5)$$

This behavior does not depend on the staggering  $\alpha$ , cf. Figs. 8 and 9. We also have analyzed the corrections to scaling in various excited states. For excitations within the continuum above  $(m, w) = (0, 0)$  but  $\tilde{m} \neq 0$  we have used the NLIEs to compute the eigenvalues. States with  $(m, w) \neq (0, 0)$  are outside the range of validity of our NLIEs. For these we have solved the Bethe equations (3.4) directly which puts a limitation on the available data to lattices with a few thousand sites. In all cases  $R(L)$  changes its sign as a function of  $L$  for certain values of the anisotropy, see e.g. Fig. 10. Therefore, even larger system sizes are needed for a quantitative analysis of the asymptotic behavior of  $R(L)$ . Based on our data, however, we find an algebraic decay consistent with the conjecture (6.5).

To interpret these findings for the staggered six-vertex model with its low energy description in terms of the Euclidean black hole sigma model CFT the considerations leading to (6.3) need to be modified as follows: first of all, the presence of a continuous component in the spectrum of critical exponents implies that the sums in (6.2) and (6.3) should be replaced by integrals and the coupling constants become functions  $g(s)$  of the continuous quantum number  $s$ . We interpret the slow crossover in the scaling behavior of  $R(L)$  from  $L^{-2}$  to  $L^{-\delta}$  observed for  $\gamma > \pi/3$  as an indication for the presence of a perturbation of the fixed point Hamiltonian by a continuum of conformal fields. Furthermore, since the ground state of the lattice model is not the (non-normalizable) vacuum of the CFT but rather the state corresponding to the lowest conformal weight  $(h_0, \tilde{h}_0)$ , first order corrections cannot be excluded to contribute to  $R_0(L)$ . This is consistent with the fact that the asymptotic behavior of *all* states considered is governed by the same exponent (6.5).

Taking into account these modifications to (6.3) would lead to the conclusion that the numerical data for  $R(L)$  for the lattice model are the first order effect of a perturbation of the fixed point Hamiltonian by a descendent of the identity operator with dimension  $X_I = 4$  and a continuum of operators starting with dimension  $X_k = 2\pi/\gamma - 2 = 2(k - 1)$ . The latter, however, is not in

