Bounded H^∞-Calculus for a Degenerate Boundary Value Problem

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Abstract

We consider a strongly elliptic second order differential operator $\mathcal{A}$ together with a degenerate boundary operator T of the form $T = \varphi_0\gamma_0 + \varphi_1\gamma_1$, where γ_0 and γ_1 denote the evaluation of a function and its exterior normal derivative, respectively, at the boundary. We assume that $\varphi_0, \varphi_1 \geq 0$ and $\varphi_0 + \varphi_1 \geq c > 0$. We show that a suitable shift of the realization A_T of $\mathcal{A}$ in $L_p(X_+)$ has a bounded H^∞-calculus whenever X_+ is a manifold with boundary and bounded geometry.

Keywords: H^∞-Calculus, no elliptic, maximal regularity

Zusammenfassung

Wir betrachten einen stark elliptischen Differentialoperator zweiter Ordnung $\mathcal{A}$ zusammen mit einem entarteten Randwertoperator T, welche als $T = \varphi_0\gamma_0 + \varphi_1\gamma_1$ gegeben ist. Hierbei sind γ_0 und γ_1 der Einschränkung der Funktion, bzw. der äußeren Normalen Ableitung, auf den Rand. Wir nehmen an, dass $\varphi_0, \varphi_1 \geq 0$ und $\varphi_0 + \varphi_1 \geq c > 0$ erfüllt sind. Unter diesen Voraussetzungen hat eine geeignete Verschiebung der $L_p(X_+)$-Realisierung A_T von $\mathcal{A}$ einen beschränkten H^∞-Kalkül, falls X_+ eine Mannigfaltigkeit mit Rand und beschränkter Geometrie ist.

Schlagworte: H^∞-Kalkül, nicht elliptisch, maximale Regularität

Contents

1 Introduction and the Main Result

Let (X_+, g) be a manifold with boundary and bounded geometry and $(\kappa, U_\kappa, V_\kappa)$ be Fermi-coordinates, for the definition see Section 2.3. We consider a second order differential operator $\mathcal{A}$ locally given by:

$$\mathcal{A}^\kappa = \sum_{1\leq i,j\leq n} a_{ij}^\kappa(x) D_i D_j + \sqrt{-1} \sum_{1\leq i\leq n} b_i^\kappa(x) D_i + c^\kappa(x). \tag{1.1}$$

The coefficients are assumed to be real valued. We call $\mathcal{A}$ M-elliptic if a constant $M > 0$ which does not depend on κ exists such that for all $x \in V_\kappa$ the following estimate holds:

$$M^{-1}|\xi|^2 \leq \sum a_{ij}^\kappa(x)\xi_i\xi_j \leq M|\xi|^2.$$

We say that $\mathcal{A}$ is sufficiently regular if a constant $C > 0$ exists which is independent of κ such that $\|a_{ij}^\kappa(x)\|_{C^\tau(V_\kappa)}$, $\|b_i^\kappa\|_{L_\infty(V_\kappa)}$, and $\|c^\kappa\|_{L_\infty(V_\kappa)}$ are bounded by that constant. After possibly enlarging M we can assume that $C \leq M$. We denote the trace operator by γ_0 and the trace of the exterior normal derivative by γ_1, for more details see Section 2.3. Given a pair of non-negative functions $\varphi_0, \varphi_1 \in C_b^\infty(\partial X_+)$ that satisfy $\varphi_0 + \varphi_1 \geq c > 0$, we define a boundary operator T of the form:

$$T = \varphi_0\gamma_0 + \varphi_1\gamma_1. \tag{1.2}$$

We obtain the classical Dirichlet problem for $\varphi_0 = 1, \varphi_1 = 0$. The choice $\varphi_0 = 0, \varphi_1 = 1$ yields Neumann boundary conditions and Robin problems correspond to the case where φ_1 is nowhere zero. These are the cases in which the Lopatinski-Shapiro ellipticity condition is satisfied, in general this is not the case. We write $\mathcal{A}_+ := r^+\mathcal{A}e^+$, where r^+ denotes the restriction in the sense of distributions and e^+ denotes the extension by zero. We define an unbounded operator A_T that acts like $\mathcal{A}_+$ on the following domain:

$$D(A_T) := \{u \in H_p^2(X_+) : Tu = 0\}.$$

The main result is that a suitable shift of A_T allows a bounded H^∞-calculus. For the definition of the H^∞-calculus see Section 3. In detail the main result is:

