Haag Duality and Jones-Kosaki-Longo Index

in

Kitaev's Quantum Double Models for Finite Abelian Groups

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Kurzzusammenfassung

Inhalt dieser Dissertation ist die Untersuchung analytischer Aspekte des thermodynamischen Limes des Kitaev Quantum Double Modells für endliche abelsche Gruppen. Es wird gezeigt, dass in der GNS Darstellung des eindeutig bestimmten translationsinvarianten Grundzustands die von Neumann Algebren der in Kegeln lokalisierten Operatoren die Eigenschaft der Haag-Dualität erfüllen. Unter Zuhilfenahme der im Verlauf des Beweises für die Haag Dualität entwickelten Techniken wird der Jones-Kosaki-Longo Index für bestimmte Inklusionen von von Neumann Algebren in der Grundzustandsdarstellung berechnet. Dies erlaubt die vollständige Charakterisierung der Superauswahlsektoren gemäß einer kürzlich entwickelten Klassifizierung solcher fundamentalen Ladungen in zweidimensionalen Quantengittersystemen. Letztere sind Äquivalenzklassen von Darstellungen der quasilokalen Algebra, welche einer DHR-artigen Auswahlregel genügen. Im Zuge dessen lässt sich eine Version der Split-Eigenschaft für von Neumann Algebren von in Paaren disjunkter Kegel lokalisierter Operatoren zeigen.Der für die Klassifizierung der Sektoren berechnete Jones-Kosaki-Longo Index ist im Quantum Double Modell für endliche abelsche Gruppen durch die totale Quantendimension der modularen Tensorkategorie gegeben, welche die lokale Anregungsstruktur charakterisiert. Darüber hinaus kann der Index als Dimension des Kodierungsraumes eines Verfahrens für Geheimnisteilung interpretiert werden. Das steht im Zusammenhang mit der operationellen Interpretation einer relativen Entropie der Inklusion, welche gleich dem Logarithmus des Index ist, als die zusätzliche Information, welche mithilfe den zusätzlichen Operationen der größeren von Neumann Algebra der Inklusion in die Kodierungszustände chiffriert werden kann. Als Resultat erhält man eine Struktur, welche auch in der Analyse der Quantum Double Modelle für endliche Systemgrößen zu finden ist.

Schlagworte: Quantum Double Modell, Superauswahlsektoren, Index Theorie

Abstract

Content of this thesis is the study of analytic aspects of the thermodynamic limit of Kitaev's quantum double models for finite abelian groups. We prove that in the GNS representation of the unique translationally invariant ground state the von Neumann algebras of observables localised in cones satisfy Haag duality. The techniques developed in the proof of Haag duality are then used to derive the Jones-Kosaki-Longo index of certain inclusions of von Neumann algebras in the representation of the translationally invariant ground state. By a recent result on the classification of charges in two-dimensional quantum many body systems, this allows to fully characterise the superselection sectors obtained from a DHR like selection criterion. In the course of this we give an explicit proof of a version of the split property, called approximate split property, for von Neumann algebras associated with pairs of disjoint cones. The Jones-Kosaki-Longo index derived for the classification of the superselection sectors is equal to the total quantum dimension of modular tensor category characterising the local excitations. Moreover, it can be interpreted as dimension of the code space of a secret sharing scheme and its logarithm is equal to a relative entropy of the inclusion giving it an operational interpretation. This resembles a structure which is also found in the analysis of the quantum double models in finite system sizes.

Keywords: Quantum Double Model, Superselection Sectors, Index Theory

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Publications

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[FNO17] Leander Fiedler, Pieter Naaijkens and Tobias J Osborne. 'Jones index, secret sharing and total quantum dimension'. *New Journal of Physics* 19.2 (Feb. 2017), p. 023039. doi: 10.1088/1367-2630/aa5c0c. arXiv: 1608.02618.

Introduction

In the past 40 years, systems that exhibit phase transitions not covered by Landau's theory of symmetry breaking have gained broad interest in the fields of condensed matter physics, quantum information and mathematical physics. In the theory of symmetry breaking, phase transitions in a quantum spin system are explained by the transition from a disordered, symmetric phase to an ordered phase in which the the state under consideration no longer obeys the symmetry of the system's dynamics [Lan08]. The phase transition can be characterised by a local order parameter, such as the magnetisation or the temperature, and a phase transition occurs if the order parameter becomes non-analyitc at some value of the system parameters [Rue69].

In contrast, exotic systems falling outside the paradigm of symmetry breaking are characterised by stability of the ground state against local perturbations of the Hamiltonian and phase transitions of such systems are characterised by non-local order parameters. Another remarkable property of these systems outside framework of symmetry breaking is that the degeneracy of the ground state depends on the topology of the manifold in which the system is embedded. Typical examples for such systems are the fractional quantum hall liquids [KL87; Hal88] and spin liquids [KT73; WWZ89; Kit03; LW05], which can break time and parity symmetry, and can be gapped or gapless.

Topological Order

Several definitions of topological order were introduced which are supposed to capture essential features of such systems. For example, the topological quantum order conditions [BHV06; NO09; BHM10; BH11] and variants thereof [RS15; JP13] address the stability against local perturbations. These conditions are influenced by intuition from quantum error correction (c.f. [KL95; BNS98]) and in fact a large class of models for topologically ordered systems, or rather states, in this sense are provided by surface codes and other topological quantum codes [Kit97; Den+02; BM06; CN08; KL09; Bom13]. In addition to the stability against local perturbations these states exhibit a degeneracy depending only on the topology in which the system is embedded. They are degenerate in the sense that for a fixed system there are multiple, distinct states which are not distinguishable by local operations but only by operations that affect large parts of the system. In fact, these states are the code states of a stabiliser code and the latter operations are given by the logical operators of this code [Kit97; Got97]. This allows for fault tolerant encoding of quantum information, hence makes them promising candidates for quantum memories. Moreover, by using the stability of these non-local operators against perturbations, these models allow for quantum computation [Den+02; KL09; Bom13] or even universal quantum computation, if the models underlying the constructions are complex enough [Moc03; Moc04]. These states also occur as ground states of local, gapped Hamiltonians, where the local terms of the Hamiltonians correspond to the stabilisers of the error correcting code [Kit03; BM08; Bre14].

Other closely related approaches follow an intuition that is derived from the theory of phase transitions by defining quantum phases as equivalence classes of ground states of homotopy equivalent Hamiltonians [CGW10]. More precisely, a quantum phase is given by those ground states whose Hamiltonians can be connected by a smooth path of Hamiltonians such that no phase transition occurs along the path. A quantum phase transition occurs, if at some point along a smooth path of Hamiltonians, the expectation values of local observables in the Hamiltonian's ground state exhibit a singularity in the thermodynamic limit of the system. As product states do not contain any quantum correlations between different particles, they are canonical representations of the class of topologically trivial states. Topologically ordered states are then those states, which cannot be connected to topological trivial states with a smooth path of Hamiltonians without undergoing a phase transition. If the Hamiltonian of one ground state is gapped, then this means that the ground state of another gapped Hamiltonian is in the same phase if they are connected by a smooth path of gapped Hamiltonians. In this case the ground state degeneracy is preserved when moving the system along this path. In a sense this approach extends Landau's theory of phase transition by symmetry breaking: The topological trivial phase can be divided further into smaller equivalence classes if one requires that symmetries are preserved when changing the system. This results in the so-called symmetry protected topological orders [CGW10; Che+13; EN14; LL13].

Yet another approach to define topological phases, and hence topological order, is by equivalence under finite depth local unitary circuits [CGW10]. This has the advantage of being more accessible for numerical investigations [Osb06; Osb07]. This approach is motivated by the observation that for a given smooth path of gapped local Hamiltonians one can construct a one-parameter family of unitary operators that connects the ground space of the Hamiltonians along the path and which preserves locality of operators up to an arbitrarily small error [HW05; Bac+11]. This shows that, as long the gap stays open, small enough local perturbations of the local Hamiltonian does not change the physics of the ground space. This is complemented by the observation that small enough local perturbations of gapped, local commuting Hamiltonians [BHM10; BH11] do not close the gap. The quasi-adiabatic continuation and the finite depth local unitary quantum circuits share the property that they spread correlations through the system only linearly w.r.t. the parameter of the continuation and the depth of the circuit, respectively [BHV06; HW05; Bac+11; BB15]. This supplies the intuition that topologically ordered states are long-range entangled in contrast to topologically trivial states which are short-ranged entangled but can contain long-range classical correlations. Several further approaches to topological order suitable for purposes of numerical investigations use, among others, projected entangled pair states [Sch+12], matrix product operators [Şah+14] and more general, tensor networks [CV13].

Classifying Topological Order

There are several immediate questions that arise. First, to which extent are the above approaches equivalent? Clearly they are related and intuitively capture much of the same physics. Up to now this seems not to be entirely answered, though some partial results are available [BHM10; Osb07]. Another and related question is, how can one classify topologically ordered phases? Does a complete classification scheme even exist? The nature of topological order indicates that phase is a property of the states alone. If the state under consideration is the ground state of a local, gapped Hamiltonian then its topological order should not depend on the details of the high energy spectrum of the Hamiltonian by the quasi-adiabatic continuation [HW05]. Moreover, since we are interested in topological order only for pure states, we can always interpolate between different Hamiltonians with the same ground states and without changing the ground space. Thus, it is not important which Hamiltonian we are considering, as long as its ground state space describe the same ground states. This suggests that quantum phases can be understood as equivalence classes of only states under a suitable equivalence relation. The underlying equivalence relations then determine how well we can distinguish different quantum phases. A classification scheme then consists in finding invariants associated to these classes which can be calculated from any representative from the class of states. Such a scheme is complete if there exists a one-to-one mapping between invariants and phases.

For one-dimension quantum spin systems it was shown that gapped Hamiltonians of finite interaction range only give rise to topological trivial phases [CGW11]. Thus the only interesting phases of one-dimensional gapped are the symmetry protected phases. Related to this, there are are classification schemes in one-dimensional systems on the level of local unitary dynamics [Gro+09; Ced+15] and for fermionic systems [Kit09].

In two-dimensional quantum spin systems, topological order is generally associated with the occurrence of anyons, i.e. particles with statistics different form the usual fermi or bose statistics. This is illustrated by Kiteav's quantum double models[Kit03] and by the string-net models [LW05]. It is an open question what the correspondence between gapped Hamiltonians in two spatial dimensions, and modular tensor categories is [Haa16], but it is expected that topological order in two dimensions is fully characterised by modular tensor categories [BN13].

One candidate for and invariant in two-dimensional systems is the topological entanglement entropy [KP06; LW06]. It is derived only from the state under consideration, but suffers certain shortcomings due Bravyi's counter example (c.f. the introduction of [Haa16]). However, it is stable under the quasi-adiabatic continuation [Mar+14]. Unfortunately, it does not provide a complete classification: For systems satisfying an area law it corresponds to a constant correction to the entanglement entropy, which, for the quantum double models and the string-net models (and other anyon models), coincides with the logarithm of the total quantum dimension of the model's underlying modular tensor category. However, the total quantum dimension does not uniquely identify a modular tensor category. Nevertheless, non-zero topological entanglement entropy is considered as a good indicator for the presence of topological order in a state. Some of its important properties are that it is considered to be stable under local perturbations that are smaller than the region it is derived from [KP06; LW05] (c.f. [Kim12]), it can be extended to systems with defects [Bro+13] and it is equivalent to an optimal sharing rate in a secret sharing scheme [KFM16].

Another approach to find invariants for 2-dimensional systems aims at computing the topological S-matrix from the ground state [ZGV14; Haa16]. This matrix is stable against local unitary transformations [Haa16], and coincides with the S-matrix of the underlying modular tensor category in the case of the toric code. This suggests that it allows to recover at least parts of the anyonic structure encoded in the state. This invariant is not complete as well, since the S-matrix itself does not uniquely determine a modular tensor category. In fact even the classification of the latter is far from being solved [RSW09].

Beyond Finite Systems Sizes

Most of the approaches mentioned in the previous sections have in common that they are taken in systems of finite size, i.e. finite volume and finite particle numbers. The account for the thermodynamic limit is then given by analysing the scaling of the quantities of interest in the limit of large volumes and particle numbers. This raises the issue of intricate and often not very precise notations to keep track of the system size. Furthermore, some assumptions on the systems, such as the the gap between the ground state energy and the higher excitations, or the observation whether a phase transition occurs, make only sense in the thermodynamic limit. An indicator that this could be problematic is given by the undecidability whether a given Hamiltonian has a gap on the basis of its local data [CPW15a; CPW15b]. In addition, there exist models for which deciding the topological order of the ground state depends on the size of the system [Bau+15]. Another example is the definition of the topological entanglement entropy, or, more precisely, the assumption of an area law. It asserts that the von Neumann entropy of the ground state when reduced to a finite region, scales with the size of the boundary of that region plus a correction that, asymptotically, i.e. in the limit of large particle numbers and large volumes, depends linearly on the system size (c.f. [KP06; LW06]).

For purposes of numerical computations, however, the thermodynamic limit is impractical and for this reason it is necessary to have analytic results in systems of finite size. Nevertheless, topological order is a global feature that does not depend on the small scale details of the states. This raises the question whether working directly in the thermodynamic limit allows one to gain a deeper understanding of this phenomena. In other words, is it possible to understand general features of topological order by abstracting possibly unnecessary details way? Can this help us to understand possible classifications of quantum phases? An advantage of transitioning to the limit of infinite volumes and particle numbers is that the mathematical theory describing this setting is already developed [BR96]. Furthermore, some difficulties arising in the discussion of finite system sizes become much simpler. For instance, relative notions to the system size become unnecessary when trying to define local operations: an observable acts locally, if it acts only on a finite number of particles.

Several methods of analysing quantum many body systems have been developed and generalised to this setting that are useful to investigate topological order, such as Lieb-Robinson bounds [NS10] and Hasting's quasi-adiabatic continuation [Bac+11]. Systems that have been analysed in this mathematical framework include the toric code [Naa11; Naa12b; Naa13a], product vacua with boundary states [BN12; Bac+15] and Kitaev's quantum double models for finite abelian groups [FN15; CNN16]. An important aspect of opening up to a different mathematical field is that it allows us to import new mathematical tools that are potentially helpful in the understanding of topological order. For instance, implementations of the program by Doplicher, Haag and Roberts (DHR) for determining the superselection sectors, or elementary charges, of a theory were used for certain two-dimensional systems with quantum symmetries [SV93], for the toric code [Naa11; Naa13a] and Kitaev's quantum double models for finite abelian groups [FN15]. Reference [Naa13a] also provides a general scheme for the classification of superselection sectors in two-dimensional lattice systems via the Jones-Kosaki-Longo index. In the case of the toric code this index is equal to the square of the total quantum dimension of the model. This suggests that it should be related to the topological entanglement entropy. Indeed, the Jones-Kosaki-Longo index is related to a secret sharing scheme, and is connected to a relative entropy [FNO17]. Hence, the index has a structure that is comparable with the topological entanglement entropy [KFM16].

Structure of the Thesis

Central to this thesis is the development of certain technical tools to analyse the structure of elementary charges in the thermodynamic limit of Kitaev's quantum double model for finite abelian groups. The quantum double models are a class of toy models introduced by Kitaev [Kit97; Kit03] and they are stabiliser error correction codes implemented on two-dimensional surfaces. The class of models is parametrised by finite groups, and the dynamics of this model is given by a four body Hamiltonian whose ground state space dimension depends only on the topology of embedding surface. They were introduced as examples for systems that can implement fault tolerant quantum computation. However, this is possible only if the groups are complex enough [Moc03; Moc04], and if the systems are kept at zero temperature. At finite temperatures the thermal states are no longer topologically ordered [Has11], for the reason that moving excitations does not cost energy. The latter makes error correction potentially difficult, however, there are methods using classical post-processing which can compensate for this [Den+02]. Nevertheless, Kitaev's quantum double models are considered very useful as toy models for topologically ordered systems; they are analytically solvable, the excitation structure is well understood [Kit03; BM08] and they exhibit many properties which are considered typical for topologically ordered systems [CC07; Bro+11].

In Chapter 1 we introduce the mathematical tools used to treat the thermodynamic limit of quantum spin systems. More precisely, we give a brief introduction in fundamental notions of the theory of C^*-algebras and von Neumann algebras. This is followed by a discussion of the index theory of inclusions of von Neumann algebras by Jones and Kosaki. As an example which we also will use in the discussion of the superselection sectors of the quantum double model, we discuss crossed products of von Neumann algebras with finite groups and review how the index theory applies there. This is followed by a discussion on how quantum spin systems are treated in the thermodynamic limit using the theory of C^*-algebras. In detail, this includes the thermodynamic limit as an inductive limit of C^*-algebras, existence of dynamics generated by local Hamiltonians and an algebraic characterisation of ground states. The last part is devoted to Drinfel'd's quantum double of finite groups. We give a review of the construction of Drinfel'd's quantum double and its representation structure.

Chapter 2 starts with a review of Kitaev's quantum double models for finite groups on the two-dimensional square lattice. The purpose of this chapter is to give a largely self-contained, technical introduction to the model with emphasis on the structure of the ribbon operators. We start with an introduction of the fundamental geometric notions such as triangles and ribbons, where we largely follow the notation in reference [BM08]. This is followed by the definition of ribbon operators which are then used to set up the Hamiltonian of the quantum double model, and the local charge projections. The properties of the local excitations above the ground state of the Kitaev Hamiltonian are determined by the commutation relations of the ribbon operators. For this reason we elaborate on the the commutation relations of ribbon operators for different configurations of ribbons. The last two sections of this chapter contains a short review of this discussion in the case that the model's underlying finite group is abelian, and a remark on the existence and uniqueness of the translationally invariant ground state of the Hamiltonian in the thermodynamic limit.

Having established the basic basic notions of the theory of operator algebras and of the quantum double model we proceed in Chapter 3 to the proof of Haag duality for cone algebras in the GNS representation of the translationally invariant ground state. These results are also published in [FN15]. Haag duality for cone algebras is a technical property which says that observables which commute with the observables localised in a cone are exactly given by the observables located in the complement of the cone. While this is a trivial statement in for finite regions, it is not obvious that it also holds for any infinite regions. In fact, it is not true for the observable algebras associated to unions of disjoint cones. Haag duality is an important tool in the analysis of the superselection sectors of the theory. While it is possible to construct representatives of some sectors explicitly without using Haag duality, it is necessary to show that the properties of the explicit representatives carry over to the whole sectors. In addition of its usefulness in the analysis of the properties of the superselection sectors, Haag duality is needed to show the completeness of the sectors.

In the first part of Chapter 4 we review a DHR-like selection criterion for superselection sectors and the construction of representatives of these sectors as presented in [FN15]. Sectors containing the representatives are shown to form a modular tensor category which is isomorphic to the modular tensor category of finite dimensional representations of Drinfel'd's quantum double of the underlying group. Adapting the construction from [Naa13a], we show in the second part that, in the ground state representation, a certain inclusion of von Neumann algebras associated to the union of disjoint cones is irreducible and related to a crossed product construction with a finite abelian group. The index of this inclusion is equal to the total quantum dimension of the modular tensor category which describes the excitation structure obtained from the ribbon operators, and can be used to show that the representatives constructed explicitly already describe all superselection sectors. Using the same techniques, we point out a version of the split property for von Neumann algebras associated to unions of disjoint and sufficiently separated cones. In the last part, we analyse the irreducible inclusions obtained in the previous sections, and point out a downward basic construction. We show that in our case the minimal Stinespring dilation of the conditional expectation associated with the original inclusion takes a very simple form, and we relate the inclusion to an error correction condition on the level of von Neumann algebras. After this we discuss how the index relates to the code space of a secret sharing scheme, and review an operational interpretation in terms of relative entropies. This structure turns out to be parallel to what is found in the finite dimensional analysis of the quantum double models. There the total quantum dimension can be obtained from the topological entanglement entropy and it can be interpreted as the maximal size of the code of a secret sharing scheme as well. However, the details of the secret sharing scheme used in this interpretation are different from ours. Nevertheless, this suggests that the underlying concept of the total quantum dimension for the thermodynamic limit is the same as for finite system sizes.

1 Preliminaries

In the following we give an introduction to most of the terminologies and tools used in this thesis. Emphasis is put on a brief and comprehensible introduction of the material. For details and further discussion relevant literature is pointed out. Very often we use notations and results from this chapter in the subsequent chapters without referring to them explicitly. To support readability there is an index and a notational index in the back matter pointing to most of the terminology used.

We give a brief overview of basic facts of the theory of C^*-algebras, including the GNS construction and the representation theory. This is complemented with a discussion on von Neumann algebras and the Murray-von Neumann classification of factors. We also give a short introduction into Jone's index classification of type II_1 subfactors [Jon83] and its generalisation to arbitrary factors by Kosaki and Longo [Kos86; Lon89; Lon91]. As an example we consider the index of a factor embedded in its crossed product with a finite group [KR97]. Following this, we review some notations and results about quantum spin systems. In particular we discuss how the thermodynamic limit is described using the language of operator algebras, how local Hamiltonians of such systems give rise to dynamics for the case of finite ranged interactions, and how ground states are described in this abstract setting [BR96]. As this thesis is mainly concerned with Kitaev's quantum double model for finite groups [Kit03] we start by introducing Drinfel'd's quantum double of a finite group [Dri88; Gou93]. Finally we discuss its representation structure, which plays a major role in the classification of excitations of the quantum double model.

1.1 C*-Algebras

This section is devoted to a very brief overview on abstract C*-algebras. Detailed account of the theory of operator algebras can be found, for instance, in the books of Takesaki [Tak79; Tak03a; Tak03b], the book of Sakai [Sak71], and, of course, Bratteli and Robinson [BR96]. We start with the very definition of a C*-algebra.

Definition 1.1.1:
A Banach space $\mathcal{A}$ over $\mathbb{C}$ is called a Banach algebra *if there exists a map $\mathcal{A} \times \mathcal{A} \ni (a, b) \mapsto ab \in \mathcal{A}$ (multiplication) which is associative, bilinear, distributive, and satisfies*

$$\forall a, b \in \mathcal{A} : \|ab\| \leqslant \|a\|\|b\|.$$

A Banach algebra $\mathcal{A}$ is called a Banach $*$-algebra[1] *if there exists an anti-linear map $\mathcal{A} \ni a \mapsto a^* \in \mathcal{A}$ (involution) that satisfies*

$$\begin{aligned} \forall a \in \mathcal{A} &: (a^*)^* = a \\ \forall a, b \in \mathcal{A} &: (ab)^* = b^* a^*. \end{aligned}$$

A C*-algebra *is a Banach-$*$-algebra where the involution satisfies the additional property*

$$\forall a \in \mathcal{A} : \|a^* a\| = \|a\|^2.$$

The topology on a C*-algebra given by its norm is usually referred to as the *uniform topology*. A C*-algebra does not necessarily have a unit, but it always possesses an approximate unit. In addition, it is always possible to extend the algebra to a C*-algebra that has a unit. A C*-algebra with unit is often called *unital*, and from now on we always assume that a C*-algebra is unital.

First we discuss some special cases of C*-algebras. The most important class of examples is given by the bounded operators $\mathcal{B}(\mathcal{H})$, and the closed $*$-subalgebras thereof, over some Hilbert space $\mathcal{H}$. We will see shortly that those already exhausts all C*-algebras. Another important class of examples is given by $C_\infty(M)$, the continuous functions vanishing at infinity over some locally compact space M. Topologised by the uniform norm $\|\cdot\|_\infty$, equipped with a multiplication given the point-wise multiplication and an involution given by the point-wise complex conjugation, this space becomes a commutative C*-algebra. In fact, one can show that every commutative C*-algebra is of this form (see e.g. [Tak79, Theorem I.4.4]).

[1] For brevity we sometimes call $\mathcal{A}$ just $*$-*algebra* and assume that it is appropriately topologised by a norm.

In quantum mechanics the description of a system requires the specification states describing the preparation of the system and observables modelling the measurements[2]. The observables are given by bounded operators on some Hilbert space $\mathcal{H}$ and the physical states of the system usually by normal states, i.e. density matrices[3] in $\mathcal{B}(\mathcal{H})$. Pure states are given by the 1-dimensional projections and mixed states by positive, affine combinations thereof. The expectation value of an operator $A \in \mathcal{B}(\mathcal{H})$ w.r.t. the state $\sigma \in \mathcal{B}(\mathcal{H})$ is given by the trace $\mathrm{tr}(\sigma A)$. Writing with slight abuse of notation $\sigma(A) \coloneqq \mathrm{tr}(\sigma A)$, this defines a continuous, linear, positive and normalised functional on $\mathcal{B}(\mathcal{H})$. As $\mathcal{B}(\mathcal{H})$ is a C^*-algebra these are taken as the defining properties for states on C^*-algebras.

Definition 1.1.2:
Let $\mathcal{A}$ be a C^*-algebra. A linear function $\omega : \mathcal{A} \to \mathbb{C}$ is called a state, *if*

- $\omega(\mathbb{1}) = 1$
- $\forall A \in \mathcal{A}$: $\omega(A^*A) \geqslant 0$.

Some of the important properties of states are summarised in the following.

Proposition 1.1.3:
A state ω on a C^*-algebras satisfies a Cauchy-Schwarz inequality, that is, for all $A, B \in \mathcal{A}$ we have

$$|\omega(A^*B)|^2 \leqslant \omega(A^*A)\omega(B^*B).$$

In addition, we have that $\omega(B^*A) = \overline{\omega(A^*B)}$ and $|\omega(A)|^2 \leqslant \omega(A^*A)\|\omega\|$ for all $A, B \in \mathcal{A}$. The norm of the state ω is given by $\|\omega\| = \omega(1)$.

The last statement implies that a state ω on a C^*-algebra $\mathcal{A}$ is continuous, hence states are elements of the positive cone of the dual $\mathcal{A}^*$ of $\mathcal{A}$. For the proof of the above statements, see for instance [Tak79]. A state ϕ on a C^*-algebra $\mathcal{A}$ is called *tracial*, if for all $A \in \mathcal{A}$ we have $\phi(AA^*) = \phi(A^*A)$. A state ϕ on $\mathcal{A}$ is called *faithful*, if $\phi(A^*A) = 0$ implies $A = 0$ for all $A \in \mathcal{A}$.

[2] This is roughly the perspective taken by quantum information theory. For introductions, see e.g. [NC09; Hol13; Key02]

[3] This is of course not true for so-called singular states such as the "eigenstates" of the position operator on $\mathcal{L}^2(\mathbb{R})$. However, one can discuss to which degree one can actually prepare such a state in the lab.

1.1.1 GNS Theorem

An important part of the theory of operator algebras is the representation theory of C^*-algebras. As mentioned earlier, C^*-algebras are, in a sense made precise below, algebras of bounded operators on a Hilbert space. The fundamental ingredient for the representation theory is the so-called *GNS-construction*. A sketch of this construction and a precise formulation of the above statements are the content of this section. We start with defining representations of C^*-algebras.

Definition 1.1.4:
Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{H}$ a Hilbert space. A representation of $\mathcal{A}$ on $\mathcal{H}$ *is a $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$.*

A *$*$-homomorphism* π between two $*$-algebras $\mathcal{A}$ and $\mathcal{B}$ is a homomorphism that preserves the involution, i.e. $\pi(ab^* + \lambda c) = \pi(a)\pi(b)^* + \lambda\pi(c)$ for all $a, b, c \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. It is not difficult to show that a $*$-homomorphism π between two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ is always continuous in the uniform topology, since one can show that $\|\pi(A)\| \leqslant \|A\|$ for all $A \in \mathcal{A}$ (c.f. [Tak79, Chapter I]). If π is a $*$-isomorphism, then in fact $\|\pi(A)\| = \|A\|$ for all $A \in \mathcal{A}$. We call a representation of a C^*-algebra $\mathcal{A}$ *faithful* if, for all $A \in \mathcal{A}$, $\pi(A) = 0$ implies $A = 0$. If $\mathcal{A}$ is a C^*-algebra acting on a Hilbert space $\mathcal{H}$, we call a vector $\psi \in \mathcal{H}$ *cyclic* for $\mathcal{A}$, if the linear space $\mathcal{A}\psi$ is dense in $\mathcal{H}$.

Let $\mathcal{A}$ be a C^*-algebra and $\omega \in \mathcal{A}^*$ be a state. The GNS construction, named after Gelfand, Naimark and Segal, shows how to construct a representation of a C^*-algebra $\mathcal{A}$ from ω. We just give a sketch of the construction. The details can be found in any textbook about operator algebras (e.g. [Tak79]). Denote by

$$\mathcal{J}_\omega := \{A \in \mathcal{A} \mid \omega(A^*A) = 0\}$$

the left kernel of ω. With the Cauchy-Schwarz inequality it is easy to see that $\mathcal{J}_\omega$ is a left ideal in $\mathcal{A}$. Now the map $\mathcal{A} \times \mathcal{A} \ni (A, B) \mapsto \omega(B^*A) \in \mathbb{C}$ defines a degenerate inner product on $\mathcal{A}$. One can check, that on the quotient space $\mathcal{A}/\mathcal{J}_\Omega$ this gives rise to a non-degenerate inner product

$$\langle [A], [B] \rangle_\omega := \omega(B^*A),$$

where $[A], [B] \in \mathcal{A}/\mathcal{J}_\omega$ are equivalence classes with representatives $A, B \in \mathcal{A}$. Hence, $\mathcal{A}/\mathcal{J}_\omega$ endowed with this, now non-degenerate, inner product becomes a pre-Hilbert space, and we denote its completion by $\mathcal{H}_\omega$. The inner product on $\mathcal{H}_\omega$

will now be denoted by $(\cdot,\cdot)$. On this space we can now define a representation π_ω of $\mathcal{A}$ by

$$\pi_\omega(A)[B] := [AB]$$

for all $A, B \in \mathcal{A}$. One can show that for each $A \in \mathcal{A}$ the linear operator $\pi_\omega(A)$ is bounded and can be extended to a bounded linear operator on $\mathcal{H}_\omega$. Furthermore, π_ω is a $*$-homomorphism from $\mathcal{A}$ into $\mathcal{B}(\mathcal{H}_\omega)$. Next one shows that ω extends to a bounded linear functional on $\mathcal{H}_\omega$, and hence there is a vector $\Omega \in \mathcal{H}$ such that for all $A \in \mathcal{A}$ we have that $\omega(A) = (\Omega, [A])$. This vector has the property $[A] = \pi_\omega(A)\Omega_\omega$ for all $A \in \mathcal{A}$, and hence Ω is cyclic for $\pi_\omega(\mathcal{A})$. In addition the vector Ω implements ω in the following sense:

$$\omega(A) = (\Omega, \pi_\omega(A)\Omega).$$

This shows that a every state on a C^*-algebras gives rise to a representation of this C^*-algebra as bounded linear operators on a Hilbert space.

Theorem 1.1.5:
Let ω be a state on a C^*-algebra $\mathcal{A}$. Then there exists a Hilbert space $\mathcal{H}_\omega$, a representation π_ω of $\mathcal{A}$ on $\mathcal{H}_\omega$, and a state $\Omega \in \mathcal{H}_\omega$ such that the following hold:

- *Ω is cyclic for $\pi_\omega(\mathcal{A})$.*
- *$\forall A \in \mathcal{A}$: $\omega(A) = \langle \Omega, \pi_\omega(A)\Omega \rangle$.*

Furthermore, if $\mathcal{H}$ is another Hilbert space and π a representation of $\mathcal{A}$ on $\mathcal{H}$, and $\psi \in \mathcal{H}$ a vector fulfilling the above properties, then there exists a unitary operator $U : \mathcal{H} \to \mathcal{H}_\omega$ such that $U\psi = \Omega$ and $U\pi(A) = \pi_\omega(A)U$ for all $A \in \mathcal{A}$.

For the uniqueness, consider another representation $(\pi'_\omega, \mathcal{H}'_\omega, \Omega')$ with the same properties as $(\pi_\omega, \mathcal{H}_\omega, \Omega)$. Setting

$$U\pi'_\omega(A)\Omega' := \pi_\omega(A)\Omega$$

for $A \in \mathcal{A}$, defines an isometry from $\pi'_\omega(A)\Omega'$ to $\pi_\omega(\mathcal{A})$, that can be extended to a unitary from $\mathcal{H}'_\omega$ to $\mathcal{H}_\omega$.

Definition 1.1.6:
The triple $(\pi_\omega, \mathcal{H}_\omega, \Omega)$ obtained from the GNS-construction is referred to as the GNS-representation*, or the* cyclic representation of $\mathcal{A}$ induced by ω*.*

The GNS theorem can then be used to show that every C^*-algebra is isomorphic to a $*$-subalgebra of the bounded operators of some Hilbert space ([Tak79]).

Theorem 1.1.7:
Let $\mathcal{A}$ be a C^*-algebra. Then there exists a Hilbert space $\mathcal{H}$ and a faithful representation π of $\mathcal{A}$ on $\mathcal{H}$.

The proof essentially consists in first showing that every representation of $\mathcal{A}$ can be written as a direct sum of cyclic representations. Then one uses the fact that for any non-zero positive operator $A \in \mathcal{A}$ there exists a state $\omega \in \mathcal{A}^*$ such that $\omega(A) \neq 0$ to construct a faithful representation by summing up over cyclic representations induced by all states on $\mathcal{A}$.

1.1.2 Von Neumann Algebras

In quantum physics, many problems can be formulated by the problem of determining the spectrum, and with it the spectral projections, of a self-adjoint, possibly unbounded operator H on some Hilbert space $\mathcal{H}$. The spectrum $\mathrm{spec}(H)$ can be interpreted as the possible outcome of a measurement the observable H in the state given by a unit vector $\psi \in \mathcal{H}$ in which the system under consideration is prepared. In general, an observable can be described by a positive operator valued measure (POVM) E, which is a map from the sigma algebra $\Omega(X)$ of some measurable space X to bounded positive operators on $\mathcal{H}$ such certain conditions are fulfilled[4]. Given that the system under consideration is prepared in the (pure) state described by a unit vector $\psi \in \mathcal{H}$, the probability of measuring an outcome contained in the set $M \in \Omega(X)$ with the observable E is given by $p_\psi(M) = \langle \psi, E(M)\psi \rangle$. A special case of this are projection valued measures (PVM), where the image of E are projections. The spectral projections $(P_M)_{M \subset \mathrm{spec}(H)}$ precisely forms such a PVM, and the probability of measuring an outcome contained in $M \subset \mathrm{specc}(H)$ if the system is prepared in the state ϕ is then $p_\psi(M) = \langle \psi, P_M \psi \rangle$. Hence, the operator H defines an observable in this sense and its spectral projections provide us with the probabilities of measurement outcomes.

If we assume for a moment that H is bounded and an element of a C^*-algebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ meant to contain all available observables of the physical system under consideration, then it is in general not true that the spectral projections of $\mathcal{H}$ are

[4]For an introduction of POVMs and a more detailed discussion see e.g. [Key02].

contained in $\mathcal{A}$, i.e. they are not physical available observables. This problem can be solved by considering von Neumann algebras[5].

In order to define von Neumann algebras, we first define commutants. Given a C^*-algebra $\mathcal{A}$ acting on a Hilbert space $\mathcal{H}$ (i.e. it is a $*$subalgebra of $\mathcal{B}(\mathcal{H})$), define the *commutant* of $\mathcal{A}$ in $\mathcal{B}(\mathcal{H})$ as

$$\mathcal{A}' := \{B \in \mathcal{B}(\mathcal{H}) \,|\, \forall A \in \mathcal{A} : [A, B] = 0\}.$$

We write $\mathcal{A}'' = (\mathcal{A}')'$, $\mathcal{A}''' = (\mathcal{A}'')'$ and so on. Note that $\mathcal{A} \subset \mathcal{A}''$, $\mathcal{A}' = \mathcal{A}'''$, and $\mathcal{A}'' = \mathcal{A}''''$, etc. A von Neumann algebra is then defined in the following way.

Definition 1.1.8:
Let $\mathfrak{M}$ be a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ and $\mathcal{H}$ some Hilbert space. Then $\mathfrak{M}$ is called a von Neumann algebra *if*

$$\mathfrak{M}'' = \mathfrak{M}.$$

A von Neumann algebra always has a unit [Tak79]. We say that a $*$-subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ is *non-degenerate*, if the closure of the space $\mathcal{A}\mathcal{H}$ is $\mathcal{H}$. It turns out that von Neumann algebras are closed in a series of topologies [Tak79; BR96]. Two of them are relevant for our discussion. Let $\mathcal{H}$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. The *weak operator topology* on $\mathcal{B}(\mathcal{H})$ is the locally convex topology induced by the family of seminorms

$$p_{\phi,\psi}(\,\cdot\,) := |\langle \phi, \,\cdot\, \psi \rangle|, \quad \phi, \psi \in \mathcal{H}.$$

The *strong operator topology* on $\mathcal{B}(\mathcal{H})$ is the locally convex topology induced by the family of seminorms

$$p_{\psi}(\,\cdot\,) := \|\,\cdot\,\psi\|, \quad \psi \in \mathcal{H}.$$

The weak operator topology is coarser than the strong operator topology, and the strong operator topology is coarser than the uniform topology. The following theorem, von Neumann's double commutant theorem, states that von Neumann algebras are characterised as non-degenerate $*$-subalgebras which are closed in the above topologies [BR96].

[5] If H is unbounded, then the spectral projections of H are contained in a von Neumann algebra $\mathfrak{M}$, if H is *affiliated* to $\mathfrak{M}$ [KR97]. This requires a von Neumann algebra as well.

Theorem 1.1.9:
Let $\mathfrak{M} \subset \mathcal{B}(\mathcal{H})$ be a non-degenerate $*$-subalgebra. Then the following statements are equivalent:

- *$\mathfrak{M}$ is a von Neumann algebra.*
- *$\mathfrak{M}$ is closed in the weak operator topology.*
- *$\mathfrak{M}$ is closed in the strong operator topology.*

Let $\mathfrak{M}$ be a von Neumann algebra on some Hilbert space $\mathcal{H}$, $A \in \mathfrak{M}$ is a self-adjoint operator, and $B \in \mathfrak{M}'$. Then B commutes also with the spectral projections of A, and hence these are also contained in $\mathfrak{M}$. As every element A in a C^*-algebra can be written as $A = A_R + iA_I$ with self-adjoint operators A_R and A_I in that algebra, this implies that the projections in $\mathfrak{M}$ span a norm-dense subspace of $\mathfrak{M}$.

A von Neumann algebra $\mathfrak{M}$ with trivial centre is called a *factor*, i.e. $\mathfrak{M}$ is a factor if

$$\mathfrak{M} \cap \mathfrak{M}' = \mathbb{C}\mathbb{1}.$$

A *subfactor* of a factor $\mathfrak{M}$ is a subalgebra $\mathfrak{N} \subset \mathfrak{M}$ such that $\mathfrak{N}$ is a factor containing the identity of $\mathfrak{M}$. A subfactor $\mathfrak{N} \subset \mathfrak{M}$ of a factor $\mathfrak{M}$ is called *irreducible* if $\mathfrak{M} \cap \mathfrak{N}' = \mathbb{C}\mathbb{1}$. In this case we call the inclusion $\mathcal{N} \subset \mathfrak{M}$ an *irreducible inclusion of factors* (see also [JS97]).

1.1.3 Type Classification of Factors

Factors of von Neumann algebras can be classified into certain types. The underlying crucial observation is that the projections of a von Neumann algebra $\mathfrak{M}$ form a complete lattice [Tak79]. This suggests that a classification of von Neumann algebras is tight to the properties of their projections, which is the starting point of the Murray-von Neumann classification of factors [MN36] (see also [Tak79]). In fact, this classification can be extended to general von Neumann algebras, but here we are only interested in factors.

We start with an equivalence relation between projections. Two projections e, f in a von Neumann algebra $\mathfrak{M}$, acting on a Hilbert space $\mathcal{H}$, are said to be *equivalent*, written $e \sim f$, if there is a partial isometry $E \in \mathfrak{M}$ such that $E^*E = e$ and $EE^* = f$. Furthermore, we there is a partial ordering on the set of projections of a von Neumann algebra. If e, f are projections in $\mathfrak{M}$ we write $e \leqslant f$ if $e\mathcal{H} \subseteq f\mathcal{H}$.

A projection $e \in \mathfrak{M}$ said to be

- *finite*, if $e \sim f$ and $e \leqslant f$ implies $e = f$,
- *infinite*, if it is not finite,
- *purely infinite*, if all non-zero projections $f \leqslant e$ are infinite.
- *abelian*, if $e\mathfrak{M}e$ is abelian.

A factor $\mathfrak{M}$ is said to be

- of *type* I, if $\mathbb{1}$ majorises a non-zero abelian projection.
- of *type* II, if $\mathfrak{M}$ has no non-zero abelian projections, and if $\mathbb{1}$ majorises a non-zero finite projection.
- of *type* III, if every non-zero projection in $\mathfrak{M}$ is infinite.

This results in the following classification of factors [Tak79].

Theorem 1.1.10:
Let $\mathfrak{M}$ be a factor. Then $\mathfrak{M}$ is either of type I, II *or* III.

It is worth noting, that this classification is not complete in the sense that there are von Neumann algebras of the same type that are not isomorphic to each other. The case of a type II factor can be split into two further classes. A type II factor $\mathfrak{M}$ is of *type* II_1, if $\mathbb{1}$ is finite, otherwise it is called *type* II_∞.

Factors of type I can be further classified into type I_n factors, where n a cardinal. Each type I_n factor is isomorphic to $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ with Hilbert space dimension n. Using the modular theory by Tomita and Takesaki, factors of type III can be classified further into factors of type III_λ with $\lambda \in [0, 1]$ (for details, see [Tak79; Tak03a]). Note, however, that factors of the same type need not necessarily be isomorphic either. For example, there exist uncountable families of non-isomorphic type II_1, II_∞ and III factors (see [Sak71] together with [Tak03a]).

For later use, we need the definition of a conditional expectation.

Definition 1.1.11:
Let $\mathfrak{M}$ be a von Neumann algebra and $\mathfrak{N} \subset \mathfrak{M}$ a sub-algebra. A conditional expectation from $\mathfrak{M}$ onto $\mathfrak{N}$ *is a linear map* $E : \mathfrak{M} \to \mathfrak{N}$ *satisfying for all* $A \in \mathfrak{M}$*:*

- $E(\mathbb{1}) = \mathbb{1}$,
- $A \geqslant 0 \implies E(A) \geqslant 0$,
- $\forall BC \in \mathfrak{N}$*:* $E(BAC) = B\mathcal{E}(A)C$,

Note that this implies that a conditional expectation $E : \mathfrak{M} \to \mathfrak{N}$ is a projection, i.e. for all $A \in \mathfrak{M}$ and $a \in \mathfrak{N}$ we have $\|E(A)\| \leqslant \|A\|$ and $\mathcal{E}(a) = a$. Furthermore, the above conditions imply that for all $A \in \mathfrak{M}$ we have that $E(A)^*E(A) \leqslant E(A^*A)$. In [Tak79, Theorem III.3.4] it is shown that in fact every projection of norm one from a C^*-algebra $\mathcal{A}$ to a $*$-subalgebra $\mathcal{B} \subset \mathcal{A}$ is a conditional expectation. As an example consider the von Neumann algebras $\mathfrak{M} = \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ and $\mathfrak{N} = \mathcal{B}(\mathcal{H}_1)$ with $\mathcal{H}_1, \mathcal{H}_2$ both finite dimensional Hilbert spaces. Then the partial trace $\mathrm{tr}_{\mathcal{H}_2}$ is a conditional expectation from $\mathfrak{M}$ to $\mathfrak{N}$. Another class of examples are states.

1.2 Crossed Products and Index Theory

This section is devoted to recall the index classification of certain inclusions of factors. In particular, we briefly review the index introduced by Jones which he used to classify inclusions of type II_1 factors [Jon83; JS97]. We then sketch a generalisation of this index which was introduced by Kosaki for inclusions of arbitrary factors [Kos86]. Both of these approaches, and a compatible one by Longo [Lon89; Lon91], are reviewed in [Kos98]. As an example we discuss crossed products of factors with finite groups.

1.2.1 Jones' Index

In [Jon83] Jones introduced an index for inclusions of type II_1 factors, and determined its possible values. We briefly sketch the definition of his index and state the results that are important for us. A detailed discussion about the Jones index can be found in the textbook by Jones and Sunders [JS97].

Let $\mathfrak{M}$ be a type II_1 factor with separable predual. Since $\mathfrak{M}$ is a finite factor there exists a unique faithful normal tracial state on $\mathfrak{M}$ [JS97, Proposition 1.2.2] (see also [Tak79]). Let $\mathrm{tr}_{\mathfrak{M}}$ be this state on $\mathfrak{M}$ and write $\mathcal{H}_1 := L^2(\mathfrak{M}, \mathrm{tr}_{\mathfrak{M}})$ for the Hilbert space obtained by the GNS construction, and we assume that $M \subset \mathcal{B}(\mathcal{H}_1)$ (for the existence and uniquencess of $\mathrm{tr}_{\mathfrak{M}}$, see [Tak79, Theorem V.2.6]). The vector $\Omega \in \mathcal{H}_1$ implementing $\mathrm{tr}_{\mathfrak{M}}$ is cyclic and, due to the faithfulness of $\mathrm{tr}_{\mathfrak{M}}$, separating for $\mathfrak{M}$. Set $\mathcal{H}_\infty := \mathcal{H}_1 \otimes \ell^2(\mathbb{N})$, and $M_\infty(\mathfrak{M}) := \mathfrak{M} \otimes \mathcal{B}(l^2(\mathbb{N}))$. Then $\mathcal{H}_\infty$ is a left $\mathfrak{M}$-module and a right $M_\infty(\mathfrak{M})$-module, and $M_\infty(\mathfrak{M})$ is a type II_∞ factor. Furthermore there exists a unique faithful normal semifinite trace Tr on $M_\infty(\mathfrak{M})$, i.e. a unique faithful normal functional Tr from the positive elements of $M_\infty(\mathfrak{M})$ to $\mathbb{R}_+$ with $\mathrm{Tr}(A^*A) = \mathrm{Tr}(AA^*)$ for all $A \in M_\infty(\mathfrak{M})$, and such that for all $A \geqslant 0$ there exists a $0 \leqslant B \leqslant A$ with $\mathrm{Tr}(B) < \infty$ (see [Tak79, Theorem V.2.34]). If $\mathcal{H}$ is a separable $\mathfrak{M}$-module there always exists a projection $p \in M_\infty(\mathfrak{M})$ such that $\mathcal{H}_\infty p \cong \mathcal{H}$, and one defines the $\mathfrak{M}$-dimension of $\mathcal{H}$ as follows [JS97].

Definition 1.2.1:
Let $\mathfrak{M}$ be a II_1 factor with separable predual, and let $\mathcal{H}$ be any separable left $\mathfrak{M}$-module. Let $p \in M_\infty(\mathfrak{M})$ be a projection such that $\mathcal{H}_\infty p \cong \mathcal{H}$ as a $\mathfrak{M}$-module. We define

$$\dim_{\mathfrak{M}} \mathcal{H} := \mathrm{Tr}(p).$$

Let $\mathfrak{N} \subset \mathfrak{M}$ be a subfactor of a type II_1 factor $\mathfrak{M}$. The index of $\mathfrak{N}$ in $\mathfrak{M}$ *is then defined as the $\mathfrak{N}$-dimension of $\mathcal{H}_1 = L^2(\mathfrak{M}, \mathrm{tr}_{\mathfrak{M}})$, i.e.*

$$[\mathfrak{M} : \mathfrak{N}] := \dim_{\mathfrak{N}}(\mathcal{H}_1). \tag{1.1}$$

The index can be directly calculated from the dimension of an $\mathfrak{M}$-module.

Proposition 1.2.2:
Let $\mathfrak{N} \subset \mathfrak{M}$ be an inclusion of type II_1 factors. Let $\mathcal{H}$ be an $\mathfrak{M}$-module with $\dim_{\mathfrak{M}} \mathcal{H} < \infty$. Then

$$[\mathfrak{M} : \mathfrak{N}] = \frac{\dim_{\mathfrak{N}} \mathcal{H}}{\dim_{\mathfrak{M}} \mathcal{H}}.$$

Furthermore, we have

$$[\mathfrak{N}' : \mathfrak{M}'] = [\mathfrak{M} : \mathfrak{N}].$$

In addition, the index is consistent with increasing the inclusion $\mathfrak{N} \subset \mathfrak{M}$ by another type II_1 factor [Jon83].

Proposition 1.2.3:
Let $\mathfrak{P} \subset \mathfrak{N} \subset \mathfrak{M}$ be a tower of II_1 factors, then

$$
\begin{aligned}
&[\mathfrak{M} : \mathfrak{M}] = 1, \\
&[\mathfrak{M} : \mathfrak{P}] \geqslant [\mathfrak{M} : \mathfrak{N}], \\
&[\mathfrak{M} : \mathfrak{P}] = [\mathfrak{M} : \mathfrak{N}][\mathfrak{N} : \mathfrak{P}], \\
&[\mathfrak{M} : \mathfrak{N}] = [\mathfrak{M} : \mathfrak{P}] \implies \mathfrak{P} = \mathfrak{N}.
\end{aligned}
$$

It is also possible to determine all possible values of the index [Jon83].

Theorem 1.2.4:
If $\mathfrak{N} \subset \mathfrak{M}$ is any inclusion of type II_1 factors, then

$$[\mathfrak{M} : \mathfrak{N}] \in \{4\cos^2 \frac{\pi}{n} \,|\, n = 3, 4, \dots\} \cup [4, \infty].$$

Furthermore, for any $\lambda \in \{4\cos^2 \frac{\pi}{n} \,|\, n = 3, 4, \dots\} \cup [4, \infty]$ there exists a type II_1 factor $\mathfrak{M}$ and a subfactor $\mathfrak{N} \subset \mathfrak{M}$ with $[\mathfrak{M} : \mathfrak{N}] = \lambda$.

The index $[\mathfrak{M} : \mathfrak{N}]$ for an inclusion $\mathfrak{N} \subset \mathfrak{M}$ of type II_1 factors can be related to a certain conditional expectation from $\mathfrak{M}$ to $\mathfrak{N}$. More precisely, consider the orthogonal projection $e_{\mathfrak{N}} : L^2(\mathfrak{M}, \mathrm{tr}) \to L^2(\mathfrak{N}, \mathrm{tr})$, where $L^2(\mathfrak{N}, \mathrm{tr})$ is identified with the subspace obtained from the closure of $\mathfrak{N}\Omega$. Recall that Ω is the cyclic and separating vector implementing tr. It is then true that $e_{\mathfrak{N}}(\mathfrak{M}\Omega) \subset \mathfrak{N}\Omega$, thus $e_{\mathfrak{N}}$ gives rise to a trace-preserving conditional expectation $E : \mathfrak{M} \to \mathfrak{N}$ satisfying

$$\forall A \in \mathfrak{M} : e_{\mathfrak{N}} A e_{\mathfrak{N}} = E(A) e_{\mathfrak{N}},$$

and, in addition, $J e_{\mathfrak{N}} = e_{\mathfrak{N}} J$, where J is the modular conjugation in $L^2(\mathfrak{M}, \mathrm{tr})$. We denote the von Neumann algebra generated by $\mathfrak{M}$ and $e_{\mathfrak{N}}$ by $\langle \mathfrak{M}, e_{\mathfrak{N}} \rangle$, and the corresponding conditional expectation onto $\mathfrak{M}$ by $\mathcal{E}_{\mathfrak{M}} : \langle \mathfrak{M}, e_{\mathfrak{N}} \rangle \to \mathfrak{M}$. The tower $\mathfrak{N} \subset \mathfrak{M} \subset \langle \mathfrak{M}, e_{\mathfrak{N}} \rangle$ is usually referred to as *basic construction* (see [JS97]).

Proposition 1.2.5 [JS97]:
Let $\mathfrak{N} \subset \mathfrak{M}$ be an inclusion of type II_1 factors. Then the following hold true.

- $e_{\mathfrak{N}} \in \mathfrak{N}'$.
- $\mathfrak{N} = \mathfrak{M} \cap \{e_{\mathfrak{N}}\}'$.
- $\langle \mathfrak{M}, e_{\mathfrak{N}} \rangle = J\mathfrak{N}'J$.
- $\langle \mathfrak{M}, e_{\mathfrak{N}} \rangle$ *is a factor if and only if* $[\mathfrak{M} : \mathfrak{N}] < \infty$.

- *If* $[\mathfrak{M} : \mathfrak{N}] < \infty$ *then* $[\langle \mathfrak{M}, e_{\mathfrak{N}} \rangle : \mathfrak{M}] = [\mathfrak{M} : \mathfrak{N}]$.
- $\mathrm{tr}_{\langle \mathfrak{M}, e_{\mathfrak{N}} \rangle}(e_{\mathfrak{N}}) = [\mathfrak{M} : \mathfrak{N}]^{-1} \mathbb{1}$.
- $\mathcal{E}_{\langle \mathfrak{M}, e_{\mathfrak{N}} \rangle}(e_{\mathfrak{N}}) = [\mathfrak{M} : \mathfrak{N}]^{-1} \mathbb{1}$.