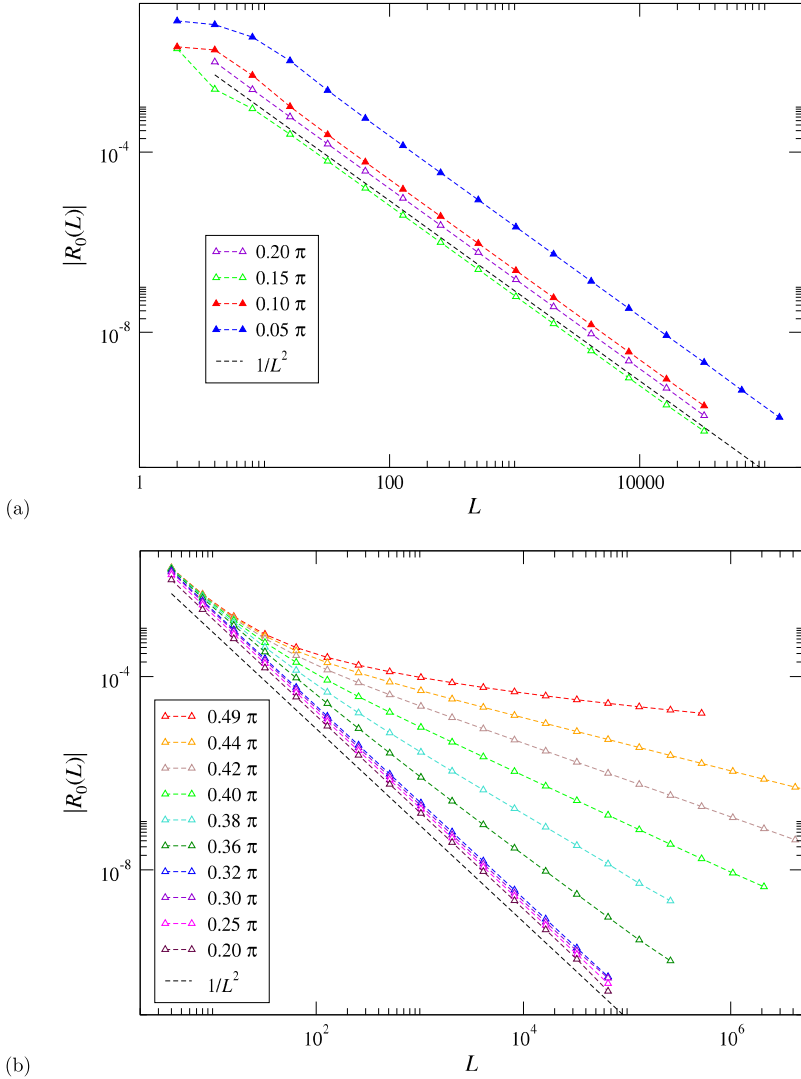


Fig. 6. The scaling correction  $R_0(L)$  to the ground state energy of the self-dual model for anisotropies (a)  $0 < \gamma \leq 0.2\pi$  and (b)  $0.2\pi \leq \gamma < \frac{\pi}{2}$ . Positive (negative)  $R_0(L)$  are depicted by filled (open) symbols.

the spectrum (5.5) of the  $SL(2\mathbb{R})/U(1)$  coset model<sup>3</sup>: while the  $\gamma$ -dependence of (6.5) could be realized by a perturbation through fields with vorticity  $w = \pm 2$  there is no sign of the divergence due to the contribution of the non-compact degree of freedom to the conformal weights as  $\gamma \rightarrow \pi/2$  (or  $k \rightarrow 2$ ).

<sup>3</sup> This is also true if one considers the normalizable operators from the series of discrete representations with real  $SL(2, \mathbb{R})$  spin  $j$  which can take values  $1/2 < -j < (k - 1)/2$  subject to a constraint on physical states relating  $j$  to the charge  $m$  and vorticity  $w$  [22,33].