Theorem 1.1. *Let (X_+, g) be a manifold with boundary and bounded geometry. Let T be as in (1.2) and A_T be the realisation given above of an M-elliptic sufficiently regular second order differential operator. Then, for every $0 < \vartheta < \pi$ a constant $\nu = \nu(M, |t|_*, \vartheta) \geq 0$ exists such that $A_T + \nu$ allows an $H^\infty(\Sigma_\vartheta)$-calculus in $L_p(X_+)$. Moreover, a constant $C = C(M, |t|_*, \vartheta) > 0$ exists such that for all $f \in H^\infty(\Sigma_\vartheta)$ the following estimate holds:*

$$\|f(A_T)\|_{\mathcal{B}(L_p(X_+))} \leq C\|f\|_{L_\infty(\Sigma_\vartheta)}.$$

The problem of providing a bounded H^∞-calculus has a long history. Let us mention some of the main results in the development and refer to the sources for further reading. The first results in this direction are in the series of papers [41], [40] and [38] by Robert

Seeley. He proves that (systems of) elliptic differential operators have bounded imaginary powers if the underlying manifold has no boundary or the operator is complemented with a boundary operator which satisfies the Lopatinski-Shapiro condition. However, the notion of a bounded H^∞-calculus was not yet established. In fact, this notion was introduced by Alan McIntosh in [29] and [11], first for Hilbert spaces and later with his co-authors for Banach spaces. In [15] and [16], Xuan Thinh Duong established the bounded H^∞-calculus under Seeley's assumptions. According to the famous result of Giovanni Dore and Alberto Venni, see [14], the existence of a bounded H^∞-calculus implies maximal regularity.
The assumption of smooth coefficients is too restrictive for applications. This led to further efforts to reduce the smoothness assumptions, see for instance [31], [5], and [12]. In [12], the existence of a bounded H^∞-calculus was established for elliptic systems on compact manifolds under the same sufficient regularity assumptions we impose here. As pointed out earlier, the boundary operator T does in general not satisfy the Lopatinski-Shapiro condition. Thus, the boundary value problem is not elliptic. Until now, the operator A_T has been known to generate an analytic semi-group, see [43]. It is well-known that this is necessary but not sufficient for the existence of a bounded H^∞-calculus.

1.1 Outline

In Section 2, we define Bessel potential and Besov spaces on euclidean (half) space and manifolds with (boundary and) bounded geometry and collect the relevant results for these spaces, including real- and complex interpolation results, existence of a bounded extension- and trace operator, and boundedness of multiplication operators. In Section 3, we introduce the notion of bounded H^∞-calculus and summarise some known perturbation results. We also sketch the connection to bounded imaginary powers and maximal regularity. The following technical result is essential for the proof of the main result, the proof is given in Section 5.4.

Theorem 1.2 (Auxiliary Result)**.** *Let $X_+ = \mathbb{R}^n_+$ and A_T be given as in Theorem 1.1. Moreover, we assume that the coefficients of A_T are smooth and bounded. Then, for every $0 < \vartheta < \pi$ a constant $\nu = \nu(|a|_*, M, |t|_*, \vartheta) \geq 0$ exists such that $A_T + \nu$ allows an $H^\infty(\Sigma_\vartheta)$-calculus in $L_p(\mathbb{R}^n_+)$. Moreover, a constant $C = C(|a|_*, M, |t|_*, \vartheta) > 0$ exists such that for all $f \in H^\infty(\Sigma_\vartheta)$ the following estimate holds:*

