Orbifold Jacobian Algebras of Isolated Singularities with Group Action

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Kurzzusammenfassung

Schlagworte: Hyperflächensingularitäten, "orbifold"-Landau-Ginzburg-Modelle, Milnorzahl, Frobenius Algebren, invertierbare Polynome, Arnolds seltsame Dualität

In der Singularitätentheorie ist die Milnorzahl eine wichtige Invariante einer Hyperflächensingularität. Sie ist die Dimension der Jacobischen Algebra, die über die partiellen Ableitungen eines Polynoms f definiert wird, welches die Singularität beschreibt. Solche Polynome mit isolierter Singularität im Ursprung werden auch in der Physik untersucht und führen auf sogenannte Landau-Ginzburg-Modelle. In dieser Arbeit befassen wir uns mit einer "orbifold"-Version hiervon. Sei f invariant unter der Wirkung einer endlichen Gruppe G. Wir definieren axiomatisch eine "orbifold" Jacobische $\mathbb{Z}/2\mathbb{Z}$-graduierte Algebra für das Paar (f, G) und zeigen die Existenz und Eindeutigkeit dieser, wenn f ein invertierbares Polynom oder ein Spitzenpolynom ist. Wir definieren auch eine "orbifold"-Milnorzahl und zeigen den Zusammenhang zu den Dimensionen der "orbifold"-Vektorräume. Wenn ein invertierbares Polynom eine ADE-Singularität oder eine exzeptionelle unimodale Singularität beschreibt, klären wir eine geometrische Bedeutung und finden einen Zusammenhang zu Arnolds seltsamer Dualität. Für die restlichen unimodalen Singularitäten, die von Spitzenpolynomen gegeben werden, finden wir einen Zusammenhang zur Gromov-Witten-Theorie von "orbifold" projektiven Geraden.

Abstract

Keywords: hypersurface singularities, orbifold Landau-Ginzburg models, Milnor number, Frobenius algebras, invertible polynomials, Arnold's strange duality

In singularity theory an important invariant of a hypersurface singularity is the Milnor number. This is the dimension of the Jacobian algebra defined by the partial derivatives of the polynomial f, which defines the singularity. Such polynomials with isolated singularity at the origin are also considered in physics, where they are called Landau-Ginzburg models. In this thesis we study this in an orbifold setting. Let f be invariant with respect to the action of a finite group G. We axiomatically define an orbifold Jacobian $\mathbb{Z}/2\mathbb{Z}$-graded algebra for the pair (f, G). We show its existence and uniqueness in the case, when f is an invertible polynomial or a cusp polynomial. We also define an orbifold Milnor number and show the connection with the dimension of the orbifold spaces. In case if an invertible polynomial defines an ADE singularity or one of the exceptional unimodal singularities, we illustrate a geometric meaning and find a connection to Arnold's strange duality. For the other unimodal singularities given by cusp polynomials we find a connection with the Gromov-Witten theory for orbifold projective lines.

Contents

1 Introduction

Singularity theory is well established in mathematics for many years (cf. [AGV85]). For almost fifty years ([Mi68]) it is known that when a function germ $f : (\mathbb{C}^n, \mathbf{0}) \to (\mathbb{C}, 0)$ has an isolated singularity at $\mathbf{0}$ there exists a local fibration over $\mathbb{C}\backslash\{0\}$ with fibre $\bar{X}_w$ and the middle Betti number μ_f called the Milnor number is equal to the dimension of the Jacobian algebra (often called the Milnor algebra) $\mathrm{Jac}(f) = \mathbb{C}[x_1, \ldots, x_n] \Big/ \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)$. Singularity theory also plays a role in physics. To a given polynomial f with isolated critical point one can associate a so called Landau-Ginzburg model. In quantum cohomology Landau-Ginzburg models and singularity theory gave some of the first examples of Frobenius manifolds. Here we are considering Frobenius algebras in more detail. It is well known that $\mathrm{Jac}(f)$ has the structure of Frobenius algebra (cf. [AGV85]). Namely by taking a nowhere vanishing holomorphic n-form there is an isomorphism $\mathrm{Jac}(f) \cong \Omega_f = \Omega^n(\mathbb{C}^n) \Big/ df \wedge \Omega^{n-1}(\mathbb{C}^n)$. It is on Ω_f, where a natural or canonical non-degenerate symmetric bilinear form, called the residue pairing, exists.

In this thesis we study pairs (f, G) of a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ with isolated singularity at the origin and a finite group G which acts on $\mathbb{C}^n$ and preserves f. Such pairs are often called orbifold Landau-Ginzburg models, in which mostly only special groups G are meant (cf. [BH95], [Kr09]). They have been studied intensively by many mathematicians and physicists working in mirror symmetry for more than twenty years since it yields important, interesting and unexpected geometric information. In particular, the so called orbifold constructions are a cornerstone. An important aspect in the approach of the physicists is the consideration of so-called twisted sectors. Roughly speaking for an orbifold version of a quotient by a group action one first defines an object for each element in the group together with a group action on this object and in the second step takes invariants of all these components. In this sense an orbifold version $\Omega_{f,G}$ as the invariant part of $\Omega'_{f,G}$ can be defined. This is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space, which also had a G-grading, and a natural non-degenerated bilinear form, called the *orbifold residue pairing*, which is a natural generalization of the residue pairing on Ω_f.

Motivated from string theory physicists defined an orbifold Euler characteristic. There are also many other equivariant Euler characteristics for spaces with an action of a finite group. First, one can consider the Euler characteristic of the quotient. Then there is defined an equivariant Euler characteristic as an element of the representation ring $R(G)$ of the group (cf. [tD79], [Wa80]) or higher generalizations of the orbifold Euler characteristic (cf. [AS89], [BF98]), which have values in the integers. A more general concept is the equivariant Euler characteristic, which is an element of the Burnside ring $B(G)$ of the group (cf. [tD79], [EG15]). The previous versions of the Euler characteristic are specializations of this one. So it is reasonable to also consider an equivariant version of the Milnor number. In this thesis we show that the *orbifold Milnor number* is the $\mathbb{Z}/2\mathbb{Z}$-graded dimension of $\Omega_{f,G}$ (Theorem 4.4.4): $\mu^{\mathrm{orb}}_{f,G} = \dim(\Omega_{f,G})_{\overline{0}} - \dim(\Omega_{f,G})_{\overline{1}}$.

The main construction in this thesis is the definition of an orbifold version of $\mathrm{Jac}(f)$. For that we restrict ourselves to subgroups of G_f, namely diagonally acting groups. This is the common restriction for *orbifold Landau-Ginzburg models* (cf. [BH95], [Kr94], [Kr09], [EG12], [FJR13]). In a joint work with Atsushi Takahashi and Alexey Basalaev [BTW16] we gave an axiomatic definition (Definition 5.2.1) of a G-twisted version of the Jacobian algebra, denoted by $\mathrm{Jac}'(f,G)$. Here we consider the pair $(\mathrm{Jac}'(f,G), \Omega'_{f,G})$ in the way it is in the classical situation when the group G is trivial. As a consequence $\mathrm{Jac}'(f,G)$ has many structures defined naturally on $\Omega'_{f,G}$, as a $\mathbb{Z}/2\mathbb{Z}$-grading, a G-grading, equivariance with respect to automorphisms of the pair (f,G), the orbifold residue pairing, and so on.

Certain works towards the definition of the Frobenius algebras associated to the pair (f,G) were also done previously by R. Kaufmann and M. Krawitz. In [Ka03], R.Kaufmann proposes a general construction of orbifolded Frobenius superalgebras of (f,G). To build such a $\mathbb{Z}/2\mathbb{Z}$-graded algebra one should make a certain non-unique choice called a “choice of a two cocycle”. A different choice of this cocycle gives indeed a different product. This construction was later used by Kaufmann in [Ka06] for the mirror symmetry purposes from the point of view of physics. In [Kr09], M. Krawitz proposes a very special construction of a commutative (not a $\mathbb{Z}/2\mathbb{Z}$-graded) algebra, for invertible polynomials (cf. [BH93]). Later this definition was improved and used in [FJJS12] to set up the mirror symmetry on the level of Frobenius algebras. However, the crucial part of it remained to be the particularly fixed product that could only be well defined for weighted-homogeneous polynomials. There is also no explanation why a particular product structure is chosen.

Mirror symmetry on the level of Frobenius algebras is a first step towards the mirror symmetry of Frobenius manifolds where the key role is played by the so-called primitive form (cf. [Sa82], [Sa83], [ST08]). From the point of view of mirror symmetry, the algebras we consider here are those in the complex geometry side, the so-called B-model side.

The main advantage of our work compared to that of Kaufmann and Krawitz is that our construction can be used as a starting point for mirror symmetry at the level of Frobenius manifolds having the notion of a primitive form (cf. [Sa82], [Sa83], [ST08]) in the definition (cf. the role of ζ in Definition 5.2.1). Secondly both Kaufmann and Krawitz predefine the product structure either by a choice of a two cocycle or a direct definition. We do not do this in our axiomatization and so we are able to consider our algebra also for not weighted-homogeneous polynomials, like cusp polynomials. Last but not least our algebra inherits a natural $\mathbb{Z}/2\mathbb{Z}$-grading from the Hodge theory associated to (f,G). This $\mathbb{Z}/2\mathbb{Z}$-grading occurs only in an abstract way in the definition of Kaufmann and was not considered at all by Krawitz.

Our Axiomatization of a G-twisted Jacobian algebra lists a minimum of conditions to be satisfied. In particular we do not predefine any product structure. The Algebra $\mathrm{Jac}(f,G)$ called the *orbifold Jacobian algebra* of the pair (f,G) will be, as usual in orbifold construction, the G-invariant subalgebra of $\mathrm{Jac}'(f,G)$. However, it is not clear in general whether such an algebra as $\mathrm{Jac}'(f,G)$ exists or not. Even if it exists it may not be unique.

The main results in this thesis are the existence and uniqueness of a G-twisted Jacobian algebra $\mathrm{Jac}'(f,G)$ for two classes of polynomials f and any subgroup of the maximal diagonal symmetry group G_f (Theorems 6.2.1 and 7.2.2). Namely it is uniquely determined up to isomorphism by our axiomatization. Moreover we show that when G is a subgroup of $\mathrm{SL}(n,\mathbb{C})$ the orbifold Jacobian algebra $\mathrm{Jac}(f,G)$ is a $\mathbb{Z}/2\mathbb{Z}$-graded commutative Frobenius algebra (Proposition 5.3.7).

The first class of polynomials are the so called *invertible polynomials*. These are weighted homogeneous polynomials with the number of monomials coinciding with the number of variables such that the weights are well defined. These polynomials were introduced in [BH93] to construct mirror pairs of Calabi-Yau manifolds. Therefore the authors considered f and the Berglund-Hübsch transpose f^T (see Definition 6.1.2). As already cited this construction was generalized to an orbifold setting in [BH95].

The second class of polynomials are the so called *cusp polynomials*. For a triplet $A = (a_1, a_2, a_3)$ of positive integers there is given the polynomial $f_A = x_1^{a_1} + x_2^{a_2} + x_3^{a_3} - q^{-1}x_1x_2x_3$ ([IST12],[ST15]).

One of the most famous examples in singularity theory is the ADE-classification of hypersurface singularities with zero modality (cf. [AGV85]). These singularities can be given by invertible polynomials. We show for this case, when f is an invertible polynomial giving an ADE-singularity and G a subgroup of $G_f \cap \mathrm{SL}(n,\mathbb{C})$, that our orbifold Jacobian algebra $\mathrm{Jac}(f,G)$ is isomorphic to the usual Jacobian algebra $\mathrm{Jac}(\overline{f})$ (Theorem 6.3.7). This result completes the results of [ET13a] where concerning a crepant resolution $\widehat{\mathbb{C}^3/G}$ of $\mathbb{C}^3/G$ it was shown that the geometry of vanishing cycles for the holomorphic map $\widehat{f} : \widehat{\mathbb{C}^3/G} \longrightarrow \mathbb{C}$ associated to f is equivalent to the one for the polynomial $\overline{f}$. Therefore, our orbifold Jacobian algebra is not only natural from the view point of algebra but also from the view point of geometry.

Also the hypersurface singularities of modality one are classified (cf.[AGV85]). The parabolic and hyperbolic singularities can be given by cusp polynomials. Moreover there are 14 exceptional families where one can again find invertible polynomials. We state a similar result, as for the ADE-singularities, for the Berglund-Hübsch transposes of these polynomials (Theorem 6.4.8).

Arnold [Ar75] observed a “strange duality” in this class of singularities, the Dolgachev numbers (a triple of algebraically defined positive integers) of one singularity are equal to the Gabrielov numbers (a triple of positive integers associated to a Coxeter-Dynkin diagram) of another one and vice versa. It is now naturally understood as one of mirror symmetry phenomena (cf. [ET11] and references therein). A corollary (Corollary 6.4.9) of the Theorem 6.4.8 shows an isomorphism $\mathrm{Jac}(f_1^T, G^{\mathrm{SL}}_{f_1^T}) \cong \mathrm{Jac}(f_2)$ if and only if the associated singularities of f_1 and f_2 are strangely dual.

Last but not least we have mentioned that our construction works as a starting point for the mirror symmetry on the level of Frobenius manifolds having the notion of a primitive form. For cusp polynomials there were given primitive forms in [ST15] and [IST12] and associated to the Gromov-Witten theory for orbifold projective lines with at most 3 orbifold points (cf. [IST15]). On the level of Frobenius algebras we associate $\mathrm{Jac}(f,G)$ to the Gromov-Witten theory for orbifold projective lines with at most r orbifold points (cf. [Sh14]) in Theorem 7.3.6.

Structure of the Thesis

This thesis starts with two introductory chapters.

In Chapter 2 we recall the basic facts about hypersurface singularities and define the algebra $\mathrm{Jac}(f)$, the Milnor number μ_f, the space Ω_f, and the residue pairing on them. We also give the definition of the Euler characteristic in Section 2.2 and the connection with the Milnor number.

In Chapter 3 we give all definitions of equivariant Euler characteristics for a space with a group action. For that we define the representation ring in Section 3.1 and the Burnside Ring in Section 3.3.

Chapter 4 first introduces the pair (f, G) and defines the action of the group G. Then we define the orbifold versions of the Milnor number, of Ω_f and of the residue pairing.

In Section 4.4 we also prove our first theorem about the correspondence between the orbifold Milnor number and the dimension of the orbifold spaces.

Chapter 5 gives the axiomatic definition of the G-twisted Jacobian algebra $\mathrm{Jac}'(f, G)$ in Section 5.2. In the setup in Section 5.1 we therefore define $\mathrm{Aut}(f, G)$, the automorphisms of the pair (f, G) which act naturally on $\Omega_{f,G}$ and $\mathrm{Jac}'(f, G)$. Then in Section 5.3 we define the orbifold Jacobian algebra $\mathrm{Jac}(f, G)$. At the end, in Section 5.4, we also give some preliminaries for the proofs in the next two chapters.

In Chapter 6 we first introduce invertible polynomials and then in Section 6.2 prove the uniqueness and existence of the G-twisted Jacobian algebra for this class of polynomials.

In Section 6.3 and 6.4 we introduce the ADE and the exceptional unimodal singularities which can be given by invertible polynomials and show a geometric meaning of the orbifold Jacobian algebra. This gives in Section 6.4 also a connection to Arnold's strange duality.

In Chapter 7 we first introduce cusp polynomials and then in Section 7.2 show the uniqueness and existence of the G-twisted Jacobian algebra for this class of polynomials.

In the last section 7.3 we associate $\mathrm{Jac}(f, G)$ for this class of polynomials to Frobenius algebras associated to the Gromov-Witten theory for orbifold projective lines.

Notation and Conventions

- We will always use the notation
  $$\mathbf{e}[\alpha] = e^{2\pi\sqrt{-1}\alpha}.$$
  So e.g. $\mathbf{e}[\frac{1}{k}]$ is a k-th root of unity.
- In this thesis we are always thinking of G as a finite group written in multiplicative way and the element $\mathrm{id} \in G$ is the neutral element.
- S_n is the symmetric group on n elements. For permutations, we use the cycle notation; i.e., we write (132) for the permutation $\left(\begin{smallmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{smallmatrix}\right) \in S_3$. Again its neutral element is denoted by $\mathrm{id} \in S_n$.
- Let the group G act on the set X. Then we denote the G-invariant part of X by
  $$X^G = \{x \in X \mid gx = x \quad \forall g \in G\}.$$
- For the disjoint union we will use $\dot\cup$. Otherwise the union need not be disjoint.
- We write $A \backslash B$ for the set A without the set B. Recognize that this is different from the next notion.
- ${}_H\backslash^G$ or G/H denote the quotient of the group G by the subgroup H.
  Normally we think of left cosets G/H, but sometimes it is relevant to consider right cosets.
- We write $|A|$ for the number of elements in the set A.
- As always $\gcd(l, m)$ is the greatest common divisor of the numbers l and m, and $\mathrm{lcm}(a_1, a_2, a_3)$ is the least common multiple of the numbers a_1, a_2, a_3.

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2 Isolated Hypersurface Singularities

In this chapter we want to introduce the fundamental and known facts about hypersurface singularities.

2.1 Milnor Number and Jacobian Algebra

Definition 2.1.1. Let n be a non-negative integer and

$$f = f(\mathbf{x}) = f(x_1, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n]$$

a complex polynomial with $f(\mathbf{0}) = 0$. f has an *isolated singularity* at $\mathbf{0}$, if the map

$$\mathrm{grad} f = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}) : \mathbb{C}^n \to \mathbb{C}^n$$

has an isolated zero at $\mathbf{0}$, i.e. there exists a neighborhood U of $\mathbf{0}$ where $\mathrm{grad} f$ has no zero in U except possibly at $\mathbf{0}$ itself.

Definition 2.1.2. The *Jacobian algebra* of f is defined as

$$\mathrm{Jac}(f) := \mathbb{C}[x_1, \ldots, x_n] \big/ (\tfrac{\partial f}{\partial x_1}, \ldots, \tfrac{\partial f}{\partial x_n}) \ .$$

When f has an isolated singularity at $\mathbf{0}$, $\mathrm{Jac}(f)$ is a finite dimensional $\mathbb{C}$-vector space. We define $\mu_f := \dim \mathrm{Jac}(f)$ the *Milnor number* of f.

Example 2.1.3.
- For $n = 0$ we have $\mathrm{Jac}(f) \cong \mathbb{C}$ and $\mu_f = 1$.
- Let be $f(x_1, x_2, x_3) = x_1^3 + x_2^3 + x_3^3 \in \mathbb{C}[x_1, x_2, x_3]$. We have $\mu_f = 8$ and

$$\mathrm{Jac}(f) = \mathbb{C}[x_1, x_2, x_3] \big/ (3x_1^2, 3x_2^2, 3x_3^2) \cong \langle 1, x_1, x_2, x_3, x_1x_2, x_1x_3, x_2x_3, x_1x_2x_3 \rangle_{\mathbb{C}} \ .$$

- Let be $f(x_1, x_2, x_3) = x_1^3 + x_2^3 x_3 + x_3^3 \in \mathbb{C}[x_1, x_2, x_3]$. We have $\mu_f = 14$ and

$$\begin{aligned}\mathrm{Jac}(f) &= \mathbb{C}[x_1, x_2, x_3] \big/ (3x_1^2, 3x_2^2 x_3, x_2^3 + 3x_3^2) \\ &\cong \left\langle 1, x_1, x_2, x_2^2, x_3, x_3^2, x_2x_3, x_2x_3^2, x_1x_2, x_1x_2^2, x_1x_3, x_1x_3^2, x_1x_2x_3, x_1x_2x_3^2 \right\rangle_{\mathbb{C}} \ .\end{aligned}$$

Definition 2.1.4. The *index* $\mathrm{ind}(\mathrm{grad} f)$ of the map $\mathrm{grad} f$ is the degree of the map

$$\frac{\mathrm{grad} f}{||\mathrm{grad} f||} : S_\varepsilon^{2n-1} \to S^{2n-1}$$

from a sufficient small sphere $||\mathbf{x}|| = \varepsilon$ in $\mathbb{C}^n$ to the unique sphere. This number is well defined, when f has an isolated singularity at $\mathbf{0}$.

Proposition 2.1.5. *We have*

$$\operatorname{ind}(\operatorname{grad} f) = \mu_f.$$

Proof. There is a good proof of this in [AGV85, sect.I.5]. □

2.2 Euler Characteristic and Milnor Fibre

Definition 2.2.1 (cf. e.g. [Eb07])**.** Let X be a topological space and

$$\Delta^k = \{\sum_{i=0}^{k} \lambda_i e_i \mid \sum \lambda_i = 1, 0 \leq \lambda_i \leq 1\} \quad e_0, \ldots, e_k \text{ standard basis of } \mathbb{R}^{k+1}$$

a standard-k-simplex. A *singular k-simplex* is a continuous map $\sigma : \Delta^k \to X$. Let $C_k(X)$ be the free abelian group of all singular k-simplices and $C_k(X) = 0$ for $k < 0$.

We define a boundary operator $\partial_k : C_k(X) \to C_{k-1}(X)$ which sends a singular k-simplex to its boundary

$$\partial_k \sigma = \sum_j (-1)^j \sigma|_{\partial \Delta_j^k},$$

where $\partial \Delta_j^k$ is the j-th face of Δ^k, which is a $(k-1)$-simplex.

Remark 2.2.2. We can calculate directly $\partial_k \partial_{k-1} = 0$ and so $(C_\bullet(X), \partial)$ is a complex (cf. e.g. [Eb07, Prop 4.8]).

Definition 2.2.3. We define the *homology groups*

$$H_k(X, \mathbb{Z}) = {\ker \partial_k}\big/{\mathrm{Im} \partial_{k+1}} .$$

We suppose that X is a topological space, s.t. each homology group is finitely generated, then we call

$$b_k(X) = \operatorname{rank} H_k(X, \mathbb{Z})$$

the *k-th Betti number.*

Definition 2.2.4. We define the *Euler characteristic* of X as

$$\chi(X) = \sum_{k=0}^{\infty} (-1)^k b_k(X).$$

We will give a well known other definition

Proposition 2.2.5. *We also have*

$$\chi(X) = \sum_{k=0}^{\infty} (-1)^k \operatorname{rank} C_k(X).$$

Proof. By the definition of ∂ and $H_k(X, \mathbb{Z})$ it is clear that we have the two short exact sequences:

$$0 \to \ker \partial_k \xrightarrow{\partial} C_k(X) \xrightarrow{\partial} \mathrm{Im}\partial_k \to 0$$

$$0 \to \mathrm{Im}\partial_{k+1} \to \ker \partial_k \to H_k(X, \mathbb{Z}) \to 0$$

So we have

$$\mathrm{rank}\, C_k(X) = \mathrm{rank}\ \ker \partial_k + \mathrm{rank}\, \mathrm{Im}\partial_k$$

and

$$\mathrm{rank}\ \ker \partial_k = \mathrm{rank}\, \mathrm{Im}\partial_{k+1} + \mathrm{rank}\, H_k(X, \mathbb{Z})$$

and so

$$\mathrm{rank}\, C_K(X) = \mathrm{rank}\, H_k(X, \mathbb{Z}) + \mathrm{rank}\, \mathrm{Im}\partial_{k+1} + \mathrm{rank}\, \mathrm{Im}\partial_k.$$

In total we get

$$\begin{aligned}
\sum_{k=0}^{\infty}(-1)^k \mathrm{rank}\, C_k(X) &= \sum_{k=0}^{\infty}(-1)^k \left(\mathrm{rank}\, H_k(X, \mathbb{Z}) + \mathrm{rank}\, \mathrm{Im}\partial_{k+1} + \mathrm{rank}\, \mathrm{Im}\partial_k\right) \\
&= \sum_{k=0}^{\infty}(-1)^k \mathrm{rank}\, H_k(X, \mathbb{Z}) + \sum_{k=0}^{\infty}(-1)^k \mathrm{rank}\, \mathrm{Im}\partial_{k+1} + \sum_{k=0}^{\infty}(-1)^k \mathrm{rank}\, \mathrm{Im}\partial_k \\
&= \chi(X) + \sum_{k=0}^{\infty}(-1)^k \left(-\mathrm{rank}\, \mathrm{Im}\partial_k + \mathrm{rank}\, \mathrm{Im}\partial_k\right) = \chi(X).
\end{aligned}$$

□

Remark 2.2.6 (cf. [Vo02])**.** The same definitions can be done for the dual complex $(C^\bullet(X), d)$ and cohomology. Of course we get the same Euler characteristic. We also get the same Euler characteristic, when we take the de Rham cohomology which is defined over the k-forms on a manifold X.

Definition 2.2.7. We define the de Rham cohomologies

$$H^k(X, \mathbb{C}) = \ker\left(\Omega^k(X) \xrightarrow{d} \Omega^{k+1}(X)\right) \Big/ \mathrm{Im}\left(\Omega^{k-1}(X) \xrightarrow{d} \Omega^k(X)\right).$$

Remark 2.2.8 ([Vo02, Thm. 0.8])**.** So we can write

$$\chi(X) = \sum_{k=0}^{\infty}(-1)^k \dim H^k(X, \mathbb{C})$$

and especially we have $H^k(X, \mathbb{C}) = H^k(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$.

Remark 2.2.9 (cf. [Fu93, p. 141-142], [Di04, Cor. 4.1.23])**.** For "good enough" spaces X, e.g. a union of cells in a finite CW-complex or a quasi-projective complex analytic variety, we can take the cohomology with compact support instead of the normal cohomology and the Euler characteristic stays the same. Then we see that the Euler characteristic is additive in the sense

$$\chi(X \dot\cup Y) = \chi(X) + \chi(Y).$$

All spaces in this thesis will be "good enough".

Now we define a fibration.

Definition 2.2.10 (cf. e.g. [Eb07])**.** A *locally trivial differentiable fibre bundle* is a tupel (E, π, B, F) where E, B, F are differentiable manifolds and $\pi : E \to B$ is a surjective differentiable map and they satisfy: Each point $b \in B$ has a neighborhood U and there exists a diffeomorphism

$$\psi : \pi^{-1}(U) \to U \times F$$

such that the following diagram commutes:

$$\begin{array}{ccc} \pi^{-1}(U) & \overset{\psi}{\to} & U \times F \\ \pi \searrow & & \swarrow \mathrm{pr}_1 \\ & U & \end{array}$$

Here pr_1 is the projection onto the first factor. E is called the *total space*, π the *projection*, B the *basis* and F the *fibre* of the bundle.

Let us now come back to a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ with an isolated singularity at the origin.

Remark 2.2.11 (cf. [AGV85])**.** We are only interested in polynomials. One can prove that each *function germ* with an isolated singularity at the origin is right-equivalent to a polynomial.

Definition 2.2.12 (cf. [Eb07])**.** An *unfolding* of f is a holomorphic function germ

$$F : \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C} \quad \text{with} \quad F(\mathbf{x}, \mathbf{0}) = f(\mathbf{x}).$$

Two unfoldings $F : \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}$ and $G : \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}$ are called *equivalent* if there is a holomorphic map germ $\psi : \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}^n$ with $\psi(\mathbf{x}, 0) = \mathbf{x}$ such that

$$G(\mathbf{x}, \mathbf{u}) = F(\psi(\mathbf{x}, \mathbf{u}), \mathbf{u}).$$

Definition 2.2.13. Let $F : \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}$ be an unfolding of f and $\phi : \mathbb{C}^l \to \mathbb{C}^m$ a holomorphic map germ. The unfolding $G : \mathbb{C}^n \times \mathbb{C}^l \to \mathbb{C}$ with

$$G(\mathbf{x}, \mathbf{t}) = F(\mathbf{x}, \phi(\mathbf{t}))$$

is called the *unfolding induced from* F. We call an unfolding $F : \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}$ of f *versal* if all unfoldings of f are equivalent to an unfolding induced from F. A versal unfolding is called *universal* if m is minimal.

Proposition 2.2.14 (cf. e.g. [Eb07, Prop. 3.17])**.** *Let $f \in \mathbb{C}[x_1, \dots, x_n]$ have an isolated singularity at* $\mathbf{0}$*. Then*

$$F : \mathbb{C}^n \times \mathbb{C}^{\mu_f} \to \mathbb{C}$$
$$(\mathbf{x}, \mathbf{u}) \mapsto f(\mathbf{x}) + \sum_{j=0}^{\mu_f - 1} \phi_j(\mathbf{x}) u_j$$

is a universal unfolding of f, where $\phi_0(\mathbf{x}) = 1, \phi_1(\mathbf{x}), \dots, \phi_{\mu_f - 1}(\mathbf{x})$ is a basis of $\mathrm{Jac}(f)$*.*

We will now define the Milnor fibration. The results were shown by Milnor [Mi68]. We will take the notations of [Eb07, 5.4], where one can also find proofs for the statements.

Remark 2.2.15. Let $f \in \mathbb{C}[x_1, \dots, x_n]$ have an isolated singularity at $\mathbf{0}$. From the implicit function theorem we know that $f^{-1}(w)$ for $w \in \mathbb{C}$, $w \neq 0$, $|w|$ small enough, is a complex manifold in the neighborhood of $0 \in \mathbb{C}$. Let $\varepsilon > 0$, we define $X = \{\mathbf{x} \in \mathbb{C}^n \mid ||\mathbf{x}|| < \varepsilon\}$ and $\Delta = \{w \in \mathbb{C} \mid |w| < \eta_0\}$ for $\eta_0 > 0$, $\eta_0 \ll \varepsilon$, such that $\mathbf{0}$ is the only critical point of f in $\bar{X} \cap f^{-1}(\bar{\Delta})$.

Definition 2.2.16. The fibration

$$f|_{\bar{X} \cap f^{-1}(\bar{\Delta}) \setminus \{0\}} : \bar{X} \cap f^{-1}(\bar{\Delta}) \setminus \{0\} \to \bar{\Delta} \setminus \{0\}$$

which exists due to [Eb07, 5.1] is called the *Milnor fibration.* The fibre

$$\bar{X}_w = f^{-1}(w) \cap \bar{X}$$

over $w \in \bar{\Delta} \setminus \{0\}$ is called the *Milnor fibre* of f. It is a $2(n-1)$-dimensional differentiable manifold with boundary and is up to diffeomorphism uniquely determined.

Theorem 2.2.17 ([Mi68])**.** *The Milnor fibre $\bar{X}_w$ of f is homotopy equivalent to a bouquet of μ_f real $(n-1)$-dimensional spheres. So we have for the dimensions of the cohomology groups $H^i(\bar{X}_w, \mathbb{C})$*

$$\dim H^i(\bar{X}_w, \mathbb{C}) = \begin{cases} 1 & \text{if } i = 0 \\ \mu_f & \text{if } i = n-1 \\ 0 & \text{otherwise} \end{cases}.$$

So for the Euler characteristic we have

$$\chi(\bar{X}_w) = 1 + (-1)^{n-1} \mu_f.$$

Here we see another meaning of the Milnor number μ_f. In [Eb07] one can find a proof of this, where the universal unfolding of f plays a role.

2.3 The Space Ω_f and the Residue Pairing

Definition 2.3.1. Let $\Omega^p(\mathbb{C}^n)$ be the $\mathbb{C}$-module of regular p-forms on $\mathbb{C}^n$. We consider the $\mathbb{C}$-module

$$\Omega_f = \Omega^n(\mathbb{C}^n)\Big/ df \wedge \Omega^{n-1}(\mathbb{C}^n)\,.$$

Remark 2.3.2. Note that Ω_f is a free $\mathrm{Jac}(f)$-module of rank 1. For a nowhere vanishing n-form $\tilde{\omega} \in \Omega^n(\mathbb{C}^n)$ we have the following isomorphism

$$\mathrm{Jac}(f) \overset{\cong}{\to} \Omega_f \quad [\phi(\mathbf{x})] \mapsto [\phi(\mathbf{x})]\omega = [\phi(\mathbf{x})\tilde{\omega}], \tag{2.1}$$

where $\omega = [\tilde{\omega}]$ is the residue class of $\tilde{\omega}$ in Ω_f. Such a class $\omega \in \Omega_f$ giving the isomorphism (2.1) is a non-zero constant multiple of the residue class of $dx_1 \wedge \cdots \wedge dx_n$.

Example 2.3.3.

- For $n = 0$ we have

  $$\Omega_f = \Omega^0(\{\mathbf{0}\})/(df \wedge \Omega^{-1}(\{\mathbf{0}\})) = \Omega^0(\{\mathbf{0}\})$$

  is the $\mathbb{C}$-module of rank one consisting of constant functions on $\{\mathbf{0}\}$.

- For $f = x_1^3 + x_2^3 + x_3^3$ we have

  $$\Omega_f = \langle dx_1 \wedge dx_2 \wedge dx_3, x_1 dx_1 \wedge dx_2 \wedge dx_3, \ldots, x_1x_2x_3 dx_1 \wedge dx_2 \wedge dx_3 \rangle\,.$$

- For $f = x_1^3 + x_2^3 x_3 + x_3^3$ we have

  $$\Omega_f = \left\langle dx_1 \wedge dx_2 \wedge dx_3, x_1 dx_1 \wedge dx_2 \wedge dx_3, \ldots, x_1x_2x_3^2 dx_1 \wedge dx_2 \wedge dx_3 \right\rangle\,.$$

Corollary 2.3.4. *As $\mathbb{C}$-modules we have*

$$\mathrm{Jac}(f) \cong \Omega_f \cong H^{n-1}(\bar{X}_w, \mathbb{C}),$$

since they all have the dimension μ_f.

Definition 2.3.5. We define the *Hessian* of f as the polynomial

$$\mathrm{hess}(f) := \det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{i,j=1,\ldots,n}\,.$$

The class of the Hessian is always a non-zero element in $\mathrm{Jac}(f)$.

Example 2.3.6.

- For $n = 0$ we define $\mathrm{hess}(f) = 1 \in \mathrm{Jac}(f) \cong \mathbb{C}$.

- For $f = x_1^3 + x_2^3 + x_3^3$ we calculate

  $$\mathrm{hess}(f) = \det\begin{pmatrix} 6x_1 & 0 & 0 \\ 0 & 6x_2 & 0 \\ 0 & 0 & 6x_3 \end{pmatrix} = 216 x_1x_2x_3 = 8 \cdot 27 x_1x_2x_3.$$

- For $f = x_1^3 + x_2^3 x_3 + x_3^3$ we calculate

$$\text{hess}(f) = \det \begin{pmatrix} 6x_1 & 0 & 0 \\ 0 & 6x_2x_3 & 3x_2^2 \\ 0 & 3x_2^2 & 6x_3 \end{pmatrix} = 216x_1x_2x_3^2 - 54x_1x_2^4.$$

So we have in $\text{Jac}(f)$ with $x_2^3 + 3x_3^2 = 0$

$$\text{hess}(f) = 216x_1x_2x_3^2 - 54x_1x_2^4 = 216x_1x_2x_3^2 + 54 \cdot 3x_1x_2x_3^2 = 14 \cdot 27x_1x_2x_3^2.$$

Definition 2.3.7. We define a $\mathbb{C}$-bilinear form, the *residue pairing* $J_f : \Omega_f \otimes \Omega_f \to \mathbb{C}$ as

$$J_f\left(\omega_1, \omega_2\right) := \text{Res}_{\mathbb{C}^n} \begin{bmatrix} \phi(\mathbf{x})\psi(\mathbf{x}) dx_1 \wedge \cdots \wedge dx_n \\ \dfrac{\partial f}{\partial x_1} \cdots \dfrac{\partial f}{\partial x_n} \end{bmatrix}$$

where $\omega_1 = [\phi(\mathbf{x}) dx_1 \wedge \cdots \wedge dx_n]$ and $\omega_2 = [\psi(\mathbf{x}) dx_1 \wedge \cdots \wedge dx_n]$ and

$$\text{Res}_{\mathbb{C}^n} \begin{bmatrix} \phi(\mathbf{x})\psi(\mathbf{x}) dx_1 \wedge \cdots \wedge dx_n \\ \dfrac{\partial f}{\partial x_1} \cdots \dfrac{\partial f}{\partial x_n} \end{bmatrix} := \frac{1}{\left(2\pi\sqrt{-1}\right)^n} \int \frac{\phi(\mathbf{x})\psi(\mathbf{x})}{\frac{\partial f}{\partial x_1} \cdots \frac{\partial f}{\partial x_n}} dx_1 \wedge \cdots \wedge dx_n$$

where the integration is along the small cycle, given by the equations $|\frac{\partial f}{\partial x_k}|^2 = \delta_k$ (see [AGV85, I.5.18]).

Proposition 2.3.8 ([AGV85, I.5.11])**.** *The bilinear form J_f on Ω_f is non-degenerate. Moreover, for $\phi(\mathbf{x}) \in \mathbb{C}[x_1, \ldots, x_n]$,*

$$J_f([\phi(\mathbf{x}) dx_1 \wedge \cdots \wedge dx_n], [\text{hess}(f) dx_1 \wedge \cdots \wedge dx_n]) \neq 0$$

if and only if $\phi(\mathbf{0}) \neq 0$. In particular, we have

$$J_f([dx_1 \wedge \cdots \wedge dx_n], [\text{hess}(f) dx_1 \wedge \cdots \wedge dx_n]) = \mu_f.$$

Example 2.3.9.

- For $n = 0$ we have $J_f\left(a, b\right) = ab$ for $a, b \in \Omega_f \cong \mathbb{C}$.
- For $f = x_1^3 + x_2^3 + x_3^3$ we calculate

$$J_f\left(dx_1 \wedge dx_2 \wedge dx_3, x_1x_2x_3 dx_1 \wedge dx_2 \wedge dx_3\right) = \frac{1}{27}$$

since $\text{hess}(f) = \mu_f \cdot 27x_1x_2x_3$.

- For $f = x_1^3 + x_2^3 x_3 + x_3^3$ we calculate

$$J_f\left(dx_1 \wedge dx_2 \wedge dx_3, x_1x_2x_3^2 dx_1 \wedge dx_2 \wedge dx_3\right) = \frac{1}{27}$$

since $\text{hess}(f) = \mu_f \cdot 27x_1x_2x_3^2$.

Definition 2.3.10. An associative $\mathbb{C}$-algebra $(A, \circ)$ is called *Frobenius* if there exists a non-degenerate bilinear form $\eta : A \otimes A \to \mathbb{C}$ such that $\eta(X \circ Y, Z) = \eta(X, Y \circ Z)$ for $X, Y, Z \in A$.

Proposition 2.3.11. *Under the isomorphism* (2.1)*, the residue pairing endows the Jacobian algebra* $\mathrm{Jac}(f)$ *with the structure of a Frobenius algebra.*

Proof. The residue pairing J_f is non-degenerate (Proposition 2.3.8) and the shifting of the multiplication can be directly seen by Definition 2.3.7. $\square$

Remark 2.3.12. If f is even defined over the real numbers, we can define everything similarly. But then $\mathrm{ind}(\mathrm{grad} f)$ need not any more be the same as the Milnor number. In this case we have the Theorem of Eisenbud-Levine-Khimshiashvili:

Theorem 2.3.13 ([EL77], [Kh77])**.** *Let* $f \in \mathbb{R}[x_1, \ldots, x_n]$ *be a polynomial with an isolated singularity at* $\mathbf{0}$*. Then we have:*

$$\mathrm{ind}(\mathrm{grad} f) = \mathrm{sign} J_f,$$

where $\mathrm{sign} J_f$ *is the signature of the symmetric bilinear form* J_f*.*

3 Equivariant Euler Characteristic

Let X be a topological space and G a finite group acting on X. In this chapter we want to discuss two equivariant versions of the Euler characteristic. The first one was introduced in [tD79, 5.1.2] and used in the way we need it in [Wa80]. It is an element of the representation ring $R(G)$. The second more general one is an element of the Burnside ring $B(G)$. This was also introduced in [tD79, 5.4.5] and used in [EG15].

3.1 The Representation Ring

Definition 3.1.1 (cf. [FH91])**.** A *representation* of a finite group G on a finite dimensional vector space V is a homomorphism

$$\rho_V : G \to \mathrm{GL}(V)$$

from G into the group of linear automorphisms of V. We will often regard V itself with a group action as a representation. A *subrepresentation* of a representation V is a linear subspace W of V which is invariant under G. A representation V is called *irreducible* if V and $\{0\}$ are the only subrepresentations of V.

Remark 3.1.2. If V and W are representations, the direct sum $V \oplus W$ and the tensor product $V \otimes W$ are also representations.

Proposition 3.1.3 (cf. [FH91, Cor. 1.6])**.** *Each representation is the direct sum of irreducible representations.*

Definition 3.1.4. We can define the *character* of a representation V. This is a class function

$$V : G \to \mathbb{C},$$

which we will also describe by V, with the value

$$V(g) = \mathrm{Tr}(\rho_V(g))$$

the trace of the linear map $\rho_V(g)$.

Remark 3.1.5. A class function is constant on conjugacy classes. So we have

$$V(hgh^{-1}) = V(g) \quad \forall g, h \in G.$$

We can calculate that

$$V(\mathrm{id}) = \dim V$$

for $\mathrm{id} \in G$ the neutral element.

Definition 3.1.6. For a representation V we define

$$V^G = \{v \in V \mid gv = v \quad \forall g \in G\}$$

the G-invariant part of V.

We can calculate the dimension of V^G, which is the multiplicity of the trivial representation in V.

Proposition 3.1.7 ([FH91, Prop. 2.8]). *The map*

$$\varphi = \frac{1}{|G|}\sum_{g\in G}\rho_V(g) \in \mathrm{GL}(V)$$

is a projection of V into V^G. So we have

$$\dim V^G = \mathrm{Tr}(\varphi) = \frac{1}{|G|}\sum_{g\in G}\mathrm{Tr}(\rho_V(g)) = \frac{1}{|G|}\sum_{g\in G}V(g).$$

Definition 3.1.8. The ring generated by isomorphism classes of representations with the operations $\oplus$ and $\otimes$ is the *representation ring* $R(G)$. With Proposition 3.1.3 it is the free abelian group of isomorphism classes of irreducible representations.

Definition 3.1.9 (cf. [FH91]). The *group algebra* $\mathbb{C}G$ of a group G is the $\mathbb{C}$-vector space with basis $\{e_g \mid g \in G\}$ and the multiplication $e_g \cdot e_h = e_{gh}$ for $g, h \in G$.

Remark 3.1.10. A representation V with $\rho_V : G \to \mathrm{GL}(V)$ can be extended to a map $\rho : \mathbb{C}G \to \mathrm{GL}(V)$ and so V becomes a $\mathbb{C}G$-module, i.e. each representation can be seen as a $\mathbb{C}G$-module.

3.2 Equivariant Euler Characteristic in $R(G)$

To introduce the equivariant Euler characteristic in $R(G)$ let X be a finite simplicial complex and we suppose that G acts in the way, that if $g \in G$ fixes one simplex, then it fixes it pointwise.

Definition 3.2.1 ([Wa80]). The *equivariant Euler characteristic* $\chi_G(X) \in R(G)$ is defined as

$$\chi_G(X) = \sum_{i=0}^{n}(-1)^i[C_i(X)] \in R(G).$$

Here we regard the chain complex

$$0 \to C_n(X) \to C_{n-1}(X) \to \cdots \to C_0(X) \to 0$$

with complex coefficients as a sequence of $\mathbb{C}G$-modules. The action of G is induced by the action on the simplices of X.

With the standard argument (cf. Proposition 2.2.5), this is the same as

$$\chi_G(X) = \sum_{i=0}^{n} (-1)^i H_i(X, \mathbb{C}),$$

where the G-action on $H_i(X, \mathbb{C})$ is induced by the one on X.

Remark 3.2.2. When we take the character of $\chi_G(X)$ we get as in Remark 3.1.5

$$\chi_G(X)(\mathrm{id}) = \chi(X).$$

Proposition 3.2.3 (cf. [Wa80])**.** *The normal Euler characteristic of the quotient ${}^{X}\!/_{G}$ can be calculated as*

$$\chi({}^{X}\!/_{G}) = \frac{1}{|G|} \sum_{g \in G} \chi(X^g)$$

where X^g is the subcomplex fixed by g.

Proof. For the character we have as Wall shows in [Wa80] $\chi_G(X)(g) = \chi(X^g)$. In the quotient ${}^{X}\!/_{G}$ each G-orbit of simplices is collapsed to one single simplex, so it follows

$$C_*({}^{X}\!/_{G}) = C_*(X) \otimes_{\mathbb{C}G} \mathbb{C}.$$

Since G is finite, $\mathbb{C}G$ is semisimple and we can identify this with the summand of $C_*(X)$ which belongs to the trivial representation, $C_*(X)^G$. So the Euler characteristic $\chi({}^{X}\!/_{G})$ is equal to the multiplicity of the trivial representation in $\chi_G(X)$, so with Proposition 3.1.7

$$\chi({}^{X}\!/_{G}) = \frac{1}{|G|} \sum_{g \in G} \chi_G(X)(g) = \frac{1}{|G|} \sum_{g \in G} \chi(X^g).$$

□

3.3 The Burnside Ring

Definition 3.3.1. Let $\operatorname{Consub} G$ be the set of all conjugacy classes of subgroups of G. This is a partially ordered set (cf. [Ha86, 2.2]) with $[K] \leq [H]$ if $\exists K \in [K], H \in [H]$ with $K \subset H$.

Remark 3.3.2 (cf. [Ha86, Thm. 2.2.1])**.** On a partially ordered set we can define the *Moebius function*

$$\mu([H],[K]) = \begin{cases} 1 & [H] = [K] \\ -\sum_{[H]<[H']\leq[K]} \mu([H'],[K]) & [H] < [K] \\ 0 & \text{otherwise} \end{cases}.$$

The Moebius inversion formula follows: Let g and f be functions on the partially ordered set. When $g([H]) = \sum_{[H]\leq[H']} f([H'])$ we have $f([H]) = \sum_{[H]\leq[H']} \mu([H],[H'])g([H'])$.

Definition 3.3.3 ([Kn73])**.** A *G-set* is a finite set with a group action on it. A G-set is called *irreducible* if the group action is transitive, i.e. it only consists of one G-orbit. A *G-map* is a map $\varphi : A \to B$ between two G-sets A and B such that for $a \in A$ we have $\varphi(ga) = g(\varphi(a))$ for all $g \in G$. Two G-sets are *isomorphic*, if there exists a G-map-isomorphism of them.

Definition 3.3.4 (cf. [Kn73])**.** The *Burnside ring* $B(G)$ is the Grothendieck ring of finite G-sets, i.e. it's the abelian group generated by the isomorphism classes of finite G-sets modulo the relation $[A\dot{\cup}B] = [A] + [B]$. The multiplication is given by the cartesian product.

Lemma 3.3.5. *The group $B(G)$ as a free group is generated by the isomorphism classes of irreducible G-sets. This isomorphism classes of irreducible G-sets are in $1:1$-correspondence with conjugacy classes in $\operatorname{Consub} G$. So we can write each element of $B(G)$ in a unique way as*

$$\sum_{[H]\in \operatorname{Consub} G} a_{[H]}[G/H] \quad \text{with } a_{[H]} \in \mathbb{Z}.$$

Proof. It is clear that each G-set is a union of irreducible G-sets. And each G-orbit, so each irreducible G-set, has $|G/H|$-many elements for one subgroup H of G. When $H, K \subset G$ are in the same conjugacy class in $\operatorname{Consub} G$, the action on G/H and G/K is the same, so we can associate to a class $[H] \in \operatorname{Consub} G$ the isomorphism class $[G/H] \in B(G)$, cf. also [Kn73]. □

3.4 Equivariant Euler Characteristic in $B(G)$

Let X be a topological space and G a finite group acting on X.

Definition 3.4.1. For each point $x \in X$ let $G_x = \{g \in G \mid gx = x\}$ be the *isotropy group* of x. Furthermore we define $X^H = \{x \in X \mid gx = x \ \forall g \in H\}$ the fixed point set of the subgroup $H \subset G$ and $X^{(H)} = \{x \in X \mid G_x = H\}$ the set of points with isotropy group H. For a conjugacy class $[H] \in \operatorname{Consub} G$ we set $X^{[H]} = \bigcup_{K\in[H]} X^K$ and $X^{([H])} = \bigcup_{K\in[H]} X^{(K)}$.

Definition 3.4.2 ([EG15])**.** The *equivariant Euler characteristic* $\chi^G(X) \in B(G)$ is defined as

$$\chi^G(X) = \sum_{[H]\in \operatorname{Consub} G} \chi(X^{([H])}/G)[G/H].$$

The *reduced equivariant Euler characteristic* of (X, G) is

$$\overline{\chi}^G(X) = \chi^G(X) - [G/G].$$

Remark 3.4.3 ([EG15])**.** The definition of the equivariant Euler characteristic in $B(G)$ is more general. For example we can see that the natural homomorphism from $B(G)$ to $R(G)$ which sends a G-set A to the vector space of functions on A, also sends the equivariant Euler characteristic $\chi^G(X) \in B(G)$ to the equivariant Euler characteristic $\chi_G(X) \in R(G)$.

Proposition 3.4.4. *We can also write*

$$\chi^G(X) = \sum_{[H]\in \operatorname{Consub} G} \frac{|H|}{|G|} \left(\sum_{[K]\in \operatorname{Consub} G} \mu([H],[K])\chi(X^{[K]}) \right) [G/H].$$

Proof. First observe that by Proposition 3.2.3 $\chi(X^{(H)}/G) = \frac{1}{|G|}\sum_{g\in G}\chi(X^{(H)^g})$. Since $X^{(H)^g} = X^{(H)}$ for $g \in H$ and $X^{(H)^g} = \emptyset$ for $g \notin H$ we have

$$\chi(X^{(H)}/G) = \frac{1}{|G|}\sum_{g\in H}\chi(X^{(H)}) = \frac{|H|}{|G|}\chi(X^{(H)}).$$

On the other hand we have $X^{([H])}$ is the disjoint union of all $X^{(H)}$ for $H \in [H]$ and so we have by the additivity from Remark 2.2.9 also

$$\chi(X^{([H])}/G) = \frac{|H|}{|G|}\chi(X^{([H])}).$$

Then we have

$$X^K = \dot{\bigcup_{K\subset H}} X^{(H)}$$

and when we take the union on both sides we also get

$$X^{[K]} = \dot{\bigcup_{K\subset H}} X^{([H])}.$$

Again by the additivity from Remark 2.2.9 we have

$$\chi(X^{[K]}) = \sum_{[K]\le[H]} \chi(X^{([H])}).$$

So with this and the Moebius inversion formula 3.3.2 we have

$$\begin{aligned}&\sum_{[H]\in \operatorname{Consub} G} \frac{|H|}{|G|} \left(\sum_{[K]\in \operatorname{Consub} G} \mu([H],[K])\chi(X^{[K]}) \right) [G/H] \\ &= \sum_{[H]\in \operatorname{Consub} G} \frac{|H|}{|G|} \left(\chi(X^{([H])})\right) [G/H] = \sum_{[H]\in \operatorname{Consub} G} \chi(X^{([H])}/G)[G/H] = \chi^G(X).\end{aligned}$$

□

3.5 The Higher Order Euler Characteristics

Definition 3.5.1 ([BF98] and cf. also [AS89])**.** Let k be a positive integer. The *k-th order Euler characteristic* of the pair (X, G) is defined as

$$\chi^{(k)}(X, G) = \frac{1}{|G|} \sum_{\substack{\mathbf{g} \in G^k \\ g_i g_j = g_j g_i}} \chi(X^{\langle g_1, g_2, \ldots, g_k \rangle}).$$

The first order Euler characteristic is nothing else but the Euler characteristic of the quotient space X/G. For us the most interesting is the second order Euler characteristic. It is called the *orbifold Euler characteristic* (cf. [DHVW] and [HH90]):

$$\chi^{\mathrm{orb}}(X, G) = \frac{1}{|G|} \sum_{gh=hg} \chi(X^{\langle g, h \rangle}).$$

Definition 3.5.2 ([EG15])**.** We define homomorphisms from $B(G)$ to $\mathbb{Z}$. The natural morphism $|\cdot|$ sends a G-set A to the number of elements $|A|$. We define the maps $r^{(k)}$ as

$$r^{(k)}([G/H]) = \chi^{(k)}([G/H], G) = \frac{1}{|G|} \sum_{\substack{\mathbf{g} \in G^k \\ g_i g_j = g_j g_i}} \left| [G/H]^{\langle g_1, g_2, \ldots, g_k \rangle} \right|.$$

The $r^{(k)}$ are homomorphisms of abelian groups and in general not ring homomorphisms.