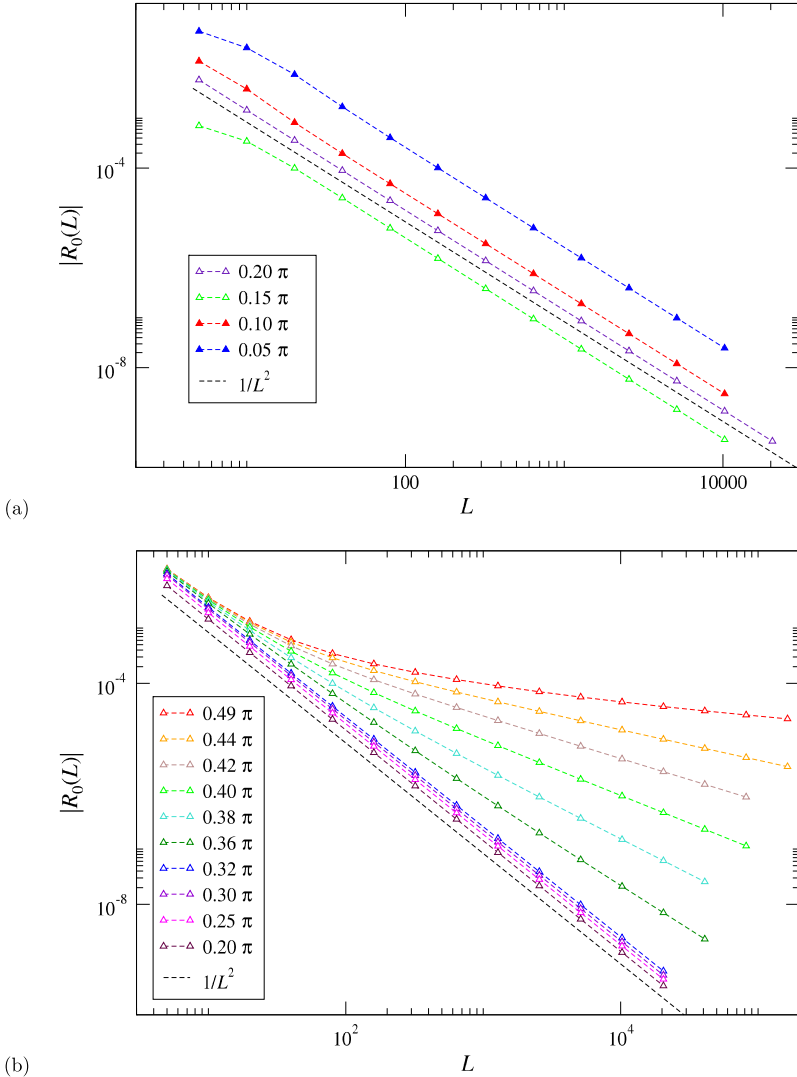


Fig. 7. Same as Fig. 6 but for the model with staggering  $\alpha(p = 3, q = 2)$ .

To resolve this discrepancy one has to consider additional ways how the regularization of the CFT in terms of the staggered six-vertex model on a finite lattice can affect the asymptotic  $L$ -dependence of the corrections to scaling, Eq. (6.3). Here one has to take into account that the latter are given – apart from the coupling constants appearing in the perturbation (6.2) – in terms of universal quantities such as the scaling dimensions and OPE coefficients of the CFT.

Let us now assume that the perturbation of the fixed point Hamiltonian  $H^*$  present in the staggered six-vertex model is given in terms of an operator from the conformal block of the identity with dimension  $X_I = 4$  and operators from the continuum of fields with quantum numbers  $(m, w) = (0, 2)$  and  $SL(2, \mathbb{R})$ -spin  $j = (-1/2 + is)$ . On a finite lattice the latter is quantized as a