Iterating the basic construction results in an infinite tower

$$\mathfrak{M}_{-1} \subset \mathfrak{M}_0 \subset \mathfrak{M}_1 \subset \dots$$

with $\mathfrak{M}_{-1} = \mathfrak{N}$, $\mathfrak{M}_0 = \mathfrak{M}$ and $\mathfrak{M}_1 = \langle \mathfrak{M}, e_{\mathfrak{N}} \rangle$. This can be used to construct the hyperfinite type II_1 factor and to find inclusions of type II_1 factors for each allowed value of the index [Jon83; JS97].

1.2.2 General Factors

With some more effort the theory of the index for type II_1 factors can be generalised for arbitrary factors [Kos86; Kos98] (for type III factors there exists an equivalent approach by Longo [Lon89; Lon91]). Let $\mathfrak{M}$ be a factor on some Hilbert space $\mathcal{H}$ and let $\mathfrak{N} \subset \mathfrak{M}$ be a subfactor. Let $P(\mathfrak{M}, \mathfrak{N})$ be the set of normal faithful semi-finite operator-valued weights $E : \mathfrak{M} \to \mathfrak{N}$. Using the Haagerup theory of weights (c.f. [Haa79a; Haa79b]) one can show that $P(\mathfrak{M}, \mathfrak{N})$ is non-empty if and only if $P(\mathfrak{N}', \mathfrak{M}')$ is non-empty (for details, see [Tak03a], especially Theorem IX.4.24). Furthermore, there exists an order-reversing bijection between $P(\mathfrak{M}, \mathfrak{N})$ and $P(\mathfrak{N}', \mathfrak{M}')$ which we write as $E \mapsto E^{-1}$.

Definition 1.2.6:
Let $\mathfrak{M}$ be a factor on a Hilbert space $\mathcal{H}$ and $\mathfrak{N}$ a subfactor of $\mathfrak{M}$. Let $E : \mathfrak{M} \to \mathfrak{N}$ be a normal conditional expectation. The index $\mathrm{Ind}(E)$ *of* E *is defined as the scalar*

$$\mathrm{Ind}(E) := E^{-1}(\mathbb{1}).$$

The definition is independent of the choice of the Hilbert space, and satisfies properties similar to the Jones index for type II_1 factors [Kos98; Kos86].

Theorem 1.2.7:
Let $\mathfrak{M}$ be a factor with subfactor $\mathfrak{N}$, both acting on a Hilbert space $\mathcal{M}$. Let $E : \mathfrak{M} \to \mathfrak{N}$ be a normal conditional expectation. Then

- *If $\tilde{\mathfrak{M}}$ is a von Neumann algebra isomorphic to $\mathfrak{M}$ and $\tilde{E} : \tilde{\mathfrak{M}} \to \tilde{\mathfrak{N}}$ the corresponding normal conditional expectation then $\mathrm{Ind}(E) = \mathrm{Ind}\tilde{E}$,*
- *if $\mathfrak{L} \subset \mathfrak{N}$ is another subfactor and $E_0 : \mathfrak{N} \to \mathfrak{L}$ is a normal conditional expectation, then $\mathrm{Ind}(E_0 \circ E) = \mathrm{Ind}(E_0)\mathrm{Ind}(E)$,*
- *$\mathrm{Ind}(E) \geqslant 1$,*
- *$\mathrm{Ind}(E) = 1$ if and only if $\mathfrak{M} = \mathfrak{N}$,*
- *$\mathrm{Ind}(E) \in \{4\cos^2 \frac{\pi}{n} \,|\, n = 3, 4, \dots\} \cup [4, \infty]$.*

In the case that $\mathfrak{N} \subset \mathfrak{M}$ are both type II_1 factors, and $E : \mathfrak{M} \to \mathfrak{N}$ is the unique normalised trace preserving conditional expectation, then $\mathrm{Ind}(E)$ is exactly Jones' index, i.e. $\mathrm{Ind}(E) = [M : N]$. Note, however, that in general the index depends on the choice of the conditional expectation E. If the inclusion $\mathfrak{N} \subset \mathfrak{M}$ is irreducible, and there exists a conditional expectation, then it is unique. If the inclusion is not irreducible, then one can minimise the index by optimising over all conditional expectations, and, if $\mathrm{Ind}(E)$ is finite for some E, then there exists a unique E_0 minimising the index [Hia88; Kos98].

Similar to the case of type II_1 factors one can obtain a basic construction in the general case as well [Kos98]. For this, consider a factor $\mathfrak{M}$ on a Hilbert space $\mathcal{H}$ and a subfactor $\mathfrak{N}$ of $\mathfrak{M}$. Let $E : \mathfrak{M} \to \mathfrak{N}$ be a normal conditional expectation with $\mathrm{Ind}(E) < \infty$. Assume that there exists a faithful normal state ϕ on $\mathfrak{N}$ and that $\mathcal{H}$ is given by $L^2(M)$ w.r.t. the state $\phi \circ E$. Let $\xi_0 \in \mathcal{H}$ be a cyclic and separating vector implementing $\phi \circ E$ on $\mathcal{H}$. Denote the corresponding modular conjugation by J. Setting

$$e_{\mathfrak{N}} x \xi_0 \coloneqq E(x)\xi_0$$

for all $x \in \mathfrak{M}$ defines a projection $e_{\mathfrak{N}} \in \mathcal{B}(\mathcal{H})$ that is independent of the choice of the state ϕ [Kos98]. The basic construction is then given by the Neumann algebra $\langle \mathfrak{M}, e_{\mathfrak{N}} \rangle$ and a conditional expectation $E_1 : \langle \mathfrak{M}, e_{\mathfrak{N}} \rangle \to \mathfrak{M}$ which can be constructed from the modular conjugation J and the conditional expectation E. These objects satisfy essentially the same properties as in the case of type II_1 factors (compare to Proposition 1.2.5).

Proposition 1.2.8:
Let $\mathfrak{M} \subset \mathfrak{N}$, $e_{\mathfrak{N}}$, and $\langle \mathfrak{M}, e_{\mathfrak{N}} \rangle$ be as above. Then we have

- $e_{\mathfrak{N}}$ *does not depend on the choice of* ϕ.
- $e_{\mathfrak{N}} \in \mathfrak{N}'$.
- *For all* $x \in \mathfrak{M}$: $e_{\mathfrak{N}} x e_{\mathfrak{N}} = E(x) e_{\mathfrak{N}}$.
- $\mathfrak{N} = \mathfrak{M} \cap \{e_{\mathfrak{N}}\}'$.
- $\langle \mathfrak{M}, e_{\mathfrak{N}} \rangle = J\mathfrak{N}'J$.
- $\mathrm{Ind}(E_1) = \mathrm{Ind}(E)$.
- $E_1(e_{\mathfrak{N}}) = (\mathrm{Ind}(E))^{-1}\mathbb{1}$.
- *Elements of the form* $a_0 + \sum_{i=1}^{n} a_i e_{\mathfrak{N}} b_i$ *with* $a_i, b_i \in \mathfrak{M}$ *densely span* $\langle \mathfrak{M}, e_{\mathfrak{N}} \rangle$.

Proof. The proofs of these statements can be found in [Kos86] and [Kos98]. □

1.2.3 Pimsner Popa Basis

Pimsner and Popa showed that for type II_1 factors the index can be characterised by a parameter optimising a certain inequality [PP86]. This is also true for arbitrary factors (see [Kos98, Theorem 3.8]). In the following we write $\mathfrak{M}_+$ for the positive elements of $\mathfrak{M}$.

Theorem 1.2.9 Pimsner-Popa inequality:
Let $\mathfrak{N} \subset \mathfrak{M}$ be an inclusion of factors. Let $E : \mathfrak{M} \to \mathfrak{N}$ be a conditional expectation with $\mathrm{Ind}(E) < \infty$. *Then we have for all* $x \in \mathfrak{M}_+$:

$$E(x) \geqslant (\mathrm{Ind}(E))^{-1} x,$$

and equality is attained for $x = e_{\mathfrak{N}}$. *If $\mathfrak{M}$ and $\mathfrak{N}$ are not of type* I, *then*

$$\sup\{\epsilon > 0 \,|\, \forall x \in \mathfrak{M}_+ : \; E(x) \geqslant \epsilon x\} = (\mathrm{Ind}(E))^{-1}.$$

In fact, the normality of E is linked to such an inequality (see [Cam+11, Appendix C]). Pimsner and Popa also showed that the index for an inclusion $\mathfrak{N} \subset \mathfrak{M}$ of type II_1 factors implies that $\mathfrak{M}$ can be interpreted as a module over $\mathfrak{N}$. More precisely, there exists a set of operators in $\mathfrak{M}$ such that elements of $\mathfrak{M}$ can be written as linear combinations of these operators with coefficients in $\mathfrak{N}$. This is also true for arbitrary factors as shown in [Kos98].

Theorem 1.2.10 Pimsner-Popa basis:
Let $\mathfrak{N} \subset \mathfrak{M}$ *be an inclusion of factors, and let* $E : \mathfrak{M} \to \mathfrak{N}$ *be a conditional expectation with finite index* $\mathrm{Ind}(E)$*. Then there exists a sequence* $\{a_i\}_{i=1}^n$ *of elements of* $\mathfrak{M}$ *such that*

$$\sum_{i=1}^{n} a_i e_{\mathfrak{N}} a_i^* = \mathbb{1}.$$

Proof. This is an adaption of the proof of [Kos86, Corollary 3.4] (see also the remark after [Kos98, Theorem 3.9]).

Let $(p_i)_{i=1}^{\infty}$ be a family of pairwise orthogonal projections in $\langle \mathfrak{M}, e_{\mathfrak{N}} \rangle$ such that $p_1 \lesssim e_{\mathfrak{N}}$ or $p_1 = 00$, $p_i \sim e_{\mathfrak{N}}$ or $p_i = 0$ for $i \geqslant 2$, and $\sum_i p_i = \mathbb{1}$. For any $i \in \mathbb{N}$, let u_i be partial isometries in $\langle \mathfrak{M}, e_{\mathfrak{N}} \rangle$ with $u_1 u_1^* = p_1$, $u_1^* u_1 = f \leqslant e_n$, and $u_i u_i^* = p_i$, $u_i^* u_i = e_{\mathfrak{N}}$ if $i \geqslant 0$. Setting $v_1 := u_1 f$ and $v_i := u_i$ for $i \geqslant 1$ we get

$$\mathbb{1} = \sum_i v_i e_{\mathfrak{N}} v_i^*.$$

By [Kos98, Lemma 3.10] there exist elements $a_i \in \mathfrak{M}, i \in \mathbb{N}$ with $v_i e_{\mathfrak{N}} = a_i e_{\mathfrak{N}}$, thus proving the claim. □

The Pimsner-Popa basis is a basis in the following sense [Kos98].

Proposition 1.2.11:
Let $\mathfrak{N} \subset \mathfrak{M}$, $E : \mathfrak{M} \to \mathfrak{N}$ *a conditional expectation with* $\mathrm{Ind}(E) < \infty$*. Let* $\{a_i\}_{i=1}^n$ *be a sequence of operators in* $\mathfrak{M}$ *with*

$$\sum_{i=1}^{n} a_i e_{\mathfrak{N}} a_i^* = \mathbb{1},$$

for all $x \in \mathfrak{M}$*. Then*

- *for all* $x \in \mathfrak{M}$ *we have that* $x = \sum_{i=1}^n a_i E(a_i x)$*, and*
- $\mathrm{Ind}(E) = \sum_{i=1}^n a_i a_i^*$.

Proof. Let $x \in \mathfrak{M}$, then by Proposition 1.2.8 we have $x e_{\mathfrak{N}} = \sum_i a_i e_{\mathfrak{N}} a_i x e_{\mathfrak{N}} = \sum_i a_i E(a_i x) e_{\mathfrak{N}}$. This implies

$$x = \sum_{i=1}^{n} a_i E(a_i x),$$

since, by assumption, there is a cyclic and separating vector $\xi_0 \in \mathcal{H}$ with $e_{\mathfrak{N}} \xi_0 = \xi_0$. (see also the definition of $e_{\mathfrak{N}}$ in the previous section).

Let $E_1 : \langle \mathfrak{M}, e_{\mathfrak{N}} \rangle \to \mathfrak{M}$ be the conditional expectation obtained from the basic construction. Then

$$\mathbb{1} = \sum_{i=1}^{n} E_1(a_i e_{\mathfrak{N}} a_i^*) = \sum_{i=1}^{n} a_i E_1(e_{\mathfrak{N}}) a_i^* = (\mathrm{Ind} E)^{-1} \sum_{i=1}^{n} a_i a_i^*.$$

□

The Pimsner-Popa basis are also the starting point for Watatani's index for C^*-subalgebras [Wat90].

1.2.4 Crossed Products

As an example for an inclusion of factors we consider the crossed product construction of a factor $\mathfrak{M}$ with a finite group G acting on $\mathfrak{M}$ as a group of automorphisms. This illustrates how the index can be understood as an analogue of the index for subgroups of groups, and it provides an explicit way to obtain the index as well as the Pimsner-Popa basis. We will encounter crossed products again when we analyse certain inclusions of von Neumann algebras in the thermodynamic limit of Kitaev'S quantum double models for finite abelian groups.

One can define the crossed products of a von Neumann algebra with a locally compact group [Tak03a; BR96; KR97], however, we will define it here only for discrete groups (e.g. [JS97; KR97; Sun87]). Crossed products play a role in the characterisation of certain representations of dynamical systems. More precisely, the covariant representations of a dynamical system are in one-to one correspondence with non-degenerate representations of the crossed product of the corresponding von Neumann algebra with the group from the dynamical system [BR96; Tak79]. Furthermore, crossed products play an important role in the classification of factors [BR96; KR97; MN36].

Definition 1.2.12:
Let $\mathfrak{M}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$, let G be a finite group, and let α be a continuous homomorphism of G into the group $\mathrm{Aut}(\mathfrak{M})$ of $*$-automorphisms of $\mathfrak{M}$. We call the triple $(\mathfrak{M}, G, \alpha)$ a von Neumann dynamical system, *or just* dynamical system.

A covariant representation *of a dynamical system $(\mathfrak{M}, G, \alpha)$ is a normal representation ρ of $\mathfrak{M}$ on some Hilbert space $\mathcal{H}$ together with a unitary representation U of G on $\mathcal{H}$ such that $\pi \circ \alpha_g(A) = U(g)\pi(A)U(g)^*$ for all $g \in G$ and $A \in \mathfrak{M}$.*

Analogously, one can also define a C^*-dynamical system, but in the remainder of the thesis we will mostly work with von Neumann algebras anyway. For a similar reason, we always assume that G is a finite group, although one can define them for more general groups as well. Dynamical systems arise for example as time evolution of physical systems. If $\mathcal{H}$ is a Hilbert space, and H a Hamiltonian with corresponding strongly continuous one-parameter group $(U_t)_{t\in\mathbb{R}}$ of unitaries, then $\alpha_t(A) := U_t A U_t^*$, with $t \in \mathbb{R}$ and $A \in \mathcal{B}(\mathcal{H})$, defines a strongly continuous on-parameter group α of automorphisms on $\mathcal{B}(\mathcal{H})$, and the triple $(\mathcal{B}(\mathcal{H}), \alpha, \mathbb{R})$ is then a dynamical system. Dynamical systems where the group is given by $\mathbb{R}$ also play a role in the further classification of type III factors [Tak03a].

Let $(\mathfrak{M}, G, \alpha)$ be a dynamical system with $\mathfrak{M}$ acting on a Hilbert space $\mathcal{H}$. Let $\ell^2(G)$ be the Hilbert space of square summable, complex valued functions on G, and let $\lambda : G \to \ell^2(G)$ be the left regular representation of G on this space, i.e. $(\lambda_g\xi)(h) = \xi(g^{-1}h)$ with $h, g \in G$ and $\xi \in \ell^2(G)$. The canonical basis $(\delta_g)_{g\in G}$ in $\ell^2(G)$ is given by $\delta_g(h) = \delta_{g,h}$ with $g, h \in G$. The crossed product of $\mathfrak{M}$ with G w.r.t. α will be defined by operators acting on the Hilbert space $\tilde{\mathcal{H}} = \ell^2(G, \mathcal{H})$ of square summable $\mathcal{H}$-valued functions on G. The Hilbert space $\tilde{\mathcal{H}}$ is unitarily equivalent to the Hilbert space $\mathcal{H} \otimes \ell^2(G)$ and $\bigoplus_{g\in G} \mathcal{H}$. The first equivalence is can be seen by checking that the map $\tilde{\mathcal{H}} \ni \xi \mapsto \sum_{g\in G} \xi(g) \otimes \delta_g \in \mathcal{H} \otimes \ell^2(G)$ is unitary. The second equivalence is given by identifying $\xi \in \tilde{\mathcal{H}}$ with square summable column vectors $(\xi(g))_{g\in G}$.

One can now define a normal faithful representation π of $\mathfrak{M}$ on $\tilde{\mathcal{H}}$ by

$$(\pi(A)\xi)(g) := \alpha_g^{-1}(A)\xi(g), \tag{1.2}$$

with $A \in \mathfrak{M}$, $\xi \in \tilde{\mathcal{H}}$ and $g \in G$, and a faithful unitary representation Λ of G on $\tilde{\mathcal{H}}$ by

$$(\Lambda(g)\xi)(h) := \xi(g^{-1}h), \tag{1.3}$$

with $g, h \in G$ and $\xi \in \tilde{\mathcal{H}}$. The faithfulness of π follows from the injectivity of α_g for each $g \in G$, and π is normal, since for any $g \in G$ α_g is an isomorphism of von Neumann algebras (see e.g. [Tak79, Corollary III.3.10]). Also, Λ is a faithful unitary representation, since the left regular representation of G is unitary and faithful.

Definition 1.2.13:
Let $(\mathfrak{M}, G, \alpha)$ *be a dynamical system, and let* $\tilde{\mathcal{H}}$, π *and* Λ *be the Hilbert space and the representations of* $\mathfrak{M}$ *and* G *on* $\tilde{\mathcal{H}}$ *as above. The* crossed product *of* $\mathfrak{M}$ *with* G *w.r.t.* α *is given by*

$$\mathfrak{M} \rtimes_\alpha G := (\pi(\mathfrak{M}) \cup \Lambda(G))'' .$$

A priori, this definition depends on the Hilbert space on which $\mathfrak{M}$ is represented. However, this is not the case, as shown in [KR97, Proposition 13.1.2] (see also [Tak03a]).

Theorem 1.2.14:
Let $(\mathfrak{M}, G, \alpha)$ *be a dynamical system and let* ρ *be any normal faithful representation of* $\mathfrak{M}$ *on some Hilbert space* $\mathcal{K}$. *Define a covariant representation* $\ell^2(G, \mathcal{K})$ *by*

$$(\rho_\alpha(A)\xi)(g) := \rho(\alpha_g^{-1}(A))\xi(g) \qquad\qquad (\Lambda_{\mathcal{K}}(h)\xi)(g) := \xi(h^{-1}g),$$

with $g, h \in G$, $A \in \mathfrak{M}$ *and* $\xi \in \ell^2(G, \mathcal{K})$. *Then there exists a* $*$*-isomorphism* $\Phi : \mathfrak{M} \rtimes_\alpha G \to \{\rho_\alpha(\mathfrak{M}) \cup \Lambda_{\mathcal{K}}(G)\}''$ *such that*

$$\Phi \circ \pi(A) = \rho_\alpha(A), \qquad\qquad \Phi(\Lambda(g)) = \Lambda_{\mathcal{K}}(g),$$

for all $g \in G$ *and* $A \in \mathfrak{M}$. *In addition, there exists a* $*$*-isomorphism* θ *from* $\mathfrak{M}$ *onto a von Neumann algebra* $\mathfrak{N}$ *such that for all* $g \in G$ *the automorphisms* $g \mapsto \theta \circ \alpha_g \circ \theta^{-1}$ *on* $\mathfrak{N}$ *are unitarily implemented.*

This shows that the crossed product $\mathcal{M} \rtimes_\alpha G$ is unique up to isomorphism of von Neumann algebras. One can also show that the crossed product is stable under perturbations by cocycles [Tak03a].

Later, it will be of some use to have the descriptions of the crossed product of $\mathfrak{M}$ with G w.r.t. α at hand for the different representations of $\ell^2(G, \mathcal{H})$. We will give a brief description here. Let $(\mathfrak{M}, G, \alpha)$ be a dynamical system, $\mathfrak{M} \rtimes_\alpha G$ its crossed product, and π and Λ as above. Let $W : l^2(G, \mathcal{H}) \to \mathcal{H} \otimes \ell^2(G)$ be the unitary given by $W\xi = \sum_{g \in G} \xi(g) \otimes \delta_g$ with $\xi \in \ell^2(G, \mathcal{H})$ and $\{\delta_g\}_{g \in G}$ the canonical basis in $\ell^2(G)$. Denote by E_g, $g \in G$, the projections onto the basis vectors $\{\delta_g\}_{g \in G}$, and by λ the left regular representation (or left shift) of G on $\ell^2(G)$. Then it is easy to check that

$$W\pi(A)W^* = \sum_{g \in G} \alpha_g^{-1}(A) \otimes E_g, \qquad\qquad W\Lambda(g)W^* = \mathbb{1} \otimes \lambda_g \tag{1.4}$$

for all $A \in \mathfrak{M}$ and $g \in G$. The crossed product $\mathfrak{M} \rtimes_\alpha G$ is then isomorphic to the von Neumann algebra generated by the linear span of products of such operators.

For the representation of $\ell^2(G, \mathcal{H})$ as $\bigoplus_{g \in G} \mathcal{H}$, let $V : l^2(G, \mathcal{H}) \to \bigoplus_{g \in G} \mathcal{H}$ be the unitary given by $V\xi = (\xi(g))_{g \in G}$ with $\xi \in \ell^2(G, \mathcal{H})$. The right hand side of the last equation can be interpreted as a column vector. This implies that any operator $B \in \mathcal{B}(\bigoplus_{g \in G} \mathcal{H})$ has, with respect to the canonical basis $(\delta_g)_{g \in G}$ of $\ell^2(G)$, a representation as a matrix $(B_{k,l})_{k,l \in G}$ with coefficients in $\mathcal{B}(\mathcal{H})$. We then have for all $A \in \mathfrak{M}$ and $g, k, l \in G$

$$(V\pi(a)V^*)_{k,l} = \alpha_l^{-1}(A)\delta_{k,l}, \qquad (V\Lambda(g)V^*)_{k,l} = \delta_{k,gl}\mathbb{1}, \tag{1.5}$$

and again, the von Neumann algebra generated by these operators is isomorphic to $\mathfrak{M} \rtimes_\alpha G$. One should mention that crossed products can also be characterised by a space of $\mathfrak{M}$-valued functions on G (for details, see [JS97, Lemma 1.3.1] and discussion thereafter, and [Tak03a]).

If we consider a dynamical system $(\mathfrak{M}, G, \alpha)$ where the automorphisms given by the representation α are spatial, i.e. implemented by unitaries on $\mathcal{B}(\mathcal{H})$ then there exists a more direct description[6] of $\mathfrak{M} \rtimes_\alpha G$. So, assume that there exists a unitary representation U of G on $\mathcal{H}$ such that for all $g \in G$ we have $\alpha_g(A) = U_g A U_g^*$. Let π and Λ be the representations of $\mathfrak{M}$ and G on $\ell^2(G, \mathcal{H})$ as defined in equations (1.2) and (1.3). The representation π then takes the form

$$(\pi(A)\xi)(g) = U_g^* A U_g \xi(g)$$

for all $A \in \mathfrak{M}$, $\xi \in \ell^2(G, \mathcal{H})$, and $g \in G$. On $\ell^2(\mathcal{H}, G)$ we define a unitary T by

$$(T\xi)(g) := U_g \xi(g)$$

for all $x \in \ell^2(G, \mathcal{H})$ and $g \in G$. It is then easy to check that $(T^*\xi)(g) = U_g^* \xi(g)$, and

$$(T\pi(A)T^*\xi)(g) = A\xi(g) \qquad (T\Lambda(h)T^*\xi)(g) = U_h \xi(h^{-1}g), \tag{1.6}$$

for any $A \in \mathfrak{M}$, $g, h \in G$ and $\xi \in \ell^2(G, \mathcal{H})$. We set $(\tilde{\pi}(A)\xi)(g) := A\xi(g)$ and $(\tilde{\Lambda}_h \xi)(g) := U_h \xi(h^{-1}g)$. The crossed product $\mathfrak{M} \rtimes_\alpha G$ is thus unitary equivalent to the von Neumann algebra $(\{\tilde{\pi}(A) \,|\, A \in \mathfrak{M}\} \cup \{\tilde{l}_h \,|\, h \in G\})''$. If $W : l^2(G, \mathcal{H}) \to \mathcal{H} \otimes \ell^2(G)$ is the unitary used in equation (1.2) we get

$$W\tilde{\pi}(A)W^* = A \otimes \mathbb{1} \qquad W\tilde{l}_h W^* = U_h \otimes \lambda_h \tag{1.7}$$

[6] In the language of [KR97] this is the *implemented crossed product*.

with $A \in \mathfrak{M}$ and $g, h \in G$. Hence, in case that α is spatial, this gives a particular clean characterisation of $\mathfrak{M} \rtimes_\alpha G$, i.e.

$$\mathfrak{M} \rtimes_\alpha G \cong (\{A \otimes \mathbb{1} \,|\, A \in \mathfrak{M}\} \cup \{U_h \otimes \lambda_h \,|\, h \in G\})'' .$$

Next we turn to the question when the crossed product of a factor with a group is again a factor. For this we need the following definition.

Definition 1.2.15:
Let $\mathfrak{M}$ be a von Neumann algebra. An automorphism θ of $\mathfrak{M}$ is said to be free *if* $\forall A \in \mathfrak{M} : (\forall B \in \mathfrak{M} : AB = \theta(B)A) \implies A = 0$.

The automorphism θ is said to be inner, *if there exists a unitary $U \in \mathfrak{M}$ such that $\theta(A) = UAU^*$ for all $A \in \mathfrak{M}$.*

If $(\mathfrak{M}, G, \alpha)$ is a dynamical system, then α is free *if for all $g \in G$ with $g \neq e$, α_g is free.*

The action α is called ergodic, *if* $\{x \in \mathfrak{M} \,|\, \forall g \in G : \alpha_g(x) = x\} = \mathbb{C}\mathbb{1}$.

This immediately implies the following.

Lemma 1.2.16:
An automorphism θ of a factor $\mathfrak{M}$ is free if and only if it is not inner.

Proof. First assume that θ is inner. Then there exists a unitary $U \in \mathfrak{M}$ such that for all $A \in \mathfrak{M}$, $\theta(A) = UAU^*$. But then, for any $B \in \mathfrak{M}$, $UB = UBU^*U = \theta(B)U$, thus θ is not free.

Assume now, that θ was not free. Then let $A \in \mathfrak{M}$ such that for all $B \in \mathfrak{M}$, $AB = \theta(B)A$. This implies that $BA = A\theta^{-1}(B)$, and thus $A^*AB = A^*\theta(B)A = BA^*A$ for all $B \in \mathfrak{M}$, and thus $A^*A \in \mathbb{C}\mathbb{1}$. Similarly, $AA^* \in \mathbb{C}\mathbb{1}$. Assume now, that $A \neq 0$, and let $A = U|A|$ be the polar decomposition of A. Then $U \in \mathfrak{M}$ is a unitary, since $AA^*, A^*A \in \mathbb{C}\mathbb{1}$, and for all $B \in \mathcal{M}$ we have, $UB = \theta(B)U$. Hence, either $A = 0$ or θ is inner. □

This leads to a characterisation of when the crossed product associated to a dynamical system is a factor [JS97, Proposition 1.4.4].

Proposition 1.2.17:
Let $(\mathfrak{M}, G, \alpha)$ be a dynamical system, and $\mathfrak{M} \rtimes_\alpha G$ its crossed product with π as in equation (1.2). *Then*

- *The action α is free if and only if $\pi(\mathfrak{M})' \cap (\mathfrak{M} \rtimes_\alpha G) = \pi(\mathfrak{M}' \cap \mathfrak{M})$.*
- *Given that α is free, then $\mathfrak{M} \rtimes_\alpha G$ is a factor if and only if the action of $\alpha \restriction_{\mathfrak{M}' \cap \mathfrak{M}}$ is ergodic.*

Here, with $\alpha \restriction_{\mathfrak{M}' \cap \mathfrak{M}}$ we mean the map obtained by the restriction of α to the subalgebra $\mathfrak{M}' \cap \mathfrak{M}$ of $\mathfrak{M}$.

Proof. For the first assertion, first consider $Y \in \pi(\mathfrak{M})' \cap (\mathfrak{M} \rtimes_\alpha G)$ with $Y = \pi(B)\Lambda(h)$ and $B \in \mathfrak{M}$, $h \in G$. Let $X = \pi(A) \in \pi(\mathfrak{M})$ with $A \in \mathfrak{M}$. Then $[X, Y] = 0$, and this is equivalent to

$$\begin{aligned}\pi(A)\pi(B) &= \pi(B)\Lambda(h)\pi(A)\Lambda(h)^* \\ &= \pi(B)\pi(\alpha_h(A)).\end{aligned}$$

Note that since π is a $*$-homomorphism we have that $\pi(\mathfrak{M}' \cap \mathfrak{M}) \subseteq \pi(\mathfrak{M})' \cap (\mathfrak{M} \rtimes_\alpha G)$.

Now, if α is free, then $\pi(A)\pi(B) = \pi(B)\pi(\alpha_h(A))$ for all $A \in \mathfrak{M}$ implies $B = 0$, if $h \neq e$. If $h = e$ this reads $\pi(AB) = \pi(BA)$, and since π is faithful, $[A, B] = 0$ for all $A \in \mathfrak{M}$, implying $B \in \mathfrak{M}' \cap \mathfrak{M}$. Thus, $\pi(\mathfrak{M})' \cap (\mathfrak{M} \rtimes_\alpha G) = \pi(\mathfrak{M}' \cap \mathfrak{M})$.

If $\pi(\mathfrak{M})' \cap (\mathfrak{M} \rtimes_\alpha G) = \pi(\mathfrak{M}' \cap \mathfrak{M})$, then $Y = \pi(B)\Lambda(h) \in \pi(\mathfrak{M} \cap \mathfrak{M}')$ and this is only possible, if $B = 0$ or $h = e$. The above equation then implies that α is free.

For the second assertion, let α be free. Note that $(\mathfrak{M} \rtimes_\alpha G) \cap (\mathfrak{M} \rtimes_\alpha G)' \subseteq (\mathfrak{M} \rtimes_\alpha G) \cap \pi(\mathfrak{M})' = \pi(\mathfrak{M} \cap \mathfrak{M}')$. Let $Y = \pi(A) \in (\mathfrak{M} \rtimes_\alpha G) \cap (\mathfrak{M} \rtimes_\alpha G)$ and $X = \pi(B)\Lambda(h) \in (\mathfrak{M} \rtimes_\alpha G)$, with $A, B \in \mathfrak{M}$ and $h \in G$. Then we get, with the same calculation as above, $\pi(A)\pi(B) = \pi(B)\pi(\alpha_h(A))$, but also $\pi(A)\pi(B) = \pi(B)\pi(A)$, since $A \in \mathfrak{M} \cap \mathfrak{M}'$. Hence, for all $B \in \mathfrak{M}$ and all $h \in G$ we have $(A - \alpha_h(A))B = 0$ and consequently, by faithfulness of π, $\alpha_h(A) = A$ for all $h \in G$. This implies $(\mathfrak{M} \rtimes_\alpha G) \cap (\mathfrak{M} \rtimes_\alpha G)' = \{\pi(A) \mid A \in \mathfrak{M} \cap \mathfrak{M}' : \forall h \in G : \alpha_h(A) = A\}$. $\square$

The following is then immediate.

Corollary 1.2.18:
Let $(\mathfrak{M}, G, \alpha)$ be a dynamical system, $\mathfrak{M}$ a factor and α free. Then $\mathfrak{M} \rtimes_\alpha G$ is a factor, and $\pi(\mathfrak{M}) \subset \mathfrak{M} \rtimes_\alpha G$ is an irreducible inclusion of factors.

If $\mathfrak{M}$ is a factor and α is free, then the crossed product $\mathfrak{M} \rtimes_\alpha G$ gives an example where one can easily calculate the index.

Proposition 1.2.19:
Let $(\mathfrak{M}, G, \alpha)$ be a dynamical system with $\mathfrak{M}$ a factor and α not inner. Then, in the notation of equations (1.2) *and* (1.3)*,*

- *the map $\mathcal{E}$ given by $\sum_{g\in G} \pi(A_g)\Lambda(g) \mapsto \pi(A_e)$ for an arbitrary indexing $(A_g)_{g\in G}$ of elements in $\mathfrak{M}$, is the unique conditional expectation $\mathfrak{M} \rtimes_\alpha G \to \pi(\mathfrak{M})$,*
- *the elements $\{\Lambda(h) \mid h \in G\}$ form a Pimsner-Popa basis for the inclusion $\pi(\mathfrak{M}) \subset \mathfrak{M} \rtimes_\alpha G$,*
- $\operatorname{Ind}(\mathcal{E}) = |G|$.

Proof. It can be checked by short calculations that $\mathcal{E}$ satisfies $\mathcal{E}(\mathbb{1}) = \mathbb{1}$, $\|\mathcal{E}(X)\| \leqslant \|X\|$, and $\mathcal{E}(aXb) = a\mathcal{E}(X)b$ for all $X \in \mathfrak{M} \rtimes_\alpha G$ and all $a, b \in \pi(\mathfrak{M})$. If $X = \sum_{g\in G} \pi(A_g)\Lambda(g) \in \mathfrak{M} \rtimes_\alpha G$, then

$$\begin{aligned} X^*X &= \sum_{g,h\in G} \Lambda(g)^*\pi(A_g^*A_h)\Lambda(h) \\ &= \sum_{g,h\in G} \pi(\alpha_{g^{-1}}(A_g^*A_{gh}))\Lambda(h), \end{aligned}$$

and therefore $\mathcal{E}(X^*X) = \sum_{g\in G} \pi(\alpha_{g^{-1}}(A_g^*A_g)) \geqslant 0$. Hence $\mathcal{E}$ is a conditional expectation, and since the inclusion $\pi(\mathfrak{M}) \subset \mathfrak{M} \rtimes_\alpha G$ is irreducible, it is unique.

Let $W : l^2(G, \mathcal{H}) \to \mathcal{H} \otimes \ell^2(G)$ be the unitary used in equation (1.4). First, we show that the Jones projection $e_{\pi(\mathfrak{M})}$ is given by

$$e_{\pi(\mathfrak{M})} = W^*\mathbb{1} \otimes E_e W$$

where E_e is the projection in $\ell^2(G)$ onto the subspace spanned by the basis vector δ_e. Indeed, with $X = \sum_{g\in G} \pi(A_g)\Lambda(g) \in \mathfrak{M} \rtimes_\alpha G$, we get

$$\begin{aligned} We_{\pi(\mathfrak{M})}Xe_{\pi(\mathfrak{M})}W^* &= \sum_{g,h\in G} (1 \otimes E_e)(\alpha_{h^{-1}}(A_g) \otimes E_h)(\mathbb{1} \otimes \lambda_g)(\mathbb{1} \otimes E_e) \\ &= \sum_{g\in G} \delta_{g,e}(A_g \otimes \mathbb{1})(\mathbb{1} \otimes \lambda_g)(\mathbb{1} \otimes E_e) \\ &= W\mathcal{E}(X)e_{\pi(\mathfrak{M})}W^*. \end{aligned}$$

Let ϕ be a faithful normal state on $\mathfrak{M}$ and, by Theorem 1.2.14, we can assume that $\mathcal{H}$ is the corresponding GNS representation of $\mathfrak{M}$. Let $\zeta_0 \in \mathcal{H}$ be cyclic and separating for $\mathfrak{M}$. Then the vector $\xi_0 := W^*\zeta_0 \otimes \delta_e \in \ell^2(G, \mathcal{H})$ cyclic and separating as well: Let $X \in \mathfrak{M} \rtimes_\alpha G$ be of the form $X = \sum_{g\in G} \pi(A_g)\Lambda(g)$ for some labelling $(A_g)_{g\in G}$ of elements in $\mathfrak{M}$. Then we have

$$\begin{aligned} W^* X\xi_0 &= \sum_{g,h\in G} (\alpha_h^{-1}(A_g) \otimes E_h)(\mathbb{1} \otimes \lambda_g)\zeta_0 \otimes \delta_e \\ &= \sum_{g\in G} \alpha_g^{-1}(A_g)\zeta_0 \otimes \delta_g, \end{aligned}$$

and hence, vectors of the form $X\xi_0$ span a dense subspace of $\ell^2(G, \mathcal{H})$. By denseness of such elements X in $\mathfrak{M} \rtimes_\alpha G$ it follows that ξ_0 is cyclic for $\mathfrak{M} \rtimes_\alpha G$.

Very similarly, let $X, Y \in \mathfrak{M} \rtimes_\alpha G$ with $X = \sum_{g\in G} \pi(A_g)\Lambda(G)$ and $Y = \sum_{g\in G} \pi(B_g)\Lambda(g)$ with A_g, B_g elements of $\mathfrak{M}$ and $g \in G$. Assume that $X\xi_0 = Y\xi_0$. Then with the same calculation in the previous equation, we find

$$\sum_{g\in G} \alpha_g^{-1}(A_g)\zeta_0 \otimes \delta_g = \sum_{g\in G} \alpha_g^{-1}(B_g)\zeta_0 \otimes \delta_g,$$

and since $(\delta_g)_{g\in G}$ is an orthonormal basis of $\ell^2(G)$, we find by comparing coefficients, that

$$A_g\zeta_0 = B_g\zeta_0,$$

for all $g \in G$. Since ζ_0 is separating for $\mathfrak{M}$, it follows that $X = Y$. Now it is easy to check that $e_{\pi(\mathfrak{M})}\xi_0 = \xi_0$, and therefore

$$e_{\pi(\mathfrak{M})} X e_{\pi(\mathfrak{M})}\xi_0 = \mathcal{E}(X) e_{\pi(\mathfrak{M})}\xi_0 = \mathcal{E}(X)\xi_0$$

for all $X \in \mathfrak{M} \rtimes_\alpha G$. Thus $e_{\pi(\mathfrak{M})}$ is the Jones projection for the conditional expectation $\mathcal{E}$.

With similar calculations as above, one can check that

$$\sum_{g\in G} \Lambda(g) e_{\pi(\mathfrak{M})} \Lambda(g)^* = \mathbb{1}.$$

Hence, the operators $(\Lambda(g))_{g\in G}$ fulfill the assumptions of Proposition 1.2.11, and thus the index $\mathrm{Ind}(\mathcal{E})$ is given by

$$\mathrm{Ind}(\mathcal{E}) = \sum_{g\in G} \Lambda(g)\Lambda(g)^* = |G|.$$

□

For crossed products there exists a correspondence between subgroups of the (discrete) group G and von Neumann subalgebras of $\mathfrak{M} \rtimes_\alpha G$ [Cho78]. This can be used to set up a Galois correspondence between von Neumann subalgebras of a von Neumann algebra and subgroups of automorphism groups acting on it [Cho78; HT72].

1.3 Quantum Spin Systems

The main objective of this thesis is the analysis of a quantum spin systems system. In this section we discuss the mathematical description of such systems in general. This includes the treatment of the thermodynamic limit in the setting of operator algebras, the existence of dynamics in the limit of infinite particle numbers and volumes, and ground states. A detailed treatment can be found in the standard reference [BR96], and an educational introduction is provided for instance by [Naa13b]. The detailed discussions of the system we analyse later can be found in the subsequent chapter.

Quantum many body systems describe the interaction between particles situated on a discrete structure, such as a lattice. Usually this framework is used as an idealised model to described crystalline structures, solid bodies and spin liquids.Here we follow the interpretation that at each point of the lattice there is a particle attached which carries degrees of freedom [BR96]. The model then describes how the degrees of freedom of the particles on the lattice interact with each other. A single particle's degrees of freedom are modelled by an n-dimensional Hilbert space. The interaction between various particles is given by a Hamiltonian and its nature depends on the type of physical system under consideration. While there are many possibilities to describe interactions, it turns out that typical physically relevant systems exhibit interactions where each single particle only interacts with the particles in close vicinity, i.e. only with few neighbours, or the coupling strength between particles decays rapidly with their relative distance. Examples for systems that are commonly used in the literature are the Ising model [Kog79; Cha87], the Heisenberg model, the AKLT model [Aff+87], the class of Kitaev's quantum double models [Kit03] and the class of Levin-Wen models [LW05]. The quantum double models can also be interpreted as lattice gauge theories [Kit03; HW05]. However, the only difference we make here is that the Hilbert spaces are not placed on the vertices of the lattice, but instead on the links (edges) between vertices (see [MM97] for an introduction in lattice gauge theories, and also [Kog79]).

1.3.1 Kinematics

The kinematics of a quantum spin system are defined in the following way. As the underlying spatial geometry we assume the square lattice $\mathbb{Z}^d$. Hereby, we regard $\mathbb{Z}^d$ as the graph containing both the vertices of $\mathbb{Z}^d$ and the edges given by tuples of neighbouring vertices. The discussion of more general lattices can be obtained in a straightforward way. The particles in the system are modelled by associating to every edge $e \in \mathbb{Z}^d$ a Hilbert space $\mathcal{H}_e$. We assume that $\mathcal{H}_e$ is a copy of $\mathbb{C}^n$ and that for any edge e the dimension n is the same. Again, the latter assumption is not strictly necessary, however it suffices for the purpose of this thesis. For finite regions $\mathcal{O} \subset \mathbb{Z}^d$ the total Hilbert space is then given as the tensor product $\bigotimes_{e\in\mathcal{O}} \mathcal{H}_e =: \mathcal{H}_\mathcal{O}$. While it is sufficient to work in the framework of Hilbert space if $\mathcal{O}$ is finite, it is more problematic when transitioning to the thermodynamic limit, that is, the limit of large particle numbers and large volumes. One could take an inductive limit of Hilbert spaces, however this requires to select a reference vector, and there is no obvious canonical choice for this [BR96]. Instead, a it is convenient to switch the perspective to bounded operators on the edge Hilbert spaces $\mathcal{A}(\{e\}) \coloneqq \mathcal{B}(\mathcal{H}_e)$, i.e. to the picture of C^*-algebras. There the limit of infinite particles and volumes is unambiguously defined [Tak79; Sak71]. For finite regions $\mathcal{O} \subset \mathbb{Z}^d$ we define the *local algebra*

$$\mathcal{A}(\mathcal{O}) \coloneqq \bigotimes_{e\in\mathcal{O}} \mathcal{A}(\{e\}) = \mathcal{B}(\mathcal{H}_\mathcal{O}).$$

For disjoint finite sets $\mathcal{O}_1$ and $\mathcal{O}_2$ we have $\mathcal{A}(\mathcal{O}_1 \cup \mathcal{O}_2) = \mathcal{B}(\mathcal{H}_{\mathcal{O}_1} \otimes \mathcal{H}_{\mathcal{O}_2})$, and $\mathcal{A}(\mathcal{O}_1)$ can be identified with the subalgebra $\mathcal{A}(\mathcal{O}_1) \otimes \mathbb{1}_{\mathcal{O}_2}$, where $\mathbb{1}_{\mathcal{O}_2}$ is the identity on $\mathcal{A}(\mathcal{O}_2)$. For finite sets $\mathcal{O} \subset \mathcal{O}'$ the algebra $\mathcal{A}(\mathcal{O})$ can be canonically embedded into $\mathcal{A}(\mathcal{O}')$ by identifying $\mathcal{A}(\mathcal{O})$ with $\mathcal{A} \otimes \mathbb{1}_{\mathcal{O}'\setminus\mathcal{O}}$. Thus, if $\mathcal{P}_f(\mathbb{Z}^d)$ denotes the family of finite subsets of $\mathbb{Z}^d$, the map

$$\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$$

defines a net of C^*-algebras. One can show that there exists an inductive limit $\mathcal{A}$, which is the smallest C^*-algebra containing all local algebras (c.f. [BR96] and [Sak71], also [Tak55]. This structure is very similar to that found in algebraic quantum field theory, see also [Emc00]).

Moreover, it can be shown that if one chooses a sequence $(\mathcal{O}_n)_{n\in\mathbb{N}}$ in $\mathcal{P}_f(\mathbb{Z}^d)$ with $\mathcal{O}_n \subset \mathcal{O}_{n+1}$ and $\bigcup_n \mathcal{O}_n = \mathbb{Z}^d$, then for the *algebra of local observables*

$$\mathcal{A}_{\text{loc}} := \bigcup_{\mathcal{O}\subset\mathbb{Z}^d} \mathcal{A}(\mathcal{O}),$$

there exists a unique norm $\|\cdot\|$ such that algebra of local observables is a norm dense subalgebra of $\mathcal{A}$, i.e. $\mathcal{A} = \overline{\mathcal{A}_{\text{loc}}}^{\|\cdot\|}$. The bar denotes the closure in the norm $\|\cdot\|$. In fact, this norm is induced by the norms on the local algebras, with the canonical identification $\mathcal{A}(\mathcal{O}) \subset \mathcal{A}(\mathcal{O}')$, if $\mathcal{O} \subset \mathcal{O}'$. Hence, operators in $\mathcal{A}$ can be approximated by local observables up to an arbitrarily small error. Due to this property the algebra $\mathcal{A}$ is usually referred to as the *quasilocal algebra*.

The local algebras have the property that for disjoint $\mathcal{O}, \mathcal{O}' \in \mathcal{P}_f(\mathbb{Z}^d)$ the operators commute, i.e. for all $A \in \mathcal{A}(\mathcal{O})$ and for all $B \in \mathcal{A}(\mathcal{O}')$ it holds that $[A, B] = 0$. This is usually referred to as *locality*. The *spatial support*, or just *support*, of a local operator $A \in \mathcal{A}_{\text{loc}}$ is the smallest set $\mathcal{O} \in \mathcal{P}_f(\mathbb{Z}^2)$ such that $A \in \mathcal{A}(\mathcal{O})$. We then write $\operatorname{supp}(A) := \mathcal{O}$.

As the local algebras are type-I factors the algebra $\mathcal{A}$ is a uniformly hyperfinite (UHF) algebram, and, as the limit of simple algebras, again simple [BR96; Sak71]. This implies that any representation π of $\mathcal{A}$ is automatically faithful.

1.3.2 Dynamics

To define the dynamics of quantum spin systems one starts out with the definition of an interaction between the particles (c.f.[BR96]). Mathematically an *interaction* is a map Φ from sets $\mathcal{O} \in \mathcal{P}_f(\mathbb{Z}^d)$ to self-adjoint operators $\Phi(\mathcal{O}) \in \mathcal{A}(\mathcal{O})$. For finite sets $\Lambda \in \mathcal{P}_f(\mathbb{Z}^d)$ the *local Hamiltonian* that generates the dynamics on $\mathcal{A}(\Lambda)$ is then given as the sum of all interaction terms of subsets of Λ, i.e.

$$H_\Phi(\Lambda) := \sum_{\mathcal{O}\subset\Lambda} \Phi(\mathcal{O}),$$

and $H(\Lambda) \in \mathcal{A}(\Lambda)$. The *range* of the interaction Φ is given by

$$d_\Phi := \inf\left\{ d \,\middle|\, d \geqslant 1 \wedge \forall \mathcal{O} \in \mathcal{P}_f(\mathbb{Z}^d) : \sup_{x,y\in\mathcal{O}} d(x,y) > d \implies \Phi(X) = 0 \right\},$$

where $d(\cdot,\cdot) : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}$ is the taxicab metric.

The local Hamiltonians generate dynamics on the local algebras. If the interaction range is finite or decays fast enough then it can be shown that the local

Hamiltonians give rise to a time evolution on the quasilocal algebra $\mathcal{A}$ in terms of a strongly continuous one-parameter group of automorphisms (see [BR96] for the standard introduction, [NS10] for an overview including recent results, and also [NOS06]). For the purpose of this thesis we only need the result for interactions with finite interaction range, i.e. $d_\Phi < \infty$. In particular this implies that for any $x \in \mathbb{Z}^d$ there are only finitely many regions $\mathcal{O} \subset \mathcal{P}_f(\mathbb{Z}^d)$ with $x \in \mathcal{O}$ and $\Phi(X) \neq 0$.

The generator of the dynamics will be a $*$-derivation. A $*$*-derivation* of a C^*-algebra $\mathcal{B}$ is a linear, possibly unbounded operator on $\mathcal{B}$ with domain $\mathcal{D}(\delta) \subset \mathcal{B}$ such that $\mathcal{D}$ is a subspace and for all $A, B \in \mathcal{D}(\delta)$ we have

$$\delta(A)^* = \delta(A^*),$$
$$\delta(AB) = \delta(A)B + A\delta(B).$$

We then have the following statement about the existence of dynamics [BR96, Example 6.2.7].

Theorem 1.3.1:
Let $\mathcal{A}$ be the quasilocal algebra of a quantum spin system on $\mathbb{Z}^d$, and let Φ be an interaction with finite range. Then there exists a norm-closable $*$-derivation δ of $\mathcal{A}$ with domain

$$\mathcal{D}(\delta) = \mathcal{A}_{\mathrm{loc}}$$

such that for all $\mathcal{O} \in \mathcal{P}_f(\mathbb{Z}^d)$ and $A \in \mathcal{A}(\mathcal{O})$ it holds

$$\delta(A) = i \sum_{\substack{\Lambda \in \mathcal{P}_f(\Gamma) \\ \Lambda \cap \mathcal{O} \neq \emptyset}} [H_\Phi(\Lambda), A].$$

Its closure $\bar{\delta}$ generates a strongly-continuous one-parameter group α of $*$-automorphisms of $\mathcal{A}$. If $(\mathcal{O}_n)_{n\in\mathbb{N}}$ is any increasing sequence of sets in $\mathcal{P}_f(\mathbb{Z}^d)$ exhausting $\mathbb{Z}^d$, then for all $t \in \mathbb{R}$ and all $A \in \mathcal{A}$

$$\lim_{n\to\infty} \|\tau_t(A) - \tau_t^{\mathcal{O}_n}(A)\| = 0$$

where the convergence is uniform for t in compacts, does not depend on the sequence $(\mathcal{O}_n)_{n\in\mathbb{N}}$, and where

$$\tau_t^{\mathcal{O}_n}(A) := e^{itH_\Phi(\mathcal{O}_n)} A e^{-itH_\Phi(\mathcal{O}_n)}.$$

1.3.3 Ground States

Most of the physically relevant Hamiltonians H have a spectrum that is bounded from below[7]. By shifting the spectrum accordingly this bound can be set to zero. This is equivalent to $\langle\psi, H\psi\rangle \geqslant 0$ for all $\psi \in \mathcal{D}(H)$, where $\mathcal{D}(H)$ is the domain on which H is self-adjoint. If the spectrum of H is discrete, or if zero is an eigenvalue of H, then vectors $\psi \in \mathcal{H}$ in the kernel of H, i.e. $H\psi = 0$, are called ground states of H.

In the description of dynamics on a C^*-algebra $\mathcal{B}$ as above, we do not *a priori* have a Hamiltonian at hand. The dynamics are given by a strongly continuous one-parameter group α of automorphisms of $\mathcal{B}$ generated by a derivation δ. However, if ω is a state that is stationary under the time evolution, i.e. $\omega \circ \alpha = \omega$, then in the GNS representation $(\pi_\omega, \mathcal{H}_\omega, \Omega)$ there exists a strongly continuous one-parameter groups of unitaries U on $\mathcal{B}(\mathcal{H}_\omega)$ such that for all $A \in \mathcal{B}$ and for all $t \in \mathbb{R}$

$$U_t \pi_\omega(A) U_t^* = \pi_\omega(\alpha_t(A)).$$

Furthermore, U can be chosen such that $U_t\Omega = \Omega$ for all $t \in \mathbb{R}$ (see also [BR96, Corollary 2.3.17]). By Stone's theorem, strongly continuous one-parameter groups of unitaries are generated by essentially self-adjoint operators (see e.g. [Sch12]), thus for the group U there exists a Hamiltonian H_ω on $\mathcal{H}_\omega$, and it satisfies $H_\omega\Omega = 0$ (c.f. [BR96, Proposition 3.2.28] and discussion afterwards). In particular, we have that

$$(\forall \psi \in \pi_\omega(\mathcal{D}(\delta))\Omega)\,(\forall A \in \mathcal{D}(\delta)) : \pi_\omega(\delta(A))\Omega = i[H_\omega, \pi_\omega(A)]\psi.$$

One now calls ω a ground state of the dynamics α if $H_\omega \geqslant 0$. This is equivalent to a more algebraic criterion (c.f. [Naa13b, Theorem 3.4.3] or [BR96]):

Theorem 1.3.2:
Let $\mathcal{A}$ be a C^*-algebra and α be a one-parameter group of automorphisms of $\mathcal{A}$ generated by a $*$-derivation δ with dense domain $\mathcal{D}(\delta)$. For a state ω of $\mathcal{A}$ the following statements are equivalent:

- *For each $t \in \mathbb{R}$ we have $\omega \circ \alpha_t = \omega$, and the Hamiltonian H_ω in the GNS representation of ω satisfies $H_\omega \geqslant 0$.*
- *For all $A \in \mathcal{D}(\delta)$ we have $-i\omega(A^*\delta(A)) \geqslant 0$.*

[7] Or they are bounded from above. However, this only differs by a sign.