Proposition 3.5.3. *We have*

$$\begin{aligned} |\chi^G(X)| &= \chi(X), \\ r^{(k)}(\chi^G(X)) &= \chi^{(k)}(X, G). \end{aligned}$$

Proof. For the first statement we can use the same formula as in Proposition 3.4.4 and its proof:

$$\begin{aligned} |\chi^G(X)| &= \sum_{[H] \in \operatorname{Consub} G} \frac{|H|}{|G|} \left(\sum_{[K] \in \operatorname{Consub} G} \mu([H], [K]) \chi(X^{[K]}) \right) |G/H| \\ &= \sum_{[H] \in \operatorname{Consub} G} \frac{|H|}{|G|} |G/H| \left(\chi(X^{([H])}) \right) \\ &= \sum_{[\{\mathrm{id}\}] \leq [H]} \chi(X^{([H])}) \\ &= \chi(X^{[\{\mathrm{id}\}]}) = \chi(X) \end{aligned}$$

For the second statement again like in the proof of Proposition 3.4.4 we first observe

$$\frac{1}{|G|} \sum_{\mathbf{g} \in G^k} \chi(X^{([H])\langle g_1, g_2, \ldots, g_k \rangle}) = \frac{1}{|G|} \sum_{\mathbf{g} \in H^k} \chi(X^{([H])}),$$

since $X^{(H)^g} = X^{(H)}$ for $g \in H$ and $X^{(H)^g} = \emptyset$ for $g \notin H$. The same we see for

$$\frac{1}{|G|} \sum_{\mathbf{g} \in G^k} \left| [G/H]^{\langle g_1, g_2, \ldots, g_k \rangle} \right| = \frac{1}{|G|} \sum_{\mathbf{g} \in H^k} |G/H|.$$

So we have

$$\begin{aligned}
& r^{(k)}(\chi^G(X)) \\
&= \sum_{[H] \in \operatorname{Consub} G} \frac{|H|}{|G|} \left(\sum_{[K] \in \operatorname{Consub} G} \mu([H],[K]) \chi(X^{[K]}) \right) \frac{1}{|G|} \sum_{\substack{\mathbf{g} \in G^k \\ g_i g_j = g_j g_i}} \left| [G/H]^{\langle g_1, g_2, \ldots, g_k \rangle} \right| \\
&= \sum_{[H] \in \operatorname{Consub} G} \frac{|H|}{|G|} \left(\chi(X^{([H])}) \right) \frac{1}{|G|} \sum_{\mathbf{g} \in H^k} |G/H| \\
&= \sum_{[\{\mathrm{id}\}] \leq [H]} \frac{1}{|G|} \sum_{\mathbf{g} \in H^k} \chi(X^{([H])}) \\
&= \sum_{[\{\mathrm{id}\}] \leq [H]} \frac{1}{|G|} \sum_{\mathbf{g} \in G^k} \chi(X^{([H])^{\langle g_1, g_2, \ldots, g_k \rangle}}) \\
&= \frac{1}{|G|} \sum_{\mathbf{g} \in G^k} \sum_{[\{\mathrm{id}\}] \leq [H]} \chi(X^{([H])^{\langle g_1, g_2, \ldots, g_k \rangle}}) \\
&= \frac{1}{|G|} \sum_{\mathbf{g} \in G^k} \chi(X^{[\{\mathrm{id}\}]^{\langle g_1, g_2, \ldots, g_k \rangle}}) \\
&= \chi^{(k)}(X, G).
\end{aligned}$$

□

Remark 3.5.4. We are able to write down also other numbers in an equivariant way. This we will do in the next chapter for the Milnor number.

4 Isolated Singularities with Group Action

4.1 About the Group Action

Let $f \in \mathbb{C}[x_1, \dots, x_n]$ be again a polynomial with isolated singularity at $\mathbf{0}$.

Definition 4.1.1. Let G be a finite group acting linearly on $\mathbb{C}^n$ which leaves f invariant. So we have for each $g \in G$ and $\mathbf{x} \in \mathbb{C}^n$

$$f(g\mathbf{x}) = f(\mathbf{x}).$$

Since G acts linearly we can identify G with a subgroup of $\mathrm{GL}(n, \mathbb{C})$.

Example 4.1.2.
- The group of maximal diagonal symmetries of f is defined as

$$G_f := \{(\lambda_1, \dots, \lambda_n) \in (\mathbb{C}^*)^n \mid f(\lambda_1 x_1, \dots, \lambda_n x_n) = f(x_1, \dots, x_n)\}.$$

- For $f = x_1^3 + x_2^3 + x_3^3$ we have

$$G_f = \left\langle (\mathbf{e}[\frac{1}{3}], 1, 1), (1, \mathbf{e}[\frac{1}{3}], 1), (1, 1, \mathbf{e}[\frac{1}{3}]) \right\rangle.$$

  Here we can also take the group $G = S_3$ permuting the coordinates.

- For $f = x_1^3 + x_2^3 x_3 + x_3^3$ we have

$$G_f = \left\langle (\mathbf{e}[\frac{1}{3}], 1, 1), (1, \mathbf{e}[\frac{1}{3}], 1) \right\rangle.$$

Definition 4.1.3. For each $g \in G$ we define the fixed locus $\mathrm{Fix}(g) := \{\mathbf{x} \in \mathbb{C}^n \mid g\mathbf{x} = \mathbf{x}\}$. G acts linearly on $\mathbb{C}^n$ so $Fix(g)$ is a linear subspace. We write $n_g = \dim \mathrm{Fix}(g)$ for its dimension and $f^g := f|_{\mathrm{Fix}(g)}$ for the restriction of f to the fixed locus of g.

Example 4.1.4.
(i) Let us consider the pair (f, G) with $f = x_1^3 + x_2^3 x_3 + x_3^3$ and the group $G = \langle g \rangle = \{\mathrm{id}, g, g^{-1}\}$ generated by one element $g = (\mathbf{e}[\frac{1}{3}], \mathbf{e}[\frac{2}{3}], 1)$. Here G is a subgroup of $\mathrm{SL}(n, \mathbb{C})$. We have $n_g = 1$ since only the third coordinate is fixed by g, and $f^g = x_3^3$.

(ii) Secondly we consider (f, G) with $f = x_1^3 + x_2^3 + x_3^3$ and the group $G = S_3 = \{\mathrm{id}, (12), (13), (23), (123), (132)\}$. We see that the fixed locus of each 2-cycle is 2 dimensional, so $n_{(12)} = 2$ since $\mathrm{Fix}((12)) = \langle (1, 1, 0), (0, 0, 1) \rangle$ and the fixed locus of each 3-cycle is 1 dimensional, so $n_{(123)} = 1$, since $\mathrm{Fix}((123)) = \langle (1, 1, 1) \rangle$. To get $f^{(12)}$ we have to think of another basis of $\mathbb{C}^3$. Let us take $\{(1, 1, 0), (1, -1, 0), (0, 0, 1)\}$. We associate the variables y_1, y_2, y_3 respectively. So we have $x_1 = \frac{1}{2}(y_1+y_2)$, $x_2 = \frac{1}{2}(y_1-y_2)$, $x_3 = y_3$. So $f = \frac{2}{8}y_1^3 + \frac{6}{8}y_1y_2^2 + y_3^3$ and so $f^{(12)} = \frac{2}{8}y_1^3 + y_3^3$ since only y_2 is not (12)-invariant. Similar we do it for the other 2-cycles. For the 3-cycles we can take a basis $\{(1,1,1),(1,\mathbf{e}[\frac{1}{3}],\mathbf{e}[\frac{2}{3}]),(1,\mathbf{e}[\frac{2}{3}],\mathbf{e}[\frac{1}{3}])\}$ of $\mathbb{C}^3$ and get e.g. $x_1 = \frac{1}{3}(y_1+y_2+y_3)$ and since only y_1 is (123)-invariant, we get $f^{(123)} = \frac{3}{27}y_1^3$.

Proposition 4.1.5 (cf. [ET13b, Prop. 5])**.** *For each $g \in G$ the restriction f^g has an isolated singularity at $\mathbf{0}$. There exists a surjective $\mathbb{C}$-algebra homomorphism $\mathrm{Jac}(f) \to \mathrm{Jac}(f^g)$. This means in particular that also the Jacobian algebra $\mathrm{Jac}(f^g)$ is finite dimensional.*

Proof. We may assume that $\mathrm{Fix}(g) = \{\mathbf{x} \in \mathbb{C}^n \mid x_{n_g+1} = \cdots = x_n = 0\}$ by a suitable coordinate transformation. Since f is invariant under G, $g \cdot x_i \neq x_i$ for $i = n_g+1, \dots, n$ and $\frac{\partial f}{\partial x_{n_g+1}}, \dots, \frac{\partial f}{\partial x_n}$ form a regular sequence, we have

$$\left(\frac{\partial f}{\partial x_{n_g+1}}, \dots, \frac{\partial f}{\partial x_n}\right) \subset \left(x_{n_g+1}, \dots, x_n\right).$$

Therefore, we have a natural surjective $\mathbb{C}$-algebra homomorphism

$$\begin{aligned} \mathrm{Jac}(f) \quad &= \quad \mathbb{C}[x_1,\dots,x_n]\Big/\left(\tfrac{\partial f}{\partial x_1},\dots,\tfrac{\partial f}{\partial x_n}\right) \\ &\longrightarrow \quad \mathbb{C}[x_1,\dots,x_n]\Big/\left(\tfrac{\partial f}{\partial x_1},\dots,\tfrac{\partial f}{\partial x_{n_g}}, x_{n_g+1},\dots,x_n\right) \\ &= \quad \mathbb{C}[x_1,\dots,x_{n_g}]\Big/\left(\tfrac{\partial f^g}{\partial x_1},\dots,\tfrac{\partial f^g}{\partial x_{n_g}}\right) = \mathrm{Jac}(f^g). \end{aligned}$$

□

Corollary 4.1.6. *For each $g \in G$, Ω_{f^g} is naturally equipped with the structure of a $\mathrm{Jac}(f)$-module.*

Proof. Since Ω_{f^g} is a free $\mathrm{Jac}(f^g)$-module of rank one (cf. (2.1)), the surjective $\mathbb{C}$-algebra homomorphism $\mathrm{Jac}(f) \longrightarrow \mathrm{Jac}(f^g)$ yields the statement. □

Remark 4.1.7. Each $g \in G$ is a bi-regular map on $\mathbb{C}^n$ and so acts also on Ω_f by the pullback g^* of differential forms. With this Ω_f is in a natural sense a $\mathbb{C}G$-module.

4.2 Equivariant Milnor Number

Definition 4.2.1 ([Wa80])**.** Let us consider $M = H^{n-1}(\bar{X}_w, \mathbb{C})$ as $\mathbb{C}G$-module. M is called the *equivariant Milnor number* in $R(G)$. Then we have like in Theorem 2.2.17

$$\chi_G(\bar{X}_w) = \mathbb{C} + (-1)^{n-1}M.$$

Theorem 4.2.2 ([Wa80, Thm. 1])**.** *$H^{n-1}(\bar{X}_w, \mathbb{C})$ and Ω_f are isomorphic as $\mathbb{C}G$-modules.*

Remark 4.2.3. So Ω_f as an element of $R(G)$ is the equivariant Milnor number $M \in R(G)$.

Definition 4.2.4. We define the *equivariant Milnor number* in $B(G)$ as

$$\mu_f^G = (-1)^{n-1}\overline{\chi}^G(\bar{X}_w).$$

So we have also defined the higher order Milnor numbers:

$$\mu_{f,G}^{(k)} = r^{(k)}(\mu_f^G)$$

and we call

$$\mu_{f,G}^{\mathrm{orb}} = \mu_{f,G}^{(2)}$$

the *orbifold Milnor number.*

Proposition 4.2.5. *We have:*

$$\begin{aligned}
|\mu_f^G| &= \mu_f \\
\mu_{f/G} := r^{(1)}(\mu_f^G) &= \frac{1}{|G|}\sum_{g\in G}(-1)^{n-n_g}\mu_{f^g} \\
\mu_{f,G}^{orb} &= \frac{1}{|G|}\sum_{gh=hg}(-1)^{n-n_{<g,h>}}\mu_{f^{<g,h>}} \\
\mu_{f,G}^{(k)} &= \frac{1}{|G|}\sum_{\substack{\mathbf{g}\in G^k \\ g_ig_j=g_jg_i}}(-1)^{n-n_{<\mathbf{g}>}}\mu_{f^{<\mathbf{g}>}}
\end{aligned}$$

Proof. With Proposition 3.5.3 and Theorem 2.2.17 we get

$$\begin{aligned}
|\mu_f^G| &= \left|(-1)^{n-1}\overline{\chi}^G(\bar{X}_w)\right| \\
&= (-1)^{n-1}\left|\chi^G(\bar{X}_w) - [G/G]\right| \\
&= (-1)^{n-1}(\chi(\bar{X}_w) - 1) \\
&= \mu_f.
\end{aligned}$$

Then observe by Theorem 2.2.17 for $\mathbf{g}\in G^k$ that $\chi(\bar{X}_w^{<\mathbf{g}>}) = 1 + (-1)^{n_{<\mathbf{g}>}-1}\mu_{f^{<\mathbf{g}>}}$ since the $<\mathbf{g}>$-invariant subspace of the Milnor fibre $\bar{X}_w$ of f is the Milnor fibre of $f^{<\mathbf{g}>} = f|_{\mathrm{Fix}(<\mathbf{g}>)}$.

Then we have with Proposition 3.5.3

$$\begin{aligned}
\mu_{f,G}^{(k)} &= r^{(k)}(\mu_f^G) \\
&= r^{(k)}\left((-1)^{n-1}\overline{\chi}^G(\bar{X}_w)\right) \\
&= (-1)^{n-1} r^{(k)}\left(\chi^G(\bar{X}_w) - [G/G]\right) \\
&= (-1)^{n-1}\left(\chi^{(k)}(\bar{X}_w, G) - r^{(k)}([G/G])\right) \\
&= (-1)^{n-1}\frac{1}{|G|}\sum_{\substack{\mathbf{g}\in G^k\\ g_i g_j = g_j g_i}} (\chi(\bar{X}_w^{<\mathbf{g}>}) - 1) \\
&= \frac{1}{|G|}\sum_{\substack{\mathbf{g}\in G^k\\ g_i g_j = g_j g_i}} (-1)^{n-n_{<\mathbf{g}>}}(-1)^{n_{<\mathbf{g}>}-1}(\chi(\bar{X}_w^{<\mathbf{g}>}) - 1) \\
&= \frac{1}{|G|}\sum_{\substack{\mathbf{g}\in G^k\\ g_i g_j = g_j g_i}} (-1)^{n-n_{<\mathbf{g}>}}\mu_{f^{<\mathbf{g}>}}.
\end{aligned}$$

This is true for all $k = 1, 2, \ldots$. □

Example 4.2.6. Let (f, G) be as in Example 4.1.4.

(i) Set $f = x_1^3 + x_2^3 x_3 + x_3^3$ and $G = \left\langle(\mathbf{e}[\frac{1}{3}], \mathbf{e}[\frac{2}{3}], 1)\right\rangle$. We have seen $f^g = x_3^3$ and so $\mu_{f^g} = 2$. Since $\mathrm{Fix}(g) = \mathrm{Fix}(g^{-1})$ we also have $\mu_{f^{g^{-1}}} = 2$ and we can calculate

$$\begin{aligned}
\mu_f &= 14, && \text{see Example 2.1.3,} \\
\mu_{f/G} &= \frac{1}{3}(14+2+2) = 6, && n - n_g = 3 - 1 \equiv 0 \mod 2, \\
\mu_{f,G}^{\mathrm{orb}} &= \frac{1}{3}((14+2+2)+(2+2+2)+(2+2+2)) = 10, && \text{since } G \text{ is abelian.}
\end{aligned}$$

(ii) Set $f = x_1^3 + x_2^3 + x_3^3$ and $G = S_3$. We have seen $f^{(12)} = \frac{2}{8}y_1^3 + y_3^3$ so $\mu_{(12)} = 4$ and similar $\mu_{(13)} = 4$ and $\mu_{(23)} = 4$. For the 3-cycles we have $\mu_{(\bullet\bullet\bullet)} = 2$. We can calculate

$$\begin{aligned}
\mu_f &= 8, \qquad \text{see Example 2.1.3,} \\
\mu_{f/G} &= \frac{1}{6}(8-4-4-4+2+2) = 0, \\
&\qquad n - n_{(\bullet\bullet)} = 3-2 \equiv 1 \mod 2, \quad n - n_{(\bullet\bullet\bullet)} = 3-1 \equiv 0 \mod 2, \\
\mu_{f,G}^{\mathrm{orb}} &= \frac{1}{6}(\underbrace{(8-4-4-4+2+2)}_{gg} + \underbrace{2(-4-4-4+2+2)}_{g\mathrm{id}\text{ and }\mathrm{id}g} + \underbrace{(2+2)}_{(\bullet\bullet\bullet)(\bullet\bullet\bullet)}) = -2.
\end{aligned}$$

4.3 Orbifold Version of Ω_f

Definition 4.3.1. We define a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}$-module $\Omega'_{f,G} = \left(\Omega'_{f,G}\right)_{\overline{0}} \oplus \left(\Omega'_{f,G}\right)_{\overline{1}}$ by

$$\left(\Omega'_{f,G}\right)_{\overline{0}} := \bigoplus_{\substack{g\in G \\ n-n_g\equiv 0 \pmod 2}} \Omega'_{f,g}, \quad \left(\Omega'_{f,G}\right)_{\overline{1}} := \bigoplus_{\substack{g\in G \\ n-n_g\equiv 1 \pmod 2}} \Omega'_{f,g},$$

where $\Omega'_{f,g} := \Omega_{f^g}$.

Remark 4.3.2. Each $g \in G$ is a bi-regular map on $\mathbb{C}^n$ and maps $\mathrm{Fix}(g^{-1}hg)$ to $\mathrm{Fix}(h)$ for each $h \in G$. So G acts naturally on $\Omega'_{f,G}$ by

$$\Omega'_{f,h} \longrightarrow \Omega'_{f,g^{-1}hg}, \quad \omega \mapsto g^*|_{\mathrm{Fix}(g)}\omega,$$

where $g^*|_{\mathrm{Fix}(g)}$ denotes the restriction of the pullback g^* of differential forms to $\mathrm{Fix}(g)$. In order to simplify the notation, for each $g \in G$, we shall denote by g^* the action of g on $\Omega'_{f,G}$.

Definition 4.3.3. Define a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}$-module $\Omega_{f,G}$ as the G-invariant part of $\Omega'_{f,G}$,

$$\Omega_{f,G} = \left(\Omega'_{f,G}\right)^G.$$

Of course we have $\Omega_{f,G} = (\Omega_{f,G})_{\overline{0}} \oplus (\Omega_{f,G})_{\overline{1}}$ where

$$(\Omega_{f,G})_{\overline{0}} := \left(\left(\Omega'_{f,G}\right)_{\overline{0}}\right)^G, \quad (\Omega_{f,G})_{\overline{1}} := \left(\left(\Omega'_{f,G}\right)_{\overline{1}}\right)^G,$$

since the dimension of $\mathrm{Fix}(g)$ is the same for all g in one conjugacy class.

Example 4.3.4. Let (f, G) be as in Example 4.1.4. We calculate $\Omega'_{f,G}$ and $\Omega_{f,G}$ (cf. also Example 2.3.3)

(i) Set $f = x_1^3 + x_2^3x_3 + x_3^3$ and $G = \left\langle (\mathbf{e}[\frac{1}{3}], \mathbf{e}[\frac{2}{3}], 1)\right\rangle$.

$$\begin{aligned}\Omega'_{f,G} = &\left\langle dx_1 \wedge dx_2 \wedge dx_3, x_1dx_1 \wedge dx_2 \wedge dx_3, \ldots, x_1x_2x_3^2dx_1 \wedge dx_2 \wedge dx_3\right\rangle \\ &\oplus \langle dx_3, x_3dx_3\rangle \oplus \langle dx_3, x_3dx_3\rangle\end{aligned}$$

and since G is abelian

$$\begin{aligned}\Omega_{f,G} = &\left\langle dx_1 \wedge dx_2 \wedge dx_3, x_3dx_1 \wedge dx_2 \wedge dx_3, x_3^2dx_1 \wedge dx_2 \wedge dx_3,\right. \\ &\left. x_1x_2dx_1 \wedge dx_2 \wedge dx_3, x_1x_2x_3dx_1 \wedge dx_2 \wedge dx_3, x_1x_2x_3^2dx_1 \wedge dx_2 \wedge dx_3\right\rangle \\ &\oplus \langle dx_3, x_3dx_3\rangle \oplus \langle dx_3, x_3dx_3\rangle.\end{aligned}$$

(ii) Set $f = x_1^3 + x_2^3 + x_3^3$ and $G = S_3$.

$$\begin{aligned}
\Omega'_{f,G} = &\langle dx_1 \wedge dx_2 \wedge dx_3, x_1 dx_1 \wedge dx_2 \wedge dx_3, \ldots, x_1x_2x_3 dx_1 \wedge dx_2 \wedge dx_3\rangle_{g=\mathrm{id}} \\
&\oplus \langle (dx_1 \wedge dx_3 + dx_2 \wedge dx_3), (x_1 + x_2)(dx_1 \wedge dx_3 + dx_2 \wedge dx_3), \\
&\quad x_3(dx_1 \wedge dx_3 + dx_2 \wedge dx_3), (x_1x_3 + x_2x_3)(dx_1 \wedge dx_3 + dx_2 \wedge dx_3)\rangle_{g=(12)} \\
&\oplus \langle (dx_1 \wedge dx_2 + dx_3 \wedge dx_2), (x_1 + x_3)(dx_1 \wedge dx_2 + dx_3 \wedge dx_2), \\
&\quad x_2(dx_1 \wedge dx_2 + dx_3 \wedge dx_2), (x_1x_2 + x_2x_3)(dx_1 \wedge dx_2 + dx_3 \wedge dx_2)\rangle_{g=(13)} \\
&\oplus \langle (dx_2 \wedge dx_1 + dx_3 \wedge dx_1), (x_2 + x_3)(dx_2 \wedge dx_1 + dx_3 \wedge dx_1), \\
&\quad x_1(dx_2 \wedge dx_1 + dx_3 \wedge dx_1), (x_1x_2 + x_1x_3)(dx_2 \wedge dx_1 + dx_3 \wedge dx_1)\rangle_{g=(23)} \\
&\oplus \langle (dx_1 + dx_2 + dx_3), (x_1 + x_2 + x_3)(dx_1 + dx_2 + dx_3)\rangle_{g=(123)} \\
&\oplus \langle (dx_1 + dx_2 + dx_3), (x_1 + x_2 + x_3)(dx_1 + dx_2 + dx_3)\rangle_{g=(132)}
\end{aligned}$$

and since here G is not abelian, we get sums for every conjugacy class of elements in G:

$$\begin{aligned}
\Omega_{f,G} = &\{0\} \\
&\oplus \langle (dx_1 \wedge dx_3 + dx_2 \wedge dx_3) + (dx_1 \wedge dx_2 + dx_3 \wedge dx_2) + (dx_2 \wedge dx_1 + dx_3 \wedge dx_1), \\
&\quad (x_1 + x_2)(dx_1 \wedge dx_3 + dx_2 \wedge dx_3) + \cdots + (x_2 + x_3)(dx_2 \wedge dx_1 + dx_3 \wedge dx_1), \\
&\quad x_3(dx_1 \wedge dx_3 + dx_2 \wedge dx_3) + x_2(dx_1 \wedge dx_2 + dx_3 \wedge dx_2) + x_1(dx_2 \wedge dx_1 + dx_3 \wedge dx_1), \\
&\quad (x_1x_3 + x_2x_3)(dx_1 \wedge dx_3 + dx_2 \wedge dx_3) + \cdots + (x_1x_2 + x_1x_3)(dx_2 \wedge dx_1 + dx_3 \wedge dx_1)\rangle \\
&\oplus \langle (dx_1 + dx_2 + dx_3) + (dx_1 + dx_2 + dx_3), \\
&\quad (x_1 + x_2 + x_3)(dx_1 + dx_2 + dx_3) + (x_1 + x_2 + x_3)(dx_1 + dx_2 + dx_3)\rangle .
\end{aligned}$$

4.4 Dimensions and Milnor Numbers

Remark 4.4.1. As we had in the section about the Milnor number, we have:

$$|\mu_f^G| = \mu_f = \dim \Omega_f$$

Proposition 4.4.2. *We have*

$$r^{(1)}(\mu_f^G) = \mu_{f/G} = \dim(\Omega_f)^G$$

is the dimension of the G-invariant part of Ω_f.

Proof. As in the proof of Proposition 3.2.3 the multiplicity of the trivial representation in $M = \Omega_f \in R(G)$ is

$$\frac{1}{|G|} \sum_{g \in G} (-1)^{n-n_g} \mu_{f^g} = \mu_{f/G}$$

and that is directly the dimension of the G-invariant part of $M = \Omega_f$. □

Example 4.4.3. Let (f, G) be as in Example 4.1.4.

(i) Set $f = x_1^3 + x_2^3 x_3 + x_3^3$ and $G = \left\langle (\mathbf{e}[\frac{1}{3}], \mathbf{e}[\frac{2}{3}], 1) \right\rangle$. As we have seen in Example 4.3.4

$$\dim \Omega_f^G = 6 \text{ and } \dim \Omega_{f,G} = 10,$$

which are $\mu_{f/G}$ and $\mu_{f,G}^{\mathrm{orb}}$ respectively, see Example 4.2.6. Here we have $\Omega_{f,G_{\bar{1}}} = \{0\}$ since $n - n_g \equiv 0$ for all $g \in G$.

(ii) Set $f = x_1^3 + x_2^3 + x_3^3$ and $G = S_3$. As we have seen in Example 4.3.4

$$\dim \Omega_f^G = 0 \text{ and } \dim \Omega_{f,G} = 4 + 2 = 6,$$

which is $\mu_{f/G}$, see Example 4.2.6. But for $\Omega_{f,G}$ we see

$$\dim (\Omega_{f,G})_{\bar{0}} = 2 \text{ and } \dim (\Omega_{f,G})_{\bar{1}} = 4$$

and then we have $\mu_{f,G}^{\mathrm{orb}} = -2 = 2 - 4$.

In general we have the following theorem.

Theorem 4.4.4. *We have*

$$\mu_{f,G}^{orb} = \dim (\Omega_{f,G})_{\bar{0}} - \dim (\Omega_{f,G})_{\bar{1}} .$$

We will first prove a restriction of this theorem.

Proposition 4.4.5. *Let G be abelian, then Theorem 4.4.4 holds.*

Proof. If G is abelian, each $h \in G$ acts on $\Omega_{f,g}$ for each $g \in G$ and we have

$$(\Omega_{f,G})_{\bar{0}} := \bigoplus_{\substack{g \in G \\ n - n_g \equiv 0 \pmod 2}} (\Omega'_{f,g})^G, \quad (\Omega_{f,G})_{\bar{1}} := \bigoplus_{\substack{g \in G \\ n - n_g \equiv 1 \pmod 2}} (\Omega'_{f,g})^G,$$

since we always get $h^{-1} g h = g$. So we have

$$\begin{aligned}
\dim (\Omega_{f,G})_{\bar{0}} - \dim (\Omega_{f,G})_{\bar{1}} &= \sum_{\substack{g \in G \\ n - n_g \equiv 0 \pmod 2}} \dim(\Omega'_{f,g})^G - \sum_{\substack{g \in G \\ n - n_g \equiv 1 \pmod 2}} \dim(\Omega'_{f,g})^G \\
{\scriptstyle \text{see Proposition 4.4.2}} &= \sum_{\substack{g \in G \\ n - n_g \equiv 0 \pmod 2}} \mu_{f^g/G} - \sum_{\substack{g \in G \\ n - n_g \equiv 1 \pmod 2}} \mu_{f^g/G} \\
&= \sum_{g \in G} (-1)^{n - n_g} \mu_{f^g/G}.
\end{aligned}$$

On the other hand we have, since $gh = hg$ for all $g, h \in G$:

$$\begin{aligned}
\mu_{f,G}^{\mathrm{orb}} &= \frac{1}{|G|} \sum_{gh = hg} (-1)^{n - n_{<g,h>}} \mu_{f^{<g,h>}} \\
&= \frac{1}{|G|} \sum_{g \in G} \sum_{h \in G} (-1)^{n - n_g + n_g - n_{<g,h>}} \mu_{f^{<g,h>}} \\
&= \sum_{g \in G} (-1)^{n - n_g} \frac{1}{|G|} \sum_{h \in G} (-1)^{n_g - n_{<g,h>}} \mu_{(f^g)^h} \\
&= \sum_{g \in G} (-1)^{n - n_g} \mu_{f^g/G}.
\end{aligned}$$

□

Now we prepare for the proof.

Definition 4.4.6. For $g \in G$ let $C(g) = \{k \in G | gk = kg\}$ be the centralizer of g and $C(g) =: \{k_1, \ldots, k_{|C(g)|}\}$. Let $[g] = \{h^{-1}gh | h \in G\}$ be the conjugacy class of g. Let $\{h_1, \ldots, h_{|[g]|}\}$ be a set, such that $[g] = \{h_1^{-1}gh_1, \ldots, h_{|[g]|}^{-1}gh_{|[g]|}\}$.

We now prove a well known fact in group theory:

Lemma 4.4.7. *There is a* $1:1$*-correspondence between* $[g]$ *and* $C(g)\backslash G$*. So we have*

$$|[g]| \cdot |C(g)| = |G|$$

and

$$\{k_j h_i \mid i = 1, \ldots, |[g]| \ ; \ j = 1, \ldots, |C(g)|\} = G.$$

Proof. We take $C(g)\backslash G = \{C(g)h \mid h \in G\}$. The map

$$[g] \to C(g)\backslash G \quad h^{-1}gh \mapsto C(g)h$$

is well defined and bijective. For $h^{-1}gh = k^{-1}gk$ we have $kh^{-1} \in C(g)$ and so we have $C(g)h = C(g)kh^{-1}h = C(g)k \in C(g)\backslash G$ and vice versa for $C(g)h = C(g)k$ we have $k = \tilde{g}h$ with $\tilde{g} \in C(g)$ and so $k^{-1}gk = (\tilde{g}h)^{-1}g(\tilde{g}h) = h^{-1}(\tilde{g}^{-1}g\tilde{g})h = h^{-1}gh$. □

Lemma 4.4.8 (cf. also [HH90])**.** *We have*

$$\mu_{f,G}^{orb} = \sum_{[g]} (-1)^{n-n_g} \mu_{f^g/C(g)},$$

where we sum over all different conjugacy classes in G.

Proof. Since Ω_{f^g} and Ω_{f^h} are isomorphic for g and h in the same conjugacy class, we also have $\mu_{f^g} = \mu_{f^h}$. So we have

$$\begin{aligned}
\mu_{f,G}^{\mathrm{orb}} &= \frac{1}{|G|} \sum_{gh=hg} (-1)^{n-n_{<g,h>}} \mu_{f^{<g,h>}} \\
&= \sum_{g \in G} \frac{1}{|G|} \sum_{h \in C(g)} (-1)^{n-n_g+n_g-n_{<g,h>}} \mu_{f^{<g,h>}} \\
&= \sum_{[g]} (-1)^{n-n_g} |[g]| \frac{1}{|G|} \sum_{h \in C(g)} (-1)^{n_g-n_{<g,h>}} \mu_{f^{<g,h>}} \\
&= \sum_{[g]} (-1)^{n-n_g} \frac{1}{|C(g)|} \sum_{h \in C(g)} (-1)^{n_g-n_{<g,h>}} \mu_{f^{<g,h>}} \\
&= \sum_{[g]} (-1)^{n-n_g} \mu_{f^g/C(g)}.
\end{aligned}$$

□

Lemma 4.4.9. *We have*

$$(\Omega_{f,G})_{\overline{0}} = \bigoplus_{\substack{[g] \\ n-n_g \equiv 0 \pmod 2}} \left(\bigoplus_{h \in [g]} \Omega'_{f,h} \right)^G, \quad (\Omega_{f,G})_{\overline{1}} = \bigoplus_{\substack{[g] \\ n-n_g \equiv 1 \pmod 2}} \left(\bigoplus_{h \in [g]} \Omega'_{f,h} \right)^G.$$

Proof. Of course we can write

$$(\Omega_{f,G})_{\overline{0}} = \left(\bigoplus_{\substack{[g] \\ n-n_g \equiv 0 \pmod 2}} \bigoplus_{h \in [g]} \Omega'_{f,h} \right)^G, \quad (\Omega_{f,G})_{\overline{1}} = \left(\bigoplus_{\substack{[g] \\ n-n_g \equiv 1 \pmod 2}} \bigoplus_{h \in [g]} \Omega'_{f,h} \right)^G,$$

since the action of $g \in G$ goes from $\Omega'_{f,h}$ to $\Omega'_{f,g^{-1}hg}$ we need to take the invariance only over the sum in one conjugacy class. $\square$

Lemma 4.4.10. *For Ω_{f^g} there exists a basis $\{v_g^1, \ldots, v_g^{\mu_{f/C(g)}}, \ldots, v_g^{\mu_{f^g}}\}$ such that $\{v_g^1, \ldots, v_g^{\mu_{f/C(g)}}\}$ is a basis of $\Omega_{f^g}^{C(g)}$.*

Proof. We can take a basis of the subspace $\Omega_{f^g}^{C(g)}$ and can extend it to a basis of Ω_{f^g}. So the statement is clear. $\square$

Lemma 4.4.11. *Let $\{v_g^1, \ldots, v_g^{\mu_{f^g}}\}$ be a basis of Ω_{f^g} as in Lemma 4.4.10. For $h \in G$ set $v^i_{h^{-1}gh} := h^*(v_g^i)$. Then $\{v^1_{h^{-1}gh}, \ldots, v^{\mu_{f^g}}_{h^{-1}gh}\}$ is a basis of $\Omega_{f^{h^{-1}gh}}$ as in Lemma 4.4.10.*

Proof. Since h induces an isomorphism from Ω_{f^g} to $\Omega_{f^{h^{-1}gh}}$, it is clear that it is a basis. We have $C(h^{-1}gh) = h^{-1}C(g)h$, so for each $k \in C(h^{-1}gh)$ we have $k = h^{-1}\tilde{k}h$ for $\tilde{k} \in C(g)$. So the basis has the property of Lemma 4.4.10:

$$\begin{aligned}
k^*(v^i_{h^{-1}gh}) &= (h^{-1}\tilde{k}h)^*(h^*(v_g^i)) \\
{\scriptstyle \text{pullback}} &= (hh^{-1}\tilde{k}h)^*(v_g^i) \\
&= h^*(\tilde{k}^*(v_g^i)) \\
&= \begin{cases} h^*(v_g^i) = v^i_{h^{-1}gh} & i \leq \mu_{f^g/C(g)} \\ h^*(k^*(v_g^i)) \neq h^*(v_g^i) & i > \mu_{f^g/C(g)} \end{cases}
\end{aligned}$$

$\square$

Lemma 4.4.12. *For each conjugacy class $[g]$ of G we have*

$$\dim \left(\bigoplus_{h \in [g]} \Omega'_{f,h} \right)^G = \mu_{f^g/C(g)}.$$

Proof. Let $\{h_1, \dots, h_{|[g]|}\}$ be as in Definition 4.4.6. We set $v^i_{h_j^{-1}gh_j} := h_j^*(v^i_g)$. Then

$$\left\{ v^1_{h_1^{-1}gh_1}, \dots, v^{\mu_{f^g}}_{h_1^{-1}gh_1}, v^1_{h_2^{-1}gh_2}, \dots, \dots, v^{\mu_{f^g}}_{h_{|[g]|}^{-1}gh_{|[g]|}} \right\}$$

is a basis of $\left(\bigoplus_{h\in[g]} \Omega'_{f,h}\right)$. Since each $v^i_{g'}$ for $i > \mu_{f^g/G}$ is not fixed by $h \in C(g')$, it is not possible to be fixed by G. So we only concentrate on $i \leq \mu_{f^g/G}$. Let $h \in G$. From Lemma 4.4.7 we know $h = k_l h_j$ for h_j as above and $k_l \in C(g)$ so we have

$$h^*(v^i_g) = (k_l h_j)^*(v^i_g) = h_j^*(k_l^*(v^i_g)) = h_j^*(v^i_g) = v^i_{h_j^{-1}gh_j}, \quad i \leq \mu_{f^g/G}.$$

And for each $m = 1, \dots, |[g]|$ we also have $h_m h \in G$ and we can again write $h_m h = k_l h_j$ from Lemma 4.4.7. So we have for $i \leq \mu_{f^g/G}$

$$h^*(v^j_{h_m^{-1}gh_m}) = h^*(h_m^*(v^i_g)) = (h_m h)^*(v^i_g) = (k_l h_j)^*(v^i_g) = h_j^*(k_l^*(v^i_g)) = h_j^*(v^i_g) = v^i_{h_j^{-1}gh_j}.$$

So each $h \in G$ sends each $v^i_\bullet$ for $i \leq \mu_{f^g/G}$ also to a $v^i_\bullet$. And since each h_m for $m = 1, \dots, |[g]|$ sends v^i_g to $v^i_{h_m^{-1}gh_m}$ only the whole sum $v^j_{h_1^{-1}gh_1} + \dots + v^j_{h_{|[g]|}^{-1}gh_{|[g]|}}$ can be invariant by all $h \in G$. So

$$\left\{ v^1_{h_1^{-1}gh_1} + \dots + v^1_{h_{|[g]|}^{-1}gh_{|[g]|}}, \dots, v^{\mu_{f^g/G}}_{h_1^{-1}gh_1} + \dots + v^{\mu_{f^g/G}}_{h_{|[g]|}^{-1}gh_{|[g]|}} \right\}$$

is a basis of the invariant part. So the dimension is as given. □

Proof of Theorem 4.4.4. As shown before we have

$$\begin{aligned}
&\dim (\Omega_{f,G})_{\overline{0}} - \dim (\Omega_{f,G})_{\overline{1}} \\
&\qquad = \sum_{\substack{[g] \\ n-n_g \equiv 0 \pmod 2}} \dim \left(\bigoplus_{h\in[g]} \Omega'_{f,h} \right)^G - \sum_{\substack{[g] \\ n-n_g \equiv 1 \pmod 2}} \dim \left(\bigoplus_{h\in[g]} \Omega'_{f,g} \right)^G \\
&\qquad = \sum_{\substack{[g] \\ n-n_g \equiv 0 \pmod 2}} \mu_{f^g/C(g)} - \sum_{\substack{[g] \\ n-n_g \equiv 1 \pmod 2}} \mu_{f^g/C(g)} \\
&\qquad = \sum_{[g]} (-1)^{n-n_g} \mu_{f^g/C(g)} \\
&\qquad = \mu^{\text{orb}}_{f,G}.
\end{aligned}$$

□

4.5 Orbifold Residue Pairing

Now we can also define a bilinear form on $\Omega'_{f,G}$ and on $\Omega_{f,G}$.

Definition 4.5.1. Since the finite group G acts linearly on $\mathbb{C}^n$ we can diagonalize each $g \in G \subset \mathrm{GL}(n, \mathbb{C})$. So each $g \in G$ is up to order uniquely isomorphic to

$$g \cong \mathrm{diag}(\mathbf{e}[\frac{a_1}{r}], \ldots, \mathbf{e}[\frac{a_n}{r}]), \quad 0 \le a_i < r,$$

where r is the order of g.

The *age* of g is defined (cf. [IR96]) as the rational number

$$\mathrm{age}(g) = \frac{1}{r} \sum_{i=1}^{n} a_i.$$

For $g \in \mathrm{SL}(n, \mathbb{C})$ we have $\mathrm{age}(g) \in \mathbb{Z}$.

Example 4.5.2. (a) If $G \subset G_f$, all g are automatically diagonal and $g = \mathrm{diag}(\mathbf{e}[\frac{a_1}{r}], \ldots, \mathbf{e}[\frac{a_n}{r}])$ is given directly and uniquely.

(b) The identity $\mathrm{id} \in G$ is of the form $\mathrm{id} = \mathrm{diag}(1, \ldots, 1) = \mathrm{diag}(\mathbf{e}[0], \ldots, \mathbf{e}[0])$. So

$$\mathrm{age}(\mathrm{id}) = 0.$$

(c) For g and g^{-1} the diagonalization can be chosen in the same way such that they preserve the same coordinates. Then we have $g \cong \mathrm{diag}(\mathbf{e}[\frac{a_1}{r}], \ldots, \mathbf{e}[\frac{a_l}{r}], \mathbf{e}[0], \ldots, \mathbf{e}[0])$ and $g^{-1} \cong \mathrm{diag}(\mathbf{e}[\frac{r-a_1}{r}], \ldots, \mathbf{e}[\frac{r-a_l}{r}], \mathbf{e}[0], \ldots, \mathbf{e}[0])$ for $l = n_g = n_{g^{-1}} \le n$. So we directly see

$$\mathrm{age}(g) + \mathrm{age}(g^{-1}) = n - n_g.$$

Example 4.5.3. Let (f, G) be as in Example 4.1.4.

(i) Set $G = \langle g \rangle = \big\langle (\mathbf{e}[\frac{1}{3}], \mathbf{e}[\frac{2}{3}], 1) \big\rangle$. So we see directly

$$\mathrm{age}(g) = 1 \text{ and } \mathrm{age}(g^{-1}) = 1.$$

(ii) Set $G = S_3 = \{\mathrm{id}, (12), (13), (23), (123), (132)\}$. In Example 4.1.4 we have seen a basis $\{(1,1,0), (1,-1,0), (0,0,1)\}$ of $\mathbb{C}^3$, s.t. (12) is diagonal on it. We have

$$(12) \cong \mathrm{diag}(\mathbf{e}[0], \mathbf{e}[\frac{1}{2}], \mathbf{e}[0])$$

and so we see

$$\mathrm{age}((12)) = \frac{1}{2} \text{ and similar } \mathrm{age}((13)) = \frac{1}{2},\ \mathrm{age}((23)) = \frac{1}{2}.$$

In the same way we saw

$$(123) \cong \mathrm{diag}(\mathbf{e}[0], \mathbf{e}[\frac{1}{3}], \mathbf{e}[\frac{2}{3}])$$

and so we see

$$\mathrm{age}((123)) = 1 \text{ and } \mathrm{age}((132)) = 1,$$

since $n - n_{(123)} = 3 - 1 = 2 = 1 + 1 = \mathrm{age}((123)) + \mathrm{age}((132))$.

Definition 4.5.4. We define the non-degenerate $\mathbb{C}$-bilinear form $J_{f,G} : \Omega'_{f,G} \otimes_{\mathbb{C}} \Omega'_{f,G} \to \mathbb{C}$, called the *orbifold residue pairing*, by

$$J_{f,G} := \bigoplus_{g \in G} J_{f,g},$$

where $J_{f,g}$ is the perfect $\mathbb{C}$-bilinear form $J_{f,g} : \Omega'_{f,g} \otimes_{\mathbb{C}} \Omega'_{f,g^{-1}} \longrightarrow \mathbb{C}$ defined by

$$J_{f,g}\left(\omega_1, \omega_2\right) := (-1)^{n-n_g} \cdot \mathbf{e}\left[-\frac{1}{2}\mathrm{age}(g)\right] \cdot |G| \cdot \mathrm{Res}_{\mathrm{Fix}(g)} \left[\begin{matrix} \phi\psi dx_{i_1} \wedge \cdots \wedge dx_{i_{n_g}} \\ \dfrac{\partial f^g}{\partial x_{i_1}} \cdots \dfrac{\partial f^g}{\partial x_{i_{n_g}}} \end{matrix}\right]$$

for $\omega_1 = [\phi dx_{i_1} \wedge \cdots \wedge dx_{i_{n_g}}] \in \Omega'_{f,g}$ and $\omega_2 = [\psi dx_{i_1} \wedge \cdots \wedge dx_{i_{n_g}}] \in \Omega'_{f,g^{-1}}$, where $x_{i_1}, \ldots, x_{i_{n_g}}$ are coordinates of $\mathrm{Fix}(g) = \mathrm{Fix}(g^{-1})$.

For each $g \in G$ with $\mathrm{Fix}(g) = \{\mathbf{0}\}$, we define

$$J_{f,g}\left(1_g, 1_{g^{-1}}\right) := (-1)^n \cdot \mathbf{e}\left[-\frac{1}{2}\mathrm{age}(g)\right] \cdot |G|,$$

where $1_g \in \Omega'_{f,g}$ and $1_{g^{-1}} \in \Omega'_{f,g^{-1}}$ denote the constant functions on $\{\mathbf{0}\}$ whose values are 1.

Example 4.5.5. Let (f, G) be as in Example 4.1.4.

(i) Set $f = x_1^3 + x_2^3 x_3 + x_3^3$ and $G = \left\langle (\mathbf{e}[\frac{1}{3}], \mathbf{e}[\frac{2}{3}], 1) \right\rangle$. We can calculate with Example 2.3.9

$$J_{f,\mathrm{id}}\left(dx_1 \wedge dx_2 \wedge dx_3, x_1 x_2 x_3^2 dx_1 \wedge dx_2 \wedge dx_3\right) = (-1)^0 \mathbf{e}\left[-\frac{1}{2} \cdot 0\right] \cdot 3 \cdot \frac{1}{27} = \frac{1}{9}.$$

With $\mu_{f^g} = 2$ and $\mathrm{hess}_{f^g} = 3 \cdot 2x_3$ we calculate

$$J_{f,g}\left(dx_3, x_3 dx_3\right) = (-1)^2 \mathbf{e}\left[-\frac{1}{2} \cdot 1\right] \cdot 3 \cdot \frac{1}{3} = -1.$$

(ii) Set $f = x_1^3 + x_2^3 + x_3^3$ and $G = S_3$. We can calculate with Example 2.3.9

$$J_{f,\mathrm{id}}\left(dx_1 \wedge dx_2 \wedge dx_3, x_1 x_2 x_3 dx_1 \wedge dx_2 \wedge dx_3\right) = (-1)^0 \mathbf{e}\left[-\frac{1}{2} \cdot 0\right] \cdot 6 \cdot \frac{1}{27} = \frac{2}{9}.$$

With $\mu_{f^{(12)}} = 4$ and $\mathrm{hess}_{f^{(12)}} = 3 \cdot 2 \cdot \frac{2}{8}(x_1 + x_2) \cdot 3 \cdot 2x_3$ we calculate

$$\begin{aligned} &J_{f,(12)}\left(dx_1 \wedge dx_3 + dx_2 \wedge dx_3, (x_1x_3 + x_2x_3)(dx_1 \wedge dx_3 + dx_2 \wedge dx_3)\right) \\ &= (-1)^1 \mathbf{e}\left[-\frac{1}{2} \cdot \frac{1}{2}\right] \cdot 6 \cdot \frac{4}{9} = (-1)(-\sqrt{-1})\frac{8}{3} = \frac{8\sqrt{-1}}{3}. \end{aligned}$$

With $\mu_{f^{(123)}} = 2$ and $\mathrm{hess}_{f^{(123)}} = 3 \cdot 2 \cdot \frac{3}{27}(x_1 + x_2 + x_3)$ we calculate

$$\begin{aligned} &J_{f,(123)}\left(dx_1 + dx_2 + dx_3, (x_1 + x_2 + x_3)(dx_1 + dx_2 + dx_3)\right) \\ &= (-1)^2 \mathbf{e}\left[-\frac{1}{2} \cdot 1\right] \cdot 6 \cdot \frac{9}{3} = (+1)(-1)18 = -18. \end{aligned}$$

Proposition 4.5.6. *The orbifold residue pairing is G-twisted $\mathbb{Z}/2\mathbb{Z}$-graded symmetric in the sense that*

$$J_{f,G}(\omega_1, \omega_2) = (-1)^{n-n_g} \cdot \mathbf{e}\left[-\mathrm{age}(g)\right] \cdot J_{f,G}(\omega_2, \omega_1)$$

for $\omega_1 \in \Omega'_{f,g}$ and $\omega_2 \in \Omega'_{f,g^{-1}}$.

Proof. We have $\mathrm{Fix}(g) = \mathrm{Fix}(g^{-1})$, and so $f^g = f^{g^{-1}}$ and $\mathrm{age}(g) + \mathrm{age}(g^{-1}) = n - n_g$, see Example 4.5.2(c). So we have

$$\begin{aligned} J_{f,G}(\omega_1, \omega_2) &= J_{f,g}\left(\omega_1, \omega_2\right) \\ &= (-1)^{n-n_g} \mathbf{e}\left[-\frac{1}{2}\mathrm{age}(g)\right] |G| \cdot \mathrm{Res}[\cdots] \\ &= \mathbf{e}\left[-\frac{1}{2}\mathrm{age}(g) + \frac{1}{2}\mathrm{age}(g^{-1})\right] (-1)^{n-n_{g^{-1}}} \mathbf{e}\left[-\frac{1}{2}\mathrm{age}(g^{-1})\right] |G| \cdot \mathrm{Res}[\cdots] \\ &= \mathbf{e}\left[-\frac{1}{2}\mathrm{age}(g) + \frac{1}{2}\mathrm{age}(g^{-1})\right] \cdot J_{f,g^{-1}}\left(\omega_2, \omega_1\right) \\ &= \mathbf{e}\left[\frac{1}{2}(\mathrm{age}(g) + \mathrm{age}(g^{-1}))\right] \mathbf{e}\left[-\mathrm{age}(g)\right] \cdot J_{f,g^{-1}}\left(\omega_2, \omega_1\right) \\ &= (-1)^{n-n_g} \cdot \mathbf{e}\left[-\mathrm{age}(g)\right] \cdot J_{f,G}\left(\omega_2, \omega_1\right). \end{aligned}$$

□

Remark 4.5.7. In [EG15] there is defined an equivariant index in $B(G)$. So we could also define some higher order indices. But since this bilinear form is $\mathbb{Z}/2\mathbb{Z}$-graded one would need a good version of the signature to find an equivariant version of Theorem 2.3.13. On the other hand for a good orbifold version of $\mathrm{Jac}(f)$ (cf. next chapter and Proposition 5.3.7) we only take $G \subset \mathrm{SL}$. And then a group in $\mathrm{SL}(n, \mathbb{R})$ would be very small, such that this is no fruitful direction.

5 Orbifold Jacobian Algebra

In a joint work with Alexey Basalaev and Atsushi Takahashi we constructed this orbifold version of $\mathrm{Jac}(f)$. The Chapters 5 and 6 are mainly an elaborated version of the paper [BTW16].

5.1 Setup

Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial with isolated singularity at $\mathbf{0}$. From now on, we shall denote by G a finite subgroup of G_f, cf. Example 4.1.2, unless otherwise stated.

Remark 5.1.1. We will restrict ourselves to subgroups of the diagonal symmetries of f, $G \subset G_f$ (cf. Example 4.1.2). For the defining axioms this is not totally necessary as we write in Remark 5.2.4. But the commutativity of the group simplifies the proofs considerably.

This is also a common assumption:

Remark 5.1.2. The pair (f, G) for a weighted homogeneous f (cf. Definition 6.1.1) and a finite subgroup $G \subset G_f$ is often called a *orbifold Landau-Ginzburg model* (cf. [BH95], [Kr94], [Kr09], [EG12], [FJR13]).

Definition 5.1.3. We will additionally define

$$G_f^{\mathrm{SL}} := G_f \cap \mathrm{SL}(n; \mathbb{C}).$$

Remark 5.1.4. We recall Example 4.5.2. Each element $g \in G_f$ has a unique expression of the form

$$g = \mathrm{diag}\left(\mathbf{e}\left[\frac{a_1}{r}\right], \ldots, \mathbf{e}\left[\frac{a_n}{r}\right]\right) \quad \text{with } 0 \leq a_i < r,$$

where r is the order of g. We use the notation $(a_1/r, \ldots, a_n/r)$ or $\frac{1}{r}(a_1, \ldots, a_n)$ for the element g. And we had defined the

$$\mathrm{age}(g) := \frac{1}{r}\sum_{i=1}^{n} a_i.$$

Note that if $g \in G_f^{\mathrm{SL}}$ then $\mathrm{age}(g) \in \mathbb{Z}$.

Definition 5.1.5. Define the group $\mathrm{Aut}(f, G)$ of automorphisms of (f, G) as

$$\mathrm{Aut}(f, G) := \{\varphi \in \mathrm{GL}(n, \mathbb{C}) \mid f(\varphi\mathbf{x}) = f(\mathbf{x}),\ \varphi^{-1}g\varphi \in G \text{ for all } g \in G\}.$$

It is obvious that G is a subgroup of $\mathrm{Aut}(f, G)$. Note that a $\varphi \in \mathrm{Aut}(f, G)$ is G-equivariant if and only if $\varphi^{-1}g\varphi = g$ for all $g \in G$.

Definition 5.1.6. For a $\mathbb{C}$-algebra R, denote by $\mathrm{Aut}_{\mathbb{C}\text{-alg}}(R)$ the group of all $\mathbb{C}$-algebra automorphisms of R. Note that $Aut(f,G)$ is identified with a subgroup of $\mathrm{Aut}_{\mathbb{C}\text{-alg}}(\mathbb{C}[x_1,\ldots,x_n])$ by the action $(\varphi^*\phi)(\mathbf{x}) = \phi(\varphi\mathbf{x})$ for $\varphi \in \mathrm{Aut}(f,G)$ and $\phi \in \mathbb{C}[x_1,\ldots,x_n]$.

Remark 5.1.7. Let $\mathbb{C}[x_1,\ldots,x_n] * G$ be the skew group ring which is the $\mathbb{C}$-vector space $\mathbb{C}[x_1,\ldots,x_n] \otimes_{\mathbb{C}} \mathbb{C}G$ with a product defined as $(\phi_1 \otimes g_1)(\phi_2 \otimes g_2) = (\phi_1 g_1^*(\phi_2)) \otimes g_1 g_2$ for any $\phi_1, \phi_2 \in \mathbb{C}[x_1,\ldots,x_n]$ and $g_1, g_2 \in G$. Then the group $\mathrm{Aut}(f,G)$ can be regarded as the subgroup of all $\varphi' \in \mathrm{Aut}_{\mathbb{C}\text{-alg}}(\mathbb{C}[x_1,\ldots,x_n]*G)$ such that $\varphi'(f\otimes\mathrm{id}) = f\otimes\mathrm{id}$. For $\varphi \in \mathrm{Aut}(f,G)$, the corresponding element in $\mathrm{Aut}_{\mathbb{C}\text{-alg}}(\mathbb{C}[x_1,\ldots,x_n] * G)$ is given by $\phi \otimes g \mapsto \varphi^*(\phi) \otimes (\varphi^{-1} g \varphi)$.