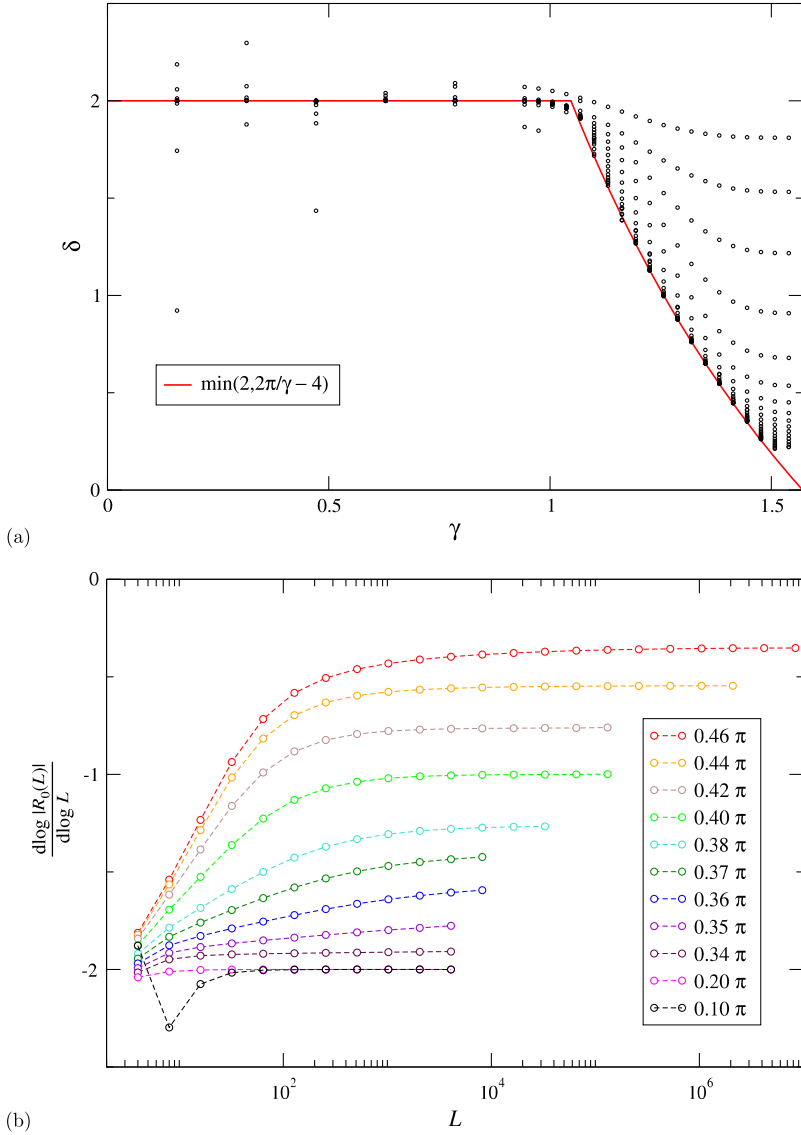


Fig. 8. (a) Conjectured  $\gamma$ -dependence (6.5) of the exponents  $\delta$  in the algebraic decay of  $R_0(L)$  (red line in the web version). Symbols denote the finite size estimates obtained for the exponents from the ground state data of the self-dual model shown in Fig. 6. (b) System size dependence of the finite size estimates  $\log(R_0(L)/R_0(L/2))/\log 2$  converging to  $-\delta$  in the large- $L$  limit.

consequence of (5.2) with  $\Delta s \simeq \pi/(2 \log L)$ . Then, for sufficiently large  $L$ , the OPE coefficients appearing to the first order expression for the corrections to scaling in the ground state of the lattice model  $(m, w) = (0, 0)$  and  $s = 0$  are

$$C_{(0,0),(0,0),(0,2)} \left( s_1 = 0, s_2 = 0, s_3 = \frac{\pi n}{2 \log L} \right), \quad n = 0, 1, 2, \dots \tag{6.6}$$