We take this theorem as the motivation for the following definition of a ground state.

Definition 1.3.3:
Let $\mathcal{B}$ be a C^*-algebra, let α be a strongly continuous one-parameter group of automorphisms on $\mathcal{B}$ generated by a derivation δ with dense domain $\mathcal{D}(\delta)$. A state ω on $\mathcal{B}$ is said to be a ground state *of α if*

$$\forall A \in \mathcal{D}(\delta) : -i\omega(A^*\delta(A)) \geqslant 0.$$

To shed some light on this condition, let $\mathcal{H}$ be a finite dimensional Hilbert space and H some Hamiltonian with $H \geqslant 0$ and ground state $\Omega \in \mathcal{H}$ satisfying $H\Omega = 0$. Then H generates a continuous one-parameter group of unitaries by $U_t = e^{iHt}$ which gives rise to an automorphism group of $\mathcal{B}(\mathcal{H})$ by $\alpha_t(A) := U_t A U_t^*$ for all $A \in \mathcal{B}(\mathcal{H})$ and $t \in \mathbb{R}$. Note, that then $U_t\Omega = \Omega$ for all $t \in \mathbb{R}$. The (bounded) derivation generating α is given by $\delta(A) = i[H, A]$. Setting $\omega(A) := \langle \Omega, A\Omega \rangle$ for $A \in \mathcal{B}(\mathcal{H})$ defines an α-invariant state, and we get

$$\omega(\delta(A)) = i\langle \Omega, (HA - AH)\Omega \rangle = 0$$

and

$$\begin{aligned} -i\omega(A^*\delta(A)) &= \langle A\Omega, (HA - AH)\Omega \rangle \\ &= \langle \Omega, A^*HA\Omega \rangle \geqslant 0, \end{aligned}$$

for all $A \in \mathcal{B}(\mathcal{H})$. The general situation in the above theorem can be proven in a very similar way. We also want to remark, that due to [BR96, Proposition 5.3.25] and Theorem 1.3.1 there always exists a ground state for quantum spin systems.

1.4 Quantum Double of Finite Groups

In this section we discuss the quantum double $\mathcal{D}(G)$ for finite groups G. It is an application of a more general scheme by Drinfel'd for constructing quasi-triangular Hopf algebras from a given finite dimensional Hopf algebra with invertible antipode [Dri88; Kas95] (for the construction of $\mathcal{D}(G)$, see also [Gou93]). Its importance for us lies in its representation theory which plays a crucial role in determining the excitation structure of Kitaev's quantum double models.

We start with introducing some notations. Consider a finite group G. Throughout this work the inverse of a group element $g \in G$ is denoted by $\bar{g}$ and we write

e for the identity of G. We denote the *centraliser* of some element $g \in G$ in G by $Z_G(g)$, i.e.

$$Z_G(g) = \{h \in G \,|\, hg = gh\},$$

which is a subgroup of G. Given $r \in G$ we denote its *conjugacy class* by

$$C \coloneqq \{hr\bar{h} \,|\, h \in G\},$$

and elements of a conjugacy class C are denoted $c_1, \ldots, c_{|C|}$. For the following note, that if $H \subseteq G$ is a subgroup, the cosets gH with $g \in G$ are either disjoint or equal. This can be seen by noting that $h \sim g$ if and only if $\bar{h}g \in H$, with $h, g \in G$, defines an equivalence relation on G. Then we have $hH = gH$ if and only if $h \sim g$. This implies the following properties.

LEMMA 1.4.1:
Let $C = \{c_1, \ldots, c_{|C|}\}$ *be a conjugacy class of* G. *Furthermore, let* $Q_C = \{q_1, \ldots, q_{|C|}\}$ *be a set of representatives of each coset of* $Z_G(r)$ *in* G *such that* $q_i r \overline{q_i} = c_i \in C$ *for all* $i = 1, \ldots, |C|$, *and where we choose* $q_i = e$ *if* $c_i = r$.

- *The elements* $c_1, \ldots, c_{|C|}$ *are in one to one correspondence with left cosets of* $Z_G(r)$ *in* G.
- *For each* $g \in G$ *there exists a unique* $n \in Z_G(r)$ *and a unique* $q \in Q_C$ *with* $g = qn$.

Proof. Concerning the first statement, note that for each $h, k \in gZ_G(r)$ for some $g \in G$, it holds that $hr\bar{h} = kr\bar{k}$. Conversely, let $h, k \in G$ such that $hr\bar{h} = kr\bar{k}$. Then $\bar{h}k \in Z_G(r)$, but this is equivalent to $hZ_G(r) = kZ_G(r)$.

For the second statement, consider a fixed $g \in G$. Then certainly $g \in gZ_G(r)$, and since the left cosets of $Z_G(r)$ are either equal or disjoint, there exists a unique $q \in Q_C$ such that $g \in qZ_G(r)$. Now let $n, m \in Z_G(r)$ be such that $g = qn = qm$. But this already implies $m = n$, thus finishing the proof. $\square$

The second statement defines a map $g \mapsto i(g)$ from elements $g \in G$ to indices $i(g) \in \{1, \ldots, |C|\}$, and a map $G \ni g \mapsto n(g) \in Z_G(r)$. We then have $g = q_{i(g)} n(g)$ for $g \in G$, and for $c_i \in C$ we have $q(c_i) = q_i$ (This notation was also used in [BM08]). For convenience, we will also sometimes write $q(g)$ for $q_{i(g)}$.

Given a subgroup $K \subset G$ let π be an *irreducible unitary representations* of K. We always assume that the representations we consider are unitary. In the following we fix a basis in the representation space of π, and for a given element $g \in G$ the matrix elements of $\pi(g)$ are denoted by $\pi_{i,j}(g)$. For convenience we denote the *character* of π (i.e. the trace of $\pi(g)$) also by π. In the discussion of ribbon operators in later sections we will use both the representation and its character in such a way, that it is clear from the context what is meant. The *trivial representation* is denoted by id.

1.4.1 The Quantum Double $\mathcal{D}(G)$

We now come to the definition of the quantum double $\mathcal{D}(G)$ of a finite group G. Let $\ell^2(G)$ be the vector space of square summable, complex valued functions on G, and $\mathbb{C}[G]$ be the group algebra of G. Note that $\ell^2(G)$ has the structure of a commutative algebra with the multiplication defined by the pointwise product. The canonical basis $(\delta_g)_{g\in G}$ of $\ell^2(G)$ is given by $\delta_g(h) := \delta_{g,h}$ with $g, h \in G$. We identify the basis of the group algebra $\mathbb{C}[G]$ with elements $x \in G$. The group algebra $\mathbb{C}[G]$ acts on $\ell^2(G)$ with the adjoint action which is determined by

$$x\delta_g\bar{x} = \delta_{xg\bar{x}}, \tag{1.8}$$

with $x, g \in G$, and extending linearly to $\ell^2(G)$. This allows to define the semi-direct product of $\ell^2(G)$ with $\mathbb{C}[G]$, which we denote by $\mathcal{D}(G)$, i.e.

$$\mathcal{D}(G) := \ell^2(G) \rtimes \mathbb{C}[G].$$

The product and the unit element of $\mathcal{D}(G)$ is given by

$$\begin{aligned}(\delta_g, x)\cdot(\delta_h, y) &= \delta_{g,xh\bar{x}}(\delta_g, xy)\\ 1 &= \sum_{g\in G}(\delta_g, e),\end{aligned}$$

with $g, h, x, y \in G$, which gives it the structure of an algebra. Note, that the product can be regarded as a map $\cdot : \mathcal{D}(G)\otimes\mathcal{D}(G) \to \mathcal{D}(G)$, and the unit element as the map $1 : \mathbb{C} \to \mathcal{D}(G)$, which is called *unit*. In addition, one can define the maps $\Delta : \mathcal{D}(G) \to \mathcal{D}(G)\otimes\mathcal{D}(G)$, $\epsilon : \mathcal{D}(G) \to \mathbb{C}$ and $S : \mathcal{D}(G) \to \mathcal{D}(G)$. by the following relations.

$$\begin{aligned}\Delta((\delta_g, x)) &:= \sum_{\substack{k,l\in G\\ kl=g}}(\delta_k, x)\otimes(\delta_l, x)\\ \epsilon((\delta_g, x)) &:= \delta_{g,e}\\ S((\delta_g, x)) &:= (\delta_{\bar{x}\bar{g}x}, \bar{x}),\end{aligned}$$

with $x, g \in G$. The map Δ is called *co-product*, ϵ is called *co-unit* and S is called *antipode*. This structure on $\mathcal{D}(G)$ makes it a *coalgebra*.

Definition 1.4.2:
The quantum double $\mathcal{D}(G)$ *of a finite group* G *is defined the outer semi-direct product*

$$\mathcal{D}(G) := \ell^2(G) \rtimes \mathbb{C}[G].$$

with respect to the action defined in Equation (1.8)*, together with the mappings* $(\cdot, 1, \Delta, \epsilon, S)$ *as defined above.*

The operations $(\cdot, 1, \Delta, \epsilon)$ defined above w.r.t. $\mathcal{D}(G)$ satisfy certain compatibility conditions and comprise what is called a *bi-algebra*. The antipode S extends this structure to that of an *Hopf algebra* structure (see also [Kas95]). In addition, $\mathcal{D}(G)$ there is also a universal R-matrix $R \in \mathcal{D}(G) \otimes \mathcal{D}(G)$ given by

$$R := \sum_{g \in G} (\delta_g, e) \otimes (1, g)$$

which makes $\mathcal{D}(G)$ a *quasi-triangular* Hopf algebra. Essentially, the existence of a universal R-matrix allows to define a *braiding* on the category of representations of $\mathcal{D}(G)$ (see for instance [Kas95]). However, in the remainder we will not make direct use of these structures and mention them here only for completeness.

1.4.2 Representations of $\mathcal{D}(G)$ in a Nutshell

The rich structure of the quantum double $\mathcal{D}(G)$ for finite groups G allows for a relatively clean classification of its irreducible representations [DPR91; Gou93] (see [Kas95] for a more general discussion). In what follows we will review the basic analysis of $\mathcal{D}(G)$. Some of this will occur later, though implicit, in the analysis of the local excitations of the quantum double model.

Let $r \in G$ be fixed from now on, and denote its corresponding conjugacy class C in G and the centraliser $Z_G(r)$ of r in G. As before, let Q_C be a set of representatives of left cosets of $Z_G(r)$ in G such that $q_i r \overline{q_i} = c_i$ for all $q_i \in Q_C$ and $c_i \in C$. There are two observations which are essential in the subsequent discussion. The first is Lemma 1.4.1 and the remark after: Each $g \in G$ has a unique decomposition $g = q(g)n(g)$ with $q(g) \in Q_C$ and $n(g) \in Z_G(r)$. Note, that this implies $\bigcup_{q \in Q_C} q Z_G(r) = G$.

The second observation is the following.

LEMMA 1.4.3:
Let C *be a conjugacy class of* G*, and let* Q_C *be as in Lemma 1.4.1. Then for each* $g \in G$ *and each* $c \in C$ *there exists a unique* $c' \in C$ *such that* $\overline{q(c)}gq(c') \in Z_G(r)$ *and* $gc\overline{g} = c'$

Proof. Let $g \in G$ and $c \in C$. By Lemma 1.4.1 there exists a unique $q' \in Q_C$, and thus a unique $c' \in C$ with $q' = q(c')$, such that $gq(c) \in q(c')Z_G(r)$, hence we have $\overline{q(c')}gq(c) \in Z_G(r)$. Using this, one can check that $gc\overline{g} = q(c')r\overline{q(c')} = c'$. □

Let now π be an irreducible representation of $Z_G(r)$ with representation space V^r_π. Then, by extending the action π of $Z_G(r)$ linearly on V^r_π, the space V^r_π is a also left $\mathbb{C}[Z_G(r)]$-module. Since $\mathbb{C}[Z_G(r)]$ also acts on $\mathbb{C}[G]$, we can take the tensor product between $\mathbb{C}[G]$ and V^r_π, and we set

$$V_{r,\pi} := \mathbb{C}[G] \otimes_{\mathbb{C}[Z_G(r)]} V^r_\pi.$$

This space is spanned by vectors of the form

$$v(c) := q(c) \otimes v$$

where $c \in C$ and $v \in V^r_\pi$, and carries an action of G given by

$$(gv)(c) := \left(\pi\left(\overline{q(gc\overline{g})}gq(c)\right)v\right)(gc\overline{g}), \tag{1.9}$$

for all $g \in G$ and $c \in C$. This action of G on $V_{\pi,r}$ can be linearly extended to an action of $\mathbb{C}[G]$ on $V_{\pi,r}$.

Now, it can be shown that the space $V_{\pi,r}$ can be decomposed as $V_{\pi,r} = \bigotimes_{c\in C} V_{\pi,r}(c)$ where $V_{\pi,r}(c) = \operatorname{linspan}\{v(c)\,|\,v \in V^r_\pi\}$, and each summand $V_{\pi,r}(c)$ carries an irreducible representation of $Z_G(c)$ [Gou93]. Thus for different choices $r, r' \in C$ the spaces $V_{\pi,r}$ and $V_{\pi,r'}$ are isomorphic. In the following we identify these spaces, and write $V_{\pi,C}$ instead of $V_{\pi,r}$. Similarly, if π' is a representation that is unitary equivalent to π, then the space $V_{\pi',C}$ is isomorphic to $V_{\pi,C}$, and we suppress the equivalence relation here as well.

Next, one defines an action of $\ell^2(G)$ on $V_{\pi,C}$ by

$$(\delta_g v)(c) := \delta_{g,c} v(c) = \delta_{g,c}\, q(c) \otimes v,$$

for all $g \in G$, $c \in C$ and $v \in V_{\pi,C}$. This, together with the $\mathbb{C}[G]$ action defined in equation 1.9, extends to an action of $\mathcal{D}(G)$ on $V_{\pi,C}$ which then reads

$$((\delta_g, h)v)(c) := \delta_{g,hc\overline{h}}\left(\pi\left(\overline{q(hc\overline{h})}hq(c)\right)v\right)(hc\overline{h}).$$

This is in fact sufficient to determine all irreducible $\mathcal{D}(G)$-modules, the proof of which can be found in [DPR91; Gou93]

Theorem 1.4.4:
Let G be a finite group, C and D conjugacy classes with representatives $r \in C$ and $s \in D$. Let π and χ be irreducible representations of $Z_G(r)$ and $Z_G(s)$, respectively. Then

- $V_{C,\pi}$ *as constructed above, is n irreducible* $\mathcal{D}(G)$*-module.*
- $V_{C,\pi} \cong V_{D,\chi}$ *if and only if* $C = D$ *and* $\pi \cong \chi$.
- *Every finite dimensional left* $\mathcal{D}(G)$*-module is completely reducible.*
- *Every irreducible left* $\mathcal{D}(G)$*-module is equivalent to one of the* $V_{C,\pi}$.

This means, that every irreducible representation of $\mathcal{D}(G)$ is labelled by a pair (C, π) where C is a conjugacy class of G and π an irreducible representation of $Z_G(r)$ for some $r \in C$.

The finite dimensional left $\mathcal{D}(G)$-modules together with the intertwiners between representations form a category, $\mathbf{Rep}_f\mathcal{D}(G)$, where f stands for finite. One can show that $\mathbf{Rep}_f\mathcal{D}(G)$ is a modular tensor category. Roughly speaking, this means that the morphism sets in $\mathbf{Rep}_f\mathcal{D}(G)$ are finite dimensional vector spaces, and the category has a tensor product, twist, subobjects, direct sums, conjugates, and invertible S-matrix (for details see [Kas95; Naa12a]).

This structure makes $\mathbf{Rep}_f\mathcal{D}(G)$ also suitable for topological quantum computation. In topologically ordered systems, where the local excitations are described by anyons, the braiding of pairs of these quasiparticles is a unitary operation on the ground state space. This can be used to implement unitary gates that are capable of performing fault tolerant quantum computations on quantum information that is encoded in the local excitations [Nay+08]. The details of the gate set depend on the details of the statistics of the anyons. However, in order to implement non-trivial gate sets one needs non-abelian anyons, i.e. where the anyons carry multiple inner degrees of freedom and where the braiding is a unitary operation on these degrees. In the case of Kitaev's quantum double model for a finite group G, the anyons are described by $\mathbf{Rep}_f(G)$. If G is complex enough this allows to implement even universal gate sets for quantum computation [Moc03; Moc04]. Note, however, that in order to do this, one needs additional ancillas to the system.

2 Kitaev's Quantum Double Model

The quantum double models are a class of analytically solvable quantum spin systems in two dimensions that were proposed by Kitaev as a generalisation of the toric code [Kit03]. Each finite group gives rise to a quantum double model and it determines the type of local excitations above the ground state, as well as the ground state degeneracy (if existent). The toric code, originally defined as a stabiliser code on a lattice embedded in a torus [Kit97], can be formulated as the quantum double model with underlying group $\mathbb{Z}_2$.

The quantum double models exhibit many features that are considered typical for topologically ordered systems and which makes them appealing as models to study topological order. The ground state(s) of the quantum double models are locally indistinguishable, i.e. in case of a ground state degeneracy, the expectation values of any operator that acts only on a small part of the system are the same for all ground states[1]. Furthermore, the gapped Hamiltonians have a ground state degeneracy depending on the topology of manifold in which the lattice is embedded. For example, the toric code has 4^g different ground states, where g is the genus of the surface in which the lattice is embedded [Kit03]. For arbitrary finite groups this degeneracy is more complicated, but the dependence on the genus is similar. This ground state degeneracy is closely related to the existence of anyons [Ein90; WDF90].

In fact, the local excitations above the ground state exhibit anyonic statistics. This means that the braiding rules, which describe the transformation behaviour of exchanging pairs of excitations, are different from those of particles with bose or fermi statistics. For instance, in the case of abelian groups the state vector de-

[1] This follows from the discussion in [Naa12a, Section 12.1]

scribing two excitations picks up a complex phase when exchanging the positions of excitations. In the case that the underlying group is non-abelian the anyons have additional inner degrees of freedom and the braiding is described by unitary transformations of the state vector. The fundamental object describing the different types of anyons, the braiding, and the fusion, which describes the transformation of the state vector if two excitations are brought together, is determined by the irreducible representations of Drinfel'd's quantum double $\mathcal{D}(G)$ of the underlying group G [Kit03]. The latter is a certain Hopf algebra constructed from the group, and its finite dimensional representations form what is called a modular tensor category. This essentially means, that there exists a consistent set of braiding and fusion rule on the category $\mathbf{Rep}_f\mathcal{D}(G)$ of finite dimensional representations of $\mathcal{D}(G)$, and each type of the anyons corresponds to an irreducible representation.

An appealing aspect of Kitaev's quantum double models is the simplicity of the local excitations. They always appear in conjugate pairs and at the endpoint of ribbons, i.e. they can be connected by certain paths on the lattice. It turns out that such pairs of excitations are independent of the ribbon connecting them, as long it does not cross any other excitations or does not change its homotopy class. There is a convenient description in terms of so-called ribbon operators acting non-trivially along ribbons. If acting on the ground state these operators create a pair of local excitations at the endpoint of the associated ribbon. Moreover, for a fixed ribbon, these operators can be labelled by irreducible representations of $\mathcal{D}(G)$ which in turn can be described by objects solely related to the group G. The study of braiding and fusion of the anyons of the model then translates to the study of the commutation relations of the ribbon operators [Kit03; BM08]. This also allows the study of the quantum double on surfaces with a boundary [BSW11], and extensions of the models to such with broken gauge symmetry [BM08].

In this chapter we review the definition of Kitaev's quantum double model for an arbitrary finite group G on the plane square lattice[2] $\mathbb{Z}^2$. Throughout the discussion we largely follow the notation of reference [BM08]. We start with introducing the underlying geometric objects and operations related to the notion of ribbons. This is followed by the definition of ribbon operators. These are then used to set up star and plaquette operators with which we define the local Hamiltonians of the quantum double models. After this we delve into the discussion of charge projections and properties of ribbon operators without making additional assumptions

[2]One can also define the model on more general 2D lattices. However, for the purpose of this thesis it suffices to consider Kitaev's quantum double models on the square lattice

on the finite group G underlying the construction. Our focus lies on the commutation relations between ribbon operators and how they transform under inverting and deforming the underlying ribbons. We devote one section to summarise the discussion for the case where G is finite and abelian and point out some additional properties which we need in later chapters. In the last section we discuss how to take the thermodynamic limit of the model and the existence of ground states of the dynamics.

2.1 Sites and Ribbons

In order to define the local Hamiltonians we need to specify certain geometric objects on $\mathbb{Z}^2$ and operators associated to them (see also [BM08] for a much more extensive treatment.) In this section we define sites and ribbons in the lattice and recall ribbon inversions and deformations.

In the following we regard $\mathbb{Z}^2$ as a graph together with a fixed orientation as in Figure 2.1. The *vertices* v of the graph are simply the points of $\mathbb{Z}^2$. The links between the vertices of $\mathbb{Z}^2$ form the *edges* of the graph. Each edge $b \in \mathbb{Z}^2$ connects two vertices $v_0, v_1 \in \mathbb{Z}^2$ and we therefore specified b by the tuple (v_0, v_1). The orientation of $\mathbb{Z}^2$ is given by the order in which the vertices appear in the tuple, hence the edge b points from v_0 to v_1. The vertices (v_0, v_1) are then the boundary of b and we write $\partial_0 b = v_0$ and $\partial_1 b = v_1$. A *face* f is the square enclosed by four consecutive edges. We say that a face f lies to the *right* (*left*) of an edge $b = (v_0, v_1)$ if it lies to the right (left) of the edge when viewed from the initial vertex v_0. If we connect the centers of two adjacent faces with each other, we obtain the dual graph $(\mathbb{Z}^2)^*$ of $\mathbb{Z}^2$. I.e. the dual graph is the graph obtained by regarding each face $f \in \mathbb{Z}^2$ as a dual vertex v^*, and connect dual vertices $v_0^*, v_1^* \in (\mathbb{Z}^2)^*$ if the corresponding faces $f_0, f_1 \in \mathbb{Z}^2$ are separated by an edge in $\mathbb{Z}^2$. Note that by this construction the vertices in $\mathbb{Z}^2$ will correspond to faces in $(\mathbb{Z}^2)^*$. Therefore, we do not distinguish between dual faces and vertices, or faces and dual vertices. Note also that for an edge $b \in \mathbb{Z}^2$ we have $(b^*)^* = b$ and so on. A (dual) edge $b^* \in (\mathbb{Z}^2)^*$ inherits an orientation from the edge $b \in \mathbb{Z}^2$ it crosses in the following way. The initial (final) dual vertex v_0^* (v_1^*) of b^* is the face to the left (right) of b. Similarly as for edges in $\mathbb{Z}^2$ we write $B^* = (v_0^*, v_1^*)$ and the order indicates the orientation.

Definition 2.1.1:
A site *in* $\mathbb{Z}^2$ *is a tuple* (v, f) *consisting of a vertex* $v \in \mathbb{Z}^2$ *and a neighboring face* $f \in \mathbb{Z}^2$*. Note that sites and dual sites coincide. We use the notation* v_s *and* f_s *to indicate the vertex and face at* s *and regard a site as the line from* v *to the center of* f*.*

A direct triangle τ *is a tuple* (s_0, s_1, b) *of sites* $s_0, s_1 \in \mathbb{Z}^2$ *and an edge* $b \in \mathbb{Z}^2$ *with the following properties: The faces of* $s_0 = (v_0, f_0)$ *and* $s_1 = (v_1, f_1)$ *coincide, i.e.* $f_0 = f_1$*, and* b *is the edge given by* (v_0, v_1) *or* (v_1, v_0)*. Regarding* s_0, s_1, b *as lines, the tuple* (s_0, s_1, b) *is a list of sides of a triangle listed in counterclockwise order.*

A dual triangle τ' *is a tuple* (s_0, s_1, b^*) *with* s_0, s_1 *being sites as above, but* $b^* \in (\mathbb{Z}^2)^*$ *is the dual edge given by* (f_0, f_1) *or* (f_1, f_0)*. The tuple* (s_0, s_1, b^*) *lists the sides of a triangle in clockwise order.*

The boundaries of both types of triangles τ *are given by* $\partial_0\tau = s_0$ *and* $\partial_1\tau = s_1$*. We say that* τ *points from* $\partial_0\tau$ *to* $\partial_1\tau$*, providing the triangles with an orientation.*

For each direct (dual) triangle $\tau = (s_0, s_1, b)$ there is a unique *complementary* triangle given by the direct (dual) triangle $\bar{\tau} = (s_0', s_1', \bar{b})$ where $\bar{b}$ obtained from b by inverting its orientation. Two triangles τ, τ' *overlap* if either $\tau = \tau'$ or, if τ is direct and τ' is dual, then for $i = 0$ or $i = 1$ it holds that $\partial_i\tau = \partial_i\tau'$.

Definition 2.1.2 [BM08]:
A strip $\rho = (\tau_0, \tau_1, \dots, \tau_n)$ *is a list of pairwise non-overlapping triangles with*

- $\forall i \in \{1, \dots, n\} : \partial_0\tau_i = \partial_1\tau_{i-1}$.

Denote by $p_\rho = (v_{\partial_0\lambda_0}, \dots, v_{\partial_1\lambda_k})$ *the ordered list of vertices of the direct triangles* λ_i *in* ρ*, and* $p_\rho^* = (f_{\partial_0\lambda_0'}, \dots, f_{\partial_1\lambda_m'})$ *the ordered list of faces of the dual triangles* λ_j' *in* ρ*. Then* ρ *is a* ribbon *if the following conditions are fulfilled:*

- $\forall \tau \in \rho : \bar{\tau} \notin \rho$,
- $\forall 0 \leqslant i \neq j \leqslant k$ *with* $i \neq 0 \vee j \neq k$ *it holds:* $v_i \neq v_j$,
- $\forall 0 \leqslant i \neq j \leqslant m$ *with* $i \neq 0 \vee j \neq m$ *it holds:* $f_i \neq f_j$.

Let ρ *be a ribbon and set* $\partial_0\rho \coloneqq \partial_0\tau_0$ *and* $\partial_1\rho \coloneqq \partial_1\tau_n$*. A ribbon is said to be* closed *if* $\partial_0\tau_0 = \partial_1\tau_n$*. Two ribbons* ρ_1, ρ_2 *are* composable *if* $\partial_1\rho_1 = \partial_0\rho_2$ *and the list* $\rho_1\rho_2$ *obtained by disjoint union is a ribbon. Two ribbons* ρ_1,ρ_2 overlap *if there exist triangles* $\tau_1 \in \rho_1$ *and* $\tau_2 \in \rho_2$ *that overlap. The ribbons* ρ_1 *and* ρ_2 *are* disjoint *if the do not overlap. We write* $\rho_1 \cap \rho_2 = \emptyset$ *if this is the case.*

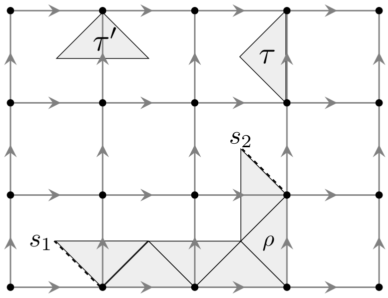

Figure 2.1: This images illustrates some of the basic geometrical objects used in the discussion of the quantum double models. The triangle τ is direct, τ' is dual and ρ is a ribbon connecting the site s_2 with s_1.

A strip can be regarded as a list of pairwise non-overlapping ribbons. Note that each triangle is a ribbon. Every ribbon ρ can be decomposed into disjoint ribbons ρ_1 and ρ_2 such that $\rho = \rho_1\rho_2$. Given a site s we denote by β_s the closed ribbon starting at s and consisting only of direct triangles. Similarly α_s is the closed ribbon starting at s and consisting only of dual triangles. Illustrations of triangles and ribbons can be found in Figure 2.1. Throughout the treatment of the quantum double model we will frequently use the following terminology. Given a site $s = (v, f)$ the *star* at s is referring to the edges that have an ending point in v, and we will denote it by $\mathrm{star}(s)$. Similarly we call the collection $\mathrm{plaq}(s)$ of the edges bounding f the *plaquette* of s.

Most of these and the following definitions can be found in [BM08] in much more detail and we will list them here for convenience. The local excitations of the quantum double models can be expressed by certain operators associated to ribbon.s The fusion and braiding of these excitations are then related to the commutation relations of such operators associated to ribbons that overlap at their ending sites or cross each other.

Definition 2.1.3:
Given ρ_1 and ρ_2 strips. We write

- $(\rho_1, \rho_2)_\prec$ (left joint)*, if there exist disjoint ribbons ρ, ρ_1', ρ_2' and a direct (dual) triangle τ_2 (τ_1) such that $\rho_i = \rho\tau_i\rho_i'$.*

- $(\rho_1, \rho_2)_\succ$ (right joint)*, if there exist disjoint ribbons ρ, ρ_1', ρ_2' and a direct (dual) triangle τ_2 (τ_1) such that $\rho_i = \rho_i'\tau_i\rho$.*

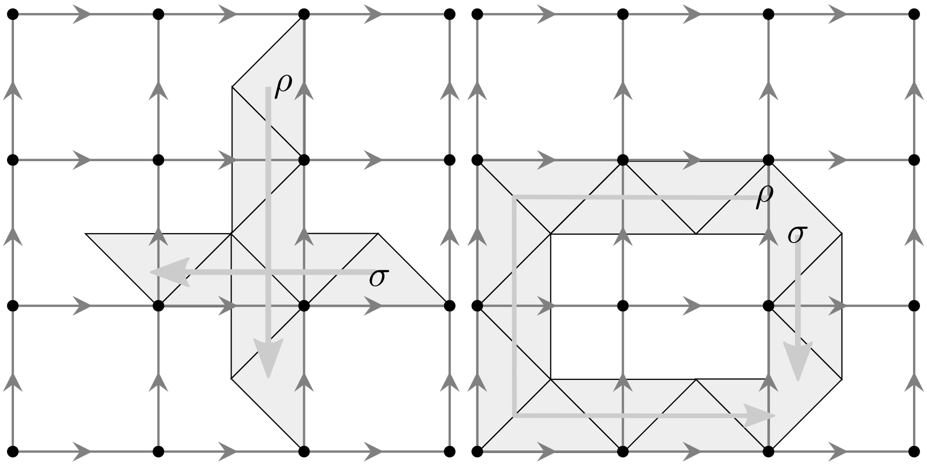

Figure 2.2: Two ribbons crossing each other, and two ribbons forming a deformation.

- $(\rho_1, \rho_2)_{\prec\succ}$ (left-right joint)*, if* $(\rho_1, \rho_2)_{\prec}$ *and* $(\rho_1, \rho_2)_{\succ}$.

- $(\rho_1, \rho_2)_{\times}$ (crossed)*, if there exist ribbons* $\sigma_1, \sigma_1', \sigma_2, \sigma_2'$ *such that* $\rho_i = \sigma_i \sigma_i'$ *(*$i = 1, 2$*),* $\sigma_1 \cap \sigma_2' = \emptyset$ *as well as* $\sigma_1' \cap \sigma_2 = \emptyset$*, and* $(\sigma_1, \sigma_2)_{\succ}$ *and* $(\sigma_2', \sigma_1')_{\prec}$.

For two ribbons ρ_1, ρ_2 with $(\rho_1, \rho_2)_{\times}$ the last conditions translate to $\partial_1 \sigma_1 = \partial_0 \sigma_2'$ and $\partial_1 \sigma_2 = \partial_0 \sigma_1'$. Note that the definition of crossings given here is different from the definition of a crossed joint in [BM08]. The latter does not describe the crossing of two ribbons but another way of joining them together.

Definition 2.1.4:
Let ρ, ρ' *be open ribbons. We say that* ρ' *is an inversion of* ρ*, written* $\bar{\rho}$*, if we have that* $\partial_0 \rho = \partial_1 \rho'$ *and* $\partial_1 \rho = \partial_0 \rho'$ *and either one of the following conditions:*

- *The set of dual edges in* ρ *coincides with the set of dual edges with inverse orientation in* ρ'*,*

- *The set of direct edges in* ρ *coincides with the set of direct edges with inverse orientation in* ρ'*.*

For the sake of later use we introduce the notations of deformations and inversions of ribbons.

Definition 2.1.5:
Let ρ, ρ' *be open ribbons. They are said to form a* simple deformation*, if*

- *no triangle in* ρ *is contained in* ρ'*,*
- $(\rho, \rho')_{\prec\succ}$*,*
- *for the dual edges* b^* *of the dual triangles in* ρ *it holds that the vertices* $\partial_1 b$ *are contained in* ρ'*,*
- *for the edges* b *of the direct triangles in* ρ' *it holds that the dual vertices* $\partial_1 b^*$ *are contained in* ρ*.*

The ribbons ρ_1, ρ_2 *form a* deformation*, if there exist ribbons* $\rho_1', \rho_2', \sigma_1, \sigma_2$ *such that* $\rho_1 = \rho_1'\sigma_1\rho_2'$ *and* $\rho_2 = \rho_1'\sigma_2\rho_2'$ *and* σ_1, σ_2 *form a simple deformation.*

We say that two ribbons ρ_1, ρ_2 *are* deformation equivalent *if there is a sequence of deformations transforming* ρ_1 *into* ρ_2*.*

For an example of crossed ribbons and ribbons forming a deformation, see Figure 2.2. The definition of deformation equivalence is slightly different that that given in [BM08]. There the equivalence was given with respect to a region in which the ribbons should be transformed into each other. Depending on the context we will however require that the deformation should be possible in a particular region, but in general we will not need this. The reason why it was introduced in [BM08] was that when ribbon operators act on a state that contains no excitations in such region then deforming the ribbon operator in that region does not change its action on the state. However, if there are excitations in that region, this is no longer the case. Later we only need the case, where the ribbon operators act on the ground state of the dynamics, so we do not have excitations to respect.

2.2 Local Hilbert Spaces

Throughout the thesis we consider Kitaev's quantum double models on the planar square lattice $\mathbb{Z}^2$. We first discuss the construction only for finite subsets of $\mathbb{Z}^2$ and assume for the following sections that we always work on a finite but large enough patch of the lattice. The reason for this assumption is that in this chapter we are only interested in the local excitations of the model. In the last section discuss how to take the limit of infinite volumes.

From now on let G be any finite group. Consider the Hilbert space of square summable, complex valued functions $\ell^2(G)$ of G. This space can be viewed as the finite finite dimensional Hilbert space with orthonormal basis of the form $(|g\rangle)_{g\in G}$ of $\mathbb{C}[G]$, i.e. $\ell^2(G) \cong \mathbb{C}^{|G|}$, where $|G|$ is the order of the group. To each edge $b \in \mathbb{Z}^2$ we now associate a copy of this Hilbert space, i.e. the edge Hilbert space $\mathcal{H}_b$ is given as $\mathcal{H}_b = \ell^2(G)$.

In the following, we use the notation $\mathcal{P}_f(\mathbb{Z}^2)$ to refer to the set of finite subsets of $\mathbb{Z}^2$. The Hilbert space describing the system on finite sets $\Lambda \in \mathcal{P}_f(\mathbb{Z}^2)$ is given as the tensor product of Hilbert spaces $\mathcal{H}_b$ taken over all edges $b \in \Lambda$, i.e. for all $\Lambda \in \mathcal{P}_f(\mathbb{Z}^2)$ we have $H_\Lambda := \bigotimes_{b\in\Lambda} \mathcal{H}_b$. Since the graph Λ is finite there are only finitely many tensor factors in $\mathcal{H}_\Lambda$ and hence the system's Hilbert space is finite dimensional. Note that neither the orientation nor the geometry of the graph play a role in this description. Both come into play when we define the local Hamiltonian for the model.

2.3 Ribbon Operators

We can use the geometrical objects introduced in the previous section to associate operators to them. These will be used to define the Hamiltonian of the quantum double model in a convenient way that allows for a precise characterisation of its ground state and the local excitation above it (see [Kit03] and [BM08] for a very detailed analysis of the ground state and its excitation structure).

Definition 2.3.1 [Kit03; BM08]:
Let τ, τ' be a direct and a dual triangle, and let e, e' be the corresponding edges in τ, τ'. Let $h, k \in G$ and let $|h\rangle$ and $|k\rangle$ be basis vectors at e and e', respectively. In case that the direction of the triangles coincide with those of their respective edges e and e' the triangle operators *are defined for $g \in G$ by:*

$$T_\tau^g |h\rangle := \delta_{g,h} |h\rangle$$
$$L_{\tau'}^g |k\rangle := |gk\rangle .$$

In case that the orientation of τ and τ' is opposite to that of e and e', repsectively, we set:

$$T_\tau^g |h\rangle := \delta_{\overline{g},h} |h\rangle$$
$$L_{\tau'}^g |k\rangle := |k\overline{g}\rangle ,$$

where we used the notation $\overline{k} = k^{-1}$ for $k \in G$. Given a ribbon ρ and any partition $\rho = \rho_1\rho_2$ into ribbons ρ_1, ρ_2, and $g, h \in G$, the ribbon operator $F_\rho^{h,g}$ *is defined* *recursively by*

$$F_{\tau}^{h,g} := T_{\tau}^{g} \qquad F_{\tau'}^{h,g} := \delta_{g,e} L_{\tau'}^{h}, \qquad F_{\epsilon}^{h,g} = \delta_{g,e}\mathbb{1},$$

for direct triangles τ, *dual triangles* τ' *and the empty ribbon* ϵ, *and*

$$F_{\rho}^{h,g} = \sum_{k \in G} F_{\rho_1}^{h,k} F_{\rho_2}^{\bar{k}hk,\bar{k}g}, \tag{2.1}$$

with $h, g \in G$ *and* $e \in G$ *the unit element. Given a site* s *we use the notation*

$$B_s^g := F_{\beta_s}^{e,g} \tag{2.2}$$

$$A_s^h := F_{\alpha_s}^{h,e} \tag{2.3}$$

for any $h, g \in G$. *Furthermore we set*

$$B_s := B_s^e$$

$$A_s := \frac{1}{|G|} \sum_{h \in G} A_s^h,$$

and refer to these as plaquette *and* star *operators.*

One can check that this definition of ribbon operators is independent of the partition of ρ into subribbons [BM08]. We sometimes refer to equation (2.1) as the *ribbon decomposition rule*. Note also that the direction of the graph enters this description. The nomenclature for the star and plaquette operators is justified by the observation that their definition does not depend on the site they are attached to, but only on the vertex, respectivley the face in this site[3]. That is, if s_0 and s_1 are two sites with common vertex, then $A_{s_0} = A_{s_1}$. Similarly if s_0 and s_1 have a common face then $B_{s_0} = B_{s_1}$.

[3] In the language of [BM08] they are rotationally invariant.

2.4 The Hamiltonian

The reason to introduce star and plaquette operators is that the local Hamiltonians of the quantum double model are defined as sums of such.

Lemma 2.4.1:
If s, s' *are distinct sites it holds that*

$$A_s^g A_s^{g'} = A_s^{gg'}, \quad B_s^h B_s^{h'} = \delta_{h,h'} B_s^h, \quad A_s^g B_s^h = B_s^{ghg^{-1}} A_s^g$$

and

$$[A_s, B_{s'}] = [A_s, A_{s'}] = [B_s, B_{s'}] = 0.$$

Furthermore the operators A_s *and* B_s *are projections.*

Definition 2.4.2:
Let $\mathcal{O} \in \mathcal{P}_f(\mathbb{Z}^2)$*. Then the* local Hamiltonian of the quantum double model *is defined as*

$$H_{\mathcal{O}} := -\sum_{s\,:\,\mathrm{star}(s)\subset\mathcal{O}} A_s - \sum_{s\,:\,\mathrm{plaq}(s)\subset\mathcal{O}} B_s \tag{2.4}$$

and the sums go over all sites whose stars respectively plaquettes are contained in $\mathcal{O}$*.*

This definition is motivated by the stabiliser code formalism [Kit03]. The star and plaquette operators in $\mathcal{O}$ form the stabilisers for a code space that will form the ground state space of the Hamiltonian. A ground state Ω of $H_{\mathcal{O}}$ must then satisfy $A_s\Omega = \Omega = B_s\Omega$ for all star and plaquette operators in $\mathcal{O}$. This also means that $H_{\mathcal{O}}$ is frustration free[4]. Important for us is that the summands in equation (2.4) all commute due to Lemma 2.4.1. Hence the ground state is stabilised by the summands of $H_{\mathcal{O}}$ and violations of any of these constraints can be interpreted as local excitations above the ground state. These excitations can be characterised by local operators that do not commute with at least one summand in the local Hamiltonian. In fact, if one acts on the ground state with some non-trivial operator O localised at a single edge e, then the pair of star or plaquette operators, or both, which also act on e do not commute with O. Hence, the operator O created a pair of excitations localised at the sites containing e, where we characterise an excitation by a local violation of the ground state condition detected by the star and plaquette operators. We will see in the following that local excitations always occur in conjugate pairs, and that a full description of local excitations is given by ribbon operators.

[4] See e.g. [BHM10] for a definition of frustration freeness.

2.5 Charge Projections

Before turning to the discussion about ribbon operators we first discuss projections onto the local excitations, which we call charge projections. As mentioned earlier, local excitations are located at sites. The building blocks for the charge projections are the operators defined in equation 2.2.

Definition 2.5.1:
Let $s \in \mathbb{Z}^2$ *be a site, and* $h, g \in G$. *We define*

$$D_s^{h,g} := B_s^h A_s^g$$

Borrowing some notation from [Kit03] we can immediately write down the commutation relations for these operators.

Lemma 2.5.2:
Given a site s *and elements* $g, h, k, l \in G$, *it holds that*

$$D_s^{h,g} D_s^{k,l} = \sum_{m,n \in G} \Omega_{(m,n)}^{(h,g),(k,l)} D_s^{m,n} \tag{2.5}$$

$$\left(D_s^{h,g}\right)^* = D_s^{\overline{g}hg,\overline{g}}, \tag{2.6}$$

with $\Omega_{(m,n)}^{(h,g),(k,l)} = \delta_{h,gk\overline{g}} \delta_{m,h} \delta_{n,gl}$.

Proof. Straightforward calculation using Lemma 2.4.1. □

For the following we recall the notations from Section 1.4. Let $C = \{c_1, ..., c_{|C|}\}$ be a conjugacy class of G, an let $r \in C$ be a representative, which we fix from now on. Let $Z_G(r)$ be the centraliser of r in G, and let $Q_C = \{q_1, \ldots, q_{|C|}\}$ be a set of representatives of the left cosets of $Z_G(r)$ in G. We choose Q_C in such a way that $q_i = e$, if $c_i = r$. Recall that for each $g \in G$ there exists by Lemma 1.4.1 a unique $n \in Z_G(r)$ and a unique $q \in Q_C$ such that $g = qn$. For convenience, we write G_{irr} for the set irreducible representations of G (up to unitary equivalence), and similar for $Z_G(r)$. We denote the set of conjugacy classes of G by G_{cj}.

Definition 2.5.3:
Let $C \in G_{\mathrm{cj}}$, $r \in C$ *a representative, and* $Q_C = \{q_1, \ldots, q_n\}$ *as above. Let* s *be a site in* $\mathbb{Z}^2$. *We define*

$$D_s^{C,\pi} := \frac{|\pi|}{|Z_G(r)|} \sum_{d \in Z_G(r)} \sum_{q \in Q_C} \overline{\pi(d)} D_s^{qr\overline{q}, qd\overline{q}}.$$

This is essentially the definition given in reference [BM08]. However, the star and plaquette terms are in different order. This does not change the properties of these projections. One can easily check that with this definition it holds that $B_s A_s = D_s^{\{e\},\mathrm{id}}$. Moreover these operators are orthogonal projections, as the next lemma shows.

Lemma 2.5.4:
Given conjugacy classes C, D *of* G*, representatives* $r \in C$ *and* $\nu \in D$*, and* Q_C, Q_D *as before. Let* $\pi \in Z_G(r)_{irr}$ *and* $\chi \in Z_G(\nu)$*. It then holds for any site* $s \in \mathbb{Z}^2$

$$\begin{aligned} D_s^{C,\pi} D_s^{D,\chi} &= \delta_{C,D}\delta_{\pi,\chi} D_s^{C,\pi} \\ \left(D_s^{C,\pi}\right)^* &= D_s^{C,\pi}. \end{aligned}$$

Proof. For the first assertion we use equation (2.5) to find

$$\begin{aligned} &D_s^{C,\pi} D_s^{D,\chi} = \\ &= c_1 c_2 \sum_{d \in Z_G(r)} \sum_{f \in Z_G(\nu)} \sum_{q \in Q_C} \sum_{p \in Q_D} \overline{\pi(d)}\,\overline{\chi(f)} D_s^{qr\overline{q}, qd\overline{q}} D_s^{p\nu\overline{p}, pf\overline{p}} \\ &= a_1 a_2 \sum_{d \in Z_G(r)} \sum_{f \in Z_G(\nu)} \sum_{q \in Q_C} \sum_{p \in Q_D} \overline{\pi(d)}\,\overline{\chi(f)} \delta_{qr\overline{q}, q\nu\overline{p}} D_s^{qr\overline{q}, qd\overline{q}pf\overline{p}} \\ &= a_1 a_2 \delta_{C,D} \sum_{d \in Z_G(r)} \sum_{f \in Z_G(r)} \sum_{q \in Q_C} \sum_{p \in Q_C} \overline{\pi(d)}\,\overline{\chi(f)} \delta_{qr\overline{q}, qr\overline{p}} D_s^{qr\overline{q}, qd\overline{q}pf\overline{p}} \\ &= a_1 a_2 \delta_{C,D} \sum_{d,f \in Z_G(r)} \sum_{q \in Q_C} \overline{\pi(d\bar{f})}\,\overline{\chi(f)} D_s^{qr\overline{q}, qd\overline{q}} \\ &= a_1 a_2 \delta_{C,D} \sum_{d,f \in Z_G(r)} \sum_{q \in Q_C} \sum_{i,s=1}^{|\pi|} \sum_{t=1}^{|\chi|} \overline{\pi_{i,s}(d)} \pi_{i,s}(f) \overline{\chi_{t,t}(f)} D_s^{qr\overline{q}, qd\overline{q}} \\ &= a_1 \delta_{C,D}\delta_{\pi,\chi} \sum_{d \in Z_G(r)} \sum_{q \in Q_C} \sum_{i,s=1}^{|\pi|} \sum_{t=1}^{|\chi|} \overline{\pi_{i,s}(d)} \delta_{i,t}\delta_{s,t} D_s^{qr\overline{q}, qd\overline{q}} \\ &= \delta_{C,D}\delta_{\pi,\chi} D_s^{C,\pi}, \end{aligned}$$

setting $a_1 = \frac{|\pi|}{|Z_G(r)|}$, $a_2 = \frac{|\chi|}{|Z_G(\nu)|}$, and using in step 3 that conjugacy classes are either disjoint or equal, and using in step 6 the orthogonality relations for irreducible representations. The second assertion is easier:

$$\left(D_s^{C,\pi}\right)^* = c_1 \sum_{d \in Z_G(r)} \sum_{q \in Q_C} \pi(d) D_s^{qr\overline{q}, q\overline{d}\overline{q}} = D_s^{C,\pi},$$

where we used equation (2.6). □

2.6 Properties of Ribbon Operators

In the following we will list some elementary and useful properties of ribbon operators.

Definition 2.6.1:
Let ρ *be a ribbon. We denote the algebra that is spanned by products of ribbon operators at* ρ *by* $\mathcal{F}_\rho$*, i.e.*

$$\mathcal{F}_\rho := \operatorname{span}\Big\{ \prod_{i=1}^{n} F_\rho^{h_i, g_i} \,|\, \forall i : h_i, g_i \in G \text{ and } n \in \mathbb{N} \Big\}.$$

In fact, $\mathcal{F}_\rho$ can be characterised by operators acting along ρ that commute with all star and plaquette operators except at the ending sites of ρ [BM08, Definition 6]. Regarding their action on the ground state this means that ribbon operators create pairs of excitations above the ground state at the ending sites of the associated ribbons. The type of an excitation can be determined by checking the commutation relations. We will come back to this at a later point.

In order to determine the excitations above the ground state of (2.4) it helps to know something about the commutation relations of ribbon operators with star and plaquette operators.

Lemma 2.6.2 [BM08]:
Let ρ *be an open ribbon and* $s_0 := \partial_0\rho$*,* $s_1 := \partial_1\rho$ *we have*

$$\forall g, h, k \in G\, \forall s \neq s_i, i = 0, 1 : [A_s^k, F_\rho^{g,h}] = 0 = [B_s^k, F_\rho^{g,h}].$$

Furthermore, for all $g, h, k \in G$*:*

$$A_{s_0}^k F_\rho^{h,g} = F_\rho^{kh\bar{k},kg} A_{s_0}^k\,, \qquad B_{s_0}^k F_\rho^{h,g} = F_\rho^{h,g} B_{s_0}^{kh}\,,$$
$$A_{s_1}^k F_\rho^{h,g} = F_\rho^{h,g\bar{k}} A_{s_1}^k\,, \qquad B_{s_1}^k F_\rho^{h,g} = F_\rho^{h,g} B_{s_1}^{\bar{g}\bar{h}gk}\,.$$

If σ *is a closed ribbon, we have for all sites* s *such that there is no triangle* τ *in* σ *with* $\partial_0\tau = s$ *(equivalently, no triangle* $\tilde{\tau} \in \sigma$ *with* $\partial_1\tilde{\tau} = s$*), that*

$$\forall g, h, k \in G : [A_s^k, F_\rho^{g,h}] = 0 = [B_s^k, F_\rho^{g,h}].$$

If s *is a site such that there is a triangle* $\tau \in \sigma$ *with* $\partial_0\tau = s$*, then for all* $g, h, k \in G$*:*

$$A_s^k F_\sigma^{h,g} = F_\sigma^{kh\bar{k},kg\bar{k}} A_s^k$$
$$B_s^k F_\sigma^{h,g} = F_\sigma^{h,g} B_s^{\bar{g}\bar{h}gkh}.$$

Proof. The proofs of these statements can be found in the appendix of [BM08]. □

Lemma 2.6.3 [BM08]:
Let ρ *and* σ *be two ribbons with* $\rho \cap \sigma = \emptyset$*, and* $g, h, k, l \in G$*. Then*

$$F_\rho^{h,g} F_\sigma^{k,l} = F_\sigma^{k,l} F_\rho^{g,h}.$$

We also have that $\sum_{g \in G} F_\rho^{1,g} = \mathbb{1}$ *and*

$$\forall g, h, k, l \in G : F_\rho^{h,g} F_\rho^{k,l} = \delta_{g,l} F_\rho^{hk,l}$$
$$\forall g, h \in G : \left(F_\rho^{h,g}\right)^* = F_\rho^{\bar{h},g}.$$

We can rewrite the first equation as

$$F_\rho^{h,g} F_\rho^{k,l} = \sum_{m,n \in G} \Lambda_{(m,n)}^{(h,g),(k,l)} F_\rho^{m,n},$$

with $\Lambda_{(m,n)}^{(h,g),(k,l)} = \delta_{g,l} \delta_{m,hk} \delta_{n,l}$.

Given ribbons ρ_1, ρ_2 *with* $\rho = \rho_1 \rho_2$ *we find for* $m, n \in G$

$$F_\rho^{m,n} = \sum_{h,g,k,l \in G} \Omega_{(m,n)}^{(h,g),(k,l)} F_{\rho_1}^{h,g} F_{\rho_2}^{k,l},$$

with the same Ω *as in Lemma 2.5.2.*

Proof. This is straightforward calculation. The first part can also be found in [BM08], the second part stems from [Kit03]. □

Many more properties were calculated in [BM08], and we will refer to them if necessary.

Proposition 2.6.4 [BM08]:
Let $C = \{c_1, \ldots, c_n\}$ *be a conjugacy class of* G*,* $r \in C$ *some representative and* π *an irreducible unitary representation of* $Z_G(r)$*, the centraliser of* r *in* G*. Choose elements* $q_1, \ldots, q_n$ *such that* $c_i = q_i r \bar{q}_i$ *for* $i = 1, \ldots, n$ *and set*

$$F_\rho^{C,\pi,i,i',j,j'} := \sum_{z \in Z_G(r)} \overline{\pi_{j,j'}(z)} F_\rho^{\bar{c}_i, q_i z \bar{q}_{i'}} \tag{2.7}$$

where $j, j' \in \{1, \ldots, |\pi|\}$ *label the matrix elements of* π *and* $i, i' \in \{1, \ldots, n\}$*. Then the linear span of these operators generate the ribbon algebra* $\mathcal{F}_\rho$*. In case that* ρ *contains both direct and dual triangles, these operators form a basis of* $\mathcal{F}_\rho$*.*

This proposition also relates the ribbon operators to the finite dimensional, irreducible representations of the quantum double $\mathcal{D}(G)$ of G. Those can be labelled by tuples (C, π) where C is a conjugacy class of G and π is an irreducible representation of $Z_G(r)$ for some $r \in C$. In the definition of $F_\rho^{C,\pi,i,i',j,j'}$ in [BM08] an additional factor $\frac{|Z_G(r)|}{|\pi|}$ was used. We omit this factor since, in the case of an abelian group G the ribbon operators are then unitary.