Remark 5.1.8. As we have said for G in Remark 4.3.2 also each $\varphi \in \mathrm{Aut}(f,G)$ is a bi-regular map on $\mathbb{C}^n$ and maps $\mathrm{Fix}(\varphi^{-1}g\varphi)$ to $\mathrm{Fix}(g)$ for each $g \in G$. Hence, the group $\mathrm{Aut}(f,G)$ acts naturally on $\Omega'_{f,G}$ by

$$\Omega'_{f,g} \longrightarrow \Omega'_{f,\varphi^{-1}g\varphi}, \quad \omega \mapsto \varphi^*|_{\mathrm{Fix}(g)}\omega,$$

where $\varphi^*|_{\mathrm{Fix}(g)}$ denotes the restriction of the pullback φ^* of differential forms to $\mathrm{Fix}(g)$. In order to simplify the notation, for each $\varphi \in \mathrm{Aut}(f,G)$, we shall denote by φ^* the action of φ on $\Omega'_{f,G}$. It also follows that $\mathrm{Aut}(f,G)$ acts naturally on $\Omega_{f,G}$.

5.2 Axioms

In order to introduce an orbifold Jacobian algebra of the pair (f,G), we first define axiomatically a G-twisted Jacobian algebra of f.

Definition 5.2.1. A *G-twisted Jacobian algebra* of f is a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}$-algebra $\mathrm{Jac}'(f,G) = \mathrm{Jac}'(f,G)_{\overline{0}} \oplus \mathrm{Jac}'(f,G)_{\overline{1}}$, $\overline{i} \in \mathbb{Z}/2\mathbb{Z}$, satisfying the following axioms:

(i) For each $g \in G$, there is a $\mathbb{C}$-module $\mathrm{Jac}'(f,g)$ isomorphic to $\Omega'_{f,g}$ as a $\mathbb{C}$-module satisfying the following conditions:

 a) For the identity id of G,

$$\mathrm{Jac}'(f,\mathrm{id}) = \mathrm{Jac}(f).$$

 b) We have

$$\mathrm{Jac}'(f,G)_{\overline{0}} = \bigoplus_{\substack{g\in G\\ n-n_g\equiv 0 \pmod 2}} \mathrm{Jac}'(f,g),$$

$$\mathrm{Jac}'(f,G)_{\overline{1}} = \bigoplus_{\substack{g\in G\\ n-n_g\equiv 1 \pmod 2}} \mathrm{Jac}'(f,g).$$

(ii) The $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}$-algebra structure $\circ$ on $\mathrm{Jac}'(f,G)$ satisfies

$$\mathrm{Jac}'(f,g)\circ\mathrm{Jac}'(f,h)\subset\mathrm{Jac}'(f,gh),\quad g,h\in G,$$

and the $\mathbb{C}$-subalgebra $\mathrm{Jac}'(f,\mathrm{id})$ of $\mathrm{Jac}'(f,G)$ coincides with the $\mathbb{C}$-algebra $\mathrm{Jac}(f)$.

(iii) The $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}$-algebra $\mathrm{Jac}'(f,G)$ is such that the $\mathbb{C}$-module $\Omega'_{f,G}$ has the structure of a $\mathrm{Jac}'(f,G)$-module

$$\vdash\colon \mathrm{Jac}'(f,G)\otimes\Omega'_{f,G}\longrightarrow\Omega'_{f,G},\quad X\otimes\omega\mapsto X\vdash\omega,$$

satisfying the following conditions:

a) For any $g,h\in G$ we have

$$\mathrm{Jac}'(f,g)\vdash\Omega'_{f,h}\subset\Omega'_{f,gh},$$

and the $\mathrm{Jac}'(f,\mathrm{id})$-module structure on $\Omega'_{f,g}$ coincides with the $\mathrm{Jac}(f)$-module structure on Ω_{f^g} given by Corollary 4.1.6.

b) By choosing a nowhere vanishing n-form, we have the following isomorphism

$$\mathrm{Jac}'(f,G)\xrightarrow{\cong}\Omega'_{f,G},\quad X\mapsto X\vdash\zeta, \tag{5.1}$$

where ζ is the residue class in $\Omega'_{f,\mathrm{id}}=\Omega_f$ of the n-form. Namely, $\Omega'_{f,G}$ is a free $\mathrm{Jac}'(f,G)$-module of rank one.

(iv) There is an induced action of $\mathrm{Aut}(f,G)$ on $\mathrm{Jac}'(f,G)$ given by

$$\varphi^*(X)\vdash\varphi^*(\zeta):=\varphi^*(X\vdash\zeta),\quad \varphi\in\mathrm{Aut}(f,G),\ X\in\mathrm{Jac}'(f,G), \tag{5.2}$$

where ζ is an element in $\Omega'_{f,\mathrm{id}}$ giving the isomorphism in Axiom (iiib). The algebra structure of $\mathrm{Jac}'(f,G)$ satisfies the following conditions:

a) It is $\mathrm{Aut}(f,G)$-invariant, namely,

$$\varphi^*(X)\circ\varphi^*(Y)=\varphi^*(X\circ Y),\quad \varphi\in\mathrm{Aut}(f,G),\ X,Y\in\mathrm{Jac}'(f,G).$$

b) It is G-twisted $\mathbb{Z}/2\mathbb{Z}$-graded commutative, namely, for any $g,h\in G$ and $X\in\mathrm{Jac}'(f,g)$, $Y\in\mathrm{Jac}'(f,h)$, we have

$$X\circ Y=(-1)^{\overline{X}\cdot\overline{Y}}g^*(Y)\circ X,$$

where $\overline{X}=n-n_g$ and $\overline{Y}=n-n_h$ are the $\mathbb{Z}/2\mathbb{Z}$-gradings of X and Y, and g^* is the induced action of g considered as an element of $\mathrm{Aut}(f,G)$.

(v) For any $g,h\in G$ and $X\in\mathrm{Jac}'(f,g)$, $\omega\in\Omega'_{f,h}$, $\omega'\in\Omega'_{f,G}$, we have

$$J_{f,G}(X\vdash\omega,\omega')=(-1)^{\overline{X}\cdot\overline{\omega}}J_{f,G}\left(\omega,((h^{-1})^*X)\vdash\omega'\right),$$

where $\overline{X}=n-n_g$ and $\overline{\omega}=n-n_h$ are the $\mathbb{Z}/2\mathbb{Z}$-gradings of X and ω, and $(h^{-1})^*$ is the induced action of h^{-1} considered as an element of $\mathrm{Aut}(f,G)$.

(vi) Let G' be a finite subgroup of G_f such that $G \subset G'$. Fix a nowhere vanishing n-form and denote by ζ its residue class in $\Omega'_{f,\mathrm{id}}$. By Axiom (iiib) for G, G', fix the isomorphisms given by ζ;

$$\mathrm{Jac}'(f,G) \xrightarrow{\cong} \Omega'_{f,G}, \quad X \mapsto X \vdash \zeta,$$
$$\mathrm{Jac}'(f,G') \xrightarrow{\cong} \Omega'_{f,G'}, \quad X' \mapsto X' \vdash \zeta.$$

Then, the injective map $\Omega'_{f,G} \longrightarrow \Omega'_{f,G'}$ induced by the identity maps $\Omega'_{f,g} \longrightarrow \Omega'_{f,g}$, $g \in G$ yields an injective map of the $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}$-modules $\mathrm{Jac}'(f,G) \to \mathrm{Jac}'(f,G')$, which is an algebra-homomorphism.

Remark 5.2.2. Such a class $\zeta \in \Omega'_{f,\mathrm{id}}$ giving the isomorphism in Axiom (iiib) is a non-zero constant multiple of the residue class of $dx_1 \wedge \cdots \wedge dx_n$. It follows that the $\mathrm{Aut}(f,G)$-action on $\mathrm{Jac}'(f,G)$ does not depend on the choice of ζ. In particular, the $\mathrm{Aut}(f,G)$-action on $\mathrm{Jac}'(f,\mathrm{id}) = \mathrm{Jac}(f)$ is nothing but the usual one which is induced by the natural $\mathrm{Aut}(f,G)$-action on $\mathbb{C}[x_1,\dots,x_n]$. For different choices of ζ we get isomorphic algebras.

Remark 5.2.3. Axioms (iva), (ivb) and (v) are naturally expected by keeping the skew group ring $\mathbb{C}[x_1,\dots,x_n] * G$ in mind (see also Remark 5.1.7). Indeed, our axioms are motivated by some intuitive properties of the "Jacobian algebra of f over the non-commutative skew group ring". Axiom (ivb) can also be found in [Ka03], while the others seem to be new in [BTW16].

Remark 5.2.4. We have not used the commutativity of $G \subset G_f$ in the axioms in Definition 5.2.1 except for the last one (vi). Instead of G_f there, by the use of the largest group like $\mathrm{Aut}(f,\{\mathrm{id}\})$ the definition can naturally be extended to the non-abelian case, namely, the case when G is any group like in Chapter 4.

5.3 Orbifold Jacobian Algebra

Lemma 5.3.1. *Let us denote by v_{id} the residue class of $1 \in \mathbb{C}[x_1,\dots,x_n]$ in $\mathrm{Jac}'(f,\mathrm{id}) = \mathrm{Jac}(f)$. v_{id} is the unit with respect to the product structure $\circ$ and v_{id} is G-invariant.*

Proof. By Axiom (v) we have

$$J_{f,G}((X \circ v_{\mathrm{id}}) \vdash \zeta, \omega) = J_{f,G}(X \vdash (v_{\mathrm{id}} \vdash \zeta), \omega) = J_{f,G}(X \vdash \zeta, \omega)$$

for all $X \in \mathrm{Jac}'(f,G)$, $\omega \in \Omega'_{f,G}$ and $\zeta \in \Omega_{f,\mathrm{id}}$ giving the isomorphism (5.1). Note also that $\varphi^*(v_{\mathrm{id}}) = v_{\mathrm{id}}$ for all $\varphi \in \mathrm{Aut}(f,G)$ since $\varphi^*(v_{\mathrm{id}}) \vdash \varphi^*(\zeta) = \varphi^*(v_{\mathrm{id}} \vdash \zeta) = \varphi^*(\zeta) = v_{\mathrm{id}} \vdash \varphi^*(\zeta)$. And so v_{id} is in particular G-invariant. □

Remark 5.3.2. By the isomorphism (5.1), it follows from Remark 5.1.8 that

$$\varphi^*(\mathrm{Jac}'(f,g)) = \mathrm{Jac}'(f,\varphi^{-1}g\varphi), \quad \varphi \in \mathrm{Aut}(f,G).$$

In particular, $g^*(\mathrm{Jac}'(f,h)) = \mathrm{Jac}'(f,g^{-1}hg)$ for $g,h \in G$. Now, G is a commutative group, we have $g^*(\mathrm{Jac}'(f,h)) = \mathrm{Jac}'(f,h)$. Since the product structure $\circ$ is also G-invariant by Axiom (iva) it follows that the G-invariant subspace of $\mathrm{Jac}'(f,G)$ has the structure of a $\mathbb{Z}/2\mathbb{Z}$-graded algebra, which is $\mathbb{Z}/2\mathbb{Z}$-graded commutative due to Axiom (ivb).

A priori there might not be a unique $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}$-algebra satisfying the axioms in Definition 5.2.1, nevertheless we expect the following:

Conjecture 5.3.3. *Let the notations be as above.*

(a) A G-twisted Jacobian algebra $\mathrm{Jac}'(f,G)$ *of f should exist.*

(b) The subalgebra $(\mathrm{Jac}'(f,G))^G$ *should be uniquely determined by (f,G) up to isomorphism.*

Definition 5.3.4. Suppose that Conjecture 5.3.3 holds for the pair (f,G). The $\mathbb{Z}/2\mathbb{Z}$-graded commutative algebra

$$\mathrm{Jac}(f,G) := (\mathrm{Jac}'(f,G))^G$$

is called the *orbifold Jacobian algebra* of (f,G).

Remark 5.3.5. Under the isomorphism in Axiom (iiib), it follows from Axiom (v) that the non-degenerate G-twisted $\mathbb{Z}/2\mathbb{Z}$-graded symmetric $\mathbb{C}$-bilinear form $J_{f,G}$ on $\Omega'_{f,G}$ equips $\mathrm{Jac}'(f,G)$ with the structure of $\mathbb{Z}/2\mathbb{Z}$-graded G-twisted Frobenius algebra.

Remark 5.3.6. Often we will have $G \subset G_f^{\mathrm{SL}}$. We don't need this from the definition of $\mathrm{Jac}(f,G)$ but only then we get a "good" orbifold Jacobian algebra. Namely only for $G \subset \mathrm{SL}(n,\mathbb{C})$ we have the following proposition.

Proposition 5.3.7. *Let $G \subset G_f^{\mathrm{SL}}$ and suppose the orbifold Jacobian Algebra exists. Then*

$$\mathrm{Jac}(f,G) \cong \Omega_{f,G}$$

as vector spaces. And the orbifold residue pairing endows $\mathrm{Jac}(f,G)$ *with the structure of a $\mathbb{Z}/2\mathbb{Z}$-graded commutative Frobenius algebra, which will be of our main interest.*

Proof. When $G \subset \mathrm{SL}(n,\mathbb{C})$ the residue class ζ is G-invariant. So we get the isomorphism by the isomorphism (5.1). Furthermore we have $\mathrm{age}(g) \in \mathbb{Z}$ for all $g \in G_f^{\mathrm{SL}}$ and so the pairing $J_{f,G}$ induces a $\mathbb{Z}/2\mathbb{Z}$-graded symmetric pairing on $\Omega_{f,G}$ due to the G-twisted $\mathbb{Z}/2\mathbb{Z}$-graded commutativity (Proposition 4.5.6). With this and Remarks 5.3.5 and 5.3.2 we see that we have here even a $\mathbb{Z}/2\mathbb{Z}$-graded commutative Frobenius algebra. $\square$

5.4 Preliminaries for the Proofs

In the next chapters we will prove Conjecture 5.3.3 (actually a stronger statement) for some classes of polynomials.

We will need some common definitions for the proofs.

Definition 5.4.1. Let $I_g := \{i_1, \ldots, i_{n_g}\}$ be a subset of $\{1, \ldots, n\}$ such that $\mathrm{Fix}(g) = \{x \in \mathbb{C}^n \mid x_j = 0, j \notin I_g\}$. In particular, $I_{\mathrm{id}} = \{1, \ldots, n\}$. Denote by I_g^c the complement of I_g in I_{id}.

Definition 5.4.2. For each $g \in G$ let us define $\omega_g \in \Omega'_{f,g}$ as

$$\omega_g := \begin{cases} \zeta & \text{if } g = \mathrm{id} \\ [dx_{i_1} \wedge \cdots \wedge dx_{i_{n_g}}] & \text{if } I_g = (i_1, \dots, i_{n_g}),\ i_1 < \cdots < i_{n_g} \\ 1_g & \text{if } \mathrm{Fix}(g) = \{\mathbf{0}\} \end{cases}.$$

Remark 5.4.3. It might not be necessary to distinguish ζ and ω_{id}, however, we regard ζ as a "primitive form" (cf. [Sa82], [Sa83], [ST08]) at the origin of the base space of the "properly-defined deformation space" of the pair (f, G) while we consider ω_{id} as just a $\mathrm{Jac}'(f, \mathrm{id})$-basis of $\Omega'_{f,\mathrm{id}}$.

We will have to proof the uniqueness and the existence.

Idea of the Uniqueness Proof

For the stronger statement we will show that for any $G \subset G_f$ the axioms in Definition 5.2.1 determine $\mathrm{Jac}'(f, G)$ uniquely up to isomorphism. We only have to show that for $g, h \in G$ the product $\circ : \mathrm{Jac}'(f, g) \otimes_{\mathbb{C}} \mathrm{Jac}'(f, h) \longrightarrow \mathrm{Jac}'(f, gh)$ is uniquely determined up to rescaling of generators of $\mathrm{Jac}(f^g)$-modules $\mathrm{Jac}'(f, g)$.

Definition 5.4.4. Let ζ be a non-zero constant multiple of the residue class of $dx_1 \wedge \cdots \wedge dx_n$. For each subgroup $G \subset G_f$, fix an isomorphism in Axiom (iii) in Definition 5.2.1

$$\vdash : \mathrm{Jac}'(f, G) \xrightarrow{\cong} \Omega'_{f,G}, \quad X \mapsto X \vdash \zeta,$$

where ζ is considered as an element in $\Omega'_{f,\mathrm{id}} = \Omega_f$ (recall Definition 4.3.1).

Definition 5.4.5. For each $g \in G$, let v_g be an element of $\mathrm{Jac}'(f, g)$, such that

$$v_g \vdash \zeta = \alpha_g \omega_g,$$

where α_g is given by a map

$$\alpha : G_f \longrightarrow \mathbb{C}^*, \quad g \mapsto \alpha_g,$$

with $\alpha_{\mathrm{id}} = 1$, which is given in more details in the different proofs.

Remark 5.4.6. We see directly that the definition of v_{id} is the same as in Lemma 5.3.1 and this says that $v_{\mathrm{id}} \circ v_g = v_g \circ v_{\mathrm{id}} = v_g$ since v_{id} is the unit.

Axiom (iiia) in Definition 5.2.1 implies that for all $Y \in \mathrm{Jac}'(f, g)$ there exists $X \in \mathrm{Jac}'(f, \mathrm{id}) = \mathrm{Jac}(f)$ represented by a polynomial in $\{x_i\}_{i \in I_g}$ such that $Y = X \circ v_g$. For any $X \in \mathrm{Jac}'(f, \mathrm{id})$, we shall often write $X \circ v_g$ as $X|_{\mathrm{Fix}(g)} v_g$ where $X|_{\mathrm{Fix}(g)}$ is the image of X under the map $\mathrm{Jac}(f) \longrightarrow \mathrm{Jac}(f^g)$.

Idea of the Existence Proof

Afterwards we will prove the existence of a G-twisted Jacobian algebra of f. We will first show this when $G = G_f$.

We will give a Definition:

Definition 5.4.7. Define a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}$-module $\mathcal{A}' = \mathcal{A}'_{\overline{0}} \oplus \mathcal{A}'_{\overline{1}}$ as follows: For each $g \in G_f$, consider a free $\mathrm{Jac}(f^g)$-module $\mathcal{A}'_g$ of rank one generated by a formal letter $\overline{v}_g$,

$$\mathcal{A}'_g = \mathrm{Jac}(f^g)\overline{v}_g.$$

and set

$$\mathcal{A}'_{\overline{0}} := \bigoplus_{\substack{g \in G_f \\ n - n_g \equiv 0 \pmod 2}} \mathcal{A}'_g, \quad \mathcal{A}'_{\overline{1}} := \bigoplus_{\substack{g \in G_f \\ n - n_g \equiv 1 \pmod 2}} \mathcal{A}'_g.$$

By definition, Axiom (i) in Definition 5.2.1 trivially holds for $\mathcal{A}'$.

Remark 5.4.8. We will then define a multiplication $\circ : \mathcal{A}' \otimes_{\mathbb{C}} \mathcal{A}' \longrightarrow \mathcal{A}'$ and a $\mathbb{C}$-bilinear map $\vdash: \mathcal{A}' \otimes_{\mathbb{C}} \Omega'_{f,G_f} \longrightarrow \mathcal{A}'$ and show all axioms of Definition 5.2.1. Where Axiom (vi) is trivially satisfied for $\mathcal{A}'$ since $G = G_f$.

And then we can get in all proofs

Proposition 5.4.9. *For each subgroup $G \subset G_f$, there exists a G-twisted Jacobian algebra of f.*

Proof. Consider the subspace $\mathcal{A}'_G$ of $\mathcal{A}'$ defined by

$$\mathcal{A}'_G := \bigoplus_{g \in G} \mathcal{A}'_g,$$

the restriction of the product structure map $\circ : \mathcal{A}' \otimes_{\mathbb{C}} \mathcal{A}' \longrightarrow \mathcal{A}'$ to $\mathcal{A}'_G \otimes_{\mathbb{C}} \mathcal{A}'_G$ and the restriction of the $\mathcal{A}'$-module structure map $\vdash: \mathcal{A}' \otimes_{\mathbb{C}} \Omega'_{f,G_f} \longrightarrow \mathcal{A}'$ to $\mathcal{A}'_G \otimes_{\mathbb{C}} \Omega'_{f,G}$. By the construction of these structures on $\mathcal{A}'$, it is almost obvious that they satisfy all the axioms in Definition 5.2.1. $\square$

6 Orbifold Jacobian Algebras for Invertible Polynomials

6.1 Invertible Polynomials

Definition 6.1.1. A polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$ is called a *weighted homogeneous polynomial* if there are positive integers $w_1, \dots, w_n$ and d such that

$$f(\lambda^{w_1} x_1, \dots, \lambda^{w_n} x_n) = \lambda^d f(x_1, \dots, x_n)$$

for all $\lambda \in \mathbb{C}^*$. We call $(w_1, \dots, w_n; d)$ a system of weights of f. A weighted homogeneous polynomial f is called *non-degenerate* if it has at most an isolated critical point at the origin in $\mathbb{C}^n$, equivalently, if the Jacobian algebra $\mathrm{Jac}(f)$ of f is finite-dimensional.

Definition 6.1.2 (cf. [BH93], [Kr94])**.** A weighted homogeneous polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$ is called *invertible* if the following conditions are satisfied.

(i) The number of variables ($= n$) coincides with the number of monomials in the polynomial f, namely,

$$f(x_1, \dots, x_n) = \sum_{i=1}^{n} a_i \prod_{j=1}^{n} x_j^{E_{ij}}$$

for some coefficients $a_i \in \mathbb{C}^*$ and non-negative integers E_{ij} for $i, j = 1, \dots, n$.

(ii) The matrix $E := (E_{ij})$ is invertible over $\mathbb{Q}$.

(iii) The polynomial f and the *Berglund-Hübsch transpose* f^T of f defined by

$$f^T(x_1, \dots, x_n) := \sum_{i=1}^{n} a_i \prod_{j=1}^{n} x_j^{E_{ji}}$$

are non-degenerate.

Remark 6.1.3. Usually a polynomial f is called invertible if only conditions (i) and (ii) are satisfied. It is called a non-degenerate invertible polynomial, if f has additionally only an isolated singularity at the origin. This is equivalent to condition (iii), see e.g. [EG12]. Here we will only say invertible polynomial, if it satisfies all three conditions.

Definition 6.1.4. Let $f(x_1, \dots, x_n) = \sum_{i=1}^n c_i \prod_{j=1}^n x_j^{E_{ij}}$ be an invertible polynomial. Define rational numbers $q_1, \dots, q_n$ by the unique solution of the equation

$$E \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Namely, set $q_i := w_i/d$, $i = 1, \dots, n$, for the system of weights $(w_1, \dots, w_n; d)$.

Example 6.1.5. Let (f, G) be as in Example 4.1.4.

(i) Set $f = x_1^3 + x_2^3 x_3 + x_3^3$ and $G = \left\langle (\mathbf{e}[\frac{1}{3}], \mathbf{e}[\frac{2}{3}], 1) \right\rangle$. f is an invertible polynomial. We have $E = \left(\begin{smallmatrix} 3 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{smallmatrix}\right)$, which is obviously invertible. So the system of weights is $(3, 2, 3; 9)$ and $q_1 = \frac{1}{3}$, $q_2 = \frac{2}{9}$, $q_3 = \frac{1}{3}$. The group is directly $G = G_f^{\mathrm{SL}}$ (cf. Example 4.1.2).

(ii) The polynomial $f = x_1^3 + x_2^3 + x_3^3$ is also an invertible polynomial. We have $E = \left(\begin{smallmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{smallmatrix}\right)$, which is obviously invertible. So the system of weights is $(1, 1, 1; 3)$ and $q_i = \frac{1}{3}$ for all $i = 1, 2, 3$. But the group S_3 is no subgroup of G_f.

Remark 6.1.6. If $f(x_1, \dots, x_n)$ is an invertible polynomial, then we have

$$G_f = \left\{ (\lambda_1, \dots, \lambda_n) \in (\mathbb{C}^*)^n \;\middle|\; \prod_{j=1}^n \lambda_j^{E_{1j}} = \dots = \prod_{j=1}^n \lambda_j^{E_{nj}} = 1 \right\},$$

and hence G_f is a finite group. It is easy to see that G_f contains an element $g_0 := (q_1, \dots, q_n)$.

It is important to note the following

Proposition 6.1.7. *The group $G_f^{\mathrm{SL}} = G_f \cap \mathrm{SL}(n; \mathbb{C})$ is a proper subgroup of G_f.*

Proof. Let f^T be the Berglund-Hübsch transpose of f. It is known by [ET11] and [Kr09] (see also Proposition 2 in [EGT16]) that

$$G_f^{\mathrm{SL}} \cong \mathrm{Hom}(G_{f^T} / \langle (\widetilde{q}_1, \dots, \widetilde{q}_n) \rangle, \mathbb{C}^*) \subsetneq \mathrm{Hom}(G_{f^T}, \mathbb{C}^*) \cong G_f,$$

where $(\widetilde{q}_1, \dots, \widetilde{q}_n)$ is the unique solution of the equation $(\widetilde{q}_1, \dots, \widetilde{q}_n)E = (1, \dots, 1)$. □

Remark 6.1.8. Let $f(x_1, \dots, x_n) = \sum_{i=1}^n c_i \prod_{j=1}^n x_j^{E_{ij}}$ be an invertible polynomial. Without loss of generality one may assume that $c_i = 1$ for $i = 1, \dots, n$ by rescaling the variables.

M. Kreuzer and H. Skarke showed the following

Proposition 6.1.9 (cf. [KS92])**.** *An invertible polynomial f can be written as a Sebastiani-Thom sum $f = f_1 \oplus \dots \oplus f_p$ of invertible polynomials (in groups of different variables) f_ν, $\nu = 1, \dots, p$ of the following types:*

(i) $x_1^{a_1} x_2 + x_2^{a_2} x_3 + \dots + x_{m-1}^{a_{m-1}} x_m + x_m^{a_m}$ *(chain type; $m \geq 1$)*

(ii) $x_1^{a_1} x_2 + x_2^{a_2} x_3 + \dots + x_{m-1}^{a_{m-1}} x_m + x_m^{a_m} x_1$ *(loop type; $m \geq 2$)*

Remark 6.1.10. In [KS92] the authors distinguished also polynomials of the so called Fermat type: $x_1^{a_1}$, which is regarded as a chain type polynomial with $m = 1$ in this thesis.

We shall use the monomial basis of the Jacobian algebra $\mathrm{Jac}(f_\nu)$.

Proposition 6.1.11 (cf. [Kr94]). *For an invertible polynomial $f_\nu = x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{m-1}^{a_{m-1}}x_m + x_m^{a_m}$ of chain type with $m \geq 1$, the Jacobian algebra $\mathrm{Jac}(f_\nu)$ has a monomial basis consisting of all the monomials $x_1^{k_1} \cdots x_m^{k_m}$ such that*

1) $0 \leq k_i \leq a_i - 1$,

2) if

$$k_i = \begin{cases} a_i - 1 \text{ for all odd } i,\ i \leq 2s-1, \\ 0 \text{ for all even } i,\ i \leq 2s-1, \end{cases}$$

then $k_{2s} = 0$.

For an invertible polynomial $f_\nu = x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{m-1}^{a_{m-1}}x_m + x_m^{a_m}x_1$ of loop type with $m \geq 2$, the Jacobian algebra $\mathrm{Jac}(f_\nu)$ has a monomial basis consisting of all the monomials $x_1^{k_1} \cdots x_m^{k_m}$ with $0 \leq k_i \leq a_i - 1$.

6.2 Theorem for Invertible Polynomials

Theorem 6.2.1. *Let f be an invertible polynomial and G a subgroup of G_f. There exists a unique G-twisted Jacobian algebra $\mathrm{Jac}'(f, G)$ of f up to isomorphism. Namely, it is uniquely characterized by the axioms in Definition 5.2.1.*

In particular, the orbifold Jacobian algebra $\mathrm{Jac}(f, G)$ of (f, G) exists.

We will first prepare some notations and then show the uniqueness and the existence as stated in Section 5.4.

Notations

Let $f = f(x_1, \ldots, x_n) = \sum_{i=1}^n \prod_{j=1}^n x_j^{E_{ij}}$ be an invertible polynomial.

In what follows, we are mostly interested in special pairs of elements of G_f.

Definition 6.2.2. (i) An ordered pair (g, h) of elements of G_f is called *spanning* if

$$I_g \cup I_h \cup I_{gh} = \{1, \ldots, n\}.$$

(ii) For a spanning pair (g, h) of elements of G_f, define $I_{g,h} := I_g^c \cap I_h^c$.

(iii) For a spanning pair (g, h) of elements of G_f, there always exist $g_1, g_2, h_1, h_2 \in G_f$ such that $g = g_1g_2$ and $h = h_1h_2$ with $g_2h_2 = \mathrm{id}$ and $I_{g_1,h_1} = \emptyset$. The tuple (g_1, g_2, h_1, h_2) is called the *factorization* of (g, h).

Remark 6.2.3. For a spanning pair (g,h) of elements of G_f, up to a reordering of the variables, we have

$$\begin{aligned} g =&(0,\dots,0,\alpha_1,\dots,\alpha_p,\beta_1,\dots,\beta_q)\\ h =&(\gamma_1,\dots,\gamma_r,0,\dots,0,1-\beta_1,\dots,1-\beta_q), \end{aligned} \tag{6.1}$$

for some rational numbers $0<\alpha_i,\beta_i,\gamma_i<1$ and integers p,q,r such that $0\le r\le n_g$ and $n_g+p+q=r+n_h+q=n$. In this presentation, we have $I_g\cap I_h=\{i_{r+1},\dots,i_{n-q-p}\}$, $I_{g,h}=\{i_{n-q+1},\dots,i_n\}$ and

$$\begin{aligned} g_1 =&(0,\dots,0,\alpha_1,\dots,\alpha_p,0,\dots,0),\\ g_2 =&(0,\dots,0,0,\dots,0,\beta_1,\dots,\beta_q),\\ h_1 =&(\gamma_1,\dots,\gamma_r,0,\dots,0,0,\dots,0),\\ h_2 =&(0,\dots,0,0,\dots,0,1-\beta_1,\dots,1-\beta_q). \end{aligned}$$

We introduce one of the most important objects in this section.

Definition 6.2.4. For each spanning pair (g,h) of elements of G_f, define a polynomial $H_{g,h}\in\mathbb{C}[x_1,\dots,x_n]$ by

$$H_{g,h}:=\begin{cases}\widetilde{m}_{g,h}\det\left(\frac{\partial^2 f}{\partial x_i\partial x_j}\right)_{i,j\in I_{g,h}} & \text{if}\quad I_{g,h}\neq\emptyset\\ 1 & \text{if}\quad I_{g,h}=\emptyset\end{cases},$$

where $\widetilde{m}_{g,h}\in\mathbb{C}^*$ is the constant uniquely determined by the following equation in $\mathrm{Jac}(f^{gh})$

$$\frac{1}{\mu_{f^{g\cap h}}}[\mathrm{hess}(f^{g\cap h})H_{g,h}]=\frac{1}{\mu_{f^{gh}}}[\mathrm{hess}(f^{gh})], \tag{6.2}$$

where $f^{g\cap h}$ is the invertible polynomial given by the restriction $f|_{\mathrm{Fix}(g)\cap\mathrm{Fix}(h)}$ of f to the locus $\mathrm{Fix}(g)\cap\mathrm{Fix}(h)$.

Remark 6.2.5. The polynomial $H_{g,h}$ is a non-zero constant multiple of the determinant of a minor of the Hessian matrix of $f(x_1,\dots,x_n)$. Since $I_g\cap I_h\subset I_{gh}$ and $I_{g,h}\subset I_{gh}$, $\mathrm{hess}(f^{g\cap h})$ and $H_{g,h}$ define elements of $\mathrm{Jac}(f^{gh})$.

Remark 6.2.6. Let (g,h) be a spanning pair of elements of G_f. Suppose that $\mathrm{Fix}(g)=\{\mathbf{0}\}$. Then $h=g^{-1}$. It is easy to check that $H_{g,h}=\frac{1}{\mu_f}[\mathrm{hess}(f)]$ by the explanation of $\widetilde{m}_{g,h}$ below. Recall also Example 2.1.3 that if $\mathrm{Fix}(g)\cap\mathrm{Fix}(h)=\{\mathbf{0}\}$ then $\mu_{f^{g\cap h}}=1$ and $\mathrm{hess}(f^{g\cap h})=1$.

Example 6.2.7. Let $f=x_1^3+x_2^3x_3+x_3^3$ and $G=\left\langle(\mathbf{e}[\frac{1}{3}],\mathbf{e}[\frac{2}{3}],1)\right\rangle$ be as in Example 4.1.4. We have that (g,g^{-1}) is a spanning pair, with $I_g=\{3\}=I_{g^{-1}}$ and so $I_{g,g^{-1}}=\{1,2\}$. We calculate

$$\det\left(\frac{\partial^2 f}{\partial x_i\partial x_j}\right)_{i,j\in\{1,2\}}=\det\left(\begin{smallmatrix}6x_1 & 0\\ 0 & 6x_2x_3\end{smallmatrix}\right)=36x_1x_2x_3$$

which is an element in $\mathrm{Jac}(f^{gg^{-1}}) = \mathrm{Jac}(f)$. With

$$\frac{1}{\mu_f}[\mathrm{hess}(f)] = \frac{1}{14} 14 \cdot 27 x_1 x_2 x_3^2$$
$$\frac{1}{\mu_{f^{g\cap g^{-1}}}}[\mathrm{hess}(f^{g\cap g^{-1}}) H_{g,g^{-1}}] = \frac{1}{2} 2 \cdot 3x_1 \cdot \widetilde{m}_{g,g^{-1}} 36 x_1 x_2 x_3$$

we see directly $\widetilde{m}_{g,g^{-1}} = \frac{1}{4}$. So we have

$$H_{g,g^{-1}} = 9x_1x_2x_3.$$

We have to show the uniqueness of $\widetilde{m}_{g,h}$. First observe:

Lemma 6.2.8. *Let (g,h) be a spanning pair of elements of G_f. Suppose that $f = f_1 \oplus \cdots \oplus f_p$ is a Sebastiani-Thom sum such that each f_ν, $\nu = 1, \ldots, p$ is either of chain type or loop type. Fix one ν. Let $I_\nu = \{i_1, \ldots, i_m\}$ be the index set of the variables of f_ν. Then, for $f_\nu = f_\nu(x_{i_1}, \ldots, x_{i_m})$, precisely one of the following holds:*

(i) f_ν is of chain type and, for some $0 \le l \le m$,

- *(a) $\{i_1, \ldots, i_m\} \subset I_g$, $\{i_1, \ldots, i_l\} \subset I_h^c$ and $\{i_{l+1}, \ldots, i_m\} \subset I_h$,*
- *(a') $\{i_1, \ldots, i_m\} \subset I_h$, $\{i_1, \ldots, i_l\} \subset I_g^c$ and $\{i_{l+1}, \ldots, i_m\} \subset I_g$,*
- *(b) $\{i_1, \ldots, i_l\} \subset I_{g,h}$ and $\{i_{l+1}, \ldots, i_m\} \subset I_g \cap I_h$.*

(ii) f_ν is of loop type and

- *(a) $\{i_1, \ldots, i_m\} \subset I_g \cap I_h$,*
- *(b) $\{i_1, \ldots, i_m\} \subset I_g \cap I_h^c$,*
- *(b') $\{i_1, \ldots, i_m\} \subset I_g^c \cap I_h$,*
- *(c) $\{i_1, \ldots, i_m\} \subset I_{g,h}$.*

Proof. From the explicit form of an invertible polynomial of each type and the group action on it the following facts are straightforward for each $g \in G_f$:

- If f_ν is of the chain type $f_\nu = x_{i_1}^{a_1} x_{i_2} + \cdots + x_{i_{m-1}}^{a_{m-1}} x_{i_m} + x_{i_m}^{a_m}$, then there exists l, $0 \le l \le m$ such that $\{i_1, \ldots, i_l\} \subset I_g^c$ and $\{i_{l+1}, \ldots, i_m\} \subset I_g$.
- If f_ν is of loop type $f_\nu = x_{i_1}^{a_1} x_{i_2} + \cdots + x_{i_{m-1}}^{a_{m-1}} x_{i_m} + x_{i_m}^{a_m} x_{i_1}$, then $I_\nu \subset I_g$ or $I_\nu \subset I_g^c$.

And so the cases above are clear. □

Lemma 6.2.9. *$\widetilde{m}_{g,h}$ exists and is uniquely determined by the equation in Definition 6.2.4.*

Proof. Suppose that $f = f_1 \oplus \cdots \oplus f_p$ is a Sebastiani-Thom sum as in Lemma 6.2.8.

Then $\mathrm{Jac}(f) = \mathrm{Jac}(f_1) \otimes \cdots \otimes \mathrm{Jac}(f_p)$ and

$$\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{i,j \in I_{\mathrm{id}}} = \prod_{\nu=1}^{p} \det\left(\frac{\partial^2 f_\nu}{\partial x_i \partial x_j}\right)_{i,j\in I_\nu}.$$

Obviously, only polynomials f_ν satisfying $I_\nu \cap I_{g,h} \neq \emptyset$ contribute non-trivially to $H_{g,h}$. Such a f_ν satisfies one of the following two by Lemma 6.2.8:

(a) $I_\nu = \{i_1, \dots, i_m\} \subset I_{g,h}$.

(b) f_ν is of the chain type and, for some $0 \le l \le m-1$, $\{i_1, \dots, i_l\} \subset I_{g,h}$ and $\{i_{l+1}, \dots, i_m\} \subset I_g \cap I_h$.

Set $\Gamma_a := \{\nu \mid f_\nu \text{ satisfies (a)}\}$ and $\Gamma_b := \{\nu \mid f_\nu \text{ satisfies (b)}\}$. Since $I_{gh} = I_{g,h} \cup (I_g \cap I_h)$, we have

$$f^{gh} = \bigoplus_{\nu_a \in \Gamma_a} f_{\nu_a} \oplus \bigoplus_{\nu_b \in \Gamma_b} f_{\nu_b} \oplus \bigoplus_{\substack{\nu \text{ s.t.} \\ I_\nu \subset I_g \cap I_h}} f_\nu,$$

where $\oplus$ denotes a Sebastiani-Thom sum and hence

$$\mathrm{Jac}(f^{gh}) = \bigotimes_{\nu_a \in \Gamma_a} \mathrm{Jac}(f_{\nu_a}) \otimes \bigotimes_{\nu_b \in \Gamma_b} \mathrm{Jac}(f_{\nu_b}) \otimes \bigotimes_{\substack{\nu \text{ s.t.} \\ I_\nu \subset I_g \cap I_h}} \mathrm{Jac}(f_\nu).$$

Consider the factorization

$$\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{i,j \in I_{g,h}} = \prod_{\nu_a \in \Gamma_a} \widetilde{H}_a^{(\nu_a)} \cdot \prod_{\nu_b \in \Gamma_b} \widetilde{H}_b^{(\nu_b)},$$

where

$$\widetilde{H}_a^{(\nu_a)} := \det\left(\frac{\partial^2 f_{\nu_a}}{\partial x_i \partial x_j}\right)_{i,j \in I_{\nu_a}}, \quad \widetilde{H}_b^{(\nu_b)} := \det\left(\frac{\partial^2 f_{\nu_b}}{\partial x_i \partial x_j}\right)_{i,j \in I_{\nu_b} \cap I_{g,h}}.$$

Suppose for simplicity that $f_{\nu_b} = x_1^{a_1} x_2 + \dots + x_{m-1}^{a_{m-1}} x_m + x_m^{a_m}$ with $I_{\nu_b} \cap I_{g,h} = \{1, \dots, l\}$. By a direct calculation, we have the following equalities in $\mathrm{Jac}(f_{\nu_b})$;

$$\left[\widetilde{H}_b^{(\nu_b)}\right] = \left(\prod_{i=1}^{l} a_i\right) \cdot \left(\sum_{j=1}^{l} (-1)^{l-j} \prod_{i=1}^{j} a_i\right) \left[x_1^{a_1-2} x_2^{a_2-1} \cdots x_l^{a_l-1} x_{l+1}\right], \tag{6.3a}$$

$$\left[\mathrm{hess}(f_{\nu_b}|_{\mathrm{Fix}(g) \cap \mathrm{Fix}(h)})\right] = \left(\prod_{i=l+1}^{m} a_i\right) \cdot \left(\sum_{j=l}^{m} (-1)^{m-j} \prod_{i=l+1}^{j} a_i\right) \left[x_{l+1}^{a_{l+1}-2} x_{l+2}^{a_{l+2}-1} \cdots x_m^{a_m-1}\right], \tag{6.3b}$$

$$[\mathrm{hess}(f_{\nu_b})] = \left(\prod_{i=1}^{m} a_i\right) \cdot \left(\sum_{j=0}^{m} (-1)^{m-j} \prod_{i=1}^{j} a_i\right) \left[x_1^{a_1-2} x_2^{a_2-1} \cdots x_m^{a_m-1}\right]. \tag{6.3c}$$

Note that

$$\mu_{f_{\nu_b}} = \sum_{j=0}^{m} (-1)^{m-j} \prod_{i=1}^{j} a_i, \quad \mu_{f_{\nu_b}|_{\mathrm{Fix}(g) \cap \mathrm{Fix}(h)}} = \sum_{j=l}^{m} (-1)^{m-j} \prod_{i=l+1}^{j} a_i.$$

Hence, it is straightforward to see the existence and the uniqueness of $\widetilde{m}_{g,h}$. □

Proposition 6.2.10. *For each spanning pair (g,h) of elements of G_f, the following holds:*

(i) The class of $H_{g,h}$ is non-zero in $\mathrm{Jac}(f^{gh})$.

(ii) If $I_{g,h}=\emptyset$, then $[H_{g,g^{-1}}H_{h,h^{-1}}]=[H_{gh,(gh)^{-1}}]$ in $\mathrm{Jac}(f)$.

(iii) For any $j\in I_{g,h}$, the class of $x_jH_{g,h}$ is zero in $\mathrm{Jac}(f^{gh})$.

Proof. Let the notations be as above. We may assume that $I_{g,h}\neq\emptyset$ since the statements are trivially true, if $I_{g,h}=\emptyset$. Part (i) is almost clear by the equation (6.2) since $[\mathrm{hess}(f^{gh})]$ is non-zero. Part (ii) follows from the normalization of $H_{g,h}$ by the equation (6.2) in view of the equations (6.3).

To prove part (iii), first note that there is ν, $1\leq\nu\leq p$, such that $j\in I_\nu$ for some f_ν satisfying either (a) or (b) above. Due to the factorization of $\mathrm{Jac}(f^{gh})$, it is enough to show that $[x_j\widetilde{H}_a^{(\nu)}]=0$ if $\nu\in\Gamma_a$ and $[x_j\widetilde{H}_b^{(\nu)}]=0$ if $\nu\in\Gamma_b$. Since the first case is almost clear, suppose that $f_\nu\in\Gamma_b$, $I_\nu=\{1,\dots,m\}$ and $I_\nu\cap I_{g,h}=\{1,\dots,l\}$. Recall again that $[\widetilde{H}_b^{(\nu)}]$ is a non-zero constant multiple of $[x_1^{a_1-2}x_2^{a_2-1}\dots x_l^{a_l-1}x_{l+1}]$. It is easy to calculate by induction that $[x_1^{a_1-1}x_2]=0$ and $[x_j^{a_j}x_{j+1}]=0$ in $\mathrm{Jac}(f_\nu)$ for $j=2,\dots,l$. Therefore, we have $[x_j\widetilde{H}_b^{(\nu)}]=0$ in $\mathrm{Jac}(f_\nu)$ for $j=1,\dots,l$ (see also the description of the monomial basis in Proposition 6.1.11). This completes part (iii) of the proposition. □

Proposition 6.2.11. *For each spanning pair (g,h) of elements of G_f, we have*

$$(n-n_g)+(n-n_h)\equiv(n-n_{gh})\ (\mathrm{mod}\ 2).$$

Moreover, if $I_{g,h}=\emptyset$ then $(n-n_g)+(n-n_h)=(n-n_{gh})$.

Proof. First of all, note that $n-n_g=|I_g^c|$. Therefore, the following equalities yield the statement:

$$n-n_g=|I_g^c\backslash I_{g,h}|+|I_{g,h}|,\quad n-n_h=|I_h^c\backslash I_{g,h}|+|I_{g,h}|,$$
$$n-n_{gh}=|I_{gh}^c|=|I_g^c\backslash I_{g,h}|+|I_h^c\backslash I_{g,h}|.$$

□

Definition 6.2.12. For each $g\in G_f$, the set $I_g\subset\{1,\dots,n\}$ and its complement I_g^c will often be regarded as a subsequence of $(1,\dots,n)$:

$$I_g=(i_1,\dots,i_{n_g}),\ i_1<\dots<i_{n_g},\quad I_g^c=(j_1,\dots,j_{n-n_g}),\ j_1<\dots<j_{n-n_g}.$$

Let $g_1,\dots,g_k$ be elements of G_f such that $I_{g_i,g_j}=\emptyset$ if $i\neq j$.

(i) Denote by $I_{g_1}^c\sqcup I_{g_2}^c$ the sequence given by adding the sequence $I_{g_2}^c$ at the end of the sequence $I_{g_1}^c$. Define inductively $I_{g_1}^c\sqcup\dots\sqcup I_{g_k}^c$ by $\left(I_{g_1}^c\sqcup\dots\sqcup I_{g_{k-1}}^c\right)\sqcup I_{g_k}^c$. Obviously, as a set, $I_{g_1}^c\sqcup\dots\sqcup I_{g_k}^c=I_{g_1\dots g_k}^c$.

(ii) Let $\sigma_{g_1,\dots,g_k}$ be the permutation which turns the sequence $I_{g_1}^c\sqcup\dots\sqcup I_{g_k}^c$ to the sequence $I_{g_1\dots g_k}^c$. Define $\widetilde{\varepsilon}_{g_1,\dots,g_k}$ as the signature $\mathrm{sgn}(\sigma_{g_1,\dots,g_k})$ of the permutation $\sigma_{g_1,\dots,g_k}$.

Remark 6.2.13. It is straightforward from the definition that

$$\widetilde{\varepsilon}_{g,\mathrm{id}} = 1 = \widetilde{\varepsilon}_{\mathrm{id},g}, \qquad g \in G_f, \tag{6.4a}$$

$$\widetilde{\varepsilon}_{g,h} = (-1)^{(n-n_g)(n-n_h)}\widetilde{\varepsilon}_{h,g}, \qquad g,h \in G_f,\ I_{g,h} = \emptyset, \tag{6.4b}$$

$$\widetilde{\varepsilon}_{g,g'}\widetilde{\varepsilon}_{gg',g''} = \widetilde{\varepsilon}_{g,g',g''} = \widetilde{\varepsilon}_{g,g'g''}\widetilde{\varepsilon}_{g',g''}, \qquad g,g',g'' \in G_f,\ I_{g,g'} = I_{g',g''} = I_{g,g''} = \emptyset. \tag{6.4c}$$

Uniqueness

Throughout this subsection, $f = f(x_1, \dots, x_n)$ denotes an invertible polynomial. And we show, as mentioned in Section 5.4, the uniqueness of $\mathrm{Jac}'(f, G)$ for any $G \subset G_f$.

Take a nowhere vanishing n-form $dx_1 \wedge \cdots \wedge dx_n$ and set $\zeta := [dx_1 \wedge \cdots \wedge dx_n] \in \Omega_f$.

Definition 6.2.14. Fix also a map

$$\alpha : G_f \longrightarrow \mathbb{C}^*, \quad g \mapsto \alpha_g,$$

such that $\alpha_{\mathrm{id}} = 1$ and

$$\alpha_g\alpha_{g^{-1}} = (-1)^{\frac{1}{2}(n-n_g)(n-n_g+1)}, \quad g \in G_f.$$

Such a map α always exists since for each g we may choose α_g as

$$\alpha_g = \mathbf{e}\left[\frac{1}{8}(n-n_g)(n-n_g+1)\right].$$

For each $g \in G$, let v_g be as in Definition 5.4.5

$$v_g \vdash \zeta = \alpha_g\omega_g.$$

Proposition 6.2.15. *For a pair (g, h) of elements of G which is not spanning, we have $v_g \circ v_h = 0 \in \mathrm{Jac}'(f, G)$.*

Proof. Denote by $[\gamma'_{g,h}(\mathbf{x})]$ the element of $\mathrm{Jac}(f^{gh})$ satisfying $v_g \circ v_h = [\gamma'_{g,h}(\mathbf{x})]v_{gh}$. Suppose that $f = f_1 \oplus \cdots \oplus f_p$ is a Sebastiani-Thom sum such that each f_ν, $\nu = 1, \dots, p$, is either of chain type or loop type. Without loss of generality, we may assume the coordinate x_k, $k \notin I_g \cup I_h \cup I_{gh}$ to be a variable of the polynomial f_1. Consider the following two cases;

(a) $f_1 = x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{m-1}^{a_{m-1}}x_m + x_m^{a_m}$ is of chain type.

(b) $f_1 = x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{m-1}^{a_{m-1}}x_m + x_m^{a_m}x_1$ is of loop type.

Case (a): First, note that $1 \notin I_g \cup I_h \cup I_{gh}$. Consider $(\frac{1}{a_1}, 0 \dots, 0) \in \mathrm{Aut}(f_1, G)$ and extend it naturally to the element $\varphi \in \mathrm{Aut}(f, G)$. Since $1 \notin I_g \cup I_h \cup I_{gh}$, we have $\varphi^*(v_{g'}) = \mathbf{e}\left[-\frac{1}{a_1}\right] v_{g'}$ (see Equation (5.2)) for $g' \in \{g, h, gh\}$. Axiom (iva) yields $\varphi^*([\gamma'_{g,h}(\mathbf{x})]) = \mathbf{e}\left[-\frac{1}{a_1}\right] [\gamma'_{g,h}(\mathbf{x})]$. On the other hand, we have $\varphi^*([\gamma'_{g,h}(\mathbf{x})]) = [\gamma'_{g,h}(\mathbf{x})]$ since $1 \notin I_{gh}$. Hence, $[\gamma'_{g,h}(\mathbf{x})] = 0$.

Case (b): First, note that $1, \dots, m \notin I_g \cup I_h \cup I_{gh}$. Choose an element of $G_{f_1} \backslash G_{f_1}^{\mathrm{SL}}$, which exists due to Proposition 6.1.7, and extend it naturally to the element $\varphi \in \mathrm{Aut}(f, G)$. There exists a complex number $\lambda_\varphi \neq 1$, the determinant of φ regarded as an element of $\mathrm{GL}(n; \mathbb{C})$, such that $\varphi^*(v_{g'}) = \lambda_\varphi^{-1} v_{g'}$ for $g' \in \{g, h, gh\}$ since $1, \dots, m \notin I_g \cup I_h \cup I_{gh}$. Axiom (iva) yields $\varphi^*([\gamma'_{g,h}(\mathbf{x})]) = \lambda_\varphi^{-1}[\gamma'_{g,h}(\mathbf{x})]$. On the other hand, we have $\varphi^*([\gamma'_{g,h}(\mathbf{x})]) = [\gamma'_{g,h}(\mathbf{x})]$ since $1, \dots, m \notin I_{gh}$. Hence, $[\gamma'_{g,h}(\mathbf{x})] = 0$. □

We consider the product $v_g \circ v_h$ for a spanning pair (g, h).

Proposition 6.2.16. *For each spanning pair (g, h) of elements of G, there exists $c_{g,h} \in \mathbb{C}$ such that*

$$v_g \circ v_h = c_{g,h}[H_{g,h}]v_{gh}.$$

Moreover, $c_{g,h}$ does not depend on the choice of the subgroup G of G_f containing g, h.

Proof. We only need to show the first statement since the second one follows from it together with Axiom (vi), the Definition 5.4.5 of v_g and the independence of $H_{g,h}$ from a particular choice of G. Based on Lemma 6.2.8, we study which variable in f_ν can appear in the product structure.

Lemma 6.2.17. *Let the notation and the cases be as in Lemma 6.2.8 above. There is a polynomial $\gamma_{g,h}(\mathbf{x}) \in \mathbb{C}[x_1, \dots, x_n]$ which doesn't depend on $x_{i_1}, \dots, x_{i_m}$ such that one of the following holds:*

(i) *(a)* $v_g \circ v_h = [\gamma_{g,h}(\mathbf{x})]v_{gh}$

(b) $$v_g \circ v_h = \begin{cases} \left[\gamma_{g,h}(\mathbf{x}) \cdot \left(x_{i_1}^{a_{i_1}-2} x_{i_2}^{a_{i_2}-1} \cdots x_{i_m}^{a_{i_m}-1}\right)\right] v_{gh} & \text{if } l = m \\ \left[\gamma_{g,h}(\mathbf{x}) \cdot \left(x_{i_1}^{a_{i_1}-2} x_{i_2}^{a_{i_2}-1} \cdots x_{i_l}^{a_{i_l}-1} x_{i_{l+1}}\right)\right] v_{gh} & \text{if } l < m \end{cases}$$

(ii) *(a)* $v_g \circ v_h = [\gamma_{g,h}(\mathbf{x})]v_{gh}$

(b) $v_g \circ v_h = [\gamma_{g,h}(\mathbf{x})]v_{gh}$

(c) $v_g \circ v_h = \left[\gamma_{g,h}(\mathbf{x}) \cdot \left(x_{i_1}^{a_{i_1}-1} x_{i_2}^{a_{i_2}-1} \cdots x_{i_m}^{a_{i_m}-1}\right)\right] v_{gh}$

Here, we denote by $[\gamma_{g,h}(\mathbf{x})]$ the class of $\gamma_{g,h}(\mathbf{x})$ in $\mathrm{Jac}(f^{gh})$.