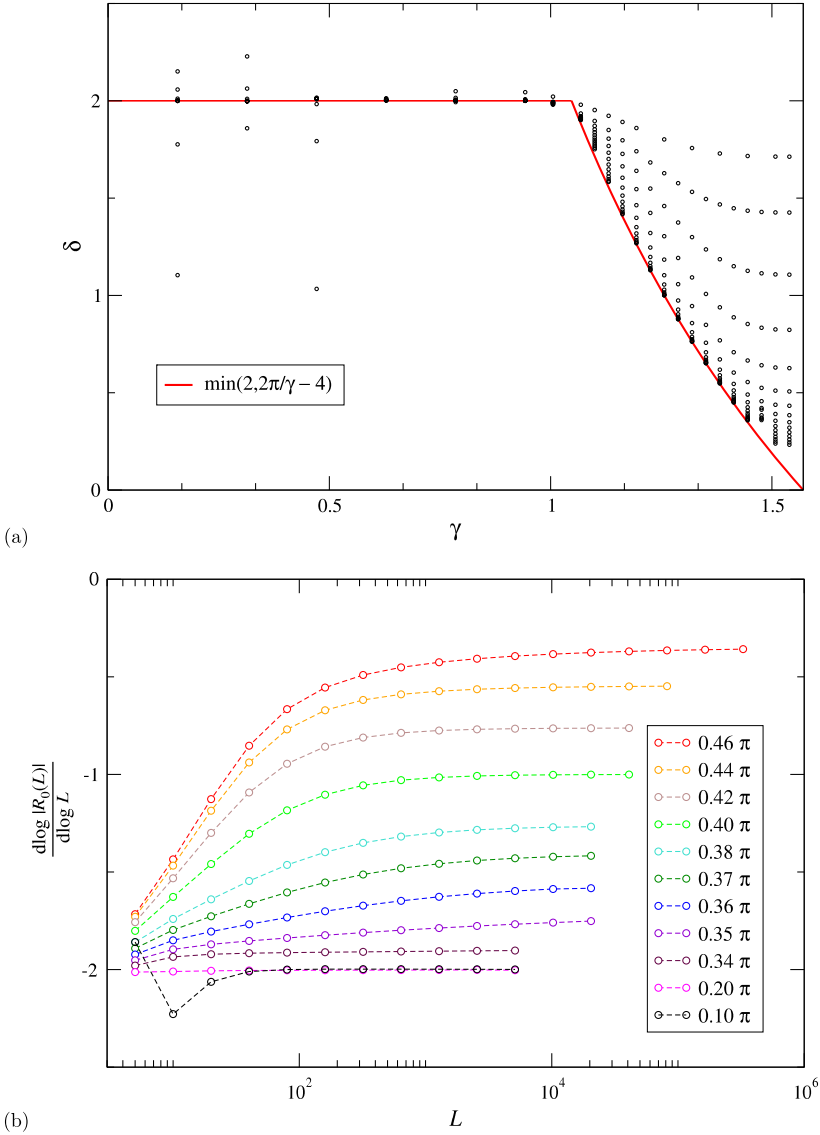


Fig. 9. Same as Fig. 8 but referring to the data with staggering  $\alpha(p = 3, q = 2)$  shown in Fig. 7.

(here we have indicated the dependence of  $C_{a,b,c}$  on the participating fields with conformal weights (5.5) through subscripts for the discrete quantum numbers ( $m_a, w_a$ ) while the continuous quantum numbers  $s_a$  appear as arguments of the OPE coefficient). Little is known about the operator product expansion in theories with a non-compact target space. For two systems related to the present model, i.e. Liouville field theory and the  $H_3^+$  Wess–Zumino–Novikov–Witten model, the OPE coefficients have been found to be given in terms of double Gamma functions depending on combinations of the spins  $j_a = (-1/2 + is_a)$  and  $k = \pi/\gamma$  [12,41,45]. With (6.6) this gives

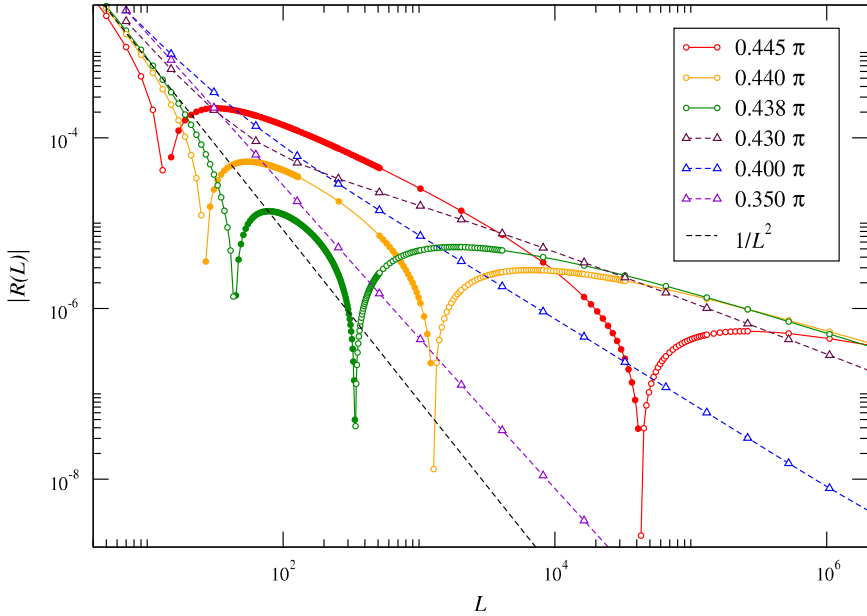


Fig. 10. Scaling correction  $R(L)$  to the excited state  $((m, w) = (0, 0)$  and  $\tilde{m} = 1$ , i.e.  $n_1 - n_2 = 1$ ) of the self-dual model. Positive (negative)  $R(L)$  are depicted by filled (open) symbols.

a rise to an additional  $L$ -dependence in the individual terms contributing to the corrections to scaling (6.3) which may account for the observed asymptotics with exponent (6.5).

Finally, we note that the exponent (6.5) vanishes as  $\gamma$  approaches  $\pi/2$  indicating the appearance of a marginal operator in the perturbation of the fixed point Hamiltonian which leads to a different low energy effective theory. In the staggered six-vertex model some of the vertex weights vanish in this limit and the lattice model has an  $OSP(2|2)$  symmetry [23].