$$\|f(A_T)\|_{\mathcal{B}(L_p(X_+))} \leq C\|f\|_{L_\infty(\Sigma_\vartheta)}.$$

In particular, we are interested in the case where A_T is homogeneous of degree two and has constant coefficients. Under these additional assumptions, we obtain the main result. Note that for the main result, the constants in the above theorem should only depend on M and not on additional seminorms $|a|_*$ of the differential operator. The details are given in Section 5.4 and the result reads as follows:

Corollary 1.3. *Let $X_+ = \mathbb{R}^n_+$ and A_T be given as in Theorem 1.1. Moreover, assume that A_T is homogeneous of degree two and has constant coefficients. Then, Theorem 1.1 holds.*

For the proof of Theorem 1.2, we proceed as follows: We give a pseudodifferential interpretation of Agmon's famous idea to consider the spectral parameter as an additional co-variable, see Section 5.1. Agmon's point of view allows us to explicitly compute the slowest decaying part of the resolvent of A_T, see Section 5.3. This computation involves the construction of a parametrix to the extended boundary value problem. In Section 5.2, we carry out the construction. This construction is divided into the construction of a parametrix to the associated Dirichlet problem and the construction of a parametrix to a pseudodifferential operator on the boundary: the well-known „Reduction to the Boundary". The assumption made on the trace operator ensures that the second parametrix exists because the resulting operator on the boundary satisfies Hörmader's hypo-ellipticity condition, see Section 5.2.2. The parametrix to the associated Dirichlet problem can be constructed in Boutet de Monvel's calculus. This construction is well-known, see Section 5.2.1. The parametrix to the extended boundary value problem is a combination of the two previously mentioned parametrices. However we have to take a technical hurdle: The parametrix on the boundary is of Hörmander type with $\delta = 1/2$, hence we need a Boutet de Monvel calculus based on such pseudodifferential operators. We did not find a source where such a calculus is treated. Therefore, in Section 4, we establish this calculus for $0 \leq \delta < 1$. The proof of Theorem 1.2 depends on explicit estimates which again rely on the results of Section 5.2 and 5.1. Once Theorem 1.2 is established, we use the technique of „freezing the coefficients" to remove the smoothness assumption, see Section 5.5.1. Theorem 1.2 implies the main result via the processes of localization and rectification, see Section 5.5.3.

In Section 6, we provide a possible application of the main result: the short time existence for the porous medium equation with general boundary condition of the form (1.2).

2 Function Spaces

In this section, we first revise some general results on function spaces. We then introduce Bessel potential and Besov spaces on $\mathbb{R}^n$, $\mathbb{R}^n_+$, and on manifolds with or without boundary which have bounded geometry. The well-known results can be found in the textbooks [47] and [46] by Hans Triebel with one exception: The recent results on manifold with boundary and bounded geometry are covered in [17].
A Fréchet space is a complete locally convex vector space whose topology is given by an increasing family of seminorms $(|\cdot|_n)_{n\in\mathbb{N}_0}$. We write $|\cdot|_*$ on the right hand side of an inequality, if the inequality holds with $|\cdot|_*$ replaced by $|\cdot|_n$ for some $n \in \mathbb{N}_0$. We write $|k|_*$ on the left hand side of an inequality, if it holds for $|\cdot|_*$ replaced by $|\cdot|_n$ for any choice of $n \in \mathbb{N}_0$. In this notation, a linear operator A between Fréchet spaces is bounded if and only if $|Au|_* \leq C|u|_*$.