Proof. (i): We may assume $f_\nu = x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_m^{a_m}$. For each $r = 1, \dots, m$, there is a unique element $\varphi_r \in \mathrm{Aut}(f_\nu, G)$ such that $\varphi_r^*(x_i) = x_i$ for all $i = r+1, \dots, m$, which is explicitly given by

$$\varphi_r^*(x_r) := \mathbf{e}\left[\frac{1}{a_r}\right] x_r,$$
$$\varphi_r^*(x_i) := \mathbf{e}\left[\frac{1}{a_i}\left(1 - \frac{1}{a_{i+1}}\left(1 - \cdots - \frac{1}{a_{r-1}}\left(1 - \frac{1}{a_r}\right)\right)\right)\right] x_i, \ 1 \le i < r.$$

Denote also by φ_r its natural extension to $\mathrm{Aut}(f, G)$ and by $\lambda_{\varphi_r} \in \mathbb{C}^*$ the determinant of φ_r as an element of $\mathrm{GL}(n; \mathbb{C})$.

(a) For each $r = 1, \ldots, m$, we have $\varphi_r^*(v_g) = v_g$, $\varphi_r^*(v_h) = \lambda_{\varphi_r}^{-1} v_h$ and $\varphi_r^*(v_{gh}) = \lambda_{\varphi_r}^{-1} v_{gh}$. Suppose that a polynomial $\gamma_{g,h}(\mathbf{x}) \in \mathbb{C}[x_1, \ldots, x_n]$ satisfies $v_g \circ v_h = [\gamma_{g,h}(\mathbf{x})] v_{gh}$. By Axiom (iva), we obtain

$$\begin{aligned}[\varphi_r^*(\gamma_{g,h}(\mathbf{x}))]v_{gh} &= \lambda_{\varphi_r}\varphi_r^*([\gamma_{g,h}(\mathbf{x})]v_{gh}) = \lambda_{\varphi_r}\varphi_r^*(v_g \circ v_h) \\ &= \lambda_{\varphi_r}\varphi_r^*(v_g) \circ \varphi_r^*(v_h) = v_g \circ v_h = [\gamma_{g,h}(\mathbf{x})]v_{gh},\end{aligned}$$

and hence $\varphi_r^*([\gamma_{g,h}(\mathbf{x})]) = [\gamma_{g,h}(\mathbf{x})]$ in $\mathrm{Jac}(f^{gh})$. In view of the above action of φ_r and Proposition 6.1.11, the polynomial $\gamma_{g,h}(\mathbf{x})$ can be chosen so that it does not depend on x_i, $i = 1, \ldots, m$.

(b) For each $r = 1, \ldots, m$, we have $\varphi_r^*(v_g) = \lambda_{\varphi_r}^{-1} v_g$, $\varphi_r^*(v_h) = \lambda_{\varphi_r}^{-1} v_h$ and $\varphi_r^*(v_{gh}) = v_{gh}$. Suppose that a polynomial $\gamma'_{g,h}(\mathbf{x}) \in \mathbb{C}[x_1, \ldots, x_n]$ satisfies $v_g \circ v_h = [\gamma'_{g,h}(\mathbf{x})] v_{gh}$. By Axiom (iva), we obtain

$$\begin{aligned}[\varphi_r^*(\gamma'_{g,h}(\mathbf{x}))]v_{gh} &= \varphi_r^*([\gamma'_{g,h}(\mathbf{x})]v_{gh}) = \varphi_r^*(v_g \circ v_h) \\ &= \varphi_r^*(v_g) \circ \varphi_r^*(v_h) = \lambda_{\varphi_r}^{-2}(v_g \circ v_h) = \lambda_{\varphi_r}^{-2}[\gamma'_{g,h}(\mathbf{x})]v_{gh},\end{aligned}$$

and hence $[\varphi_r^*(\gamma'_{g,h}(\mathbf{x}))] = \lambda_{\varphi_r}^{-2}[\gamma'_{g,h}(\mathbf{x})]$ in $\mathrm{Jac}(f^{gh})$. In view of the above action of φ_r and Proposition 6.1.11, the polynomial $\gamma'_{g,h}(\mathbf{x})$ can be chosen so that it is divisible by $x_1^{a_1-2}x_2^{a_2-1}\cdots x_m^{a_m-1}$ if $l = m$ and by $x_1^{a_1-2}x_2^{a_2-1}\cdots x_l^{a_l-1}x_{l+1}$ if $l < m$.

(ii): We may assume $f_\nu = x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_m^{a_m}x_1$. For each element $\varphi \in G_{f_\nu} \subset \mathrm{Aut}(f_\nu, G)$, denote also by φ its natural extension to $\mathrm{Aut}(f, G)$. Let $\lambda_\varphi \in \mathbb{C}^*$ be the determinant of φ as an element of $\mathrm{GL}(n; \mathbb{C})$. Note that if $\varphi \neq \mathrm{id}$ then $\varphi^*(x_i) \neq x_i$ for all $i = 1, \ldots, m$.

(a) For all $\varphi \in G_{f_\nu}$, we have $\varphi^*(v_g) = v_g$, $\varphi^*(v_h) = v_h$ and $\varphi^*(v_{gh}) = v_{gh}$. Suppose that a polynomial $\gamma_{g,h}(\mathbf{x}) \in \mathbb{C}[x_1, \ldots, x_n]$ satisfies $v_g \circ v_h = [\gamma_{g,h}(\mathbf{x})]v_{gh}$. By Axiom (iva), we obtain

$$\begin{aligned}[\varphi^*(\gamma_{g,h}(\mathbf{x}))]v_{gh} &= \varphi^*(\gamma_{g,h}(\mathbf{x})v_{gh}) = \varphi^*(v_g \circ v_h) \\ &= \varphi^*(v_g) \circ \varphi^*(v_h) = v_g \circ v_h = [\gamma_{g,h}(\mathbf{x})]v_{gh},\end{aligned}$$

and hence $[\varphi^*(\gamma_{g,h}(\mathbf{x}))] = [\gamma_{g,h}(\mathbf{x})]$ in $\mathrm{Jac}(f^{gh})$. In view of Proposition 6.1.11, the polynomial $\gamma_{g,h}(\mathbf{x})$ can be chosen so that it does not depend on x_i, $i = 1, \ldots, m$.

(b) Suppose that a polynomial $\gamma_{g,h}(\mathbf{x}) \in \mathbb{C}[x_1, \ldots, x_n]$ satisfies $v_g \circ v_h = [\gamma_{g,h}(\mathbf{x})]v_{gh}$. Since $1, \ldots, m$ do not belong to $I_g \cap I_h$ nor $I_{g,h}$, it is obvious that the polynomial $\gamma_{g,h}(\mathbf{x})$ can be chosen so that it does not depend on x_i, $i = 1, \ldots, m$.

(c) For all $\varphi \in G_{f_\nu}$, we have $\varphi^*(v_g) = \lambda_\varphi^{-1}v_g$, $\varphi^*(v_h) = \lambda_\varphi^{-1}v_h$ and $\varphi^*(v_{gh}) = v_{gh}$. Suppose that a polynomial $\gamma'_{g,h}(\mathbf{x}) \in \mathbb{C}[x_1, \ldots, x_n]$ satisfies $v_g \circ v_h = [\gamma'_{g,h}(\mathbf{x})]v_{gh}$. By Axiom (iva), we obtain

$$\begin{aligned}[\varphi^*(\gamma'_{g,h}(\mathbf{x}))]v_{gh} &= \varphi^*(\gamma'_{g,h}(\mathbf{x})v_{gh}) = \varphi^*(v_g \circ v_h) \\ &= \varphi^*(v_g) \circ \varphi^*(v_h) = \lambda_\varphi^{-2}(v_g \circ v_h) = \lambda_\varphi^{-2}[\gamma'_{g,h}(\mathbf{x})]v_{gh},\end{aligned}$$

and hence $[\varphi^*(\gamma'_{g,h}(\mathbf{x}))] = \lambda_\varphi^{-2}[\gamma'_{g,h}(\mathbf{x})]$ in $\mathrm{Jac}(f^{gh})$. In view of Proposition 6.1.11, the polynomial $\gamma'_{g,h}(\mathbf{x})$ can be chosen so that it is divisible by $x_1^{a_1-1}x_2^{a_2-1}\cdots x_m^{a_m-1}$. $\square$

Now the first statement of the proposition is a direct consequence of Lemma 6.2.17, since $H_{g,h}$ is a constant multiple of the product of the monomials in the round brackets there. We have finished the proof of the proposition. $\square$

By Proposition 6.2.16, we may assume that $G = G_f$. We give some properties of $c_{g,h}$.

Lemma 6.2.18. *For each $g \in G_f$, we have*

$$c_{g,g^{-1}} = (-1)^{\frac{1}{2}(n-n_g)(n-n_g-1)} \cdot \mathbf{e}\left[-\frac{1}{2}\mathrm{age}(g)\right].$$

Proof. We have

$$\begin{aligned}
\frac{1}{\mu_{f^g}} J_{f,g}([\mathrm{hess}(f^g)]v_g \vdash \zeta, v_{g^{-1}} \vdash \zeta) &= \frac{\alpha_g \alpha_{g^{-1}}}{\mu_{f^g}} J_{f,g}([\mathrm{hess}(f^g)]\omega_g, \omega_{g^{-1}}) \\
&= (-1)^{\frac{1}{2}(n-n_g)(n-n_g-1)} \cdot \mathbf{e}\left[-\frac{1}{2}\mathrm{age}(g)\right] \cdot |G|.
\end{aligned}$$

On the other hand, by Axiom (v) and normalization (6.2) of $H_{g,h}$, we have

$$\begin{aligned}
\frac{1}{\mu_{f^g}} J_{f,g}([\mathrm{hess}(f^g)]v_g \vdash \zeta, v_{g^{-1}} \vdash \zeta) &= \frac{1}{\mu_{f^g}} J_{f,id}(\omega_{\mathrm{id}}, [\mathrm{hess}(f^g)]v_g \circ v_{g^{-1}} \vdash \zeta) \\
&= \frac{1}{\mu_{f^g}} J_{f,id}(\omega_{\mathrm{id}}, c_{g,g^{-1}}[\mathrm{hess}(f^g)H_{g,g^{-1}}]\omega_{\mathrm{id}}) \\
&= \frac{c_{g,g^{-1}}}{\mu_f} J_{f,id}(\omega_{\mathrm{id}}, [\mathrm{hess}(f)]\omega_{\mathrm{id}}) \\
&= c_{g,g^{-1}}|G|.
\end{aligned}$$

$\square$

Lemma 6.2.19. *For each pair (g,h) of elements of G_f such that $I_{g,h} = \emptyset$, we have*

$$c_{g,h}c_{h^{-1},g^{-1}} = (-1)^{(n-n_g)(n-n_h)}.$$

In particular it follows that $c_{g,h} \neq 0$.

Remark 6.2.20. If $I_{g,h} = \emptyset$ for a pair (g,h) of elements of G_f, it is spanning.

Proof. We have

$$\begin{aligned}
&v_g \circ (v_h \circ v_{h^{-1}}) \circ v_{g^{-1}} \\
&\quad = (-1)^{\frac{1}{2}(n-n_g)(n-n_g-1)+\frac{1}{2}(n-n_h)(n-n_h-1)} \cdot \mathbf{e}\left[-\frac{1}{2}\mathrm{age}(g) - \frac{1}{2}\mathrm{age}(h)\right][g^*(H_{h,h^{-1}})H_{g,g^{-1}}]v_{\mathrm{id}}, \\
&(v_g \circ v_h) \circ (v_{h^{-1}} \circ v_{g^{-1}}) \\
&\quad = (-1)^{\frac{1}{2}(n-n_{gh})(n-n_{gh}-1)}\mathbf{e}\left[-\frac{1}{2}\mathrm{age}(gh)\right] c_{g,h}c_{h^{-1},g^{-1}}[H_{gh,(gh)^{-1}}]v_{\mathrm{id}}.
\end{aligned}$$

The proposition follows from the facts that the product $\circ$ is associative, $g^*(H_{h,h^{-1}}) = H_{h,h^{-1}}$ since $I_{g,h} = \emptyset$, $[H_{g,g^{-1}}H_{h,h^{-1}}] = [H_{gh,(gh)^{-1}}]$ in $\mathrm{Jac}(f)$, $\mathrm{age}(g) + \mathrm{age}(h) = \mathrm{age}(gh)$ since $I_{g,h} = \emptyset$, and $(n-n_g)+(n-n_h) \equiv (n-n_{gh}) \pmod 2$ by Proposition 6.2.11. $\square$

Corollary 6.2.21. *Let (g, h) be a spanning pair of elements of G_f with the factorization (g_1, g_2, h_1, h_2). The complex numbers c_{g_1,h_2}, c_{g_2,h_1} and c_{g_1,h_1} are non-zero.*

Proof. It follows from the fact that $I_{g_1,h_2} = \emptyset$, $I_{g_2,h_1} = \emptyset$ and $I_{g_1,h_1} = \emptyset$. □

Proposition 6.2.22. *Let (g, h) be a spanning pair of elements of G_f with the factorization (g_1, g_2, h_1, h_2). We have*

$$c_{g,h} = (-1)^{\frac{1}{2}(n-n_{g_2})(n-n_{g_2}-1)} \cdot \mathbf{e}\left[-\frac{1}{2}\mathrm{age}(g_2)\right] \cdot \frac{c_{g_1,h_1}}{c_{g_1,g_2}c_{h_2,h_1}}.$$

In particular, $c_{g,h} \neq 0$.

Proof. We have

$$\begin{aligned} v_{g_1} \circ (v_{g_2} \circ v_{h_2}) \circ v_{h_1} &= (-1)^{\frac{1}{2}(n-n_{g_2})(n-n_{g_2}-1)} \cdot \mathbf{e}\left[-\frac{1}{2}\mathrm{age}(g_2)\right] \cdot v_{g_1} \circ [H_{g_2,g_2^{-1}}]v_{\mathrm{id}} \circ v_{h_1} \\ &= (-1)^{\frac{1}{2}(n-n_{g_2})(n-n_{g_2}-1)} \cdot \mathbf{e}\left[-\frac{1}{2}\mathrm{age}(g_2)\right] \cdot c_{g_1,h_1}[H_{g_2,g_2^{-1}}]v_{gh}. \end{aligned}$$

On the other hand, we get:

$$\begin{aligned} (v_{g_1} \circ v_{g_2}) \circ (v_{h_2} \circ v_{h_1}) &= c_{g_1,g_2}v_{g_1g_2} \circ c_{h_2,h_1}v_{h_1h_2} \\ &= c_{g_1,g_2}c_{h_2,h_1}c_{g,h}[H_{g,h}]v_{gh}. \end{aligned}$$

Note that $H_{g,h} = H_{g_2,g_2^{-1}} = H_{h_2,h_2^{-1}}$ by the definition of the factorization (g_1, g_2, h_1, h_2). By Corollary 6.2.21, we know that c_{g_1,g_2} and c_{h_2,h_1} are non-zero, which gives the statement. □

Hence, by this proposition, we only have to determine $c_{g,h}$ for all pairs (g, h) of elements of G_f such that $I_{g,h} = \emptyset$.

Remark 6.2.23. Suppose that $f = f_1 \oplus \cdots \oplus f_p$ is a Sebastiani-Thom sum such that each f_ν, $\nu = 1, \ldots, p$, is either of chain type or loop type. Then, we have a natural isomorphism $G_f \cong G_{f_1} \times \cdots \times G_{f_p}$. Therefore, it follows that each $g \in G_f$ has a unique expression $g = g_1 \cdots g_p$ such that $g_i \in G_{f_i}$ for all $i = 1, \ldots, p$, hence $I_{g_i,g_j} = \emptyset$ if $i \neq j$ and $I_g^c = I_{g_1}^c \dot\cup \ldots \dot\cup I_{g_p}^c$.

Definition 6.2.24. With this notation, define $\widetilde{v}_g$ by

$$\widetilde{v}_g := \widetilde{\varepsilon}_{g_1,\ldots,g_p} v_{g_1} \circ \cdots \circ v_{g_p}.$$

Obviously, $\widetilde{v}_g$ is a non-zero constant multiple of v_g for all $g \in G_f$.

Remark 6.2.25. It is also easy to see that $\widetilde{v}_g$ does not depend on the choice of ordering in the Sebastiani-Thom sum and by having equation (6.4) in mind for a pair (g, h) of elements of G_f with $I_{g,h} = \emptyset$ we have

$$\widetilde{v}_g \circ \widetilde{v}_h = \frac{1}{\widetilde{\varepsilon}_{g,h}} \widetilde{v}_{gh}.$$

Proposition 6.2.26. *For each $g \in G$, we have*

$$\widetilde{v}_g \circ \widetilde{v}_{g^{-1}} = (-1)^{\frac{1}{2}(n-n_g)(n-n_g-1)} \cdot \mathbf{e}\left[-\frac{1}{2}\mathrm{age}(g)\right] \cdot [H_{g,g^{-1}}]\widetilde{v}_{\mathrm{id}}.$$

Proof. There is an inductive presentation of $\widetilde{v}_g$ given by

$$\widetilde{v}_g = \begin{cases} v_{g_1} & \text{if} \quad g = g_1 \\ \widetilde{\varepsilon}_{g_1 \dots g_i, g_{i+1}} \widetilde{v}_{g_1 \dots g_i} \circ v_{g_{i+1}} & \text{if} \quad g = g_1 \dots g_i g_{i+1}, \ i = 1, \dots, p-1 \end{cases}.$$

The statement follows by induction from the following calculation:

$$\begin{aligned}
\widetilde{v}_g \circ \widetilde{v}_{g^{-1}} &= (\widetilde{\varepsilon}_{g_1 \dots g_i, g_{i+1}} \widetilde{v}_{g_1 \dots g_i} \circ v_{g_{i+1}}) \circ (\widetilde{\varepsilon}_{g_1^{-1} \dots g_i^{-1}, g_{i+1}^{-1}} \widetilde{v}_{g_1^{-1} \dots g_i^{-1}} \circ v_{g_{i+1}^{-1}}) \\
&= (-1)^{(n-n_{g_1^{-1} \dots g_i^{-1}})(n-n_{g_{i+1}})} \cdot (\widetilde{v}_{g_1 \dots g_i} \circ \widetilde{v}_{g_1^{-1} \dots g_i^{-1}}) \circ (v_{g_{i+1}} \circ v_{g_{i+1}^{-1}}) \\
&= (-1)^{(n-n_{g_1 \dots g_i})(n-n_{g_{i+1}}) + \frac{1}{2}(n-n_{g_1 \dots g_i})(n-n_{g_1 \dots g_i}-1) + \frac{1}{2}(n-n_{g_{i+1}})(n-n_{g_{i+1}}-1)} \\
&\quad \cdot \mathbf{e}\left[-\frac{1}{2}\mathrm{age}(g_1 \dots g_i) - \frac{1}{2}\mathrm{age}(g_{i+1})\right] \cdot [H_{g_1 \dots g_i, g_1^{-1} \dots g_i^{-1}} H_{g_{i+1}, g_{i+1}^{-1}}]\widetilde{v}_{\mathrm{id}} \\
&= (-1)^{\frac{1}{2}(n-n_g)(n-n_g-1)} \cdot \mathbf{e}\left[-\frac{1}{2}\mathrm{age}(g)\right] \cdot [H_{g,g^{-1}}]\widetilde{v}_{\mathrm{id}}
\end{aligned}$$

□

This proposition states that by replacing the map $\alpha : G_f \longrightarrow \mathbb{C}^*$ by a suitable one we have a new basis $\{\widetilde{v}_g\}_{g \in G_f}$ instead of $\{v_g\}_{g \in G_f}$. To summarize, we finally obtain the following:

Corollary 6.2.27. *Let (g, h) be a spanning pair of elements of G_f with the factorization (g_1, g_2, h_1, h_2). We have*

$$\widetilde{v}_g \circ \widetilde{v}_h = (-1)^{\frac{1}{2}(n-n_{g_2})(n-n_{g_2}-1)} \cdot \mathbf{e}\left[-\frac{1}{2}\mathrm{age}(g_2)\right] \cdot \frac{\widetilde{\varepsilon}_{g_1,g_2}\widetilde{\varepsilon}_{h_2,h_1}}{\widetilde{\varepsilon}_{g_1,h_1}}[H_{g,h}]\widetilde{v}_{gh}.$$

In particular, for any subgroup G of G_f, if a G-twisted Jacobian algebra of f exists, then it is uniquely determined by the axioms in Definition 5.2.1 up to isomorphism.

Existence

Throughout this subsection, $f = f(x_1, \dots, x_n)$ denotes an invertible polynomial. And we show, as mentioned in Section 5.4, the existence of $\mathrm{Jac}'(f, G)$ for any $G \subset G_f$. Let $\mathcal{A}'$ be as in Definition 5.4.7.

Definition 6.2.28. For a spanning pair (g, h) of elements of G_f with the factorization (g_1, g_2, h_1, h_2), set

$$\overline{c}_{g,h} := (-1)^{\frac{1}{2}(n-n_{g_2})(n-n_{g_2}-1)} \cdot \mathbf{e}\left[-\frac{1}{2}\mathrm{age}(g_2)\right] \cdot \frac{\widetilde{\varepsilon}_{g_1,g_2}\widetilde{\varepsilon}_{h_2,h_1}}{\widetilde{\varepsilon}_{g_1,h_1}}. \tag{6.5}$$

Remark 6.2.29. It is easy to see that

$$\overline{c}_{g,\mathrm{id}} = 1 = \overline{c}_{\mathrm{id},g}, \qquad g \in G_f,$$
$$\overline{c}_{g,g^{-1}} = (-1)^{\frac{1}{2}(n-n_g)(n-n_g-1)} \cdot \mathbf{e}\left[-\frac{1}{2}\mathrm{age}(g)\right], \qquad g \in G_f,$$
$$\overline{c}_{g,h} = \widetilde{\varepsilon}_{g,h}^{-1}, \qquad g,h \in G_f,\ I_{g,h} = \emptyset.$$

Definition 6.2.30. For each $g, h \in G_f$, define an element of $\mathcal{A}'_{gh}$ by

$$\overline{v}_g \circ \overline{v}_h := \begin{cases} \overline{c}_{g,h}\,[H_{g,h}]\,\overline{v}_{gh} & \text{if the pair } (g,h) \text{ is spanning} \\ 0 & \text{otherwise} \end{cases}.$$

It is clear that $\overline{v}_{\mathrm{id}} \circ \overline{v}_g = \overline{v}_g = \overline{v}_g \circ \overline{v}_{\mathrm{id}}$ since $I_{\mathrm{id},g} = I_{g,\mathrm{id}} = \emptyset$ and hence $[H_{\mathrm{id},g}] = [H_{g,\mathrm{id}}] = 1$.

Proposition 6.2.31. *For a spanning pair (g,h) of elements of G_f with the factorization (g_1, g_2, h_1, h_2), we have*

$$\overline{c}_{g,h} = (-1)^{(n-n_g)(n-n_h)} \cdot \mathbf{e}\left[-\mathrm{age}(g_2)\right] \cdot \overline{c}_{h,g}.$$

Hence, we have

$$\overline{v}_g \circ \overline{v}_h = (-1)^{(n-n_g)(n-n_h)} \cdot \left(\mathbf{e}\left[-\mathrm{age}(g_2)\right]\overline{v}_h \circ \overline{v}_g\right).$$

Proof. We have

$$\begin{aligned}\overline{c}_{g,h} =& (-1)^{\frac{1}{2}(n-n_{g_2})(n-n_{g_2}-1)} \cdot \mathbf{e}\left[-\frac{1}{2}\mathrm{age}(g_2)\right] \cdot \frac{\widetilde{\varepsilon}_{g_1,g_2}\widetilde{\varepsilon}_{h_2,h_1}}{\widetilde{\varepsilon}_{g_1,h_1}} \\ =& (-1)^{(n-n_{g_1})(n-n_{g_2})+(n-n_{h_1})(n-n_{h_2})-(n-n_{g_1})(n-n_{h_1})+(n-n_{g_2})} \cdot \mathbf{e}\left[-\mathrm{age}(g_2)\right] \\ & \cdot (-1)^{\frac{1}{2}(n-n_{h_2})(n-n_{h_2}-1)} \cdot \mathbf{e}\left[-\frac{1}{2}\mathrm{age}(h_2)\right] \cdot \frac{\widetilde{\varepsilon}_{h_1,h_2}\widetilde{\varepsilon}_{g_2,g_1}}{\widetilde{\varepsilon}_{h_1,g_1}} \\ =& (-1)^{(n-n_g)(n-n_h)} \cdot \mathbf{e}\left[-\mathrm{age}(g_2)\right] \cdot \overline{c}_{h,g},\end{aligned}$$

where we used that $h_2 = g_2^{-1}$, $n - n_{g_2} = \mathrm{age}(g_2) + \mathrm{age}(h_2)$ and Proposition 6.2.11. □

Proposition 6.2.32. *For each $g, g', g'' \in G_f$, we have*

$$(\overline{v}_g \circ \overline{v}_{g'}) \circ \overline{v}_{g''} = \overline{v}_g \circ (\overline{v}_{g'} \circ \overline{v}_{g''}). \tag{6.6}$$

Proof. First, we show the following

Lemma 6.2.33. *For $g, g', g'' \in G_f$, suppose that (g, g') and (gg', g'') are spanning pairs with $I_{g,g'} \subset I_{g''}$.*

(i) *There exist $g_1, g_2, g_3, g'_1, g'_2, g'_3, g''_1, g''_2, g''_3 \in G_f$ such that*

$$g = g_1 g_2 g_3,\ g' = g'_1 g'_2 g'_3,\ g'' = g''_1 g''_2 g''_3, \quad g'_1 g''_1 = \mathrm{id}, g_2 g''_2 = \mathrm{id},\ g_3 g'_3 = \mathrm{id},$$

and $(g_1 g_2, g_3, g'_1 g'_2, g'_3)$ is the factorization of (g, g') and $(g_1 g'_2, g_2 g'_1, g''_3, g''_1 g''_2)$ is the factorization of (gg', g'').

(ii) *The pairs* (g', g'') *and* $(g, g'g'')$ *are spanning such that* $I_{g',g''} \subset I_g$ *and* $(g'_2 g'_3, g'_1, g''_2 g''_3, g''_1)$ *is the factorization of* (g', g'') *and* $(g_1, g_2 g_3, g'_2 g''_3, g''_2 g'_3)$ *is the factorization of* $(g, g'g'')$.

Proof. (i) Similarly to the presentation of (6.1), the elements g, g', g'' satisfying the conditions can be expressed, in the multiplicative form, as follows:

$$\begin{array}{lcccccccccccc} g & = & g_1 & \cdot & g_2 & \cdot & \mathrm{id} & \cdot & \mathrm{id} & \cdot & g_3 & \cdot & \mathrm{id} \\ g' & = & \mathrm{id} & \cdot & \mathrm{id} & \cdot & g'_1 & \cdot & g'_2 & \cdot & g'_3 & \cdot & \mathrm{id} \\ g'' & = & \mathrm{id} & \cdot & g''_2 & \cdot & g''_1 & \cdot & \mathrm{id} & \cdot & \mathrm{id} & \cdot & g''_3 \end{array}$$

(ii) By the above presentation, it is easy to see that (g, g') and (gg', g'') are spanning pairs with the given factorization. It follows from $g'_1 g''_1 = \mathrm{id}$ that $I_{g',g''} \subset I_g$. □

Lemma 6.2.34. *The LHS of* (6.6) *is non-zero if and only if the RHS of* (6.6) *is non-zero.*

Proof. By Proposition 6.2.10 (iii), the LHS of (6.6) is non-zero only if both pairs (g, g') and (gg', g'') are spanning and $I_{g,g'} \subset I_{g''}$ and the RHS of (6.6) is non-zero only if both pairs $(g, g'g'')$ and (g', g'') are spanning and $I_{g',g''} \subset I_g$. Lemma 6.2.33 together with Proposition 6.2.31 yields the statement. □

Lemma 6.2.35. *Let the notations be as above. We have*

$$H_{g,g'} = H_{g_3,g'_3},\ H_{gg',g''} = H_{g_2 g'_1, g''_2 g''_1},\ H_{g,g'g''} = H_{g_2 g_3, g''_2 g'_3},\ H_{g',g''} = H_{g'_1,g''_1},$$

and hence $[H_{g,g'} H_{gg',g''}] = [H_{g,g'g''} H_{g',g''}]$ *in* $\mathrm{Jac}(f^{gg'g''})$.

Proof. The first statement follows from the definition of $H_{g,h}$ and the second one does from Proposition 6.2.10 (ii). □

Therefore, we only have to show the following

Lemma 6.2.36. *Let the notations be as above. We have*

$$\overline{c}_{g,g'} \overline{c}_{gg',g''} = \overline{c}_{g,g'g''} \overline{c}_{g',g''}.$$

Proof. It follows from the definition (6.5) that

$$\overline{c}_{g,g'} = (-1)^{\frac{1}{2}(n-n_{g_3})(n-n_{g_3}-1)} \cdot \mathbf{e}\left[-\frac{1}{2}\mathrm{age}(g_3)\right] \cdot \frac{\widetilde{\varepsilon}_{g_1 g_2, g_3} \widetilde{\varepsilon}_{g'_3, g'_1 g'_2}}{\widetilde{\varepsilon}_{g_1 g_2, g'_1 g'_2}},$$

$$\overline{c}_{gg',g''} = (-1)^{\frac{1}{2}(n-n_{g_2 g'_1})(n-n_{g_2 g'_1}-1)} \cdot \mathbf{e}\left[-\frac{1}{2}\mathrm{age}(g_2 g'_1)\right] \cdot \frac{\widetilde{\varepsilon}_{g_1 g'_2, g_2 g'_1} \widetilde{\varepsilon}_{g''_2 g''_1, g''_3}}{\widetilde{\varepsilon}_{g_1 g'_2, g''_3}},$$

$$\overline{c}_{g,g'g''} = (-1)^{\frac{1}{2}(n-n_{g_2 g_3})(n-n_{g_2 g_3}-1)} \cdot \mathbf{e}\left[-\frac{1}{2}\mathrm{age}(g_2 g_3)\right] \cdot \frac{\widetilde{\varepsilon}_{g_1, g_2 g_3} \widetilde{\varepsilon}_{g''_2 g'_3, g'_2 g''_3}}{\widetilde{\varepsilon}_{g_1, g'_2 g''_3}},$$

$$\overline{c}_{g',g''} = (-1)^{\frac{1}{2}(n-n_{g'_1})(n-n_{g'_1}-1)} \cdot \mathbf{e}\left[-\frac{1}{2}\mathrm{age}(g'_1)\right] \cdot \frac{\widetilde{\varepsilon}_{g'_2 g'_3, g'_1} \widetilde{\varepsilon}_{g''_1, g''_2 g''_3}}{\widetilde{\varepsilon}_{g'_2 g'_3, g''_2 g''_3}}.$$

Since all $I^c_{g_i}$, $I^c_{g'_i}$ and $I^c_{g''_i}$ are mutually disjoint, we get

$$
\begin{aligned}
\overline{c}_{g,g'}\overline{c}_{gg',g''} =&(-1)^{\frac{1}{2}(n-n_{g_3})(n-n_{g_3}-1)+\frac{1}{2}(n-n_{g_2g'_1})(n-n_{g_2g'_1}-1)} \\
&\cdot \mathbf{e}\left[-\frac{1}{2}\mathrm{age}(g_3)-\frac{1}{2}\mathrm{age}(g_2g'_1)\right]\cdot\frac{\widetilde{\varepsilon}_{g_1g_2,g_3}\widetilde{\varepsilon}_{g'_3,g'_1g'_2}}{\widetilde{\varepsilon}_{g_1g_2,g'_1g'_2}}\frac{\widetilde{\varepsilon}_{g_1g'_2,g_2g'_1}\widetilde{\varepsilon}_{g''_2g''_1,g''_3}}{\widetilde{\varepsilon}_{g_1g'_2,g''_3}} \\
=&(-1)^{\frac{1}{2}(n-n_{g_3})(n-n_{g_3}-1)+\frac{1}{2}(n-n_{g_2}+n-n_{g'_1})(n-n_{g_2}+n-n_{g'_1}-1)} \\
&\cdot \mathbf{e}\left[-\frac{1}{2}\mathrm{age}(g_3)-\frac{1}{2}\mathrm{age}(g_2)-\frac{1}{2}\mathrm{age}(g'_1)\right] \\
&\cdot\frac{\widetilde{\varepsilon}_{g_1,g_2}\widetilde{\varepsilon}_{g_1,g_2,g_3}\widetilde{\varepsilon}_{g'_3,g'_1,g'_2}\widetilde{\varepsilon}_{g'_1,g'_2}\widetilde{\varepsilon}_{g_1,g'_2}\widetilde{\varepsilon}_{g_1,g'_2,g'_1,g_2}\widetilde{\varepsilon}_{g'_1,g_2}\widetilde{\varepsilon}_{g''_1,g''_2}\widetilde{\varepsilon}_{g''_1,g''_2,g''_3}}{\widetilde{\varepsilon}_{g_1,g_2}\widetilde{\varepsilon}_{g_1,g_2,g'_1,g'_2}\widetilde{\varepsilon}_{g'_1,g'_2}\widetilde{\varepsilon}_{g_1,g'_2}\widetilde{\varepsilon}_{g_1,g'_2,g''_3}} \\
=&(-1)^{\frac{1}{2}\left((n-n_{g_3})^2-(n-n_{g_3})+(n-n_{g_2})^2-(n-n_{g_2})+(n-n_{g'_1})^2-(n-n_{g'_1})+2(n-n_{g_2})(n-n_{g'_1})\right)} \\
&\cdot \mathbf{e}\left[-\frac{1}{2}\mathrm{age}(g_3)-\frac{1}{2}\mathrm{age}(g_2)-\frac{1}{2}\mathrm{age}(g'_1)\right] \\
&\cdot\frac{\widetilde{\varepsilon}_{g_1,g_2,g_3}\widetilde{\varepsilon}_{g'_3,g'_1,g'_2}\widetilde{\varepsilon}_{g_1,g'_2,g'_1,g_2}\widetilde{\varepsilon}_{g'_1,g_2}\widetilde{\varepsilon}_{g''_1,g''_2}\widetilde{\varepsilon}_{g''_1,g''_2,g''_3}}{\widetilde{\varepsilon}_{g_1,g_2,g'_1,g'_2}\widetilde{\varepsilon}_{g_1,g'_2,g''_3}},
\end{aligned}
$$

and

$$
\begin{aligned}
\overline{c}_{g,g'g''}\overline{c}_{g',g''} =&(-1)^{\frac{1}{2}(n-n_{g_2g_3})(n-n_{g_2g_3}-1)+\frac{1}{2}(n-n_{g'_1})(n-n_{g'_1}-1)} \\
&\cdot \mathbf{e}\left[-\frac{1}{2}\mathrm{age}(g_2g_3)-\frac{1}{2}\mathrm{age}(g'_1)\right]\cdot\frac{\widetilde{\varepsilon}_{g_1,g_2g_3}\widetilde{\varepsilon}_{g''_2g'_3,g'_2g''_3}}{\widetilde{\varepsilon}_{g_1,g'_2g''_3}}\frac{\widetilde{\varepsilon}_{g'_2g'_3,g'_1}\widetilde{\varepsilon}_{g''_1,g''_2g''_3}}{\widetilde{\varepsilon}_{g'_2g'_3,g''_2g''_3}} \\
=&(-1)^{\frac{1}{2}(n-n_{g_2}+n-n_{g_3})(n-n_{g_2}+n-n_{g_3}-1)+\frac{1}{2}(n-n_{g'_1})(n-n_{g'_1}-1)} \\
&\cdot \mathbf{e}\left[-\frac{1}{2}\mathrm{age}(g_3)-\frac{1}{2}\mathrm{age}(g_2)-\frac{1}{2}\mathrm{age}(g'_1)\right] \\
&\cdot\frac{\widetilde{\varepsilon}_{g_1,g_2,g_3}\widetilde{\varepsilon}_{g_2,g_3}\widetilde{\varepsilon}_{g''_2,g'_3}\widetilde{\varepsilon}_{g''_2,g'_3,g'_2,g''_3}\widetilde{\varepsilon}_{g'_2,g''_3}\widetilde{\varepsilon}_{g'_2,g'_3}\widetilde{\varepsilon}_{g'_2,g'_3,g'_1}\widetilde{\varepsilon}_{g''_1,g''_2,g''_3}\widetilde{\varepsilon}_{g''_2,g''_3}}{\widetilde{\varepsilon}_{g_1,g'_2,g''_3}\widetilde{\varepsilon}_{g'_2,g''_3}\widetilde{\varepsilon}_{g'_2,g'_3}\widetilde{\varepsilon}_{g'_2,g'_3,g''_2,g''_3}\widetilde{\varepsilon}_{g''_2,g''_3}} \\
=&(-1)^{\frac{1}{2}\left((n-n_{g_3})^2-(n-n_{g_3})+(n-n_{g_2})^2-(n-n_{g_2})+(n-n_{g'_1})^2-(n-n_{g'_1})+2(n-n_{g_2})(n-n_{g_3})\right)} \\
&\cdot \mathbf{e}\left[-\frac{1}{2}\mathrm{age}(g_3)-\frac{1}{2}\mathrm{age}(g_2)-\frac{1}{2}\mathrm{age}(g'_1)\right] \\
&\cdot\frac{\widetilde{\varepsilon}_{g_1,g_2,g_3}\widetilde{\varepsilon}_{g_2,g_3}\widetilde{\varepsilon}_{g''_2,g'_3}\widetilde{\varepsilon}_{g''_2,g'_3,g'_2,g''_3}\widetilde{\varepsilon}_{g'_2,g'_3,g'_1}\widetilde{\varepsilon}_{g''_1,g''_2,g''_3}}{\widetilde{\varepsilon}_{g_1,g'_2,g''_3}\widetilde{\varepsilon}_{g'_2,g'_3,g''_2,g''_3}}.
\end{aligned}
$$

Therefore, we only have to show that

$$
\begin{aligned}
(-1)^{(n-n_{g_2})(n-n_{g'_1})}&\cdot\frac{\widetilde{\varepsilon}_{g_1,g_2,g_3}\widetilde{\varepsilon}_{g'_3,g'_1,g'_2}\widetilde{\varepsilon}_{g_1,g'_2,g'_1,g_2}\widetilde{\varepsilon}_{g'_1,g_2}\widetilde{\varepsilon}_{g''_1,g''_2}\widetilde{\varepsilon}_{g''_1,g''_2,g''_3}}{\widetilde{\varepsilon}_{g_1,g_2,g'_1,g'_2}\widetilde{\varepsilon}_{g_1,g'_2,g''_3}} \\
&= (-1)^{(n-n_{g_2})(n-n_{g_3})}\cdot\frac{\widetilde{\varepsilon}_{g_1,g_2,g_3}\widetilde{\varepsilon}_{g_2,g_3}\widetilde{\varepsilon}_{g''_2,g'_3}\widetilde{\varepsilon}_{g''_2,g'_3,g'_2,g''_3}\widetilde{\varepsilon}_{g'_2,g'_3,g'_1}\widetilde{\varepsilon}_{g''_1,g''_2,g''_3}}{\widetilde{\varepsilon}_{g_1,g'_2,g''_3}\widetilde{\varepsilon}_{g'_2,g'_3,g''_2,g''_3}}.
\end{aligned}
$$

Since $g'_1 g''_1 = \mathrm{id}$, $g_2 g''_2 = \mathrm{id}$ and $g_3 g'_3 = \mathrm{id}$, we have $I^c_{g'_1} = I^c_{g''_1}$, $I^c_{g_2} = I^c_{g''_2}$ and $I^c_{g_3} = I^c_{g'_3}$. We also have that $\widetilde{\varepsilon}^2_\bullet = 1$ for any expression $\bullet$. Hence, the problem is reduced to showing the following equation:

$$\begin{aligned}(-1)^{(n-n_{g_2})(n-n_{g'_1})} &\cdot \frac{\widetilde{\varepsilon}_{g_1,g_2,g_3}\widetilde{\varepsilon}_{g_3,g'_1,g'_2}\widetilde{\varepsilon}_{g_1,g'_2,g'_1,g_2}\widetilde{\varepsilon}_{g'_1,g_2,g''_3}}{\widetilde{\varepsilon}_{g_1,g_2,g'_1,g'_2}\widetilde{\varepsilon}_{g_1,g'_2,g''_3}} \\ &= (-1)^{(n-n_{g_2})(n-n_{g_3})} \cdot \frac{\widetilde{\varepsilon}_{g_1,g_2,g_3}\widetilde{\varepsilon}_{g_2,g_3,g'_2,g''_3}\widetilde{\varepsilon}_{g'_2,g_3,g'_1}\widetilde{\varepsilon}_{g'_1,g_2,g''_3}}{\widetilde{\varepsilon}_{g_1,g'_2,g''_3}\widetilde{\varepsilon}_{g'_2,g_3,g_2,g''_3}}\end{aligned}$$

Recall also that $\widetilde{\varepsilon}_\bullet$ is the signature of a permutation $\sigma_\bullet$ based on the expression $\bullet$ (see Definition 6.2.12), and hence we get a suitable sign by interchanging two indices, for example, $\widetilde{\varepsilon}_{g_3,g'_1,g'_2} = (-1)^{(n-n_{g'_1})(n-n_{g'_2})}\widetilde{\varepsilon}_{g_3,g'_2,g'_1}$. The LHS of the above equation is given by

$$\begin{aligned}(-1)^{(n-n_{g_2})(n-n_{g'_1})} &\cdot \frac{\widetilde{\varepsilon}_{g_1,g_2,g_3}\widetilde{\varepsilon}_{g_3,g'_1,g'_2}\widetilde{\varepsilon}_{g_1,g'_2,g'_1,g_2}\widetilde{\varepsilon}_{g'_1,g_2,g''_3}}{\widetilde{\varepsilon}_{g_1,g_2,g'_1,g'_2}\widetilde{\varepsilon}_{g_1,g'_2,g''_3}} \\ &= (-1)^{(n-n_{g_2})(n-n_{g'_1})} \cdot \frac{\widetilde{\varepsilon}_{g_1,g_2,g_3}(-1)^{(n-n_{g'_1})(n-n_{g'_2})}\widetilde{\varepsilon}_{g_3,g'_2,g'_1}\widetilde{\varepsilon}_{g'_1,g_2,g''_3}}{\widetilde{\varepsilon}_{g_1,g'_2,g''_3}} \\ &\quad \cdot \frac{(-1)^{(n-n_{g_2})(n-n_{g'_1})+(n-n_{g_2})(n-n_{g'_2})+(n-n_{g'_2})(n-n_{g'_1})}\widetilde{\varepsilon}_{g_1,g_2,g'_1,g'_2}}{\widetilde{\varepsilon}_{g_1,g_2,g'_1,g'_2}} \\ &= (-1)^{(n-n_{g_2})(n-n_{g'_2})} \cdot \frac{\widetilde{\varepsilon}_{g_1,g_2,g_3}\widetilde{\varepsilon}_{g_3,g'_2,g'_1}\widetilde{\varepsilon}_{g'_1,g_2,g''_3}}{\widetilde{\varepsilon}_{g_1,g'_2,g''_3}},\end{aligned}$$

while the RHS is given by

$$\begin{aligned}(-1)^{(n-n_{g_2})(n-n_{g_3})} &\cdot \frac{\widetilde{\varepsilon}_{g_1,g_2,g_3}\widetilde{\varepsilon}_{g_2,g_3,g'_2,g''_3}\widetilde{\varepsilon}_{g'_2,g_3,g'_1}\widetilde{\varepsilon}_{g'_1,g_2,g''_3}}{\widetilde{\varepsilon}_{g_1,g'_2,g''_3}\widetilde{\varepsilon}_{g'_2,g_3,g_2,g''_3}} \\ &= (-1)^{(n-n_{g_2})(n-n_{g_3})} \cdot \frac{\widetilde{\varepsilon}_{g_1,g_2,g_3}(-1)^{(n-n_{g'_2})(n-n_{g_3})}\widetilde{\varepsilon}_{g_3,g'_2,g'_1}\widetilde{\varepsilon}_{g'_1,g_2,g''_3}}{\widetilde{\varepsilon}_{g_1,g'_2,g''_3}} \\ &\quad \cdot \frac{(-1)^{(n-n_{g_2})(n-n_{g_3})+(n-n_{g_2})(n-n_{g'_2})+(n-n_{g'_2})(n-n_{g_3})}\widetilde{\varepsilon}_{g'_2,g_3,g_2,g''_3}}{\widetilde{\varepsilon}_{g'_2,g_3,g_2,g''_3}} \\ &= (-1)^{(n-n_{g_2})(n-n_{g'_2})} \cdot \frac{\widetilde{\varepsilon}_{g_1,g_2,g_3}\widetilde{\varepsilon}_{g_3,g'_2,g'_1}\widetilde{\varepsilon}_{g'_1,g_2,g''_3}}{\widetilde{\varepsilon}_{g_1,g'_2,g''_3}},\end{aligned}$$

which coincides with the LHS. □

We have finished the proof of the proposition. □

Now it is possible to equip $\mathcal{A}'$ with the structure of a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}$-algebra.

Definition 6.2.37. Define a $\mathbb{C}$-bilinear map $\circ : \mathcal{A}' \otimes_{\mathbb{C}} \mathcal{A}' \longrightarrow \mathcal{A}'$ by setting, for each $g, h \in G_f$ and $\phi(\mathbf{x}), \psi(\mathbf{x}) \in \mathbb{C}[x_1, \dots, x_n]$,

$$([\phi(\mathbf{x})]\overline{v}_g) \circ ([\psi(\mathbf{x})]\overline{v}_h) := \overline{c}_{g,h}\,[\phi(\mathbf{x})\psi(\mathbf{x})H_{g,h}]\,\overline{v}_{gh}.$$

It is easy to see that the map $\circ$ is well defined by Proposition 6.2.10 (iii).

Proposition 6.2.38. *The map* $\circ$ *equips* $\mathcal{A}'$ *with the structure of a* $\mathbb{Z}/2\mathbb{Z}$*-graded* $\mathbb{C}$*-algebra with the identity* $\overline{v}_{\mathrm{id}}$*, which satisfies Axiom* (ii)*.*

Proof. The associativity of the product follows from Proposition 6.2.32. It is obvious by Proposition 6.2.11 that $\mathcal{A}'_{\bar{i}} \circ \mathcal{A}'_{\bar{j}} \subset \mathcal{A}'_{\overline{i+j}}$ for all $\bar{i}, \bar{j} \in \mathbb{Z}/2\mathbb{Z}$. It is also clear by the definition of the map $\circ$ above that the natural surjective maps $\mathrm{Jac}(f) \longrightarrow \mathrm{Jac}(f^g)$, $g \in G_f$, equip $\mathcal{A}'$ with the structure of a $\mathrm{Jac}(f)$-module, which coincides with the product map $\circ : \mathcal{A}'_{\mathrm{id}} \otimes_{\mathbb{C}} \mathcal{A}'_g \longrightarrow \mathcal{A}'_g$. $\square$

Definition 6.2.39. Take a nowhere vanishing n-form $dx_1 \wedge \cdots \wedge dx_n$ and set $\zeta := [dx_1 \wedge \cdots \wedge dx_n] \in \Omega_f$. Define a $\mathbb{C}$-bilinear map $\vdash : \mathcal{A}' \otimes_{\mathbb{C}} \Omega'_{f,G_f} \longrightarrow \Omega'_{f,G_f}$ by setting, for each $g, h \in G_f$ and $\phi(\mathbf{x}), \psi(\mathbf{x}) \in \mathbb{C}[x_1, \dots, x_n]$,

$$([\phi(\mathbf{x})]\overline{v}_g) \vdash ([\psi(\mathbf{x})]\omega_h) := \frac{\overline{\alpha}_{gh}\overline{c}_{g,h}}{\overline{\alpha}_h} [\phi(\mathbf{x})\psi(\mathbf{x})H_{g,h}]\, \omega_{gh},$$

where $\overline{\alpha} : G \longrightarrow \mathbb{C}^*$, $g \mapsto \overline{\alpha}_g$ is the map given by

$$\overline{\alpha}_g := \mathbf{e}\left[\frac{1}{8}(n - n_g)(n - n_g + 1)\right]. \tag{6.7}$$

Remark 6.2.40. The map $\overline{\alpha} : G \longrightarrow \mathbb{C}^*$ satisfies $\overline{\alpha}_{\mathrm{id}} = 1$ and

$$\overline{\alpha}_g \overline{\alpha}_{g^{-1}} = (-1)^{\frac{1}{2}(n-n_g)(n-n_g+1)}, \quad g \in G_f.$$

Proposition 6.2.41. *The map* $\vdash : \mathcal{A}' \otimes_{\mathbb{C}} \Omega'_{f,G_f} \longrightarrow \Omega'_{f,G_f}$ *satisfies Axiom* (iii) *in Definition 5.2.1.*

Proof. The map $\vdash$ induces an isomorphism $\vdash \zeta : \mathcal{A}' \longrightarrow \Omega'_{f,G_f}$ of $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}$-modules:

$$\vdash \zeta : \mathcal{A}'_g \longrightarrow \Omega'_{f,g}, \quad [\phi(\mathbf{x})]\overline{v}_g \mapsto [\phi(\mathbf{x})]\overline{v}_g \vdash \zeta = \overline{\alpha}_g [\phi(\mathbf{x})]\omega_g.$$

So we directly see Axiom (iiib). Then we can show for each $g, h \in G_f$ and $\phi(\mathbf{x}), \psi(\mathbf{x}) \in \mathbb{C}[x_1, \dots, x_n]$

$$\begin{aligned}([\phi(\mathbf{x})]\overline{v}_g) \vdash ([\psi(\mathbf{x})]\overline{v}_h \vdash \zeta) &= ([\phi(\mathbf{x})]\overline{v}_g) \vdash (\overline{\alpha}_h[\psi(\mathbf{x})]\omega_h) \\ &= \overline{\alpha}_{gh}\overline{c}_{g,h}\, [\phi(\mathbf{x})\psi(\mathbf{x})H_{g,h}]\, \omega_{gh} \\ &= \overline{c}_{g,h}\, [\phi(\mathbf{x})\psi(\mathbf{x})H_{g,h}]\, \overline{v}_{gh} \vdash \zeta \\ &= (([\phi(\mathbf{x})]\overline{v}_g) \circ ([\psi(\mathbf{x})]\overline{v}_h)) \vdash \zeta.\end{aligned}$$

So we have seen the $\mathcal{A}'$-module structure of $\Omega'_{f,G}$. Axiom (iiia) is clear from the definition. $\square$

On $\mathcal{A}'$ we have the action of $\varphi \in \mathrm{Aut}(f, G)$ induced by the isomorphism $\vdash \zeta : \mathcal{A}' \longrightarrow \Omega'_{f,G_f}$, which is denoted by φ^*. We also use the notation of Remark 5.1.8.

Proposition 6.2.42. *Axiom* (iv) *in Definition 5.2.1 is satisfied by* $\mathcal{A}'$*, namely, Axioms* (iva) *and* (ivb) *hold.*

Proof. Let (g,h) be a spanning pair of elements of G_f with the factorization (g_1,g_2,h_1,h_2) and $\varphi\in\mathrm{Aut}(f,G)$. For simplicity, set $g':=\varphi^{-1}g\varphi$, $h':=\varphi^{-1}h\varphi$, $g'_i:=\varphi^{-1}g_i\varphi$ and $h'_i:=\varphi^{-1}h_i\varphi$ for $i=1,2$. Note that the pair (g',h') is a spanning pair with the factorization (g'_1,g'_2,h'_1,h'_2) since φ induces a bi-regular map $\varphi:\mathrm{Fix}(g'_i)\longrightarrow\mathrm{Fix}(g_i)$. It also follows that there exist $\lambda_\varphi,\lambda_{\varphi_{g_i}},\lambda_{\varphi_{h_i}}\in\mathbb{C}^*$, $i=1,2$ such that

$$\varphi^*(\omega_{\mathrm{id}})=\lambda_\varphi\omega_{\mathrm{id}},\quad \varphi^*(\omega_{g_i})=\lambda_{\varphi_{g_i}}\omega_{g'_i},\ \varphi^*(\omega_{h_i})=\lambda_{\varphi_{h_i}}\omega_{h'_i},\ i=1,2,$$

and that, by (6.7), $\overline{\alpha}_{g'}=\overline{\alpha}_g$, $\overline{\alpha}_{h'}=\overline{\alpha}_h$, $\overline{\alpha}_{g'_i}=\overline{\alpha}_{g_i}$ and $\overline{\alpha}_{h'_i}=\overline{\alpha}_{h_i}$ for $i=1,2$.

For each $\phi(\mathbf{x})\in\mathbb{C}[x_1,\dots,x_n]$, we have

$$\varphi^*([\phi(\mathbf{x})]\overline{v}_g)=[\varphi^*\phi(\mathbf{x})]\varphi^*(\overline{v}_g),$$

since

$$\begin{aligned}&\varphi^*([\phi(\mathbf{x})]\overline{v}_g)\vdash\varphi^*(\zeta)=\varphi^*([\phi(\mathbf{x})]\overline{v}_g\vdash\zeta)=\varphi^*(\overline{\alpha}_g[\phi(\mathbf{x})]\omega_g)\\&=\overline{\alpha}_g[\varphi^*\phi(\mathbf{x})]\varphi^*(\omega_g)=\frac{\overline{\alpha}_g}{\overline{\alpha}_{g'}}([\varphi^*\phi(\mathbf{x})]\varphi^*(\overline{v}_g))\vdash\varphi^*(\zeta)=([\varphi^*\phi(\mathbf{x})]\varphi^*(\overline{v}_g))\vdash\varphi^*(\zeta).\end{aligned}$$

Therefore, we only need to show that $\varphi^*(\overline{v}_g)\circ\varphi^*(\overline{v}_h)=\varphi^*(\overline{v}_g\circ\overline{v}_h)$.