Note that this choice of ribbon operators is closed under taking the adjoint, as the next lemma shows.

LEMMA 2.6.5:
Let $C = \{c_1, \dots, c_n\}$ *be a conjugacy class of* G, $r \in C$ *some representative and* π *an irreducible unitary representation of* $Z_G(r)$, *the centraliser of* r *in* G. *Choose elements* $q_1, \dots, q_n$ *such that* $c_i = q_i r \overline{q}_i$ *for* $i = 1, \dots, n$. *Then*

$$(F_\rho^{C,\pi,i,i',j,j'})^* = F_\rho^{\overline{C},\overline{\pi},i,i',j,j'},$$

where $\overline{C}$ *is the conjugacy class containing* $\bar{r}$, *and* $\overline{\pi}$ *is the conjugate representation to* π.

Proof. Let C and π be as in the assumptions. Then the conjugate representation $\overline{\pi}$ to π is given by $\overline{\pi} = \mathrm{Ad}_J \circ \pi$ with the complex conjugation J that is diagonal in the basis chosen to represent π. Note also, that for $r \in C$ it holds that $Z_G(r) = Z_G(\bar{r})$. Furthermore, for $z \in Z_G(r)$, $r \in C$, the matrix elements of $\overline{\pi}(z)$ are given by $\overline{\pi}_{j,j'}(z) = \overline{\pi_{j,j'}(z)}$. Hence, for any ribbon ρ we have

$$\begin{aligned}
(F_\rho^{C,\pi,i,i',j,j'})^* &= \sum_{z \in Z_G(r)} \pi_{j,j'}(z) F_\rho^{c_i, q_i z \overline{q}_{i'}} \\
&= \sum_{z \in Z_G(\bar{r})} \overline{\overline{\pi}_{j,j'}(z)} F_\rho^{c_i, q_i z \overline{q}_{i'}} \\
&= F_\rho^{\overline{C},\overline{\pi},i,i',j,j'}
\end{aligned}$$

□

The next lemma shows that at the endpoints of a ribbon these operators create excitations from the vacuum that are conjugate to each other. This was stated already in [BM08], but we will prove it here for completeness and for illustrating the handling of ribbon operators. In addition our definition of the charge projections is slightly different than in the reference, which doesn't change the overall result.

Lemma 2.6.6:
Given conjugacy classes $C, D \in G_{cj}$, representatives $r \in C$ and $\nu \in D$, and Q_C, Q_D as in the foregoing lemma. Let $\pi \in Z_G(r)_{irr}$ and $\chi \in Z_G(\nu)_{irr}$. Let j, j' be indices that run through $1, \dots, |\pi|$. Let ρ be a ribbon, and set $s_0 \coloneqq \partial_0\rho$ and $s_1 \coloneqq \partial_1\rho$. We then have

$$
\begin{aligned}
D_{s_0}^{D,\chi}F_\rho^{C,\pi,i,i',j,j'}D_{s_0}^{\{e\},\mathrm{id}} &= \delta_{C,D}\delta_{\pi,\chi}F_\rho^{C,\pi,i,i',j,j'}D_{s_0}^{\{e\},\mathrm{id}}\\
D_{s_1}^{D,\chi}F_\rho^{C,\pi,i,i',j,j'}D_{s_1}^{\{e\},\mathrm{id}} &= \delta_{\overline{C},D}\delta_{\overline{\pi},\chi}F_\rho^{C,\pi,i,i',j,j'}D_{s_1}^{\{e\},\mathrm{id}}
\end{aligned}
$$

Proof. The proof of these statements is very similar to the ones of Lemma 2.5.4. Note that, given a conjugacy class C with Q_C and $r \in C$ as above, then for any $g \in G$ there exist a unique $q_i \in Q_C$ and a unique $n \in Z_G(r)$ such that $g = q_i n$. Adapting the notation from [BM08] we set $i(g) \coloneqq i$ and $n(g) \coloneqq n$. We only prove the first statement, since the second one can be proven in complete analogy. We start with C, π, Q_C, r and D, χ, Q_D, ν as in the preamble, and set $a \coloneqq \frac{|\chi|}{|Z_G(\nu)|}$. We then have by Lemma 2.6.2 for $h, g \in G$

$$
\begin{aligned}
D_{s_0}^{h,g}F_\rho^{C,\pi,i,i',j,j'} &= \sum_{n\in Z_G(r)} \overline{\pi_{j,j'}(n)}D_s^{h,g}F_\rho^{\overline{c_i},q_i n\overline{q_i'}}\\
&= \sum_{n\in Z_G(r)} \overline{\pi_{j,j'}(n)}F_\rho^{g\overline{c_i}\overline{g},gq_i n\overline{q_i'}}D_{s_0}^{hg\overline{c_i}\overline{g},g}\\
&= \sum_{n\in Z_G(r)} \overline{\pi_{j,j'}(n)}F_\rho^{\overline{c_{i(gq_i)}},q_{i(gq_i)}n(gq_i)n\overline{q_i'}}D_{s_0}^{hg\overline{c_i}\overline{g},g}\\
&= \sum_{n\in Z_G(r)} \overline{\pi_{j,j'}(\overline{n(gq_i)}n)}F_\rho^{\overline{c_{i(gq_i)}},q_{i(gq_i)}n\overline{q_i'}}D_{s_0}^{hg\overline{c_i}\overline{g},g}\\
&= \sum_{n\in Z_G(r)}\sum_{s=1}^{|\pi|} \pi_{s,j}(n(gq_i)\overline{\pi_{s,j'}(n)}F_\rho^{\overline{c_{i(gq_i)}},q_{i(gq_i)}n\overline{q_i'}}D_{s_0}^{hg\overline{c_i}\overline{g},g}\\
&= \sum_{s=1}^{|\pi|} \pi_{s,j}(n(gq_i))F_\rho^{C,\pi,i(gq_i),i',s,j'}D_{s_0}^{hg\overline{c_i}\overline{g},g}.
\end{aligned}
\tag{2.8}
$$

Very similarly we get

$$
D_{s_1}^{h,g}F_\rho^{C,\pi,i,i',j,j'} = \sum_{s=1}^{|\pi|} \overline{\pi_{s,j'}(n(gq_{i'}))}F_\rho^{C,\pi,i,i(gq_{i'}),j,s}D_{s_1}^{gc_{i'}\overline{g}h,g}. \tag{2.9}
$$

Inserting the definition of $D_{s_0}^{D,\chi}$ in equation (2.8), and using Lemma 2.4.1 we find

$$\begin{aligned}
&D_{s_0}^{D,\chi} F_\rho^{C,\pi,i,i',j,j'} D_{s_0}^{e,id} = \\
&\quad = a \sum_{\substack{d\in Z_G(\nu)\\ p\in Q_D}} \sum_{s=1}^{|\pi|} \overline{\chi(d)} \pi_{s,j}(n(pd\overline{p}q_i)) \delta_{\overline{p}c_i p,\nu} F_\rho^{C,\pi,i(pd\overline{p}q_i),i',s,j'} D_{s_0}^{e,id} \\
&\quad = a\delta_{C,D} \sum_{d\in Z_G(r)} \sum_{s=1}^{|\pi|} \overline{\chi(d)} \pi_{s,j}(n(q_i d)) F_\rho^{C,\pi,i(q_i d),i',s,j'} D_{s_0}^{e,id},
\end{aligned}$$

where we used that $\delta_{qc_i\overline{q},r}$ is non-zero if and only if $q = q_i$. Now $n(q_i d) = d$ and $i(q_i d) = i$, therefore:

$$\begin{aligned}
D_{s_0}^{D,\chi} F_\rho^{C,\pi,i,i',j,j'} D_{s_0}^{e,id} &= a\delta_{C,D} \sum_{d\in Z_G(r)} \sum_{s=1}^{|\pi|} \overline{\chi(d)} \pi_{s,j}(d) F_\rho^{C,\pi,i,i',s,j'} D_{s_0}^{e,id} \\
&= a\delta_{C,D} \sum_{d\in Z_G(r)} \sum_{s=1}^{|\pi|} \sum_{t=1}^{|\chi|} \overline{\chi_{t,t}(d)} \pi_{s,j}(d) F_\rho^{C,\pi,i,i',s,j'} D_{s_0}^{e,id} \\
&= \delta_{C,D}\delta_{\chi,\pi} \sum_{s,t=1}^{|\pi|} \delta_{t,s}\delta_{t,j} F_\rho^{C,\pi,i,i',s,j'} D_{s_0}^{e,id} \\
&= \delta_{C,D}\delta_{\chi,\pi} F_\rho^{C,\pi,i,i',j,j'} D_{s_0}^{e,id},
\end{aligned}$$

proving the first assertion. The second statement follows analogously. □

LEMMA 2.6.7:
Let ρ be an open ribbon, $C \in G_{cj}$, $r \in C$, $\pi \in Z_G(r)_{irr}$ and corresponding indices i, i', j, j'. With $s = \partial_0\rho$ or $s = \partial_1\rho$ it then holds

$$[F_\rho^{C,\pi,i,i',j,j'}, B_s] = 0 \iff C = \{e\}. \tag{2.10}$$

Furthermore, if C consists of a central element $c_0 \in G$ it holds

$$[F_\rho^{C,\pi,0,0,j,j'}, A_s] = 0 \iff \pi = \mathrm{id}. \tag{2.11}$$

Proof. First note, that $F_\rho^{C,\pi,i,i',j,j'} \neq 0$ as well as A_s and B_s, since the ground state is not contained in the respective kernels. Using equation (2.8) we find that

$$\begin{aligned}
A_{s_0} F_\rho^{C,\pi,i,i',j,j'} &= \frac{1}{|G|} \sum_{g\in G} \sum_{s=1}^{|\pi|} \pi_{s,j}(n(gq_i)) F_\rho^{C,\pi,i(gq_i),i',s,j'} A_{s_0}^g \\
B_{s_0} F_\rho^{C,\pi,i,i',j,j'} &= F_\rho^{C,\pi,i,i',j,j'} B_{s_0}^{\overline{c_i}}
\end{aligned}$$

and by equation (2.9)

$$
\begin{aligned}
A_{s_1} F_\rho^{C,\pi,i,i',j,j'} &= \frac{1}{|G|}\sum_{g\in G}\sum_{s=1}^{|\pi|} \overline{\pi_{s,j'}(n(gq_{i'}))} F_\rho^{C,\pi,i,i(gq_{i'}),j,s} A_{s_1}^{g} \\
B_{s_1} F_\rho^{C,\pi,i,i',j,j'} &= F_\rho^{C,\pi,i,i',j,j'} B_{s_1}^{c_{i'}}.
\end{aligned}
$$

Therefore, if in the assertion the right hand sides of equations (2.10) and (2.11) are true then the respective left hand sides are implied.

Now assume $F_\rho^{C,\pi,i,i',j,j'}$ commutes with B_{s_0} and that $C \neq \{e\}$. Then, by the previous expressions we have

$$
F_\rho^{C,\pi,i,i',j,j'} B_{s_0} = B_{s_0} F_\rho^{C,\pi,i,i',j,j'} B_{s_0} = \delta_{e,c_i} F_\rho^{C,\pi,i,i',j,j'} B_{s_0} = 0.
$$

By [BM08, Appendix B, Lemma 5] this contradicts that $F_\rho^{C,\pi,i,i',j,j'}$ is non-zero. Hence C must be equal to $\{e\}$. The case for B_{s_1} is proven in full analogy.

Next, assume that $C = \{c_0\}$ with $c_0 \in G$ central, $F_\rho^{C,\pi,0,0,j,j'}$ commutes with A_{s_0} and that π is a non-trivial irreducible representation of $Z_G(r)$. Then

$$
\begin{aligned}
F_\rho^{C,\pi,0,0,j,j'} A_{s_0} &= \frac{1}{|G|}\sum_{g\in G}\sum_{s=1}^{|\pi|} \pi_{s,j}(g) F_\rho^{C,\pi,0,0,s,j'} A_{s_0} \\
&= \delta_{\pi,\mathrm{id}} F_\rho^{C,\pi,0,0,j,j'} A_{s_0},
\end{aligned}
$$

thus proving the claim. Again, the statement at the site s_1 is proven analogously. □

Lemma 2.6.8:
Let ρ be an open ribbon, $C \in G_{cj}$, $r \in C$, $\pi \in Z_G(r)_{irr}$ and corresponding indices i, i', j, j'. With $s = \partial_0\rho$ or $s = \partial_1\rho$, it then holds

$$
[F_\rho^{C,\pi,i,i',j,j'}, D_s^{\{e\},\mathrm{id}}] = 0 \iff (C,\pi) = (\{e\}, \mathrm{id}).
$$

Proof. The direction from left to right is obvious, since $F_\rho^{\{e\},,\mathrm{id}} = \mathbb{1}$. For the other direction assume that $[F_\rho^{C,\pi,i,i',j,j'}, D_s^{\{e\},\mathrm{id}}] = 0$. Then

$$
D_s^{\{e\},\mathrm{id}} F_\rho^{C,\pi,i,i',j,j'} = D_s^{\{e\},\mathrm{id}} F_\rho^{C,\pi,i,i',j,j'} D_s^{\{e\},\mathrm{id}}.
$$

By Lemma 2.6.6 and [BM08, Appendix B, Lemma 5] this is only possible if $C = \{e\}$ and $\pi = \mathrm{id}$. □

The ribbon operators as given in Proposition 2.6.4 obeys a decomposition rule when splitting the corresponding ribbons.

Lemma 2.6.9:

Let $C = \{c_1, \ldots, c_n\}$ *be a conjugacy class of* G*,* $r \in C$ *some representative and* π *an irreducible unitary representation of* $Z_G(r)$*, the centraliser of* r *in* G*. Choose elements* $q_1, \ldots, q_n$ *such that* $c_i = q_i r \overline{q}_i$ *for* $i = 1, \ldots, n$*, and let* ρ, ρ_1, ρ_2 *be ribbons with* $\rho = \rho_1 \rho_2$*. Then*

$$F_\rho^{C,\pi,I,J} = \sum_K F_{\rho_1}^{C,\pi,I,K} F_{\rho_2}^{C,\pi,K,J},$$

where we set $I = (i,j)$*,* $J = (i',j')$ *and* K *is an according such tuple.*

Proof. By definition

$$\begin{aligned}
F_\rho^{C,\pi,i,i',j,j'} &= \sum_{z \in Z_G(r)} \overline{\pi_{j,j'}(z)} F_\rho^{\overline{c}_i, q_i z \overline{q}_{i'}} \\
&= \sum_{g \in G} \sum_{z \in Z_G(r)} \overline{\pi_{j,j'}(z)} F_{\rho_1}^{\overline{c}_i, g} F_{\rho_2}^{\overline{g}\overline{c}_i g, \overline{g} q_i z \overline{q}_{i'}} \\
&= \sum_{g \in G} \sum_{z \in Z_G(r)} \overline{\pi_{j,j'}(z)} F_{\rho_1}^{\overline{c}_i, g} F_{\rho_2}^{\overline{g} q_i \overline{r} \overline{q}_i g, \overline{g} q_i z \overline{q}_{i'}}.
\end{aligned}$$

Now using that for each $g \in G$ there exist unique $q \in Q_C$ and $n \in Z_G(r)$ such that $g = qn$ we get, with the notation from the proof of Lemma 2.6.6 and setting $q(g) := q_{i(g)}$,

$$F_\rho^{C,\pi,i,i',j,j'} = \sum_{g,h \in G} \sum_{z \in Z_G(r)} \delta_{h, \overline{g} q_i} \overline{\pi_{j,j'}(z)} F_{\rho_1}^{\overline{c}_i, q_i \overline{n(h)}\,\overline{q(h)}} F_{\rho_2}^{q(h) \overline{r}\, \overline{q(h)}, q(h) n(h) z \overline{q}_{i'}}.$$

Ordering the summation and using the properties of π we get

$$\begin{aligned}
F_\rho^{C,\pi,i,i',j,j'} &= \sum_{g,h \in G} \sum_{z \in Z_G(r)} \delta_{h, \overline{g} q_i} \overline{\pi_{j,j'}(\overline{n(h)} z)} F_{\rho_1}^{\overline{c}_i, q_i \overline{n(h)}\,\overline{q(h)}} F_{\rho_2}^{c_{i(h)}, q(h) z \overline{q}_{i'}} \\
&= \sum_{g,h \in G} \sum_{t=1}^{|\pi|} \delta_{h, \overline{g} q_i} \overline{\pi_{j,t}(\overline{n(h)})} F_{\rho_1}^{\overline{c}_i, q_i \overline{n(h)}\,\overline{q(h)}} F_{\rho_2}^{C,\pi,i(h),i',t,j'} \\
&= \sum_{g \in G} \sum_{t=1}^{|\pi|} \overline{\pi_{j,t}(\overline{n(g)})} F_{\rho_1}^{\overline{c}_i, q_i \overline{n(g)}\,\overline{q(g)}} F_{\rho_2}^{C,\pi,i(g),i',t,j'}.
\end{aligned}$$

Using that Q_C is a set of representatives of the elements of $G/Z_G(r)$ we finally get

$$\begin{aligned}
F_\rho^{C,\pi,i,i',j,j'} &= \sum_{z \in Z_G(r)} \sum_{t=1}^{|\pi|} \sum_{s=1}^{|C|} \overline{\pi_{j,t}(z)} F_{\rho_1}^{\overline{c}_i, q_i z \overline{q_s}} F_{\rho_2}^{C,\pi,s,i',t,j'} \\
&= \sum_{t=1}^{|\pi|} \sum_{s=1}^{|C|} F_{\rho_1}^{C,\pi,i,s,j,t} F_{\rho_2}^{C,\pi,s,i',t,j'}.
\end{aligned}$$

□

We can use the notation from the previous lemma to express the following in an elegant way.

Lemma 2.6.10:
Given a ribbon ρ*, and given a conjugacy classe* C *of* G *with representative* $r \in C$*, centraliser* $Z_G(r)$ *and* Q_C *as above. Then, with the notation from Lemma 2.6.9, we have*

$$\sum_{I} \left(F_\rho^{C,\pi,I,J}\right)^* F_\rho^{C,\pi,I,K} = \delta_{J,K}\mathbb{1}$$

$$\sum_{J} F_\rho^{C,\pi,I,J} \left(F_\rho^{C,\pi,K,J}\right)^* = \delta_{I,K}\mathbb{1}$$

Proof. This follows from Lemma 2.6.3, Lemma 2.6.5 and the definition of the ribbon operators in Proposition 2.6.4:

$$\begin{aligned}
\sum_{(i,j)} (F_\rho^{C,\pi,i,k,j,k'})^* F_\rho^{C,\pi,i,l,j,l'} &= \sum_{(i,j)} F_\rho^{\overline{C},\overline{\pi},i,k,j,k'} F_\rho^{C,\pi,i,l,j,l'} \\
&= \sum_{(i,j)} \sum_{y,z \in Z_G(r)} \pi_{j,k'}(y)\overline{\pi_{j,l'}(z)} F_\rho^{c_i,q_i y q_k} F_\rho^{\bar{c}_i,q_i z \bar{q}_l} \\
&= \sum_{(i,j)} \sum_{y,z \in Z_G(r)} \delta_{q_i y \bar{q}_k, q_i z \bar{q}_l} \pi_{j,k'}(y)\overline{\pi_{j,l'}(z)} F_\rho^{e,q_i y \bar{q}_k} \\
&= \sum_{i} \sum_{y,z \in Z_G(r)} \delta_{y\bar{q}_k, z\bar{q}_l} \pi_{l',k'} \bar{z} y F_\rho^{e,q_i y \bar{q}_k}
\end{aligned}$$

Since Q_C is a set of representatives of the left cosets of $Z_G(r)$ in G, we find

$$\begin{aligned}
\sum_{(i,j)} (F_\rho^{C,\pi,i,k,j,k'})^* F_\rho^{C,\pi,i,l,j,l'} &= \sum_{i} \sum_{z \in Z_G(r)} \delta_{k,l} \pi_{l',k'} e F_\rho^{e,q_i y q_k} \\
&= \sum_{i} \sum_{z \in Z_G(r)} \delta_{k,l} \delta_{l',k'} F_\rho^{e,q_i y q_k} \\
&= \sum_{g \in G} \delta_{k,l} \delta_{l',k'} F_\rho^{e,g} \\
&= \delta_{k,l}\delta_{l',k'}\mathbb{1},
\end{aligned}$$

proving the first equation. The second equation follows from an analogous calculation. □

The commutation relations of ribbons that overlap at their ends and of ribbons that cross each other relate to the fusion and braiding structure of the modular tensor category of finite dimensional representations of $\mathcal{D}(G)$ [Kit03]. In the following we give explicit expressions for these commutation relations.

Lemma 2.6.11:

Let C, D *be conjugacy classes of* G *and* π, χ *irreducible representations of the resepective centralisers* $Z_G(r), Z_G(v)$ *with* $r \in C$ *and* $v \in D$. *Denote by* q_k *the elements of* Q_C *and by* p_l *those of* Q_D. *With the notation used previously we denote the decomposition of an element* $g \in G$ *according to* Q_D *by* $g = p(g)m(g)$ *and use the index notation* $s(g)$ *defined by* $p(g) = p_{s(g)}$. *Let* ρ, σ *be ribbons.*

If $(\rho, \sigma)_{\prec}$ *then*

$$F_\rho^{C,\pi,i,i',j,j'} F_\sigma^{D,\chi,s,s',t,t'} = \\ = \sum_{l=1}^{|\chi|} \chi_{l,t}(m(\bar{c}_i p_s)) F_\sigma^{D,\chi,s(\bar{c}_i p_s),s',l,t'} F_\rho^{C,\pi,i,i',j,j'}.$$

If $(\rho, \sigma)_{\succ}$ *we have*

$$F_\rho^{C,\pi,i,i',j,j'} F_\sigma^{D,\chi,s,s',t,t'} = \\ = \sum_{l=1}^{|\chi|} \overline{\chi_{l,t'}(m(\bar{c}_{i'} p_{s'}))} F_\sigma^{D,\chi,s,s(\bar{c}_{i'} p_{s'}),s',t,l} F_\rho^{C,\pi,i,i',j,j'}.$$

Proof. Starting with the first assertion, i.e. $(\rho, \sigma)_{\prec}$, we have for $hg, k, l \in G$ [BM08]:

$$F_\rho^{h,g} F_\sigma^{k,l} = F_\sigma^{hk\bar{h},hl} F_\rho^{h,g}.$$

Using this and setting $m_{i,s} := m(\bar{c}_i p_s)$, we get

$$\begin{aligned} F_\rho^{C,\pi,i,i',j,j'} F_\sigma^{D,\chi,s,s',t,t'} &= \\ &= \sum_{\substack{n \in Z_G(r) \\ m \in Z_G(v)}} \overline{\pi_{j,j'}(n)} \overline{\chi_{t,t'}(m)} F_\sigma^{\bar{c}_i \bar{d}_s c_i, \bar{c}_i p_s m \bar{p}_{s'}} F_\rho^{\bar{c}_i, q_i n \bar{q}_{i'}} \\ &= \sum_{m \in Z_G(v)} \overline{\chi_{t,t'}(\bar{m}_{i,s} m)} F_\sigma^{\bar{d}_{s(\bar{c}_i p_s))}, p(\bar{c}_i p_s)) m \bar{p}_{s'}} F_\rho^{C,\pi,i,i',j,j'} \\ &= \sum_{l=1}^{|\chi|} \chi_{l,t}(m_{i,s}) F_\sigma^{D,\chi,s(\bar{c}_i p_s),s',l,t'} F_\rho^{C,\pi,i,i',j,j'}. \end{aligned}$$

For the second assertion, i.e. for $(\rho, \sigma)_{\succ}$, we have with $h, g, k, l \in G$ [BM08]

$$F_\rho^{h,g} F_\sigma^{k,l} = F_\sigma^{k,l\bar{g}\bar{h}g} F_\rho^{h,g}.$$

Thus

$$
\begin{aligned}
F_\rho^{C,\pi,i,i',j,j'} F_\sigma^{D,\chi,s,s',t,t'} &= \\
&= \sum_{\substack{n \in Z_G(r) \\ m \in Z_G(\nu)}} \overline{\pi_{j,j'}(n)}\,\overline{\chi_{t,t'}(m)} F_\sigma^{\bar{d}_s, p_s m \bar{p}_{s'} c_{i'}} F_\rho^{\bar{c}_i, q_i n \bar{q}_{i'}} \\
&= \sum_{m \in Z_G(\nu)} \overline{\chi_{t,t'}(m m(\bar{c}_{i'} p_{s'}))} F_\sigma^{\bar{d}_s, p_s m \overline{p(\bar{c}_{i'} p_{s'})}} F_\rho^{C,\pi,i,i',j,j'} \\
&= \sum_{l=1}^{|\chi|} \overline{\chi_{l,t'}(m(\bar{c}_{i'} p_{s'}))} F_\sigma^{D,\chi,s,s(\bar{c}_{i'} p_{s'}),s',t,l} F_\rho^{C,\pi,i,i',j,j'}.
\end{aligned}
$$

□

As an immediate consequence of this lemma we can calculate the commutation relations of two ribbons crossing each other.

Lemma 2.6.12:
Let C, π, Q_C *and* D, χ, Q_D *as usual, with* $r \in C$ *and* $\nu \in D$*, and let* ρ, σ *be ribbons with* $(\rho, \sigma)_\times$*. Then*

$$
\begin{aligned}
F_\rho^{C,\pi,i,i',j,j'} F_\sigma^{D,\chi,s,s',t,t'} = \sum_{i_1=1}^{|C|} \sum_{\substack{j_1=1 \\ j_2=1}}^{|\pi|} \sum_{s_1=1}^{|D|} \sum_{\substack{t_1=1 \\ t_2=1}}^{|\chi|} \overline{\chi_{t_1,t_2}(m_{i_1,s_1})} \pi_{j_1,j_2}(n_{s_1,i_1}) \times \\
\times F_{\sigma_1}^{D,\chi,s,s_{i_1,s_1},t,t_2} F_{\sigma_2}^{D,\chi,s_1,s',t_1,t'} F_{\rho_1}^{C,\pi,i,i_1,j,j_1} F_{\rho_2}^{C,\pi,i_{s_1,i_1},i',j_2,j'},
\end{aligned}
$$

with

$$
\begin{aligned}
m_{i_1,s_1} &= m(\overline{c_{i_1}} p_{s_1}), & n_{s_1,i_1} &= n(d_{s_1} q_{i_1}), \\
s_{i_1,s_1} &= s(\overline{c_{i_1}} p_{s_1}), & i_{s_1,i_1} &= i(d_{s_1,q_{i_1}}),
\end{aligned}
$$

and we used the decomposition $\rho = \rho_1 \rho_2$ *and* $\sigma = \sigma_1 \sigma_2$ *from the definition of* $(\rho, \sigma)_\times$.

Proof. This is a direct consequence of the Lemmata 2.6.11 and 2.6.9. □

Ribbon operators create excitations only at their endpoints, which follows from their commutation relations with the star and plaquette operators (see Lemma 2.6.2). Thus it is expected that the action of ribbon operators does not depend on the geometry of the ribbon but just on its endpoints. This was already mentioned in the seminal paper by Kitaev [Kit03] and elaborated on in more detail in [BM08]. We summarise this in the following lemma.

Lemma 2.6.13:
Let ρ, ρ' *be deformation equivalent, open ribbons. Let* $R \subset \mathbb{Z}^2 \cup (\mathbb{Z}^2)^*$ *be a region such that* ρ, ρ' *lie in* R *and such that all vertices and faces in* ρ, ρ' *are contained in* R*. Then it holds for all* $h, g \in G$

$$F_\rho^{h,g} \prod_{f,\nu \in R} A_\nu B_f = F_{\rho'}^{h,g} \prod_{f,\nu \in R} A_\nu B_f$$

Proof. This follows from the discussion in [BM08, Appendix C]. □

In addition to deforming ribbons it is also possible to invert ribbons. Since the excitations created by a ribbon operator at the ends of the ribbon are conjugate to each other, inversions should then exchange these excitations. This is indeed the case [BM08].

Lemma 2.6.14:
Let ρ *be an open ribbon and let* $\bar{\rho}$ *be an inversion of* ρ*. Let* $R \subset \mathbb{Z}^2 \cup (\mathbb{Z}^2)^*$ *be a region such that* $\rho, \bar{\rho}$ *lie in* R *and such that all vertices and faces in* $\rho, \bar{\rho}$ *are contained in* R*. Then, for all* $h, g \in G$ *it holds*

$$F_\rho^{h,g} \prod_{f,\nu \in R} A_\nu B_f = F_{\bar{\rho}}^{\bar{g}\bar{h}g,\bar{g}} \prod_{f,\nu \in R} A_\nu B_f$$

Proof. This follows from the discussion in [BM08, Appendix C]. □

2.7 The Abelian Model

Much of the structures of ribbon operators becomes significantly simpler when considering finite *abelian* groups. Later in the proof of Haag duality for the quantum double model for finite abelian groups we will need some of these results. We will repeat some of the results from the previous sections here without proof, since we feel it enhances the readability of later sections, and add some further structural lemmas. In the following we will always assume that the group G is finite and abelian. Note that now conjugacy classes of G are given by the elements $c \in G$, and the centraliser $Z_G(c)$ coincides with G. Hence irreducible representations of $\mathcal{D}(G)$ are labelled by tuples (c, χ) where $c \in G$ and χ is an irreducible representation of G. Note also that χ is now necessarily one-dimensional. In the following we will often write for such tuples (χ, c) with χ an irreducible representation of G and $c \in G$.

Apart from obvious changes the transition to abelian groups does not result in very insightful differences in the shape of plaquette and star operators, as well as that of ribbon operators. The charge projections in Definition 2.5.3 become slightly simpler to read. With $s \in \mathbb{Z}^2$ being a site, $c \in G$ and $\chi \in (G)_{irr}$ they become

$$D_s^{\chi,c} = \frac{1}{|G|} \sum_{g \in G} \overline{\chi(g)} D_s^{c,g}. \tag{2.12}$$

The basis of ribbon operators Proposition 2.6.4 is now

$$F_\rho^{\chi,c} = \frac{1}{|G|} \sum_g \overline{\chi(g)} F_\rho^{\bar{c},g},$$

with some ribbon ρ, and

$$(F_\rho^{\chi,c})^* = F_\rho^{\bar{\chi},\bar{c}}, \tag{2.13}$$

by Lemma 2.6.5-

Lemma 2.7.1:
Let ρ be an open ribbon, $c \in G$ and $\chi \in G_{irr}$. With $s = \partial_0 \rho$ or $s = \partial_1 \rho$ it then holds

$$[F_\rho^{\chi,c}, B_s] = 0 \iff C = \{e\},$$

and

$$[F_\rho^{\chi,c}, A_s] = 0 \iff \chi = \mathrm{id}.$$

Lemma 2.7.2:
Let ρ_1, ρ_2, ρ and σ be ribbons with $\rho = \rho_1 \rho_2$. Consider elements $c, d \in G$ and irreducible representations $\chi, \xi \in G_{irr}$. Then we have the following:

$$F_\rho^{\chi,c} = F_{\rho_1}^{\chi,c} F_{\rho_2}^{\chi,c} \qquad \textit{and} \qquad F_\rho^{\chi,c} F_\rho^{\xi,d} = F_\rho^{\chi\xi,cd}.$$

Furthermore,

- *if* $(\rho, \sigma)_\prec$ *then* $F_\rho^{\chi,c} F_\sigma^{\xi,d} = \overline{\xi(c)} F_\sigma^{\xi,d} F_\rho^{\chi,c}$,
- *if* $(\rho, \sigma)_\succ$ *then* $F_\rho^{\chi,c} F_\sigma^{\xi,d} = \xi(c) F_\sigma^{\xi,d} F_\rho^{\chi,c}$,
- *and, if* $(\rho, \sigma)_\times$ *then* $F_\rho^{\chi,c} F_\sigma^{\xi,d} = \xi(c)\chi(d) F_\sigma^{\xi,d} F_\rho^{\chi,c}$.

The commutation relation between ribbon operators and plaquette and star operators suggest that we can regard the ribbon operators as operations on the ground state that create pairs of conjugate excitations at the ending points of ribbons. Furthermore, we can fuse such excitations by considering multiple ribbons starting at the same site. The excitation obtained by this procedure should be the result of the fusion rules of the underlying object that describes these excitations. I.e. in our case, the modular tensor category of finite dimensional representation of Drinfeld's quantum double $\mathcal{D}(G)$ of the group G. For finite groups this has a particular simple structure, as the next lemma illustrates. More precisely we show that given n ribbon connecting with one end to a common site s we can, when acting on the ground state, express the product of the according ribbon operators by one ribbon operators associated to a ribbon that connects to s, multiplied with $n-1$ operators associated to ribbon connecting the remaining ending points of the original ribbons.

Lemma 2.7.3:
Let $\rho_1, \dots, \rho_n$ be open ribbons and s be some site. Assume that

$$\forall i \in \{1, \dots, n\} \exists! j \in \{0, 1\} : \partial_j \rho_i = s.$$

This gives a map $\{1, \dots, n\} \ni i \mapsto j_i \in \{0, 1\}$. Furthermore assume that for all $i, i' \in \{1, \dots, n\}$ it holds that $\partial_{1-j_i} \rho_i \neq \partial_{1-j_{i'}} \rho_{i'}$. Let $\chi_i, i = 1, \dots, n$ be irreducible representations of G and elements $c_i \in G$, $i = 1, \dots, n$. Set $\chi := \chi_1 \cdots \chi_n$ and $c := c_1 \cdots c_n$.

Then there are ribbons $\sigma_1, \dots, \sigma_{n-1}$ with $\{\partial_0 \sigma_k, \partial_1 \sigma_k \,|\, k = 1, \dots, n-1\} = \{\partial_{1-j_i} \rho_i \,|\, i = 1, \dots, n\}$, a ribbon γ with $\partial_0 \gamma = s$ and $\partial_1 \gamma = \partial_{1-j_i} \rho_i$ for some $i \in \{1, \dots, n\}$, and irreducible representations $\xi_1, \dots, \xi_n$ of G and elements $d_1, \dots, d_{n-1} \in G$ such that

$$F_{\rho_1}^{\chi_1, c_1} \cdots F_{\rho_n}^{c_n} \Omega = z F_{\sigma_1}^{\xi_1, d_1} \cdots F_{\sigma_{n-1}}^{\xi_{n-1}, d_{n-1}} F_{\gamma}^{\chi, c} \Omega$$

where $z \in \mathbb{C}$ and $|z| = 1$.

Proof. The proof works by induction over the number of ribbons. By means of inversions of ribbons, i.e. Lemma 2.6.14, we can assume w.o.l.g. that $j(\{1, \dots, n\}) = \{0\}$ for any $n > 0$. In other words we assume that all ribbons involved have their starting point at s since otherwise we could invert them due to the aforementioned lemma.

If $n = 1$ the claim is trivial. We will elaborate on the case $n = 2$ since this illustrates the basic idea of the proof. Let ρ_1, ρ_2 be ribbons as in the assumptions.

Let χ_1, χ_2 be irreducible representations of G and $c_1, c_2 \in G$. Let $\overline{\rho}_1$ be an inversion of ρ_1 such that $\rho_2\overline{\rho}_1$ is a ribbon. Then by Corollary 2.8.2 and Lemma 2.6.14 we have

$$\begin{aligned} F^{\chi_1,c_1}_{\rho_1} F^{\chi_2,c_2}_{\rho_2}\Omega &= F^{\chi_1,c_1}_{\rho_1} F^{\chi_2,c_2}_{\rho_2} F^{\chi_2,c_2}_{\overline{\rho}_1} F^{\chi_2,c_2}_{\rho_1}\Omega \\ &= zF^{\chi_1\chi_2,c_1c_2}_{\rho_1} F^{\chi_2,c_2}_{\rho_2\overline{\rho}_1}\Omega \end{aligned}$$

where z is the factor given by the commutation relations in Lemma 2.7.2. Now let σ be a deformation of $\rho_2\overline{\rho}_1$ such that $s \notin \sigma$. We then have

$$\begin{aligned} F^{\chi_1,c_1}_{\rho_1} F^{\chi_2,c_2}_{\rho_2}\Omega &= zF^{\chi_1\chi_2,c_1c_2}_{\rho_1} F^{\chi_2,c_2}_{\sigma}\Omega \\ &= \tilde{z}F^{\chi_2,c_2}_{\sigma} F^{\chi_1\chi_2,c_1c_2}_{\rho_1}\Omega \end{aligned}$$

as claimed, wnd where $\tilde{z}$ is the phase factor obtained by the last swap.

Now let $\rho_1, \ldots, \rho_n$ be ribbons as in the preamble of the Lemma and assume that the claim holds for all any $n-1$ such ribbons. Let $\chi_1, \ldots, \chi_n$ be irreducible representation of G and $c_1, \ldots, c_n \in G$. Set $\xi := \chi_2 \cdots \chi_n$ and $d := c_2 \cdots c_n$. Then

$$F^{\chi_1,c_1}_{\rho_1} \ldots F^{\chi_n,c_n}_{\rho_n}\Omega = zF^{\chi_1,c_1}_{\rho_1} F^{\xi_2,d_2}_{\sigma_2} \ldots F^{\xi_{n-1},d_{n-1}}_{\sigma_{n-1}} F^{\xi,d}_{\gamma}\Omega$$

where the ribbons γ, σ_k, irreducible representations ξ_k and $c_k \in G$ are corresponding to the claim. Let $\overline{\gamma}$ be an inversion of γ such that $\rho_1\overline{\gamma}$ is a ribbon. Let σ_1 be a deformation of $\rho_1\overline{\gamma}$. Again, using the same Lemmas as above we have

$$\begin{aligned} F^{\chi_1,c_1}_{\rho_1} \ldots F^{\chi_n,c_n}_{\rho_n}\Omega &= \tilde{z}F^{\xi_2,d_2}_{\sigma_2} \ldots F^{\xi_{n-1},d_{n-1}}_{\sigma_{n-1}} F^{\chi_1,c_1}_{\rho_1} F^{\xi,d}_{\gamma}\Omega \\ &= yF^{\xi_2,d_2}_{\sigma_2} \ldots F^{\xi_{n-1},d_{n-1}}_{\sigma_{n-1}} F^{\chi_1,c_1}_{\rho_1} F^{\xi,d}_{\gamma} F^{\chi_1,c_1}_{\overline{\gamma}} F^{\chi_1,c_1}_{\gamma}\Omega \\ &= \tilde{y}F^{\xi_2,d_2}_{\sigma_2} \ldots F^{\xi_{n-1},d_{n-1}}_{\sigma_{n-1}} F^{\xi\chi_1,cc_1}_{\gamma} F^{\chi_1,c_1}_{\rho_1\overline{\gamma}}\Omega \\ &= \tilde{y}F^{\xi_2,d_2}_{\sigma_2} \ldots F^{\xi_{n-1},d_{n-1}}_{\sigma_{n-1}} F^{\xi\chi_1,cc_1}_{\gamma} F^{\chi_1,c_1}_{\sigma_1}\Omega \\ &= \hat{y}F^{\xi_2,d_2}_{\sigma_2} \ldots F^{\xi_{n-1},d_{n-1}}_{\sigma_{n-1}} F^{\chi_1,c_1}_{\sigma_1} F^{\xi\chi_1,cc_1}_{\gamma}\Omega. \end{aligned}$$

The factors $\tilde{z}, y, \tilde{y}$ and $\hat{y}$ are products with z and phase factors resulting from the commutation relations of the ribbon operators. The last expression is of the form as in the claim. □

The next lemmas show that under certain circumstances we can add triangles to the beginning or the end of a ribbon without changing the action of the ribbon operator on the ground state.

Lemma 2.7.4:
Let ρ *be an open ribbon and denote* $s_0 := \partial_0\rho$ *and* $s_1 := \partial_1\rho$*. Pick* $c \in G$ *and an irreducible representation* χ *of* G*. If there is a direct triangle* τ *such that* $\tau\rho$ *is a ribbon the following holds:*

$$[F_\rho^{\chi,c}, A_{s_0}] = 0 \implies F_\rho^{\chi,c} = F_{\tau\rho}^{\chi,c}$$

The analogue statement holds true if $\rho\tau$ *is a ribbon and the ribbon operator commutes with the star operator at* s_1*.*

If there is a dual triangle τ' *such that* $\tau'\rho$ *is a ribbon then*

$$[F_\rho^{\chi,c}, B_{s_0}] = 0 \implies F_\rho^{\chi,c} = F_{\tau'\rho}^{\chi,c}$$

and again an analogue statement holds true if $\rho\tau'$ *is a ribbon.*

Proof. By Lemma 2.6.7 $[F_\rho^{\chi,c}, A_{s_0}] = 0$ implies $\chi = \mathrm{id}$. Hence $F_\rho^{\chi,c} = F_\rho^{\mathrm{id},c}$ and therefore

$$F_{\tau\rho}^{\mathrm{id},c} = \sum_{g,k\in G} T_\tau^g F_\rho^{\bar{c},\bar{g}k} = F_\rho^{\mathrm{id},c}$$

since $\sum_{g\in G} T_\tau^g = I$. Analogously the other case. For if $\tau'\rho$ is a ribbon $[F_\rho^{\chi,c}, B_{s_0}] = 0 \implies c = e$ and

$$F_{\tau\rho}^{\chi,e} = \sum_{g,k\in G} \overline{\chi(k)} L_{\tau'}^{e} \delta_{g,e} F_\rho^{e,\bar{g}k} = F_\rho^{\chi,e}$$

and again analogously for the second case. □

Since G is abelian we also have that ribbon operators of closed ribbons commute with all star and plaquette operators.

Lemma 2.7.5:
Let ρ *be any closed ribbon. Then for all* $h, g, k \in G$

$$[F_\rho^{h,g}, A^k] = 0 = [F_\rho^{h,g}, B^k].$$

The proof can be found in [BM08, Appendix B.5]. A somewhat weaker statement of this is also true if we remove one triangle from a closed ribbon.

Lemma 2.7.6:
Let ρ be an open ribbon such that there is a direct triangle τ with $\tau\rho$ is a closed ribbon. Then, with χ, c, s_0 as above, we have

$$[A_{s_0}, F^{\chi,c}_\rho] = 0 \implies [B_{s_0}, F^{\chi,c}_\rho] = 0$$

Given instead that there is a dual triangle τ' such that $\tau'\rho$ is a closed ribbon. Then

$$[B_{s_0}, F^{\chi,c}_\rho] = 0 \implies [A_{s_0}, F^{\chi,c}_\rho] = 0$$

Proof. 1.) The premises imply, by Lemma 2.7.4, that $F^{\chi,c}_\rho = F^{\chi,c}_{\tau\rho}$, and since $\tau\rho$ is a closed ribbon the claim follows.

2.) The premises imply by Lemma 2.7.4, that $F^{\chi,c}_\rho = F^{\chi,c}_{\tau'\rho}$, and since $\tau'\rho$ is a closed ribbon the claim follows. □

2.8 The Thermodynamic Limit

Recall that we denote the collection of finite subsets of $\mathbb{Z}^2$ by $\mathcal{P}_f(\mathbb{Z}^2)$. This is a partially ordered set with partial order given by the inclusion of sets. For finite sets $\mathcal{O} \in \mathbb{P}_f(\mathbb{Z}^2)$ we set $\mathcal{H}_\mathcal{O} = \bigotimes_{e\in\mathcal{O}} \mathcal{H}_e$. As described in Section 1.3 such systems are described from the perspective of operator algebras. We briefly recall the necessary notions here. For each edge $e \in \mathbb{Z}^2$ we set $\mathcal{A}_e = \mathcal{B}(l^2(G))$ and define the algebra of local observables in $\mathcal{O} \in \mathcal{P}_f(\mathbb{Z}^2)$ by $\mathcal{A}(\mathcal{O}) := \bigotimes_{e\in\mathcal{O}} \mathcal{A}_e$. This results in a net $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$ of C^*-algebras and the algebra of local observables in $\mathbb{Z}^2$ is given by $\mathcal{A}_{\mathrm{loc}} = \bigcup_{\mathcal{O}\in\mathcal{P}_f(\mathbb{Z}^2)} \mathcal{A}(\mathcal{O})$.

The quasilocal algebra $\mathcal{A}$ of the system is defined as the inductive limit of this net, and it can be described by $\mathcal{A}_{\mathrm{loc}}$ as follows. There unique norm $\|\cdot\|$ such that $\mathcal{A} = \overline{\mathcal{A}_{\mathrm{loc}}}^{\|\cdot\|}$ and $\mathcal{A}$ is a C^*-algebra. Note, that if $\Lambda \subset \mathbb{Z}^2$ is an infinite set, then in a similar way we obtain the algebra of observables localised in Λ by

$$\mathcal{A}(\Lambda) = \overline{\bigcup_{\substack{\mathcal{O}\in\mathcal{P}f(\mathbb{Z}^2)\\ \mathcal{O}\subset\Lambda}} \mathcal{A}(\mathcal{O})}^{\|\cdot\|}. \tag{2.14}$$

By Theorem 1.3.1 the dynamics are given by a strongly continuous 1-parameter group of automorphisms α of $\mathcal{A}$ with generator δ. For Kitaev's quantum double models finding a translational invariant ground state of the dynamics is a well-understood problem [Naa12a]. For abelian groups it is even possible to characterise the ground state that are not translational invariant [CNN16].

Theorem 2.8.1 [Naa12a; FN15]:
In the thermodynamic limit of Kitaev's quantum double model for finite groups on the planar square lattice there exists a unique translational invariant ground state ω_0 of the dynamics. This state is uniquely determined by

$$\omega_0(A_s) = \omega_0(B_s) = 1,$$

for any site s in $\mathbb{Z}^2$. Moreover this state is pure.

In the following we use ω_0 to refer to the translational invariant ground state and the tuple $(\pi_0, \mathcal{H}_0, \Omega)$ to refer to the cyclic representation obtained by the GNS construction for ω_0. We often call this particular triple $(\pi_0, \mathcal{H}_0, \Omega)$ the *ground state representation*. Since π_0 is faithful and to simplify notation we usually identify $\mathcal{A}$ with its image $\pi_0(\mathcal{A})$ and deviate from this convention when we want to emphasise the dependency on the chosen representation.

From the properties of the ground state ω_0 we can conclude some additional properties from the discussion in Section 2.6.

Corollary 2.8.2:
Let ρ, ρ' and σ, σ' be deformation equivalent ribbons. Let ω_0 be the unique translational invariant ground state of the Kitaev Hamiltonian. Then for all $A \in \mathcal{A}$ and all $g, h, k, l \in G$ it holds

$$\omega_0(F_\rho^{h,g} A F_\sigma^{l,k}) = \omega_0(F_{\rho'}^{h,g} A F_{\sigma'}^{l,k}).$$

Proof. Since ω_0 is the translationally invariant ground state of (2.4) it is invariant under all star and plaquette operators, i.e. $\forall s \in \mathbb{Z}^2 : \omega(A_s) = 1 = \omega(B_s)$ (see [FN15; Naa12a]). Thus the assertion is a direct consequence of Lemma 2.6.13. □

Corollary 2.8.3:
Let ρ, σ be open ribbons, and let $\overline{\rho}, \overline{\sigma}$ be inverses thereof. Then for all $A \in \mathcal{A}$ and all $g, h, k, l \in G$ it holds

$$\omega_0(F_\rho^{h,g} A F_\sigma^{l,k}) = \omega_0(F_{\overline{\rho}}^{\overline{g}\overline{h}g,\overline{g}} A F_{\overline{\sigma}}^{\overline{k}\overline{l}k,\overline{k}}).$$

Proof. Similar to the previous corollary this is a direct consequence of Lemma 2.6.14. □

The following lemma will be of use in the following chapters as well.

Lemma 2.8.4:
Given ribbons ρ and ρ'. Then, if $(\rho,\rho')_{\prec}$ and $(\rho,\rho')_{\succ}$ then

$$\forall g, g'h, h' \in G : \omega_0((F_\rho^{h,g})^* F_{\rho'}^{h',g'}) = \frac{1}{|G|}\delta_{h,h'}\delta_{g,g'}.$$

If at least one of the conditions on ρ and ρ' is not fulfilled then we have

$$\omega_0((F_\rho^{h,g})^* F_{\rho'}^{h',g'}) = \frac{1}{|G|^2}\delta_{h,e}\delta_{h',e},$$

for all $h, h', g, g' \in G$.

Proof. The proof for the case $\rho = \rho'$ can also be found in [BSW11]. Let ρ and ρ' be as above. Then by definition $\partial_i\rho = \partial_i\rho'$ for $i = 0, 1$. Let $h, h', g, g' \in G$. Then, by Lemma 2.8.2 we can deform ρ' to ρ to get

$$\omega_0((F_\rho^{h,g})^* F_{\rho'}^{h',g'}) = \omega_0(F_\rho^{\bar{h},g} F_\rho^{h',g'}) = \delta_{g,g'}\omega_0(F_\rho^{\bar{h}h',g}).$$

Now we can use that the plaquette operators are stabilisers, i.e. set $s := \partial_0\rho$, then by Lemma 2.6.2

$$\omega_0(F_\rho^{\bar{h}h',g}) = \omega_0(B_s F_\rho^{\bar{h}h',g}) = \omega_0(F_\rho^{\bar{h}h',g} B_s^{\bar{h}h'}) = \delta_{h,h'}\omega_0(F_\rho^{\bar{h}h',g}).$$

Similarly we get

$$\omega_0(F_\rho^{e,g}) = \omega_0(A_s F_\rho^{e,g}) = \frac{1}{|G|}\sum_{k\in G}\omega_0(F_\rho^{e,kg}A_s^k) = \frac{1}{|G|}\sum_{k\in G}\omega_0(F_\rho^{e,kg}) = \frac{1}{|G|},$$

where we used that $\sum_{k\in G} F_\rho^{e,k} = \mathbb{1}$ and $A_s = A_s^k A_s$ (see [BM08]). Thus, putting everything together, we get

$$\omega_0((F_\rho^{h,g})^* F_{\rho'}^{h',g'}) = \frac{1}{|G|}\delta_{h,h'}\delta_{g,g'},$$

as claimed.

For the second claim, note that $\partial_i\rho \neq \partial_i\rho'$ for at least one $i \in \{0,1\}$. Assume w.l.o.g. that $\partial_0\rho \neq \partial_0\rho'$. Set $s := \partial_0\rho$ and $s' = \partial_0\rho'$. Then, as above

$$\begin{aligned}\omega_0((F_\rho^{h,g})^* F_{\rho'}^{h',g'}) &= \omega_0(F_\rho^{\bar{h},g} F_{\rho'}^{h',g'} B_s^h B_{s'}^{h'}) \\ &= \delta_{h,e}\delta_{h',e}\omega_0((F_\rho^{h,g})^* F_{\rho'}^{h',g'}).\end{aligned}$$

Analogous to the first claim we get

$$\omega_0((F_\rho^{e,g})^* F_{\rho'}^{e,g'}) = \frac{1}{|G|^2}\sum_{k,k'\in G}\omega_0(F_\rho^{e,kg}F_{\rho'}^{e,k'g'}) = \frac{1}{|G|^2}.$$

□

2.9 Notation

In order to keep notation as clean as possible we will use some conventions which are complemented by the notational index in the backmatter of this thesis. We will use the greek letters ρ, σ to denote ribbons and ϵ for the trivial ribbon or certain small constants. The letter τ will always refer to a triangle. If τ and τ' occur at the same instance then τ is a direct triangle and τ' always a dual triangle, if not mentioned otherwise.

Lower case letters from the middle of the latin alphabet are usually used for elements of the group G and e is always the identity. As indices group elements will always appear on the upper right of the indexed object. An exception is the Kronecker delta, where they appear in the lower right. For group elements $g \in G$ we use the notation $\overline{g} = g^{-1}$ to make some equations more readable. If $z \in \mathbb{C}$ the symbol $\bar{z}$ will always mean the complex conjugation of z.

The greek letters χ, ξ, φ and π denote irreducible representations of finite groups. Conjugacy classes of the group G are denoted by letters C and D. We will use π also for representations of C^*-algebras, but it will be clear from the context when this is this the case and we then separate it from the use for representations of groups. Ribbon operators are usually denoted by a capital F decorated by some indices according to the definition. The notation A_s^g and B_s^g will always refer to a star and a plaquette operator at site s. Triangle operators are always denoted by T and L.