The inductive limit of Fréchet spaces is defined as follows: Let $(E_j)_{j\in\mathbb{N}_0}$ be a sequence of Fréchet spaces such that $E_j \hookrightarrow E_{j'}$ if $j \leq j'$. We equip the vector space $E := \cup_{j\in\mathbb{N}_0} E_j$ with the finest locally convex topology such that the natural embedding $E_j \subset E$ is continuous for all $j \in \mathbb{N}_0$. It is well-known that a linear operator A from E into a locally convex space F is continuous if and only if the restriction to E_j is for all $j \in \mathbb{N}_0$. Furthermore, we need the projective limit of Fréchet spaces: Let $(F_j)_{j\in\mathbb{N}_0}$ be a sequence of Fréchet spaces such that $E_j \hookleftarrow E_{j'}$ if $j \leq j'$. We equip the vector space $E := \cap_{j\in\mathbb{N}_0} E_j$ with the coarsest locally convex topology such that the embedding $F \subset F_j$ is continuous for each $j \in \mathbb{N}_0$. It is well-known that a linear operator A that maps a Banach space E into a projective limit of Fréchet spaces F is bounded if and only if $A \in \mathcal{B}(E, F_j)$ for all $j \in \mathbb{N}_0$. For more details on the projective and inductive limit, see [26], [33], and [44].

We recall the projective topological tensor product: Let E and F be locally convex spaces and $E \otimes F$ the algebraic tensor product. We consider this space with the projective topology, with respect to the map $E \times F \ni (x, y) \mapsto x \otimes y \in E \otimes F$. Let $(p_i)_{i\in\mathbb{N}_0}$ and $(q_j)_{j\in\mathbb{N}_0}$ be families of seminorms on E and F which define the topologies. Then, the topology of $E \otimes F$ is given by the following family of seminorms:

$$[p_i \otimes q_i](u) := \inf \left\{ \sum_{k=1}^{n} p_i(x_k) q_j(y_k) : u = \sum_{k=1}^{n} x_k \otimes y_k \right\}.$$

By $E\hat{\otimes}_\pi F$, we denote the completion of the above space. This completion is necessary because the tensor product of complete space does, in general, not have this property. The subscript π refers to the choice of the topology, but this is not the only reasonable choice. For more details and the next result, we refer to [33].

Theorem 2.1 (Structure of Tensor Products)**.** *Let E and F be Fréchet spaces. Then, for any $u \in E\hat{\otimes}_\pi F$, sequences $(c_k) \in l_1(\mathbb{N}_0)$, $(x_k) \in c_0(\mathbb{N}_0; E)$, and $(y_k) \in c_0(\mathbb{N}_0; F)$ exist such that u admits the following decomposition:*

$$u = \sum_{k=1}^{\infty} c_k x_k \otimes y_k.$$

The sum converges absolutely and $[p_i \otimes q_j](u) \le \sum_{k=1}^{\infty} |c_k| p_i(x_k) q_j(y_k)$ *for all* p_i *and* q_j*.*

We will use the following notation for interpolation theory: Let E_1, E_2 be Banach spaces which are subspaces of a common (Hausdorff) topological vector space. Then, we say that (E_1, E_2) is a compatible couple. In this situation, $E_1 \cap E_2$ with norm $\|\cdot\|_{E_1 \cap E_2} := \max\{\|\cdot\|_{E_1}, \|\cdot\|_{E_2}\}$ is a Banach space, as well as $E_1 + E_2$ with norm $\|x\| = \inf\{\|x_1\|_{E_1} + \|x_2\|_{E_2} : x_1 + x_2 = x\}$. These couples form a category. The morphisms are bounded linear maps on the sum which have bounded restriction to the components. We use two functors to the category of Banach space. By $[E_1, E_2]_\theta$, we denote the complex interpolation functor, here $0 \le \theta \le 1$. We write $[E_1, E_2]_{\theta,q}$ for the real interpolation functor, with $0 \le \theta \le 1$ and $1 \le q$. For the construction of these functors, we refer to [8]. We write $*$ instead of θ or θ, p, if a statement holds for the real and complex interpolation functor. The images of those functors are interpolation spaces, i.e., $E_1 \cap E_2 \hookrightarrow [E_1, E_2]_* \hookrightarrow E_1 + E_2$ and $T : [E_1, E_2]_* \to [E_1', E_2']_*$ is a bounded linear operator, if T is a morphism between the couples (E_1, E_2) and (E_1', E_2'). Let E and F be Banach spaces. We say that F is a retract of E, if bounded operators $R \in \mathcal{B}(E, F)$ exists and $S \in \mathcal{B}(F, E)$ such that $RS = 1$. The operator R is called a retraction and the operator S is the coretraction.