It easily follows that

$$\varphi^*(\overline{v}_{\mathrm{id}})=\overline{v}_{\mathrm{id}},\quad \varphi^*(\overline{v}_{g_i})=\frac{\lambda_{\varphi_{g_i}}}{\lambda_\varphi}\overline{v}_{g'_i},\quad \varphi^*(\overline{v}_{h_i})=\frac{\lambda_{\varphi_{h_i}}}{\lambda_\varphi}\overline{v}_{h'_i},\ i=1,2,$$

since $\varphi^*(\overline{v}_{\mathrm{id}})\vdash\varphi^*(\zeta)=\varphi^*(\overline{v}_{\mathrm{id}}\vdash\zeta)=\varphi^*(\zeta)$ and

$$\begin{aligned}(\lambda_{\varphi_{g_i}}\overline{v}_{g'_i})\vdash\zeta&=\lambda_{\varphi_{g_i}}\overline{\alpha}_{g'_i}\omega_{g'_i}=\varphi^*(\overline{\alpha}_{g_i}\omega_{g_i})=\varphi^*(\overline{v}_{g_i})\vdash\varphi^*(\zeta)=(\lambda_\varphi\varphi^*(\overline{v}_{g_i}))\vdash\zeta,\\(\lambda_{\varphi_{h_i}}\overline{v}_{h'_i})\vdash\zeta&=\lambda_{\varphi_{h_i}}\overline{\alpha}_{h'_i}\omega_{h'_i}=\varphi^*(\overline{\alpha}_{h_i}\omega_{h_i})=\varphi^*(\overline{v}_{h_i})\vdash\varphi^*(\zeta)=(\lambda_\varphi\varphi^*(\overline{v}_{h_i}))\vdash\zeta.\end{aligned}$$

Lemma 6.2.43. *We have*

$$\varphi^*(\omega_g)=\frac{\lambda_{\varphi_{g_1}}\lambda_{\varphi_{g_2}}}{\lambda_\varphi}\cdot\frac{\widetilde{\varepsilon}_{g_1,g_2}}{\widetilde{\varepsilon}_{g'_1,g'_2}}\cdot\omega_{g'},\quad \varphi^*(\omega_h)=\frac{\lambda_{\varphi_{h_1}}\lambda_{\varphi_{h_2}}}{\lambda_\varphi}\cdot\frac{\widetilde{\varepsilon}_{h_1,h_2}}{\widetilde{\varepsilon}_{h'_1,h'_2}}\cdot\omega_{h'},$$

which implies

$$\varphi^*(\overline{v}_g)=\frac{\lambda_{\varphi_{g_1}}\lambda_{\varphi_{g_2}}}{\lambda_\varphi^2}\cdot\frac{\widetilde{\varepsilon}_{g'_1,g'_2}}{\widetilde{\varepsilon}_{g_1,g_2}}\cdot\overline{v}_{g'},\quad \varphi^*(\overline{v}_h)=\frac{\lambda_{\varphi_{h_1}}\lambda_{\varphi_{h_2}}}{\lambda_\varphi^2}\cdot\frac{\widetilde{\varepsilon}_{h'_1,h'_2}}{\widetilde{\varepsilon}_{h_1,h_2}}\cdot\overline{v}_{h'}.$$

Proof. Let $\mathcal{T}_{\mathbb{C}^n}$ be the tangent sheaf on $\mathbb{C}^n$. For each $g''\in G_f$, define a poly-vector field $\widetilde{\theta}_{g''}\in\Gamma(\mathbb{C}^n,\wedge^{n-n_{g''}}\mathcal{T}_{\mathbb{C}^n})$ by

$$\widetilde{\theta}_{g''}:=\begin{cases}\frac{\partial}{\partial x_{j_1}}\wedge\cdots\wedge\frac{\partial}{\partial x_{j_{n-n_{g''}}}} & \text{if } I^c_{g''}=(j_1,\dots,j_{n-n_{g''}}),\ j_1<\cdots<j_{n-n_{g''}}\\ 1 & \text{if } I^c_{g''}=\emptyset\end{cases}.$$

Since we have $\varphi^*(\omega_{\mathrm{id}}) = \lambda_\varphi \omega_{\mathrm{id}}$ and $\varphi^*(\omega_{g_i}) = \lambda_{\varphi_{g_i}} \omega_{g'_i}$ for $i = 1, 2$, the poly-vector field $\widetilde{\theta}_{g_i}$ transforms under φ as

$$\widetilde{\theta}_{g_i} \mapsto \frac{\lambda_{\varphi_{g_i}}}{\lambda_\varphi} \cdot \frac{\widetilde{\varepsilon}_{g_i}}{\widetilde{\varepsilon}_{g'_i}} \cdot \widetilde{\theta}_{g'_i}, \quad i = 1, 2,$$

where $\widetilde{\varepsilon}_{g_i}$ is the signature of the permutation $I_{\mathrm{id}} \longrightarrow I^c_{g_i} \sqcup I_{g_i}$ and $\widetilde{\varepsilon}_{g'_i}$ is the signature of the permutation $I_{\mathrm{id}} \longrightarrow I^c_{g'_i} \sqcup I_{g'_i}$. Suppose that $\varphi^*(\omega_g) = \lambda_{\varphi_g}\omega_{g'}$ for some $\lambda_{\varphi_g} \in \mathbb{C}^*$ and let $\widetilde{\varepsilon}_g$ be the signature of the permutation $I_{\mathrm{id}} \longrightarrow I^c_g \sqcup I_g$ and $\widetilde{\varepsilon}_{g'}$ be signature of the permutation $I_{\mathrm{id}} \longrightarrow I^c_{g'} \sqcup I_{g'}$. Then, $\widetilde{\theta}_g$ transforms under φ as

$$\widetilde{\theta}_g \mapsto \frac{\lambda_{\varphi_g}}{\lambda_\varphi} \cdot \frac{\widetilde{\varepsilon}_g}{\widetilde{\varepsilon}_{g'}} \cdot \widetilde{\theta}_{g'}.$$

Note that $\widetilde{\theta}_g = \widetilde{\varepsilon}_{g_1,g_2}\widetilde{\theta}_{g_1} \wedge \widetilde{\theta}_{g_2}$ and $\widetilde{\theta}_{g'} = \widetilde{\varepsilon}_{g'_1,g'_2}\widetilde{\theta}_{g'_1} \wedge \widetilde{\theta}_{g'_2}$. Hence, we have

$$\frac{\lambda_{\varphi_g}}{\lambda_\varphi} \cdot \frac{\widetilde{\varepsilon}_g \widetilde{\varepsilon}_{g'_1,g'_2}}{\widetilde{\varepsilon}_{g'} \widetilde{\varepsilon}_{g_1,g_2}} = \frac{\lambda_{\varphi_{g_1}}\lambda_{\varphi_{g_2}}}{\lambda_\varphi^2} \cdot \frac{\widetilde{\varepsilon}_{g_1}\widetilde{\varepsilon}_{g_2}}{\widetilde{\varepsilon}_{g'_1}\widetilde{\varepsilon}_{g'_2}}.$$

Therefore, the statement is reduced to showing that

$$\frac{\widetilde{\varepsilon}_{g_1}\widetilde{\varepsilon}_{g_2}}{\widetilde{\varepsilon}_g} = \frac{\widetilde{\varepsilon}_{g'_1}\widetilde{\varepsilon}_{g'_2}}{\widetilde{\varepsilon}_{g'}}.$$

However, by calculating the number of elements less than j in the sequences $I^c_{g_1}$, $I^c_{g_2}$ and I^c_g for each element j in $I^c_{g_1}$ or $I^c_{g_2}$, it turns out that the LHS of the above equation is equal to $(-1)^{(n-n_{g_1})(n-n_{g_2})}$. Similarly, the RHS is equal to $(-1)^{(n-n_{g'_1})(n-n_{g'_2})}$. They coincide since we have $n_{g_1} = n_{g'_1}$ and $n_{g_2} = n_{g'_2}$. $\square$

Lemma 6.2.44. *We have*

$$[\varphi^* H_{g,h}] = \frac{\lambda^2_{\varphi_{g_2}}}{\lambda^2_\varphi}[H_{g',h'}].$$

Proof. Recall Definition 6.2.4, where $H_{g,h}$ is defined as a non-zero constant multiple of $\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{i,j\in I_{g,h}}$. Now, $I_{g,h} = I^c_{g_2} = I^c_{h_2}$, $I_{g',h'} = I^c_{g'_2} = I^c_{h'_2}$. This is nothing but the transformation rule of the determinant under the automorphism φ. $\square$

Since $g_2h_2 = \mathrm{id}$ and $g'_2h'_2 = \mathrm{id}$ by definition of the factorizations,

$$n_{g_2} = n_{h_2} = n_{h'_2} = n_{g'_2}, \quad \lambda_{\varphi_{g_2}} = \lambda_{\varphi_{h_2}},$$

where we identify ω_{h_2} with ω_{g_2} under $\Omega_{f,h_2} = \Omega_{f,g_2}$ and $\omega_{h'_2}$ with $\omega_{g'_2}$ under $\Omega_{f,h'_2} = \Omega_{f,g'_2}$.

By the above lemma, it follows that

$$
\begin{aligned}
&\varphi^*(\overline{v}_g) \circ \varphi^*(\overline{v}_h)\\
&= \frac{\lambda_{\varphi_{g_1}}\lambda_{\varphi_{g_2}}\lambda_{\varphi_{h_1}}\lambda_{\varphi_{h_2}}}{\lambda_\varphi^4} \cdot \frac{\widetilde{\varepsilon}_{g_1,g_2}}{\widetilde{\varepsilon}_{g_1',g_2'}} \cdot \frac{\widetilde{\varepsilon}_{h_1,h_2}}{\widetilde{\varepsilon}_{h_1',h_2'}} \cdot \overline{v}_{g'} \circ \overline{v}_{h'}\\
&= \frac{\lambda_{\varphi_{g_1}}\lambda_{\varphi_{g_2}}\lambda_{\varphi_{h_1}}\lambda_{\varphi_{h_2}}}{\lambda_\varphi^4} \cdot \frac{\widetilde{\varepsilon}_{g_1,g_2}}{\widetilde{\varepsilon}_{g_1',g_2'}} \cdot \frac{\widetilde{\varepsilon}_{h_1,h_2}}{\widetilde{\varepsilon}_{h_1',h_2'}}\\
&\qquad \cdot (-1)^{\frac{1}{2}(n-n_{g_2'})(n-n_{g_2'}-1)} \cdot \mathbf{e}\left[-\frac{1}{2}\mathrm{age}(g_2')\right] \cdot \frac{\widetilde{\varepsilon}_{g_1',g_2'}\widetilde{\varepsilon}_{h_2',h_1'}}{\widetilde{\varepsilon}_{g_1',h_1'}} \cdot [H_{g',h'}]\, \overline{v}_{g'h'}\\
&= (-1)^{\frac{1}{2}(n-n_{g_2})(n-n_{g_2}-1)} \cdot \mathbf{e}\left[-\frac{1}{2}\mathrm{age}(g_2)\right] \cdot \frac{\widetilde{\varepsilon}_{g_1,g_2}\widetilde{\varepsilon}_{h_2,h_1}}{\widetilde{\varepsilon}_{g_1,h_1}}\\
&\qquad \cdot \left(\frac{\lambda_{\varphi_{g_2}}^2}{\lambda_\varphi^2}\,[H_{g',h'}]\right)\left(\frac{\lambda_{\varphi_{g_1}}\lambda_{\varphi_{h_1}}}{\lambda_\varphi^2} \cdot \frac{\widetilde{\varepsilon}_{g_1,h_1}}{\widetilde{\varepsilon}_{g_1',h_1'}}\overline{v}_{g'h'}\right)\\
&= c_{g,h}\,[\varphi^* H_{g,h}]\,\varphi^*(\overline{v}_{gh}) = \varphi^*(\overline{v}_g \circ \overline{v}_h),
\end{aligned}
$$

where we also used that

$$
\widetilde{\varepsilon}_{h_1,h_2} = (-1)^{(n-n_{h_1})(n-n_{h_2})}\widetilde{\varepsilon}_{h_2,h_1}, \quad \widetilde{\varepsilon}_{h_1',h_2'} = (-1)^{(n-n_{h_1'})(n-n_{h_2'})}\widetilde{\varepsilon}_{h_2',h_1'}.
$$

Hence, we proved that the algebra structure $\circ$ of $\mathcal{A}'$ is $\mathrm{Aut}(f, G)$-invariant.

The G-twisted $\mathbb{Z}/2\mathbb{Z}$-graded commutativity, Axiom (ivb), is a direct consequence of Proposition 6.2.31 since $H_{g,h} = H_{h,g}$ and $g^*(\overline{v}_h) = \mathbf{e}[-\mathrm{age}(g_2)] \cdot \overline{v}_h$ which follows from the calculation

$$
\begin{aligned}
&g^*(\overline{v}_h) \vdash \zeta = g^*(\overline{v}_h) \vdash (\mathbf{e}\,[-\mathrm{age}(g)]\, g^*(\zeta)) = \mathbf{e}\,[-\mathrm{age}(g)] \cdot g^*(\overline{\alpha}_h \omega_h)\\
&\quad = \mathbf{e}\,[-\mathrm{age}(g_2)] \cdot (\overline{\alpha}_h \omega_h) = (\mathbf{e}[-\mathrm{age}(g_2)] \cdot \overline{v}_h) \vdash \zeta.
\end{aligned}
$$

We have finished the proof of the proposition. □

We show the invariance of the bilinear form $J_{f,G}$ with respect to the product structure of $\mathcal{A}'$. We use the notation in Definition 6.2.4.

Proposition 6.2.45. *For a spanning pair (g, h) of elements of G_f, we have*

$$
\begin{aligned}
&J_{f,gh}\left(\overline{v}_g \vdash \omega_h, \left[\frac{1}{\mu_{f^{g\cap h}}}\mathrm{hess}(f^{g\cap h})\right] \omega_{(gh)^{-1}}\right)\\
&\qquad = (-1)^{(n-n_g)(n-n_h)} J_{f,h}\left(\omega_h, \left((h^{-1})^*\overline{v}_g\right) \vdash \left(\left[\frac{1}{\mu_{f^{g\cap h}}}\mathrm{hess}(f^{g\cap h})\right] \omega_{(gh)^{-1}}\right)\right).
\end{aligned}
\tag{6.8}
$$

As a consequence, the algebra $\mathcal{A}'$ satisfies Axiom (v) *in Definition 5.2.1.*

Proof. Let (g_1, g_2, h_1, h_2) be the factorization of the spanning pair (g, h). The LHS of the equation (6.8) is calculated as

$$\begin{aligned}
& J_{f,gh}\left(\overline{v}_g \vdash \omega_h, \left[\frac{1}{\mu_{f^{g\cap h}}}\mathrm{hess}(f^{g\cap h})\right]\omega_{(gh)^{-1}}\right) \\
&= \frac{1}{\overline{\alpha}_h}\cdot J_{f,gh}\left((\overline{v}_g \circ \overline{v}_h) \vdash \zeta, \left[\frac{1}{\mu_{f^{g\cap h}}}\mathrm{hess}(f^{g\cap h})\right]\omega_{(gh)^{-1}}\right) \\
&= \frac{\overline{\alpha}_{gh}\overline{c}_{g,h}}{\overline{\alpha}_h}\cdot J_{f,gh}\left(\omega_{gh}, \left[\frac{1}{\mu_{f^{g\cap h}}}\mathrm{hess}(f^{g\cap h})H_{g,h}\right]\omega_{(gh)^{-1}}\right) \\
&= \frac{\overline{\alpha}_{gh}}{\overline{\alpha}_h}\cdot(-1)^{\frac{1}{2}(n-n_{g_2})(n-n_{g_2}-1)}\cdot \mathbf{e}\left[-\frac{1}{2}\mathrm{age}(g_2)\right]\cdot\frac{\widetilde{\varepsilon}_{g_1,g_2}\widetilde{\varepsilon}_{h_2,h_1}}{\widetilde{\varepsilon}_{g_1,h_1}} \\
&\quad\cdot(-1)^{n-n_{gh}}\cdot\mathbf{e}\left[-\frac{1}{2}\mathrm{age}(gh)\right]\cdot|G| \\
&= \frac{\overline{\alpha}_{gh}}{\overline{\alpha}_h}\cdot(-1)^{\frac{1}{2}(n-n_{g_2})(n-n_{g_2}-1)+(n-n_{gh})} \\
&\quad\cdot\mathbf{e}\left[-\frac{1}{2}\mathrm{age}(g_1)-\frac{1}{2}\mathrm{age}(h_1)-\frac{1}{2}\mathrm{age}(g_2)\right]\cdot\frac{\widetilde{\varepsilon}_{g_1,g_2}\widetilde{\varepsilon}_{h_2,h_1}}{\widetilde{\varepsilon}_{g_1,h_1}}\cdot|G|.
\end{aligned}$$

On the other hand, the RHS of the equation (6.8) can be calculated by having in mind that $(h^{-1})^*\overline{v}_g = \mathbf{e}[-\mathrm{age}(h_2^{-1})]\overline{v}_g$ by Equation (5.2), since only h_2^{-1} acts on variables not fixed by g:

$$\begin{aligned}
& (-1)^{(n-n_g)(n-n_h)}\cdot J_{f,h}\left(\omega_h, \left((h^{-1})^*\overline{v}_g\right)\vdash\left(\left[\frac{1}{\mu_{f^{g\cap h}}}\mathrm{hess}(f^{g\cap h})\right]\omega_{(gh)^{-1}}\right)\right) \\
&= \frac{1}{\overline{\alpha}_{(gh)^{-1}}}(-1)^{(n-n_g)(n-n_h)}\cdot\mathbf{e}\left[-\mathrm{age}(h_2^{-1})\right] \\
&\quad\cdot J_{f,h}\left(\omega_h, \left(\left[\frac{1}{\mu_{f^{g\cap h}}}\mathrm{hess}(f^{g\cap h})\right]\overline{v}_g\circ\overline{v}_{(gh)^{-1}}\right)\vdash\zeta\right) \\
&= \frac{\overline{\alpha}_{h^{-1}}c_{g,(gh)^{-1}}}{\overline{\alpha}_{(gh)^{-1}}}(-1)^{(n-n_g)(n-n_h)}\cdot\mathbf{e}\left[-\mathrm{age}(g_2)\right]\cdot J_{f,h}\left(\omega_h, \left[\frac{1}{\mu_{f^{g\cap h}}}\mathrm{hess}(f^{g\cap h})\right]\vdash\omega_{h^{-1}}\right) \\
&= \frac{\overline{\alpha}_{h^{-1}}}{\overline{\alpha}_{(gh)^{-1}}}(-1)^{(n-n_g)(n-n_h)+\frac{1}{2}(n-n_{g_1})(n-n_{g_1}-1)}\cdot\mathbf{e}\left[-\frac{1}{2}\mathrm{age}(g_1)-\mathrm{age}(g_2)\right]\cdot\frac{\widetilde{\varepsilon}_{g_2,g_1}\widetilde{\varepsilon}_{g_1^{-1},h_1}}{\widetilde{\varepsilon}_{g_2,h_1}} \\
&\quad\cdot(-1)^{n-n_h}\cdot\mathbf{e}\left[-\frac{1}{2}\mathrm{age}(h)\right]\cdot|G| \\
&= \frac{\overline{\alpha}_{h^{-1}}}{\overline{\alpha}_{(gh)^{-1}}}(-1)^{(n-n_g+1)(n-n_h)+\frac{1}{2}(n-n_{g_1})(n-n_{g_1}-1)-(n-n_{g_2})+(n-n_{g_1})(n-n_{g_2})} \\
&\quad\cdot\mathbf{e}\left[-\frac{1}{2}\mathrm{age}(g_1)-\frac{1}{2}\mathrm{age}(h_1)-\frac{1}{2}\mathrm{age}(g_2)\right]\cdot\frac{\widetilde{\varepsilon}_{g_1,g_2}\widetilde{\varepsilon}_{h_2,h_1}}{\widetilde{\varepsilon}_{g_1,h_1}}\cdot|G|,
\end{aligned}$$

where we used that $\widetilde{\varepsilon}^{-1}_{g_2^{-1},h_1} = \widetilde{\varepsilon}_{g_2^{-1},h_1} = \widetilde{\varepsilon}_{h_2,h_1}$ and $\widetilde{\varepsilon}_{g_1^{-1},h_1} = \widetilde{\varepsilon}_{g_1,h_1} = \widetilde{\varepsilon}^{-1}_{g_1,h_1}$. We have $\overline{\alpha}_{gh}\overline{\alpha}_{(gh)^{-1}} = (-1)^{\frac{1}{2}(n-n_{gh})(n-n_{gh}+1)}$ and $\overline{\alpha}_h\overline{\alpha}_{h^{-1}} = (-1)^{\frac{1}{2}(n-n_h)(n-n_h+1)}$ by Remark 6.2.40. Hence, it follows from a direct calculation by the use of

$$
\begin{aligned}
&n-n_g=(n-n_{g_1})+(n-n_{g_2}), \quad n-n_h=(n-n_{h_1})+(n-n_{h_2}),\\
&n-n_{gh}=n-n_{g_1g_2}=(n-n_{g_1})+(n-n_{h_1}), \quad n_{g_2}=n_{h_2},
\end{aligned}
$$

(cf. Proposition 6.2.11) that

$$
\begin{aligned}
&\frac{1}{2}(n-n_{gh})(n-n_{gh}+1)+\frac{1}{2}(n-n_{g_2})(n-n_{g_2}-1)+(n-n_{gh})\\
&-\frac{1}{2}(n-n_h)(n-n_h+1)+(n-n_g+1)(n-n_h)\\
&+\frac{1}{2}(n-n_{g_1})(n-n_{g_1}-1)-(n-n_{g_2})+(n-n_{g_1})(n-n_{g_2})\\
\equiv\ & 0 \pmod 2,
\end{aligned}
$$

which gives the equation (6.8)

For $X \in \mathcal{A}'_g$, $\omega \in \Omega'_{f,h}$, $\omega' \in \Omega'_{f,G}$, $J_{f,G}(X \vdash \omega, \omega')$ is non-zero only if $\omega' \in \Omega'_{f,(gh)^{-1}}$ and the pair (g,h) is a spanning pair. Note that $I_g \cup I_h \cup I_{gh} = I_{\mathrm{id}}$ if and only if $I_h \cup I_{(gh)^{-1}} \cup I_{g^{-1}} = I_{\mathrm{id}}$, which means the pair (g,h) is a spanning pair if and only if the pair $(h,(gh)^{-1})$ is so. Therefore, $J_{f,G}(X \vdash \omega, \omega')$ is non-zero if and only if $J_{f,G}(\omega, (h^{-1})^*X \vdash \omega')$ is so. It follows that Axiom (v) can be reduced to the equation (6.8). $\square$

So we have shown all axioms and with Proposition 5.4.9 we have finished the proof of Theorem 6.2.1.

Example 6.2.46. Let $f = x_1^3 + x_2^3x_3 + x_3^3$ and $G = \left\langle (\mathbf{e}[\frac{1}{3}], \mathbf{e}[\frac{2}{3}], 1) \right\rangle$ be as in Example 4.1.4. With Example 4.3.4 we see

$$
\mathrm{Jac}(f,G) \cong \left\langle v_{\mathrm{id}}, [x_3], [x_3]^2, [x_1x_2], [x_1x_2][x_3], [x_1x_2][x_3]^2 \right\rangle \oplus \langle v_g, [x_3]v_g \rangle \oplus \langle v_{g^{-1}}, [x_3]v_{g^{-1}} \rangle
$$

with the following relations

$$
[x_1x_2]^2 = 0,\ [x_3]^3 = 0,\ v_g^2 = 0,\ v_{g^{-1}}^2 = 0,\ v_g \circ v_{g^{-1}} = 9[x_1x_2x_3],
$$

see the normal multiplication in $\mathrm{Jac}(f)^G = \mathrm{Jac}(f, \mathrm{id})$, Proposition 6.2.15, Example 6.2.7 and observe that

$$
\begin{aligned}
\overline{c}_{g,g^{-1}} &= (-1)^{\frac{1}{2}(n-n_g)(n-n_g-1)} \cdot \mathbf{e}\left[-\frac{1}{2}\mathrm{age}(g)\right] \cdot \frac{\widetilde{\varepsilon}_{\mathrm{id},g}\widetilde{\varepsilon}_{g^{-1},\mathrm{id}}}{\widetilde{\varepsilon}_{\mathrm{id},\mathrm{id}}}\\
&= (-1)^{\frac{1}{2}(3-1)(3-1-1)} \cdot \mathbf{e}\left[-\frac{1}{2}1\right] \cdot \frac{1\cdot 1}{1} = (-1)(-1) = 1.
\end{aligned}
$$

6.3 Orbifold Jacobian Algebras for ADE Singularities

Definition 6.3.1. The classification of invertible polynomials in three variables giving ADE singularities and the subgroups of their maximal diagonal symmetries preserving the holomorphic volume form is given in Table 6.1 (see also [ET13a, Sec. 8 Table 3]).

Type	$f(x_1, x_2, x_3)$	G_f^{SL}	Singularity Type
I	$x_1^{2k+1} + x_2^2 + x_3^2, \quad k \geq 1$	$\left\langle \frac{1}{2}(0,1,1) \right\rangle$	A_{2k}
	$x_1^{2k} + x_2^2 + x_3^2, \quad k \geq 1$	$\left\langle \frac{1}{2}(0,1,1), \frac{1}{2}(1,0,1) \right\rangle$	A_{2k-1}
	$x_1^3 + x_2^3 + x_3^2$	$\left\langle \frac{1}{3}(1,2,0) \right\rangle$	D_4
	$x_1^4 + x_2^3 + x_3^2$	$\left\langle \frac{1}{2}(1,0,1) \right\rangle$	E_6
	$x_1^5 + x_2^3 + x_3^2$	$\{1\}$	E_8
II	$x_1^2 + x_2^2 + x_2 x_3^{2k}, \quad k \geq 1$	$\left\langle \frac{1}{2}(1,0,1) \right\rangle$	A_{4k-1}
	$x_1^2 + x_2^2 + x_2 x_3^{2k+1}, \quad k \geq 1$	$\left\langle \frac{1}{2}(0,1,1) \right\rangle$	A_{4k+1}
	$x_1^2 + x_2^{k-1} + x_2 x_3^2, \quad k \geq 4$	$\left\langle \frac{1}{2}(1,0,1) \right\rangle$	D_k
	$x_1^3 + x_2^2 + x_2 x_3^2$	$\{1\}$	E_6
	$x_1^2 + x_2^3 + x_2 x_3^3$	$\{1\}$	E_7
III	$x_1^2 + x_3 x_2^2 + x_2 x_3^{k+1}, \quad k \geq 1$	$\{1\}$	D_{2k+2}
IV	$x_1^k + x_1 x_2 + x_2 x_3^l, \quad k, l \geq 2$	$\{1\}$	A_{kl-1}
	$x_1^2 + x_1 x_2^k + x_2 x_3^2, \quad k \geq 2$	$\{1\}$	D_{2k+1}
V	$x_1 x_2 + x_2^k x_3 + x_3^l x_1, \quad k, l \geq 1$	$\{1\}$	A_{kl}

Table 6.1: Classification of invertible polynomials giving ADE singularities and the groups of their diagonal symmetries preserving the holomorphic volume form.

Remark 6.3.2. As it is explained in Section 8 in [ET13a], one can describe explicitly the geometry of vanishing cycles for the holomorphic map $\widehat{f} : \widehat{\mathbb{C}^3/G} \longrightarrow \mathbb{C}$. Here, $\widehat{\mathbb{C}^3/G}$ is a crepant resolution of $\mathbb{C}^3/G$ and $\widehat{f}$ is the convolution of the resolution map $\widehat{\mathbb{C}^3/G} \longrightarrow \mathbb{C}^3/G$ and the induced one $f : \mathbb{C}^3/G \longrightarrow \mathbb{C}$. Note that $\widehat{\mathbb{C}^3/G}$ is covered by some charts all isomorphic to $\mathbb{C}^3$.

Remark 6.3.3. When G respects one coordinate we only need to look at the resolutions of $\mathbb{C}^2$ given in [BK91]. For $G \cong \mathbb{Z}/2\mathbb{Z}$ acting $(x_i, x_j) \mapsto (-x_i, -x_j)$, we have $\mathbb{C}^3/G \cong \mathbb{C} \times \{z^2 = xy\} \subset \mathbb{C}^4$ by $x = x_i^2, y = x_j^2, z = x_i x_j$ and we have the two charts $\mathbb{C}^3 \to \mathbb{C}^4$:

$$(t, u, v) \mapsto (t, u, uv^2, uv) \text{ and } (t, u, v) \mapsto (t, u^2 v, v, uv)$$

For $G \cong \mathbb{Z}/3\mathbb{Z} = \langle g \rangle$ acting by $g^* x_i = \mathbf{e}[\frac{1}{3}] x_i,\ g^* x_j = \mathbf{e}[\frac{2}{3}] x_j$, we have $\mathbb{C}^3/G \cong \mathbb{C} \times \{z^3 = xy\} \subset \mathbb{C}^4$ by $x = x_i^3, y = x_j^3, z = x_i x_j$ and we have the three charts $\mathbb{C}^3 \to \mathbb{C}^4$:

$$(t, u, v) \mapsto (t, u, u^2 v^3, uv) \ , \ (t, u, v) \mapsto (t, u^2 v, uv^2, uv) \text{ and } (t, u, v) \mapsto (t, u^3 v^2, v, uv)$$

Example 6.3.4. We will calculate the restriction of $\widehat{f}$ on each chart based on the classification in Table 6.1.
1. For the pair

$$f := x_1^{k+1} + x_2^2 + x_3^2 \ (k \geq 1), \quad G := \left\langle \frac{1}{2}(0,1,1) \right\rangle,$$

we have in the two charts

$$\widehat{f}(t,u,v) = t^{k+1} + u + uv^2 \text{ and } \widehat{f}(t,u,v) = t^{k+1} + u^2 v + v.$$

Critical points of $\widehat{f}$ are on the intersection of the two charts.
2. For the pair

$$f := x_1^{2k} + x_2^2 + x_3^2 \ (k \geq 1), \quad G := \left\langle \frac{1}{2}(1,0,1) \right\rangle,$$

we have in the two charts

$$\widehat{f}(t,u,v) = t^2 + u^k + uv^2 \text{ and } \widehat{f}(t,u,v) = t^2 + u^{2k} v^k + v.$$

Critical points of $\widehat{f}$ are on the first chart.
3. See Example 6.3.5.
4. For the pair

$$f := x_1^3 + x_2^3 + x_3^2, \quad G := \left\langle \frac{1}{3}(1,2,0) \right\rangle,$$

we have in the three charts

$$\widehat{f}(t,u,v) = t^2 + u + u^2 v^3, \ \widehat{f}(t,u,v) = t^2 + u^2 v + uv^2 \text{ and } \widehat{f}(t,u,v) = t^2 + u^3 v^2 + v.$$

Critical points of $\widehat{f}$ are on the second chart.
5. For the pair

$$f := x_1^4 + x_2^3 + x_3^2, \quad G := \left\langle \frac{1}{2}(1,0,1) \right\rangle,$$

we have in the two charts

$$\widehat{f}(t,u,v) = t^3 + u^2 + uv^2 \text{ and } \widehat{f}(t,u,v) = t^3 + u^4 v^2 + v.$$

Critical points of $\widehat{f}$ are on the first chart.
6. For the pair

$$f := x_1^2 + x_2^2 + x_2 x_3^{2k} \ (k \geq 1), \quad G := \left\langle \frac{1}{2}(1,0,1) \right\rangle,$$

we have in the two charts

$$\widehat{f}(t,u,v) = t^2 + tu^k v^{2k} + u \text{ and } \widehat{f}(t,u,v) = t^2 + tv^k + vu^2.$$

Critical points of $\widehat{f}$ are on the second chart.

7. For the pair

$$f := x_1^2 + x_2^2 + x_2x_3^{2k+1}\ (k \geq 1), \quad G := \left\langle \frac{1}{2}(0,1,1) \right\rangle,$$

we have in the two charts

$$\widehat{f}(t,u,v) = t^2 + u + u^{k+1}v^{2k+1} \text{ and } \widehat{f}(t,u,v) = t^2 + vu^2 + uv^{k+1}.$$

Critical points of $\widehat{f}$ are on the second chart.
8. For the pair

$$f := x_1^2 + x_2^{k-1} + x_2x_3^2\ (k \geq 4), \quad G := \left\langle \frac{1}{2}(1,0,1) \right\rangle,$$

we have in the two charts

$$\widehat{f}(t,u,v) = t^{k-1} + tuv^2 + u \text{ and } \widehat{f}(t,u,v) = t^{k-1} + tv + vu^2.$$

Critical points of $\widehat{f}$ are on the second chart.

Example 6.3.5. For $k \geq 1$, set

$$f := x_1^{2k} + x_2^2 + x_3^2\ (k \geq 1), \quad G := \left\langle \frac{1}{2}(0,1,1), \frac{1}{2}(1,0,1) \right\rangle.$$

Here, since the resolution is not unique, we take A-Hilb $\mathbb{C}^3$ of [CR02] where $A = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. We have $\mathbb{C}^3/G \cong \{z^2 = wxy\} \subset \mathbb{C}^4$ by $w = x_1^2, x = x_2^2, y = x_3^2, z = x_1x_2x_3$ and we have four charts $\mathbb{C}^3 \to \mathbb{C}^4$:

$$(t,u,v) \mapsto (t,u,tuv^2,tuv)\ ,\ (t,u,v) \mapsto (t,tu^2v,v,tuv)\ ,$$

$$(t,u,v) \mapsto (t^2uv,u,v,tuv) \text{ and } (t,u,v) \mapsto (tu,uv,tv,tuv)$$

Then we have in the four charts

$$\widehat{f}(t,u,v) = t^k + u + tuv^2\ ,\ \widehat{f}(t,u,v) = t^k + tu^2v + v,$$

$$\widehat{f}(t,u,v) = t^{2k}u^kv^k + u + v \text{ and } \widehat{f}(t,u,v) = t^ku^k + uv + tv.$$

Critical points of $\widehat{f}$ are on the fourth chart.

Remark 6.3.6. To summarize, we observed that critical points of the map $\widehat{f}$ are contained in one chart isomorphic to $\mathbb{C}^3$. The restriction of $\widehat{f}$ to the chart is given by $\overline{f}$ defined in Table 6.2.

Concerning the geometry of vanishing cycles, the pair (f,G) is equivalent to the pair $(\overline{f}, \{\mathrm{id}\})$. Then, it is quite natural to expect that the orbifold Jacobian algebra $\mathrm{Jac}(f,G)$ of (f,G) is isomorphic to the one $\mathrm{Jac}(\overline{f}, \{\mathrm{id}\})$ of the pair $(\overline{f}, \{\mathrm{id}\})$, the usual Jacobian algebra $\mathrm{Jac}(\overline{f})$ of $\overline{f}$, which is the following theorem.

	$f(x_1,x_2,x_3)$	G	$\overline{f}(y_1,y_2,y_3)$
1.	$x_1^{k+1}+x_2^2+x_3^2, \quad k\geq 1$	$\left\langle \frac{1}{2}(0,1,1)\right\rangle$	$y_1^{k+1}+y_2+y_2y_3^2$
2.	$x_1^{2k}+x_2^2+x_3^2, \quad k\geq 1$	$\left\langle \frac{1}{2}(1,0,1)\right\rangle$	$y_1^2+y_2^k+y_2y_3^2$
3.	$x_1^{2k}+x_2^2+x_3^2, \quad k\geq 1$	$\left\langle \frac{1}{2}(0,1,1),\frac{1}{2}(1,0,1)\right\rangle$	$y_1^ky_2^k+y_1y_3+y_2y_3$
4.	$x_1^3+x_2^3+x_3^2$	$\left\langle \frac{1}{3}(1,2,0)\right\rangle$	$y_1^2+y_3y_2^2+y_2y_3^2$
5.	$x_1^4+x_2^3+x_3^2$	$\left\langle \frac{1}{2}(1,0,1)\right\rangle$	$y_1^3+y_2^2+y_2y_3^2$
6.	$x_1^2+x_2^2+x_2x_3^{2k}, \quad k\geq 1$	$\left\langle \frac{1}{2}(1,0,1)\right\rangle$	$y_1^2+y_1y_2^k+y_2y_3^2$
7.	$x_1^2+x_2^2+x_2x_3^{2k+1}, \quad k\geq 1$	$\left\langle \frac{1}{2}(0,1,1)\right\rangle$	$y_1^2+y_3y_2^2+y_2y_3^{k+1}$
8.	$x_1^2+x_2^{k-1}+x_2x_3^2, \quad k\geq 4$	$\left\langle \frac{1}{2}(1,0,1)\right\rangle$	$y_1^{k-1}+y_1y_2+y_2y_3^2$

Table 6.2: $(f,G)\cong(\overline{f})$

Theorem 6.3.7. *There is an isomorphism of Frobenius algebras*

$$\mathrm{Jac}(f,G)\cong\mathrm{Jac}(\overline{f})$$

for all f and $\overline{f}$ in Table 6.2.

Proof. We shall give a proof of this theorem based on the classification in Table 6.2. Let the notation be as in the sections before. For each $g\in G$ let K_g be the maximal subgroup of G fixing $\mathrm{Fix}(g)$. Let $v_g\in Jac'(f,G)$ be the elements with $v_g\vdash\zeta=\alpha_g\omega_g$, cf. Definition 5.4.5. We will now define $e_g\in\mathrm{Jac}(f,G)$ by $e_g:=\frac{\alpha_g^{-1}}{|K_g|}v_g$, which is a more natural element than v_g.
1. For $k\geq 1$, set

$$f:=x_1^{k+1}+x_2^2+x_3^2,\quad G:=\langle g\rangle\,,\ g:=\frac{1}{2}(0,1,1),$$

$$\overline{f}:=y_1^{k+1}+y_2+y_2y_3^2.$$

The Jacobian algebra $\mathrm{Jac}(f)$ can be calculated as

$$\mathrm{Jac}(f)=\mathbb{C}[x_1,x_2,x_3]\big/\big((k+1)x_1^k,2x_2,2x_3\big)\cong\left\langle 1,x_1,\ldots,x_1^{k-1}\right\rangle_{\mathbb{C}},$$

so $\mu_f=k$. With $\mathrm{hess}(f)=k(k+1)\cdot 2\cdot 2\cdot x_1^{k-1}$ we can calculate the bilinear form J_f on Ω_f

$$J_f\left([dx_1\wedge dx_2\wedge dx_3],[x_1^{k-1}dx_1\wedge dx_2\wedge dx_3]\right)=\frac{1}{4(k+1)}.$$

As a $\mathbb{C}$-module, the orbifold Jacobian algebra $\mathrm{Jac}(f,G)$ is of the following form:

$$\mathrm{Jac}(f,G)\cong\left\langle e_{\mathrm{id}},[x_1],\ldots,[x_1]^{k-1}\right\rangle_{\mathbb{C}}\oplus\left\langle e_g,[x_1]e_g,\ldots,[x_1]^{k-1}e_g\right\rangle_{\mathbb{C}}.$$

Note that $\dim_{\mathbb{C}} \mathrm{Jac}(f,G) = 2k$. The bilinear form $J_{f,G}$ on $\Omega_{f,G}$ can be calculated as

$$\begin{aligned} J_{f,\mathrm{id}}\left(e_{\mathrm{id}} \vdash \zeta, [x_1]^{k-1} \vdash \zeta\right) &= J_{f,\mathrm{id}}\left([dx_1 \wedge dx_2 \wedge dx_3], [x_1^{k-1} dx_1 \wedge dx_2 \wedge dx_3]\right) \\ &= 2 \cdot \frac{1}{4(k+1)} = \frac{1}{2(k+1)} \end{aligned}$$

and with $\mu_{f^g} = k$, $\mathrm{hess}(f^g) = (k+1)kx_1^{k-1}$

$$\begin{aligned} J_{f,g}\left(e_g \vdash \zeta, [x_1]^{k-1} e_g \vdash \zeta\right) &= \frac{1}{4} J_{f,g}\left([dx_1], [x_1^{k-1} dx_1]\right) \\ &= \frac{1}{4} \cdot (-1) \cdot 2 \cdot \frac{1}{k+1} = -\frac{1}{2(k+1)}, \end{aligned}$$

which imply the following relations in the orbifold Jacobian algebra $\mathrm{Jac}(f,G)$:

$$[x_1]^k = 0, \quad e_g^2 = -e_{\mathrm{id}}.$$

The first relation is clear from the relation in $\mathrm{Jac}(f,\mathrm{id}) = \mathrm{Jac}(f)^G$. The second one we get by having Axiom (v) in mind. We see that $e_g \circ [x_1]^{k-1} e_g = -[x_1]^{k-1}$ from the calculation of the orbifold residue pairings. And since $e_g \circ [x_1]^{k-1} e_g = [x_1]^{k-1} \circ e_g^2$, we see the relation $e_g^2 = -e_{\mathrm{id}}$ with Remark 5.4.6 in mind. In similar way we always get the relations for the other calculations.

On the other hand, the Jacobian algebra $\mathrm{Jac}(\overline{f})$ is given by

$$\begin{aligned} \mathrm{Jac}(\overline{f}) &= \mathbb{C}[y_1, y_2, y_3] \big/ \left((k+1)y_1^k, 1 + y_3^2, 2y_2y_3\right) \\ &\cong \mathbb{C}[y_1, y_3] \big/ \left(y_1^k, y_3^2 + 1\right) . \end{aligned}$$

Note that $\dim_{\mathbb{C}} \mathrm{Jac}(\overline{f}) = 2k$. Therefore, we have an algebra isomorphism

$$\mathrm{Jac}(\overline{f}) \xrightarrow{\cong} \mathrm{Jac}(f,G), \quad y_1 \mapsto [x_1], \ y_3 \mapsto e_g,$$

which is, moreover, an isomorphism of Frobenius algebras since we have
$\mathrm{hess}(\overline{f}) = -(k+1)k \cdot 2 \cdot 2y_1^{k-1} y_3^2 = 2(k+1) \cdot 2k y_1^{k-1}$ and so

$$J_{\overline{f}}\left([dy_1 \wedge dy_2 \wedge dy_3], [y_1^{k-1} dy_1 \wedge dy_2 \wedge dy_3]\right) = \frac{1}{2(k+1)}.$$

2. For $k \geq 1$, set

$$f := x_1^{2k} + x_2^2 + x_3^2, \quad G := \langle g \rangle, \ g := \frac{1}{2}(1,0,1),$$

$$\overline{f} := y_1^2 + y_2^k + y_2 y_3^2.$$

The Jacobian algebra $\mathrm{Jac}(f)$ can be calculated as

$$\mathrm{Jac}(f) = \mathbb{C}[x_1, x_2, x_3] \big/ \left(2kx_1^{2k-1}, 2x_2, 2x_3\right) \cong \left\langle 1, x_1, \ldots, x_1^{2k-2} \right\rangle_{\mathbb{C}},$$

so $\mu_f = 2k-1$. With $\mathrm{hess}(f) = 2k(2k-1)\cdot 2\cdot 2\cdot x_1^{2k-2} = 8k(2k-1)x_1^{2k-2}$ we can calculate the bilinear form J_f on Ω_f

$$J_f\left([dx_1\wedge dx_2\wedge dx_3],[x_1^{2k-2}dx_1\wedge dx_2\wedge dx_3]\right) = \frac{1}{8k}.$$

As a $\mathbb{C}$-module, the orbifold Jacobian algebra $\mathrm{Jac}(f,G)$ is of the following form:

$$\mathrm{Jac}(f,G)\cong\left\langle e_{\mathrm{id}},[x_1^2],\dots,[x_1^2]^{k-1}\right\rangle_{\mathbb{C}}\oplus\langle e_g\rangle_{\mathbb{C}}\,.$$

Note that $\dim_{\mathbb{C}}\mathrm{Jac}(f,G) = k+1$. The bilinear form $J_{f,G}$ on $\Omega_{f,G}$ can be calculated as

$$\begin{aligned} J_{f,\mathrm{id}}\left(e_{\mathrm{id}}\vdash\zeta,[x_1^2]^{k-1}\vdash\zeta\right) &= J_{f,\mathrm{id}}\left([dx_1\wedge dx_2\wedge dx_3],[x_1^{2k-2}dx_1\wedge dx_2\wedge dx_3]\right)\\ &= 2\cdot\frac{1}{8k} = \frac{1}{4k}, \end{aligned}$$

and with $\mu_{f^g}=1$, $\mathrm{hess}(f^g)=2$

$$\begin{aligned} J_{f,g}\left(e_g\vdash\zeta,e_g\vdash\zeta\right) &= \frac{1}{4}J_{f,g}\left([dx_2],[dx_2]\right)\\ &= \frac{1}{4}\cdot(-1)\cdot 2\cdot\frac{1}{2} = -\frac{1}{4}, \end{aligned}$$

which imply the following relations in the orbifold Jacobian algebra $\mathrm{Jac}(f,G)$:

$$[x_1^2]\circ e_g = 0,\quad e_g^2 = -k[x_1^2]^{k-1}.$$

Here the first relation is clear since x_1 is not fixed by g. The second one is again as in the last calculation directly seen by the residue pairings.

On the other hand, the Jacobian algebra $\mathrm{Jac}(\overline{f})$ is given by

$$\begin{aligned} \mathrm{Jac}(\overline{f}) &= \mathbb{C}[y_1,y_2,y_3]\left/\left(2y_1,ky_2^{k-1}+y_3^2,2y_2y_3\right)\right.\\ &\cong \mathbb{C}[y_2,y_3]\left/\left(ky_2^{k-1}+y_3^2,y_2y_3\right)\right.\,. \end{aligned}$$

Note that $\dim_{\mathbb{C}}\mathrm{Jac}(\overline{f}) = k+1$. Therefore, we have an algebra isomorphism

$$\mathrm{Jac}(\overline{f})\xrightarrow{\cong}\mathrm{Jac}(f,G),\quad y_2\mapsto[x_1^2],\ y_3\mapsto e_g,$$

which is, moreover, an isomorphism of Frobenius algebras since we have
$\mathrm{hess}(\overline{f}) = 2\cdot k(k-1)\cdot 2y_2^{k-1}-2\cdot 2\cdot 2y_3^2 = 4k(k-1)y_2^{k-1}+8\cdot ky_2^{k-1} = 4k(k-1+2)y_2^{k-1}\in\mathrm{Jac}(\overline{f})$ and so

$$J_{\overline{f}}\left([dy_1\wedge dy_2\wedge dy_3],[y_2^{k-1}dy_1\wedge dy_2\wedge dy_3]\right) = \frac{1}{4k}.$$

3. For $k \geq 1$, set

$$f := x_1^{2k} + x_2^2 + x_3^2, \quad G := \langle g, h \rangle\,,\ g := \frac{1}{2}(0,1,1),\ h := \frac{1}{2}(1,0,1),$$

$$\overline{f} := y_1^k y_2^k + y_1 y_3 + y_2 y_3.$$

The Jacobian algebra $\mathrm{Jac}(f)$ can be calculated as

$$\mathrm{Jac}(f) = \mathbb{C}[x_1, x_2, x_3] \big/ \big(2kx_1^{2k-1}, 2x_2, 2x_3\big) \cong \big\langle 1, x_1, \ldots, x_1^{2k-2} \big\rangle_{\mathbb{C}}\,,$$

so $\mu_f = 2k-1$. With $\mathrm{hess}(f) = 2k(2k-1) \cdot 2 \cdot 2 \cdot x_1^{2k-2} = 8k(2k-1)x_1^{2k-2}$ we can calculate the bilinear form J_f on Ω_f

$$J_f\left([dx_1 \wedge dx_2 \wedge dx_3], [x_1^{2k-2} dx_1 \wedge dx_2 \wedge dx_3]\right) = \frac{1}{8k}.$$

As a $\mathbb{C}$-module, the orbifold Jacobian algebra $\mathrm{Jac}(f, G)$ is of the following form:

$$\mathrm{Jac}(f, G) \cong \big\langle e_{\mathrm{id}}, [x_1^2], \ldots, [x_1^2]^{k-1} \big\rangle_{\mathbb{C}} \oplus \big\langle e'_g, [x_1^2]e'_g, \ldots, [x_1^2]^{k-2} e'_g \big\rangle_{\mathbb{C}}\,,$$

where $e'_g := [x_1]e_g$ since $\mathrm{Jac}(f, h) = \{0\}$ and $\mathrm{Jac}(f, gh) = \{0\}$. Note that $\dim_{\mathbb{C}} \mathrm{Jac}(f, G) = 2k-1$. The bilinear form $J_{f,G}$ on $\Omega_{f,G}$ can be calculated as

$$\begin{aligned} J_{f,\mathrm{id}}\left(e_{\mathrm{id}} \vdash \zeta, [x_1^2]^{k-1} \vdash \zeta\right) &= J_{f,\mathrm{id}}\left([dx_1 \wedge dx_2 \wedge dx_3], [x_1^{2k-2} dx_1 \wedge dx_2 \wedge dx_3]\right) \\ &= 4 \cdot \frac{1}{8k} = \frac{1}{2k}, \end{aligned}$$

and with $\mu_{f^g} = 2k-1$, $\mathrm{hess}(f^g) = 2k(2k-1)x_1^{2k-2}$

$$\begin{aligned} J_{f,g}\left(e'_g \vdash \zeta, [x_1^2]^{k-2} e'_g \vdash \zeta\right) &= \frac{1}{4} J_{f,g}\left([x_1 dx_1], [x_1^{2k-3} dx_1]\right) \\ &= \frac{1}{4} \cdot (-1) \cdot 4 \cdot \frac{1}{2k} = -\frac{1}{2k}, \end{aligned}$$

which imply the following relations in the orbifold Jacobian algebra $\mathrm{Jac}(f, G)$:

$$[x_1^2]^{k-1} \circ e'_g = 0, \quad (e'_g)^2 = -[x_1^2].$$

On the other hand, the Jacobian algebra $\mathrm{Jac}(\overline{f})$ is given by

$$\mathrm{Jac}(\overline{f}) = \mathbb{C}[y_1, y_2, y_3] \big/ \big(ky_1^{k-1} y_2^k + y_3, ky_1^k y_2^{k-1} + y_3, y_1 + y_2\big)\,.$$

Note that $\dim_{\mathbb{C}} \mathrm{Jac}(\overline{f}) = 2k-1$. Therefore, we have an algebra isomorphism

$$\mathrm{Jac}(\overline{f}) \xrightarrow{\cong} \mathrm{Jac}(f, G), \quad y_1 y_2 \mapsto [x_1^2],\ y_1 \mapsto e'_g,$$

which is, moreover, an isomorphism of Frobenius algebras since we have
$\mathrm{hess}(\overline{f}) = k^2 y_1^{k-1} y_2^{k-1} + k^2 y_1^{k-1} y_2^{k-1} - k(k-1) y_1^{k-2} y_2^{k} - k(k-1) y_1^{k} y_2^{k-2}$
$= 2k^2 y_1^{k-1} y_2^{k-1} - k(k-1) y_1^{k-2} y_2^{k-2}(y_2^2 + y_1^2) = 2k^2 y_1^{k-1} y_2^{k-1} - k(k-1) y_1^{k-2} y_2^{k-2}((y_2+y_1)^2 - 2y_1y_2)$
$= (2k^2 + 2k(k-1)) y_1^{k-1} y_2^{k-1} = 2k(2k-1) y_1^{k-1} y_2^{k-1} \in \mathrm{Jac}(\overline{f})$ and so

$$J_{\overline{f}}\left([dy_1 \wedge dy_2 \wedge dy_3], [y_1^{k-1} y_2^{k-1} dy_1 \wedge dy_2 \wedge dy_3]\right) = \frac{1}{2k}.$$

4. Set

$$f := x_1^3 + x_2^3 + x_3^2, \quad G := \langle g \rangle\,, \ g := \frac{1}{3}(1, 2, 0),$$

$$\overline{f} := y_1^2 + y_3 y_2^2 + y_2 y_3^2.$$

The Jacobian algebra $\mathrm{Jac}(f)$ can be calculated as

$$\mathrm{Jac}(f) \cong \mathbb{C}[x_1, x_2, x_3] \big/ \big(3x_1^2, 3x_2^2, 2x_3\big) \cong \langle 1, x_1, x_2, x_1x_2 \rangle_{\mathbb{C}}\,,$$

so $\mu_f = 4$. With $\mathrm{hess}(f) = 6 \cdot 6 \cdot 2 \cdot x_1 x_2 = 4 \cdot 18 x_1 x_2$ we can calculate the bilinear form J_f on Ω_f

$$J_f\left([dx_1 \wedge dx_2 \wedge dx_3], [x_1 x_2 dx_1 \wedge dx_2 \wedge dx_3]\right) = \frac{1}{18}.$$

As a $\mathbb{C}$-module, the orbifold Jacobian algebra $\mathrm{Jac}(f, G)$ is of the following form:

$$\mathrm{Jac}(f, G) \cong \langle e_{\mathrm{id}}, [x_1 x_2] \rangle_{\mathbb{C}} \oplus \langle e_g, e_{g^{-1}} \rangle_{\mathbb{C}}\,.$$

Note that $\dim_{\mathbb{C}} \mathrm{Jac}(f, G) = 4$. The bilinear form $J_{f,G}$ on $\Omega_{f,G}$ can be calculated as

$$\begin{aligned} J_{f,\mathrm{id}}\left(e_{\mathrm{id}} \vdash \zeta, [x_1 x_2] \vdash \zeta\right) &= J_{f,\mathrm{id}}\left([dx_1 \wedge dx_2 \wedge dx_3], [x_1 x_2 dx_1 \wedge dx_2 \wedge dx_3]\right) \\ &= 3 \cdot \frac{1}{18} = \frac{1}{6}, \end{aligned}$$

and with $\mu_{f^g} = 1$, $\mathrm{hess}(f^g) = 2$

$$\begin{aligned} J_{f,g}\left(e_g \vdash \zeta, e_{g^{-1}} \vdash \zeta\right) &= \frac{1}{9} J_{f,g}\left([dx_3], [dx_3]\right) \\ &= \frac{1}{9} \cdot (-1) \cdot 3 \cdot \frac{1}{2} = -\frac{1}{6}, \end{aligned}$$

which imply the following relations in the orbifold Jacobian algebra $\mathrm{Jac}(f, G)$:

$$e_g^2 = 0, \quad e_{g^{-1}}^2 = 0, \quad e_g \circ e_{g^{-1}} = -[x_1 x_2].$$

The first two relations are proven in Proposition 6.2.15.