Capital calligraphic letters from the beginning of the latin alphabet are used for C^*-algebras, and capital letters from the middle of the alphabet for von-Neumann algebras with the exception that for certain von Neumann algebras we use calligraphic letters and for all other fractal letters. An exception is the symbol $\mathcal{F}$ which is used for algebras generated by ribbons. Automorphisms of C^*- and von-Neumann algebras are usually referred to by the greek letters α, β and τ, if no confusion arises. We use the letter Λ to denote cones, and the letter $\mathcal{O}$ for finite subgraphs of the graph $\mathbb{Z}^2$.

3 Haag Duality in the Quantum Double Model for Finite Abelian Groups

In this chapter we prove Haag duality for cone algebras in the GNS representation of the translational invariant ground state of Kitaev's quantum double model for finite abelian groups. Before entering the proof let us explain what Haag duality is an what underlying ideas are which we use in the subsequent. The contents of this chapter are published in [FN15].

We consider Kitaev's quantum double model for a finite abelian groups defined on the square lattice embedded in the two-dimensional plane. We work in the thermodynamic limit of this model. I.e. we describe the system by the quasilocal algebra $\mathcal{A}$ generated by the local algebras $\mathcal{A}(\mathcal{O})$, where $\mathcal{O}$ are finite regions in $\mathbb{Z}^2$, together with the dynamics generated by the local Hamiltonians introduced in Section 2.4 and its unique translationally invariant ground state ω_0. Recall the ground state representation $(\mathcal{H}, \pi, \Omega)$ given by the GNS representation of ω_0.

By construction the local algebras $\mathcal{A}(\mathcal{O})$ with $\mathcal{O} \in \mathcal{P}_f(\mathbb{Z}^2)$ satisfy *locality*: If $\mathcal{O}' \in \mathcal{P}_f(\mathbb{Z}^2)$ is any other finite region disjoint to $\mathcal{O}$, then the operators in $\mathcal{A}(\mathcal{O})$ commute with all operators in $\mathcal{A}(\mathcal{O}')$. In the ground state representation locality can be written as $\pi(\mathcal{A}(\mathcal{O})) \subset \pi(\mathcal{A}(\mathcal{O}^c))'$, where $\mathcal{O}^c$ is the complement of $\mathcal{O}$. In general this inclusion is strict. If, however, Λ is a *cone*, we will prove that in the ground state representation a stronger statement is true, namely $\pi(\mathcal{A}(\Lambda))'' = \pi(\mathcal{A}(\Lambda^c)'$. This property is called *Haag duality*, and it can be used to characterise the superselection sectors of the quantum double models [Naa13a; FN15].

We start with defining cones and introduce the necessary notations for boundaries and sites being parts of cones. In the second section we prove Haag duality. The idea there is to understand how the Hilbert space of the ground state representation is obtained from local excitations above the ground state. This allows to decompose the Hilbert space into a direct sum of a Hilbert space containing excitations inside a cone and a space containing excitations in the cone's complement. Concerning the first summand in this decomposition the ground state vector is in fact cyclic with respect to the cone algebra. The reason is that local excitations can be created by acting with ribbon operators on the ground state. This structure allows us to reduce the problem of showing Haag duality on $\mathcal{H}$ to a commutation problem on this smaller Hilbert space using a result by Rieffel and van Daele [RD75].

3.1 Cones

The main motivation to consider cone-like regions is given by the localisation regions of single excitations of the ground state. These turn out to be suitably described by *cones*. How these cones are defined and which properties we need them to fulfill is described in the following. We will state a list of requirements as a definition and then give a family of regions which fulfill this list. Some of these requirements originate in the localisation properties of excitations sitting at the end of ribbons. Others are motivated as a technical requirement for proving a weaker form of the split property. Most importantly cones should be "ribbon connected" in the sense that we can connect any site inside the cone with ribbons without leaving the cone. Furthermore it should be possible to translate any finite subset of the lattice into the cone using some lattice translation.

First we discuss what we mean by the boundary of a subset of $\Gamma = \mathbb{Z}^2$, where we view Γ as a graph as in Section 2.1. We regard edges as a pair of vertices which are connected by an oriented edge. If we remove a vertex we also discard the edges that contain this vertex. If we remove an edge we do not remove the corresponding vertices. We say that a vertex is *blank* if it does not belong to an edge. The following is a slight refinement of the definitions in reference [FN15]

Definition 3.1.1:
Let $\Lambda \subset \Gamma$ be a collection of edges and associated vertices and by Λ^c the set by first removing form Γ all edges in Λ and then removing all blank vertices. For this, we write $\Lambda^c = \Gamma \setminus \Lambda$. The interior $\mathrm{int}(\Lambda^c)$ *of Λ^c is defined by the collection of edges and vertices in Λ^c obtained by removing all vertices from Γ that are contained in Λ. The* boundary $\partial\Lambda^c$ *of Λ^c is then defined to be $\partial\Lambda^c := \Gamma \setminus (\Lambda \cup \mathrm{int}(\Lambda^c))$ and we set $\partial\Lambda := \partial\Lambda^c$.*

Note that the definition of $\partial\Lambda$ is symmetric under the exchange of Λ and $\mathrm{int}(\Lambda^c)$. Furthermore $\Lambda \cup \mathrm{int}(\Lambda^c)$ is a proper subset of Γ. That is to say $\partial\Lambda$ is the "gap" between Λ and the interior of Λ^c. We also have the following.

Lemma 3.1.2:
Let $\Lambda \subset \mathbb{Z}^2$ be a set of edges and associated vertices. We then have

$$(\mathrm{int}(\Lambda^c))^c = \Lambda \cup \partial\Lambda.$$

Proof. Let $\Lambda \subset \mathbb{Z}^2$ as in the premises. Let E_Λ and V_Λ be the set of edges and vertices in Λ, respectively, and Λ is completely determined by these sets. The set of edges of Λ^c is then

$$E_{\Lambda^c} = \{e \in \mathbb{Z}^2 \,|\, e \notin \Lambda\},$$

and the set of vertices in Λ^c is given by

$$V_{\Lambda^c} = \{v \in \mathbb{Z}^2 \,|\, \exists e \in E_{\Lambda^c} : v \in e\}.$$

Using the definition we get for the correspondent sets of $\mathrm{int}(\Lambda^c)$,

$$\begin{aligned} E_{\mathrm{int}(\Lambda^c)} &= \{e \in \mathbb{Z}^2 \,|\, \partial_0 e \notin V_\Lambda \wedge \partial_1 e \notin E_\Lambda\} \\ V_{\mathrm{int}(\Lambda^c)} &= \{v \in \mathbb{Z}^2 \,|\, v \notin E_\Lambda\}. \end{aligned}$$

Using these we can determine $(\mathrm{int}(\Lambda^c))^c$:

$$\begin{aligned} E_{(\mathrm{int}(\Lambda^c))^c} &= \{e \in \mathbb{Z}^2 \,|\, e \notin \mathrm{int}(\Lambda^c)\} \\ &= \{e \in \mathbb{Z}^2 \,|\, \partial_0 e \in V_\Lambda \vee \partial_1 e \in V_\Lambda\}, \end{aligned}$$

and

$$V_{(\mathrm{int}(\Lambda^c))^c} = \{v \in \mathbb{Z}^2 \,|\, \exists e \in E_(\mathrm{int}(\Lambda^c))^c : v \in e\}.$$

The boundary $\partial\Lambda$ is then described by

$$\begin{aligned} E_{\partial\Lambda} &= \{e \in \mathbb{Z}^2 \,|\, e \notin E_{\mathrm{int}(\Lambda^c)} \wedge e \notin E_\Lambda\} \\ &= \{e \in \mathbb{Z}^2 \,|\, (\partial_0 e \in V_\Lambda \vee \partial_1 e V_\Lambda) \wedge \neg(\partial_0 e \in V_\Lambda \wedge \partial_1 e \in V_\Lambda)\} \\ &= \{e \in \mathbb{Z}^2 \,|\, \partial_0 e \in V_\Lambda \dot{\vee} \partial_1 V_\Lambda\} \end{aligned}$$

and the vertices are given by

$$V_{\partial\Lambda} = \{v \in \mathbb{Z}^2 \,|\, \exists e \in E_{\partial\Lambda} : v \in e\}.$$

Combining this, leads us to the following:

$$
\begin{aligned}
E_{\partial\Lambda} \cup E_{\Lambda} &= \{e \in \mathbb{Z}^2 \mid \partial_0 e \in V_{\Lambda} \vee \partial_1 e \in V_{\Lambda}\} \\
&= E_{(\mathrm{int}(\Lambda^c))^c},
\end{aligned}
$$

and

$$
V_{\partial\Lambda} \cup V_{\Lambda} = V_{(\mathrm{int}(\Lambda^c))^c}.
$$

□

The next step consists in specifying when a triangle and a ribbon belong to a set.

Definition 3.1.3:
Given a subset $\Lambda \subset \Gamma$, a triangle $\tau \subset \Gamma$ and a ribbon $\rho \subset \Gamma$. We say that τ belongs to *or* is contained in Λ *if the edge of τ is in Λ. Similarly we say ρ* belongs to Λ *if all triangles of ρ belong to Λ. If this is the case we write $\tau \subset \Lambda$ and $\rho \subset \Lambda$.*

As we saw in Lemma 2.6.2, excitations above the ground states are localised at sites and can be detected by star and plaquette operators. Therefore, in order to distinguish whether an excitation is contained inside an area or not, we have to specify when a site is, which is rather obvious. Less clear on the other hand is the specification of a site sitting at the boundary of an area. For our purposes and keeping in mind Lemma 2.7.4 we use the following notion.

Definition 3.1.4:
Let $\Lambda \subset \Gamma$ be again a subset and let $s = (v, f)$ be any site. Then s is considered to be contained in Λ, *writing $s \in \Lambda$, whenever for any edge $e \in \Gamma$ with $\partial e = v$ it holds $e \in \Lambda$.*

We say that s is contained in $\partial\Lambda$, *writing $s \in \partial\Lambda$ whenever $s \notin \Lambda$ and there are edges $e \in \Lambda$ and $e' \in \Lambda^c$ which bound f or are contain v.*

In other words $s = (v, f) \in \Lambda$ if the star at v is contained in Λ, and $s \in \partial\Lambda$ if the star or the plaquette has non-empty intersection with Λ and if $s \notin \Lambda$ (c.f. Figure 3.1 (b)).

Note that the definition of $s \in \partial\Lambda$ is in fact symmetric[1] under swapping the roles of Λ and $\mathrm{int}(\Lambda^c)$. The reason is Lemma 3.1.2, namely the boundary of Λ is contained in $(\mathrm{int}(\Lambda^c))^c$. To see this, note that $\mathrm{int}\left((\mathrm{int}(\Lambda^c))^c\right) = \Lambda$.

[1]This was overlooked in reference [FN15].

The boundary of $\mathrm{int}\left((\mathrm{int}(\Lambda^c))^c\right)$ is then given by

$$\partial(\mathrm{int}(\Lambda^c))^c = \mathbb{Z}^2 \setminus ()\mathrm{int}(\Lambda^c) \cup \Lambda) = \partial\Lambda.$$

Note, however, that there might be sites that are contained in $\partial\Lambda$ that have empty intersection with Λ^c

Definition 3.1.4 allows us to distinguish stars and plaquettes that are contained in $\mathrm{int}(\Lambda^c)$ from those having non-trivial intersection with Λ. We will use this later on to move excitations that sit on the boundary of cones into the interior of the respective cone.

Lemma 3.1.5:
Let $\Lambda \subset \Gamma$ be some subset and let $s = (\nu, f) \in \mathrm{int}(\Lambda^c)$ be some site. Then for all edges e ending at ν or bounding f it holds $e \in \Lambda^c$.

Proof. Assume that there was an edge $e \in \Lambda$ ending at ν or bounding f. Then in case it ends in ν we have $s \notin \mathrm{int}(\Lambda^c)$. In case e bounds f but does not end in ν we have that both $\partial_0 e, \partial_1 e \in \Lambda$. But then there is at least one edge e' ending at ν and one of $\partial_0 e, \partial_1 e$ and hence $e' \in \partial\Lambda$. But then $s \notin \mathrm{int}(\Lambda^c)$. □

Finally the straightforward definition of a ribbon ρ *starting or ending at* $\partial\Lambda$ is given by requiring that the starting and ending sites $\partial_{0/1}\rho$ are contained in $\partial\Lambda$. With this definition we have that a ribbon $\rho \subset \Lambda^c$ with, say, $\partial_0\rho \in \partial\Lambda$, is at most one triangle apart from Λ in the following sense. There is a ribbon $\rho_0 \subset \Lambda^c$ with $\partial_0\rho_0 \in \partial\Lambda$ such that $\rho_0\rho$ is a ribbon and ρ_0 is either a single triangle or a trivial ribbon. (Here we have again Lemma 2.7.4 in mind.). This situation is depicted in Figure 3.1 (a).

We now come to the definition of cones. For any subset $\mathcal{O} \subset \Gamma$ and any point $y \in \mathbb{Z}^2$ we denote by $y + \mathcal{O}$ the subset in Γ obtained by translating all pairs of vertices corresponding to edges in $\mathcal{O}$ by y.

Definition 3.1.6:
A subset $\Lambda \subset \Gamma$ is called cone *if it satisfies all of the following criteria.*

1. *For any finite subset $\mathcal{O} \subset \Gamma$ there is a point $y \in \mathbb{Z}^2$ such that $y + \mathcal{O} \subset \Lambda$.*

2. *For any pair of sites $s_0, s_1 \in \Lambda$ there is a ribbon $\rho \subset \Lambda$ with $\partial_{0/1}\rho = s_{0/1}$.*

3. *For any pair of sites $s_0, s_1 \in \partial\Lambda$ there are ribbons $\rho_0, \rho_1 \subset \Lambda^c$ and $\rho \subset \Lambda$ such that $\rho_0\rho\rho_1$ is a ribbon, and for $i = 0, 1$ we have that $\partial_i\rho_i = s_i$, and ρ_i is a single triangle or trivial.*

4. *For any pair of sites $s_0, s_1 \in \partial\Lambda$ there is a ribbon $\rho \subset \Lambda^c$ such that $\partial_i\rho = s_i$ with $i = 0, 1$.*

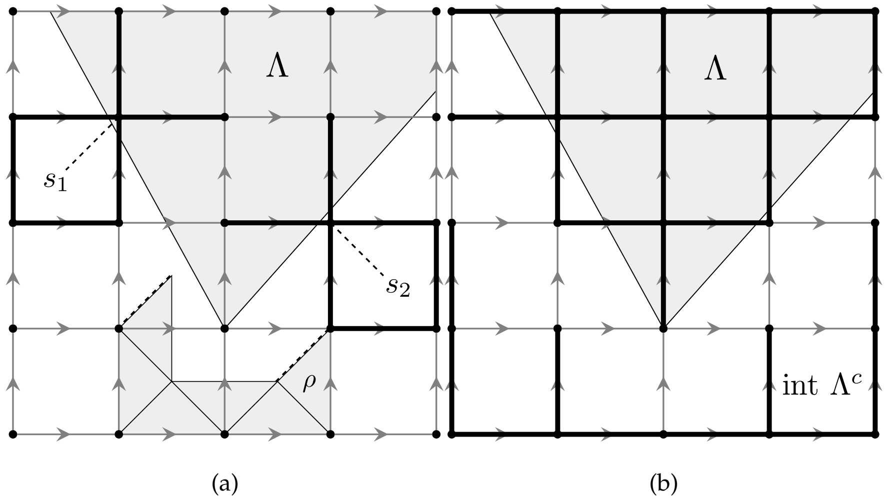

Figure 3.1: In both pictures the grey shaded region Λ indicates a cone. (a): Dotted lines indicate sites, especially $s_1 \in \Lambda$ and $s_2 \in \partial\Lambda$. The black lines highlight the edges belonging to the stars and plaquettes at s_1 and s_2. The ribbon ρ connects a site at Λ with a site in $\partial\Lambda$. (b): Edges that are drawn black are either contained in Λ or $\mathrm{int}(\Lambda^c)$ The grey bonds form the boundary $\partial\Lambda$.

The first condition is of technical nature and plays a role when proving that the weak closures of cone algebras in the vacuum representation are type II_∞ or type III factors. We refer to the details to [Naa11, Theorem 5.2].

The second and the third condition express a kind of connectedness: Any pair of sites inside a cone Λ can be connected with a ribbon, and sites at the boundary can be connected by ribbons that are contained in Λ up to single triangles at the ends. Both of them do not prohibit Λ having holes inside, they just make sure that it is sufficiently connected in the aforementioned sense. The last condition ensures that that the complement Λ^c is properly connected so that there are no holes in Λ.

As a result we can choose whether we want to connect sites at the boundary of the cone by ribbons that run in the exterior or in the interior of the cone up to triangles at the endpoints of the ribbon. In particular for any ribbon $\rho \subset \Lambda^c$ with $\partial_i\rho \in \partial\Lambda, i = 0, 1$ there exist ribbons $\rho_0, \rho_1 \subset \Gamma$ and $\tilde{\rho} \subset \Lambda$ such that $\rho_0\tilde{\rho}\rho_1$ is a ribbon, $\partial_0\rho_0 = \partial_1\rho$, $\partial_1\rho_1 = \partial_0\rho$ and ρ_0, ρ_1 are trivial ribbons or single triangles. Furthermore, by condition 1, any cone is an infinite set.

Examples of cones can be generated by those in $\mathbb{R}^2$: let $l_1 \neq l_2$ be two semi-infinite lines in $\mathbb{R}^2$ emanating from a common point in $\mathbb{Z}^2$ and enclosing an angle smaller than π. Denote by Λ the set of edges that are contained in area enclosed by or have non-empty intersection with the two lines (see also Figure 3.1 (b)). It can be easily checked that Λ is a cone, and in the following, if not specified otherwise, Λ will be a cone.

3.2 Preliminaries

In this section we lay out the necessary notions and structure we will use in the subsequent sections to proof Haag duality.

For the start consider a cone $\Lambda \subset \Gamma$, denote the associated cone algebra by $\mathcal{A}(\Lambda)$ and by $\mathcal{A}(\Lambda^c)$ the one of the complement (see equation (2.14)). Recall that the tuple $(\pi_0, \Omega, \mathcal{H}_0)$ denotes the GNS representation of the translationally invariant ground state ω_0. For any region $\mathcal{O} \subset \Gamma$ we denote the weak closure of $\mathcal{A}(\mathcal{O})$ by $\mathcal{R}_\mathcal{O} := \pi_0(\mathcal{A}(\mathcal{O}))''$. As sketched above we aim at finding a subspace $\mathcal{H}_\Lambda \subset \mathcal{H}_0$ such that Ω is cyclic for $\mathcal{R}_\Lambda$. Since $\mathcal{A}$ is simple, we will identify operators $A \in \mathcal{A}$ with their image under π_0.

Let ρ be a ribbon and let again $\mathcal{F}_\rho := \{F_\rho^{h,g} \,|\, h, g \in G\}$ be the algebra linearly generated by all ribbon operators at ρ. Note that the inclusion $\mathcal{F}_\rho \subseteq \bigotimes_{e\in\rho} \mathcal{A}_e$ is usually proper since $\mathcal{F}_\rho$ can be viewed as the subset of elements of the right hand side singled out by the commutation relations given by Lemma 2.6.2 (c.f.[BM08, B.8]). For cones Λ we denote by $\mathcal{F}_\Lambda := \bigcup_{\rho\subset\Lambda} \mathcal{F}_\rho$ the algebra of ribbon operators localised in Λ. Analogously we denote $\mathcal{F}_{\Lambda^c}$ the algebra of ribbon operators localised in Λ^c.

The first observation is that products of operators in $\mathcal{F}_\Lambda$ and $\mathcal{F}_{\Lambda^c}$ generate a norm-dense subspace of $\mathcal{H}_0$ when applied to Ω (compare also [Naa12b]).

Lemma 3.2.1:
Given that $\Lambda \subset \Gamma$ is a cone we have with the notation from above:

$$\overline{\mathcal{F}_\Lambda \mathcal{F}_{\Lambda^c} \Omega}^{\|\cdot\|} = \mathcal{H}_0.$$

Proof. Single triangle operators are contained in $\mathcal{F}_\Lambda$ and $\mathcal{F}_{\Lambda^c}$. Since they form a basis of the edge algebras, operators in $\mathcal{A}_{\text{loc}}(\Lambda)$ and $\mathcal{A}_{\text{loc}}(\Lambda^c)$ are contained in $\mathcal{F}_\Lambda$ and $\mathcal{F}_{\Lambda^c}$, respectively. But those are norm-dense in $\mathcal{A}(\Lambda)$ and $\mathcal{A}(\Lambda^c)$, respectively, and together with cyclicity of Ω we arrive at the claim. $\square$

Next, we define the subspace $\mathcal{H}_\Lambda$ containing the excitations in the cone Λ.

Definition 3.2.2:
Let $\Lambda \subset \Gamma$ be a cone. We set $\mathcal{H}_\Lambda := \overline{\mathcal{F}_\Lambda \Omega}^{\|\cdot\|} \subset \mathcal{H}_0$ and write P_Λ for the projection onto $\mathcal{H}_\Lambda$.

This subspace turns out to be left invariant by observables localised in the cone. Furthermore such observables are completely determined by their restriction to this space. The proof of this is the same as in [Naa12b, Lemma 3.5] and we won't repeat it here.

Lemma 3.2.3:
For any cone $\Lambda \subset \Gamma$ the subspace $\mathcal{H}_\Lambda \subset \mathcal{H}_0$ is invariant under $\mathcal{A}(\Lambda)$, i.e. $\mathcal{A}(\Lambda)\mathcal{H}_\Lambda \subset \mathcal{H}_\Lambda$. Furthermore any element $A \in \mathcal{R}_\Lambda$ is completely determined by its restriction to $\mathcal{H}_\Lambda$.

As a consequence we have that $P_\Lambda \in \mathcal{R}_\Lambda'$. One basic observation in the proof is that $\mathcal{F}_\Lambda$ is dense in $\mathcal{A}(\Lambda)$ in the uniform topology.

3.3 Three Lemmas

In the following we prove three Lemmas that are essential in gaining a better understanding of the Hilbert space $\mathcal{H}_\Lambda$. Essentially we show that a similar but less obvious statement as in Lemma 3.2.3 holds true for operators commuting with those localised in Λ^c. I.e. we show that observables in the commutant of $\mathcal{A}(\Lambda^c)$ leave $\mathcal{H}_\Lambda$ invariant. The main idea is to show that we can characterise $\mathcal{H}_\Lambda^\perp$ by certain ribbon operators in $\mathcal{F}_{\Lambda^c}$ namely those which create non-trivial excitations in $\mathrm{int}(\Lambda^c)$. In order to do so we are essentially relying on the properties of ribbon operators discussed in Chapter 2.

The basic idea is the same as that of the proof of [Naa12b, Lemma 3.6]: We can characterize vectors of the form $F_1 \cdots F_n \Omega$ to lie either in $\mathcal{H}_\Lambda$ or in $\mathcal{H}_\Lambda^\perp$ where $F_1, \ldots, F_n \in \mathcal{A}$ are ribbon operators. Namely if $F_1 \cdots F_n \Omega$ contains non-trivial excitations in $\mathrm{int}(\Lambda^c)$ then belongs to $\mathcal{H}_\Lambda^\perp$. If there are no excitations in $\mathrm{int}(\Lambda^c)$ contained in this vector, it belongs to $\mathcal{H}_\Lambda$. The next two lemmas show this in a stronger sense, namely that the orthogonal relation in the first case holds even if we apply any operator from $\mathcal{A}(\Lambda^c)'$ to the vector.

The idea is to detect excitations with star and plaquette operators acting on the ending sites of the corresponding ribbons. For this recall the definition of the projections $D_s^{\chi,c}$ in Definition 2.5.1 acting at a site s. To say that there is a charge in $\mathrm{int}(\Lambda^c)$, which is created by some ribbon operator, amounts to seeing that there is some site $s \in \mathrm{int}(\Lambda^c)$, such that $D_s^{\mathrm{id},e}$ does not commute with this operator. Note that this follows from Lemma 2.6.6.

The first lemma shows that if a vector in $\mathcal{H}$ contains only excitations in the *interior* of the complement of the cone Λ, then it is actually contained in $\mathcal{H}_\Lambda^\perp$.

LEMMA 3.3.1:
Let $\hat{F} := \hat{F}_1 \cdots \hat{F}_n \in \mathcal{F}_{\Lambda^c}$ *be a product of ribbon operators associated to ribbons in* Λ^c*. Then the following holds:*

$$\begin{aligned}&\Big(\exists s \in \mathrm{int}(\Lambda^c) : [A_s, \hat{F}] \neq 0 \vee [B_s, \hat{F}] \neq 0\Big)\\ &\quad\Longrightarrow \Big((\forall F, C \in \mathcal{F}_\Lambda)(\forall X \in \mathcal{A}(\Lambda^c)') : (\hat{F}F\Omega, XC\Omega) = 0\Big)\,.\end{aligned} \tag{3.1}$$

Especially the left hand side implies $\hat{F}\Omega \in \mathcal{H}_\Lambda^\perp$*.*

Proof. First note that because of Lemma 3.1.5 $s \in \mathrm{int}(\Lambda^c)$, implies $A_s, B_s \in \mathcal{F}_{\Lambda^c}$. The proof works by repeated use of the lemmas of the discussion in Section 2.6.

It is sufficient to work with ribbon operators labelled by irreducible representations of $\mathcal{D}(G)$ as defined in Lemma 2.6.3. Consider arbitrary such ribbon operators $\hat{F}_1, \dots, \hat{F}_n \in \mathcal{F}_{\Lambda^c}$ and let $C, F \in \mathcal{F}_\Lambda$ be some operators. By definition of $\mathcal{F}_\Lambda$ the operators C and F are sums of products of ribbon operators localised in Λ. For convenience we set $\eta := \hat{F}_1 \cdots \hat{F}_n F\Omega \in \mathcal{F}_{\Lambda^c}\mathcal{F}_\Lambda\Omega$ and $\zeta := C\Omega \in \mathcal{H}_\Lambda$.

Now for the proof of implication (3.1), namely, that if there are excitations in η created by $\hat{F}_1, \dots, \hat{F}_n \in \mathcal{F}_{\Lambda^c}$ then η is orthogonal to $X\zeta$ for all $C, F \in \mathcal{F}_\Lambda$ and all $X \in \mathcal{A}(\Lambda^c)'$.

Assume there exists a site $s \in \mathrm{int}(\Lambda^c)$ whose star operator A_s does not commute with $\hat{F}_1 \cdots \hat{F}_n$. Then, by equation (2.8) and locality, we have

$$\begin{aligned}(\eta, X\zeta) &= \frac{1}{|G|}\sum_{k\in G}(\hat{F}_1 \cdots \hat{F}_n F A_s^k \Omega, X\zeta)\\ &= \frac{1}{|G|}\sum_{k\in G}\hat{\chi}_1(k)\cdots\hat{\chi}_n(k)(\eta, X\zeta)\end{aligned}$$

where $\hat{\chi}_j(k)$ either coincides with the corresponding term of the non-trivial representation of $\hat{F}_j$ if it doesn't commute with A_s, or $\hat{\chi}_j(k) = 1$. Since for abelian groups the tensor product of irreducible representations is again irreducible (they are all 1-dimensional), the right hand side equals 0 since the appearing product representation is non-trivial. If the product representation was trivial then $[A_s, \hat{F}_1 \cdots \hat{F}_n] = 0$ and hence would contradict the assumptions (see Lemma 2.6.7). Thus we arrive at $(\eta, X\zeta) = 0$.

Assume that there is a site $s \in \mathrm{int}(\Lambda^c)$ such that the associated plaquette operator B_s does not commute with $\hat{F}_1 \cdots \hat{F}_n$. Then there is at least one $j \in \{1, \dots, n\}$ with $[B_s, \hat{F}^{\chi,c}_{\rho_j}] \neq 0$ implying $c \neq e$ due to the commutation relations, see Lemma 2.6.7. More general there is a $k \in G$ with $k \neq e$ such that

$$\hat{F}_1 \cdots \hat{F}_n B_s = B^k_s \hat{F}_1 \cdots \hat{F}_n$$

giving

$$(\eta, X\zeta) = (B^k_s \hat{F}_1 \cdots \hat{F}_n C\Omega, \zeta) = (\eta, XFB^k_s \Omega) = 0.$$

□

We complement the previous lemma with the following, where we show that, if a vector in $\mathcal{H}$ contains no excitations in the interior of the complement Λ^c of the cone Λ, then this vector belongs to $\mathcal{H}_\Lambda$.

Lemma 3.3.2:
Let $\hat{F} := \hat{F}_1 \cdots \hat{F}_n \in \mathcal{F}_{\Lambda^c}$ *be a product of ribbon operators associated to ribbons in* Λ^c. *Then the following holds:*

$$\Big(\forall s \in \mathrm{int}(\Lambda^c) : [A_s, \hat{F}] = 0 \wedge [B_s, \hat{F}] = 0\Big) \implies \hat{F}\Omega \in \mathcal{H}_\Lambda \tag{3.2}$$

Proof. Again, as in the previous proof, it is sufficient to work with ribbon operators labelled by irreducible representations of $\mathcal{D}(G)$. First some remarks about some general simplifications we are allowed to make. In case two ribbons ρ, σ have the same starting and ending sites then, by Corollary 2.8.2, one of them can be deformed into the other, giving

$$\hat{F}^{\chi,c}_\rho \hat{F}^{\tau,d}_\sigma \Omega = \hat{F}^{\chi\tau,cd}_\rho \Omega. \tag{3.3}$$

We always can assume that there are non-trivial and non-closed ribbons in the product $\hat{F}_1 \cdots \hat{F}_n$. If ribbon operators associated to closed ribbons appeared then we simply could commute them past the other operators in C to Ω where they leave Ω invariant. This can be seen by noting that if ρ is a closed ribbon and

$\rho = \rho_1\rho_2$ is a partition into ribbons then by Corollary 2.8.3 and Corollary 2.8.2 we have $\hat{F}^{\chi,c}_{\rho}\Omega = \hat{F}^{\chi,c}_{\rho_1}\hat{F}^{\overline{\chi},\overline{c}}_{\overline{\rho}_2}\Omega = \Omega$. Here $\overline{\rho}_2$ is an inversion of ρ_2 which, by construction, starts and ends at the same sites as ρ_1. Due to the commutation relations of ribbons, see the discussion in Section 2.6, we may pick up some phase factors which will not be important here.

In case that there are two ribbon operators $\hat{F}_1, \hat{F}_2$ associated to open ribbons ρ_1, ρ_2 such that $\rho_1\rho_2$ is a closed ribbon we can write them as a product of a ribbon operator of a closed ribbon and an operator associated to an open ribbon. To see this we move $\hat{F}_1$ and $\hat{F}_2$ to each other using the commutation relations of ribbons. Then we use Lemma 2.7.2 to find

$$\hat{F}^{\chi,c}_{\rho_1}\hat{F}^{\xi,d}_{\rho_2} = \hat{F}^{\chi,c}_{\rho_1}\hat{F}^{\overline{\xi},\overline{d}}_{\rho_1}\hat{F}^{\xi,d}_{\rho_1}\hat{F}^{\xi,d}_{\rho_2} = \hat{F}^{\chi\overline{\xi},c\overline{d}}_{\rho_1}\hat{F}^{\xi,d}_{\rho_1\rho_2}. \tag{3.4}$$

By the same lemma we can always assume that ribbons just appear at most once in each product.

Now we turn to the claim of the lemma, equation (3.2). We are performing an induction over the number of ribbon operators in $\hat{F}_1 \cdots \hat{F}_n$, i.e. over the number of ribbon operators outside Λ. Let's start with $n = 1$ and let $\hat{F}_1 \in \mathcal{F}_{\Lambda^c}$ be a ribbon operator. Then we have that the ribbon $\rho \subset \Lambda^c$, which $\hat{F}_1$ is associated to, is either of one of the following forms: It connects two sites in $\partial\Lambda$ or at least one ending site of ρ is contained in $\mathrm{int}(\Lambda^c)$.

Consider the case that ρ connects two sites in $\partial\Lambda$. Taking a look at Definition 3.1.4 we see that there are at most two triangles $\tau, \tilde{\tau} \subset \Lambda^c$ such that $\tau\rho\tilde{\tau} \subset \Lambda^c$ is a ribbon. By assumption and Lemma 2.7.4 we have that $\hat{F}_1\Omega = \hat{F}_{\tau\rho\tilde{\tau}}\Omega$. But then we can invoke Corollary 2.8.3 and Corollary 2.8.2 to obtain a ribbon $\overline{\rho} \subset \Lambda$ with $\hat{F}_{\tau\rho\tilde{\tau}}\Omega = \hat{F}_{\overline{\rho}}\Omega$ and $\hat{F}_{\overline{\rho}} \in \mathcal{A}(\Lambda)$. In case that ρ has at least one ending site contained in $\mathrm{int}(\Lambda^c)$ Lemma 2.6.7 (or an analogue calculation with Lemma 2.6.2) implies that $\hat{F}_1 = \mathbb{1}$. Hence in either case the vector is contained in $\mathcal{H}_\Lambda$.

Now let $n > 1$ be arbitrary but fixed and assume that equation (3.2) holds for all $\hat{F}_1, \ldots, \hat{F}_{n-1} \in \mathcal{A}(\Lambda^c)$. Let therefore $\hat{F}_1, \ldots, \hat{F}_n \in \mathcal{A}(\Lambda^c)$ be ribbon operators associated to ribbons in Λ^c and set $\eta := \hat{F}_1 \cdots \hat{F}_n\Omega$. The remainder of the proof can be subdivided into different cases corresponding to the different configurations ribbons. We will relate some of them to each other and proof the remaining cases. The two main cases are the following: Firstly, there could be $k \leqslant n$ ribbons that start and end at $\partial\Lambda$. Secondly, there could be several ribbons having at least one end in $\mathrm{int}(\Lambda^c)$. See also Figure 3.2.

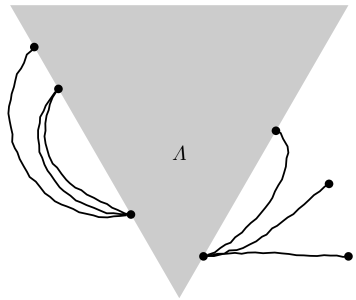

Figure 3.2: The two main cases in Lemma 3.3.2 depicted in one image: On the left hand side of the cone Λ is the case where only ribbons occur that connect sites of $\partial\Lambda$ whith each other. On the right hand side is the case with ribbons having ending sites in $\partial\Lambda$.

The first main case can be handled as follows. Assume that there is a ribbon that connects two sites at $\partial\Lambda$, say ρ_k, $1 \leqslant k \leqslant n$. Then we can commute the associated ribbon operator $\hat{F}_k$ in η to the right in front of Ω thereby possibly obtaining a phase factor due to Lemma 2.7.2. But then, by using the argument from above, we can replace $\hat{F}_k$ with some operator $F_k \in \mathcal{A}(\Lambda)$ leaving a product of $n-1$ operators in $\mathcal{A}(\Lambda^c)$ in front of $F_k\Omega$.

The second main case is a bit more involved. Consider that there is no such ribbon as in the first main case. If there is a ribbon ρ having at least one ending site inside $\mathrm{int}(\Lambda^c)$ the following scenarios are possible. Firstly, one ending site of ρ which is contained in $\mathrm{int}(\Lambda^c)$ does not coincide with an ending site of another ribbon occurring in η. Secondly, ρ connects a site on $\partial\Lambda$ with a site in $\mathrm{int}(\Lambda^c)$ at which $k \geqslant 1$ other ribbons start or end.

In the first case we find, by Lemma 2.6.7, that the associated ribbon operator $\hat{F}_\rho$ must be the identity operator. This reduces the product $\hat{F}_1 \cdots \hat{F}_n \subset \mathcal{A}(\Lambda^c)$ in η to a product of $n-1$ ribbon operators in $\mathcal{A}(\Lambda^c)$.

In the second case we can assume that every of these k ribbons connects to $\partial\Lambda$, since otherwise, we can just pick one of them that doesn't and use the previous procedure to remove it. Remember that we don't have to consider closed ribbons any more as well as open ribbons forming a closed loop. Now consider the ribbon operator $\hat{F}_\rho$ associated to ρ. We can safely assume that $\partial_1\rho$ is the site of interest. The other case can be treated in complete analogy. If there is a ribbon ρ_l with $\partial_0\rho_l = \partial_1\rho$ then we first can deform ρ into a ribbon $\tilde{\rho}$ such that $\tilde{\rho}\rho_l$ is a ribbon. On the level of ribbon operators this means first commuting the associated ribbon operator $\hat{F}_\rho$ in η to the right in front of Ω and then using Corollary 2.8.2 to replace it with an operator $\hat{F}_{\tilde{\rho}}$. After that we use the commutation relations of ribbon operators again to move $\hat{F}_{\rho_l}$ to $\hat{F}_{\tilde{\rho}}$. We then can invoke equation (3.4) to obtain

$$\hat{F}^{\chi,c}_{\tilde{\rho}}\hat{F}^{\xi,d}_{\rho_l} = \hat{F}^{\chi\bar{\xi},c\bar{d}}_{\tilde{\rho}}F^{\xi,d}_{\tilde{\rho}\rho_l}.$$

The ribbon $\tilde{\rho}\rho_l$ connects two sites at $\partial\Lambda$ and we can use a previous argument to replace $\hat{F}_{\tilde{\rho}\rho_l}$ in η by a ribbon operator in $\mathcal{A}(\Lambda)$.

If there is no ribbon ρ_l with $\partial_0\rho_l = \partial_1\rho$ we pick one ribbon ρ_l and apply Corollary 2.8.3 to replace it with a ribbon operator associated to a ribbon $\bar{\rho}_l$ with $\partial_0\bar{\rho}_l = \partial_1\rho$. But then we can proceed as before. Note that we also could have applied Lemma 2.7.3 instead to conclude the same for the second case.

By induction we now can conclude that for any $n \in \mathbb{N}$ and any product of ribbon operators $\hat{F}_1, \cdots, \hat{F}_n \in \mathcal{A}(\Lambda^c)$ the relation in equation (3.2) holds true. □

We can now combine the previous lemmas to show that operators commuting with all operators from $\mathcal{A}(\Lambda^c)$ leave the Hilbert space $\mathcal{H}_\Lambda$ invariant. This can be understood as such operators do not create excitations contained in Λ^c if acting on the ground state.

Lemma 3.3.3:
For any cone $\Lambda \subset \Gamma$ it holds $\mathcal{A}(\Lambda^c)'\mathcal{H}_\Lambda \subset \mathcal{H}_\Lambda$, hence $P_\Lambda \in \mathcal{R}_{\Lambda^c}$.

Proof. Let $\hat{F} := \hat{F}_1 \cdots \hat{F}_n$ be a product of ribbon operators $\hat{F}_1, \dots, \hat{F}_n \in \mathcal{F}_{\Lambda^c}$. Furthermore let $F, C \in \mathcal{F}_\Lambda$ and $X \in \mathcal{A}(\Lambda^c)'$ be any, non-zero, operators. For convenience set $\eta := \hat{F}F\Omega$ and $\xi := C\Omega$. Recall the definition of D_s in equation (2.12).

By Lemma 3.3.1 we have that if $(\eta, X\xi) \neq 0$ holds for all $F, C \in \mathcal{F}_\Lambda$ and $X \in \mathcal{A}(\Lambda^c)'$ then for any $s \in \mathrm{int}(\Lambda^c)$ the operator $\hat{F}$ commutes with D_s, i.e. $[F, D_s] = 0$. Now by Lemma 3.3.2 this implies $\eta \in \mathcal{H}_\Lambda$. To see this note that

$$\hat{F}\Omega \in \mathcal{H}_\Lambda \iff \left(\forall F \in \mathcal{F}_\Lambda : \hat{F}F\Omega \in \mathcal{H}_\Lambda\right)$$

since $\hat{F}F = F\hat{F}$ and $\mathcal{F}_\Lambda\mathcal{H}_\Lambda \subseteq \mathcal{H}_\Lambda$. The other direction of this equivalence can be seen by assuming that the right hand side was true while the left hand was not which immediately leads to a contradiction since $\mathbb{1} \in \mathcal{F}_\Lambda$. Summarizing this we obtain

$$(\eta, X\xi) \neq 0 \implies \eta \in \mathcal{H}_\Lambda \tag{3.5}$$

for all $\eta = \hat{F}F\Omega$, $\xi = C\Omega$ and $\hat{F}, F, C, X$ as above.

By definition $\mathcal{F}_{\Lambda^c}$ contains all matrix units of the edge algebras $\mathcal{A}_e$ for $e \in \Lambda^c$ since the former are products of triangle operators. Hence products of ribbon operators $\hat{F}_1, \ldots, \hat{F}_n$ form a generating system of $\mathcal{F}_{\Lambda^c}$. Thus, by Lemma 3.2.1, the linear span of the set

$$\{\hat{F}_1 \cdots \hat{F}_n F\Omega \,|\, \hat{F}_1, \ldots \hat{F}_n \in \mathcal{F}_{\Lambda^c} \text{ ribbon operators }, F \in \mathcal{F}_\Lambda, n \in \mathbb{N}\}$$

is a dense subspace of $\mathcal{H}$. From this we conclude that equation (3.5) holds for any $\eta \in \mathcal{H}$ and $\xi \in \mathcal{H}_\Lambda$. Therefore

$$(\forall \psi \in \mathcal{H}) : \psi \in \mathcal{H}_\Lambda^\perp \implies ((\forall \phi \in \mathcal{H}_\Lambda)(\forall X \in \mathcal{A}(\Lambda^c)') : (\psi, X\phi) = 0)$$

and we arrive at $\mathcal{A}(\Lambda^c)'\mathcal{H}_\Lambda \perp \mathcal{H}_\Lambda^\perp$. □

3.4 The Reduced Commutation Problem

As the next step we want to consider the restrictions of the von Neumann algebras $\mathcal{R}_\Lambda$ and $\mathcal{R}_{\Lambda^c}$ to $\mathcal{H}_\Lambda$. By [Tak79, Proposition II.3.10] both restrictions are again von Neumann algebras.

Definition 3.4.1:
For any cone $\Lambda \subset \Gamma$ we write $\mathcal{A}_\Lambda := P_\Lambda \mathcal{R}_\Lambda P_\Lambda \restriction_{\mathcal{H}_\Lambda}$ and $\mathcal{B}_\Lambda = P_\Lambda \mathcal{R}_{\Lambda^c} P_\Lambda \restriction_{\mathcal{H}_\Lambda}$ as subalgebras of $\mathcal{B}(\mathcal{H}_\Lambda)$.

By using similar techniques as in the proof of the lemmas 3.3.1 and 3.3.2 we show that elements of the form $A_s + iB_s$ with $A_s \in \mathcal{A}_s$ and $B_s \in \mathcal{B}_s$ already generate $\mathcal{H}_\Lambda$ when applied to the ground state vector Ω. Here $\mathcal{A}_s$ is the self-adjoint part of $\mathcal{A}_\Lambda$, and similarly for $\mathcal{B}_s$.

Lemma 3.4.2:
Let $\mathcal{A}_s$ be the self-adjoint part of $\mathcal{A}_\Lambda$ and $\mathcal{B}_s$ that of $\mathcal{B}_\Lambda$. Then the space $\mathcal{A}_s\Omega + i\mathcal{B}_s\Omega$ is dense in $\mathcal{H}_\Lambda$.

Proof. First note that since both $\mathcal{A}_s$ and $\mathcal{B}_s$ are real vector spaces it suffices to show that $F\Omega$ and $iF\Omega$ are contained in $\mathcal{A}_s\Omega + i\mathcal{B}_s\Omega$ for operators $F \in \mathcal{F}_\Lambda$. In order to do so we first show this to hold if F is a finite product of ribbon operators in $\mathcal{F}_\Lambda$ and then conclude for general operators $F \in \mathcal{F}_\Lambda$ by a density argument. The structure of the vector space $\mathcal{H}_\Lambda$ that we elaborated on earlier in Lemma 3.2.1 and in the proofs of Lemma 3.3.1 and 3.3.2 is essential for this proof. This is to say that finite products of ribbon operators in $\mathcal{F}_\Lambda$ applied to the vacuum vector Ω sufficiently describe $\mathcal{H}_\Lambda$ and certain ribbon operators in $\mathcal{F}_{\Lambda^c}$ map Ω to vectors in $\mathcal{H}_\Lambda$ and can be expressed as the images of Ω of certain elements of $\mathcal{F}_\Lambda$.

Throughout the proof we consider ribbon operators labelled by irreducible representations of the quantum double model $\mathcal{D}(G)$. We can assume that these labels are nontrivial, for they were trivial we just obtained the identity operator. Again we will use the charge projections $D_s^{\chi,c}$ introduced in equation (2.12) which project onto the excitation given by (χ, c) at site s. In particular, recall that we have $D_s = D_s^{\mathrm{id},e} = A_s B_s$.

Now let $F_1, \dots, F_n \in \mathcal{F}_\Lambda$ be ribbon operators with $n > 0$ and set $F := F_1 \cdots F_n$. The idea is to construct self-adjoint elements of $\mathcal{A}_s$ and $\mathcal{B}_s$ by taking linear combinations of products of projections A_s, B_s and products of ribbon operators in $\mathcal{F}_\Lambda$ and $\mathcal{F}_{\Lambda^c}$. These self-adjoint operators are chosen in such a way that they map the state vector to the same vector as F. Again, as in previous proofs, we will work with an induction over the number of ribbon operators in F. With the same argument as in the proof of Lemma 3.3.2 we can assume that there are no ribbon operators associated to closed ribbons or trivial ribbons in F.

Let $n = 1$ and let ρ denote the corresponding ribbon. In case that both the star and the plaquette at least one of the ending sites of ρ, denoted by s, are contained in Λ we set

$$\tilde{F} := FD_s + D_s F^* \qquad \text{and} \qquad \hat{F} := i(FD_s - D_s F^*).$$

Obviously these operators are selfadjoint hence contained in $\mathcal{A}_s$ and it can easily be checked that $\tilde{F}\Omega = F\Omega$ and $\hat{F}\Omega = iF\Omega$. Therefore $F\Omega$ and $iF\Omega$ belong to $\mathcal{A}_s\Omega$.

Assume that at both ends of ρ are contained in Λ but the plaquettes at both sites are not contained in Λ. Then the stars are still contained in Λ, by definition (c.f. Definition 3.1.4 and the discussion after) and the star operators are elements of $\mathcal{A}_s$. In case $[F, A_s] \neq 0$, with $s = \partial_0\rho$ or $s = \partial_1\rho$, it suffices to take

$$\tilde{F} := FA_s + A_s F^* \qquad \text{and} \qquad \hat{F} := i(FA_s - A_s F^*),$$

since then $\tilde{F}\Omega = F\Omega + \delta_{\chi,\mathrm{id}}\Omega = F\Omega$ an analogously $\hat{F}\Omega = iF\Omega$ where χ is part of the label of F. These operators are selfadjoint and $\tilde{F}, \hat{F} \in \mathcal{A}_s$ hence $F\Omega, iF\Omega \in \mathcal{A}_s\Omega$.

If, however, $[F, A_s] = 0$ we can use Lemma 2.7.4 to extend ρ with triangles $\tau, \tilde{\tau}$ such that $\tilde{\rho} := \tau\rho\tilde{\tau}$ is a ribbon, and $\partial_0\tilde{\rho}, \partial_1\tilde{\rho} \in \partial\Lambda$. Furthermore we then have $F_{\tilde{\rho}}\Omega = F\Omega$. But now we can invoke Corollary 2.8.3 and Corollary 2.8.2 to find a ribbon $\overline{\rho} \subset \Lambda^c$ such that $F\Omega = F^*_{\overline{\rho}}\Omega$, and we set

$$\tilde{F} := \frac{1}{2}(F + F^*) + i\left(\frac{i}{2}(F_{\overline{\rho}} - F^*_{\overline{\rho}})\right)$$

and it can easily be checked that the "real part" of $\tilde{F}$ is an element of $\mathcal{A}_s$ and the "imaginary part" one of $\mathcal{B}_s$, hence $\tilde{F} \in \mathcal{A}_s + i\mathcal{B}_s$. By construction $F\Omega = \tilde{F}\Omega \in \mathcal{A}_s\Omega + i\mathcal{B}_s\Omega$. Similarly

$$\hat{F} := \frac{i}{2}(F - F^*) + \frac{i}{2}(F_{\overline{\rho}} + F^*_{\overline{\rho}})$$

and $iF\Omega = \hat{F}\Omega \in \mathcal{A}_s\Omega + i\mathcal{B}_\Omega$.

We now proceed by induction. Let $n > 0$ be arbitrary but fixed and assume that the assertion holds for all $F_1, \ldots, F_{n-1} \in \mathcal{F}_\Lambda$. Let $F_1, \ldots, F_n \in \mathcal{F}_\Lambda$ be any non-trivial ribbon operators. If one of them was trivial then we could remove it and obtained $n-1$ factors. Again we have different cases to treat. First of all we handle the case where we can remove or combine ribbon operators leaving us with $n-1$ factors in the product. More precisely, consider that there are two ribbon operators associated to ribbons ρ_i, ρ_k with $1 \leqslant i, k \leqslant n$ such that they start and end at the same site. Then either $\partial_j\rho_i = \partial_j\rho_k, j = 0, 1$ or $\rho_i\rho_k$ is a closed ribbon. In either case in the product $F_1 \cdots F_n$ we can bring F_i and F_k to the right by using the commutation relations of ribbon operators. Then we can use equation (3.3) and Lemma 2.7.2 to replace F_iF_j in front of Ω with a single ribbon operator. If $\rho_i\rho_k$ is closed then we have $F_i^{\chi,c}F_j^{\xi,d}\Omega = F_{\rho_i}^{\chi\overline{\xi},c\overline{d}}\Omega$. In case $\partial_j\rho_i = \partial_j\rho_k, j = 0, 1$ we have $F_i^{\chi,c}F_j^{\xi,d}\Omega = F_i^{\chi\xi,cd}\Omega$. Again (χ, c) and (ξ, d) are irreducible representations of $\mathcal{D}(G)$. That is, in both cases we end up with a product of $n-1$ ribbon operators in front of Ω. This allows us to assume in the rest of the proof that in $F_1 \cdots F_n$ each ribbon involved there is appearing exactly once.

The rest of the proof can be divided into three main cases. Let again $F_1 \cdots F_n$ be the product of non-trivial ribbon operators in $\mathcal{F}_\Lambda$. Assume that there are no such ribbons as in the previous case. Then there are three possibilities: (I) either there exists a ribbon ρ involved in the product such that $D_{\partial_i\rho} \in \mathcal{F}_\Lambda$ for at least one $i = 0, 1$, or (II) all ribbons end at $\partial\Lambda$, or (III) neither of both, i.e. for any ribbon involved in $F_1 \cdots F_n$ it holds that $\partial_i\rho \notin \partial\Lambda$ and $D_{\partial_i\rho} \notin \mathcal{F}_\Lambda$ for both $i = 1, 2$.

Consider case (I), namely that $D_{\partial_i\rho} \in \mathcal{F}_\Lambda$ for $i = 0$ or $i = 1$ for at least one ribbon involved in $F_1 \cdots F_n$. We set $s := \partial_0\rho$ and without loss of generality we can assume that $F_\rho = F_n$ and $i = 0$. If the ribbon operator was not F_n we could use the commutation relations of ribbon operators to move this operator to the last place in the product. We can divide the treatment of this case into two different cases. The first case is that there is a site $s \in \Lambda$ such that $[F_1 \cdots F_n, D_s] \neq 0$. In the other case we have that for all sites $s' \in \Lambda$ with $D_{s'} \in \mathcal{F}_\Lambda$ it holds $[F_1 \cdots F_n, D_{s'}] = 0$.

Now for the first subcase of case (I). If there is a site $s \in \Lambda$ with $[F_1 \cdots F_n, D_s] \neq 0$ we can set

$$\tilde{F} := F_1 \cdots F_n D_s + D_s F_1^* \cdots F_n^* \qquad \text{and} \qquad \hat{F} := iF_1 \cdots F_n D_s - iD_s F_1^* \cdots F_n^*.$$

Then $\tilde{F}, \hat{F} \in \mathcal{A}_s$ and it holds $F_1 \cdots F_n \Omega = \tilde{F}\Omega$ and similarly $iF_1 \cdots F_n \Omega = \hat{F}\Omega$.