### 7. Discussion

We have investigated the finite size spectrum of the staggered six-vertex model for the range of parameters  $0 \leq \gamma < \alpha < \pi - \gamma$ . As has been noted in previous works the continuous component of this spectrum leads to a strong logarithmic size dependence [17,23,27]. Therefore both a formulation of the spectral problem allowing for numerical studies of large system sizes and insights into the parametrization of the low energy degrees of freedom in terms of the parameters of the lattice model are needed. For the self-dual model,  $\alpha = \pi/2$ , these points have been addressed before and provided evidence for the proposal that the critical theory of the model is the  $SL(2\mathbb{R})/U(1)$  sigma model at level  $k = \pi/\gamma > 2$  describing a two-dimensional Euclidean black hole [9,24].

We have derived a set of coupled NLIEs (4.3) and (4.4) which generalize the ones obtained previously for the self-dual case  $\alpha = \pi/2$  [9] to the range of staggering given above. The kernel functions appearing in the NLIEs used here are regular in Fourier space. As a consequence this formulation is particularly suitable for their numerical solution: we can compute the energies of the ground state and in the continuum above it for chains with up to  $10^6$  lattice sites for arbitrary staggering  $\gamma < \alpha \leq \pi/2$ . Based on our numerical data we have extended the proposal [24] for

the quantum number for the continuous part of the spectrum in terms of the conserved quasi-momentum of the vertex model for staggering away from the self-dual line, Eq. (5.3). With this input we were able to compute the density of states of the model from the finite size spectrum obtained by numerical solution of the NLIEs. Together with the existing data for the finite size spectrum [17] this shows that the model is in the same universality class as the self-dual model independent of the staggering  $\gamma < \alpha < \pi - \gamma$ . Both the finite size spectrum and the density of states agree with what is known about the Euclidean black hole sigma model.

Finally, we have extended previous studies of the finite size spectrum [17,23] by considering the corrections to scaling due to irrelevant perturbations of the fixed point Hamiltonian appearing in the lattice model. Such perturbations are expected to lead to subleading power-laws in the finite size spectrum which can provide additional information on the operator content of the continuum model and insights into the emergence of the continuum of critical exponents in the thermodynamic limit of the lattice model. Again, our numerical data suggest that the variation of the staggering parameter does not change the critical theory: different values  $\alpha$  only lead to small changes in the non-universal coupling constants  $g$  in (6.2). As for the interpretation of our numerical results summarized in the conjecture (6.5) for the asymptotic algebraic decay of the corrections to scaling, however, we find that the known predictions for theories with purely discrete spectrum have to be modified here. These modifications appear to be closely related to the way how the non-compact degree of freedom is dealt with in the regularization of the field theory leading to the staggered six-vertex model. To make progress in this direction additional work from the CFT side is called for, in particular with respect to the operator product expansion in theories with non-compact target space.

A natural extension to our work would be the finite size scaling analysis of the lattice model in sectors with non-zero magnetization, i.e. with  $m = n_1 + n_2 - L \neq 0$ , or non-zero vorticity  $w$ . In the derivation of the corresponding NLIEs this amounts to consider hole-type solutions of the Bethe equations (3.4) appearing inside the integration contours which lead to additional logarithmic driving terms. Another direction for future work is to consider more general lattice models which develop a continuous spectrum of critical exponents in the thermodynamic limit. Known examples with such a behavior are the supersymmetric vertex models based on alternating representations of  $U_q[gl(2|1)]$ . These models are known to contain the staggered six-vertex model studied in the present work as a subsector [15–17]. Apart from general insights into the critical properties of quantum spin chains based on super Lie algebras and conformal field theories with non-compact target space this may also provide a basis for an improved understanding of some topical problems in condensed matter physics, e.g. the quantum phase transitions in two-dimensional disordered systems [19,21,35,48] or possibly the deconfinement of  $U(1)$  gauge fields coupled to the Fermi surface of a two-dimensional system [32,36], in the context of an exactly solvable model.

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