On the other hand, the Jacobian algebra $\mathrm{Jac}(\overline{f})$ is given by

$$\begin{aligned}\mathrm{Jac}(\overline{f}) &= \mathbb{C}[y_1, y_2, y_3] \,/\, \big(2y_1, 2y_3y_2 + y_3^2, y_2^2 + 2y_2y_3\big) \\ &\cong \mathbb{C}[y_2, y_3] \,/\, \big(2y_3y_2 + y_3^2, y_2^2 + 2y_2y_3\big) \ .\end{aligned}$$

Note that $\dim_{\mathbb{C}} \mathrm{Jac}(\overline{f}) = 4$. Therefore, we have an algebra isomorphism

$$\begin{aligned}&\mathrm{Jac}(\overline{f}) \xrightarrow{\cong} \mathrm{Jac}(f, G), \\ &y_2 \mapsto \mathbf{e}[\frac{1}{3}]e_g + \mathbf{e}[\frac{2}{3}]e_{g^{-1}}, \quad y_3 \mapsto \mathbf{e}[\frac{2}{3}]e_g + \mathbf{e}[\frac{1}{3}]e_{g^{-1}},\end{aligned}$$

which is, moreover, an isomorphism of Frobenius algebras since we have
$\mathrm{hess}(\overline{f}) = 2(2\cdot 2\cdot y_2y_3 - (2y_2+2y_3)^2) = 2(4y_2y_3 - 4y_2^2 - 8y_2y_3 - 4y_3^2) = 2(-4y_2y_3 + 8y_2y_3 + 8y_2y_3)$
$= 4 \cdot 6y_2y_3 \in \mathrm{Jac}(\overline{f})$ and so

$$J_{\overline{f}}\left([dy_1 \wedge dy_2 \wedge dy_3], [y_2y_3dy_1 \wedge dy_2 \wedge dy_3]\right) = \frac{1}{6}.$$

5. Set

$$f := x_1^4 + x_2^3 + x_3^2, \quad G := \langle g \rangle\,,\ g := \frac{1}{2}(1, 0, 1),$$

$$\overline{f} := y_1^3 + y_2^2 + y_2y_3^2.$$

The Jacobian algebra $\mathrm{Jac}(f)$ can be calculated as

$$\mathrm{Jac}(f) = \mathbb{C}[x_1, x_2, x_3] \,/\, \big(4x_1^3, 3x_2^2, 2x_3\big) \cong \big\langle 1, x_1, x_2, x_1^2, x_1x_2, x_1^2x_2 \big\rangle_{\mathbb{C}}\,,$$

so $\mu_f = 6$. With $\mathrm{hess}(f) = 12 \cdot 6 \cdot 2 \cdot x_1^2x_2 = 24 \cdot 6x_1^2x_2$ we can calculate the bilinear form J_f on Ω_f

$$J_f\left([dx_1 \wedge dx_2 \wedge dx_3], [x_1^2x_2dx_1 \wedge dx_2 \wedge dx_3]\right) = \frac{1}{24}.$$

As a $\mathbb{C}$-module, the orbifold Jacobian algebra $\mathrm{Jac}(f, G)$ is of the following form:

$$\mathrm{Jac}(f, G) \cong \big\langle e_{\mathrm{id}}, [x_2], [x_1^2], [x_1^2][x_2] \big\rangle_{\mathbb{C}} \oplus \langle e_g, [x_2]e_g \rangle_{\mathbb{C}}\,.$$

Note that $\dim_{\mathbb{C}} \mathrm{Jac}(f, G) = 6$. The bilinear form $J_{f,G}$ on $\Omega_{f,G}$ can be calculated as

$$\begin{aligned}J_{f,\mathrm{id}}\big(e_{\mathrm{id}} \vdash \zeta, [x_1^2][x_2] \vdash \zeta\big) &= J_{f,\mathrm{id}}\big([dx_1 \wedge dx_2 \wedge dx_3], [x_1^2x_2dx_1 \wedge dx_2 \wedge dx_3]\big) \\ &= 2 \cdot \frac{1}{24} = \frac{1}{12},\end{aligned}$$

and with $\mu_{f^g} = 2$, $\mathrm{hess}(f^g) = 3 \cdot 2x_2$

$$\begin{aligned}J_{f,g}\left(e_g \vdash \zeta, [x_2]e_g \vdash \zeta\right) &= \frac{1}{4}J_{f,g}\left([dx_2], [x_2dx_2]\right) \\ &= \frac{1}{4} \cdot (-1) \cdot 2 \cdot \frac{1}{3} = -\frac{1}{6},\end{aligned}$$

which imply the following relations in the orbifold Jacobian algebra $\mathrm{Jac}(f,G)$:

$$[x_2]^2=0,\quad [x_1^2]\circ e_g=0,\quad e_g^2=-2[x_1^2].$$

On the other hand, the Jacobian algebra $\mathrm{Jac}(\overline{f})$ is given by

$$\mathrm{Jac}(\overline{f})=\mathbb{C}[y_1,y_2,y_3]\,/\big(3y_1^2,2y_2+y_3^2,2y_2y_3\big)\ .$$

Note that $\dim_{\mathbb{C}}\mathrm{Jac}(\overline{f})=6$. Therefore, we have an algebra isomorphism

$$\mathrm{Jac}(\overline{f})\xrightarrow{\cong}\mathrm{Jac}(f,G),\quad y_1\mapsto[x_2],\ y_2\mapsto[x_1^2],\ y_3\mapsto e_g,$$

which is, moreover, an isomorphism of Frobenius algebras since we have $\mathrm{hess}(\overline{f})=6\cdot2\cdot2\cdot y_1y_2-6\cdot2\cdot2\cdot y_1y_3^2=24y_1y_2+24\cdot2y_1y_2=6\cdot12y_1y_2\in\mathrm{Jac}(\overline{f})$ and so

$$J_{\overline{f}}\left([dy_1\wedge dy_2\wedge dy_3],[y_1y_2dy_1\wedge dy_2\wedge dy_3]\right)=\frac{1}{12}.$$

6. For $k\geq1$, set

$$f:=x_1^2+x_2^2+x_2x_3^{2k},\quad G:=\langle g\rangle\,,\ g:=\frac{1}{2}(1,0,1),$$

$$\overline{f}:=y_1^2+y_1y_2^k+y_2y_3^2.$$

The Jacobian algebra $\mathrm{Jac}(f)$ can be calculated as

$$\begin{aligned}\mathrm{Jac}(f)&=\mathbb{C}[x_1,x_2,x_3]\,/\big(2x_1,2x_2+x_3^{2k},2kx_2x_3^{2k-1}\big)\\&\cong\big\langle1,x_3,\ldots,x_3^{2k-1},x_2,x_2x_3,\ldots,x_2x_3^{2k-2}\big\rangle_{\mathbb{C}}\,,\end{aligned}$$

so $\mu_f=4k-1$. With $\mathrm{hess}(f)=2\cdot2\cdot2k(2k-1)x_2x_3^{2k-2}-2\cdot2k\cdot2k\cdot x_3^{4k-2}$ $=8k(2k-1)x_2x_3^{2k-2}+8k^2\cdot2x_2x_3^{2k-2}=8k(4k-1)x_2x_3^{2k-2}\in\mathrm{Jac}(f)$ we can calculate the bilinear form J_f on Ω_f

$$J_f\left([dx_1\wedge dx_2\wedge dx_3],[x_2x_3^{2k-2}dx_1\wedge dx_2\wedge dx_3]\right)=\frac{1}{8k}.$$

As a $\mathbb{C}$-module, the orbifold Jacobian algebra $\mathrm{Jac}(f,G)$ is of the following form:

$$\mathrm{Jac}(f,G)\cong\big\langle e_{\mathrm{id}},[x_3^2],\ldots,[x_3^2]^{k-1},[x_2],[x_2][x_3^2],\ldots,[x_2][x_3^2]^{k-1}\big\rangle_{\mathbb{C}}\oplus\langle e_g\rangle_{\mathbb{C}}\,.$$

Note that $\dim_{\mathbb{C}}\mathrm{Jac}(f,G)=2k+1$. The bilinear form $J_{f,G}$ on $\Omega_{f,G}$ can be calculated as

$$\begin{aligned}J_{f,\mathrm{id}}\left(e_{\mathrm{id}}\vdash\zeta,[x_2][x_3^2]^{k-1}\vdash\zeta\right)&=J_{f,\mathrm{id}}\left([dx_1\wedge dx_2\wedge dx_3],[x_2x_3^{2k-2}dx_1\wedge dx_2\wedge dx_3]\right)\\&=2\cdot\frac{1}{8k}=\frac{1}{4k},\end{aligned}$$

and with $\mu_{f^g} = 1$, $\mathrm{hess}(f^g) = 2$

$$J_{f,g}\left(e_g \vdash \zeta, e_g \vdash \zeta\right) = \frac{1}{4} J_{f,g}\left([dx_2],[dx_2]\right)$$
$$= \frac{1}{4}\cdot(-1)\cdot 2\cdot\frac{1}{2} = -\frac{1}{4},$$

which imply the following relations in the orbifold Jacobian algebra $\mathrm{Jac}(f,G)$:

$$[x_3^2]^k = -2[x_2], \quad [x_3^2]\circ e_g = 0, \quad e_g^2 = -k[x_2][x_3^2]^{k-1}.$$

On the other hand, the Jacobian algebra $\mathrm{Jac}(\overline{f})$ is given by

$$\mathrm{Jac}(\overline{f}) = \mathbb{C}[y_1,y_2,y_3]\big/\left(2y_1 + y_2^k, ky_1y_2^{k-1} + y_3^2, 2y_2y_3\right).$$

Note that $\dim_{\mathbb{C}}\mathrm{Jac}(\overline{f}) = 2k+1$. Therefore, we have an algebra isomorphism

$$\mathrm{Jac}(\overline{f}) \xrightarrow{\cong} \mathrm{Jac}(f,G), \quad y_1 \mapsto [x_2],\ y_2 \mapsto [x_3^2],\ y_3 \mapsto e_g,$$

which is, moreover, an isomorphism of Frobenius algebras since we have
$\mathrm{hess}(\overline{f}) = 2\cdot k(k-1)\cdot 2\cdot y_1y_2^{k-1} - 2\cdot 2\cdot 2\cdot y_3^2 - 2\cdot k\cdot ky_2^{2k-1}$
$= 4k(k-1)y_1y_2^{k-1} + 8\cdot ky_1y_2^{k-1} + 2k^2\cdot 2y_1y_2^{k-1} = 4k(2k+1)y_1y_2^{k-1} \in \mathrm{Jac}(\overline{f})$ and so

$$J_{\overline{f}}\left([dy_1\wedge dy_2\wedge dy_3],[y_1y_2^{k-1}dy_1\wedge dy_2\wedge dy_3]\right) = \frac{1}{4k}.$$

7. For $k \geq 1$, set

$$f := x_1^2 + x_2^2 + x_2x_3^{2k+1}, \quad G := \langle g\rangle,\ g := \frac{1}{2}(0,1,1),$$

$$\overline{f} := y_1^2 + y_3y_2^2 + y_2y_3^{k+1}.$$

The Jacobian algebra $\mathrm{Jac}(f)$ can be calculated as

$$\mathrm{Jac}(f) = \mathbb{C}[x_1,x_2,x_3]\big/\left(2x_1, 2x_2 + x_3^{2k+1}, (2k+1)x_2x_3^{2k}\right)$$
$$\cong \left\langle 1, x_3, \ldots, x_3^{2k}, x_2, x_2x_3, \ldots, x_2x_3^{2k-1}\right\rangle_{\mathbb{C}},$$

so $\mu_f = 4k+1$. With $\mathrm{hess}(f) = 2\cdot 2\cdot(2k+1)2kx_2x_3^{2k-1} - 2\cdot(2k+1)\cdot(2k+1)\cdot x_3^{4k}$
$= 8k(2k+1)x_2x_3^{2k-1} + 2(2k+1)^2\cdot 2x_2x_3^{2k-1} = 4(2k+1)(4k+1)x_2x_3^{2k-1} \in \mathrm{Jac}(f)$ we can calculate the bilinear form J_f on Ω_f

$$J_f\left([dx_1\wedge dx_2\wedge dx_3],[x_2x_3^{2k-1}dx_1\wedge dx_2\wedge dx_3]\right) = \frac{1}{4(2k+1)}.$$

As a $\mathbb{C}$-module, the orbifold Jacobian algebra $\mathrm{Jac}(f,G)$ is of the following form:

$$\mathrm{Jac}(f,G) \cong \left\langle e_{\mathrm{id}}, [x_3^2], \ldots, [x_3^2]^k, [x_2x_3], [x_2x_3][x_3^2], \ldots, [x_2x_3][x_3^2]^{k-1}\right\rangle_{\mathbb{C}} \oplus \langle e_g\rangle_{\mathbb{C}}.$$

Note that $\dim_{\mathbb{C}} \mathrm{Jac}(f,G) = 2(k+1)$. The bilinear form $J_{f,G}$ on $\Omega_{f,G}$ can be calculated as

$$\begin{aligned} J_{f,\mathrm{id}}\left(e_{\mathrm{id}} \vdash \zeta, [x_2x_3][x_3^2]^{k-1} \vdash \zeta\right) &= J_{f,\mathrm{id}}\left([dx_1 \wedge dx_2 \wedge dx_3], [x_2x_3^{2k-1}dx_1 \wedge dx_2 \wedge dx_3]\right) \\ &= 2 \cdot \frac{1}{4(2k+1)} = \frac{1}{2(2k+1)}, \end{aligned}$$

and with $\mu_{f^g} = 1$, $\mathrm{hess}(f^g) = 2$

$$\begin{aligned} J_{f,g}\left(e_g \vdash \zeta, e_g \vdash \zeta\right) &= \frac{1}{4} J_{f,g}\left([dx_1], [dx_1]\right) \\ &= \frac{1}{4} \cdot (-1) \cdot 2 \cdot \frac{1}{2} = -\frac{1}{4}, \end{aligned}$$

which imply the following relations in the orbifold Jacobian algebra $\mathrm{Jac}(f,G)$:

$$[x_3^2]^{k+1} = -2[x_2x_3], \quad [x_3^2] \circ e_g = 0, \quad e_g^2 = -\frac{2k+1}{2}[x_2x_3][x_3^2]^{k-1}.$$

On the other hand, the Jacobian algebra $\mathrm{Jac}(\overline{f})$ is given by

$$\begin{aligned} \mathrm{Jac}(\overline{f}) &= \mathbb{C}[y_1,y_2,y_3] \big/ \left(2y_1, 2y_3y_2 + y_3^{k+1}, y_2^2 + (k+1)y_2y_3^k\right) \\ &\cong \mathbb{C}[y_2,y_3] \big/ \left(2y_3y_2 + y_3^{k+1}, y_2^2 + (k+1)y_2y_3^k\right) . \end{aligned}$$

Note that $\dim_{\mathbb{C}} \mathrm{Jac}(\overline{f}) = 2(k+1)$. Therefore, we have an algebra isomorphism

$$\mathrm{Jac}(\overline{f}) \xrightarrow{\cong} \mathrm{Jac}(f,G), \quad y_2 \mapsto e_g - \frac{1}{2}[x_3^2]^k, \ y_3 \mapsto [x_3^2],$$

which is, moreover, an isomorphism of Frobenius algebras since we have
$\mathrm{hess}(\overline{f}) = 2 \cdot 2 \cdot (k+1)k \cdot y_2y_3^k - 2 \cdot (2y_2 + (k+1)y_3^k)^2$
$= 4k(k+1)y_2y_3^k - 2(4y_2^2 + 4(k+1)y_2y_3^k + (k+1)^2y_3^{2k})$
$= 4k(k+1)y_2y_3^k - 2(-4(k+1)y_2y_3^k + 4(k+1)y_2y_3^k - (k+1)^2 \cdot 2y_2y_3^k)$
$= (4k(k+1) + 4(k+1)^2)y_2y_3^k = 2(k+1)(4k+2)y_2y_3^k \in \mathrm{Jac}(\overline{f})$ and so

$$J_{\overline{f}}\left([dy_1 \wedge dy_2 \wedge dy_3], [y_2y_3^k dy_1 \wedge dy_2 \wedge dy_3]\right) = \frac{1}{2(2k+1)}.$$

8. For $k \geq 4$, set

$$f := x_1^2 + x_2^{k-1} + x_2x_3^2, \quad G := \langle g \rangle, \ g := \frac{1}{2}(1,0,1),$$

$$\overline{f} := y_1^{k-1} + y_1y_2 + y_2y_3^2.$$

The Jacobian algebra $\mathrm{Jac}(f)$ can be calculated as

$$\begin{aligned} \mathrm{Jac}(f) &= \mathbb{C}[x_1,x_2,x_3] \big/ \left(2x_1, (k-1)x_2^{k-2} + x_3^2, 2x_2x_3\right) \\ &\cong \left\langle 1, x_2, \ldots, x_2^{k-2}, x_3 \right\rangle_{\mathbb{C}}, \end{aligned}$$

so $\mu_f = k$. With $\mathrm{hess}(f) = 2\cdot(k-1)(k-2)\cdot 2\cdot x_2^{k-2} - 2\cdot 2\cdot 2\cdot x_3^2 = 4(k-1)(k-2)x_2^{k-2} + 8\cdot(k-1)x_2^{k-2}$ $= 4(k-1)(k)x_2^{k-2} \in \mathrm{Jac}(f)$ we can calculate the bilinear form J_f on Ω_f

$$J_f\left([dx_1 \wedge dx_2 \wedge dx_3], [x_2^{k-2} dx_1 \wedge dx_2 \wedge dx_3]\right) = \frac{1}{4(k-1)}.$$

As a $\mathbb{C}$-module, the orbifold Jacobian algebra $\mathrm{Jac}(f, G)$ is of the following form:

$$\mathrm{Jac}(f, G) \cong \left\langle e_{\mathrm{id}}, [x_2], \ldots, [x_2]^{k-2}\right\rangle_{\mathbb{C}} \oplus \left\langle e_g, [x_2]e_g, \ldots, [x_2]^{k-3} e_g\right\rangle_{\mathbb{C}}.$$

Note that $\dim_{\mathbb{C}} \mathrm{Jac}(f, G) = 2k-3$. The bilinear form $J_{f,G}$ on $\Omega_{f,G}$ can be calculated as

$$\begin{aligned} J_{f,\mathrm{id}}\left(e_{\mathrm{id}} \vdash \zeta, [x_2]^{k-2} \vdash \zeta\right) &= J_{f,\mathrm{id}}\left([dx_1 \wedge dx_2 \wedge dx_3], [x_2^{k-2} dx_1 \wedge dx_2 \wedge dx_3]\right) \\ &= 2 \cdot \frac{1}{4(k-1)} = \frac{1}{2(k-1)}, \end{aligned}$$

and with $\mu_{f^g} = k-2$, $\mathrm{hess}(f^g) = (k-1)(k-2)x_2^{k-3}$

$$\begin{aligned} J_{f,g}\left(e_g \vdash \zeta, [x_2]^{k-3} e_g \vdash \zeta\right) &= \frac{1}{4} J_{f,g}\left([dx_2], [x_2^{k-3} dx_2]\right) \\ &= \frac{1}{4} \cdot (-1) \cdot 2 \cdot \frac{1}{k-1} = -\frac{1}{2(k-1)}, \end{aligned}$$

which imply the following relations in the orbifold Jacobian algebra $\mathrm{Jac}(f, G)$:

$$[x_2]^{k-2} \circ e_g = 0, \quad e_g^2 = -[x_2].$$

On the other hand, the Jacobian algebra $\mathrm{Jac}(\overline{f})$ is given by

$$\mathrm{Jac}(\overline{f}) = \mathbb{C}[y_1, y_2, y_3] \big/ \left((k-1)y_1^{k-2} + y_2, y_1 + y_3^2, 2y_2 y_3\right).$$

Note that $\dim_{\mathbb{C}} \mathrm{Jac}(\overline{f}) = 2k-3$. Therefore, we have an algebra isomorphism

$$\mathrm{Jac}(\overline{f}) \xrightarrow{\cong} \mathrm{Jac}(f, G), \quad y_1 \mapsto [x_2], \; y_2 \mapsto -(k-1)[x_2]^{k-2}, \; y_3 \mapsto e_g,$$

which is, moreover, an isomorphism of Frobenius algebras since we have
$\mathrm{hess}(\overline{f}) = -(k-1)(k-2) \cdot 2 \cdot 2 \cdot y_1^{k-3} y_3^2 - 2y_2 = 4(k-1)(k-2)y_1^{k-2} + 2 \cdot (k-1)y_1^{k-2}$
$= 2(k-1)(2k-4+1)y_1^{k-2} \in \mathrm{Jac}(\overline{f})$ and so

$$J_{\overline{f}}\left([dy_1 \wedge dy_2 \wedge dy_3], [y_1^{k-2} dy_1 \wedge dy_2 \wedge dy_3]\right) = \frac{1}{2(k-1)}.$$

We finished the proof of Theorem 6.3.7. $\square$

6.4 Orbifold Jacobian Algebras for Exceptional Unimodal Singularities

Definition 6.4.1 ([AGV85, p. 247])**.** There is a list of 14 exceptional families of unimodal isolated hypersurface singularities. In the notation of Arnold they are called E_{12}, E_{13}, E_{14}, Z_{11}, Z_{12}, Z_{13}, W_{12}, W_{13}, Q_{10}, Q_{11}, Q_{12}, S_{11}, S_{12} and U_{12}. One can give invertible polynomials in three variables belonging to these families. In Table 6.3 there are listed all possible choices of an invertible polynomial, representing an exceptional unimodal singularity (see [RN16, Table 1]).

Singularity Type	f (v1)	f (v2)	f (v3)
E_{12}	$x^2+y^3+z^7$	-	-
E_{13}	$x^2+y^3+yz^5$	-	-
E_{14}	$x^3+y^2+yz^4$	$x^2+y^3+z^8$	-
Z_{11}	$x^2+y^3z+z^5$	-	-
Z_{12}	$x^2+y^3z+yz^4$	-	-
Z_{13}	$x^2+xy^3+yz^3$	$x^2+y^3z+z^6$	-
W_{12}	$x^5+y^2+yz^2$	$x^2+y^4+z^5$	-
W_{13}	$x^2+xy^2+yz^4$	$x^2+y^4+yz^4$	-
Q_{10}	$x^3+y^2z+z^4$	-	-
Q_{11}	$x^2y+y^3z+z^3$	-	-
Q_{12}	$x^3+y^2z+yz^3$	$x^3+y^2z+z^5$	-
S_{11}	$x^2y+y^2z+z^4$	-	-
S_{12}	$x^3y+y^2z+xz^2$	-	-
U_{12}	$x^4+y^2z+yz^2$	$x^3+y^3+z^4$	$x^4+y^3+yz^2$

Table 6.3: All invertible polynomials, representing the exceptional unimodal singularities.

Remark 6.4.2. In some cases G_f^{SL} is not the trivial group. We can try to do the same as in Remark 6.3.2 and consider a crepant resolution of $\mathbb{C}^3/G_f^{\mathrm{SL}}$. We observe:

Only in the cases, where f and f^T are of the same singularity type, the critical points of $\widehat{f}$ are isolated and contained in one chart isomorphic to $\mathbb{C}^3$.

Example 6.4.3. For E_{14} and the pair (f, G)

$$f := x_1^2 + x_2^3 + x_3^8, \quad G := \left\langle \frac{1}{2}(1,0,1) \right\rangle,$$

we have in the two charts

$$\widehat{f}(t,u,v) = t^3 + u + (uv^2)^4 \text{ and } \widehat{f}(t,u,v) = t^3 + u^2v + v^4.$$

Critical points of $\widehat{f}$ are on the second chart.

We see $f = f^T$ and the restriction of $\widehat{f}$ to the chart is given by $\overline{f} = y_1^4 + y_2^3 + y_1 y_3^2$ which is of singularity type Q_{10} which is strangely dual (cf. [Ar75]) to E_{14}.

For Z_{13} and the pair (f, G)

$$f := x_1^3 x_2 + x_2^6 + x_3^2, \quad G := \left\langle \frac{1}{2}(1,1,0) \right\rangle,$$

we have in the two charts

$$\widehat{f}(t,u,v) = t^2 + uuv + (uv^2)^3 \text{ and } \widehat{f}(t,u,v) = t^2 + u^2 vuv + v^3.$$

There are no isolated singularities in the first chart.

Proposition 6.4.4 ([ET11, Table 9]). *For the polynomials (v1) of Table 6.3 we always have $G_{f^T}^{\mathrm{SL}} = \{\mathrm{id}\}$. And we get:*

When f (v1) is of one singularity type, f^T is of the singularity type of the strangely dual in the sense of Arnold (c.f. [Ar75]).

Proof. One can easily see that there are no elements in G_f that are also in $\mathrm{SL}(3,\mathbb{C})$ for all polynomials (v1). The second statement is shown in [ET11]. $\square$

Remark 6.4.5. From Remark 6.4.2 and Proposition 6.4.4 it is straightforward to consider the pairs $(f^T, G_{f^T}^{\mathrm{SL}})$.

Example 6.4.6. With Remark 6.3.3 we will calculate the restriction of $\widehat{f^T}$ for all f in Table 6.3 with $G_{f^T}^{\mathrm{SL}} \supsetneq \{\mathrm{id}\}$.

1. For E_{14} and the pair (f^T, G)

$$f^T := x_1^2 + x_2^3 + x_3^8, \quad G := \left\langle \frac{1}{2}(1,0,1) \right\rangle,$$

see Example 6.4.3.

2. For Z_{13} and the pair (f^T, G)

$$f^T := x_1^2 + x_2^3 + x_2 x_3^6, \quad G := \left\langle \frac{1}{2}(1,0,1) \right\rangle,$$

we have in the two charts

$$\widehat{f^T}(t,u,v) = t^3 + u + t(uv^2)^3 \text{ and } \widehat{f^T}(t,u,v) = t^3 + u^2 v + tv^3.$$

Critical points of $\widehat{f^T}$ are on the second chart.

3. For W_{12} and the pair (f^T, G)

$$f^T := x_1^2 + x_2^4 + x_3^5, \quad G := \left\langle \frac{1}{2}(1,1,0) \right\rangle,$$

we have in the two charts

$$\widehat{f^T}(t,u,v) = t^5 + u + (uv^2)^2 \text{ and } \widehat{f^T}(t,u,v) = t^5 + u^2v + v^2.$$

Critical points of $\widehat{f^T}$ are on the second chart.

4. For W_{13} and the pair (f^T, G)

$$f^T := x_1^2 + x_2^4x_3 + x_3^4, \quad G := \left\langle \frac{1}{2}(1,1,0) \right\rangle,$$

we have in the two charts

$$\widehat{f^T}(t,u,v) = t^4 + u + t(uv^2)^2 \text{ and } \widehat{f^T}(t,u,v) = t^4 + u^2v + tv^2.$$

Critical points of $\widehat{f^T}$ are on the second chart.

5. For Q_{12} and the pair (f^T, G)

$$f^T := x_1^3 + x_2^2 + x_2x_3^5, \quad G := \left\langle \frac{1}{2}(0,1,1) \right\rangle,$$

we have in the two charts

$$\widehat{f^T}(t,u,v) = t^3 + u + uv(uv^2)^2 \text{ and } \widehat{f^T}(t,u,v) = t^3 + u^2v + uvv^2.$$

Critical points of $\widehat{f^T}$ are on the second chart.

6. For U_{12} and the pair (f^T, G)

$$f^T := x_1^3 + x_2^3 + x_3^4, \quad G := \left\langle \frac{1}{3}(1,2,0) \right\rangle,$$

we have in the three charts

$$\widehat{f^T}(t,u,v) = t^4 + u + (u^2v^3), \quad \widehat{f^T}(t,u,v) = t^4 + u^2v + uv^2$$
$$\text{and} \quad \widehat{f^T}(t,u,v) = t^4 + u^3v^2 + v.$$

Critical points of $\widehat{f^T}$ are on the second chart.

7. For U_{12} and the pair (f^T, G)

$$f^T := x_1^4 + x_2^3x_3 + x_3^2, \quad G := \left\langle \frac{1}{2}(0,1,1) \right\rangle,$$

we have in the two charts

$$\widehat{f^T}(t,u,v) = t^4 + uuv + (uv^2) \text{ and } \widehat{f^T}(t,u,v) = t^4 + u^2vuv + v.$$

Critical points of $\widehat{f^T}$ are on the first chart.

Remark 6.4.7. Here we observed that critical points of the map $\widehat{f^T}$ are contained in one chart isomorphic to $\mathbb{C}^3$. The restriction of $\widehat{f^T}$ to the chart is given by $\overline{f}$ defined in Table 6.4.

	Type of f	f^T	$G^{\mathrm{SL}}_{f^T}$	$\overline{f}$	Type of $\overline{f}$
1.	E_{14}	$x_1^2 + x_2^3 + x_3^8$	$\langle\frac{1}{2}(1,0,1)\rangle$	$y_1^3 + y_2^2y_3 + y_3^4$	Q_{10}
2.	Z_{13}	$x_1^2 + x_2^3 + x_2x_3^6$	$\langle\frac{1}{2}(1,0,1)\rangle$	$y_1^2y_2 + y_2^3y_3 + y_3^3$	Q_{11}
3.	W_{12}	$x_1^2 + x_2^4 + x_3^5$	$\langle\frac{1}{2}(1,1,0)\rangle$	$y_1^5 + y_2^2 + y_2y_3^2$	W_{12}
4.	W_{13}	$x_1^2 + x_2^4x_3 + x_3^4$	$\langle\frac{1}{2}(1,1,0)\rangle$	$y_1^2y_2 + y_2^2y_3 + y_3^4$	S_{11}
5.	Q_{12}	$x_1^3 + x_2^2 + x_2x_3^5$	$\langle\frac{1}{2}(0,1,1)\rangle$	$y_1^3 + y_2^2y_3 + y_2y_3^3$	Q_{12}
6.	U_{12}	$x_1^3 + x_2^3 + x_3^4$	$\langle\frac{1}{3}(1,2,0)\rangle$	$y_1^4 + y_2^2y_3 + y_2y_3^2$	U_{12}
7.	U_{12}	$x_1^4 + x_2^3x_3 + x_3^2$	$\langle\frac{1}{2}(0,1,1)\rangle$	$y_1^4 + y_2^2y_3 + y_2y_3^2$	U_{12}

Table 6.4: $(f^T, G^{\mathrm{SL}}_{f^T}) \cong (\overline{f})$

It is again natural to expect that the orbifold Jacobian algebra $\mathrm{Jac}(f^T, G^{\mathrm{SL}}_{f^T})$ of $(f^T, G^{\mathrm{SL}}_{f^T})$ is isomorphic to the usual Jacobian algebra $\mathrm{Jac}(\overline{f})$ of $\overline{f}$, which is the following theorem.

Theorem 6.4.8. *There is an isomorphism of Frobenius algebras*

$$\mathrm{Jac}(f^T, G^{\mathrm{SL}}_{f^T}) \cong \mathrm{Jac}(\overline{f})$$

for all f^T and $\overline{f}$ in Table 6.4.

Proof. We give a proof of this theorem based on the classification in Table 6.4. Let the notation be as in the sections before and again $e_g := \frac{\alpha_g^{-1}}{|K_g|}v_g \in \mathrm{Jac}(f,G)$ the element already mentioned in the proof of Theorem 6.3.7.
1. Set

$$f^T := x_1^2 + x_2^3 + x_3^8, \quad G := \langle g \rangle\,, \; g := \frac{1}{2}(1,0,1),$$

$$\overline{f} := y_1^3 + y_2^2y_3 + y_3^4.$$

The Jacobian algebra $\mathrm{Jac}(f^T)$ can be calculated as

$$\mathrm{Jac}(f^T) = \mathbb{C}[x_1, x_2, x_3] \,/ \left(2x_1, 3x_2^2, 8x_3^7\right) \cong \left\langle 1, x_2, x_3, \ldots, x_3^6, x_2x_3, \ldots, x_2x_3^6 \right\rangle_{\mathbb{C}},$$

so $\mu_{f^T} = 14$. With $\mathrm{hess}(f^T) = 2 \cdot 6 \cdot 56 \cdot x_2x_3^6 = 14 \cdot 48 x_2x_3^6$ we can calculate the bilinear form J_{f^T} on Ω_{f^T}

$$J_{f^T}\left([dx_1 \wedge dx_2 \wedge dx_3], [x_2x_3^6 dx_1 \wedge dx_2 \wedge dx_3]\right) = \frac{1}{48}.$$

As a $\mathbb{C}$-module, the orbifold Jacobian algebra $\mathrm{Jac}(f^T, G)$ is of the following form:

$$\mathrm{Jac}(f^T, G) \cong \left\langle e_{\mathrm{id}}, [x_2], [x_3^2], [x_3^2]^2, [x_3^2]^3, [x_2][x_3^2], [x_2][x_3^2]^2, [x_2][x_3^2]^3 \right\rangle_{\mathbb{C}} \oplus \langle e_g, [x_2]e_g \rangle_{\mathbb{C}}.$$

Note that $\dim_{\mathbb{C}} \mathrm{Jac}(f^T, G) = 10$. The bilinear form $J_{f^T,G}$ on $\Omega_{f^T,G}$ can be calculated as

$$\begin{aligned} J_{f^T,\mathrm{id}}\left(e_{\mathrm{id}} \vdash \zeta, [x_2][x_3^2]^3] \vdash \zeta\right) &= J_{f^T,\mathrm{id}}\left([dx_1 \wedge dx_2 \wedge dx_3], [x_2x_3^6 dx_1 \wedge dx_2 \wedge dx_3]\right) \\ &= 2 \cdot \frac{1}{48} = \frac{1}{24}, \end{aligned}$$

and with $\mu_{f^{Tg}} = 2$, $\mathrm{hess}(f^{Tg}) = 3 \cdot 2x_2$

$$\begin{aligned} J_{f^T,g}\left(e_g \vdash \zeta, [x_2]e_g \vdash \zeta\right) &= \frac{1}{4} J_{f^T,g}\left([dx_2], [x_2 dx_2]\right) \\ &= \frac{1}{4} \cdot (-1) \cdot 2 \cdot \frac{1}{3} = -\frac{1}{6}, \end{aligned}$$

which imply the following relations in the orbifold Jacobian algebra $\mathrm{Jac}(f^T, G)$:

$$[x_2]^2 = 0, \quad [x_3^2] \circ e_g = 0, \quad e_g^2 = -4[x_3^2]^3.$$

On the other hand, the Jacobian algebra $\mathrm{Jac}(\overline{f})$ is given by

$$\mathrm{Jac}(\overline{f}) = \mathbb{C}[y_1, y_2, y_3] \,/ \left(3y_1^2, 2y_2y_3, y_2^2 + 4y_3^3\right).$$

Note that $\dim_{\mathbb{C}} \mathrm{Jac}(\overline{f}) = 10$. Therefore, we have an algebra isomorphism

$$\mathrm{Jac}(\overline{f}) \xrightarrow{\cong} \mathrm{Jac}(f^T, G), \quad y_1 \mapsto [x_2], \; y_2 \mapsto e_g, \; y_3 \mapsto [x_3^2],$$

which is, moreover, an isomorphism of Frobenius algebras since we have
$\mathrm{hess}(\overline{f}) = 6 \cdot 2 \cdot 12 \cdot y_1y_3^3 - 6 \cdot 2 \cdot 2 \cdot y_1y_2^2 = 24 \cdot 6y_1y_3^3 + 24 \cdot 4y_1y_3^3 = 24 \cdot (10)y_1y_3^3 \in \mathrm{Jac}(\overline{f})$ and so

$$J_{\overline{f}}\left([dy_1 \wedge dy_2 \wedge dy_3], [y_1y_3^3 dy_1 \wedge dy_2 \wedge dy_3]\right) = \frac{1}{24}.$$

2. Set

$$f^T := x_1^2 + x_2^3 + x_2x_3^6, \quad G := \langle g \rangle, \; g := \frac{1}{2}(1, 0, 1),$$

$$\overline{f} := y_1^2 y_2 + y_2^3 y_3 + y_3^3.$$

The Jacobian algebra $\mathrm{Jac}(f^T)$ can be calculated as

$$\begin{aligned}\mathrm{Jac}(f^T) &= \mathbb{C}[x_1,x_2,x_3] \big/ \big(2x_1, 3x_2^2 + x_3^6, 6x_2x_3^5\big)\\ &\cong \big\langle 1, x_2, x_2^2, x_3, \ldots, x_3^5, x_2x_3, \ldots, x_2x_3^4, x_2^2x_3, \ldots, x_2^2x_3^4\big\rangle_{\mathbb{C}},\end{aligned}$$

so $\mu_{f^T} = 16$. With $\mathrm{hess}(f^T) = 2\cdot 6\cdot 30\cdot x_2^2x_3^4 - 2\cdot 6\cdot 6\cdot x_3^{10} = 36\cdot 10x_2^2x_3^4 + 2\cdot 36\cdot 3x_2^2x_3^4$ $= 36\cdot 16x_2^2x_3^4 \in \mathrm{Jac}(f^T)$ we can calculate the bilinear form J_{f^T} on Ω_{f^T}

$$J_{f^T}\big([dx_1\wedge dx_2\wedge dx_3], [x_2^2x_3^4 dx_1\wedge dx_2\wedge dx_3]\big) = \frac{1}{36}.$$

As a $\mathbb{C}$-module, the orbifold Jacobian algebra $\mathrm{Jac}(f^T, G)$ is of the following form:

$$\begin{aligned}\mathrm{Jac}(f^T,G) \cong\, &\big\langle e_{\mathrm{id}}, [x_2], [x_2]^2, [x_3^2], [x_3^2]^2, [x_2][x_3^2], [x_2][x_3^2]^2, [x_2]^2[x_3^2], [x_2]^2[x_3^2]^2\big\rangle_{\mathbb{C}}\\ &\oplus \langle e_g, [x_2]e_g\rangle_{\mathbb{C}}\,.\end{aligned}$$

Note that $\dim_{\mathbb{C}} \mathrm{Jac}(f^T,G) = 11$. The bilinear form $J_{f^T,G}$ on $\Omega_{f^T,G}$ can be calculated as

$$\begin{aligned}J_{f^T,\mathrm{id}}\big(e_{\mathrm{id}} \vdash \zeta, [x_2]^2[x_3^2]^2 \vdash \zeta\big) &= J_{f^T,\mathrm{id}}\big([dx_1\wedge dx_2\wedge dx_3], [x_2^2x_3^4 dx_1\wedge dx_2\wedge dx_3]\big)\\ &= 2\cdot\frac{1}{36} = \frac{1}{18},\end{aligned}$$

and with $\mu_{f^{Tg}} = 2$, $\mathrm{hess}(f^{Tg}) = 3\cdot 2x_2$

$$\begin{aligned}J_{f^T,g}\left(e_g \vdash \zeta, [x_2]e_g \vdash \zeta\right) &= \frac{1}{4} J_{f^T,g}\left([dx_2], [x_2dx_2]\right)\\ &= \frac{1}{4}\cdot(-1)\cdot 2\cdot\frac{1}{3} = -\frac{1}{6},\end{aligned}$$

which imply the following relations in the orbifold Jacobian algebra $\mathrm{Jac}(f^T,G)$:

$$[x_3^2]^3 = -3[x_2]^2, \quad [x_3^2]e_g = 0, \quad e_g^2 = -3[x_2x_3^4].$$

On the other hand, the Jacobian algebra $\mathrm{Jac}(\overline{f})$ is given by

$$\mathrm{Jac}(\overline{f}) = \mathbb{C}[y_1,y_2,y_3] \big/ \big(2y_1y_2, y_1^2 + 3y_2^2y_3, y_2^3 + 3y_3^2\big)\,.$$

Note that $\dim_{\mathbb{C}} \mathrm{Jac}(\overline{f}) = 11$. Therefore, we have an algebra isomorphism

$$\mathrm{Jac}(\overline{f}) \xrightarrow{\cong} \mathrm{Jac}(f^T,G), \quad y_1 \mapsto e_g,\ y_2 \mapsto [x_3^2],\ y_3 \mapsto [x_2],$$

which is, moreover, an isomorphism of Frobenius algebras since we have
$\mathrm{hess}(\overline{f}) = 2\cdot 6\cdot 6\cdot y_2^2y_3^2 - 2\cdot 3\cdot 3\cdot y_2^5 - 2\cdot 2\cdot 6\cdot y_1^2y_3 = 72y_2^2y_3^2 + 18\cdot 3y_2^2y_3^2 + 24\cdot 6y_2^2y_3^2$ $= 18(4+3+4)y_2^2y_3^2 \in \mathrm{Jac}(\overline{f})$ and so

$$J_{\overline{f}}\big([dy_1\wedge dy_2\wedge dy_3], [y_2^2y_3^2 dy_1\wedge dy_2\wedge dy_3]\big) = \frac{1}{18}.$$

3. Set

$$f^T := x_1^2 + x_2^4 + x_3^5, \quad G := \langle g \rangle\,,\ g := \frac{1}{2}(1,1,0),$$

$$\overline{f} := y_1^5 + y_2^2 + y_2 y_3^2.$$

The Jacobian algebra $\mathrm{Jac}(f^T)$ can be calculated as

$$\mathrm{Jac}(f^T) = \mathbb{C}[x_1,x_2,x_3]\big/\big(2x_1, 4x_2^3, 5x_3^4\big) \cong \big\langle 1, x_2, x_2^2, x_3, x_3^2, x_3^3, x_2x_3, \ldots, x_2^2x_3^3 \big\rangle_{\mathbb{C}}\,,$$

so $\mu_{f^T} = 12$. With $\mathrm{hess}(f^T) = 2\cdot 12\cdot 20\cdot x_2^2x_3^3 = 12\cdot 40x_2^2x_3^3$ we can calculate the bilinear form J_{f^T} on Ω_{f^T}

$$J_{f^T}\big([dx_1\wedge dx_2\wedge dx_3], [x_2^2x_3^3dx_1\wedge dx_2\wedge dx_3]\big) = \frac{1}{40}.$$

As a $\mathbb{C}$-module, the orbifold Jacobian algebra $\mathrm{Jac}(f^T,G)$ is of the following form:

$$\begin{aligned}\mathrm{Jac}(f^T,G) \cong\, & \big\langle e_{\mathrm{id}}, [x_2^2], [x_3], [x_3]^2, [x_3]^3, [x_2^2][x_3], [x_2^2][x_3]^2, [x_2^2][x_3]^3 \big\rangle_{\mathbb{C}} \\ & \oplus \big\langle e_g, [x_3]e_g, [x_3]^2e_g, [x_3]^3e_g \big\rangle_{\mathbb{C}}\,.\end{aligned}$$

Note that $\dim_{\mathbb{C}} \mathrm{Jac}(f^T,G) = 12$. The bilinear form $J_{f^T,G}$ on $\Omega_{f^T,G}$ can be calculated as

$$\begin{aligned} J_{f^T,\mathrm{id}}\big(e_{\mathrm{id}}\vdash\zeta, [x_2^2][x_3]^3\vdash\zeta\big) &= J_{f^T,\mathrm{id}}\big([dx_1\wedge dx_2\wedge dx_3], [x_2^2x_3^3dx_1\wedge dx_2\wedge dx_3]\big) \\ &= 2\cdot\frac{1}{40} = \frac{1}{20},\end{aligned}$$

and with $\mu_{f^{Tg}} = 4$, $\mathrm{hess}(f^{Tg}) = 5\cdot 4x_3^3$

$$\begin{aligned} J_{f^T,g}\big(e_g\vdash\zeta, [x_3]^3e_g\vdash\zeta\big) &= \frac{1}{4}J_{f^T,g}\big([dx_3], [x_3^3dx_3]\big) \\ &= \frac{1}{4}\cdot(-1)\cdot 2\cdot\frac{1}{5} = -\frac{1}{10},\end{aligned}$$

which imply the following relations in the orbifold Jacobian algebra $\mathrm{Jac}(f^T,G)$:

$$[x_3]^4 = 0, \quad [x_2^2]\circ e_g = 0, \quad e_g^2 = -2[x_2^2].$$

On the other hand, the Jacobian algebra $\mathrm{Jac}(\overline{f})$ is given by

$$\mathrm{Jac}(\overline{f}) = \mathbb{C}[y_1,y_2,y_3]\big/\big(5y_1^4, 2y_2+y_3^2, 2y_2y_3\big)\,.$$

Note that $\dim_{\mathbb{C}} \mathrm{Jac}(\overline{f}) = 12$. Therefore, we have an algebra isomorphism

$$\mathrm{Jac}(\overline{f}) \xrightarrow{\cong} \mathrm{Jac}(f^T,G), \quad y_1\mapsto [x_3],\ y_2\mapsto [x_2^2],\ y_3\mapsto e_g,$$

which is, moreover, an isomorphism of Frobenius algebras since we have $\mathrm{hess}(\overline{f}) = 20\cdot 2\cdot 2\cdot y_1^3y_2 - 20\cdot 2\cdot 2\cdot y_1^3y_3^2 = 80y_1^3y_2 + 80\cdot 2y_1^3y_2 = 20\cdot 12y_1^3y_2 \in \mathrm{Jac}(\overline{f})$ and so

$$J_{\overline{f}}\left([dy_1\wedge dy_2\wedge dy_3],[y_1^3y_2dy_1\wedge dy_2\wedge dy_3]\right) = \frac{1}{20}.$$

4. Set

$$f^T := x_1^2 + x_2^4x_3 + x_3^4,\quad G := \langle g\rangle,\ g := \frac{1}{2}(1,1,0),$$

$$\overline{f} := y_1^2y_2 + y_2^2y_3 + y_3^4.$$

The Jacobian algebra $\mathrm{Jac}(f^T)$ can be calculated as

$$\begin{aligned}\mathrm{Jac}(f^T) &= \mathbb{C}[x_1,x_2,x_3]\,/\left(2x_1, 4x_2^3x_3, x_2^4+4x_3^3\right)\\ &\cong \left\langle 1, x_2, x_2^2, x_2^3, x_3, x_3^2, x_3^3, x_2x_3, x_2x_3^2, x_2x_3^3, x_2^2x_3, x_2^2x_3^2, x_2^2x_3^3\right\rangle_{\mathbb{C}},\end{aligned}$$

so $\mu_{f^T} = 13$. With $\mathrm{hess}(f^T) = 2\cdot 12\cdot 12\cdot x_2^2x_3^3 - 2\cdot 4\cdot 4\cdot x_2^6 = 288x_2^2x_3^3 + 32\cdot 4x_2^2x_3^3 = 32(9+4)x_2^2x_3^3 \in \mathrm{Jac}(f^T)$ we can calculate the bilinear form J_{f^T} on Ω_{f^T}

$$J_{f^T}\left([dx_1\wedge dx_2\wedge dx_3],[x_2^2x_3^3dx_1\wedge dx_2\wedge dx_3]\right) = \frac{1}{32}.$$

As a $\mathbb{C}$-module, the orbifold Jacobian algebra $\mathrm{Jac}(f^T,G)$ is of the following form:

$$\begin{aligned}\mathrm{Jac}(f^T,G) \cong &\left\langle e_{\mathrm{id}}, [x_2^2], [x_3], [x_3]^2, [x_3]^3, [x_2^2][x_3], [x_2^2][x_3]^2, [x_2^2][x_3]^3\right\rangle_{\mathbb{C}}\\ &\oplus \left\langle e_g, [x_3]e_g, [x_3]^2e_g\right\rangle_{\mathbb{C}}.\end{aligned}$$

Note that $\dim_{\mathbb{C}} \mathrm{Jac}(f^T,G) = 11$. The bilinear form $J_{f^T,G}$ on $\Omega_{f^T,G}$ can be calculated as

$$\begin{aligned}J_{f^T,\mathrm{id}}\left(e_{\mathrm{id}}\vdash\zeta, [x_2^2][x_3]^3\vdash\zeta\right) &= J_{f^T,\mathrm{id}}\left([dx_1\wedge dx_2\wedge dx_3],[x_2^2x_3^3dx_1\wedge dx_2\wedge dx_3]\right)\\ &= 2\cdot\frac{1}{32} = \frac{1}{16},\end{aligned}$$

and with $\mu_{f^{Tg}} = 3$, $\mathrm{hess}(f^{Tg}) = 4\cdot 3x_3^2$

$$\begin{aligned}J_{f^T,g}\left(e_g\vdash\zeta, [x_3]^2e_g\vdash\zeta\right) &= \frac{1}{4}J_{f^T,g}\left([dx_3],[x_3^2dx_3]\right)\\ &= \frac{1}{4}\cdot(-1)\cdot 2\cdot\frac{1}{4} = -\frac{1}{8},\end{aligned}$$

which imply the following relations in the orbifold Jacobian algebra $\mathrm{Jac}(f^T,G)$:

$$[x_2^2]^2 = -4[x_3]^3,\quad [x_2^2]\circ e_g = 0,\quad e_g^2 = -2[x_2^2x_3].$$

On the other hand, the Jacobian algebra $\mathrm{Jac}(\overline{f})$ is given by

$$\mathrm{Jac}(\overline{f}) = \mathbb{C}[y_1,y_2,y_3]\,/\left(2y_1y_2, y_1^2+2y_2y_3, y_2^2+4y_3^3\right).$$

Note that $\dim_{\mathbb{C}} \mathrm{Jac}(\overline{f}) = 11$. Therefore, we have an algebra isomorphism

$$\mathrm{Jac}(\overline{f}) \xrightarrow{\cong} \mathrm{Jac}(f^T, G), \quad y_1 \mapsto e_g,\ y_2 \mapsto [x_2^2],\ y_3 \mapsto [x_3],$$

which is, moreover, an isomorphism of Frobenius algebras since we have
$\mathrm{hess}(\overline{f}) = 2 \cdot 2 \cdot 12 \cdot y_2 y_3^3 - 2 \cdot 2 \cdot 2 \cdot y_2^3 - 2 \cdot 2 \cdot 12 \cdot y_1^2 y_3^2 = 48 y_2 y_3^3 + 8 \cdot 4 y_2 y_3^3 + 48 \cdot 2 y_2 y_3^3$
$= 16(3+2+6) y_2 y_3^3 \in \mathrm{Jac}(\overline{f})$ and so

$$J_{\overline{f}}\left([dy_1 \wedge dy_2 \wedge dy_3], [y_2 y_3^3 dy_1 \wedge dy_2 \wedge dy_3]\right) = \frac{1}{16}.$$

5. Set

$$f^T := x_1^3 + x_2^2 + x_2 x_3^5, \quad G := \langle g \rangle\,,\ g := \frac{1}{2}(0,1,1),$$

$$\overline{f} := y_1^3 + y_2^2 y_3 + y_2 y_3^3.$$

The Jacobian algebra $\mathrm{Jac}(f^T)$ can be calculated as

$$\begin{aligned}\mathrm{Jac}(f^T) &= \mathbb{C}[x_1, x_2, x_3] \,/\, \left(3x_1^2, 2x_2 + x_3^5, 5x_2 x_3^4\right)\\ &\cong \left\langle 1, x_1, x_3, x_3^2, x_3^3, x_3^4, x_3^5, x_3^6, x_3^7, x_3^8, x_1 x_3, \ldots, x_1 x_3^8 \right\rangle_{\mathbb{C}},\end{aligned}$$

so $\mu_{f^T} = 18$. With $\mathrm{hess}(f^T) = 6 \cdot 2 \cdot 20 \cdot x_1 x_2 x_3^3 - 6 \cdot 5 \cdot 5 \cdot x_1 x_3^8 = -15 \cdot 16 \cdot \frac{1}{2} x_1 x_3^8 - 15 \cdot 10 x_1 x_3^8$
$= -15 \cdot 18 x_1 x_3^8 \in \mathrm{Jac}(f^T)$ we can calculate the bilinear form J_{f^T} on Ω_{f^T}

$$J_{f^T}\left([dx_1 \wedge dx_2 \wedge dx_3], [x_1 x_3^8 dx_1 \wedge dx_2 \wedge dx_3]\right) = \frac{-1}{15}.$$

As a $\mathbb{C}$-module, the orbifold Jacobian algebra $\mathrm{Jac}(f^T, G)$ is of the following form:

$$\begin{aligned}\mathrm{Jac}(f^T, G) \cong &\left\langle e_{\mathrm{id}}, [x_1], [x_3^2], [x_3^2]^2, [x_3^2]^3, [x_3^2]^4, [x_1][x_3^2], [x_1][x_3^2]^2, [x_1][x_3^2]^3, [x_1][x_3^2]^4 \right\rangle_{\mathbb{C}}\\ &\oplus \left\langle e_g, [x_1] e_g \right\rangle_{\mathbb{C}}.\end{aligned}$$