The case that for all sites $s' \in \Lambda$ with $D_{s'} \in \mathcal{F}_\Lambda$ it holds $[F_1 \cdots F_n, D_{s'}] = 0$ can be treated as follows. Since we assumed that there is at least one ribbon ρ involved in the product, the corresponding ribbon operator is either trivial, by Lemma 2.6.7, or there is at least one additional ribbon ending or starting at one of the endpoints of ρ. We excluded the first case by assumption so we have to treat the second one. Therefore consider the situation where there are k ribbons $\rho_{n-k}, \ldots, \rho_n$ in $F_1 \cdots F_n$ ending at s. By Lemma 2.6.7 the condition that the operators commute with the charge projector is equivalent to $\chi_{n-k} \cdots \chi_n = \mathrm{id}$ and $c_{n-k} \cdots c_n = e$, where χ_i are irreducible representations of G and $c_i \in G$ with $i = n-k, \ldots, n$. But by Lemma 2.7.3 we have that there are ribbons $\sigma_{n-k}, \ldots, \sigma_{n-1}$ such that they do not cross the site s, a ribbon γ having s as an ending site, irreducible representations $\xi_{n-k}, \ldots, \xi_{n-1}$ of G and elements $d_{n-k}, \ldots, d_{n-1} \in G$ such that

$$F\Omega = zF_{\rho_1}^{\chi_1, c_1} \cdots F_{\rho_{n-k-1}}^{\chi_{n-k-1}, c_{n-k-1}} F_{\sigma_{n-k}}^{\xi_{n-k}, d_{n-k}} \cdots F_{\sigma_{n-1}}^{\xi_{n-1}, d_{n-1}} F_{\gamma}^{\chi, c} \Omega$$

where $z \in \mathbb{C}, |z| = 1$, $\chi = \chi_{n-k} \cdots \chi_n$, $c = c_{n-k} \cdots c_n$, and $F = F_{\rho_1}^{\chi_1, c_1} \cdots F_{\rho_n}^{\chi_n, c_n}$. The commutation relation with the charge projection now tells us that $\xi = \mathrm{id}$ and $c = e$, hence $F_\gamma^{\chi, c} = \mathbb{1}$. This gives an expression with $n-1$ ribbon operators acting on Ω and we are done for this case.

Let's turn case (II), where in the product $F_1 \cdots F_n \in \mathcal{F}_\Lambda$ there are only ribbons $\rho_i, i \in \{1, \ldots, n\}$ involved whose ending sites are contained in $\partial\Lambda$. By definition, c.f. Definition 3.1.4, it holds for all $i \in \{1, \ldots, n\}$ that $D_{\partial_k \rho_i} \neq \mathcal{F}_\Lambda, k = 0, 1$ so we cannot treat this in the manner as the first main case. In the proof of Lemma 3.3.2 we used that we can replace ribbon operators associated to ribbons, which are contained in Λ^c and which connect sites on $\partial\Lambda$, to ribbon operators of ribbons which are contained in Λ and which connect the same sites, without changing the image of Ω under these operators. Of course, this works the other way round, too.

So choosing

$$\tilde{F} := \frac{1}{2}\left(F_{\rho_1}\cdots F_{\rho_n} + F^*_{\rho_n}\cdots F^*_{\rho_1}\right) + i\left(\frac{i}{2}(F_{\tilde{\rho}_1}\cdots F_{\tilde{\rho}_n} - F^*_{\tilde{\rho}_n}\cdots F^*_{\tilde{\rho}_1})\right) \tag{3.6}$$

and

$$\hat{F} := \frac{i}{2}\left(F_{\rho_1}\cdots F_{\rho_n} - F^*_{\rho_n}\cdots F^*_{\rho_1}\right) + \frac{i}{2}\left(F_{\tilde{\rho}_1}\cdots F_{\tilde{\rho}_n} + F^*_{\tilde{\rho}_n}\cdots F^*_{\tilde{\rho}_1}\right) \tag{3.7}$$

will do the job. We used the notation F_{ρ_i} instead of $F_i, i = 1, \dots, n$ to indicate the dependence on the ribbon. As above $\tilde{\rho}_i$ indicates the ribbon obtained by extending ρ_i by triangles corresponding to Lemma 2.7.4 if necessary, and inverting it using Corollary 2.8.3. Then $\tilde{F}, \hat{F} \in \mathcal{A}_s + i\mathcal{B}_s$ and it can easily be verified that $\tilde{F}\Omega = F_1 \cdots F_n\Omega$ and $\hat{F}\Omega = iF_1 \cdots F_n\Omega$.

It remains to treat case (III). Consider there is no ribbon ρ involved in F such that it falls under the two previous main cases. I.e. for any $\rho \subset \Lambda$ appearing in F both of the ending sites $s_i := \partial_i\rho, i = 0, 1$ are such that $\mathcal{D}_{s_i} \notin \mathcal{F}_\Lambda$ for $i = 1, 2$ and there are ribbons ρ involved in F with $s_i \notin \partial\Lambda$ for at least one $i = 1, 2$. This means that any ending site s of any ribbon occurring in F is either contained in $\partial\Lambda$ or close enough to the boundary of Λ that the plaquette at s is not contained in Λ any more. By Definition 3.1.4, however, if $s \notin \Lambda$ then the star at s will nevertheless be still contained in Λ. Let now ρ be any ribbon in F for which at least one ending site s is not contained in the boundary of Λ, i.e. $s \notin \partial\Lambda$. Without loss of generality $s = \partial_0\rho$. Then, by construction of Λ and by Definition 3.1.4, $A_s \in \mathcal{F}_\Lambda$.

There are two cases appearing here, namely either $[A_s, F] \neq 0$, or for any ending sites s of ribbons ρ in F with $s \notin \partial\Lambda$ it holds $[A_s, F] = 0$. In case $[A_s, F] \neq 0$ for one such ribbon ρ we simply set

$$\tilde{F} := F_1 \cdots F_n A_s + A_s F_1^* \cdots F_n^* \quad \text{and} \quad \hat{F} := iF_1 \cdots F_n A_s - iA_s F_1^* \cdots F_n^*. \tag{3.8}$$

Then $\tilde{F}, \hat{F} \in \mathcal{A}_s$ and it holds $F_1 \cdots F_n\Omega = \tilde{F}\Omega$ and similarly $iF_1 \cdots F_n\Omega = \hat{F}\Omega$.

In case $[A_s, F] = 0$ for any ending site s of ρ with $s \notin \partial\Lambda$ we proceed as follows. Let ρ be such a ribbon, and without loss of generality let $s = \partial_0\rho \notin \partial\Lambda$. By Lemma 2.7.4, there is a direct triangle $\tau \in \Lambda$ such that $\tau\rho$ is a ribbon with $F_{\tau\rho}\Omega = F_\rho\Omega$. Then by the discussion after Definition 3.1.4 and the definition of cones, $\partial_0\sigma \in \partial\Lambda$. The analogue statement holds true, if $s = \partial_1\rho$ with $\sigma = \rho\tau$ and τ according to Lemma 2.7.4. If we apply this to each ribbon in F we end up at the situation in the second main case from where we can proceed accordingly. □

With these preparations we are finally in a position to prove the main theorem. In particular, the last lemma allows us to use the result of Rieffel and Van Daele mentioned before.

THEOREM 3.4.3:
Cone algebras of the quantum double model for finite abelian groups on the infinite square lattice satisfy Haag duality in the vacuum representation.

More precisely, if $\Lambda \subset \Gamma$ *is a cone then*

$$\pi_0\left(\mathcal{A}(\Lambda^c)\right)' = \pi_0\left(\mathcal{A}(\Lambda)\right)''.$$

Proof. The argument is exactly the same as that given in reference [Naa12b]. For the convenience of the reader, we will restate it here.

It remains to prove $\mathcal{A}(\Lambda^c)' \subset \mathcal{A}(\Lambda)''$ since, by locality, the other direction already holds. By construction it holds that $\mathcal{A}_\Lambda \subset \mathcal{B}'_\Lambda$ (as sub-algebras of $\mathcal{B}(\mathcal{H}_\Lambda)$) and both, $\mathcal{A}_\Lambda$ and $\mathcal{B}_\Lambda$, are von-Neumann algebras on the same Hilbert space $\mathcal{H}_\Lambda$. Hence, by [RD75, Theorem 2], the statement of Lemma 3.4.2 is equivalent to $\mathcal{A}_\Lambda = \mathcal{B}'_\Lambda$.

Furthermore, by [Tak79, Proposition II.3.10], it holds that $\mathcal{B}'_\Lambda = P_\Lambda \mathcal{R}'_{\Lambda^c} P_\Lambda \restriction_{\mathcal{H}_\Lambda}$. Now let $B \in \mathcal{R}'_{\Lambda^c}$ and denote $B_\Lambda \coloneqq P_\Lambda B P_\Lambda \restriction_{\mathcal{H}_\Lambda} \in \mathcal{B}'_\Lambda$. Then $B_\Lambda \in \mathcal{A}_\Lambda$ and, by Lemma 3.2.3, there exists a unique element $A \in \mathcal{R}_\Lambda$ such that $B_\Lambda = P_\Lambda A P_\Lambda \restriction_{\mathcal{H}_\Lambda}$.

To proof the claim it suffices to show that $B = A$. Pick any $\hat{F} \in \mathcal{F}_{\Lambda^c}$ and $F \in \mathcal{F}_\Lambda$. Then

$$B\hat{F}F\Omega = \hat{F}BF\Omega = \hat{F}B_\Lambda F\Omega = \hat{F}AF\Omega = A\hat{F}F\Omega$$

giving $A = B$, by Lemma 3.2.1, and consequently $B \in \mathcal{R}_\Lambda$. $\square$

4 Jones-Kosaki-Longo Index and Further Structure

4.1 Superselection Sectors

As we have seen in Section 2.6 acting with local observables on the unique translationally invariant ground state ω_0 always creates pairs of excitations. The excitations can be detected using the charge projections from Definition 2.5.3, or using operators associated to closed ribbon encircling the excitation [BM08]. This implies that if $\psi \in \mathcal{H}$ is a state containing only local excitations above the ground state we can use local observables to remove these excitations. I.e. globally the state ψ is the ground state plus some local excitations. This suggests that to create single excitations, if they exist at all, one cannot use local observables.

One way to obtain single excitations would be the following. Consider a ribbon $\rho \subset \mathbb{Z}^2$ together with a ribbon operator F which will create a pair of conjugate excitations at the end sites of ρ. The idea now is to take one end of the ribbon ρ to infinity while keeping the other end fixed. If applied to the ground state vector Ω the resulting operator then creates an excitation at the fixed end of the ribbon and at infinity. It is still possible to decide which excitation has been created by measuring at the fixed end of ρ but there is no local operator that can remove this excitation from the ground state. It is also clear form the preceding section that we can move such an excitation on the lattice using local operations. Assuming now for a moment that we could apply the above procedure to obtain two states, ψ and ϕ, that contain different excitations, then by construction one can check that $\langle\psi| A |\phi\rangle = 0$ for all local observables A, i.e. $A \in \mathcal{A}_{\mathrm{loc}}$. Hence these states obey a *superselection rule* [WWW52]. That is, physical observables can distinguish the states ϕ and ψ but it is not possible to change them into each other.

There are some problems with this approach. The limit of a sequence of ribbon operators, where for growing index the length of the ribbon goes to infinity, does not converge in the uniform topology of $\mathcal{A}$ (neither does the sequence of vectors). Therefore such operations are not part of $\mathcal{A}$. Furthermore, this procedure somewhat depends on the ground state representation, since it was implicitly assumed that we can act on the ground state vector. In addition, the ground state Ω is cyclic for $\pi_0(\mathcal{A})$, hence $\mathcal{H}_0$ consists only of states that can be created by acting with local observables on the ground state (compare also to Lemma 3.2.1).

It turns out, however, that this approach is the right way to go and that it is possible to make such a construction rigorous. However, instead of taking limit of ribbon operators with ribbons extending eventually to infinity, one considers a sequence of automorphism of $\mathcal{A}$ which converge strongly to an automorphism of $\mathcal{A}$ [Naa11; FN15]. In fact this gives rise to equivalence classes of representations, where states obtained from ω_0 by these automorphisms are inequivalent to ω_0. Even more, the GNS representations of these states satisfy a selection criterion for cone-like localised excitations, and inequivalent representations belong to distinct equivalence classes of representations. It also follows that each representative of these equivalence classes satisfy a DHR-like selection criterion. If one extends the selection criterion to allow more general representations of $\mathcal{A}$, then one obtains a family of equivalence classes, or *superselection sectors* (or *sectors*), which reflect the charge content in the thermodynamic limit of the model. This realises the DHR theory of superselection sectors [DHR74] in the sense of the BF framework for string-like excitations [BF82]. The construction in this chapter itself in principle allows us to only construct a few sectors, namely those corresponding to the single excitations created by semi-infinite ribbons. The analysis in the subsequent chapter shows that these are in fact all sectors satisfying this criterion, i.e we already obtain a full classification of the superselection sectors of the quantum double models for finite abelian groups.

In what follows we review the construction in [FN15]. We put slight emphasis on the construction of the charge transporters that carry charges from one cone into another, since we will use them in the subsequent sections.

4.1.1 Semi-infinite ribbons

Consider a cone $\Lambda \subset \mathbb{Z}^2$ which we fix from now on. Let ρ be a semi-infinite ribbon in Λ, i.e. one end, $\partial_0\rho$ is fixed while the other end is sent to infinity. We denote by $(\rho_n)_{n\in\mathbb{N}}$ a sequence of ribbons in Λ with $\rho_n \subset \rho$, ρ_{n-1} properly contained in ρ_n, and $\partial_0\rho_n = \partial_0\rho$ for all $n \in \mathbb{N}$. By Lemma 2.6.6 each ribbon operator $F_n^{\chi,c} := F_{\rho_n}^{\chi,c}$, with $(\chi, c) \in \hat{G} \times G$, creates a pair of excitations of type (χ, c) at its ending sites when acting on the ground state. On the level of observables, the resulting state is given by $\omega_{\rho_n}^{(\chi,c)}(\cdot) = \omega_0(F_n^{\chi,c} \cdot (F_n^{\chi,c})^*)$. This motivates the definition of

$$\alpha_n^{\chi,c}(A) := F_n^{\chi,c} A (F_n^{\chi,c})^*, \quad A \in \mathcal{A},$$

which is an automorphism, by Lemma 2.7.2 and equation (2.13).

Proposition 4.1.1 [FN15]:
Let ρ be a semi-infinite ribbon, and let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence of ribbons contained in ρ with $\rho_n \subset \rho$, ρ_{n-1} properly contained in ρ_n, and $\partial_0\rho_n = \partial_0\rho$ for all $n \in \mathbb{N}$. Let $(\chi, c) \in \hat{G} \times G$. Then for each $A \in \mathcal{A}$ the limit

$$\alpha_\rho^{\chi,c}(A) := \lim_{n\to\infty} \alpha_n^{\chi,c}(A)$$

converges uniformly and defines an automorphism $\alpha_\rho^{\chi,c} : \mathcal{A} \to \mathcal{A}$ fulfilling the following properties:

1. $\alpha_\rho^{\chi,c}(A) = A$ *for* $A \in \mathcal{A}$ *whenever* $\mathrm{supp}(A) \cap \rho = \emptyset$.
2. $\forall A \in \mathcal{A}_{\mathrm{loc}}$: $\alpha_\rho^{\chi,c}(A) = \alpha_{\hat{\rho}}^{\chi,c}(A)$ *for any ribbon* $\hat{\rho} \subset \rho$ *with* $\mathrm{supp}(A) \cap \rho \subset \hat{\rho}$.

The first property implies that $\alpha_\rho^{\chi,c}$ is localised in any cone Λ containing ρ, i.e. for all $A \in \mathcal{A}(\Lambda^c)$ it holds $\alpha_\rho^{\chi,c}(A) = A$. For reasons that will become apparent later we define localisation for $*$-morphisms of $\mathcal{A}$ into $\mathcal{B}(\mathcal{H}_0)$.

Definition 4.1.2:
A $*$-morphism α of $\mathcal{A}$ into $\mathcal{B}(\mathcal{H}_0)$ is said to be localised *in the cone $\Lambda \subset \mathbb{Z}^2$ if for all $A \in \mathcal{A}(\Lambda^c)$ we have $\alpha(A) = A$.*

Composing $\alpha_\rho^{\chi,c}$ with the ground state ω_0 gives a state which describes a single excitation at $\partial_0\rho$. Note that the automorphism $\alpha_\rho^{\chi,c}$ still depends on the ribbon ρ. However, this will not be a problem as we are interested in representations describing a single charges and it turns out later that for different ribbons ρ and ρ' the corresponding automorphisms $\alpha_\rho^{\chi,c}$ and $\alpha_{\rho'}^{\chi,c}$ give rise to equivalent representations and hence belong to the same sector. For different labels (χ, c) and (ξ, d) in $\hat{G} \times G$ however, the representations are inequivalent as was shown in [FN15].

Proposition 4.1.3:
Given $(\chi, c), (\xi, d) \in \hat{G} \times G$ *and a semi-infinite ribbon* ρ. *If* $(\chi, c) \neq (\xi, d)$ *the corresponding localised automorphisms* $\alpha_\rho^{\chi,c}$ *and* $\alpha_\rho^{\xi,d}$ *belong to different superselection sectors.*

Consider now the state $\omega_s^{\chi,c} := \omega_0 \circ \alpha_\rho^{\chi,c}$ with $s = \partial_0\rho$, and denote its GNS representation by $\pi_s^{\chi,c}$. Using Corollary 2.8.2 and Lemma 2.7.2 one can see that $\omega_s^{\chi,c}$, and hence $\pi_s^{\chi,c}$, only depends on s and not on the particular ribbon ρ (For the corresponding statement in the toric code, see [Naa11, Lemma 3.6]). If s' is another site and $\omega_{s'}^{\chi,c}$ is the state obtained from a semi-infinite ribbon starting at s', we can choose a ribbon $\hat{\rho}$ with $\partial_0\hat{\rho} = s'$ and such that the composition $\hat{\rho}\rho$ is a ribbon. Since $\omega_{s'}^{\chi,c}$ does not depend on the ribbon we have that $\omega_{s'}^{\chi,c} = \omega_0 \circ \alpha_{\hat{\rho}\rho}^{\chi,c}$. Since $\alpha_{\hat{\rho}\rho}^{\chi,c} = \mathrm{Ad}_{F_{\hat{\rho}}^{\chi,c}} \circ \alpha_\rho^{\chi,c}$ the representations $\pi_s^{\chi,c}$ and $\pi_{s'}^{\chi,c}$ are unitarily equivalent as well. This and the localisation properties of the automorphisms $\alpha_\rho^{\chi,c}$ allow us to prove that representations $\pi_s^{\chi,c}$ can be localised in cones as well.

Proposition 4.1.4:
Let ρ *be a semi-infinite ribbon with* $s := \partial_0\rho$, *and let* $(\chi, c) \in \hat{G} \times G$. *Let* $\pi_s^{\chi,c}$ *be the GNS representation of the state* $\omega_s^{\chi,c} := \omega_0 \circ \alpha_\rho^{\chi,c}$. *Then for any cone* $\Lambda \subset \mathbb{Z}^2$ *we have*

$$\pi_0 \restriction_{\mathcal{A}(\Lambda^c)} \cong \pi_s^{\chi,c} \restriction_{\mathcal{A}(\Lambda^c)},$$

where $\pi \restriction_{\mathcal{A}(\Lambda^c)}$ *denotes the restriction of* π *to the algebra* $\mathcal{A}(\Lambda^c)$.

Proof. The proof is the same as that of the corresponding that statement in [Naa11, Theorem 3.7]. Let ρ and $(\chi, c) \in \hat{G} \times G$ as above. The representation $(\pi_0 \circ \alpha_\rho^{\chi,c}, \mathcal{H}_0, \Omega)$ is a cyclic representation of the state $\omega_s^{\chi,c}$ and hence $\pi_s^{\chi,c} \cong \pi_0 \circ \alpha_\rho^{\chi,c}$.

Now let Λ be a cone containing $\partial_0\rho$ and, by the preceding discussion, we can assume that $\rho \subset \Lambda$. Since $\alpha_\rho^{\chi,c}$ is localised in Λ, we have $\pi_0 \circ \alpha_\rho^{\chi,c}(A) = \pi_0(A)$ for any $A \in \mathcal{A}(\Lambda^c)$, and the claim follows.

Now consider a cone Λ with $s \notin \Lambda$. Then by the preceding discussion, $\pi_s^{\chi,c}$ is unitarily equivalent to $\pi_{s'}^{\chi,c}$. Thus the claim follows here as well. □

This shows that the representations $\pi_s^{\chi,c}$ satisfy a DHR like selection criterion for string-like localised excitations (c.f. [DHR74; BF82]). Note that this is automatically true for the representation $\pi_0 \circ \alpha_\rho^{\chi,c}$. In fact, this is true for any representation π of $\mathcal{A}$ that is equivalent to $\pi_s^{\chi,c}$.

In order to see this consider any cone Λ, let $\mathcal{H}$ is the Hilbert space on which $\mathcal{A}$ is represented by π, and let $\mathcal{H}_s^{\chi,c}$ be the Hilbert space obtained by the GNS representation of $\omega_s^{\chi,c}$. Let $U : \mathcal{H} \to \mathcal{H}_s^{\chi,c}$ be the unitary operator with $U\pi(A) = \pi_s^{\chi,c}(A)U$ for all $A \in \mathcal{A}$, and let $V : \mathcal{H}_s^{\chi,c} \to \mathcal{H}_0$ be the unitary operator with $V\pi_s^{\chi,c}(A) = AV$ for all $A \in \mathcal{A}(\Lambda^c)$. Using Haag duality, it is now easy to check that VU sets up the equivalence $\pi \restriction_{\mathcal{A}(\Lambda^c)} \cong \pi_0 \restriction_{\mathcal{A}(\Lambda^c)}$, and $VU\pi(\mathcal{A})U^*V^* \subset \mathcal{A}(\Lambda)''$.

It is therefore tempting to define a superselection criterion for cone-like localised excitations in the following way.

Definition 4.1.5:
We say that a representation π of $\mathcal{A}$ satisfies the selection criterion *if for all cones $\Lambda \subset \mathbb{Z}^2$ it follows that*

$$\pi \restriction_{\mathcal{A}(\Lambda^c)} \cong \pi_0 \restriction_{\mathcal{A}(\Lambda^c)} . \tag{4.1}$$

Another consequence of the previous discussion is that whenever we derive properties of the automorphisms $\alpha_\rho^{\chi,c}$ they are automatically true for the representations in the same equivalence class[1].

4.1.2 Transportability

Next we turn to transportability of sectors, which implies that for different semi-infinite ribbons ρ and ρ' the automorphisms $\alpha_\rho^{\chi,c}$ and $\alpha_{\rho'}^{\chi,c}$ belong to the same sector.

Take a semi-infinite ribbon ρ, $(\chi, c) \in \hat{G} \times G$ and $\alpha_\rho^{\chi,c}$ as above, which is thus localised in the cone Λ. Transportability now means that for any other cone $\Lambda' \subset \mathbb{Z}^2$ there exists an automorphism β of $\mathcal{A}$ localised in Λ' such that $\alpha_\rho^{\chi,c}$ is unitarily equivalent to β. The unitary implementing this equivalence is in general not included in $\mathcal{A}$.

Proposition 4.1.6:
Let ρ and ρ' be semi-infinite ribbons and $(\chi, c) \in \hat{G} \times G$. Then the corresponding localised automorphisms of $\mathcal{A}$, $\alpha_\rho^{\chi,c}$ and $\alpha_{\rho'}^{\chi,c}$, are unitarily equivalent.

Sketch of proof. For the full proof we refer to reference [FN15]. First we assume that ρ and ρ' start at the same site. Then the two states obtained by composing the ground state with the automorphisms are equal by Corollary 2.8.2 and hence the representations obtained from composing π_0 with $\alpha_\rho^{\chi,c}$ and $\alpha_{\rho'}^{\chi,c}$ are cyclic

[1] We implicitly identified $\alpha_\rho^{\chi,c}$ with $\pi_0 \circ \alpha_\rho^{\chi,c}$ since we already identified $\pi_0(\mathcal{A})$ with $\mathcal{A}$.

representations of the same state, thus unitarily equivalent. Note that this unitary U can be chosen such that $U\Omega = \Omega$ and this fixes U uniquely. If the ribbons ρ and ρ' do not start at the same site one can extend ρ to $\partial_0\rho'$ by a ribbon. Since the ribbon operator corresponding to the ribbon extending ρ is a unitary operator this gives rise to an automorphism which is unitarily equivalent to $\alpha_\rho^{\chi,c}$, and also unitarily equivalent to $\alpha_{\rho'}^{\chi,c}$ by the previous remark. $\square$

If Λ' is a cone containing both ρ and ρ' then the unitary V implementing the equivalence between $\alpha_\rho^{\chi,c}$ and $\alpha_{\rho'}^{\chi,c}$ is contained in $\mathcal{A}(\Lambda')''$. To see this consider $A \in \mathcal{A}((\Lambda')^c)$. Localisation then implies $VA = V\alpha_\rho^{\chi,c}(A) = \alpha_{\rho'}^{\chi,c}V = AV$, hence $V \in \mathcal{A}((\Lambda')^c)' = \mathcal{A}(\Lambda')''$ by Haag duality. The unitary V can be interpreted as transporting the single excitation at $\partial_0\rho$ to an excitation at $\partial_0\rho'$.

More generally, if π is a representation of $\mathcal{A}$ satisfying the selection criterion, then it gives rise to transportable $*$-morphisms. Take cones Λ, $\tilde{\Lambda}$ and unitaries W, $\tilde{W}$ implementing the corresponding equivalences in the selection criterion. Then $\alpha := \mathrm{Ad}_W \circ \pi$ and $\beta := \mathrm{Ad}_{\tilde{W}} \circ \pi$ are $*$-morphisms of $\mathcal{A}$ that are localised in Λ respectively $\tilde{\Lambda}$ and $\tilde{W}W^*$ is a unitary from from α to β. Note however, that $\alpha(\mathcal{A}(\Lambda)) \subset \mathcal{A}(\Lambda)''$ and similarly for β. Thus in general α and β are not $*$-morphisms of $\mathcal{A}$ into itself. We return to this later.

The above observations motivate the following definition.

Definition 4.1.7:
Let Λ be a cone and α a $*$-morphism of $\mathcal{A}$ into $\mathcal{B}(\mathcal{H}_0)$ localised in Λ. We say that α is transportable *if for any cone Λ' there exists a $*$-morphism β of $\mathcal{A}$ into $\mathcal{B}(\mathcal{H}_0)$ localised in Λ' that is unitarily equivalent to α. The unitary operator V is referred to as a* charge transporter*.*

In our case an explicit construction of the charge transporters between $\alpha_\rho^{\chi,c}$ and $\alpha_{\rho'}^{\chi,c}$ can be given [FN15].

Proposition 4.1.8:
Let ρ and ρ' be semi-infinite ribbons and $(\chi, c) \in \hat{G} \times G$, and let V be an intertwiner of $\alpha_\rho^{\chi,c}$ and $\alpha_{\rho'}^{\chi,c}$. Let $(\rho_n)_{n\in\mathbb{N}}$ and $(\rho'_n)_{n\in\mathbb{N}}$ be sequences as in Proposition 4.1.1. For each $n \in \mathbb{N}$ consider a ribbon σ_n with $\partial_0\sigma_n = \partial_1\rho_n$ and $\partial_1\sigma_n = \partial_1\rho'_n$ such that $\rho_n\sigma_n$ is a ribbon and the distance of σ_n to $\partial_0\rho$ and $\partial_0\rho'$ goes to infinity as $n \to \infty$. Then we have that

$$V = \operatorname*{w-lim}_{n\to\infty} F_{\rho'_n}^{\chi,c} F_{\rho_n\sigma_n}^{\overline{\chi},\overline{c}}.$$

Furthermore, $V\alpha_\rho^{\chi,c}(A) = \alpha_{\rho'}^{\chi,c}(A)V$ for all $A \in \mathcal{A}$.

Proof. The proof is given in [FN15], however we recall it here for convenience. With the notation from above we define a sequence of unitaries by $V_n := F^{X,c}_{\rho'_n} F^{\overline{X},\overline{c}}_{\rho_n\sigma_n}$. We need to show that this sequence converges weakly and that it indeed is a charge transporter of $\alpha^{X,c}_{\rho}$ and $\alpha^{X,c}_{\rho'}$. Consider first that $\partial_0\rho = \partial_0\rho'$. By applying Lemma 2.7.2, Corollary 2.8.2 and Corollary 2.8.3 it follows that for each $n \in \mathbb{N}$ we have $V_n\Omega = \Omega$.

Furthermore, for each $A \in \mathcal{A}_{\mathrm{loc}}$ there is an $n \in \mathbb{N}$ such that $V_n\alpha^{X,c}_{\rho}(A) = \alpha^{X,c}_{\rho'}(A)V_n$. This can be seen by choosing n big enough so that $F^{\overline{X},\overline{c}}_{\sigma_n}A = AF^{\overline{X},\overline{c}}_{\sigma_n}$ and, by Proposition 4.1.1, both, $\alpha^{X,c}_{\rho}(A) = \alpha^{X,c}_{\rho_n}(A)$ and $\alpha^{X,c}_{\rho'}(A) = \alpha^{X,c}_{\rho'_n}(A)$. Hence $V_n\alpha^{X,c}_{\rho}(A) = V_n\alpha^{X,c}_{\rho_n}(A) = \alpha^{X,c}_{\rho'_n}(A)V_n = \alpha^{X,c}_{\rho'}(A)V_n$. Considering $V\Omega = \Omega$ it follows that for any $A, B \in \mathcal{A}_{\mathrm{loc}}$ we have

$$\begin{aligned}(\alpha^{X,c}_{\rho}(A)\Omega, V\alpha^{X,c}_{\rho}(B)\Omega) &= (\alpha^{X,c}_{\rho}(A)\Omega, \alpha^{X,c}_{\rho'}(B)V_n\Omega)\\ &= (\alpha^{X,c}_{\rho}(A)\Omega, V_n\alpha^{X,c}_{\rho}(B)\Omega).\end{aligned}$$

Consequently this is true for any $A, B \in \mathcal{A}$ and since the sequence $(V_n)_{n\in\mathbb{N}}$ is uniformly bounded it converges weakly to V. Furthermore, the limit does not depend on the sequences $(\rho_n)_{n\in\mathbb{N}}$, $(\rho'_n)_{n\in\mathbb{N}}$ and $(\sigma_n)_{n\in\mathbb{N}}$ as long as the properties in Proposition 4.1.1 are fulfilled.

Now assume that ρ and ρ' start at different sites. We can choose a ribbon $\hat{\rho}$ starting at $\partial_0\rho'$ and ending at $\partial_0\rho$ such that $\hat{\rho}\rho$ is a ribbon. Furhtermore, let σ_n as in the premises of the theorem. The ribbon operator $F^{X,c}_{\hat{\rho}}$ is unitary and we have that $\alpha^{X,c}_{\hat{\rho}\rho} = \mathrm{Ad}_{F^{X,c}_{\hat{\rho}}} \circ \alpha^{X,c}_{\rho}$.

Consider now the intertwiner $\hat{V}$ of $\alpha^{X,c}_{\rho'}$ and $\alpha^{X,c}_{\hat{\rho}\rho}$. Let $((\hat{\rho}\rho)_n)_{n\in\mathbb{N}}$ be the sequence given by composing for each n the ribbon $\hat{\rho}$ with ρ_n. Then $\hat{V} = \text{w-lim}_{n\to\infty} F^{X,c}_{\rho'_n} F^{\overline{X},\overline{c}}_{(\hat{\rho}\rho)_n\sigma_n}$. Next we set $V_n := F^{X,c}_{\rho'_n} F^{\overline{X},\overline{c}}_{(\hat{\rho}\rho)_n\sigma_n} F^{X,c}_{\hat{\rho}} = F^{X,c}_{\rho'_n} F^{\overline{X},\overline{c}}_{\rho_n\sigma_n}$ and with the same arguments as above it can be checked that $V_n\alpha^{X,c}_{\rho}(A) = \alpha_{\rho'}(A)V_n$ for $A \in \mathcal{A}_{\mathrm{loc}}$ and n big enough. Hence $\text{w-lim}_{n\to\infty} V_n = V$ and $V = \hat{V}F^{X,c}_{\hat{\rho}}$ is the intertwiner for $\alpha^{X,c}_{\rho}$ and $\alpha^{X,c}_{\rho'}$. □

We already noted that the intertwiner V for $\alpha^{X,c}_{\rho}$ and $\alpha^{X,c}_{\rho'}$ is contained in the von-Neumann algebra $\mathcal{A}(\Lambda)''$, where Λ is a cone containing both ρ and ρ'. This is problematic in so far as that in order to define fusion and braiding one needs that the intertwiners are contained in the same algebra the $\alpha^{X,c}_{\rho}$ act on. Moreover, if π is a representation satisfying the selection criterion, Λ a cone and U the unitary operator setting up the equivalence $\pi \restriction_{\mathcal{A}(\Lambda^c)} \cong \pi_0 \restriction_{\mathcal{A}(\Lambda^c)}$, it follows for any $A \in \mathcal{A}$ that $U\pi(A)U^* \in \mathcal{A}(\Lambda)''$. To see this let $B \in \mathcal{A}(\Lambda^c)$. Then

$\pi_0(B)U\pi(A)U^* = U\pi(BA)U^* = U\pi(B)U^*\pi_0(A)$ and by Haag duality the claim follows. Thus, as mentioned earlier, the localised $*$-morphisms $\mathrm{Ad}_U \circ \pi$ are in general not automorphisms of $\mathcal{A}$.

These problems can be cured by considering an auxiliary algebra $\mathcal{A}^a$ which contains $\mathcal{A}''$ and where these $*$-morphisms actually become endomorphisms (c.f. [BF82; Naa11; FN15]), and where the charge transporters are contained in. This was not necessary up to now, since the localised morphisms we constructed were actually automorphisms of $\mathcal{A}$. It can be shown, however, that they can be extended to weakly continuous endomorphisms of $\mathcal{A}^a$ in the same way as in [BF82]. We omit the definition of the auxiliary algebra and the discussion of the extension of the automorphisms to it. The reason is that this mainly of technical issue and we are primarily interested in the overall structure. Let us note, however, that for the definition of $\mathcal{A}^a$ it is necessary to fix a cone Λ^a. The important property of $\mathcal{A}^a$ is that for each cone Λ with $\Lambda \subset (\Lambda^a + x)^c$ for some $x \in \mathbb{Z}^2$, we have $\mathcal{A}(\Lambda)'' \subset \mathcal{A}^a$ [FN15]. In the following we largely suppress the auxiliary in our notion and only refer to it if necessary. The discussion in this subsection can now be summarised as follows.

Theorem 4.1.9:
Let G be a finite abelian group and let π_0 be the GNS representation of the unique translational invariant ground state of the quantum double model for G. Then for each pair (χ, c), where χ is a character of G and $c \in G$, there is an equivalence class of representations satisfying the selection criterion (4.1). The representation $\pi_0 \circ \alpha_\rho^{\chi,c}$, where $\alpha_\rho^{\chi,c}$ is localised in some cone Λ and constructed as above, is a representative of such an equivalence class. The equivalence classes corresponding to distinct pairs (χ, c) are disjoint.

4.1.3 Braiding and Fusion

Fusion describes the particle content of the outcome of two particles merging. More abstractly, it describes in which irreducible sectors the tensor product of two sectors decomposes. As this is a property of the equivalence classes of representations we define the tensor product for the automorphisms $\alpha_\rho^{\chi,c}$ from the previous section.

Braiding, on the other hand, describes the outcome of moving one particle around another. In terms of tensor products it is given by the map that interchanges the factors in the tensor product of two sectors. In the following we define both for the equivalence classes of which the representations $\pi_0 \circ \alpha_\rho^{\chi,c}$ are representatives.

In order to define the tensor product between two sectors we need a definition of both the tensor product of automorphisms and the intertwiners between different tensor products. As mentioned before, in order to define the intertwiners it is necessary to transition to a slightly larger algebra $\mathcal{A}^a$, for the intertwiners of the automorphisms are not contained in $\mathcal{A}$. However, we are not going too much into the details here since it does not add much insight (the construction is essentially the same as in [Naa11] with the obvious modifications).

Let (χ, c) and (ξ, d) be elements of $\hat{G} \times G$, and let ρ, ρ' be semi-infinite ribbons. The tensor product is then defined by [FN15]

$$\alpha_\rho^{\chi,c} \otimes \alpha_{\rho'}^{\xi,d} := \alpha_\rho^{\chi,c} \circ \alpha_{\rho'}^{\xi,d}.$$

Note that if $\rho' = \rho$ we have that $\alpha_\rho^{\chi,c} \otimes \alpha_\rho^{\xi,d} = \alpha_\rho^{\chi\xi,cd}$. This can be seen by evaluating the expression on the left hand side on local observables and then using Proposition 4.1.1 and Lemma 2.7.2. For localised $*$-morphisms α and β of $\mathcal{A}$ the tensor product is defined in a similar way by

$$\alpha \otimes \beta := \alpha^a \circ \beta,$$

where α^a is the extension of α to the auxiliary algebra $\mathcal{A}^a$. Now let α, α', β, and β' be localised $*$-morphisms with V and W intertwiners between α and α', and β and β'. Then it can be checked that

$$V \otimes W := V\alpha^a(W)$$

is an intertwiner between $\alpha \otimes \beta$ and $\alpha' \otimes \beta'$. Given $\alpha_\rho^{\chi,c}$ and $\alpha_{\rho'}^{\xi,d}$ as above this implies that

$$\alpha_\rho^{\chi,c} \otimes \alpha_{\rho'}^{\xi,d} \cong \alpha_\rho^{\chi\xi,cd}$$

by transportability. I.e. fusing two sectors with labels (χ, c) and (ξ, d) results in the sector with label $(\chi\xi, cd)$.

The braiding is defined by an intertwiner between $\alpha \otimes \beta$ and $\beta \otimes \alpha$. For this one needs to introduce an ordering relation between cones that allows to speak about the relative position between excitations [FN15]. More precisely, we want to be able to say that a cone is to the left to another one. This can be achieved by fixing a reference cone, say the cone Λ^a used for the auxiliary algebra $\mathcal{A}^a$. One defines a total ordering on the set of cones as follows. Take two cones Λ_1, Λ_2 and choose $x \in \mathbb{Z}^2$ such that $\Lambda_1, \Lambda_2 \subset (\Lambda^a + x)^c$. Then we say $\Lambda_1 < \Lambda_2$, if Λ_1 can be rotated counterclockwise around its apex until it has non-empty intersection with $\Lambda^a + x$ such that for any intermediate angle it is disjoint from Λ_2 (see also [Naa11]).

Now let ρ and ρ' be semi-infinite ribbons located in cones Λ_1 and Λ_2, respectively, and assume that there is a cone Λ with $\Lambda \supset \Lambda_1 \cup \Lambda_2$. Let α and β be transportable $*$-morphisms localised in Λ_1 and Λ_2, respectively. Then one chooses a cone $\hat{\Lambda}_2 < \Lambda_1$ and an intertwiner V such that $\mathrm{Ad}_V \circ \beta$ is localised in $\hat{\Lambda}_2$ and $V \in \mathcal{A}^a$. Setting

$$\epsilon_{\alpha,\beta} := (V \otimes \mathbb{1}_\alpha)^*(\mathbb{1}_\alpha \otimes V) = V^* \alpha^a(V)$$

defines an intertwiner between $\alpha \otimes \beta$ and $\beta \otimes \alpha$. It then follows that $\epsilon_{\alpha,\beta}$ does not depend on the chosen cones, but only on the relation $\hat{\Lambda}_2 < \Lambda_1$. Furthermore it satisfies the braid equations and it is natural (see [Hal07] or [Naa11, Lemma 4.8]) and hence it is a braiding in the categorical sense. For the automorphisms $\alpha_\rho^{\chi,c} \otimes \alpha_{\rho'}^{\xi,d}$ the braiding can then be calculated to be [FN15]

$$\epsilon_{\alpha_\rho^{\chi,c},\alpha_{\rho'}^{\xi,d}} = \overline{\chi}(d)\overline{\xi}(c)\mathbb{1}.$$

Similarly to the case of the toric code [Naa11] one can now show that the sectors of the automorphisms $\alpha_\rho^{\chi,c}$ form a braided monoidal category and are equivalent as such to the braided monoidal category of finite dimensional representations of $\mathcal{D}(G)$ (see also [Hal07]). We will later see in Theorem 4.2.11, that these are in fact all superselection sectors of the quantum double model for the group G.

4.2 Approximate Split Property and Jones-Kosaki-Longo index

When constructing superselection sectors of a theory an immediate question that arises is whether all sectors have been constructed. Here this can be answered by calculating an index associated to the inclusion of certain von-Neumann algebras. This is very similar to the situation in algebraic quantum field theory where such an index is related to the statistical dimension of the superselection sectors [Lon91] (see [Hal07] for an overview). This index we are considering here is the one introduced by Kosaki [Kos86] for arbitrary factors which coincides with the one in [DHR74] and which is related to Jone's index for type-II_1 [Jon83; JS97].

Analogously to the toric code [Naa13a] we show that for abelian finite groups G the cone index is given by $|G|^2$ which determines the number of super-selection sectors and implies that the category of super-selection sectors is equivalent to the modular tensor category of finite dimensional representations of $\mathcal{D}(G)$. The statements are proved in the same manner as in [Naa13a] while using the techniques that were developed in the proof of Haag duality for the cone algebras in the vacuum representation.

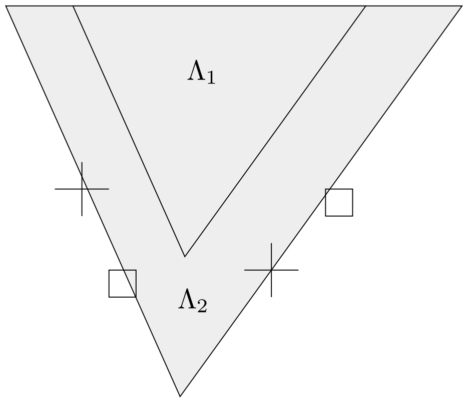

Figure 4.1: Two cones with $\Lambda_1 \ll \Lambda_2$. Note that stars and plaquettes in Λ_2^c have empty intersection with Λ_1. Similarly, stars and plaquettes in Λ_1 have empty intersection with Λ_2^c.

The basic idea is to show that for pairs of disjoint cones Λ_1, Λ_2 which are far enough separated from each other the von-Neumann algebras $\widehat{\mathcal{R}}(\Lambda)$ and $\mathcal{R}(\Lambda) \vee \{V_g \mid g \in H\}$ coincide, where $H = G \times \widehat{G}$ and G is an abelian group. Very similarly to the proof of Haag duality we break down the problem to one on a sub-Hilbert space of $\mathcal{H}$.

We first start with introducing the necessary notions. Let $\Lambda_1, \Lambda_2 \subset \mathbb{Z}_2$ be two cones. We say $\Lambda_1 \ll \Lambda_2$ if there is no star and no plaquette in $\Lambda_1 \cup \Lambda_2^c$ that has non-empty intersection with both cones. We denote by $\mathcal{C}^2$ be the collection of sets $\Gamma = \Lambda_1 \cup \Lambda_2 \subset \mathbb{Z}_2$ where Λ_1 and Λ_2 are disjoint cones such that there is a cone $\Lambda_1 \ll \Lambda$ and $\Lambda_2 \subset \Lambda^c$. An illustration of this can be found in Figure 4.1.

Definition 4.2.1:
Fix $\Gamma = \Lambda_1 \cup \Lambda_2 \in \mathcal{C}^2$ and let ρ_0 be a fixed ribbon with $\partial_0 \in \Lambda_1$ and $\partial_1 \in \Lambda_2$. Denote the set of finite ribbons in Γ by $\mathfrak{p}(\Gamma)$. We set

$$\mathcal{F}(\Gamma) := \{F_\rho \in \mathcal{A}(\Gamma) \mid \rho \in \mathfrak{p}(\Gamma)\},$$
$$\mathcal{F}(\bar{\Gamma}) := \{F_\rho \in \mathcal{A}(\Gamma) \mid \rho \in \mathfrak{p}(\Gamma) \cup \{\rho_0\}\}.$$

Moreover we define

$$\mathfrak{F}(\Gamma) := \{F_1 \cdot F_n \mid F_1, \dots, F_n \in \mathcal{F}(\Gamma)\},$$
$$\mathfrak{F}(\bar{\Gamma}) := \{F_1 \cdot F_n \mid F_1, \dots, F_n \in \mathcal{F}(\bar{\Gamma})\}.$$

Set $\mathfrak{A} := \mathcal{R}(\Gamma) \vee \{V_g \mid g \in H\}$, $\mathfrak{B} := \mathcal{A}(\Gamma^c)$ *and* $\mathcal{H}_{\bar{\Gamma}} := \overline{\mathfrak{A}\Omega}^{\|\cdot\|}$.

First we show that the Hilbert space $\mathcal{H}_{\bar{\Gamma}}$ is densely generated from the ground state by the ribbon operators in $\mathfrak{F}(\bar{\Gamma})$.

Lemma 4.2.2:
Let $\Gamma \in \mathcal{C}^2$ and consider $\mathcal{H}_{\bar{\Gamma}}$ and $\mathfrak{F}(\bar{\Gamma})$ as defined above. We then have

$$\mathcal{H}_{\bar{\Gamma}} = \overline{\operatorname{span}\left(\mathfrak{F}(\bar{\Gamma})\Omega\right)}^{\|\cdot\|}.$$

Proof. The proof for this statement completely goes along the lines of that of [Naa13a, Lemma 4.4] with suitable adjustments. We repeat the arguments here while making the suitable modifications. First recall that the unitary operators V_g with $g \in H$ are constructed by considering a pair of semi-infinite ribbons $\rho_1 \subset \Lambda_1$ and $\rho_2 \subset \Lambda_2$.

As in the proof of [Naa13a, Lemma 4.4] we consider ribbons ξ_1 and ξ_2 with $\partial_0\xi_1 = \partial_0\rho_1$, $\partial_1\xi_1 = \partial_0\rho_0$, $\partial_0\xi_2 = \partial_1\rho_0$ and $\partial_1\xi_2 = \partial_0\rho_2$. Hence the ribbon $\xi_1\rho_0\xi_2$ connects the ribbons ρ_1 and ρ_2 through Γ^c. Now fiy any $g \in H$. Furthermore, by the construction of V_g we have $V_g\Omega = F^{\bar{g}}_{\xi_1\rho_0\xi_2}\Omega$. Since V_g is the weak limit of ribbon operators F^g_n of finite lengths we can calculate the commutation relations between V_g and any element $F \in \mathcal{F}(\Gamma)$, i.e. $F^g_n F = \alpha(g) F F_n$ where $\alpha(g)$ is some phase depending on g. If n is large enough $\alpha(g)$ does not depend on n any more and therefore $V_g F = \alpha(g) F V_g$ by separate weak continuity. Hence $V_g F\Omega = \alpha(g) F V_g \Omega = \alpha(g) F F^{\bar{g}}_{\xi_1\rho_0\xi_2}\Omega \in \mathfrak{F}(\bar{\Gamma})\Omega$. By denseness of the linear span of $\mathcal{F}(\Gamma)$ in $\mathcal{A}(\Gamma)$ the claim follows. □

Lemma 4.2.3:
With the notation from above we have for all $\Gamma \in \mathcal{C}^2$ that

$$\mathfrak{A}\mathcal{H}_{\bar{\Gamma}} \subset \mathcal{H}_{\bar{\Gamma}}\,,$$
$$\mathfrak{B}'\mathcal{H}_{\bar{\Gamma}} \subset \mathcal{H}_{\bar{\Gamma}}\,,$$

and elements from both algebras are uniquely determined by their restriction to $\mathcal{H}_{\bar{\Gamma}}$.

Proof. Again, the idea of the proof is essentially the same as that of [Naa13a, Lemma 4.5]. However, we need to employ the techniques developed in the proofs of Lemmas 3.3.1, 3.3.2 and 3.3.3.

First, we treat the claims for $\mathfrak{A}$. The statement $\mathfrak{A}\mathcal{H}_{\bar{\Gamma}} \subset \mathcal{H}_{\bar{\Gamma}}$ is obvious and follows from the definition of $\mathcal{H}_{\bar{\Gamma}}$. For the second claim for $\mathfrak{A}$, consider $A, A' \in \mathfrak{A}$ such that $A\eta = A'\eta$ for any $\eta \in \mathcal{H}_{\bar{\Gamma}}$. We have to show that this implies $A\xi = A'\xi$ for all $\xi \in \mathcal{H}$, since this then implies that $A = A'$. However, using the same argument as in the proof of Lemma 3.2.1, we have that $\mathfrak{F}(\Gamma^c)\mathfrak{F}(\Gamma)\Omega$ is norm dense in $\mathcal{H}$. Thus, it suffices to show that $A\hat{F}F\Omega = A'\hat{F}F\Omega$ for all $\hat{F} \in \mathfrak{F}(\Gamma^c)$ and $F \in \mathfrak{F}(\Gamma)$. So, take $\hat{F} \in \mathfrak{F}(\Gamma^c)$ and $F \in \mathfrak{F}(\Gamma)$. Since $\mathfrak{A}$ is generated by $\mathcal{A}(\Gamma)''$ and the unitary operators V_g with $g \in G$, the operator $\hat{F}$ commutes with A and A'. Thus, $A\hat{F}F\Omega = \hat{F}AF\Omega = \hat{F}A'F\Omega = A'\hat{F}F\Omega$. Using a density argument, this implies $A\xi = A'\xi$ for all $\xi \in \mathcal{H}$ and thus $A = A'$.

To show that $\mathfrak{B}'\mathcal{H}_{\bar{\Gamma}} \subset \mathcal{H}_{\bar{\Gamma}}$ we first prove analogous statements to Lemma 3.3.1 and Lemma 3.3.1. More precisely, given $\hat{F} \in \mathfrak{F}(\Gamma^c)$ we show that

$$\begin{aligned} &\Big(\exists s \in \operatorname{int}(\Gamma^c) : [A_s, \hat{F}] \neq 0 \vee [B_s, \hat{F}] \neq 0\Big) \\ &\implies \Big((\forall F, C \in \mathfrak{F}(\bar{\Gamma}))(\forall X \in \mathfrak{B}') : (\hat{F}F\Omega, XC\Omega) = 0\Big), \end{aligned} \tag{4.2}$$

and

$$\Big(\forall s \in \operatorname{int}(\Gamma^c) : [A_s, \hat{F}] = 0 \wedge [B_s, \hat{F}] = 0\Big) \implies \hat{F}\Omega \in \mathcal{H}_{\bar{\Gamma}}. \tag{4.3}$$

Having established that, it then follows, as in Lemma 3.3.3, that $(\hat{F}F\Omega, XC\Omega) \neq 0 \implies \hat{F}F\Omega \in \mathcal{H}_{\bar{\Gamma}}$ for all $\hat{F} \in \mathfrak{F}(\Gamma^c)$ and all $F, C \in \mathfrak{F}(\bar{\Gamma})$. By Lemma 4.2.2 the linear span of $\mathfrak{F}(\bar{\Gamma})\Omega$ is dense in $\mathcal{H}_{\bar{\Gamma}}$. Furthermore, vectors of the form $\hat{F}F\Omega$ with $\hat{F} \in \mathfrak{F}(\Gamma^c)$ and $F \in \mathfrak{F}(\bar{\Gamma})$ span a dense subspace of $\mathcal{H}$. Therefore we find

$$(\forall \psi \in \mathcal{H}_{\bar{\Gamma}}^{\perp})(\forall \eta \in \mathcal{H}_{\bar{\Gamma}})(\forall X \in \mathcal{B}) : (\psi, X\eta) = 0.$$

Now for the proof of equations (4.2) and (4.3). Since this will essentially be a modification of the ones of Lemma 3.3.1 and Lemma 3.3.2 we will recall their structure while focusing on the modifications and refer for the details to the aforementioned proofs.

We start with equation (4.2). Recall from the proof of Lemma 4.2.2 that for $F \in \mathfrak{F}(\Gamma)$ and $g \in H$ it follows that $V_g F\Omega = \alpha(g)FF^g_{\rho_0}$ where $\alpha(g)$ is a phase depending on g and ρ_0 is the ribbon fixed at the beginning of this section which connects Λ_1 with Λ_2. Also note, that for $s \in \operatorname{int}(\Gamma^c)$ we have that $[A_s, V_g] = 0 = [B_s, V_g]$ for all $g \in H$. Together with locality and Lemma 4.2.2 this implies that for any $\eta \in \mathcal{H}_{\bar{\Gamma}}$ we have that $A_s\eta = \eta = B_s\eta$ for such sites. Consequently, we can just restate the proof of Lemma 3.3.1 here. I.e. let $\hat{F} = \hat{F}_1 \cdots \hat{F}_n \in \mathfrak{F}(\Gamma^c)$, $C, F \in \mathfrak{F}(\bar{\Gamma})$ and $X \in \mathfrak{B}'$. Consider the ribbon operators $\hat{F}_1, \ldots \hat{F}_n$ being labelled by irreducible representations of $\mathcal{D}(G)$, i.e. by tuples (χ_i, k_i), $i = 1, \ldots, n$ where χ_i are characters of G and $k_i \in G$. Assume first that there is a site $s \in \Gamma^c$ such that $[A_s, \hat{F}] \neq 0$. Then

$$(\hat{F}F\Omega, XC\Omega) = (\hat{F}F\Omega, A_s XC\Omega) = \frac{1}{|G|} \sum_{k \in G} \hat{\chi}_1(k) \cdots \hat{\chi}_n(k) (\hat{F}FA_s^k \Omega, XC\Omega)$$
$$= \frac{1}{|G|} \sum_{k \in G} \hat{\chi}_1(k) \cdots \hat{\chi}_n(k) (\hat{F}F\Omega, XC\Omega) = 0,$$

where $\hat{\chi}_i(k)$ is either zero or given by $\chi_i(k)$. Since A_s does not commute with $\hat{F}$ at least one of the factors $\hat{\chi}_i(k)$ will be non-zero, and since $\hat{G}$ is a group, the sum vanishes.