Note that $\dim_{\mathbb{C}} \mathrm{Jac}(f^T, G) = 12$. The bilinear form $J_{f^T,G}$ on $\Omega_{f^T,G}$ can be calculated as

$$\begin{aligned}J_{f^T,\mathrm{id}}\left(e_{\mathrm{id}} \vdash \zeta, [x_1][x_3^2]^4 \vdash \zeta\right) &= J_{f^T,\mathrm{id}}\left([dx_1 \wedge dx_2 \wedge dx_3], [x_1 x_3^8 dx_1 \wedge dx_2 \wedge dx_3]\right)\\ &= 2 \cdot \frac{-1}{15} = \frac{-2}{15},\end{aligned}$$

and with $\mu_{f^{Tg}} = 2$, $\mathrm{hess}(f^{Tg}) = 3 \cdot 2 x_1$

$$\begin{aligned}J_{f^T,g}\left(e_g \vdash \zeta, [x_1] e_g \vdash \zeta\right) &= \frac{1}{4} J_{f^T,g}\left([dx_1], [x_1 dx_1]\right)\\ &= \frac{1}{4} \cdot (-1) \cdot 2 \cdot \frac{1}{3} = -\frac{1}{6},\end{aligned}$$

which imply the following relations in the orbifold Jacobian algebra $\mathrm{Jac}(f^T, G)$:

$$[x_1]^2 = 0, \quad [x_3^2] \circ e_g = 0, \quad e_g^2 = \frac{5}{4}[x_3^8].$$

On the other hand, the Jacobian algebra $\mathrm{Jac}(\overline{f})$ is given by

$$\begin{aligned}\mathrm{Jac}(\overline{f}) &= \mathbb{C}[y_1, y_2, y_3] \big/ \big(3y_1^2, 2y_2y_3 + y_3^3, y_2^2 + 3y_2y_3^2\big) \\ &\cong \big\langle 1, y_1, y_2, y_3, y_3^2, y_3^3, y_3^4, y_1y_2, y_1y_3, y_1y_3^2, y_1y_3^3, y_1y_3^4 \big\rangle_{\mathbb{C}}.\end{aligned}$$

Note that $\dim_{\mathbb{C}} \mathrm{Jac}(\overline{f}) = 12$ and we have $y_2^2 = -3y_2y_3^2 = (-3)(\frac{-1}{2}y_3^4) = \frac{3}{2}y_3^4 \in \mathrm{Jac}(\overline{f})$. Therefore, we have an algebra isomorphism

$$\mathrm{Jac}(\overline{f}) \xrightarrow{\cong} \mathrm{Jac}(f^T, G), \quad [y_1] \mapsto [x_1],\ [y_2] \mapsto \frac{\sqrt{6}}{\sqrt{5}} e_g,\ [y_3] \mapsto [x_3^2],$$

which is, moreover, an isomorphism of Frobenius algebras since we have $\mathrm{hess}(\overline{f}) = 6y_1(2 \cdot 6y_2y_3^2 - (2y_2 + 3y_3^2)^2) = 6y_1(12y_2y_3^2 - 4y_2^2 - 12y_2y_3^2 - 9y_3^4) = 6y_1(-4 \cdot \frac{3}{2}y_3^4 - 9y_3^4) = -6(15)y_1y_3^4 \in \mathrm{Jac}(\overline{f})$ and so

$$J_{\overline{f}}\left([dy_1 \wedge dy_2 \wedge dy_3], [y_1y_3^4 dy_1 \wedge dy_2 \wedge dy_3]\right) = \frac{-2}{15}.$$

6. and 7. Set

$$f_1^T := x_1^3 + x_2^3 + x_3^4, \quad G_1 := \langle g \rangle,\ g := \frac{1}{3}(1, 2, 0),$$

$$f_2^T := z_1^4 + z_2^3 z_3 + z_3^2, \quad G_2 := \langle h \rangle,\ h := \frac{1}{2}(0, 1, 1),$$

$$\overline{f} := y_1^4 + y_2^2 y_3 + y_2 y_3^2.$$

The Jacobian algebra $\mathrm{Jac}(f_1^T)$ can be calculated as

$$\begin{aligned}\mathrm{Jac}(f_1^T) &\cong \mathbb{C}[x_1, x_2, x_3] \big/ \big(3x_1^2, 3x_2^2, 4x_3^3\big) \\ &\cong \big\langle 1, x_1, x_2, x_1x_2, x_3, x_1x_3, x_2x_3, x_1x_2x_3, x_3^2, x_1x_3^2, x_2x_3^3, x_1x_2x_3^2 \big\rangle_{\mathbb{C}},\end{aligned}$$

so $\mu_{f_1^T} = 12$. With $\mathrm{hess}(f_1^T) = 6 \cdot 6 \cdot 12 \cdot x_1x_2x_3^2$ we can calculate the bilinear form $J_{f_1^T}$ on $\Omega_{f_1^T}$

$$J_{f_1^T}\left([dx_1 \wedge dx_2 \wedge dx_3], [x_1x_2x_3^2 dx_1 \wedge dx_2 \wedge dx_3]\right) = \frac{1}{36}.$$

As a $\mathbb{C}$-module, the orbifold Jacobian algebra $\mathrm{Jac}(f_1^T, G_1)$ is of the following form:

$$\begin{aligned}\mathrm{Jac}(f_1^T, G_1) \cong &\big\langle e_{\mathrm{id}}, [x_1x_2], [x_3], [x_3]^2, [x_1x_2][x_3], [x_1x_2][x_3]^2 \big\rangle_{\mathbb{C}} \\ &\oplus \big\langle e_g, [x_3]e_g, [x_3]^2 e_g \big\rangle_{\mathbb{C}} \oplus \big\langle e_{g^{-1}}, [x_3]e_{g^{-1}}, [x_3]^2 e_{g^{-1}} \big\rangle_{\mathbb{C}}.\end{aligned}$$

Note that $\dim_{\mathbb{C}} \mathrm{Jac}(f_1^T, G_1) = 12$. The bilinear form $J_{f_1^T, G_1}$ on $\Omega_{f_1^T, G_1}$ can be calculated as

$$\begin{aligned} J_{f_1^T,\mathrm{id}}\left(e_{\mathrm{id}} \vdash \zeta, [x_1x_2][x_3]^2 \vdash \zeta\right) &= J_{f_1^T,\mathrm{id}}\left([dx_1 \wedge dx_2 \wedge dx_3], [x_1x_2x_3^2 dx_1 \wedge dx_2 \wedge dx_3]\right) \\ &= 3 \cdot \frac{1}{36} = \frac{1}{12}, \end{aligned}$$

and with $\mu_{f_1^{Tg}} = 3$, $\mathrm{hess}(f_1^{Tg}) = 4 \cdot 3x_3^2$

$$\begin{aligned} J_{f_1^T,g}\left(e_g \vdash \zeta, [x_3]^2 e_{g^{-1}} \vdash \zeta\right) &= \frac{1}{9} J_{f_1^T,g}\left([dx_3], [x_3^2 dx_3]\right) \\ &= \frac{1}{9} \cdot (-1) \cdot 3 \cdot \frac{1}{4} = -\frac{1}{12}, \end{aligned}$$

which imply the following relations in the orbifold Jacobian algebra $\mathrm{Jac}(f_1^T, G)$:

$$[x_3]^3 = 0, \quad e_g^2 = 0, \quad e_{g^{-1}}^2 = 0, \quad e_g \circ e_{g^{-1}} = -[x_1x_2].$$

Secondly the Jacobian algebra $\mathrm{Jac}(f_2^T)$ can be calculated as

$$\begin{aligned} \mathrm{Jac}(f_2^T) &\cong \mathbb{C}[z_1, z_2, z_3] \big/ \left(4z_1^3, 3z_2^2 z_3, z_2^3 + 2z_3\right) \\ &\cong \left\langle 1, z_1, z_1^2, z_2, z_2^2, z_3, z_2z_3, z_1z_2, z_1z_2^2, z_1z_3, z_1z_2z_3, z_1^2z_2, z_1^2z_2^2, z_1^2z_3, z_1^2z_2z_3, \right\rangle_{\mathbb{C}}, \end{aligned}$$

so $\mu_{f_2^T} = 15$. With $\mathrm{hess}(f_2^T) = 12 \cdot 6 \cdot 2 \cdot z_1^2 z_2 z_3 - 12 \cdot 3 \cdot 3 \cdot z_1^2 z_2^4 = 144 z_1^2 z_2 z_3 + 108 \cdot 2 z_1^2 z_2 z_3$ $= 24(6+9) z_1^2 z_2 z_3 \in \mathrm{Jac}(f_2^T)$ we can calculate the bilinear form $J_{f_2^T}$ on $\Omega_{f_2^T}$

$$J_{f_2^T}\left([dz_1 \wedge dz_2 \wedge dz_3], [z_1^2 z_2 z_3 dz_1 \wedge dz_2 \wedge dz_3]\right) = \frac{1}{24}.$$

As a $\mathbb{C}$-module, the orbifold Jacobian algebra $\mathrm{Jac}(f_2^T, G_2)$ is of the following form:

$$\begin{aligned} \mathrm{Jac}(f_2^T, G_2) \cong &\left\langle e_{\mathrm{id}}, [z_1], [z_1]^2, [z_2^2], [z_2z_3], [z_1][z_2^2], [z_1][z_2z_3], [z_1]^2[z_2^2], [z_1]^2[z_2z_3] \right\rangle_{\mathbb{C}} \\ &\oplus \left\langle e_h, [z_1]e_h, [z_1]^2 e_h \right\rangle_{\mathbb{C}}. \end{aligned}$$

Note that $\dim_{\mathbb{C}} \mathrm{Jac}(f_2^T, G_2) = 12$. The bilinear form $J_{f_2^T, G_2}$ on $\Omega_{f_2^T, G_2}$ can be calculated as

$$\begin{aligned} J_{f_2^T,\mathrm{id}}\left(e_{\mathrm{id}} \vdash \zeta, [z_1]^2[z_2z_3] \vdash \zeta\right) &= J_{f_2^T,\mathrm{id}}\left([dz_1 \wedge dz_2 \wedge dz_3], [z_1^2 z_2 z_3 dz_1 \wedge dz_2 \wedge dz_3]\right) \\ &= 2 \cdot \frac{1}{24} = \frac{1}{12}, \end{aligned}$$

and with $\mu_{f_2^{Th}} = 3$, $\mathrm{hess}(f_2^{Th}) = 4 \cdot 3z_1^2$

$$\begin{aligned} J_{f_2^T,h}\left(e_h \vdash \zeta, [z_1]^2 e_{h^{-1}} \vdash \zeta\right) &= \frac{1}{4} J_{f_2^T,h}\left([dz_1], [z_1^2 dz_1]\right) \\ &= \frac{1}{4} \cdot (-1) \cdot 2 \cdot \frac{1}{4} = -\frac{1}{8}, \end{aligned}$$

which imply the following relations in the orbifold Jacobian algebra $\mathrm{Jac}(f_2^T, G)$:

$$[z_1]^3 = 0, \quad [z_2^2] \circ e_h = 0, \quad [z_2^2]^2 = -2[z_2 z_3], \quad e_h^2 = \frac{-3}{2}[z_2 z_3].$$

Therefore, we have an algebra isomorphism

$$\mathrm{Jac}(f_1^T, G_1) \xrightarrow{\cong} \mathrm{Jac}(f_2^T, G_2),$$

$$[x_1 x_2] \mapsto [z_2 z_3],\ [x_3] \mapsto [z_1],\ e_g \mapsto \frac{\sqrt{-1}}{2}[z_2^2] + \frac{1}{\sqrt{3}} e_h,\ e_{g^{-1}} \mapsto \frac{-\sqrt{-1}}{2}[z_2^2] + \frac{1}{\sqrt{3}} e_h,$$

which is, moreover, an isomorphism of Frobenius algebras since we have

$$J_{f_1^T, G_1}\left(e_{\mathrm{id}} \vdash \zeta, [x_1 x_2 x_3^2] \vdash \zeta\right) = J_{f_2^T, G_2}\left(e_{\mathrm{id}} \vdash \zeta, [z_1^2 z_2 z_3] \vdash \zeta\right) = \frac{1}{12}.$$

On the other hand, the Jacobian algebra $\mathrm{Jac}(\overline{f})$ is given by

$$\mathrm{Jac}(\overline{f}) = \mathbb{C}[y_1, y_2, y_3] \big/ \left(4y_1^3, 2y_3 y_2 + y_3^2, y_2^2 + 2y_2 y_3\right).$$

Note that $\dim_{\mathbb{C}} \mathrm{Jac}(\overline{f}) = 12$. Therefore, we have an algebra isomorphism

$$\mathrm{Jac}(\overline{f}) \xrightarrow{\cong} \mathrm{Jac}(f_1^T, G_1),$$

$$y_1 \mapsto [x_3], \quad y_2 \mapsto \mathbf{e}[\frac{1}{3}] e_g + \mathbf{e}[\frac{2}{3}] e_{g^{-1}}, \quad y_3 \mapsto \mathbf{e}[\frac{2}{3}] e_g + \mathbf{e}[\frac{1}{3}] e_{g^{-1}},$$

which is, moreover, an isomorphism of Frobenius algebras since we have
$\mathrm{hess}(\overline{f}) = 12y_1(2 \cdot 2 \cdot y_2 y_3 - (2y_2 + 2y_3)^2) = 12y_1^2(4y_2 y_3 - 4y_2^2 - 8y_2 y_3 - 4y_3^2)$
$= 12y_1^2(-4y_2 y_3 + 4 \cdot 2y_2 y_3 + 4 \cdot 2y_2 y_3) = 12(12)y_1^2 y_2 y_3 \in \mathrm{Jac}(\overline{f})$ and so

$$J_{\overline{f}}\left([dy_1 \wedge dy_2 \wedge dy_3], [y_1^2 y_2 y_3 dy_1 \wedge dy_2 \wedge dy_3]\right) = \frac{1}{12}.$$

Then it is clear that there is also an algebra isomorphism

$$\mathrm{Jac}(\overline{f}) \xrightarrow{\cong} \mathrm{Jac}(f_2^T, G_2)$$

by the composition of the last two isomorphisms.

We finished the proof of Theorem 6.4.8. □

Corollary 6.4.9 (cf. also our note [BTW17]). *Let f_1 and f_2 be invertible polynomials defining exceptional unimodal singularities see Table 6.3. There is an isomorphism of Frobenius algebras*

$$\mathrm{Jac}(f_1^T, G_{f_1^T}^{\mathrm{SL}}) \cong \mathrm{Jac}(f_2)$$

if and only if the associated singularities of f_1 and f_2 are strangely dual to each other in the sense of Arnold.

Proof. It is clear that for two polynomials of the same singularity type the normal Jacobian algebras are isomorphic. So the statement is clear, if one can show it for one polynomial f_2 which is strangely dual to f_1. For all polynomials f_1 with $G_{f_1^T}^{\mathrm{SL}} = \{\mathrm{id}\}$ the statement is clear from Proposition 6.4.4. The rest follows from Theorem 6.4.8 since there $\overline{f}$ is always the strangely dual to f. □

7 Orbifold Jacobian Algebras for Cusp Polynomials

7.1 Cusp Polynomials

Let A be a triplet (a_1, a_2, a_3) of positive integers such that $a_1 \leq a_2 \leq a_3$. Set

$$\mu_A = a_1 + a_2 + a_3 - 1$$

and

$$\chi_A = \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} - 1.$$

Definition 7.1.1. A polynomial $f_A \in \mathbb{C}[x_1, x_2, x_3]$ given by

$$f_A = x_1^{a_1} + x_2^{a_2} + x_3^{a_3} - q^{-1}x_1x_2x_3$$

for some $q \in \mathbb{C}\backslash\{0\}$ is called a *cusp polynomial of type A*.

Definition 7.1.2. We have three cases for f_A:

(i) If $\chi_A > 0$ we call f_A an *affine cusp polynomial*.

(ii) If $\chi_A = 0$ we have the following three cases:
   a) $f_A = x_1^2 + x_2^3 + x_3^6 - q^{-1}x_1x_2x_3$
   b) $f_A = x_1^2 + x_2^4 + x_3^4 - q^{-1}x_1x_2x_3$
   c) $f_A = x_1^3 + x_2^3 + x_3^3 - q^{-1}x_1x_2x_3$

(iii) If $\chi_A < 0$ f_A defines a *cusp singularity*.

Remark 7.1.3. In case (i) the polynomial has many singularities and the Milnor fibre at $\mathbf{0}$ is not the right one to consider. So we will only concentrate on cusp polynomials with $\chi_A \leq 0$. These are the parabolic (case (ii)) and hyperbolic (case (iii)) unimodal singularities (cf. [AGV85, p.146])

Lemma 7.1.4. *In case* (iii) *for all $q \in \mathbb{C}\backslash\{0\}$ the polynomial f_A has an isolated singularity at* $\mathbf{0}$. *In case* (ii) *we exclude $q^{-6} = 432$ in* (iia), *$q^{-4} = 64$ in* (iib), *$q^{-3} = 27$ in* (iic) *respectively. For all other $q \in \mathbb{C}\backslash\{0\}$ the polynomial f_A has an isolated singularity at* $\mathbf{0}$.

Proof. This is an easy computation. We see that $\mathrm{Jac}(f_A)$ has always a finite dimension over $\mathbb{C}$. See also Definition 7.1.8. $\square$

Definition 7.1.5 (cf. [ST15])**.** We can consider the universal unfolding of f_A (cf. Proposition 2.2.14). A holomorphic function $F_A(\mathbf{x};\mathbf{s},s_{\mu_A})$ defined on a neighborhood of $(\mathbf{0};\mathbf{0},q)$ of $\mathbb{C}^3 \times (\mathbb{C}^{\mu_A-1} \times \mathbb{C}\backslash\{0\})$ is given as follows:

$$F_A(\mathbf{x};\mathbf{s},s_{\mu_A}) = x_1^{a_1} + x_2^{a_2} + x_3^{a_3} - s_{\mu_A}^{-1} x_1x_2x_3 + s_1 \cdot 1 + \sum_{i=1}^{3}\sum_{j=1}^{a_i-1} s_{i,j}x_i^j.$$

Of course we have

$$F_A(\mathbf{x},\mathbf{0},q) = f_A(\mathbf{x}).$$

Remark 7.1.6. In [ST15] and [IST12] it was shown that for a cusp polynomial a good primitive form (cf. [Sa82], [Sa83], [ST08]) is given by

$$\zeta_A = [s_{\mu_A}^{-1} dx_1 \wedge dx_2 \wedge dx_3].$$

In [IST12] it is even done for $\chi_A > 0$. There is defined an algebra as an $\mathcal{O}_M$ module for $M = (\mathbb{C}^{\mu_A-1} \times \mathbb{C}\backslash\{0\})$, or for $\bar{M} = (\mathbb{C}^{\mu_A-1} \times \mathbb{C})$.

We will still use our normal $\mathbb{C}$-module $\mathrm{Jac}(f_A)$. But we will always use this primitive form $\zeta = [q^{-1}dx_1 \wedge dx_2 \wedge dx_3]$ for the isomorphism (2.1).

Definition 7.1.7. We can calculate the hessian of f_A:

$$\begin{aligned} \mathrm{hess}(f_A) = {} & a_1(a_1-1)a_2(a_2-1)a_3(a_3-1)x_1^{a_1-2}x_2^{a_2-2}x_3^{a_3-2} \\ & -(2+a_1-1+a_2-1+a_3-1)q^{-3}x_1x_2x_3 \end{aligned} \quad \in \mathrm{Jac}(f_A)$$

We define $\kappa = 1$ for $\chi_A < 0$ and $\kappa = 1 - 432q^6$ for (iia), $\kappa = 1 - 64q^4$ for (iib), $\kappa = 1 - 27q^3$ for (iic) respectively for $\chi_A = 0$. So we get

$$\mathrm{hess}(f_A) = -\kappa\mu_{f_A}q^{-3}x_1x_2x_3 \in \mathrm{Jac}(f_A).$$

Definition 7.1.8. The Jacobiam algebra $\mathrm{Jac}(f_A)$ has the monomial basis

- 1
- $x_1, x_1^2, \ldots, x_1^{a_1-1}$
- $x_2, x_2^2, \ldots, x_2^{a_2-1}$
- $x_3, x_3^2, \ldots, x_3^{a_3-1}$
- $\kappa q^{-1}x_1x_2x_3$.

So we have $\mu_{f_A} = (2 + a_1 - 1 + a_2 - 1 + a_3 - 1) = \mu_A$.

Remark 7.1.9. We see

$$J_{f_A}(\zeta, \kappa q^{-1}x_1x_2x_3\zeta) = J_{f_A}(q^{-1}dx_1 \wedge dx_2 \wedge dx_3, \kappa q^{-1}x_1x_2x_3q^{-1}dx_1 \wedge dx_2 \wedge dx_3) = -1.$$

That is the reason for this monomial basis.

Remark 7.1.10. If $f_A(x_1, x_2, x_3)$ is a cusp polynomial, then we have

$$G_f = G_f^{\mathrm{SL}},$$

and hence $\mathrm{age}(g)$ is an integer for all $g \in G_f$.

Let now be G again a subgroup of G_f.

Definition 7.1.11. For $i = 1, 2, 3$ let K_i be the maximal subgroup of G fixing the i-th coordinate x_i, whose order $|K_i|$ is denoted by n_i.

Proposition 7.1.12 ([ET14, Cor. 2])**.** *We have*

$$|G| = 1 + 2j_G + \sum_{i=1}^{3}(n_i - 1),$$

where j_G is the number of elements $g \in G$ such that $\mathrm{age}(g) = 1$ *and* $n_g = 0$.

Remark 7.1.13. From this we also see directly that each group K_i, $i = 1, 2, 3$ can only have the form $K_i = \mathbb{Z}/n_i\mathbb{Z}$ and we can choose generators for these cyclic groups. All elements $g \in G$, that are not in one K_i, $i = 1, 2, 3$ have $n_g = 0$. And we directly have always pairs g, g^{-1}. When we have $\mathrm{age}(g) = 1$ we get $\mathrm{age}(g^{-1}) = 2$.

Remark 7.1.14. From Remark 7.1.6 we see that q plays an important role for cusp polynomials. We have to consider this as an additional variable. So we will define $\mathrm{Aut}(f, G)$ in a little different way.

Definition 7.1.15. For a cusp polynomial $f_A = x_1^{a_1} + x_2^{a_2} + x_3^{a_3} - q^{-1}x_1x_2x_3$ and a group $G \subset G_{f_A}$ we define

$$\mathrm{Aut}(f_A, G) := \{\varphi \in \mathrm{GL}(3+1, \mathbb{C}) \mid F_A(\varphi(\mathbf{x}; \mathbf{0}, q)) = F_A(\mathbf{x}; \mathbf{0}, q),\ \varphi^{-1}g\varphi \in G \text{ for all } g \in G\}.$$

Here we see $G \subset G_{f_A}$ as a subgroup of $\mathrm{GL}(3+1, \mathbb{C})$ which leaves q invariant. Then it is again obvious that G is a subgroup of $\mathrm{Aut}(f_A, G)$.

7.2 Theorem for Cusp Polynomials

We cannot give the uniqueness in total for all cusp polynomials, for the following pair we cannot give the uniqueness:

Definition 7.2.1. Let $f = x_1^{a_1} + x_2^{a_2} + x_3^{a_3} - q^{-1}x_1x_2x_3$ be a cusp polynomial and G be a group of diagonal symmetries of f, such that there exists a $\mathrm{id} \neq g \in G$ and $i \in \{1, 2, 3\}$ with $x_i \in \mathrm{Fix}(g)$ and $a_i = 3$. Such a pair (f, G) is called of *bad type*.

Theorem 7.2.2. *Let $f = x_1^{a_1} + x_2^{a_2} + x_3^{a_3} - q^{-1}x_1x_2x_3$ be a cusp polynomial and G a subgroup of G_f. There exists a G-twisted Jacobian algebra $\mathrm{Jac}'(f,G)$ of f. Furthermore when (f,G) is not of bad type it is a unique G-twisted Jacobian algebra $\mathrm{Jac}'(f,G)$ of f up to isomorphism. Namely, it is uniquely characterized by the axioms in Definition 5.2.1.*

In particular, the orbifold Jacobian algebra $\mathrm{Jac}(f,G)$ of (f,G) exists.

We will first define some elements and then show the uniqueness and the existence as stated in Section 5.4.

Definition 7.2.3. We choose a generator g_1 of K_1.

Let $\varphi_{ij} \in \mathrm{GL}(3,\mathbb{C})$ be the automorphism which interchanges the i-th and j-th coordinate.

If $(\varphi_{1j}^{-1} g_1 \varphi_{1j})$ is a generator for K_j, $j = 2,3$, we choose $g_j = (\varphi_{1j}^{-1} g_1 \varphi_{1j})$. Otherwise we choose other generators g_j for K_j, $j = 2,3$.

Definition 7.2.4. Let $\varphi_i \in \mathrm{Aut}(f,G)$ be the element, which sends x_i to $\mathbf{e}[\frac{1}{a_i}]x_i$ and q to $\mathbf{e}[\frac{1}{a_i}]q$ and preserves the other coordinates.

Uniqueness

Throughout this subsection, $f = x_1^{a_1} + x_2^{a_2} + x_3^{a_3} - q^{-1}x_1x_2x_3$ denotes a cusp polynomial. And we show, as mentioned in Section 5.4, the uniqueness of $\mathrm{Jac}'(f,G)$ for any $G \subset G_f$, such that (f,G) is not of bad type.

Take the nowhere vanishing 3-form $q^{-1}dx_1 \wedge dx_2 \wedge dx_3$ and set $\zeta := [q^{-1}dx_1 \wedge dx_2 \wedge dx_3] \in \Omega_f$.

Definition 7.2.5. Fix also a map

$$\alpha : G_f \longrightarrow \mathbb{C}^*, \quad g \mapsto \alpha_g,$$

such that $\alpha_{\mathrm{id}} = 1$ and

$$\alpha_g \alpha_{g^{-1}} = 1, \quad g \in G_f.$$

Such a map α always exists since for each g we may choose $\alpha_g = 1$. For each $g \in G$, let v_g be as in Definition 5.4.5

$$v_g \vdash \zeta = \alpha_g \omega_g.$$

Proposition 7.2.6. *For $g,h \in G$ with $g,h,gh \neq \mathrm{id}$ and $Fix(g) = \{0\}$ or $Fix(h) = \{0\}$, we have $v_g \circ v_h = 0 \in \mathrm{Jac}'(f,G)$.*

Proof. W.l.o.g. $Fix(g) = \{0\}$. Denote by $[\gamma'_{g,h}(\mathbf{x})]$ the element of $\mathrm{Jac}(f^{gh})$ satisfying $v_g \circ v_h = [\gamma'_{g,h}(\mathbf{x})] v_{gh}$. We have four cases:

(i) $Fix(h) = \{0\}$, $\mathrm{Fix}(gh) = \{0\}$

(ii) $Fix(h) = \{0\}$, $gh \in K_i$ for one $i \in \{1,2,3\}$

(iii) $h \in K_i$ for one $i \in \{1,2,3\}$, $Fix(gh) = 0$

(iv) $h \in K_i$ for one $i \in \{1,2,3\}$, $gh \in K_j$ for one $i \neq j \in \{1,2,3\}$

We prove it in every case:

(i) Here we have $v_g, v_h, v_{gh} \in \mathrm{Jac}'(f,G)_{\bar{1}}$ so we get zero by the $\mathbb{Z}/2\mathbb{Z}$-grading.

(ii) We have $\varphi_i^*(v_g) = v_g$ and $\varphi_i^*(v_h) = v_h$, but $\varphi_i^*(v_{gh}) = \mathbf{e}[\frac{1}{a_i}]v_{gh}$. Axiom (iva) yields $\varphi^*([\gamma'_{g,h}(\mathbf{x})]) = \mathbf{e}[\frac{-1}{a_i}][\gamma'_{g,h}(\mathbf{x})]$, so $[\gamma'_{g,h}(\mathbf{x})]$ has to be a constant multiple of $x_i^{a_i-1}$ or of q^{-1}. We have $x_i^{a_i-1} = 0$ in $\mathrm{Jac}(f^{gh})$. And for q^{-1} we get a contradiction by taking φ_j, $j \neq i$, which leaves all v_g, v_h, v_{gh} invariant.

(iii) When we take $g \in G \subset Aut(f,G)$, we have $g^*(v_g) = v_g$, $g^*(v_{gh}) = v_{gh}$, but $g^*(v_h) = \beta v_h$ for $\beta \neq 1 \in \mathbb{C}$, since ζ is G-invariant. Axiom (iva) yields $g^*([\gamma'_{g,h}(\mathbf{x})]) = \beta[\gamma'_{g,h}(\mathbf{x})]$, so $[\gamma'_{g,h}(\mathbf{x})] = 0$ since $\mathrm{Jac}(f^{gh}) \cong \mathbb{C}$.

(iv) Here we have $v_g \in \mathrm{Jac}'(f,G)_{\bar{1}}$ and $v_h, v_{gh} \in \mathrm{Jac}'(f,G)_{\bar{0}}$ so we get zero by the $\mathbb{Z}/2\mathbb{Z}$-grading.

□

Proposition 7.2.7. *For $g, h \in G$ with $g \in K_i$ and $h \in K_j$, $i \neq j$, we have $v_g \circ v_h = 0 \in \mathrm{Jac}'(f,G)$.*

Proof. Denote by $[\gamma'_{g,h}(\mathbf{x})]$ the element of $\mathrm{Jac}(f^{gh})$ satisfying $v_g \circ v_h = [\gamma'_{g,h}(\mathbf{x})]v_{gh}$. We have again two cases:

(i) $\mathrm{Fix}(gh) = \{0\}$

(ii) $gh \in K_k$ for one $k \in \{1,2,3\}\backslash\{i,j\}$

We prove it in every case:

(i) Here we have $v_g \in \mathrm{Jac}'(f,G)_{\bar{0}}$ and $v_h, v_{gh} \in \mathrm{Jac}'(f,G)_{\bar{1}}$ so we get zero by the $\mathbb{Z}/2\mathbb{Z}$-grading.

(ii) As in the second case of Proposition 7.2.6, we get with φ_k that $[\gamma'_{g,h}(\mathbf{x})]$ has to be a constant multiple of $x_k^{a_k-1}$ or of q^{-1}, which is zero or gives a contradiction by taking φ_i or φ_j.

□

So we only have to consider $g, h \in K_i$ for each $i \in \{1,2,3\}$ and g, g^{-1} with $\mathrm{Fix}(g) = \{0\}$.

Proposition 7.2.8. *For $g \in G$ with $\mathrm{Fix}(g) = \{0\}$ we have*

$$v_g \circ v_{g^{-1}} = (-1)^{\mathrm{age}(g)} \kappa q^{-1} x_1 x_2 x_3 v_{\mathrm{id}}$$

with the κ of Definition 7.1.7.

Proof. Since $\alpha_g \alpha_{g^{-1}} = 1$, we have

$$\begin{aligned} J_{f,\mathrm{id}}\left(\zeta, v_g \circ v_{g^{-1}} \vdash \zeta\right) &= J_{f,g}\left(v_g \vdash \zeta, v_{g^{-1}} \vdash \zeta\right) \\ &= (-1)^3 \cdot \mathbf{e}\left[-\frac{1}{2}\mathrm{age}(g)\right] \cdot |G| \\ &= -(-1)^{-\mathrm{age}(g)}|G| \end{aligned}$$

and on the other hand

$$\begin{aligned} &J_{f,\mathrm{id}}\left(\zeta, \kappa q^{-1} x_1 x_2 x_3 v_{\mathrm{id}} \vdash \zeta\right) \\ &\quad = |G| J_f\left(q^{-1} dx_1 \wedge dx_2 \wedge dx_3, \kappa q^{-1} x_1 x_2 x_3 q^{-1} dx_1 \wedge dx_2 \wedge dx_3\right) \\ &\quad = -|G|. \end{aligned}$$

□

For each $i = 1, 2, 3$, we take the generators g_i of the group $K_i \cong \mathbb{Z}/n_i\mathbb{Z}$ (cf. Definition 7.2.3). We define the elements $w_{g_i^l}$ of $\mathrm{Jac}'(f, g_i^l)$ as $w_{g_i^l} = v_{g_i^l}$ for each $l \in \mathbb{Z}$ with $l \notin n_i\mathbb{Z}$ and we set the element $w_{g_i^l} = x_i v_{\mathrm{id}} \in \mathrm{Jac}'(f, \mathrm{id})$ for each $l \in n_i\mathbb{Z}$.

Lemma 7.2.9. *For $i \neq j \in \{1, 2, 3\}$ and all $l_i, l_j \in \mathbb{Z}$, we have the following equality in* $\mathrm{Jac}'(f, g_i^{l_i} g_j^{l_j})$

$$w_{g_i^{l_i}} \circ w_{g_j^{l_j}} = \begin{cases} a_k q x_k^{a_k - 1} v_{\mathrm{id}} & l_i \in n_i\mathbb{Z} \text{ and } l_j \in n_j\mathbb{Z},\ k \in \{1,2,3\}\backslash\{i,j\} \\ 0 & \text{otherwise} \end{cases}.$$

Proof. If $l_i \in n_i\mathbb{Z}, l_j \in n_j\mathbb{Z}$, then $w_{g_i^0} \circ w_{g_j^0} = x_i v_{\mathrm{id}} \circ x_j v_{\mathrm{id}} = a_k q x_k^{a_k-1} v_{\mathrm{id}}$ in $\mathrm{Jac}'(f, \mathrm{id}) \cong \mathrm{Jac}(f)$. If $l_i \in n_i\mathbb{Z}$ and $l_j \notin n_i\mathbb{Z}$, then $w_{g_i^0} \circ w_{g_j^{l_j}} = x_i v_{\mathrm{id}} \circ v_{g_j^{l_j}} = x_i v_{g_j^{l_j}} = 0$ in $\mathrm{Jac}'(f, g_j^{l_j})$ and vice versa. If $l_i \notin n_i\mathbb{Z}$ and $l_j \notin n_j\mathbb{Z}$, we are in the case of Proposition 7.2.7. □

Proposition 7.2.10. *For each pair $l, m \in \mathbb{Z}$ there exists $c_{l,m} \in \mathbb{C}$ such that*

$$w_{g_i^l} \circ w_{g_i^m} = c_{l,m} x_i w_{g_i^{l+m}} \in \mathrm{Jac}'(f, g_i^{l+m}).$$

Remark 7.2.11. The $c_{l,m} \in \mathbb{C}$ can depend on q in the case $\chi_A = 0$ as we will see in Lemma 7.2.14.

Proof. Denote by $[\gamma'_{l,m}(\mathbf{x})]$ the element of $\mathrm{Jac}(f^{g_i^{l+m}})$ satisfying $w_{g_i^l} \circ w_{g_i^m} = [\gamma'_{l,m}(\mathbf{x})] w_{gh}$. With the $\mathrm{Aut}(f, G)$-element φ_i we get $\varphi_i(w_{g_i^l}) = \mathbf{e}[\frac{1}{a_i}] w_{g_i^l}$ and so since the multiplication is $\mathrm{Aut}(f, G)$-invariant, we get $\varphi_i^*([\gamma'_{l,m}(\mathbf{x})]) = \mathbf{e}[\frac{1}{a_i}][\gamma'_{l,m}(\mathbf{x})]$ so it has to be a multiple of x_i or of q. For $j \neq i$ we have $\varphi_j^*(w_{g_i^l}) = w_{g_i^l}$, so a constant multiple of x_i is the only possibility. □

Remark 7.2.12. In the proof of Proposition 7.2.10 we assumed that $w_{g_i^l} \circ w_{g_i^m}$ is always a multiple of $w_{g_i^{l+m}}$ which is a priori not clear for $l + m \in n_i\mathbb{Z}$. But even there we can only have that $w_{g_i^l} \circ w_{g_i^m}$ is a constant multiple of $q^2 x_j^{a_j-2} x_k^{a_k-2}$ this is not zero only for $a_j = a_k = 3$ but then this is not possible, since we would have $0 = J_{f,g}(x_i w_{g_i^l}, w_{g_i^m}) = J_{f,\mathrm{id}}(v_{\mathrm{id}}, x_i w_{g_i^l} \circ w_{g_i^m}) = J_{f,\mathrm{id}}(v_{\mathrm{id}}, c q^2 x_i x_j x_k) \neq 0$. So we get a contradiction.

We give some properties of $c_{l,m}$.

It is obvious that $c_{l,m+n_i} = c_{l,m} = c_{l+n_i,m}$. If $l \in n_i\mathbb{Z}$ or $m \in n_i\mathbb{Z}$ it is clear that $c_{0,m} = c_{l,0} = 1$ because of Axiom (iiia) and the definition of $w_{g_i^0}$. Nevertheless for $a_i = 2$ the multiplication need not to be nonzero.

Lemma 7.2.13. *We have $c_{l,m} = c_{m,l}$.*

Proof. Since $n - n_g \equiv 0 \mod 2$ for g_i^l for all $l \in \mathbb{Z}$, this multiplication is in $\mathrm{Jac}'(f,G)_{\overline{0}}$, the commutative subalgebra. □

Lemma 7.2.14. *We have $c_{l,-l} = \kappa$ from Definition 7.1.7 for $l \in \mathbb{Z}$, $l \notin n_i\mathbb{Z}$. So for $\chi_A < 0$ we even have $c_{l,-l} = 1$ for all $l \in \mathbb{Z}$.*

Proof. For $l \in n_i\mathbb{Z}$ it is clear that $c_{l,-l} = 1$. For $l \notin n_i\mathbb{Z}$ we have

$$\begin{aligned} J_{f,g_i^l}(x_i^{a_i-2}v_{g_i^l} \vdash \zeta, v_{g_i^{-l}} \vdash \zeta) &= \alpha_{g_i^l}\alpha_{g_i^{-l}} J_{f,g}(x_i^{a_i-2}\omega_g, \omega_{g^{-1}}) \\ &= 1 \cdot (-1)^{3-1}\mathbf{e}\left[-\frac{1}{2}\mathrm{age}(g_i^l)\right] \cdot |G| \cdot \frac{1}{a_i} \\ &= -\frac{|G|}{a_i}. \end{aligned}$$

On the other hand, by Axiom (v), we have

$$\begin{aligned} J_{f,g}(x_i^{a_i-2}v_{g_i^l} \vdash \zeta, v_{g_i^{-l}} \vdash \zeta) &= J_{f,id}(\omega_{\mathrm{id}}, x_i^{a_i-2}v_{g_i^l} \circ v_{g_i^{-l}} \vdash \zeta) \\ &= J_{f,id}(\omega_{\mathrm{id}}, c_{l,-l}x_i^{a_i-1}v_{g_i^0} \vdash \zeta) \\ &= c_{l,-l}J_{f,id}(\omega_{\mathrm{id}}, x_i^{a_i}\omega_{\mathrm{id}}) \\ &= c_{l,-l}J_{f,id}(\omega_{\mathrm{id}}, \frac{1}{a_i}q^{-1}x_1x_2x_3\omega_{\mathrm{id}}) \\ &= c_{l,-l}\frac{1}{a_i} \cdot |G| J_f(q^{-1}dx_1 \wedge dx_2 \wedge dx_3, q^{-1}x_1x_2x_3q^{-1}dx_1 \wedge dx_2 \wedge dx_3) \\ &= c_{l,-l}\frac{1}{a_i} \cdot |G|(\frac{-1}{\kappa}). \end{aligned}$$

□

Remark 7.2.15. Note that if $a_i = 2$ then $w_{g_i^l} \circ w_{g_i^m} \neq 0$ if and only if $l + m \in n_i\mathbb{Z}$ and the product structure is uniquely determined by these lemmata.

Also for the three polynomials with $\chi_A = 0$ and $a_i \neq 2$ we can only have $K_i \cong \mathbb{Z}/2\mathbb{Z}$ when K_i is not trivial. For the polynomial with $a_i = 3$ for all $i = 1, 2, 3$ we would otherwise have a pair of bad type (cf. Definition 7.2.1). So for all three polynomials with $\chi_A = 0$ the product structure is also uniquely determined by these lemmata.

So from now on we can assume $\chi_A < 0$ and $a_i \geq 4$, since for $a_i = 3$ we would have a pair of bad type.

Lemma 7.2.16. *Assume that $a_i \geq 4$. We have $c_{l,m}c_{l+m,n} = c_{l,m+n}c_{m,n}$ for all $l, m, n \in \mathbb{Z}$.*

Proof. We have

$$(v_{g_i^l} \circ v_{g_i^m}) \circ v_{g_i^n} = \left(c_{l,m} x_i v_{g_i^{l+m}}\right) \circ v_{g_i^n} = c_{l,m} c_{l+m,n} x_i^2 v_{g_i^{l+m+n}},$$

$$v_{g_i^l} \circ (v_{g_i^m} \circ v_{g_i^n}) = v_{g_i^l} \circ \left(c_{m,n} x_i v_{g_i^{m+n}}\right) = c_{l,m+n} c_{m,n} x_i^2 v_{g_i^{l+m+n}}.$$

The associativity of $\circ$ yields the statement. $\square$

Lemma 7.2.17. *Assume that $a_i \geq 4$. For all $l, m \in \mathbb{Z}$, we have $c_{l,m} c_{-l,-m} = 1$, in particular, $c_{l,m} \neq 0$ for all $l, m \in \mathbb{Z}$.*

Proof. We have

$$\begin{aligned} v_{g_i^l} \circ v_{g_i^m} \circ v_{g_i^{-l}} \circ v_{g_i^{-m}} &= \left(c_{l,m} x_i v_{g_i^{l+m}}\right) \circ \left(c_{-l,-m} x_i v_{g_i^{-l-m}}\right) \\ &= c_{l,m} c_{-l,-m} c_{l+m,-l-m} x_i^3 v_{g_i^0} = c_{l,m} c_{-l,-m} x_i^4 v_{\mathrm{id}}, \\ v_{g_i^l} \circ v_{g_i^{-l}} \circ v_{g_i^m} \circ v_{g_i^{-m}} &= \left(c_{l,-l} x_i v_{g_i^0}\right) \circ \left(c_{m,-m} x_i v_{g_i^0}\right) \\ &= c_{l,-l} c_{m,-m} x_i^2 x_i^2 v_{\mathrm{id}} = x_i^4 v_{\mathrm{id}}. \end{aligned}$$

The statement follows from the associativity and the commutativity of the product $\circ$ with Lemma 7.2.14. $\square$

Lemma 7.2.18. *Assume that $a_i \geq 4$. For all $l, m \in \mathbb{Z}_{\geq 1}$, we have*

$$c_{l,m} = \frac{\prod_{d=0}^{l+m-1} c_{1,d}}{(\prod_{a=0}^{l-1} c_{1,a})(\prod_{b=0}^{m-1} c_{1,b})}.$$

Proof. By Lemma 7.2.16 and Lemma 7.2.17, we have

$$c_{l,m} = c_{l+1,m-1} \frac{c_{1,l}}{c_{1,m-1}},$$

and hence

$$c_{l,m} = c_{l+m-1,1} \frac{c_{1,l+m-2}}{c_{1,1}} \cdots \frac{c_{1,l+1}}{c_{1,m-2}} \frac{c_{1,l}}{c_{1,m-1}} = \frac{\prod_{d=0}^{l+m-1} c_{1,d}}{(\prod_{a=0}^{l-1} c_{1,a})(\prod_{b=0}^{m-1} c_{1,b})}.$$

$\square$

For each $l \in \mathbb{Z}_{\geq 1}$, set

$$\widetilde{c}_l := \left(\prod_{a=0}^{n_i-1} c_{1,a}\right)^{-\frac{l}{n_i}} \left(\prod_{a=0}^{l-1} c_{1,a}\right).$$

Lemma 7.2.19. *Assume that $a_i \geq 4$. For all $l \in \mathbb{Z}_{\geq 1}$, we have*

$$\widetilde{c}_{l+n_i} = \widetilde{c}_l.$$

Proof. Note that $\prod_{a=l}^{l+n_i-1} c_{1,a} = \prod_{a=0}^{n_i-1} c_{1,a}$ since $c_{1,b} = c_{1,b+n_i}$ for all $b \in \mathbb{Z}$. Then the statement follows from the following equation.

$$\widetilde{c}_{l+n_i} = \left(\prod_{a=0}^{n_i-1} c_{1,a}\right)^{-\frac{l+n_i}{n_i}} \left(\prod_{a=0}^{l+n_i-1} c_{1,a}\right) = \widetilde{c}_l \left(\prod_{a=0}^{n_i-1} c_{1,a}\right)^{-1} \left(\prod_{a=l}^{l+n_i-1} c_{1,a}\right) = \widetilde{c}_l.$$

□

By this Lemma, for all $l \in \mathbb{Z}$ we can define $\widetilde{c}_l$ as $\widetilde{c}_{l+n_iN}$ by choosing a positive integer N such that $l + n_iN \geq 0$, which is independent of the choice of such an N.

Lemma 7.2.20. *We have:*

$$\widetilde{c}_{n_i} = 1$$

$$\widetilde{c}_l\widetilde{c}_{n_i-l} = 1$$

Proof. $\widetilde{c}_{n_i} = \left(\prod_{a=0}^{n_i-1} c_{1,a}\right)^{-\frac{n_i}{n_i}} \left(\prod_{a=0}^{n_i-1} c_{1,a}\right) = 1$ and
$\widetilde{c}_l\widetilde{c}_{n_i-l} = \left(\prod_{a=0}^{n_i-1} c_{1,a}\right)^{-\frac{l}{n_i}} \left(\prod_{a=0}^{l-1} c_{1,a}\right) \left(\prod_{a=0}^{n_i-1} c_{1,a}\right)^{-\frac{n_i-l}{n_i}} \left(\prod_{a=0}^{n_i-l-1} c_{1,a}\right)$
$= \left(\prod_{a=0}^{l-1} c_{1,a}\right) \left(\prod_{a=0}^{n_i-l-1} c_{1,a}\right) \left(\prod_{a=0}^{n_i-1} c_{1,a}\right)^{-1} = c_{l,n_i-l}^{-1} = 1.$ □

For each $l \in \mathbb{Z}$, set $\widetilde{w}_{g_i^l} := \widetilde{c}_l w_{g_i^l}$.

Lemma 7.2.21. *Assume that $a_i \geq 4$. In $\mathrm{Jac}'(f, g_i^{l+m})$, we have the following equality*

$$\widetilde{w}_{g_i^l} \circ \widetilde{w}_{g_i^m} = x_i\widetilde{w}_{g_i^{l+m}}.$$

Proof. It follows directly from Lemma 7.2.18. □

This lemma states that by replacing the map $\alpha : G_f \longrightarrow \mathbb{C}^*$ by a suitable one we have a new basis $\{\widetilde{v}_g\}_{g\in G_f}$ instead of $\{v_g\}_{g\in G_f}$. To summarize, we finally obtain the following

Corollary 7.2.22. *Let $g, h \in G$ and (f, G) not of bad type. We have*

$$\widetilde{v}_g \circ \widetilde{v}_h = \begin{cases} \widetilde{v}_g & \textit{if } h = \mathrm{id} \\ \widetilde{v}_h & \textit{if } g = \mathrm{id} \\ x_i\widetilde{w}_{gh} & \textit{if } \mathrm{Fix}(g) = \mathrm{Fix}(h) = \{x_i\} = \mathrm{Fix}(gh) \\ \kappa x_i\widetilde{w}_{gh} & \textit{if } \mathrm{Fix}(g) = \mathrm{Fix}(h) = \{x_i\},\ gh = \mathrm{id} \\ (-1)^{\mathrm{age}(g)}\kappa q^{-1}x_1x_2x_3 & \textit{if } \mathrm{Fix}(g) = \mathrm{Fix}(h) = \{0\},\ gh = \mathrm{id} \\ 0 & \textit{otherwise} \end{cases}$$

with the κ from Definition 7.1.7.

In particular, for any subgroup G of G_f and (f, G) not of bad type, if a G-twisted Jacobian algebra of f exists, then it is uniquely determined by the axioms in Definition 5.2.1 up to isomorphism.

Existence

Throughout this subsection, $f = x_1^{a_1} + x_2^{a_2} + x_3^{a_3} - q^{-1}x_1x_2x_3$ denotes a cusp polynomial. And we show, as mentioned in Section 5.4, the existence of $\mathrm{Jac}'(f, G)$ for any $G \subset G_f$. Let $\mathcal{A}'$ be as in Definition 5.4.7.

Definition 7.2.23. For each $g, h \in G_f$, define an element of $\mathcal{A}'_{gh}$ by

$$\overline{v}_g \circ \overline{v}_h = \begin{cases} \overline{v}_g & \text{if } h = \mathrm{id} \\ \overline{v}_h & \text{if } g = \mathrm{id} \\ x_i \overline{v}_{gh} & \text{if } \mathrm{Fix}(g) = \mathrm{Fix}(h) = \mathrm{Fix}(gh) = \{x_i\} \\ \kappa x_i^2 \overline{v}_{\mathrm{id}} & \text{if } \mathrm{Fix}(g) = \mathrm{Fix}(h) = \{x_i\},\ gh = \mathrm{id} \\ (-1)^{\mathrm{age}(g)} \kappa q^{-1} x_1 x_2 x_3 \overline{v}_{\mathrm{id}} & \text{if } \mathrm{Fix}(g) = \mathrm{Fix}(h) = \{0\},\ gh = \mathrm{id} \\ 0 & \text{otherwise} \end{cases}$$

with the κ from Definition 7.1.7.

Lemma 7.2.24. *For $g, h \in G_f$ we have*

$$\overline{v}_g \circ \overline{v}_h = (-1)^{(n-n_g)(n-n_h)} \cdot (\overline{v}_h \circ \overline{v}_g)\,.$$

Proof. This is clear from the definition, since only for $\mathrm{Fix}(g) = \{0\}$ we have $n - n_g \equiv 1 \mod 2$ and so in this case we have if $g = (\frac{a_1}{r}, \frac{a_2}{r}, \frac{a_3}{r})$ is an element of age 1 with $0 < a_i < r$, $i = 1, 2, 3$, then $g^{-1} = (\frac{r-a_1}{r}, \frac{r-a_2}{r}, \frac{r-a_3}{r})$ is an element of age 2 and vice versa (cf. Proposition 7.1.12). $\square$

Proposition 7.2.25. *For each $g, g', g'' \in G_f$, we have*

$$(\overline{v}_g \circ \overline{v}_{g'}) \circ \overline{v}_{g''} = \overline{v}_g \circ (\overline{v}_{g'} \circ \overline{v}_{g''}).$$

Proof. We only do not get zero on both sides, if one of g, g', g'' is the identity, or if $\mathrm{Fix}(g) = \mathrm{Fix}(g') = \mathrm{Fix}(g'') = \{x_i\}$ for one $i \in \{1, 2, 3\}$. If one of g, g', g'' is the identity, this is trivially satisfied since $\overline{v}_g \circ \overline{v}_{\mathrm{id}} = \overline{v}_g$. For the other case we define the elements $\overline{w}_{g_i^l}$ of $\mathrm{Jac}'(f, g_i^l)$ as $\overline{w}_{g_i^l} = v_{g_i^l}$ for each $l \in \mathbb{Z}$ with $l \notin n_i\mathbb{Z}$ and we set the element $\overline{w}_{g_i^l} = x_i v_{\mathrm{id}} \in \mathrm{Jac}'(f, \mathrm{id})$ for each $l \in n_i\mathbb{Z}$. Then we have for $\chi_A < 0$ and so $\kappa = 1$: $(\overline{w}_g \circ \overline{w}_{g'}) \circ \overline{w}_{g''} = x_i \overline{w}_{gg'} \circ \overline{w}_{g''} = x_i^2 \overline{w}_{gg'g''} = x_i \overline{w}_g \circ \overline{w}_{g'g''} = \overline{w}_g \circ (\overline{w}_{g'} \circ \overline{w}_{g''})$. For $\chi_A = 0$ we either have $a_i = 2$ and so both sides are zero or we have $a_i = 3$ for all $i = 1, 2, 3$. Then we could have $K_i = \mathbb{Z}/3\mathbb{Z}$ and then we get either $gg'g'' = \mathrm{id}$ and so $g = g' = g''$ so it is clear or $gg'g'' \neq \mathrm{id}$. Then we get on both sides a multiple of $x_i \overline{v}_{gg'g''}$ which is zero in $\mathrm{Jac}(f, gg'g'')$. $\square$

Now it is possible to define a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}$-algebra structure on $\mathcal{A}'$.

Definition 7.2.26. Define a $\mathbb{C}$-bilinear map $\circ : \mathcal{A}' \otimes_{\mathbb{C}} \mathcal{A}' \longrightarrow \mathcal{A}'$ by setting, for each $g, h \in G_f$ and $\phi(\mathbf{x}),\ \psi(\mathbf{x}) \in \mathbb{C}[x_1, x_2, x_3]$,

$$([\phi(\mathbf{x})]\overline{v}_g) \circ ([\psi(\mathbf{x})]\overline{v}_h)$$
$$:= \begin{cases} [\phi(\mathbf{x})\psi(\mathbf{x})]\overline{v}_g & \text{if } h = \mathrm{id} \\ [\phi(\mathbf{x})\psi(\mathbf{x})]\overline{v}_h & \text{if } g = \mathrm{id} \\ [\phi(\mathbf{x})\psi(\mathbf{x})x_i]\overline{v}_{gh} & \text{if } \mathrm{Fix}(g) = \mathrm{Fix}(h) = \mathrm{Fix}(gh) = \{x_i\} \\ \kappa[\phi(\mathbf{x})\psi(\mathbf{x})x_i^2]\overline{v}_{\mathrm{id}} & \text{if } \mathrm{Fix}(g) = \mathrm{Fix}(h) = \{x_i\},\ gh = \mathrm{id} \\ (-1)^{\mathrm{age}(g)}\kappa[\phi(\mathbf{x})\psi(\mathbf{x})q^{-1}x_1x_2x_3]\overline{v}_{\mathrm{id}} & \text{if } \mathrm{Fix}(g) = \mathrm{Fix}(h) = \{0\},\ gh = \mathrm{id} \\ 0 & \text{otherwise} \end{cases}$$

with the κ from Definition 7.1.7.

Proposition 7.2.27. *The map $\circ$ equips $\mathcal{A}'$ with the structure of a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}$-algebra with the identity $\overline{v}_{\mathrm{id}}$, which satisfies Axiom* (ii) *in Definition 5.2.1.*

Proof. The associativity of the product follows from Proposition 7.2.25. By the definition 7.2.23 it is obvious that $\mathcal{A}'_{\overline{i}} \circ \mathcal{A}'_{\overline{j}} \subset \mathcal{A}'_{\overline{i+j}}$ since we always have zero, when $(n - n_g) + (n - n_h) \not\equiv (n - n_{gh}) \mod 2$. It is also clear by the definition of the map $\circ$ above that the natural surjective maps $\mathrm{Jac}(f) \longrightarrow \mathrm{Jac}(f^g)$, $g \in G_f$, equip $\mathcal{A}'$ with the structure of a $\mathrm{Jac}(f)$-module, which coincides with the product map $\circ : \mathcal{A}'_{\mathrm{id}} \otimes_{\mathbb{C}} \mathcal{A}'_g \longrightarrow \mathcal{A}'_g$. ☐

Definition 7.2.28. Take the nowhere vanishing 3-form $q^{-1}dx_1 \wedge dx_2 \wedge dx_3$ and set $\zeta := [q^{-1}dx_1 \wedge dx_2 \wedge dx_3] \in \Omega_f$. Define a $\mathbb{C}$-bilinear map $\vdash: \mathcal{A}' \otimes_{\mathbb{C}} \Omega'_{f,G_f} \longrightarrow \Omega'_{f,G_f}$ by setting, for each $g, h \in G_f$ and $\phi(\mathbf{x}),\ \psi(\mathbf{x}) \in \mathbb{C}[x_1, x_2, x_3]$,

$$([\phi(\mathbf{x})]\overline{v}_g) \vdash ([\psi(\mathbf{x})]\omega_h)$$
$$:= \begin{cases} \frac{\overline{\alpha}_{gh}}{\overline{\alpha}_h}[\phi(\mathbf{x})\psi(\mathbf{x})]\omega_g & \text{if } h = \mathrm{id} \\ \frac{\overline{\alpha}_{gh}}{\overline{\alpha}_h}[\phi(\mathbf{x})\psi(\mathbf{x})]\omega_h & \text{if } g = \mathrm{id} \\ \frac{\overline{\alpha}_{gh}}{\overline{\alpha}_h}[\phi(\mathbf{x})\psi(\mathbf{x})x_i]\omega_{gh} & \text{if } \mathrm{Fix}(g) = \mathrm{Fix}(h) = \mathrm{Fix}(gh) = \{x_i\} \\ \frac{\kappa\overline{\alpha}_{gh}}{\overline{\alpha}_h}[\phi(\mathbf{x})\psi(\mathbf{x})x_i^2]\omega_{\mathrm{id}} & \text{if } \mathrm{Fix}(g) = \mathrm{Fix}(h) = \{x_i\},\ gh = \mathrm{id} \\ (-1)^{\mathrm{age}(g)}\frac{\kappa\overline{\alpha}_{gh}}{\overline{\alpha}_h}[\phi(\mathbf{x})\psi(\mathbf{x})q^{-1}x_1x_2x_3]\omega_{\mathrm{id}} & \text{if } \mathrm{Fix}(g) = \mathrm{Fix}(h) = \{0\},\ gh = \mathrm{id} \\ 0 & \text{otherwise} \end{cases}$$

with the κ from Definition 7.1.7 and $\overline{\alpha} : G \longrightarrow \mathbb{C}^*$, $g \mapsto \overline{\alpha}_g$ is a map we will define now:

Definition 7.2.29.

$$\overline{\alpha}_g := 1 \text{ if } \mathrm{Fix}(g) = \{0\}.$$

All other $g \in G_f$ can be written as g_i^l for the generators g_i of K_i, $i = 1, 2, 3$. We define for $i \in \{1, 2, 3\}$ and $m \in \mathbb{Z}_{\geq 0}$ the numbers:

$$c_m^i := \begin{cases} 1 & m \equiv 0 \mod n_i \\ 1 & m \equiv n_i - 1 \mod n_i \\ \sqrt{-1} & \textit{otherwise} \end{cases}$$

Then we define

$$\overline{\alpha}_{g_i^l} = \left(\prod_{m=0}^{n_i-1} c_m^i\right)^{-\frac{l}{n_i}} \left(\prod_{m=0}^{l-1} c_m^i\right).$$

Lemma 7.2.30. *This is well defined since $\overline{\alpha}_{g_i^l} = \overline{\alpha}_{g_i^{l+n_i}}$ and $\overline{\alpha}_{g_i^{n_i}} = 1$ for all $i = 1, 2, 3$.*

Proof. Note that $\prod_{m=l}^{l+n_i-1} c_m^i = \prod_{m=0}^{n_i-1} c_m^i$ since in both products we have twice a 1 and (n_i-2)-times a $\sqrt{-1}$. Then the statement follows from the following equation.

$$\overline{\alpha}_{g_i^{l+n_i}} = \left(\prod_{m=0}^{n_i-1} c_m^i\right)^{-\frac{l+n_i}{n_i}} \left(\prod_{m=0}^{l+n_i-1} c_m^i\right) = \overline{\alpha}_{g_i^l} \left(\prod_{m=0}^{n_i-1} c_m^i\right)^{-1} \left(\prod_{m=l}^{l+n_i-1} c_m^i\right) = \overline{\alpha}_{g_i^l}.$$

For all $i = 1, 2, 3$ we have: $\overline{\alpha}_{g_i^{n_i}} = \left(\prod_{m=0}^{n_i-1} c_m^i\right)^{-\frac{n_i}{n_i}} \left(\prod_{m=0}^{n_i-1} c_m^i\right) = 1.$ □

Lemma 7.2.31. *The map $\overline{\alpha} : G \longrightarrow \mathbb{C}^*$ satisfies $\overline{\alpha}_{\mathrm{id}} = 1$ and*

$$\overline{\alpha}_g \overline{\alpha}_{g^{-1}} = 1, \quad g \in G_f.$$

Proof. For all $i = 1, 2, 3$ we have $\overline{\alpha}_{\mathrm{id}} = \overline{\alpha}_{g_i^{n_i}} = 1$. For $\mathrm{Fix}(g) = \{0\}$ and for $g = \mathrm{id}$ the second statement is trivially satisfied. In the other cases for $i \in \{1, 2, 3\}$ we can take $0 < l < n_i$ and have

$$\begin{aligned}
\overline{\alpha}_g \overline{\alpha}_{g^{-1}} = \overline{\alpha}_{g_i^l} \overline{\alpha}_{g_i^{n_i-l}} &= \left(\prod_{m=0}^{n_i-1} c_m^i\right)^{-\frac{l}{n_i}} \left(\prod_{m=0}^{l-1} c_m^i\right) \left(\prod_{m=0}^{n_i-1} c_m^i\right)^{-\frac{n_i-l}{n_i}} \left(\prod_{m=0}^{n_i-l-1} c_m^i\right) \\
&= \left(\prod_{m=0}^{l-1} c_m^i\right) \left(\prod_{m=0}^{n_i-l-1} c_m^i\right) \left(\prod_{m=0}^{n_i-1} c_m^i\right)^{-1} = \left(\prod_{m=1}^{l-1} \sqrt{-1}\right) \left(\prod_{m=1}^{n_i-l-1} \sqrt{-1}\right) \left(\prod_{m=1}^{n_i-2} \sqrt{-1}\right)^{-1} \\
&= \sqrt{-1}^{\,l-1+n_i-l-1-(n_i-2)} = \sqrt{-1}^{\,0} = 1.
\end{aligned}$$

□

The map $\vdash$ induces an isomorphism $\vdash \zeta : \mathcal{A}' \longrightarrow \Omega'_{f,G_f}$ of $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}$-modules:

$$\vdash \zeta : \mathcal{A}'_g \longrightarrow \Omega'_{f,g}, \quad [\phi(\mathbf{x})]\overline{v}_g \mapsto [\phi(\mathbf{x})]\overline{v}_g \vdash \zeta = \overline{\alpha}_g [\phi(\mathbf{x})]\omega_g,$$

Note that for each $g, h \in G_f$ and $\phi(\mathbf{x})$, $\psi(\mathbf{x}) \in \mathbb{C}[x_1, x_2, x_3]$ we have

$$([\phi(\mathbf{x})]\overline{v}_g) \vdash ([\psi(\mathbf{x})]\overline{v}_h \vdash \zeta) = (([\phi(\mathbf{x})]\overline{v}_g) \circ ([\psi(\mathbf{x})]\overline{v}_h)) \vdash \zeta,$$

by which we obtain the following

Proposition 7.2.32. *The map $\vdash: \mathcal{A}' \otimes_{\mathbb{C}} \Omega'_{f,G_f} \longrightarrow \Omega'_{f,G_f}$ satisfies Axiom* (iii) *in Definition 5.2.1.*

On $\mathcal{A}'$ we have the action of $\varphi \in \mathrm{Aut}(f,G)$ induced by the isomorphism $\vdash \zeta : \mathcal{A}' \longrightarrow \Omega'_{f,G_f}$, which is denoted by φ^*. We also use the notation of Remark 5.1.8.

Proposition 7.2.33. *Axiom* (iv) *in Definition 5.2.1 is satisfied by $\mathcal{A}'$, namely, Axioms* (iva) *and* (ivb) *hold.*

Proof. Let $g \in G_f$. For simplicity, set $g' = \varphi^{-1} g \varphi$. There exist λ_φ and λ_{φ_g} such that

$$\varphi^*(\widetilde{\omega}_{\mathrm{id}}) = \lambda_\varphi \widetilde{\omega}_{\mathrm{id}}, \quad \varphi^*(\widetilde{\omega}_g) = \lambda_{\varphi_g} \widetilde{\omega}_{g'}$$

First note that $\lambda_\varphi = \pm 1$, since all $\varphi \in \mathrm{Aut}(f,G)$ preserve f and so also preserve $q^{-1}x_1x_2x_3$ and so they leave $\omega_{\mathrm{id}} = [q^{-1}dx_1 \wedge dx_2 \wedge dx_3]$ invariant except perhaps the order of the dx_i.

For each $\phi(\mathbf{x}) \in \mathbb{C}[x_1,x_2,x_3]$, we have

$$\varphi^*([\phi(\mathbf{x})]\overline{v}_g) = [\varphi^*\phi(\mathbf{x})]\varphi^*(\overline{v}_g),$$

since

$$\begin{aligned}
&\varphi^*([\phi(\mathbf{x})]\overline{v}_g) \vdash \varphi^*(\zeta) = \varphi^*([\phi(\mathbf{x})]\overline{v}_g \vdash \zeta) = \varphi^*(\overline{\alpha}_g[\phi(\mathbf{x})]\omega_g) = [\varphi^*\phi(\mathbf{x})]\varphi^*(\overline{\alpha}_g\omega_g) \\
&= [\varphi^*\phi(\mathbf{x})] \vdash \varphi^*(\overline{\alpha}_g\omega_g) = [\varphi^*\phi(\mathbf{x})] \vdash \varphi^*(\overline{v}_g \vdash \zeta) = [\varphi^*\phi(\mathbf{x})] \vdash (\varphi^*(\overline{v}_g) \vdash \varphi^*(\zeta)) \\
&= ([\varphi^*\phi(\mathbf{x})]\varphi^*(\overline{v}_g)) \vdash \varphi^*(\zeta).
\end{aligned}$$

Therefore, we only need to show that $\varphi^*(\overline{v}_g) \circ \varphi^*(\overline{v}_h) = \varphi^*(\overline{v}_g \circ \overline{v}_h)$.

It easily follows that

$$\varphi^*(\overline{v}_{\mathrm{id}}) = \overline{v}_{\mathrm{id}}, \quad \varphi^*(\overline{v}_g) = \frac{\overline{\alpha}_g \lambda_{\varphi_g}}{\overline{\alpha}_{g'}\lambda_\varphi}\overline{v}_{g'},$$

since $\varphi^*(\overline{v}_{\mathrm{id}}) \vdash \varphi^*(\zeta) = \varphi^*(\overline{v}_{\mathrm{id}} \vdash \zeta) = \varphi^*(\zeta)$ and

$$\begin{aligned}
(\lambda_{\varphi_g}\overline{v}_{g'}) \vdash \zeta &= \lambda_{\varphi_g}\overline{\alpha}_{g'}\omega_{g'} = \overline{\alpha}_{g'}\varphi^*(\omega_g) = \overline{\alpha}_{g'}\varphi^*(\frac{1}{\overline{\alpha}_g}\overline{v}_g \vdash \zeta) \\
&= \frac{\overline{\alpha}_{g'}}{\overline{\alpha}_g}\varphi^*(\overline{v}_g) \vdash \varphi^*(\zeta) = \frac{\overline{\alpha}_{g'}}{\overline{\alpha}_g}\lambda_\varphi\varphi^*(\overline{v}_g) \vdash \zeta,
\end{aligned}$$

So for the multiplication with $\overline{v}_{\mathrm{id}}$ the $\mathrm{Aut}(f,G)$-invariance is clear.

Since the fixed loci of g and g' have the same dimension, we only have to show the $\mathrm{Aut}(f,G)$-invariance for each case of Definition 7.2.23.

For $\mathrm{Fix}(g) = \mathrm{Fix}(h) = \{0\}$ and $gh = \mathrm{id}$, we have $\overline{\alpha}_g = \overline{\alpha}_h = 1$ and also $\lambda_{\varphi_g} = \lambda_{\varphi_h} = 1$ since $\omega_g = 1_g$ and $\varphi^*(1_g) = 1_g$. So we have to show that

$$\varphi^*(\overline{v}_g) \circ \varphi^*(\overline{v}_h) = \frac{\overline{\alpha}_g\lambda_{\varphi_g}}{\overline{\alpha}_{g'}\lambda_\varphi}\frac{\overline{\alpha}_h\lambda_{\varphi_h}}{\overline{\alpha}_{h'}\lambda_\varphi}\overline{v}_{g'} \circ \overline{v}_{h'} = \frac{1}{\lambda_\varphi^2}(-1)^{\mathrm{age}(g')}\kappa q^{-1}x_1x_2x_3\overline{v}_{\mathrm{id}}$$

is the same as

$$\varphi^*(\overline{v}_g \circ \overline{v}_h) = \varphi^*((-1)^{\mathrm{age}(g)}\kappa q^{-1}x_1x_2x_3\overline{v}_{\mathrm{id}}) = (-1)^{\mathrm{age}(g)}\varphi^*(\kappa)q^{-1}x_1x_2x_3\overline{v}_{\mathrm{id}}.$$

Since $\lambda_\varphi^2 = 1$ we only have to show $\varphi^*(\kappa) = \kappa$. For $\chi_A < 0$ we have $\kappa = 1$ and this is clear. In the other three cases κ depends on a multiple $q^{\mathrm{lcm}(a_1,a_2,a_3)}$, see Definition 7.1.7. So we see directly, when $\varphi \in \mathrm{Aut}(f,G) \subset GL(3+1,\mathbb{C})$ leaves $f = x_1^{a_1} + x_2^{a_2} + x_3^{a_3} - q^{-1}x_1x_2x_3$ invariant it also leaves $q^{\mathrm{lcm}(a_1,a_2,a_3)}$ and so κ invariant.

For $\mathrm{Fix}(g) = \mathrm{Fix}(h) = \{x_i\}$ for a fixed $i \in \{1,2,3\}$ we have $\varphi^*(x_i) = \lambda_{\varphi_{g_i}} x_j$ for one $j \in \{1,2,3\}$, since $\varphi \in \mathrm{Aut}(f,G)$ and $\omega_{g_i} = [dx_i]$. So it is also clear that $\lambda_{\varphi_{g_i^l}} = \lambda_{\varphi_{g_i}}$ for all $l \in \mathbb{Z}\backslash n_i\mathbb{Z}$. By the definition of the g_i we have for $\lambda_\varphi = -1$ that ${g_i}' = g_j^{-1}$ and for $\lambda_\varphi = 1$ that ${g_i}' = g_j^a$ for one $a \in \mathbb{Z}$ with $gcd(a,n_j) = 1$. That is because φ is G-equivariant and we also have $n_i = n_j$. (Usually we have $i = j$ but it it also possible that $i \neq j$.) So we have to show for $0 < l_1, l_2 < n_i$ that

$$\begin{aligned}
\varphi^*(\overline{v}_{g_i^{l_1}}) \circ \varphi^*(\overline{v}_{g_i^{l_2}}) &= \frac{\overline{\alpha}_{g_i^{l_1}} \lambda_{\varphi_{g_i^{l_1}}}}{\overline{\alpha}_{(g_i')^{l_1}} \lambda_\varphi} \frac{\overline{\alpha}_{g_i^{l_2}} \lambda_{\varphi_{g_i^{l_2}}}}{\overline{\alpha}_{(g_i')^{l_2}} \lambda_\varphi} \overline{v}_{(g_i')^{l_1}} \circ \overline{v}_{(g_i')^{l_2}} \\
&= \begin{cases} \dfrac{\overline{\alpha}_{g_i^{l_1}} \overline{\alpha}_{g_i^{l_2}} \lambda^2_{\varphi_{g_i}}}{\overline{\alpha}_{(g_i')^{l_1}} \overline{\alpha}_{(g_i')^{l_2}} \lambda_\varphi^2} \kappa x_j^2 \overline{v}_{\mathrm{id}} & \text{if } l_1 + l_2 = n_i \\ \dfrac{\overline{\alpha}_{g_i^{l_1}} \overline{\alpha}_{g_i^{l_2}} \lambda^2_{\varphi_{g_i}}}{\overline{\alpha}_{(g_i')^{l_1}} \overline{\alpha}_{(g_i')^{l_2}} \lambda_\varphi^2} x_j \overline{v}_{(g_i')^{l_1+l_2}} & \text{if } l_1 + l_2 \neq n_i \end{cases}
\end{aligned}$$

is the same as

$$\begin{aligned}
\varphi^*(\overline{v}_{g_i^{l_1}} \circ \overline{v}_{g_i^{l_2}}) &= \begin{cases} \varphi^*(\kappa x_i^2) & \text{if } l_1 + l_2 = n_i \\ \varphi^*(x_i \overline{v}_{g_i^{l_1+l_2}}) & \text{if } l_1 + l_2 \neq n_i \end{cases} \\
&= \begin{cases} \lambda^2_{\varphi_{g_i}} \kappa x_j^2 & \text{if } l_1 + l_2 = n_i \\ \dfrac{\overline{\alpha}_{g_i^{l_1+l_2}} \lambda_{\varphi_{g_i}}}{\overline{\alpha}_{(g_i')^{l_1+l_2}} \lambda_\varphi} \lambda_{\varphi_{g_i}} x_j \overline{v}_{(g_i')^{l_1+l_2}} & \text{if } l_1 + l_2 \neq n_i \end{cases}.
\end{aligned}$$

For $l_1 + l_2 = n_i$ this is clear because $\overline{\alpha}_g \overline{\alpha}_{g^{-1}} = 1$. So we only have to show, for $0 < l_1, l_2 < n_i$, $l_1 + l_2 \neq n_i$:

$$\frac{\overline{\alpha}_{g_i^{l_1}} \overline{\alpha}_{g_i^{l_2}}}{\overline{\alpha}_{(g_i')^{l_1}} \overline{\alpha}_{(g_i')^{l_2}} \lambda_\varphi} = \frac{\overline{\alpha}_{g_i^{l_1+l_2}}}{\overline{\alpha}_{(g_i')^{l_1+l_2}}}$$

For $\lambda_\varphi = 1$ this is $1 = 1$ since $\overline{\alpha}_{g_i^l} = \overline{\alpha}_{(g_i')^l}$.

For $\lambda_\varphi = -1$ we have $n_i = n_j$ and so $c^i_\bullet = c^j_\bullet$. So we can calculate the LHS as

$$\begin{aligned}
\frac{\overline{\alpha}_{g_i^{l_1}}\overline{\alpha}_{g_i^{l_2}}}{\overline{\alpha}_{(g_i')^{l_1}}\overline{\alpha}_{(g_i')^{l_2}}\lambda_\varphi} &= -\frac{\overline{\alpha}_{g_i^{l_1}}\overline{\alpha}_{g_i^{l_2}}}{\overline{\alpha}_{g_j^{n_j-l_1}}\overline{\alpha}_{g_j^{n_j-l_2}}} \\
&= -\frac{\left(\prod_{m=0}^{n_i-1} c^i_m\right)^{-\frac{l_1}{n_i}}\left(\prod_{m=0}^{l_1-1} c^i_m\right)\left(\prod_{m=0}^{n_i-1} c^i_m\right)^{-\frac{l_2}{n_i}}\left(\prod_{m=0}^{l_2-1} c^i_m\right)}{\left(\prod_{m=0}^{n_i-1} c^i_m\right)^{-\frac{n_i-l_1}{n_i}}\left(\prod_{m=0}^{n_i-l_1-1} c^i_m\right)\left(\prod_{m=0}^{n_i-1} c^i_m\right)^{-\frac{n_i-l_2}{n_i}}\left(\prod_{m=0}^{n_i-l_2-1} c^i_m\right)} \\
&= -\frac{\left(\prod_{m=0}^{n_i-1} c^i_m\right)^{-\frac{l_1+l_2}{n_i}}\left(\sqrt{-1}^{l_1-1}\right)\left(\sqrt{-1}^{l_2-1}\right)}{\left(\prod_{m=0}^{n_i-1} c^i_m\right)^{-\frac{2n_i-l_1-l_2}{n_i}}\left(\sqrt{-1}^{n_i-l_1-1}\right)\left(\sqrt{-1}^{n_i-l_2-1}\right)} \\
&= \frac{\left(\prod_{m=0}^{n_i-1} c^i_m\right)^{-\frac{l_1+l_2}{n_i}}}{\left(\prod_{m=0}^{n_i-1} c^i_m\right)^{-\frac{2n_i-l_1-l_2}{n_i}}}(-1)\sqrt{-1}^{2l_1+2l_2-2n_i} \\
&= \frac{\left(\prod_{m=0}^{n_i-1} c^i_m\right)^{-\frac{l_1+l_2}{n_i}}}{\left(\prod_{m=0}^{n_i-1} c^i_m\right)^{-\frac{2n_i-l_1-l_2}{n_i}}}(-1)^{1+l_1+l_2-n_i}.
\end{aligned}$$

The RHS is given by

$$\begin{aligned}
\frac{\overline{\alpha}_{g_i^{l_1+l_2}}}{\overline{\alpha}_{(g_i')^{l_1+l_2}}} = \frac{\overline{\alpha}_{g_i^{l_1+l_2}}}{\overline{\alpha}_{g_j^{n_j-l_1-l_2}}} &= \frac{\left(\prod_{m=0}^{n_i-1} c^i_m\right)^{-\frac{l_1+l_2}{n_i}}\left(\prod_{m=0}^{l_1+l_2-1} c^i_m\right)}{\left(\prod_{m=0}^{n_i-1} c^i_m\right)^{-\frac{n_i-l_1+n_i-l_2}{n_i}}\left(\prod_{m=0}^{n_i-l_1+n_i-l_2-1} c^i_m\right)} \\
&= \begin{cases} \dfrac{\left(\prod_{m=0}^{n_i-1} c^i_m\right)^{-\frac{l_1+l_2}{n_i}}\left(\sqrt{-1}^{l_1+l_2-1}\right)}{\left(\prod_{m=0}^{n_i-1} c^i_m\right)^{-\frac{2n_i-l_1-l_2}{n_i}}\left(\sqrt{-1}^{2n_i-l_1-l_2-1-2}\right)} & l_1+l_2 < n_i \\[2ex] \dfrac{\left(\prod_{m=0}^{n_i-1} c^i_m\right)^{-\frac{l_1+l_2}{n_i}}\left(\sqrt{-1}^{l_1+l_2-1-2}\right)}{\left(\prod_{m=0}^{n_i-1} c^i_m\right)^{-\frac{2n_i-l_1-l_2}{n_i}}\left(\sqrt{-1}^{2n_i-l_1-l_2-1}\right)} & l_1+l_2 > n_i \end{cases} \\
&= \begin{cases} \dfrac{\left(\prod_{m=0}^{n_i-1} c^i_m\right)^{-\frac{l_1+l_2}{n_i}}}{\left(\prod_{m=0}^{n_i-1} c^i_m\right)^{-\frac{2n_i-l_1-l_2}{n_i}}}\sqrt{-1}^{2l_1+2l_2-2n_i+2} & l_1+l_2 < n_i \\[2ex] \dfrac{\left(\prod_{m=0}^{n_i-1} c^i_m\right)^{-\frac{l_1+l_2}{n_i}}}{\left(\prod_{m=0}^{n_i-1} c^i_m\right)^{-\frac{2n_i-l_1-l_2}{n_i}}}\sqrt{-1}^{2l_1+2l_2-2n_i-2} & l_1+l_2 > n_i \end{cases}
\end{aligned}$$

which coincides with the LHS.

Hence, we proved the algebra structure $\circ$ of $\mathcal{A}'$ is $\mathrm{Aut}(f,G)$-invariant.

The G-twisted $\mathbb{Z}/2\mathbb{Z}$-graded commutativity (ivb) is a direct consequence of Lemma 7.2.24 since $g^*(\overline{v}_h) = \overline{v}_h$ for $\mathrm{Fix}(g) = \mathrm{Fix}(h)$ or $g = id$, $h = \mathrm{id}$ and in all other cases our multiplication is zero.

We have finished the proof of the proposition. □

We show the invariance of the bilinear form $J_{f,G}$ with respect to the product structure of $\mathcal{A}'$.

Proposition 7.2.34. *For each $g, h \in G_f$, we have*

$$J_{f,gh}\left(\overline{v}_g \vdash \omega_h, [\phi(\mathbf{x})]\,\omega_{(gh)^{-1}}\right) = (-1)^{(n-n_g)(n-n_h)} J_{f,h}\left(\omega_h, (h^{-1})^{*}\overline{v}_g \vdash \left([\phi(\mathbf{x})]\,\omega_{(gh)^{-1}}\right)\right)$$

for a suitable $\phi(\mathbf{x})$ that this is not zero. As a consequence, the algebra $\mathcal{A}'$ satisfies Axiom (v) *in Definition 5.2.1.*

Proof. We only have to look at the cases for g, h of Definition 7.2.28.

If g or h are the identity the statement is directly clear.

For $\mathrm{Fix}(g) = \mathrm{Fix}(h) = \{x_i\}$ for $i \in \{1, 2, 3\}$ and $l_1, l_2, l_1 + l_2 \notin n_i\mathbb{Z}$ we have: (In this case $a_i \geq 3$, otherwise we directly have $\overline{v}_{g_i^{l_1}} \vdash \omega_{g_i^{l_2}} = 0$.)

$$\begin{aligned} J_{f,g_i^{l_1+l_2}}\left(\overline{v}_{g_i^{l_1}} \vdash \omega_{g_i^{l_2}}, x_i^{a_i-3}\omega_{g_i^{n_i-l_1-l_2}}\right) &= J_{f,g_i^{l_1+l_2}}\left(x_i\omega_{g_i^{l_1+l_2}}, x_i^{a_i-3}\omega_{g_i^{n_i-l_1-l_2}}\right) \\ &= (-1)\frac{1}{a_i}|G| \end{aligned}$$

and

$$\begin{aligned} (-1)^{(3-1)(3-1)} J_{f,g_i^{l_2}} & \left(\omega_{g_i^{l_2}}, (g_i^{n_i-l_2})^{*}\overline{v}_{g_i^{l_1}} \vdash \left(x_i^{a_i-3}\omega_{g_i^{n_i-l_1-l_2}}\right)\right) \\ &= J_{f,g_i^{l_2}}\left(\omega_{g_i^{l_2}}, \overline{v}_{g_i^{l_1}} \vdash \left(x_i^{a_i-3}\omega_{g_i^{n_i-l_1-l_2}}\right)\right) \\ &= J_{f,g_i^{l_2}}\left(\omega_{g_i^{l_2}}, x_i^{a_i-2}\omega_{g_i^{n_i-l_2}}\right) = (-1)\frac{1}{a_i}|G| \end{aligned}$$

For $l_1 + l_2 \in n_i\mathbb{Z}$ we have:

$$\begin{aligned} J_{f,id}\left(\overline{v}_{g_i^{l_1}} \vdash \omega_{g_i^{l_2}}, x_i^{a_i-2}\omega_{\mathrm{id}}\right) &= J_{f,\mathrm{id}}\left(\kappa x_i^2\omega_{\mathrm{id}}, x_i^{a_i-2}\omega_{\mathrm{id}}\right) \\ &= \frac{-1}{a_i}|G| \end{aligned}$$

and

$$\begin{aligned} (-1)^{(3-1)(3-1)} J_{f,g_i^{l_2}} & \left(\omega_{g_i^{l_2}}, (g_i^{n_i-l_2})^{*}\overline{v}_{g_i^{l_1}} \vdash \left(x_i^{a_i-2}\omega_{\mathrm{id}}\right)\right) \\ &= J_{f,g_i^{l_2}}\left(\omega_{g_i^{l_2}}, \overline{v}_{g_i^{l_1}} \vdash \left(x_i^{a_i-2}\omega_{\mathrm{id}}\right)\right) \\ &= J_{f,g_i^{l_2}}\left(\omega_{g_i^{l_2}}, x_i^{a_i-2}\omega_{g_i^{l_1}}\right) = (-1)\frac{1}{a_i}|G| \end{aligned}$$

Let $Fix(g) = \{0\}$ and $h = g^{-1}$ then we have

$$\begin{aligned} J_{f,\mathrm{id}}\left(\overline{v}_g \vdash \omega_h, \omega_{(gh)^{-1}}\right) &= J_{f,\mathrm{id}}\left((-1)^{\mathrm{age}(g)}\kappa q^{-1}x_1x_2x_3\omega_{\mathrm{id}}, \omega_{\mathrm{id}}\right) \\ &= -(-1)^{\mathrm{age}(g)}|G| \end{aligned}$$

and

$$\begin{aligned} (-1)^{(3-0)(3-0)} J_{f,g^{-1}}\left(\omega_h, (h^{-1})^{*}\overline{v}_g \vdash (\omega_{\mathrm{id}})\right) &= (-1)J_{f,g^{-1}}\left(\omega_h, \overline{v}_g \vdash (\omega_{\mathrm{id}})\right) \\ &= (-1)(-1)^{3-0-\mathrm{age}(h)}|G| \\ &= (-1)^{-\mathrm{age}(h)}|G| \end{aligned}$$

and $(-1)^{-\mathrm{age}(h)} = -(-1)^{-\mathrm{age}(g)} = -(-1)^{\mathrm{age}(g)}$ since $h = g^{-1}$. $\square$

So we have shown all axioms and with Proposition 5.4.9 we have finished the proof of Theorem 7.2.2.

Remark 7.2.35. We have shown the existence for all cusp polynomials, even for those pairs (f, G) of bad type (Definition 7.2.1). The crucial reason why we cannot prove the uniqueness there can be seen in Lemma 7.2.17. Namely, we cannot prove that the $c_{l,m}$ are not zero. If we take a zero multiplication there, we would also satisfy the axioms.

7.3 Frobenius Algebras Associated to the Gromov-Witten Theory for Orbifold Projective Lines

Remark 7.3.1. In [ST15] and [IST12] it was shown that the Frobenius manifold associated to the pair of a cusp singularity f_A and its canonical primitive form ζ is isomorphic to the one constructed from the Gromov-Witten theory for an orbifold projective line with at most three orbifold points.

We are only interested in the Frobenius algebra $\mathrm{Jac}(f_A)$. The proofs in [ST15] and [IST12] were done with the uniqueness theorem for Frobenius manifolds of orbifold projective lines from [IST15]. The interesting facts for the Frobenius algebra are:

Proposition 7.3.2 (cf. [ST15])**.** *For $A = (a_1, a_2, a_3)$ the Frobenius algebra $\mathrm{Jac}(f_A)$ has dimension*

$$\mu_A = \sum_{i=1}^{3}(a_i - 1) + 2$$

and a basis $\{1, y_{\mu_A}, y_{i,j} \mid i = 1, 2, 3\ ;\ j = 1, 2, \ldots, a_i - 1\}$. The bilinear form J_{f_A} satisfies

$$J_{f_A}(1, y_{\mu_A}) = -1$$
$$J_{f_A}(y_{i_1,j_1}, y_{i_2,j_2}) = \begin{cases} \frac{-1}{a_i} & \text{if } i_1 = i_2 = i \text{ and } j_1 + j_2 = a_i \\ 0 & \text{otherwise} \end{cases}$$

(cf. Condition (ii) of [IST15, Thm 3.1], where we only have another scaling and -1 instead of $+1$).

In the limit $q \to 0$ the Frobenius algebra is isomorphic to

$$\mathrm{Jac}(f_A)|_{q\to 0} \cong \mathbb{C}[y_1, y_2, y_3] \big/ (y_1y_2, y_2y_3, y_3y_1, a_1y_1^{a_1} - a_2y_2^{a_2}, a_2y_2^{a_2} - a_3y_2^{a_3}, a_3y_3^{a_3} - a_1y_1^{a_1})$$

where $y_{i,j} \mapsto y_i^j$ and $y_{\mu_A} \mapsto a_iy_i^{a_i}$ (cf. Condition (v) of [IST15, Thm 3.1]).

Proof. This is an easy computation when we take the basis

$$\{1, \kappa q^{-1}x_1x_2x_3, (\sqrt[a_i]{\kappa}x_i)^j \mid i = 1, 2, 3\ ;\ j = 1, 2, \ldots, a_i - 1\}$$

(cf. Definition 7.1.8) of $Jac(f_A)$. We have $q^{-1}x_1x_2x_3 = a_ix_i^{a_i} \in \mathrm{Jac}(f_A)$ and so $\kappa q^{-1}x_1x_2x_3 = a_i(\sqrt[a_i]{\kappa}x_i)^{a_i} \in \mathrm{Jac}(f_A)$. Then we can take the limit $q \to 0$. For $q \to 0$ we even have $\kappa \to 1$. $\square$

Remark 7.3.3. A uniqueness theorem for Frobenius manifolds for orbifold projective lines with r orbifold points, where r is an arbitrary positive integer was given in [Sh14].

Definition 7.3.4 (cf. [ET13a, Thm. 5.12])**.** Let f_A be the cusp polynomial of the tuple $A = (a_1, a_2, a_3)$ and $G \subset G_{f_A}$. For $i = 1, 2, 3$ let be K_i be the subgroup of G preserving the i-th coordinate with $|K_i| = n_i$. We define

$$a_i' = \frac{a_i}{|G/K_i|}.$$

Define a tuple $B = (b_1, \ldots, b_r)$ by

$$(b_1, \ldots, b_r) = (a_i' * n_i, i = 1, 2, 3)$$

where $u * v = (\underbrace{u, u, \ldots, u}_{v\text{-times}})$. So we have $r = \sum_{i=1}^{3} n_i$.

Remark 7.3.5. We are only interested in the commutative part $\mathrm{Jac}(f_A, G)_{\overline{0}}$ of our orbifold Jacobian algebra. For G with $j_G = 0$ (cf. Proposition 7.1.12) this is the total orbifold Jacobian algebra.

We will now prove a similar statement as Proposition 7.3.2 for $\mathrm{Jac}(f_A, G)_{\overline{0}}$:

Theorem 7.3.6. *Let $B = (b_1, \ldots, b_r)$ be as in Definition 7.3.4. The Frobenius algebra $\mathrm{Jac}(f_A, G)_{\overline{0}}$ has dimension*

$$\mu_B = \sum_{i=1}^{r} (b_i - 1) + 2$$

and a basis $\{1, y_{\mu_B}, y_{i,j} \mid i = 1, 2, \ldots, r \ ; \ j = 1, 2, \ldots, b_i - 1\}$. The bilinear form $J_{f_A,G}$ satisfies

$$J_{f_A,G}(1, y_{\mu_B}) = -1$$
$$J_{f_A,G}(y_{i_1,j_1}, y_{i_2,j_2}) = \begin{cases} \frac{-1}{b_i} & \text{if } i_1 = i_2 = i \text{ and } j_1 + j_2 = b_i \\ 0 & \text{otherwise} \end{cases}$$

(cf. Condition (ii) of [Sh14, Thm 3.1], where we only have another scaling and -1 instead of $+1$).

In the limit $q \to 0$ the Frobenius algebra is isomorphic to

$$\mathrm{Jac}(f_A, G)_{\overline{0}}|_{q \to 0} \cong \mathbb{C}[y_1, \ldots, y_r] \big/ (y_i y_j, b_i y_i^{b_i} - b_j y_j^{b_j})_{1 \leq i \neq j \leq r}$$

where $y_{i,j} \mapsto y_i^j$ and $y_{\mu_B} \mapsto b_i y_i^{b_i}$ (cf. Condition (v) of [Sh14, Thm 3.1]).

We will prove the first statement and then give some definitions to prove the remaining parts.

Lemma 7.3.7. *We have*

$$\dim \mathrm{Jac}(f_A, G)_{\overline{0}} = \mu_B = \sum_{i=1}^{r} (b_i - 1) + 2.$$

Proof. We have

$$\begin{aligned}\mu_B &= \sum_{i=1}^{r} (b_i - 1) + 2 = \sum_{i=1}^{3} \sum_{j=1}^{n_i} (a_i' - 1) + 2 \\ &= \sum_{i=1}^{3} n_i \left(\frac{a_i}{|G/K_i|} - 1\right) + 2 = \frac{1}{|G|} \left(\sum_{i=1}^{3} n_i^2 a_i - \sum_{i=1}^{3} n_i |G| + 2|G|\right)\end{aligned}$$

On the other hand from Theorem 4.4.4 or since G is abelian from Proposition 4.4.5 we know:

$$\dim (\Omega_{f,G})_{\overline{0}} = \sum_{\substack{g \in G \\ n - n_g \equiv 0 \pmod 2}} \mu_{f^g/G} = \frac{1}{|G|} \sum_{\substack{g \in G \\ n - n_g \equiv 0 \pmod 2}} \sum_{h \in G} (-1)^{n_g - n_{<g,h>}} \mu_{f^{<g,h>}}$$

Since $G \subset \mathrm{SL}(n, \mathbb{C})$ this is also the dimension of $\mathrm{Jac}(f, G)_{\overline{0}}$. We have

$$\mu_{f^{<\mathrm{id},\mathrm{id}>}} = \mu_A = \sum_{i=1}^{3} (a_i - 1) + 2, \qquad \mu_{f^{<\mathrm{id},g>}} = 1 \quad \text{if } g \notin K_i \ \forall i = 1, 2, 3,$$
$$\mu_{f^{<g,h>}} = (a_i - 1) \ \text{ if } g, h \in K_i, \qquad \mu_{f^{<g,h>}} = 1 \ \text{ if } g \in K_i, h \notin K_i.$$

So we calculate with $|G| = 1 + \sum_{i=1}^{3} (n_i - 1) + 2 j_G$, cf. Proposition 7.1.12:

$$\begin{aligned}\dim \mathrm{Jac}(f, G)_{\overline{0}} &= \frac{1}{|G|} \sum_{\substack{g \in G \\ n - n_g \equiv 0 \pmod 2}} \sum_{h \in G} (-1)^{n_g - n_{<g,h>}} \mu_{f^{<g,h>}} \\ &= \frac{1}{|G|} \left(\sum_{h \in G} (-1)^{n - n_h} \mu_{f^{<\mathrm{id},h>}} + \sum_{i=1}^{3} \sum_{g \in K_i \setminus \{\mathrm{id}\}} \sum_{h \in G} (-1)^{n_g - n_{<g,h>}} \mu_{f^{<g,h>}} \right) \\ &= \frac{1}{|G|} \left(\mu_A + \sum_{i=1}^{3} (n_i - 1)(a_i - 1) - 2 j_G \cdot 1 + \sum_{i=1}^{3} (n_i - 1) \left(n_i (a_i - 1) - |G \backslash K_i| \cdot 1 \right) \right) \\ &= \frac{1}{|G|} \left(\sum_{i=1}^{3} (a_i - 1) + 2 + \sum_{i=1}^{3} (n_i - 1)(a_i - 1) - 2 j_G + \sum_{i=1}^{3} (n_i - 1) \left(n_i a_i - |G| \right) \right) \\ &= \frac{1}{|G|} \left(\sum_{i=1}^{3} n_i^2 a_i - |G| \sum_{i=1}^{3} n_i + 3|G| - |G| \right) = \mu_B\end{aligned}$$

□

Remark 7.3.8. Now we want to give a basis. For this let $v_g \in Jac'(f_A, G)$ be the elements with $v_g \vdash \zeta = \alpha_g \omega_g$, cf. Definition 5.4.5. We will define $e_g \in \mathrm{Jac}'(f_A, G)$ by $e_g := \frac{1}{|K_g|} v_g$, which is the more natural element as stated in the proof of Theorem 6.3.7. So here since we are only interested in the commutative part, we can write each $g \in G$ with $\mathrm{Fix}(g) \neq \{0\}$ as g_i^l for $i = 1, 2, 3$ and $l \in \mathbb{Z}$ as in the last section.

We will now define suitable elements.

Let us first consider the case $\chi_A < 0$: So we have $e_{g_i^l} := \frac{1}{n_i} v_{g_i^l}$, $l \notin n_i\mathbb{Z}$ and $e_{\mathrm{id}} = v_{\mathrm{id}}$, since $K_{g_i} = K_i$. We will additionally define $e_{g_i^l} = \frac{1}{n_i} x_i v_{\mathrm{id}}$ for $l \in n_i\mathbb{Z}$ (cf. w in Lemma 7.2.9).

From our last section we know

$$e_{g_i^l} \circ e_{g_j^m} = \begin{cases} \frac{1}{n_i} x_i e_{g_i^{l+m}} & i = j \\ a_k q x_k^{a_k - 1} e_{\mathrm{id}} & i \neq j \ l \in n_i\mathbb{Z} \text{ and } m \in n_j\mathbb{Z},\ k \in \{1,2,3\} \backslash \{i,j\} \\ 0 & \text{otherwise} \end{cases}. \tag{7.1}$$

$e_{g_i^l}$ need not be in $\mathrm{Jac}(f_A, G)$ because it is not necessarily G-invariant. But $x_i^{|G/K_i|-1} e_{g_i^l}$ is G-invariant for all $l \in \mathbb{Z}$.

Definition 7.3.9. For each $i = 1, 2, 3$ and $k = 1, \dots, n_i$, put

$$[x_{i,k}] := \sum_{l=0}^{n_i - 1} \mathbf{e}\left[\frac{(k-1)l}{n_i}\right] x_i^{|G/K_i|-1} e_{g_i^l}.$$

It is straightforward that all $[x_{i,k}]$ are G-invariant.

Lemma 7.3.10. *In* $\mathrm{Jac}(f_A, G)$, *we have the following equalities*

$$[x_{i,k}] \circ [x_{i,k}] = x_i^{|G/K_i|} [x_{i,k}], \quad i = 1, 2, 3,\ k = 1, \dots, n_i,$$

in particular,

$$a_i' [x_{i,k}]^{a_i'} = \frac{1}{|G|} a_i x_i^{a_i}.$$

And

$$[x_{i,k_1}] \circ [x_{i,k_2}] = 0, \quad i = 1, 2, 3,\ k_1 \neq k_2,$$

Proof. By direct calculation we get:

$$\begin{aligned} [x_{i,k}] \circ [x_{i,k}] &= \sum_{m=0}^{n_i-1} \sum_{l=0}^{n_i-1} \mathbf{e}\left[\frac{(k-1)(l+m)}{n_i}\right] \frac{1}{n_i} x_i^{2|G/K_i|-1} e_{g_i^{l+m}} \\ &= n_i \sum_{l=0}^{n_i-1} \mathbf{e}\left[\frac{(k-1)(l)}{n_i}\right] \frac{1}{n_i} x_i^{2|G/K_i|-1} e_{g_i^l} \\ &= \frac{n_i}{n_i} x_i^{|G/K_i|} [x_{i,k}]. \end{aligned}$$

So we have

$$a'_i[x_{i,k}]^{a'_i} = a'_i x_i^{(a'_i-1)|G/K_i|}[x_{i,k}] = \frac{a'_i}{n_i} x_i^{a'_i|G/K_i|} + 0 = \frac{a_i}{|G/K_i|n_i} x_i^{\frac{a_i}{|G/K_i|}|G/K_i|} = \frac{a_i}{|G|} x_i^{a_i}.$$

For $k_1 \neq k_2$ we always have a sum of all different $e[\frac{m}{n_i}]$ in each summand and we know that the sum over all roots of unity is zero. $\square$

Remark 7.3.11. For $\chi_A = 0$ we define $e_{g_i^l} = \frac{1}{n_i}(\sqrt[a_i]{\kappa} x_i)v_{\text{id}}$ (cf. Proof of Proposition 7.3.2). Here we have

$$e_{g_i^l} \circ e_{g_i^m} = \begin{cases} \frac{1}{n_i} x_i e_{g_i^{l+m}} & l, m, l+m \notin n_i\mathbb{Z} \\ \frac{1}{n_i} \sqrt[a_i]{\kappa} x_i e_{g_i^{l+m}} & l \in n_i\mathbb{Z} \text{ or } m \in n_i\mathbb{Z} \\ \frac{1}{n_i}(\sqrt[a_i]{\kappa})^{a_i-1} x_i e_{g_i^{l+m}} & l, m \notin n_i\mathbb{Z} \text{ and } l+m \in n_i\mathbb{Z} \end{cases}.$$

So we can calculate

$$\left(\sum_{l=0}^{n_i-1} \mathbf{e}\left[\frac{(k-1)l}{n_i}\right] (\sqrt[a_i]{\kappa} x_i)^{|G/K_i|-1} e_{g_i^l}\right)^{a'_i} = \phi(\kappa)\frac{1}{n_i} x_i^{a_i}$$

where $\phi(\kappa)$ is a complex number which can depend on κ and we always have $\phi(1) = 1$.

Example 7.3.12. We will calculate $\phi(\kappa)$ explicitly for $A = (3,3,3)$ and $G = K_1 \cong \mathbb{Z}/3\mathbb{Z} = \{\text{id}, g_1, g_1^2\}$. So we have $a'_1 = \frac{3}{|K_1/K_1|} = 3$ and $a'_i = \frac{3}{|K_1/\{\text{id}\}|} = 1$ for $i = 2,3$. So we can calculate: e.g. for $k = 2$

$$\begin{aligned}
&\left(\sum_{l=0}^{2} \mathbf{e}\left[\frac{(2-1)l}{3}\right] (\sqrt[3]{\kappa} x_1)^{1-1} e_{g_1^l}\right)^3 = \left(\frac{1}{3}\sqrt[3]{\kappa} x_1 + \mathbf{e}\left[\frac{1}{3}\right] e_{g_1} + \mathbf{e}\left[\frac{2}{3}\right] e_{g_1^2}\right)^3 \\
&= \left(\frac{1}{9}\sqrt[3]{\kappa}^2 x_1^2 + 2\mathbf{e}\left[\frac{1}{3}\right]\frac{1}{3}\sqrt[3]{\kappa} x_1 e_{g_1} + 2\mathbf{e}\left[\frac{2}{3}\right]\frac{1}{3}\sqrt[3]{\kappa} x_1 e_{g_1^2} + \mathbf{e}\left[\frac{2}{3}\right]\frac{1}{3} x_1 e_{g_1^2} + 2\mathbf{e}\left[\frac{3}{3}\right]\frac{1}{9}\kappa x_1^2\right. \\
&\quad \left. + \mathbf{e}\left[\frac{4}{3}\right]\frac{1}{3} x_1 e_{g_1^4}\right) \circ \left(\frac{1}{3}\sqrt[3]{\kappa} x_1 + \mathbf{e}\left[\frac{1}{3}\right] e_{g_1} + \mathbf{e}\left[\frac{2}{3}\right] e_{g_1^2}\right) \\
&= \left((\frac{1}{3} + \frac{2}{3}\sqrt[3]{\kappa})\frac{1}{3}\sqrt[3]{\kappa}^2 x_1^2 + (\frac{2}{3} + \frac{1}{3\sqrt[3]{\kappa}})\mathbf{e}\left[\frac{1}{3}\right]\sqrt[3]{\kappa} x_1 e_{g_1} + (\frac{2}{3} + \frac{1}{3\sqrt[3]{\kappa}})\mathbf{e}\left[\frac{2}{3}\right]\sqrt[3]{\kappa} x_1 e_{g_1^2}\right) \\
&\quad \circ \left(\frac{1}{3}\sqrt[3]{\kappa} x_1 + \mathbf{e}\left[\frac{1}{3}\right] e_{g_1} + \mathbf{e}\left[\frac{2}{3}\right] e_{g_1^2}\right) \\
&= (\frac{1}{3} + \frac{2}{3}\sqrt[3]{\kappa})\frac{1}{9}\sqrt[3]{\kappa}^3 x_1^3 + 0 + 0 + 0 + 0 \\
&\quad + (\frac{2}{3} + \frac{1}{3\sqrt[3]{\kappa}})\mathbf{e}\left[\frac{3}{3}\right]\sqrt[3]{\kappa}^2 x_1 \frac{1}{9}\kappa x_1^2 + 0 + (\frac{2}{3} + \frac{1}{3\sqrt[3]{\kappa}})\mathbf{e}\left[\frac{3}{3}\right]\sqrt[3]{\kappa} x_1 \frac{1}{9}\kappa x_1^2 + 0 \\
&= (\frac{1}{3} + \frac{2}{3}\sqrt[3]{\kappa})(\frac{1}{9}\sqrt[3]{\kappa}^3 x_1^3 + \frac{1}{9}\kappa x_1^3 + \frac{1}{9}\kappa x_1^3) = (\frac{1}{3} + \frac{2}{3}\sqrt[3]{\kappa})\frac{1}{3}\kappa x_1^3
\end{aligned}$$

So here we have $\phi(\kappa) = \frac{1}{3} + \frac{2}{3}\sqrt[3]{\kappa}$.

Definition 7.3.13. For each $i = 1, 2, 3$ and $k = 1, \dots, n_i$, we define

$$[x_{i,k}] := \frac{1}{\sqrt[a'_i]{\phi(\kappa)}} \sum_{l=0}^{n_i-1} \mathbf{e}\left[\frac{(k-1)l}{n_i}\right] (\sqrt[a_i]{\kappa} x_i)^{|G/K_i|-1} e_{g_i^l}.$$

Lemma 7.3.14. *In* $\mathrm{Jac}(f_A, G)$*, we have the following equalities*

$$a'_i [x_{i,k}]^{a'_i} = \frac{1}{|G|} a_i \kappa x_i^{a_i},$$

$$[x_{i,k_1}] \circ [x_{i,k_2}] = 0, \quad i = 1, 2, 3, \ k_1 \neq k_2.$$

Proof. The first equation is clear from the definition of $\phi(\kappa)$ and the second one is the same as in Lemma 7.3.10. $\square$

Remark 7.3.15. Note that for $q \to 0$ we have $\kappa \to 1$ and so the Definitions 7.3.9 and 7.3.13 coincide in the limit.

Lemma 7.3.16. *In the limit* $q \to 0$ *we have for all* $1 \leq k_i \leq n_i$, $i = 1, 2, 3$ *the following equalities in* $\mathrm{Jac}(f_A, G)|_{q \to 0}$

$$[x_{i,k_i}] \circ [x_{j,k_j}] = 0 \textit{ for } i \neq j.$$

Proof. In the limit we have $x_i x_j = 0 \in \mathrm{Jac}(f_A, \mathrm{id})|_{q \to 0} = \mathrm{Jac}(f_A)|_{q \to 0}$ and from Equation (7.1) $e_{g_i^l} \circ e_{g_j^m} = 0 \in \mathrm{Jac}'(f_A, G)|_{q \to 0}$ for $i \neq j$. $\square$

Proof of Theorem 7.3.6. Let us rewrite

$$(b_1, \dots, b_r) \text{ as } (b_{1,1}, \dots, b_{1,n_1}, b_{2,1}, \dots, b_{2,n_2}, b_{3,1}, \dots, b_{3,n_3}).$$

So we have $b_{i,k} = a'_i$ for $i = 1, 2, 3$, $k = 1, \dots, n_i$.

We take the basis $\{1, \frac{1}{|G|}\kappa q^{-1} x_1 x_2 x_3, [x_{i,k}]^j \mid i = 1, 2, 3 \ ; \ k = 1, \dots, n_i \ ; \ j = 1, \dots, a'_i - 1\}$. Therefore the lemmata above yield Theorem 7.3.6. $\square$

Problem 7.3.17. For future research it might also be possible to associate a Frobenius manifold to the pair (f_A, G) and the canonical primitive form ζ and show that it is isomorphic to the one constructed from the Gromov-Witten theory for an orbifold projective line with at most r orbifold points.

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