Similarly, if for some $s \in \mathrm{int}(\Gamma^c)$ we have $[B_s, \hat{F}] \neq 0$ then there is a $k \in G$ with $k \neq e$ such that

$$(\hat{F}F\Omega, XC\Omega) = (\hat{F}FB_s^k \Omega, XC\Omega) = 0.$$

Now let us turn to equation (4.3). Again, let $\hat{F} = \hat{F}_1 \cdots \hat{F}_n \in \mathfrak{F}(\Gamma^c)$ and consider the ribbon operators $\hat{F}_1, \dots, \hat{F}_n$ being labelled by irreducible representations of $\mathcal{D}(G)$. Without losing generality we can make some simplifying assumptions. We assume that none of these operators is trivial or corresponds to a closed ribbon. Furthermore we assume that no ribbon in the product can be joined together to another ribbon (see Lemma 2.6.9) and that every ribbon occurs just once in the product (see Lemma 2.6.11). As in Lemma 3.3.2 we prove the statement by induction over the number of ribbon operators in $\hat{F}$.

Assume now that for any $s \in \mathrm{int}(\Gamma^c)$ it follows that $[A_s, \hat{F}] = 0 = [B_s, \hat{F}]$. Take $n = 1$, i.e. $\hat{F} = \hat{F}_1$. Then $\hat{F}$ is only non-trivial if its corresponds to a ribbon ξ whose endpoints are at the boundary $\partial\Gamma$ of Γ. By Corollaries 2.8.2 and 2.8.3 can deform and invert ribbons without changing the action of the corresponding ribbon operator on the ground state. Therefore, if ξ connects Λ_1 with Λ_2, we can find (possibly trivial) ribbons $\rho_1 \subset \Lambda_1$ and $\rho_2 \subset \Lambda_2$ such that $\hat{F}\Omega = \hat{F}_{\rho_1} \hat{F}_{\rho_0} \hat{F}_{\rho_2} \Omega$. Otherwise, we get $\hat{F}\Omega = \hat{F}_{\rho_i}\Omega, i = 1, 2$ for ribbons $\rho_1 \subset \Lambda_1$ and $\rho_2 \subset \Lambda_2$ (see also Lemma 3.3.2). In both cases, we find $\hat{F}\Omega \in \mathcal{H}_{\overline{\Gamma}}$.

Now assume that for $n = k-1$ the statement is true. Let $\hat{F} = \hat{F}_1 \cdots \hat{F}_k \in \mathcal{F}(\Gamma^c)$ as above and assume that $[A_s, \hat{F}] = 0 = [B_s, \hat{F}]$ for any $s \in \mathrm{int}(\Gamma^c)$. There are two possibilities to consider now: Case (I), there is at least one ribbon occurring in the product $\hat{F}$ that has both ends at the boundary $\partial\Gamma^c$. Case (II), the product contains only ribbons that have at least one end in the bulk $\mathrm{int}(\Gamma^c)$.

By using the commutation relations between ribbon operators, case (I) can be handled as in the case $n = 1$ to replace one ribbon operator with both ends on $\partial\Gamma^c$ with a ribbon operator from $\mathcal{F}(\bar{\Gamma})$. By locality this leaves a product of $k-1$ ribbon operators from $\mathcal{F}(\Gamma^c)$ in front of Ω.

In case (II) we can assume that for each site $s \in \mathrm{int}(\Gamma^c)$ there are either zero or at least two ribbons in $\hat{F}$ that end at s. If there was a site in $\mathrm{int}(\Gamma^c)$ with just one ribbon end, the ribbon operator corresponding to this ribbon must necessarily be trivial by the assumed commutation relations with the star and plaquette operators at this side. Now safely assuming that this is not the case, we can always clean up a site s at which ribbons end according to Lemma 2.7.3 leaving us with one ribbon ending at s and a trivial ribbon operator associated to this ribbon. Hence, when acting on Ω there are only $k-1$ ribbon operators in $\hat{F}$ left. □

This lemma also implies that the projection $P_{\hat{\Gamma}} : \mathcal{H} \to \mathcal{H}_{\hat{\Gamma}}$ is contained in $\mathfrak{B}''$. Denote by $\mathfrak{A}_{\bar{\Gamma}} := P_{\bar{\Gamma}}\mathfrak{A}P_{\bar{\Gamma}}$ and $\mathfrak{B}_{\bar{\Gamma}} := P_{\bar{\Gamma}}\mathfrak{B}''P_{\bar{\Gamma}}$ the restrictions of $\mathfrak{A}$ and $\mathfrak{B}''$ to $\mathcal{H}_{\bar{\Gamma}}$. Denote the self-adjoint cones of the restricted algebras by $\mathfrak{A}_s$ and $\mathfrak{B}_s$.

Lemma 4.2.4:
The space $(\mathfrak{A}_s + i\mathfrak{B}_s)\,\Omega$ *is dense in* $\mathcal{H}_{\hat{\Gamma}}$.

Proof. We follow the ideas of the proof of [Naa13a, Lemma 4.6] and employ the techniques of the proof of Lemma 3.4.2. More precisely, we show in the following that for any $\hat{F} \in \mathfrak{F}(\hat{\Gamma})$ there are operators $A, A' \in \mathfrak{A}_s$ and $B, B' \in \mathfrak{B}_s$ such that $F\Omega = (A + iB)\Omega$ and $iF\Omega = (A' + iB')\Omega$. Since by Lemma 4.2.2 the linear span of $\mathfrak{F}(\bar{\Gamma})\Omega$ is dense in $\mathcal{H}_{\hat{\Gamma}}$ this then proves the statement.

The proof will again be an induction over the number of ribbon operators in $\hat{F}$. Analogously to the previous proof, we may assume that there are no closed ribbons appearing in $\hat{F}$ and no trivial ribbon operators. Furthermore we can assume that no ribbon in $\hat{F}$ appears more than once, and that there are no pairs of ribbons that join to a ribbon. If this were the case we could clean them up using the techniques described in subsections 2.6 and 2.7. Note that whenever there is a ribbon operator in $\hat{F}$ that corresponds to ρ_0 we can replace the operator by V_g for an appropriate element $g \in G$. Given a site $s \in \mathbb{Z}^2$ we use the notation D_s for the operator $D_s^{\mathrm{id},e}$, where the latter is defined as in equation (2.12).

Now let $n = 1$. There are several possibilities. First, $\hat{F} = F_{\rho_0}$. Then $\hat{F}_R := V_g D_s + D_s V_g^*$ and $\hat{F}_I := i\left(V_g D_s - D_s V_g^*\right)$ with $s = \partial_0\rho_0$ or $s = \partial_1\rho_0$ fulfill $\hat{F}_R, \hat{F}_I \in \mathfrak{A}_s$ and $\hat{F}_R\Omega = \hat{F}\Omega = -i\hat{F}_I\Omega$. Secondly, $\hat{F}$ contains a ribbon operator that does not commute with $D_s \in \mathcal{F}(\Gamma)$. Then we can define $F_R, F_I \in \mathfrak{A}_s$ the same way as in the first case. Thirdly, $\hat{F}$ does commute with any $D_s \in \mathcal{F}(\Gamma)$. This is now completely analogous to the situation in the proof of Lemma 3.4.2 and there exists $A, A' \in \mathfrak{A}_s$ and $B, B' \in \mathfrak{B}_s$ satisfying $\hat{F}\Omega = (A + iB)\Omega = -i(A' + iB')\Omega$. Essentially this is proved by observing that the ribbon in $\hat{F}$ must create an excitation near the boundary of Γ. This excitation can either be detected with a star operator, in which case we can proceed as in the preceding case. Or, when the ribbon does not commute with any star operator, the ribbon can be replaced with a ribbon in Γ^c having the same starting and ending sites. The details for this can be found in the proof of Lemma 3.4.2.

We can now turn to assuming that the statement was true for $n = k - 1$ ribbon operators and let $\hat{F} \in \mathfrak{F}(\bar{\Gamma})$ with $\hat{F} = \hat{F}_1 \cdots \hat{F}_k$. First note, that by assumption $\hat{F}$ does not contain closed ribbons and and no pairs of ribbons that join to a ribbon. Furthermore, there are no trivial ribbon operators and no multiple ribbon operators associated to the same ribbon in $\hat{F}$.. Again, as in the proof of Lemma 3.4.2 there are three cases to consider. Case (I) is that there is a ribbon ρ occurring in $\hat{F}$ for which $D_{\partial_i \rho} \in \mathcal{F}(\Gamma)$ for at least one $i = 0$ or $i = 1$. Case (II) is that all ribbons in $\hat{F}$ end in $\partial\Gamma$. The third case, (III), is that there are ribbons ρ occurring in $\hat{F}$ with at least one ending site $s \notin \partial\Lambda$ but also for all ending sites s of all ribbons in $\hat{F}$ it holds it holds $D_s \notin \Lambda$.

Case (I) is fully analogous to the situation in Lemma 3.4.2. Either ρ is the only ribbon which ends at site s, in which case we can proceed by setting $F_R := D_s\hat{F}^* + \hat{F}D_s$ and $F_I := i\left(\hat{F}D_s - \hat{F}^*D_s\right)$, or there are multiple ribbons in $\hat{F}$ ending at s. In the latter case we can use Lemma 2.7.3 to clean up that site such that there is only one ribbon left connected to this site. In case there is a ribbon ρ with $\rho_0 \subset \rho$ we split it into three parts $\rho = \rho_1\rho_0\rho_2$ and factor the ribbon operator F_ρ into $F_{\rho_1}F_{\rho_0}F_{\rho_2}$ according to Lemma 2.6.9. Using the commutation relations of ribbons, we can bring F_{ρ_0} in $\hat{F}\Omega$ in front of Ω and replace it there with V_g for some suitable $g \in H$.

In case (II), where all ribbons in $\hat{F}$ end at $\partial\Gamma$ we can essentially replace each ribbon with one in Γ^c that starts and ends at the same sites as its counterpart in Γ. If necessary some ribbons need to be extended by triangle operators, see Lemma 2.7.4. Let $\rho_1, \ldots, \rho_k \subset \Gamma \cup \{\rho_0\}$ be the ribbons involved in $\hat{F}$ and $\tilde{\rho}_1, \ldots, \tilde{\rho}_k$ their counterparts in Γ^c. Then $\hat{F}_R$ and $\hat{F}_I$ can be defined as in equations (3.6) and (3.7), respectively.

In case (III), each ending site s of any ribbon in $\hat{F}$ is either contained in $\partial\Lambda$ or very close to Λ. By very close, we mean that the star at s is contained in Γ whereas the plaquette is not. For details, see Definition 3.1.4 and the discussion afterwards. In case there is a ribbon in $\hat{F}$ with one ending site $s \notin \partial\Lambda$ and with $[A_s, F] \neq 0$, we can set F_R and F_I as in equation (3.8). Note that in this case $A_s \in \mathfrak{F}(\Gamma)$. In case that all ribbons with at least one ending site $s \notin \partial\Lambda$ satisfy $[A_s, F] = 0$ for all such sites s, we can extend the ribbons with direct triangles by Lemma 2.7.4 in such a way that their action on the ground state Ω does not change and the ending sites of the extended ribbons are contained in $\partial\Lambda$. Replacing all ribbon operators in $\hat{F}$ by such ribbons we obtain an operator $\hat{F}' \in \mathfrak{F}(\overline{\Gamma})$ with $\hat{F}'\Omega = \hat{F}\Omega$, and all ribbons in $\hat{F}'$ end in $\partial\Lambda$. Hence we find ourselves in the second case. □

Very similarly to the proof of Haag duality for cone algebras in Theorem 3.4.3 and in the proof of [Naa13a, Lemma 4.1] we can now apply a result by Rieffel and van Daele [RD75] (or [Tak79, Lemma IV.5.7]), which states that Lemma 4.2.4 is equivalent to $\mathfrak{A}_{\overline{\Gamma}} = \mathfrak{B}'_{\overline{\Gamma}}$ as von-Neumann algebras on $\mathcal{H}_{\overline{\Gamma}}$.

THEOREM 4.2.5:
Let $\Gamma = \Lambda_1 \cup \Lambda_2 \in \mathcal{C}^2$ and let $\mathcal{R}(\Gamma) = \pi_0(\mathcal{A}(\Gamma))''$ and $\widehat{\mathcal{R}}(\Gamma) = \pi_0(\mathcal{A}(\Gamma^c)')'$. Choose charge transporters $\{V_g \mid g \in H\}$ between Λ_1 and Λ_2. Then

$$\widehat{\mathcal{R}}(\Gamma) = \mathcal{R}(\Gamma) \vee \{V_g \mid g \in H\}.$$

Proof. Let $\Gamma \in \mathcal{C}^2$ and $\mathcal{R}(\Gamma), \widehat{\mathcal{R}}(\Gamma)$ as in the assertion. Set $\mathfrak{A} := \mathcal{R}(\Gamma) \vee \{V_g \mid g \in H\}$ and $\mathfrak{B} := \pi_0(\mathcal{A}(\Gamma^c))$, and let $\mathcal{H}_{\overline{\Gamma}} := \overline{\mathfrak{A}\Omega}^{\|\cdot\|}$ with projection $P_{\overline{\Gamma}} : \mathcal{H} \to \mathcal{H}_{\overline{\Gamma}}$. Then by Lemma 4.2.3 both algebras, $\mathfrak{A}$ and $\mathfrak{B}'$, leave $\mathcal{H}_{\overline{\Gamma}}$ invariant, and by Lemma 4.2.4 and [RD75, Theorem 2] their restrictions $\mathfrak{A}_{\overline{\Gamma}} = P_\Gamma \mathfrak{A} P_\Gamma$ and $\mathfrak{B}_{\overline{\Gamma}} = P_{\overline{\Gamma}} \mathfrak{B}'' P_{\overline{\Gamma}}$ satisfy $A_{\overline{\Gamma}} = B'_{\overline{\Gamma}}$. Note, that by [Tak79, Proposition II.3.10] $P_{\overline{\Gamma}} \mathfrak{B}' P_{\overline{\Gamma}} = (P_{\overline{\Gamma}} \mathfrak{B} P_{\overline{\Gamma}})'$. Hence, by Lemma 4.2.3, it follows that $\mathfrak{A} = \mathfrak{B}'$ which completes the proof. □

Next we want to prove that for any pair of cones $\Gamma \in \mathcal{C}^2$ the von-Neumann algebra $\widehat{\mathcal{R}}(\Gamma)$ is isomorphic to the crossed product $\mathcal{R}(\Gamma) \rtimes_\alpha H$ where α is a representation of H as automorphisms on $\mathcal{R}(\Gamma)$ given by $\alpha_g(A). = V_g A V_g^*, g \in H, A \in \mathcal{R}(\Gamma)$. In order to do so we need some prerequisites. These essentially stem from an explicit proof of the approximate split property in [Naa12b] and we state a generalised version here for the case of abelian groups G. We adapt the following from [Naa12b] and use the tools from Chapter 2.

First we introduce some notations. Let Λ_1 and Λ be cones with $\Lambda_1 \ll \Lambda$. Let $\Lambda_0 := (\Lambda_1 \cup \Lambda^c)^c = \Lambda_1^c \cap \Lambda$. We define the interior of Λ_0 by $\mathrm{int}(\Lambda_0) := \Lambda_0 \setminus \partial\Lambda_1$

(recall that $\partial\Lambda_1 \subset \Lambda_1^c$ and similarly for $\partial\Lambda$, see also Definition 3.1.1.) We pick a fixed ribbon $\rho_0 \subset \Lambda_0$ with $\partial_0\rho_0 \in \partial\Lambda_1$ and $\partial_1\rho_0 \in \partial\Lambda$, and by the discussion after Definition 3.1.4 we may choose ρ_0 such that there are ribbons $\sigma \subset \Lambda_1$ and $\sigma' \subset \Lambda^c$ such that $\sigma\rho_0\sigma'$ is again a ribbon. In addition to this we choose a site $s_0 \in \mathrm{int}(\Lambda_0)$ and a ribbon $\rho_{s_0} \subset \Lambda_0$ such that $\partial_0\rho_{s_0} = s_0$ and $\partial_1\rho_{s_0} \in \partial\Lambda_1$. Again, we require that ρ_{s_0} is chosen such that there exists a ribbon $\sigma \subset \Lambda_1$ such that $\rho_{s_0}\sigma$ is a ribbon. For each site $s \in \mathrm{int}(\Lambda_0)$ fix a ribbon ρ_s with $\partial_0\rho_s = s$ and $\partial_1\rho_s = s_0$. Define $\Theta := \{\rho_s \mid s \in \mathrm{int}(\Lambda_0) \cup \{s_0\}\}$.

As before we let $\mathcal{H}_{\Lambda_1}$ and $\mathcal{H}_{\Lambda^c}$ be the Hilbert spaces obtained by taking the closure of the linear span of products of ribbons in Λ_1 and Λ^c, respectively, acting on the ground state. Furthermore set $\mathfrak{F}_0 := \{F_{\rho_1} \cdots F_{\rho_n} \mid \rho_1, \dots, \rho_n \in \Theta \cup \{\rho_0\}\}$. We denote by $\mathcal{H}_0$ the Hilbert space given by the closure of the linear span of $\mathfrak{F}_0\Omega$. With the notation $\mathfrak{F}(\Lambda_1)$ and $\mathfrak{F}(\Lambda^c)$ as above we find the following.

Lemma 4.2.6:
The space $\mathrm{span}\,\mathfrak{F}(\Lambda_1)\mathfrak{F}_0\mathfrak{F}(\Lambda^c)\Omega$ *norm is dense in* $\mathcal{H}$.

Proof. By Lemma 3.2.1 products of ribbon operators with ribbons in $\mathbb{Z}^2$ span a dense subspace of $\mathcal{H}$. Hence, what we will show is that for each ribbon operator $F_\rho \in \mathcal{A}$ with ribbon ρ there are operators $F_1 \in \mathfrak{F}(\Lambda_1)$, $F_0 \in \mathfrak{F}_0$ and $F_2 \in \mathfrak{F}(\Lambda^c)$ such that $F_\rho\Omega = F_1F_0F_2\Omega$. The claim then immediately follows, since in a product of ribbon operators, we can always commute operators with each other at the cost of complex phases.

So, let F_ρ be as above. Essentially what we are going to do is to separate the parts in ρ that lie in Λ_0 and express the resulting factors of the ribbon operators via operators in $\mathfrak{F}_0$. This is possible, since, when acting on Ω we can deform and invert ρ without affecting the vector $F_\rho\Omega$ (see e.g. Lemmas 2.6.13 and 2.6.14). The first four cases are analogous to those in the proof of [Naa12b, Lemma 4.3]. It is obvious that, if $\partial_i\rho$ are both contained either in Λ_1 or in Λ^c, there is an operator $F \in \mathfrak{F}(\Lambda_1)$ or $F \in \mathfrak{F}(\Lambda^c)$ such that $F\Omega = F_\rho\Omega$. In the following we assume that ρ is not a closed loop, for if this was the case, $F_\rho\Omega = \Omega$. Set $s' := \partial_0\rho$ and $s := \partial_1\rho$.

Assume that $s' \in \Lambda_1$ and $s \in \mathrm{int}(\Lambda_0)$. Let $\rho_s \in \Theta$ be the corresponding ribbon to s and consider ρ_{s_0} as chosen before. Let $\sigma \in \Lambda_1$ such that $\partial_0\sigma = \partial_1\rho_{s_0}$ and $\partial_1\sigma = s'$. Then, by Lemmas 2.6.13, 2.6.14 and 2.7.2, there are a ribbon operator $F_1 = F_\sigma \in \mathfrak{F}(\Lambda_1)$, a product of ribbon operators $F_0 = F_{\rho_s}F_{\rho_{s_0}} \in \mathfrak{F}_0$ and a complex phase $\alpha \in \mathbb{C}$ such that $F_\rho\Omega = \alpha F_1F_0\Omega$.

Now, assume that $s' \in \Lambda^c$ and $s \in \mathrm{int}\Lambda_0$. Let $\rho_s \in \Theta$ and let $\sigma \subset \Lambda^c$ be a ribbon with $\partial_0\sigma' = \partial_1\rho_0$ and $\partial_1\sigma' = s'$. Furthermore, let $\sigma \subset \Lambda_1$ be a ribbon with $\partial_0\sigma = \partial_1\rho_{s_0}$ and $\partial_1\sigma = \partial_0\rho_0$. Then there are ribbon operators $F_{\rho_0}, F_{\rho_{s_0}}, F_{\rho_s} \in \mathfrak{F}_0$, $F_{\sigma'} \in \mathfrak{F}(\Lambda^c)$ and $F_\sigma \in \mathfrak{F}(\Lambda)$, and a complex phase $a \in \mathbb{C}$ such that $F_\rho\Omega = cF_\sigma F_{\rho_0} F_{\rho_{s_0}} F_{\rho_s} F_{\sigma'}\Omega$.

Consider now the case in which $s' \in \Lambda_1$ and $s \in \Lambda^c$. Then we can pick a deformation $\tilde{\rho}$ of ρ such that $\tilde{\rho} = \sigma\rho_0\sigma'$ with $\sigma \in \Lambda_1$ and $\sigma' \in \Lambda^c$. By Lemma 2.7.2 it follows that $F\Omega$ has the desired form.

If $s, s' \in \mathrm{int}(\Lambda_0)$ we pick $\rho_s, \rho_{s'} \in \Theta$ and find ribbon operators $F_{\rho_s}, F_{\rho_{s'}} \in \mathfrak{F}_0$ with $F_\rho\Omega = F_{\rho_s} \cdot F_{\rho_{s'}}\Omega$.

Now for the last three more difficult cases. We only treat one of them, since the others can be treated in an analogous way using the arguments from above. Consider first $s' \in \partial\Lambda_1$ and $s \in \mathrm{int}(\Lambda_0)$. By the remark after Lemma 3.1.5 the ribbon ρ is at most one triangle apart from Λ_1, that is, either there is at most one bond in Λ_0 between ρ and Λ_1. This leaves us with two sub-cases to consider.

First, assume that there is no such bond. Hence we can pick a ribbon $\sigma \subset \Lambda_1$ such that $\partial_0\sigma = s'$ and $\partial_1\sigma = \partial_0\rho_0$. Then, with $\rho_s \in \Theta$, we can find ribbon operators $F_\sigma \in \mathfrak{F}(\Lambda_1), F_{\rho_0}, F_{\rho_s} \in \mathfrak{F}_0$ such that $F_\rho\Omega = F_\sigma F_{\rho_0} F_{\rho_s}\Omega$.

For the second sub-case consider that there is a triangle $\tau \in \Lambda_0$ such that $\tau\rho$ is a ribbon and $\partial_0\tau \in \partial\Lambda_1$. W.l.o.g. we can assume that $F_\rho = F_\rho^{\chi,c}$ with $(\chi, c) \in \hat{G} \times G$. By Lemma 2.7.2 we have that $F_\rho^{\chi,c} = F_\rho^{\chi,e} F_\rho^{\mathrm{id},c}$ and these factors commute. Assume that τ is direct. Then by Lemmas 2.7.4 and 2.6.7 it holds that $F_\rho^{\mathrm{id},c} = F_{\tau\rho}^{\mathrm{id},c}$ and we can apply the procedure from the preceding paragraph to $F_{\tau\rho}^{\mathrm{id},c}$. We still have to treat the factor $F_\rho^{\chi,e}$. In order to do so, let τ' be the first triangle in ρ, i.e. $\partial_0\tau' = s'$. If necessary, we first deform ρ in Λ_0 such that removing τ' from ρ results in a ribbon $\tilde{\rho}$ with $\partial_0\tilde{\rho} \in \mathrm{int}(\Lambda_0)$. If τ' is dual we find by Lemmas 2.7.4 and 2.6.7 that $F_{\tilde{\rho}}^{\chi,e}\Omega = F_\rho^{\chi,e}\Omega$ and we can treat $F_{\tilde{\rho}}^{\chi,e}$ as above. If, however, τ' is direct, we can append a dual triangle τ'' to $\tilde{\rho}$ such that $\tau''\tilde{\rho}$ is a ribbon and $F_{\tilde{\rho}}^{\chi,e} = F_{\tau''\tilde{\rho}}^{\chi,e}$. Then there are three possibilities. Either $\partial_0\tau'' \in \mathrm{int}(\Lambda_0)$, $\partial_0\tau'' \in \partial\Lambda_1$ with no bond distance to Λ_1, or $\partial_0\tau'' \in \partial\Lambda_1$ with one bond distance to Λ_1. The first possibilities can be treated as before, and the last one belongs to the case where the first triangle τ in ρ is dual, which can be treated analogously.

The remaining difficult cases are given when $s' \in \mathrm{int}(\Lambda_0)$ and $s \in \partial(\Lambda^c)$, and when $s' \in \partial\Lambda_1$ and $s \in \partial(\Lambda^c)$. However, these can be treated in an analogous manner as the preceding case together with arguments from the first part of this proof. $\square$

Proposition 4.2.7:
Let $\Lambda_1 \ll \Lambda$ be cones and $\mathcal{H}_{\Lambda_1}, \mathcal{H}_0$ and $\mathcal{H}_{\Lambda^c}$ as before. Then there exists a unitary $U : \mathcal{H} \to \mathcal{H}_{\Lambda_1} \otimes \mathcal{H}_{\Lambda^c} \otimes \mathcal{H}_0$ which is determined by

$$UF_1F_0F_2\Omega = F_1\Omega \otimes F_2\Omega \otimes F_0\Omega, \tag{4.4}$$

with $F_1 \in \mathfrak{F}(\Lambda_1), F_0 \in \mathfrak{F}_0$ and $F_2 \in \mathfrak{F}(\Lambda^c)$.

Proof. The proof of the previous lemma shows that for any product F of ribbon operators in $\mathcal{A}$ there exist operators $F_1 \in \mathfrak{F}(\Lambda_1), F_0 \in \mathfrak{F}_0$ and $\mathfrak{F}_2 \in \mathfrak{F}(\Lambda^c)$ such that $F\Omega = F_1F_0F_2\Omega$. Hence, equation (4.4) defines a linear map from a dense subspace of $\mathcal{H}$ to $\mathcal{H}_{\Lambda_1} \otimes \mathcal{H}_{\Lambda^c} \otimes \mathcal{H}_0$ with dense image. Let $F_1, F_1' \in \mathfrak{F}(\Lambda_1)$, $F_0, F_0' \in \mathfrak{F}_0$ and $F_2, F_2' \in \mathfrak{F}(\Lambda^c)$. We just need to show that U is an isometry, i.e. $(UF_1F_0F_2\Omega, UF_1'F_0'F_2'\Omega) = (F_1F_0F_2\Omega, F_1'F_0'F_2'\Omega)$. Note that

$$(UF_1F_0F_2\Omega, UF_1'F_0'F_2'\Omega) = (F_1\Omega, F_1'\Omega)(F_2\Omega, F_2'\Omega)(F_0\Omega, F_0'\Omega).$$

Assume now that $F_0 \neq F_0'$. Then there exists a site $s \in \mathbb{Z}^2$ such that the star or the plaquette operator at s does not commute with F_0 or F_0'. Here, "or" is not exclusive. Then it is easy to see that $(UF_1F_0F_2\Omega, UF_1'F_0'F_2'\Omega) = 0$. So it remains to show that $(F_1F_0F_2\Omega, F_1'F_0'F_2'\Omega) = 0$. There are two possibilities now. The first one, case (I), is given by $s \in \mathrm{int}(\Lambda_0)$ and the second, case (II), consists of the case where the star and plaquette operators at all sites in $\mathrm{int}(\Lambda_0)$ commute with F_0 and F_0'.

Consider first case (I), where there is a site $s \in \mathrm{int}(\Lambda_0)$ such that the star or plaquette operator does not commute with F_0 or F_0'. By locality the star and plaquette operators commute with F_1, F_2, F_1', F_2' and it immediately follows that $(F_1F_0F_2\Omega, F_1'F_0'F_2'\Omega) = 0$.

Let us turn to case (II), that is, the plaquette and star operators at all sites in $\mathrm{int}(\Lambda_0)$ commute with both operators, F_0 and F_0'. This implies that both operators cannot contain non-trivial ribbon operators except for those associated to ρ_0, since by construction ribbon operators in F_0 and F_0' must either be associated to ρ_0 or to ribbons that end at the site $s_0 \in \mathrm{int}(\Lambda_0)$. Furthermore, ρ_0 was chosen such that $\rho_0 \subset \Lambda_0$ and therefore F_0 and F_0' commute with F_1, F_2, F_1' and F_2'. Since local operators can only create an even number of excitations above the ground state (see Lemma 3.2.1), and F_0, F_0' each create different single excitations at both $\partial\Lambda_1$ and $\partial\Lambda^c$ it follows that there is no operator F in $\mathfrak{F}(\Lambda_1)$ or $\mathfrak{F}(\Lambda^c)$ such that $FF_0\Omega = \Omega$ or $FF_0'\Omega = \Omega$. Hence $(F_1F_0F_2\Omega, F_1'F_0'F_2'\Omega) = 0$.

Now consider the case where $F_0 = F_0'$, hence $(F_0\Omega, F_0'\Omega) = 1$. By unitarity of ribbon operators and by locality we have that $(F_1F_0F_2\Omega, F_1'F_0'F_2'\Omega) = (F_1F_2\Omega, F_1'F_0^*F_0'F_2'\Omega) = (F_1F_2\Omega, F_1'F_2'\Omega) = ((F_1')^*F_1\Omega, F_2^*F_2'\Omega)$. Hence, the only interesting case here is if there is a site in $\mathbb{Z}^2$ such that the star or plaquette operator does not commute with at least one of the operators $(F_1')^*F_1$ or $F_2^*F_2'$. If there was no such site, then these operators would commute with all star and plaquette operators. Hence these operators would only contain trivial ribbon operators or ribbon operators associated to closed ribbons and thus $(F_i\Omega, F_i'\Omega) = 1$ and $(F_1F_0F_2\Omega, F_1'F_0'F_2'\Omega) = 1$. Now let $s \in \mathbb{Z}_2$ be such that the star or plaquette operator, or both, do not commute with $(F_1')^*F_1$ or $F_2^*F_2'$, or both. Then $((F_1')^*F_1\Omega, \Omega) = (F_1\Omega, F_1'\Omega) = 0$ and/or $(F_2^*\Omega, F_2\Omega) = 0$. And also $((F_1')^*F_1\Omega, F_2^*F_2'\Omega) = 0$ by locality and therefore $(F_1F_0F_2\Omega, F_1'F_0'F_2'\Omega) = 0$.

From all this, it follows that $(UF_1F_0F_2\Omega, UF_1'F_0'F_2'\Omega) = (F_1F_0F_2\Omega, F_1'F_0'F_2'\Omega)$ for all $F_1, F_1' \in \mathfrak{F}(\Lambda_1)$, $F_2, F_2' \in \mathfrak{F}(\Lambda^c)$ and $F_0, F_0' \in \mathfrak{F}_0$. By Lemma 4.2.6 it then follows that U is in fact a unitary from $\mathcal{H}$ to $\mathcal{H}_{\Lambda_1} \otimes \mathcal{H}_{\Lambda^c} \otimes \mathcal{H}_0$. □

For completeness we state the result that the ground state ω satisfies the approximate split property for cones. The proof can be taken verbatim from the one for [Naa12b, Theorem 4.5].

Theorem 4.2.8:

The translationally invariant ground state ω of Kitaev's quantum double model for finite abelian groups satisfies the approximate split property. I.e. let $\Lambda_1 \ll \Lambda$ be cones and $\mathcal{R}(\Lambda_1)$ and $\mathcal{R}(\Lambda)$ the associated von-Neumann algebras in the ground state representation. Then there exists a type-I factor $\mathcal{N}$ such that

$$\mathcal{R}(\Lambda_1) \subset \mathcal{N} \subset \mathcal{R}(\Lambda).$$

Note that this can be proven without the above construction (see [Naa12a, Theorem 11.3] and [FN15])[2]. However, the construction provides an explicit realisation of the intermediate type-I factor. Note also, that the approximate split property implies that $\mathcal{R}(\Lambda_1) \vee \mathcal{R}(\Lambda^c)$ is unitarily equivalent to $\mathcal{R}(\Lambda_1) \otimes \mathcal{R}(\Lambda^c)$. The unitary in Proposition 4.2.7 provides an explicit implementation of this fact. Another advantage of this direct construction is that it allows us to prove that the von Neumann algebra $\widehat{\mathcal{R}}(\Gamma) = \mathcal{R}(\Gamma^c)'$ is isomorphic to the crossed product of $\mathcal{R}(\Gamma)$ with the group $H = G \times \hat{G}$, where $\Gamma = (\Lambda_1 \cup \Lambda_2) \in \mathcal{C}^2$ is a pair of cones. This was proven to be true for the toric code where $G = \mathbb{Z}_2$ [Naa13a], and now we are able to show that this is also true for arbitrary finite abelian groups.

[2]These references use a combination of results of references [Tak58] and [DL83].

Given a pair of cones $\Gamma = (\Lambda_1 \cup \Lambda_2) \in \mathcal{C}^2$, and the von Neumann algebras $\widehat{\mathcal{R}}(\Gamma) = \mathcal{R}(\Gamma^c)'$ and $\mathcal{R}(\Gamma)$. Let $\{V_g \,|\, g \in H\}$ be a set of charge transporters between Λ_1 and Λ_2, and let α be the group of automorphisms on $\mathcal{R}(\Gamma)$ defined by $\alpha_g(A) := V_g A V_g^*, g \in H, A \in \mathcal{R}(\Gamma)$. Let $\mathcal{K} := \mathcal{H} \otimes \ell^2(H)$, where $\ell^2(H)$ is the Hilbert space of square summable complex-valued functions of the group H. Then, by the discussion after Definition 1.2.13 which leads to equation (1.7), the crossed product $\mathcal{R}(\Gamma) \rtimes_\alpha H$ is isomorphic to the von Neumann algebra spanned by $\{A \otimes \mathbb{1}, V_g \otimes \lambda_g \,|\, A \in \mathcal{R}(\Gamma), h, g \in H\}$, where λ is the left regular representation of G on $\ell^2(H)$.

In our case it is *a priori* not clear, whether the automorphisms α_g with $g \in H$ and $g \neq e$ are free.

Theorem 4.2.9:
Given a pair of cones $\Gamma = (\Lambda_1 \cup \Lambda_2) \in \mathcal{C}^2$, the map given by

$$\Phi(X) = \sum_{g \in H} A_g V_g,$$

with $X = \sum_{g \in H} (A_g \otimes \mathbb{1})(V_g \otimes \lambda_g) \in \mathcal{R}(\Gamma) \rtimes_\alpha H$ defines a $*$-isomorphism

$$\Phi : \mathcal{R}(\Gamma) \rtimes_\alpha H \to \widehat{\mathcal{R}}(\Gamma)$$

such that $\Phi(\mathcal{R}(\Gamma)) = \mathcal{R}(\Gamma)$.

Proof. The proof is mostly along the same lines as the corresponding one for the toric code in [Naa13a]. We recall it here with the necessary modifications.

Clearly, Φ is a $*$-homomorphism. Furthermore $\mathcal{R}(\Gamma)$ can be identified with the diagonal part of $\mathcal{R}(\Gamma) \rtimes_\alpha H$, and therefore $\Phi(\mathcal{R}(\Gamma)) \subseteq \mathcal{R}(\Gamma)$.

We start by showing that $\Phi : \mathcal{R}(\Gamma) \rtimes_\alpha H \to \widehat{\mathcal{R}}(\Gamma)$ is surjective. We do this by first showing that Φ is normal, i.e. for normal states φ on $\widehat{\mathcal{R}}(\Gamma)$ the states $\phi \circ \Phi$ on $\widehat{\mathcal{R}}(\Gamma) \rtimes_\alpha H$ are normal as well. If Φ is normal, then its image is a von Neumann algebra in $\widehat{\mathcal{R}}(\Gamma)$ containing both $\mathcal{R}(\Gamma)$ and the unitaries $\{V_g \,|\, g \in H\}$. Since by Theorem 4.2.5 $\widehat{\mathcal{R}}(\Gamma)$ is already the smallest von Neumann algebra containing these, this implies that Φ is surjective.

Let $\varphi \in \widehat{\mathcal{R}}(\Gamma)_*$ be a state. Then there exists a sequence $(\psi_n)_{n\in\mathbb{N}}$ of states in $\mathcal{K}$ such that $\sum_{n=1}^{\infty} \|\psi_n\|^2 < \infty$ and $\varphi(X) = \sum_{n=1}^{\infty} (\psi, X\psi)$ for all $X \in \widehat{\mathcal{R}}(\Gamma)$ (see e.g. [Tak79, Proposition II.3.20]). Let $(\delta_g)_{g\in H}$ be the canonical basis in $\ell^2(H)$ with $\delta_g(k) = \delta_{g,k}$. Define vectors $\xi_n := \sum_{g\in H} \psi_n \otimes \delta_g \in \mathcal{K}$ and $\zeta_n := \psi_n \otimes \delta_e \in \mathcal{K}$ with $n \in \mathbb{N}$ and e the identity element in H. Then $\phi(X) := \sum_n (\xi_n, X\zeta_n)$ defines a normal functional on $\mathcal{R}(\Gamma) \rtimes_\alpha H$ and for all X it follows that

$$\begin{aligned}\phi(X) &= \sum_n (\xi_n, X\zeta_n) = \sum_{n\in\mathbb{N}} \sum_{g,h,k\in H} (\psi_n \otimes \delta_k, (A_g \otimes \mathbb{1})(V_g \otimes \lambda_g)\psi_n \otimes \delta_e) \\ &= \sum_{n\in\mathbb{N}} \sum_{g\in H} (\psi_n, A_g V_g \psi_n).\end{aligned}$$

Hence $\phi = \varphi \circ \Phi$ is a normal state.

Now for injectivity of Φ, i.e. $X = 0$ whenever $\Phi(X) = 0$. By definition of $\Gamma = \Lambda_1 \cup \Lambda_2$ we can choose a cone Λ with $\Lambda_1 \ll \Lambda$ and $\Lambda_2 \subset \Lambda^c$. Let $\mathcal{H}_{\Lambda_1}$ and $\mathcal{H}_{\Lambda^c}$ be Hilbert spaces as defined in Definition 3.2.2, and let $\mathcal{H}_0$ be the Hilbert space defined before Lemma 4.2.6. The definition of $\mathcal{H}_0$ required the specification of a ribbon $\rho_0 \subset \Lambda_0 = (\Lambda_1^c \cap \Lambda) \setminus \partial\Lambda_1$ with $\partial_0\rho_0 \in \partial\Lambda_1$ and $\partial_1\rho_0 \in \partial\Lambda^c$. Furthermore, there exists a ribbon ρ with $\partial_0\rho \in \Lambda_1$ and $\partial_1\rho \in \Lambda_2$ such that for all $g \in H$ we have $V_g\Omega = F_\rho^g\Omega$. We can choose ρ_0 such that $\rho_0 \subset \rho$. Let $U : \mathcal{H} \to \mathcal{H}_{\Lambda_1} \otimes \mathcal{H}_{\Lambda^c} \otimes \mathcal{H}_0$ as in Proposition 4.2.7. Hence, $U\mathcal{R}(\Lambda_1)U^* \subset \mathcal{B}(\mathcal{H}_{\Lambda_1})$ and $U\mathcal{R}(\Lambda_2)U^* \subset \mathcal{B}(\Lambda_2)$.

For every $g \in H$ we define a set of orthonormal states by $\Omega_g := F_{\rho_0}^g\Omega$. Let $P_{\Omega_g}, g \in H$ be the one-dimensional projections associated with these vectors, and set $P_g := UP_{\Omega_g}U^* = \mathbb{1} \otimes \mathbb{1} \otimes P_{\Omega_g}$. Note that $P_g \in (U\mathcal{R}(\Gamma)U^*)'$ for all $g \in H$. Given $X \in \mathcal{R}(\Gamma) \rtimes_\alpha H$ and $k \in H$, then

$$\begin{aligned}P_k U\Phi(X)U^* &= P_k U \sum_{g\in H} A_g V_g U^* \\ &= \sum_{g\in H} UA_g U^* P_k U V_g U^*.\end{aligned}$$

Next, show that $P_k UV_g U^*\zeta \otimes \Omega = \delta_{k,g} UV_g U^*\zeta \otimes \Omega$ for any $g, h \in H$ and any $\zeta \in \mathcal{H}_{\Lambda_1} \otimes \mathcal{H}_{\Lambda^c}$. From this it follows, that if $\Phi(X)\zeta \otimes \Omega = 0$ for any $\zeta \in \mathcal{H}_{\Lambda_1} \otimes \mathcal{H}_{\Lambda^c}$ then $A_g U^*\zeta = 0$ for all $g \in H$, where $X = \sum_{g\in H} A_g V_g$. Hence, by Lemma 4.2.3, also $A_g = 0$ for all $g \in H$.

By the definition of $\mathcal{H}_{\Lambda_1}$ and $\mathcal{H}_{\Lambda^c}$ we only need to show this for products of ribbon operators, i.e. for operators $F \in \mathfrak{F}(\Gamma)$. Let $F \in \mathfrak{F}(\Gamma)$, and $\xi = F\Omega$. Note that $U\xi = \zeta \otimes \Omega$ with $\zeta \in \mathcal{H}(\Lambda_1) \otimes \mathcal{H}(\Lambda^c)$. Then $UV_g U^*U\xi = UV_g F\Omega = \alpha(g)UFF_\rho^g\Omega$, with $\alpha(g) \in \mathbb{C}$ and $|\alpha(g)| = 1$, where we used the commutation relations between ribbon operators. In addition, we used that the unitary operators V_g are weak limits of sequences of ribbon operators, and that the multiplication is separate weakly continuous (as in the proof of Lemma 4.2.2).

Since $\rho_0 \subset \rho$ there are ribbons ρ_1 and ρ_2 such that $\rho = \rho_1\rho_0\rho_2$. Hence

$$\begin{aligned} UV_gU^*U\xi &= \alpha(g)UFF^g_{\rho_1}F^g_{\rho_0}F^g_{\rho_2}\Omega \\ &= \alpha(g)UFU^*\left(F^g_{\rho_1}\Omega \otimes F^g_{\rho_2}\Omega \otimes F_{\rho_0}\Omega\right). \end{aligned}$$

Using the definition of P_k we find $P_kUV_gU^*\zeta \otimes \Omega = \delta_{k,g}UV_gU^*\zeta \otimes \Omega$. Since vectors of the form $F\Omega$ with $F \in \mathfrak{F}(\Gamma)$ span a dense subspace of $\mathcal{H}_\Gamma$ this holds for all $\zeta \in \mathcal{H}_{\Lambda_1} \otimes \mathcal{H}_{\Lambda^c}$. Hence, let $X \in \mathcal{R}(\Gamma) \rtimes_\alpha H$ with $X = \sum_{g,h}(A_g \otimes \mathbb{1})(V_g \otimes E_{h,gh})$, and assume that $\Phi(X) = 0$.

Then for every $k \in H$ and every $\zeta \in \mathcal{H}_{\Lambda_1} \otimes \mathcal{H}_{\Lambda^c}$ it follows that

$$0 = P_kU\Phi(X)U^*\zeta \otimes \Omega = UA_kV_kU^*\zeta \otimes \Omega,$$

hence, by Lemma 4.2.2, $A_k = 0$ and thus $X = 0$. □

An immediate consequence of this theorem is that the unitary operators V_g with $g \in G$ and $g \neq e$ are not elements of $\mathcal{R}(\Gamma)$. Furthermore, we can apply the discussion of Section 1.2.4 to our situation.

Corollary 4.2.10:
$\mathcal{R}(\Gamma) \subseteq \widehat{\mathcal{R}}(\Gamma)$ is an irreducible inclusion of factors with index $|G|$. The action α of G on $\widehat{\mathcal{R}}(\Gamma)$ is not inner.

Proof. As a UHF algebra $\mathcal{A}$ is simple (see e.g. [BR96]), thus π is faithful and, by purity of ω (see [FN15, Theorem 3.5]), irreducible. Therefore, by locality $\mathcal{R}(\Gamma) \vee \mathcal{R}(\Gamma^c) = \mathcal{B}(\mathcal{H})$ for regions Λ. It follows that $\mathcal{R}(\Gamma)' \cap \widehat{\mathcal{R}}(\Gamma) = \mathbb{C}$ (see also [Naa13a, Lemma 3.2]). Since $\mathcal{R}(\Gamma) \subseteq \widehat{\mathcal{R}}(\Gamma)$ this implies that $\mathcal{R}(\Gamma)$ is a factor. Similarly, we have $\widehat{\mathcal{R}}(\Gamma)' \vee \widehat{\mathcal{R}}(\Gamma) \supseteq (\widehat{\mathcal{R}}(\Gamma) \cup \mathcal{R}(\Gamma)')' = \mathcal{B}(\mathcal{H})$, and therefore $\widehat{\mathcal{R}}(\Gamma)$ is a factor as well. Consequently, by Proposition 1.2.17, the automorphisms α_g for $g \in H$ and $g \neq e$ are free, and therefore not inner. By Proposition 1.2.19 $\{V_g\}_{g\in H}$ is a Pimsner-Popa basis and the unique conditional expectation $\mathcal{E} : \widehat{\mathcal{R}}(\Gamma) \to \mathcal{R}(\Gamma)$ given by $\mathcal{E}(\sum_{g\in H} A_gV_g) = A_0$, with some indexing $(A_g)_{g\in H}$ of elements in $\widehat{\mathcal{R}}(\Gamma)$, has index $|G|$. □

This result allows us to conclude that the number of superselection sectors in the quantum double models for finite abelian groups is given by $|G|^2$, as expected. For the toric code this is proved in [Naa13a, Theorem 4.9]. In our situation the proof carries over verbatim so that we need not to prove it here.

Theorem 4.2.11:
In Kitaev's quantum double model for finite abelian groups G *on the infinite 2D lattice, the cone index* $[\widehat{\mathcal{R}}(\Gamma) : \mathcal{R}(\Gamma)]$ *is equal to* $|G|^2$. *This implies that there are exactly* $|G|^2$ *distinct superselection sectors. Furthermore the sectors are not degenerate and each sector has statistical dimension one.*

4.3 Further Structure of the Inclusion

The previous sections we singled out some properties of cone algebras in the representation of the translational invariant ground state of Kitaev's quantum double model for a finite group G on an infinite plane. Of particular interest in this section is the result that the commutants of von Neumann algebras associated with pairs of cones can be identified with the crossed product of the von Neumann algebras of the same cones and the group $H = G \times \hat{G}$. In the following we further analyse this structure.

Recall that $\mathcal{R}(\Gamma) = \pi(\mathcal{A}(\Gamma))''$ and $\widehat{\mathcal{R}}(\Gamma) = \pi(\mathcal{A}(\Gamma^c))'$, where $(\pi, \Omega, \mathcal{H})$ is the cyclic representation of the translational invariant ground state ω, and $\Gamma = \Lambda_1 \cup \Lambda_2 \in \mathcal{C}^2$ is a pair of disjoint, and sufficiently separated cones. Let α be the automorphic action of $H = G \times \hat{G}$ on $\mathcal{R}(\Gamma)$ given by the charge transporters by $\alpha_g(A) := V_g A V_g^*$, $A \in \mathcal{R}(\Gamma)$, $g \in G$, and the group G is considered finite and abelian. The structure obtained in the previous section was summarised in Corollary 4.2.10, and we recall it here for convenience.

Proposition 4.3.1:
The algebras $\mathcal{R}(\Gamma) \subseteq \widehat{\mathcal{R}}(\Gamma)$ *is an irreducible inclusion of factors. Furthermore,* $\widehat{\mathcal{R}}(\Gamma)$ *is isomorphic to* $\mathcal{R}(\Gamma) \rtimes_\alpha H$ *and the action of* α *is free. The map* $\mathcal{E} : \widehat{\mathcal{R}}(\Gamma) \to \mathcal{R}(\Gamma)$ *defined by*

$$\mathcal{E}(X) = A_e, \quad X = \sum_{g \in H} A_g V_g \in \widehat{\mathcal{R}}(\Gamma),$$

is the unique normal conditional expectation and has index $\operatorname{Ind}(\mathcal{E}) = \frac{1}{|G|^2}$.

We now want to point out a downward basic construction onto the α-invariant subalgebra of $\mathcal{R}(\Gamma)$. Since this is a structure that mainly depends on the properties of $\widehat{\mathcal{R}}(\Gamma)$ as a crossed product, we first discuss the more general case of arbitrary crossed products and then apply it to $\widehat{\mathcal{R}}(\Gamma)$.

We start with considering a dynamical system $(\mathfrak{M}, K, \alpha)$ on some Hilbert space $\mathcal{H}$ as in Definition 1.2.12 and with K now some arbitrary abelian finite group. We assume that $\mathfrak{M}$ is a factor and that the action α is free on $\mathfrak{M}$. In addition, we require that there are unitary operators $\{V_g \mid g \in K\}$ on $\mathcal{H}$ implementing α. Then $\mathfrak{M} \rtimes_\alpha K$ is a factor as well, by Proposition 1.2.17. By the discussion after Definition 1.2.13 the crossed product $\mathfrak{M} \rtimes_\alpha K$ is isomorphic to the von Neumann algebra on $\mathcal{H} \times \ell^2(G)$ generated by

$$\{A \otimes \mathbb{1} \mid A \in \mathfrak{M}\} \cup \{V_g \otimes \lambda_g \mid g \in K\},$$

where $\lambda : G \to \ell^2(K)$ is the left regular representation of K. The canonical embedding $\pi : \mathfrak{M} \to \mathfrak{M} \rtimes_\alpha K$ is given by

$$\pi(A) := A \otimes 1,\ A \in \mathfrak{M},$$

and for all $g \in K$ the operators

$$\Lambda(g) = V_g \otimes \lambda_g,$$

define a faithful unitary representation of G on $\mathcal{H} \otimes \ell^2(K)$ (see also equation 1.7). The normal conditional expectation $E : \mathfrak{M} \rtimes_\alpha G \to \pi(\mathfrak{M})$ is given by

$$E(X) := A_e,$$

where $X = \frac{1}{|K|} \sum_{g \in K} A_g \Lambda(g)$, and $(A_g), g \in G$ is some labelling of elements in $\pi(\mathfrak{M})$. We denote the fixed point algebra under the action α by

$$\mathfrak{M}_0 := \{A \in \mathfrak{M} \mid \forall g \in K : \alpha_g(A) = A\} \tag{4.5}$$

which is again a von Neumann algebra. This algebra is accompanied by a normal conditional expectation $E_0 : \mathfrak{M} \to \mathfrak{M}_0$ given by

$$E_0(A) := \frac{1}{|K|} \sum_{g \in K} V_g A V_g^*$$

It is simple to see that $E_0(\mathfrak{M}) = \mathfrak{M}_0$, and that E_0 satisfies the criteria of a conditional expectation. Next, we define for each $\chi \in \hat{K}$ an orthogonal projection

$$Q_\chi := \sum_{g \in K} \overline{\chi(g)} \Lambda(g), \tag{4.6}$$

and it can be checked that $\Lambda(g) Q_\chi = \chi(g) Q_\chi$, $\Lambda(g) = \sum_{\chi \in \hat{K}} \chi(g) Q_\chi$, and $Q_\chi Q_\xi = \delta_{\chi,\xi} P_\chi$ for any $g \in K$ and any $\chi, \xi \in \hat{K}$.

Lemma 4.3.2:
The projections $P_\chi, \chi \in \hat{K}$ *as defined in equation* (4.6) *satisfy the following properties. For all* $\chi \in \hat{K}$ *we have*

- $\forall A \in \pi(\mathfrak{M}) : E_0(A)P_\chi = Q_\chi A Q_\chi$,
- $\forall A \in \pi(\mathfrak{M}) : E_0(A) = \sum_{\chi \in \hat{K}} Q_\chi A Q_\chi$,
- $E(Q_\chi) = \frac{1}{|K|}\mathbb{1}$,
- $\forall X \in \mathfrak{M} \rtimes_\alpha K : E(X)Q_\chi = \frac{1}{|K|} X Q_\chi$.
- $\mathfrak{M}_0 = (\mathfrak{M}' \cup \{P_\chi\})'$.

Proof. The first assertion follows from straightforward calculation, and the second is implied by the first, considering $P_\chi P_\xi = \delta_{\chi,\xi} P_\chi$ for any $\chi, \xi, \in \hat{K}$. The third is obvious from the definitions of E and the projections, and the fourth is again a simple calculation. It is clear that $\mathfrak{M}_0 \subseteq (\mathfrak{M}' \cup \{P_\chi\})'$. Conversely, let $X \in (\mathfrak{M}' \cup \{P_\chi\})'$. Note that $\mathfrak{M}' \subseteq \mathfrak{M}_0'$ and thus, for all $Y \in \mathfrak{M}_0'$ we have $[X, Y] = 0$, hence $X \in \mathfrak{M}_0$. □

For all $g \in G$ and $A \in \mathfrak{M}$ we have $\Lambda(g)A \otimes \mathbb{1})\Lambda(g)^* = \alpha_g(A) \otimes \mathbb{1}$, so this implies that

$$\pi(\mathfrak{M}_0) = \{X \in \pi(\mathfrak{M}_0) \,|\, \forall g \in K : \Lambda(g)X = X\Lambda(g)\}.$$

By faithfulness of both π and Λ, the normal conditional expectation $\tilde{E}_0 := \pi \circ \mathcal{E}_0 \circ \pi^{-1}$ satisfies the same properties as listed in Lemma 4.3.2 with respect to Q_χ and $\pi(\mathfrak{M}_0)$.

Proposition 4.3.3:
Let $(\mathfrak{M}, K, \alpha)$ *be a dynamical system,* $\mathfrak{M}$ *a factor, and* α *unitarily implemented on* $\mathcal{H}$*. Let* $Q_\chi, \chi \in \hat{K}$ *be the projections as in equation 4.6. Then for any* $\chi \in \hat{K}$

$$\langle \pi(\mathfrak{M}_0), Q_\chi \rangle \cong \mathfrak{M} \rtimes_\alpha K.$$

Furthermore, $\pi(\mathfrak{M}_0)$*, and hence* $\mathfrak{M}_0$ *is a factor. The index of* E_0 *is* $\mathrm{Ind}(E_0) = \frac{1}{|K|}$.

Proof. The existence of a $*$-isomorphism between $\langle \pi(\mathfrak{M}_0), Q_\chi \rangle$ and $\mathfrak{M} \rtimes_\alpha K$ follows from Lemma 4.3.2 and [Kos98, Proposition 3.12]. This also implies that $\mathrm{Ind}(E_0) = \frac{1}{|K|}$. The same proposition also implies that $\pi(\mathfrak{M}_0)$ is a factor, and so is $\mathfrak{M}_0$ by faithfulness of the representation π. □

This proposition singles out the algebra $\pi(\mathfrak{M}_0)$ and the conditional expectation E_0 as a downward basic construction of the inclusion $\pi(\mathfrak{M}) \subseteq \mathfrak{M} \rtimes_\alpha K$. For the next steps we interject a small technical lemma.

Lemma 4.3.4:
Let $\mathfrak{N}_1, \mathfrak{N}_2$ and $\mathfrak{N}_3$ be von Neumann algebras over some common Hilbert space $\mathcal{H}$. Then

$$\mathfrak{N}_1 \cap (\mathfrak{N}_2 \vee \mathfrak{N}_3) = (\mathfrak{N}_1 \cap \mathfrak{N}_2) \vee (\mathfrak{N}_1 \cap \mathfrak{N}_3).$$

Proof. We have that

$$\mathfrak{N}_1 \cap (\mathfrak{N}_2 \cup \mathfrak{N}_3) = (\mathfrak{N}_1 \cap \mathfrak{N}_2) \cup (\mathfrak{N}_1 \cap \mathfrak{N}_3).$$

With $(\mathfrak{N}_1 \vee \mathfrak{N}_2) = (\mathfrak{N}_1 \cup \mathfrak{N}_2)'' = (\mathfrak{N}_1' \cap \mathfrak{N}_2')'$, the bicommutants of each side are then

$$\begin{aligned}\mathfrak{N}_1 \cap (\mathfrak{N}_2 \cup \mathfrak{N}_3)'' &= (\mathfrak{N}_1' \vee (\mathfrak{N}_2 \cup \mathfrak{N}_3)')' \\ &= (\mathfrak{N}_1 \cap (\mathfrak{N}_2 \vee \mathfrak{N}_3)),\end{aligned}$$

and

$$\begin{aligned}((\mathfrak{N}_1 \cap \mathfrak{N}_2) \cup (\mathfrak{N}_1 \cap \mathfrak{N}_3))'' &= (\mathfrak{N}_1 \cap \mathfrak{N}_2)'' \vee (\mathfrak{N}_1 \cap \mathfrak{N}_3)'' \\ &= (\mathfrak{N}_1 \cap \mathfrak{N}_2) \vee (\mathfrak{N}_1 \cap \mathfrak{N}_3).\end{aligned}$$

□

Lemma 4.3.5:
Let $(\mathfrak{M}, K, \alpha)$ be as in the previous proposition. Then the inclusion $\mathfrak{M}_0 \subseteq \mathfrak{M}$ is irreducible, i.e.

$$\mathfrak{M}_0' \cap \mathfrak{M} = \mathbb{C}\mathbb{1}.$$

Proof. Note that $\mathfrak{M}_0 = \mathfrak{M} \cap \{V_g \,|\, g \in K\}'$ by construction. Using that $\mathfrak{M}$ is a factor and that the operators V_g with $g \in K$ and $g \neq e$ are not contained in $\mathfrak{M}$, we find

$$\begin{aligned}\mathfrak{M}_0' \cap \mathfrak{M} &= (\mathfrak{M}' \cup \{V_g \,|\, g \in K\})'' \cap \mathfrak{M} \\ &= \mathfrak{M} \cap \{V_g \,|\, g \in K\}'' \\ &= \mathbb{C}\mathbb{1}.\end{aligned}$$

□

The structure of the embedding $\pi(\mathfrak{M}_0) \subset \pi(\mathfrak{M})$ is illustrated as follows.

Lemma 4.3.6:
The map $\pi(\mathfrak{M}_0) \ni X \mapsto XQ_\chi \in Q_\chi(\mathfrak{M} \rtimes_\alpha K)Q_\chi$ *is a* $*$*-isomorphism for any choice of* $\chi \in \widehat{K}$.

Proof. Fix any $\chi \in \widehat{K}$. For convenience let l_χ denote the map $\pi(\mathfrak{M}_0) \ni X \mapsto XQ_\chi \in Q_\chi \mathfrak{M} \rtimes_\alpha KQ_\chi$. This map is non-zero by $l_\chi(\mathbb{1}) = Q_\chi$. Furthermore it is a $*$-homomorphism since $l_\chi(XY) = XYQ_\chi = XQ_\chi YQ_\chi = l_\chi(X)l_\chi(Y)$. Since $(\mathfrak{M}_0)$ is a factor and l_χ is non-trivial, l_χ is injective, by [JS97, Corollary A.3.2].

To prove surjectivity consider $X \in \mathfrak{M} \rtimes_\alpha K$. Then $X = \sum_{g\in G} A_g\Lambda(g)$ with $(A_g)_{g\in G}$ some labelling of elements of $\pi(\mathfrak{M})$, and, using Lemma 4.3.2, we have

$$\begin{aligned} Q_\chi X Q_\chi &= Q_\chi \sum_{g\in K} A_g\Lambda(g)Q_\chi = \sum_{g\in K} \chi(g)Q_\chi A_g Q_\chi \\ &= \sum_{g\in K} \chi(g)\tilde{E}_0(A_g)Q_\chi \end{aligned}$$

which is of the form YQ_χ with $Y \in \pi(\mathfrak{M}_0)$. □

By [KR97, Proposition 5.5.6] $Q_\chi(\mathfrak{M} \rtimes_\alpha K)Q_\chi$ viewed as a von Neumann algebra on $Q_\chi\mathcal{H}\otimes\ell^2(K)$ is a factor. The lemma implies that, when restricted to the subspaces $[P_\chi\mathcal{H} \otimes \ell^2(K)]$, the von Neumann algebra $\mathfrak{M} \rtimes_\alpha K$ becomes a copy of $\mathfrak{M}_0$.

We can also give a separate proof for that $\mathfrak{M}_0$ is a factor if $\mathfrak{M}$ is a factor.

Proposition 4.3.7:
Given a dynamical system $(\mathfrak{M}, K, \alpha)$ *on some Hilbert space* $\mathcal{H}$ *with* α *free and implemented on* $\mathcal{H}$ *by unitaries* $\{V_g \mid g \in K\}$. *Then*

$$\mathfrak{M}_0 \cap \mathfrak{M}_0' = \mathfrak{M} \cap \mathfrak{M}' \cap \{V_g \mid g \in K\}'$$

Especially, if $\mathfrak{M}$ *is a factor, then* $\mathfrak{M}_0$ *is a factor.*

Proof. First of all, note that $\mathfrak{M}_0 = \mathfrak{M} \cap \{V_g \mid g \in K\}'$. By the previous lemma we have for the center of $\mathfrak{M}_0$:

$$\begin{aligned} \mathfrak{M}_0 \cap \mathfrak{M}_0' &= (\mathfrak{M} \cap \{V_g \mid g \in K\}') \cap (\mathfrak{M}' \vee \{V_g \mid g \in K\}'') \\ &= (\mathfrak{M} \cap \mathfrak{M}' \cap \{V_g \mid g \in K\}') \vee (\mathfrak{M} \cap \{V_g \mid g \in K\}' \cap \{V_g \mid g \in K\}'') \end{aligned}$$

Since the action α is free, we have $\mathfrak{M} \cap \{V_g \mid g \in K\}' \cap \{V_g \mid g \in K\}'' = \mathbb{C}\mathbb{1}$, and thus

$$\mathfrak{M}_0 \cap \mathfrak{M}_0' = \mathfrak{M} \cap \mathfrak{M}' \cap \{V_g \mid g \in K\}',$$

which is the claim. □

Now we turn back to the setting of the previous section. Let $\Gamma \in \mathcal{C}^2$ and $\{V_g \mid g \in H\}$ be the charge transporters of the quantum double model with $H = G \times \hat{G}$ and G a finite abelian group. Consider the algebras $\mathcal{R}(\Gamma)$ and $\widehat{\mathcal{R}}(\Gamma)$ as in the previous section, and define

$$\mathcal{R}_0 := \{A \in \mathcal{R}(\Gamma) \mid \forall g \in H : V_g A V_g^* = 0\}$$

and

$$\mathcal{E}_0(A) := \frac{1}{|G^2|} \sum_{g \in H} V_g A V_g^* \tag{4.7}$$

for all $A \in \mathcal{R}(\Gamma)$. Recall the definition

$$P_\chi = \frac{1}{|G^2|} \sum_{g \in H} \overline{\chi(g)} V_g$$

for $\chi \in \hat{H}$, and for convenience we write $P_0 := P_{\mathrm{id}}$.

Lemma 4.3.8:
With the notation from above we have $\mathcal{R}_0 = \mathcal{R}(\Gamma) \cap \{P_0\}'$.

Proof. First, we have that $\mathcal{R}_0 \subset \mathcal{R}(\Gamma) \cap \{P_0\}'$, as, given $A \in \mathcal{R}(\Gamma)$, $\alpha_g(A) = A$ for all $g \in H$ if and only if $V_g A = A V_g$ for all $g \in H$.

To show $\mathcal{R}_0 \supset \mathcal{R}(\Gamma) \cap \{P_0\}'$ we need the structure of the $\mathcal{R}(\Gamma)$. We only need to consider products of ribbon operators, since such products span a dense subset of $\mathcal{R}(\Gamma)$. Let $F \in \mathcal{R}(\Gamma)$ be a product of ribbon operators with ribbons localised in Γ. Then for each $g \in H$ we have that $V_g F = \gamma(g) F V_g$ where $\gamma(g) \in \mathbb{C}$ is a phase (see [FN15]). Assume now that F commutes with P_0, i.e. $\sum_{g \in H} [F, V_g] = 0$. But this is only possible if F commutes with all summands and hence $F \in \mathcal{R}_0$. To see this note that $\sum_{g \in H} F V_g = \sum_{g \in H} \gamma(g) F V_g$. Since ribbon operators in F are unitary this implies $\sum_{g \in H} V_g = \sum_{g \in H} \gamma(g) V_g$ and hence $\alpha(g) = 1$ for all $g \in H$. By induction we find that if F is a sum of products of ribbon operators then $[P_0, F] = 0$ implies $[V_g, F] = 0$ for each $g \in H$. $\square$

The previous discussion then implies the following.

PROPOSITION 4.3.9:
With $\Gamma \in \mathbb{C}^2$ and $\mathcal{R}(\Gamma)$, $\widehat{\mathcal{R}}(\Gamma)$, $\mathcal{R}_0$, and $\mathcal{E}_0$ as before, we have that

- $\mathcal{E}_0 : \mathcal{R}(\Gamma) \to \mathcal{R}_0$ *is a normal conditional expectation with index* $\mathrm{Ind}(\mathcal{E}_0)$,
- $\mathcal{R}_0$ *is a factor, and* $\mathcal{R}_0 \subseteq \mathcal{R}(\Gamma)$ *is irreducible,*
- *for all* $\chi \in \hat{H}$ *we have* $\widehat{\mathcal{R}}(\Gamma) \cong \langle \mathcal{R}(\Gamma), P_\chi \rangle$,
- *for any* $\chi \in \hat{H}$ *the map* $\mathcal{R}_0 \ni A \mapsto AP_\chi \in P_\chi \widehat{\mathcal{R}}(\Gamma) P_\chi$ *is a* $*$*-isomorphism.*

Summarising this, $\mathcal{R}_0 \subseteq \mathcal{R}(\Gamma) \subseteq \widehat{\mathcal{R}}(\Gamma)$ *with the normal conditional expectations* $\mathcal{E}$ *and* $\mathcal{E}_0$ *constitute a basic construction.*

Proof. All points directly follow from the preceding discussion except for the basic construction. But this follows from [Kos98, Theorem 3.12]. □

The normal conditional expectation $\mathcal{E} : \widehat{\mathcal{R}}(\Gamma) \to \mathcal{R}(\Gamma)$ is a normal, linear, completely positive [NTU60] and unit preserving map.

DEFINITION 4.3.10:
A linear map $E : \mathfrak{M} \to \mathfrak{N}$ *between von Neumann algebras is called a* channel, *if* E *is normal, unit preserving and completely positive.*

Thus the conditional expectation $\mathcal{E}$ is a channel[3] with input system $\mathcal{R}(\Gamma)$ and output $\widehat{\mathcal{R}}(\Gamma)$. Channels from a von Neumann algebra $\mathfrak{M}$ into the bounded operators $\mathcal{B}(\mathcal{K})$ of some Hilbert space $\mathcal{K}$ can be represented as a composition of an isometric embedding of $\mathfrak{M}$ into a larger Hilbert space $\mathcal{L}$ and a normal representation of $\mathfrak{M}$ on $\mathcal{L}$. This is the content of a theorem by Stinespring ([Pau13, Theorem 4.1], see also [Tak79, Theorem IV.3.6] for von Neumann algebras).

THEOREM 4.3.11:
Let $\mathcal{A}$ *be a unital* C^**-algebra and let* $E : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ *be a completely positive and unit preserving map. Then there exists a Hilbert space* $\mathcal{L}$*, a representation* $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{L})$ *and an isometry* $U : \mathcal{K} \to \mathcal{L}$ *such that for all* $A \in \mathcal{A}$

$$E(A) = U^* \pi(A) U. \tag{4.8}$$

[3] For a physical motivation for the definition of channels, see [NC09; Wil11; Key02]. The definition for von Neumann algebras can be found, for instance, in [Kri+06].

If $\pi(\mathcal{A})U\mathcal{K}$ is dense in $\mathcal{L}$, then the triple $(\pi, U, \mathcal{L})$ is unique up to unitaries, i.e. if $(\pi', U', \mathcal{L}')$ is another such triple satisfying equation (4.8)*, then there exists a unitary $V : \mathcal{L} \to \mathcal{L}'$ such that $VU = U'V$ and $U\pi(A) = \pi'(A)U'$ for all $A \in \mathcal{A}$.*

This representation of channels (or more general, completely positive maps) can be understood as a generalisation of the GNS representation.

Definition 4.3.12:
Let $E : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ be a completely positive and unit preserving map. A triple $(\pi, U, \mathcal{L})$ obtained from Theorem 4.3.11 is called Stinespring dilation*. The triple $(\pi, U, \mathcal{L})$ is called* minimal Stinespring dilation*, if $\pi(\mathcal{A})U\mathcal{K}$ is norm dense in $\mathcal{L}$.*

Let $E : \mathfrak{M} \to \mathcal{B}(\mathcal{K})$ be a channel, and let $E_* : \mathcal{B}(\mathcal{K})_* \to \mathfrak{M}_*$ be the unique completely positive map given by $E_*(\omega)(A) = \omega(E(A))$ for all $A \in \mathfrak{M}$ and all $\omega \in \mathcal{B}(\mathcal{K})_*$. If ω is a normal state on $\mathcal{B}(\mathcal{K})$ describing the preparation of a physical system, then $E_*(\omega)$ is the state of the system after undergoing the evolution given by the channel. If ω was initially pure, this is not necessarily true anymore for $E_*(\omega)$. From a more information theoretical point of view, the quantum channel E describes how information is transmitted from the input system $\mathcal{B}(\mathcal{H})$ to the output system $\mathfrak{M}$. The Stinespring dilation theorem can now be interpreted as that this evolution can be understood as an isometric interaction of the system with an environment, i.e. the evolution is given by the coupling of the system to the environment $\mathcal{L}$ via the isometry U. Due to the coupling to the environment information can be lost to the environment. The channel describing this is called the complementary channel.

Definition 4.3.13:
Let $E : \mathfrak{M} \to \mathcal{B}(\mathcal{K})$ be a quantum channel, and let $(\pi, U, \mathcal{L})$ be a Stinespring dilation of E. The channel $E^c_{(\pi,U,\mathcal{L})} : \pi(\mathfrak{M})' \to \mathcal{B}(\mathcal{K})$ given by

$$E^c_{(\pi,U,\mathcal{L})}(A) := UAU^*,$$

for all $A \in \pi(\mathfrak{M})'$ is called complementary channel of E w.r.t. $(\pi, U, \mathcal{L})$.

For conditional expectations $E : \mathfrak{M} \to \mathfrak{N}$, with $\mathfrak{N} \subseteq \mathfrak{M}$ acting on $\mathcal{H}$, the Stinespring dilation was characterised in [NTU60].

Now we come back to the situation on Kitaev's quantum double model for a finite abelian group G. In our case, the $*$-isomorphism $\Phi : \mathcal{R}(\Gamma) \rtimes_\alpha H \to \widehat{\mathcal{R}}(\Gamma)$ constructed in Theorem 4.2.9 provides us with the minimal Stinespring dilation.. Recall, that the Hilbert space $\mathcal{H}$ on which $\mathcal{R}(\Gamma)$ and $\widehat{\mathcal{R}}(\Gamma)$ act is the Hilbert space obtained by the GNS construction from the unique translationally invariant ground space ω. Let $U : \mathcal{H} \to \mathcal{H} \otimes \ell^2(H)$ be given by

$$U\xi = \xi \otimes \delta_e,$$

where e is the unit of G, and $\xi \in \mathcal{H}$. Then one can check that U is an isometry, and the adjoint is given by

$$U^*(\xi \otimes \delta_g) = \delta_{g,e}\xi,$$

with $\xi \in \mathcal{H}$.

Proposition 4.3.14:
Let $\Gamma \in \mathcal{C}^2$, and let $\mathcal{E} : \widehat{\mathcal{R}}(\Gamma) \to \mathcal{R}(\Gamma)$ and $U : \mathcal{H} \to \mathcal{H} \otimes \ell^2(H)$ as before. Let $\Phi : \mathcal{R}(\Gamma) \rtimes_\alpha H \to \widehat{\mathcal{R}}(\Gamma)$ be the $*$-isomorphism constructed in Theorem 4.2.9. Then the tuple $(V, \mathcal{H}', \Phi^{-1})$ is a minimal Stinespring dilation for $\mathcal{E}$, i.e.

- $\forall X \in \widehat{\mathcal{R}} : \mathcal{E}(X) = U^*\Phi^{-1}(X)U,$
- *the space $\Phi^{-1}(\widehat{\mathcal{R}})U\mathcal{H}$ is dense in $\mathcal{H}'$.*

Proof. First of all note, that $\mathcal{E}$ is a channel [NTU60]. Next, we recall the definition of Φ. Given $X \in \widehat{\mathcal{R}}$ we can write it as $X = \sum_{g\in G} A_g V_g$ with $A_g \in \mathcal{R}$ and the unitaries V_g from the unitary representation $V : G \to \mathcal{B}(\mathcal{H})$. The corresponding outer action of G on $\mathcal{R}$ is denoted by α, i.e. for $A \in \mathcal{R}$ and $g \in G$ it is given by $\alpha_g(A) = V_g A V_g^*$.

The crossed product $\mathcal{R}(\Gamma) \rtimes_\alpha H$ on $\mathcal{H} \otimes \ell^2(H)$ is spanned by elements of the form $X = \sum_{g\in H}(A_g \otimes \mathbb{1})(V_g \otimes \lambda_g)$ where $(A_g)_{g\in H}$ is some indexing of elements of $\mathcal{R}(\Gamma)$ and λ is the left regular representation of H on $\ell^2(H)$. The $*$-isomorphism Φ is then given as

$$\Phi(\sum_{g\in H}(A_g \otimes \mathbb{1})(V_g \otimes \lambda_g)) = \sum_{g\in H} A_g V_g,$$

and its inverse is given by

$$\Phi^{-1}(\sum_{g\in H} A_g V_g) = \sum_{g\in H}(A_g \otimes \mathbb{1})(V_g \otimes \lambda_g).$$

With $Y = \sum_{g\in H} A_g V_g$ we then have

$$\Phi^{-1}(Y)U\xi = \sum_{g\in H}(A_g V_g \xi) \otimes \delta_g \tag{4.9}$$

and thus

$$U^*\Phi^{-1}(Y)U\xi = A_e\xi.$$

This holds for all $\xi \in \mathcal{H}$ and since elements of the form $\sum_{g\in H} A_g V_g$ form a dense subalgebra of $\widehat{\mathcal{R}}(\Gamma)$ we arrive at

$$\mathcal{E}(Y) = U^*\Phi^{-1}(Y)U$$

for all $Y \in \widehat{\mathcal{R}}(\Gamma)$. The denseness of $\Phi^{-1}(\widehat{\mathcal{R}})V\mathcal{H}$ in $\mathcal{H}'$ directly follows from equation (4.9). □

This structure allows to draw a connection to error correction[4] and private subspaces. The following definition is due to reference [Cra+15].

DEFINITION 4.3.15:
Let $\mathcal{K}$ be a Hilbert space, $\mathfrak{M}$ a von Neumann algebra, $E : \mathfrak{M} \to \mathcal{B}(\mathcal{K})$ a quantum channel and $P \in \mathcal{B}(\mathcal{K})$ an orthogonal projection. A von Neumann algebra $\mathfrak{N} \subset \mathcal{B}(P\mathcal{K})$ is called private for E w.r.t. P, *if*

$$\mathrm{Ad}_P \circ E(\mathfrak{M}) \subseteq \mathfrak{N}'.$$

The von Neumann algebra $\mathfrak{N}$ is said to be correctable for E w.r.t. P, *if there exists a quantum channel $R : \mathfrak{N} \to \mathfrak{M}$ such that*

$$\mathrm{Ad}_P \circ E \circ R = \mathrm{id}_{\mathfrak{N}}.$$

We say that $\mathfrak{N}$ is private (correctable) for E, *if $P = \mathbb{1}$.*

Proposition 4.3.14 allows to apply [Cra+15, Theorem 4.7] to the present situation.

PROPOSITION 4.3.16:
Given $\mathcal{E} : \widehat{\mathcal{R}}(\Gamma) \to \mathcal{R}(\Gamma)$ as before, and let $(V, \mathcal{H}', \Phi^{-1})$ be the minimal Stinespring dilation from Theorem 4.3.14. Then the von Neumann algebra $\mathcal{R}(\Gamma)'$ is private for $\mathcal{E}$. Moreover, $\mathcal{R}(\Gamma)'$ is correctable for the complementary channel $\mathcal{E}^c$ w.r.t. this dilation, and the correction map $\mathcal{R} : \mathcal{R}(\Gamma)' \to \mathcal{B}(\mathcal{H}')$ is given by $\mathcal{R}(X) = X \otimes \mathbb{1}$, $X \in \mathcal{R}(\Gamma)'$.

Proof. By definition, the von Neumann subalgebra $\mathcal{R}(\Gamma)' \subset \mathcal{B}(\mathcal{H})$ is private for the channel $\mathcal{E}$ if $\mathcal{E}(\widehat{\mathcal{R}}) \subset \mathcal{R}(\Gamma)''$. Since $\mathcal{E}(\widehat{\mathcal{R}}) = \mathcal{R}(\Gamma)$ this is fulfilled. By [Cra+15,

[4] for further reading on error correction and its generalisation to von Neumann algebras, see [KL95; BNS98; KLP05; BKK07b; BKK07a; Chu+11]

Theorem 4.7] $\mathcal{R}(\Gamma)'$ is then also correctable for the complementary channel $\mathcal{E}^c$ of $\mathcal{E}$ w.r.t. $(V, \mathcal{H}', \Phi^{-1})$. That is, there exists a channel $\mathcal{R} : \mathcal{R}(\Gamma)' \to \mathcal{B}(\mathcal{H}')$ such that $\mathcal{E}^c \circ \mathcal{R} = \mathrm{id}_{\mathcal{R}(\Gamma)'}$. It is now easy to see that the channel $\mathcal{R}(\Gamma) : \mathcal{R}(\Gamma)' \to \mathcal{B}(\mathcal{H}')$ defined by $\mathcal{R}(X) = X \otimes \mathbb{1}$ fulfills exactly this criterion. □

4.4 Total Quantum Dimension and Secret Sharing

In what follows we discuss the structure of the inclusion $\mathcal{R}(\Gamma) \subseteq \widehat{\mathcal{R}}(\Gamma)$ from an operational point of view. In particular it is possible to show that the Jones-Kosaki-Longo index $\mathrm{Ind}(\mathcal{E})$ is associated to a structure that is very similar to the structure found for the total quantum dimension in the analysis of the quantum double models for finite system sizes. These results are published in [FNO17] and the purpose of this section is to give a brief review of the important statements.

The conditional expectation $\mathcal{E} : \mathcal{R}(\Gamma) : \widehat{\mathcal{R}}(\Gamma)$ is unique, satisfies $\mathcal{E}(X^*X) \geqslant \frac{1}{|G|^2}X^*X$ for all $X \in \widehat{\mathcal{R}}(\Gamma)$, and its index is $\mathrm{Ind}(\mathcal{E}) = |G|^2$. From Theorem 4.2.11 (for more details, see the discussion after Definition 3.5 in [Naa13a]) it follows that the index of $\mathcal{E}$ is equal to the square of the total quantum dimension $\mathcal{D}^2$ of $\mathcal{D}(G)$, i.e.

$$\mathrm{Ind}(\mathcal{E}) = \sum_a d(\rho_a)^2,$$

where the sum goes over the charge sectors, and $d(\rho_a)$ is the statistical dimension of representatives of the sectors.

The structure of the inclusion $\mathcal{R}(\Gamma) \subset \widehat{\mathcal{R}}(\Gamma)$ is related to the occurrence of a *secret sharing scheme* [FNO17]. These schemes share information between several parties in a way that the information is hidden from an adversary. Equivalently, secret sharing schemes describe how to share a secret between several shareholders such that each shareholder alone is not capable of recovering the secret from their share, but it needs a minimal number of shareholders to collaborate in order to decode the secret [CGL99; Got00].

In quantum systems such schemes are described by a partition of the system into subsystems, *part*, a subset of system's Hilbert space, the *code space*, and a conditions on the parties and on the vectors of the code space [Got00]. Each party has only access to observables on their part of the system. Furthermore, parts are distinguished into authorised parts, which can access the secret using operations on their part of the system, and unauthorised parts, which cannot decode the information with local operations on their part of the system. Decoding operations is understood as being able to distinguish the code states. Depending on whether the information encoded in the code space is quantum or classical, the code space is a subspace of the system's Hilbert space, or a set of orthogonal states. This also means, that if we enlarge an unauthorised part by adding an authorise one, the resulting new part is authorised.

In our case, the authorised parts are given by the cones Λ_1 and Λ_2, where $\Gamma = \Lambda_1 \cup \Lambda_2 \in \mathcal{C}^2$. There is only one unauthorised part which is Γ^c. The local operations of each of theses systems are $\mathcal{R}(\Lambda_1)$, $\mathcal{R}(\Lambda_2)$, and $\mathcal{R}(\Gamma^c)$. The code states are given by classes of vectors $\mathcal{R}(\Gamma)V_g\Omega$, where Ω is the cyclic state implementing the ground state ω_0, and $V_{g_{g\in H}}$ is a fixed set of charge transporters from Λ_1 to Λ_2. Note, that for each $g, h \in H$ and $A, B \in \mathcal{R}(\Gamma)$ we have

$$(AV_g\Omega, BV_h\Omega) = \delta_{h,g}(AV_g\Omega, BV_g\Omega),$$

by the construction of the charge transporters and the commutation relations of ribbon operators. This implies that the vectors $AV_g\Omega$ and $BV_h\Omega$ are not superposable with respect to observables in $\mathcal{R}(\Lambda_1)$ and $\mathcal{R}(\Lambda_2)$: If ϕ is a phase, and $\psi := \frac{1}{\sqrt{2}}(AV_g\Omega + e^{i\phi}BV_h\Omega)$, then

$$2(\psi, C\psi) = (AV_g\Omega, CAV_g\Omega) + (BV_h\Omega, CBV_h\Omega)$$

for all $C \in \mathcal{R}(\Lambda_1)$, and the same holds true for all $C \in \mathcal{R}(\Lambda_2)$. In addition, for each $A, B \in \mathcal{R}(\Gamma)$, there are projections in $\mathcal{R}(\Lambda_1)$ and $\mathcal{R}(\Lambda_2)$ which can distinguish the vectors $AV_g\Omega$ and $BV_h\Omega$ for each $g, h \in H$: To construct the charge transporters one needs to specify fixed sites $s_i \in \Lambda_i$, $i = 1, 2$. If $A, B = \mathbb{1}$ it suffices to choose the charge projections in Definition 2.5.1 at s_1 and s_2. If A, B are products of ribbon operators, then can choose loops around the regions in Λ_1 and Λ_2 in which A and B act non-trivially and takes the projections onto the charge in these regions (for such operators, c.f. [BM08]). Finally, by locality we have for any operator $E \in \mathcal{R}(\Gamma^c)$, any $g, h \in H$ and $A, B \in \mathcal{R}(\Gamma)$:

$$(AV_g\Omega, EBV_h\Omega) = \delta_{h,g}(AV_g\Omega, EBV_g\Omega).$$

This means that the code states are correctable with respect to errors imposed by operators in $\mathcal{R}(\Gamma^c)$ [Got00]. Hence, encoding information by applying the charge transporters to vectors in $\mathcal{R}(\Gamma)\Omega$ results in vectors from the code space. These states cannot be distinguished by an adversary having only access to observables in $\mathcal{R}(\Gamma^c)$. Hence, if restricted to the algebra $\mathcal{R}(\Gamma^c)$ these states coincide.

The above discussion allows to conclude that the cones Λ_1 and Λ_2 can be viewed as authorised parts of a secret sharing scheme for classical information, and Γ^c as unauthorised. The code states are in fact the classes of vectors $\mathcal{R}(\Gamma)V_g\Omega$ with $g \in H$. The classical information is encoded on states from $\mathcal{R}(\Gamma)\Omega$ by using the charge transporters $\{V_g \mid g \in H\}$. The fact that $\widehat{\mathcal{R}}(\Gamma) \cong \mathcal{R}(\Gamma) \rtimes_\alpha H$, or $\mathrm{Ind}(\mathcal{E}) = |G|^2$ implies that this choice of code states is maximal in the sense, that there are no additional code states we can add without violating the conditions on the secret sharing scheme [FNO17]. In addition, we have that $\mathcal{R}(\Gamma^c) \subset \mathcal{R}(\Gamma)'$ and this inclusion is strict, since $\mathcal{R}(\Gamma^c) = \widehat{\mathcal{R}}(\Gamma)$. For instance, the projections onto the total charges in Λ_1 and Λ_2, respectively, are contained in $\mathcal{R}(\Gamma)'$, but not in $\mathcal{R}(\Gamma^c)$. An adversary having access only to the observables in the algebra $\mathcal{R}(\Gamma^c)$ therefore cannot determine the total charge in either cone Λ_1 and Λ_2. However, it is not so clear to connect the privacy of $\mathcal{R}(\Gamma)'$ with respect to the channel $\mathcal{E}$ directly to this picture. However, in both situations we find, as discussed below, that the index characterised the amount of information that can be hidden from an adversary. For private subspaces in secret sharing schemes in finite dimensions this is much clearer [KKS08].

There is yet another facet to this. The logarithm of the index of the inclusion $\mathcal{R}(\Gamma) \subset \widehat{\mathcal{R}}(\Gamma)$ can be related to a relative entropy [PP86; Hia90]. This relative entropy can be interpreted as a the amount of information that can be hidden in the secret sharing scheme [FNO17]. We briefly review the according discussion in reference [FNO17] to which we refer for the details. Given a normal state ϕ on $\widehat{\mathcal{R}}(\Gamma)$, then the relative entropy between $\widehat{\mathcal{R}}(\Gamma)$ and $\mathcal{R}(\Gamma)$ w.r.t. ϕ is given as

$$H_\phi(\widehat{\mathcal{R}}(\Gamma)|\mathcal{R}(\Gamma)) = \sup_{\phi_i} \sum_i [S(p_i\phi_i, \phi) - S(p_i\phi_i \restriction \mathcal{R}(\Gamma), \phi \restriction \mathcal{R}(\Gamma))],$$

where the supremum is taken over all finite families $(\phi_i)_{i\in I}$ of normal states on $\widehat{\mathcal{R}}(\Gamma)$ such that $\phi = \sum_i p_i\phi_i$, where $(p_i)_{i\in I}$ is a probability distribution over the finite index set I. The relative entropy $S(p_i\phi_i, \phi)$ used here is a generalisation of the relative entropy of finite dimensional systems (c.f. [OP93])[5]. It can be shown that $H_\phi(\widehat{\mathcal{R}}(\Gamma)|\mathcal{R}(\Gamma))$ is non-negative and can be interpreted as the additional amount of information an encode in the state ϕ, if we use operations of $\widehat{\mathcal{R}}(\Gamma)$ instead of $\mathcal{R}(\Gamma)$ [FNO17]. The relative entropy between $\widehat{\mathcal{R}}(\Gamma)$ and $\mathcal{R}(\Gamma)$ w.r.t. to the conditional

[5]The order of the arguments in $S(p_i\phi_i, \phi)$ is reversed comparted to the definition in [OP93].

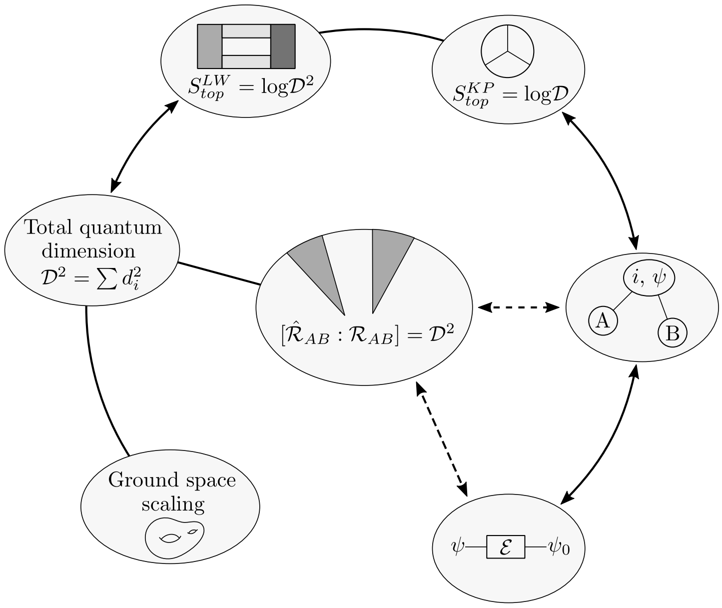

Figure 4.2: Relation between different concepts related to the total quantum dimension in Kitaev's quantum double model for finite abelian groups. The dotted lines are the relations that are established in [FNO17]. The total quantum dimension can be obtained from combinations entanglement entropies of certain regions [KP06; LW05]. It also can be obtained from the Jones-Kosaki-Longo index. Both, the topological entanglement entropy and the logarithm of the index can be interpreted in terms of a secret sharing scheme as the amount of information that can be hidden from an adversary.

expectation $\mathcal{E}$ is defined as

$$H_{\mathcal{E}}(\widehat{\mathcal{R}}(\Gamma), \mathcal{R}(\Gamma)) := \sup_{\phi} H_{\phi}(\widehat{\mathcal{R}}(\Gamma), \mathcal{R}(\Gamma)),$$

and the supremum is taken over all faithful normal states ϕ on $\widehat{\mathcal{R}}(\Gamma)$ such which fulfill $\mathcal{E} \circ \phi = \phi$. It then follows that

$$H_{\mathcal{E}}(\widehat{\mathcal{R}}(\Gamma), \mathcal{R}(\Gamma)) = \log\left(\mathrm{Ind}(\mathcal{E})\right),$$

which also proves that there are states ϕ on $\widehat{\mathcal{R}}(\Gamma)$ such that $H_{\phi}(\widehat{\mathcal{R}}(\Gamma)|\mathcal{R}(\Gamma)) > 0$. The relative entropy $H_{\mathcal{E}}(\widehat{\mathcal{R}}(\Gamma), \mathcal{R}(\Gamma))$, and therefore the total quantum dimension $\mathcal{D}$, give us an upper bound on the amount of information we can possibly encode in a state on $\widehat{\mathcal{R}}(\Gamma)$ if we use the additional operators in this algebra which are not contained in the algebra $\mathcal{R}(\Gamma)$. Together with the discussion of the secret sharing scheme, the total quantum dimension (more precisely, its logarithm) can be interpreted as the maximum amount of information that can be hidden using the charge transporters.

Let us summarise this discussion. The index of the inclusion $\mathcal{R}(\Gamma) \subset \widehat{\mathcal{R}}(\Gamma)$ is equal to the square of total quantum dimension $\mathcal{D}$ of the modular tensor category of finite dimensional representations of $\mathcal{D}(G)$. At the same time, it can be interpreted as the dimension of the code space of a secret sharing scheme for classical information. Furthermore, the relative entropy of the inclusion $\mathcal{R}(\Gamma) \subset \widehat{\mathcal{R}}(\Gamma)$ with respect to the channel $\mathcal{E}$ is equal to the logarithm of the index, i.e. $H_{\mathcal{E}}(\widehat{\mathcal{R}}(\Gamma), \mathcal{R}(\Gamma)) = \log \mathcal{D}^2$, hence giving an upper bound on the information that can be hidden in the secret sharing scheme. This is strikingly similar to the properties one finds in the analysis of the quantum double models in finite system sizes. There the total quantum dimension of $\mathrm{Rep}_f(\mathcal{D}(G))$ can be obtained from the the topological entanglement entropy S_{top}. The topological entanglement entropy of the quantum double model is a zero order correction in the area law of the ground state. It can be interpreted as the reduction of the von Neumann entropy of the ground state restricted to a finite region due to the presence of anyonic excitations [KP06; LW06]. For the ground state of Kitaev's quantum double models the topological entanglement entropy is proportional to $\mathcal{D}$ [KP06; LW06]. Moreover, it was shown, that S_{top} can be interpreted as the rate of a secret sharing scheme [KFM16], though the scheme is different from ours. The relation between these different concepts discussed above is depicted in Figure 4.2.

5 Summary and Outlook

Due to their stability against local perturbations and their potential to implement unitary gates for quantum computation, topologically ordered systems have gained broad interest as possible candidates for quantum memories and fault tolerant quantum computation. Among the different models that are usually investigated to better understand topological order, the quantum double models proposed by Kitaev are the most prominent examples whose structure is relatively simple and well understood, and which combine many of the properties topological ordered systems ought to have [Kit03]. It is this structure which makes this class of models also appealing as toy models in the study of topologically ordered systems in the thermodynamic limit.

The name "quantum double models" stems from the fact that the local excitations above the ground state are characterised by the irreducible representations of Drinfel'd's quantum double $\mathcal{D}(G)$ of a finite group G that enters the construction of the model [Kit03]. Thus, the local excitations are anyons and their statistics is very different from fermions and bosons. If the group underlying the construction is finite and abelian, an assumption we make in large parts of this thesis, the irreducible representations are then labelled by elements of the group $H = G \times \hat{G}$, where $\hat{G}$ is the group of irreducible characters of G. The fusion and braiding rules are given by the multiplication rules of the group H [FN15]. Depending on the topology of the surface in which Kitaev's quantum double model is embedded, the ground state space is degenerate and the number of ground states depends only on the surface's genus. On the plain square lattice, i.e. in the thermodynamic limit, there only exists one translationally invariant ground state, and this state is pure [FN15; Naa12a]. There are other ground states of the dynamics which are associated with single excitations, but are neither translationally invariant nor necessarily pure [CNN16].

Since the structure of local excitations in these models is well understood in terms of the ribbon operators, this allows for a straightforward construction of states in the thermodynamic limit that contain single excitations: Local excitations always occur in conjugate pairs at the end of ribbons, and a single excitation can be obtained by fixing one end and moving the other end to infinity. This results in states containing cone-like localised excitations, and one can show that the representations of the quasilocal algebra $\mathcal{A}$ associated to these states do not depend on the ribbon or the position of the excitations up to unitary equivalence. In fact, these representations are representatives of equivalence classes of representations of $\mathcal{A}$ that are selected by a DHR-like superselection criterion.

The equivalence classes that are distinguished by the selection criterion are called superselection sectors and describe the elementary charges of the model. A representation obeys the superselection criterion, if it is equivalent to the representation of the translationally invariant ground state outside of cones. It then turns out that the explicitly constructed representations that contain single excitations are representatives of distinct superselection sectors, if the label of the excitations are different. However, the representations constructed in this way are *a priori* representatives of only some classes. In order to fully classify the sectors one needs an additional technical property called Haag duality for cone algebras in the ground state representation. It allows to show that the properties of an explicitly constructed representation carry over to its whole equivalence class. In addition, Haag duality can be used to define braiding and fusion on the superselection sectors, which then can be used to show that the sectors containing the explicitly constructed representations form a braided monoidal category. The latter can then be shown to be isomorphic to the modular tensor category of irreducible representations of $\mathcal{D}(G)$.

By locality, operators localised inside a cone commute with the operators localised in the complement of the cone. Haag duality is the converse statement: operators that commute with all operators localised in the complement of a cone are in fact themselves localised inside the cone. A large part of this thesis is devoted to show that Haag duality holds for cone algebras in the GNS representation of the translationally invariant ground state of the quantum double model in the case of finite abelian groups. This is achieved by understanding the structure of the local excitations created inside and outside of a cone Λ. In particular, the proof relies on the characterisation of a certain Hilbert subspace $\mathcal{H}_\Lambda$ of the Hilbert space $\mathcal{H}$. Here $\mathcal{H}$ is the Hilbert space obtained from the GNS construction of the translationally invariant ground state ω_0. This subspace $\mathcal{H}_\Lambda$ contains excitations created only inside the cone Λ; excitations localised outside Λ are contained in its orthogonal complement. This allows to restrict both the algebra $\pi_0(\mathcal{A}(\Lambda))$ of operators localised inside the cone and the algebra $\pi_0(\mathcal{A}(\Lambda^c))'$ of operators commuting with those in the complement of Λ, to $\mathcal{H}_\Lambda$. Here the prime denotes the commutant of the algebra in $\mathcal{B}(\mathcal{H})$ and π_0 is the GNS representation of ω_0. A result by Rieffel and van Daele [RD75] allows then to conclude Haag duality for the cone algebras, i.e. the equality $\pi(\mathcal{A}(\Lambda))'' = \pi(\mathcal{A}(\Lambda^c))'$.

That Haag duality is not necessarily fulfilled for any subset of the square lattice is illustrated by the algebra of observables associated to the union $\Gamma = \Lambda_1 \cup \Lambda_2$ of two disjoint cones Λ_1, Λ_2. There Haag duality fails: the commutant of the algebra of observables localised in Γ^c is strictly larger than the algebra of observables localised in Γ. A recent result [Naa13a] shows that the Jones-Kosaki-Longo index of the corresponding inclusion of von Neumann algebras allows to give an upper bound on the number of superselection sectors which the theory supports. The second part of this thesis is devoted to the analysis of the algebra $\pi_0(\mathcal{A}(\Gamma^c))'$. This algebra is generated by $\pi_0(\mathcal{A}(\Gamma))''$ and the charge transporters from Λ_1 to Λ_2. The charge transporters form a unitary representation of the group $H = G \times \widehat{G}$, and as in the toric code [Naa13a], the algebra $\pi_0(\mathcal{A}(\Gamma^c))'$ is shown to be isomorphic to the crossed product of $\pi_0(\mathcal{A}(\Gamma))''$ with the group H. The proof relies on the tools and the understanding of the local excitations developed in the proof of Haag duality. These tools can also be used to conclude that the algebra $\pi_0(\mathcal{A}(\Gamma))''$ is isomorphic to the tensor product $\pi_0(\mathcal{A}(\Lambda_1))'' \otimes \pi_0(\mathcal{A}(\Lambda_2))$ if the cones Λ_1 and Λ_2 are sufficiently separated from each other. This property is called *approximate split property* and is proved by explicitly constructing a unitary which implements the equivalence. As mentioned earlier, the von Neumann algebra $\pi_0(\mathcal{A}(\Gamma^c))'$ is isomorphic to the crossed product of $\pi_0(\mathcal{A}(\Gamma))''$ with the group H. This implies that the Jones-Kosaki-Longo index of the inclusion $\pi_0(\mathcal{A}(\Gamma))'' \subset \pi_0(\mathcal{A}(\Gamma^c))'$ is equal to $|G|^2$, which in turn is the square of total quantum dimension $\mathcal{D}$ of $\mathcal{D}(G)$. As this gives an upper bound to the number of superselection sectors [Naa13a], this proves that *all* superselection sectors are completely characterised by the finite dimensional representations of $\mathcal{D}(G)$ (see also [KLM01] for a similar result in the setting of conformal field theory).

The inclusion $\pi_0(\mathcal{A}(\Gamma))'' \subset \pi_0(\mathcal{A}(\Gamma^c))'$ carries more structure: The conditional expectation implementing the restriction of the larger algebra $\pi_0(\mathcal{A}(\Gamma^c))'$ to $\pi_0(\mathcal{A}(\Gamma))''$ is in fact a quantum channel. The commutant $\pi_0(\mathcal{A}(\Gamma))'$ of the observables localised in Γ is then a private subalgebra with respect to $\mathcal{E}$. This seems to be related to the existence of a secret sharing scheme for classical information associated with the inclusion $\pi_0(\mathcal{A}(\Gamma))'' \subset \pi_0(\mathcal{A}(\Gamma^c))'$. If Ω is the vector implementing the translationally invariant ground state, the charge transporters contained in the larger algebra can be used to prepare states from the ground state vector Ω that are indistinguishable for operators localised in the complement of Γ^c, i.e. contained in the algebra $\pi_0(\mathcal{A}(\Gamma^c))''$. On the other hand, these states can be distinguished by operators localised in the cones Λ_1 and Λ_2. The number of states, or rather classes of states, that can be prepared in this way is given by the index, hence the total quantum dimension $\mathcal{D}$. This is complemented by the observation that the logarithm of the Jones-Kosaki-Longo index equal to a relative entropy of the inclusion with respect to $\mathcal{E}$ [Hia90; PP86]. This relative entropy can be interpreted as the additional information one can encode in a state using operators from $\pi_0(\mathcal{A}(\Gamma^c))'$ compared to only using operators from the smaller algebra $\pi_0(\mathcal{A}(\Gamma))''$ [FNO17].

The structure found for the index is very similar to the structure found when analysing the quantum double models in finite system sizes. There the logarithm of the total quantum dimension can be obtained from the topological entanglement entropy, where it characterises a correction of the von Neumann entropy of the ground state restricted to a finite region, due to the presence of anyons in the system [KP06; LW06]. In addition, the topological entanglement entropy, and hence the logarithm of $\mathcal{D}$, can be interpreted as the sharing rate of a secret sharing scheme for certain regions [KFM16]. This suggests that the underlying concept of the total quantum dimension for the thermodynamic limit is the same as for finite system sizes, and sheds some new light on the structure of the total quantum dimension.

5.1 Outlook

As remarked in Section 4.4 the Jones-Kosaki-Longo index in Kitaev's quantum double model for finite abelian groups is accompanied by a structure that is very similar to that found in the analysis of this model in finite system sizes. Judging from the situation in finite system sizes, where it is known that the area law, and hence the topological entanglement entropy, is stable under the quasi-adiabatic continuation [Mar+14], one would expect that the index is stable against perturbations of the Hamiltonian with local interactions. As long as the gap above the ground state energy in the spectrum of the Hamiltonian is not closed, such perturbations do not lead to a phase transition. Furthermore, one would expect that the superselection sectors are invariant. A useful tool in this analysis could be the generalisation of the quasi-adiabatic continuation to the thermodynamic limit [Bac+11]. More precisely, in reference [Bac+11] it was shown that ground states of such perturbed Hamiltonian are automorphically equivalent to the ground states of the original Hamiltonian by a so-called spectral flow. This flow satisfies a Lieb-Robinson bound and therefore maps strictly localised observables to observables that are only quasi-local. As a consequence, single excitations of the resulting deformed Kitaev model can no longer be expected to be cone-like localised. This also suggests that selection criterion in Definition 4.1.5 in this strictly local form is then not valid anymore, and should be replaced by a more suitable one.

It seems possible, but laborious, to proof Haag duality for cone algebras in the ground state representation in the non-abelian case as well. This requires a more detailed understanding of the commutation relations of ribbon operators with focus on relating the appearing sums and coefficients to the fusion coefficients and R-matrix elements of $\mathbf{Rep}_f\mathcal{D}(G)$. Furthermore, it should be possible to use some of the techniques developped for the corresponding proof in the abelian case. We have seen for the quantum doubles model for finite abelian groups that the category of localised endomorphisms is equivalent to the modular tensor category of finite dimensional representations of $\mathcal{D}(G)$. It is expected that this is also the case if the underlying group is non-abelian. However, it is not so clear how to describe the superselection sectors and what the corresponding objects are that play, for instance, the role of the inclusion $\pi_0(\mathcal{A}(\Gamma))'' \subset \pi_0(\mathcal{A}(\Gamma^c))'$ in the abelian case. One problem that arises here is, that the irreducible representations of $\mathcal{D}(G)$ are no longer necessarily one-dimensional, and thus describe non-abelian anyons. Hence, the anyons now have inner degrees of freedom which transform non-trivially under the fusion and braiding. This is indicated by the form of the ribbon operators when labelled by irreducible representations of $\mathcal{D}(G)$ as well as their fusion and braiding rules (c.f. Proposition 2.6.4 and Lemmata 2.6.11 and 2.6.14). A promising approach is to use amplimorphism of the quasilocal algebra $\mathcal{A}$ into the tensor product of $\mathcal{A}$ with finite dimensional matrix algebras. This is discussed in reference [Naa15] in more detail, where such amplimorphisms where constructed. Similar approaches for one-dimensional spin chains can be found in the references [SV93; NS97].

Inclusions $\mathfrak{N} \subset \mathfrak{N}$ of von Neumann algebras with conditional expectation $E : \mathfrak{M} \to \mathfrak{N}$ of finite index provide a large class of examples of von Neumann algebras that are correctable in the sense of reference [Cra+15]: the von Neumann algebra $\mathfrak{N}'$ is private for E, and therefore $\mathfrak{N}'$ is correctable for any complementary channel E^c of E. In the case of crossed products there is more structure. If $\mathfrak{N}$ is a factor, and $\mathfrak{M} = \mathfrak{N} \rtimes_\alpha G$ for some finite group G acting freely on $\mathfrak{N}$, then the subgroups of G are in one-to-one correspondence with intermediate subfactors of $\mathfrak{N} \rtimes_\alpha G$, and for each such subfactor $\mathfrak{L}$ there exists a conditional expectation $E_H : \mathfrak{N} \rtimes_\alpha G \to \mathfrak{L}$ [Cho78]. The commutant $\mathfrak{L}'$ is then private for the associated conditional expectation E_H. One question here is, whether there exists a projection P such that $\mathfrak{N}'$ is private for E_H with respect to P. The observation that the relative entropy of the inclusion $\mathcal{R}(\Gamma) \subset \widehat{\mathcal{R}}(\Gamma)$ with respect to $\mathcal{E}$ characterises the amount of information that is erased from the private subalgebra under the complementary channel, suggests that a similar interpretation exists for general inclusion $\mathfrak{N} \subset \mathfrak{M}$ if there exists a conditional expectation $E : \mathfrak{M} \to \mathfrak{N}$ of finite index. Furthermore, it should be possible to translate this discussion to channels from von Neumann algebras into bounded operators acting on some Hilbert space, since, by [Pau13, Theorem 3.18], channels give rise to conditional expectations onto the algebra of fixed points of the channel.

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