Theorem 2.2. *Let* (E_1, E_2) *and* (F_1, F_2) *be interpolation couples of Banach spaces. Moreover,* F_1 *and* F_2 *are retracts of* E_1 *respectively* E_2*, with common retraction* R *and coretraction* S*. Then,* $[F_1, F_2]_\theta = R[E_1, E_2]_\theta$ *and* $[F_1, F_2]_{\theta,p} = R[E_1, E_2]_{\theta,p}$*.*

2.1 Function Space on Euclidean Space

In the theory of differential equation, the use of multi indices is common. We denote the partial derivatives acting on distributions by $D_{x_i} := -i\frac{\partial}{\partial x_i}$. These operators commute. Thus, the following notion is defined:

$$D_x^\alpha := D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n} \text{ and } x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n} \text{ for } \alpha, \beta \in \mathbb{N}_0^n.$$

A distribution $u \in \mathcal{D}'(\mathbb{R}^n)$ is a rapidly decreasing function, if $x^\beta D^\alpha u \in C_b(\mathbb{R}^n)$ for any choice of $\alpha, \beta \in \mathbb{N}_0^n$. We denote the space of these functions by $\mathcal{S}(\mathbb{R}^n)$, called the Schwartz space. The topology of this space is defined by one of the following families of seminorms, with the index set $(\alpha, \beta) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$:

$$|u|^1_{\alpha,\beta} := \|x^\alpha D^\beta u\|_{L_1(\mathbb{R}^n)}, \quad |u|^2_{\alpha,\beta} := \|x^\alpha D^\beta u\|_{L_2(\mathbb{R}^n)} \text{ or } |u|^\infty_{\alpha,\beta} := \|x^\alpha D^\beta u\|_{L_\infty(\mathbb{R}^n)}.$$

In the literature, the latter family is most commonly used. However, these families are equivalent. The following family of seminorms is increasing and induces the same topology as those mentioned above:

$$|u|_n := \sup_{|\alpha|, |\beta| \le n} \{|u|^1_{\alpha,\beta}, |u|^2_{\alpha,\beta}, |u|^\infty_{\alpha,\beta}\} \text{ for } n \in \mathbb{N}_0.$$

$\mathcal{S}(\mathbb{R}^n)$ is complete and thus a Fréchet space. Additionally, $\mathcal{S}(\mathbb{R}^n)$ is invariant under the Fourier transform. We use the following convention:

$$\mathcal{F}u = \left[\xi \mapsto \int e^{-i\xi x}u(x)dx\right] \;\; and \;\; \mathcal{F}^{-1}u = \left[x \mapsto \int e^{i\xi x}u(\xi)đ\xi\right], \;\; \text{with} \;\; đ\xi := (2\pi)^{-n}d\xi.$$

The Fourier transform of a tempered distribution $u \in \mathcal{S}'(\mathbb{R}^n)$ is still a tempered distribution. It is defined by $[\mathcal{F}u](\phi) := u(\mathcal{F}\phi)$ for all $\phi \in \mathcal{S}(\mathbb{R}^n)$, as is $\mathcal{F}^{-1}$. We will use the following properties of the Fourier transform:

(i) $\mathcal{F}$ and $\mathcal{F}^{-1}$ are a linear and bounded on $\mathcal{S}(\mathbb{R}^n)$ resp. $\mathcal{S}'(\mathbb{R}^n)$. Moreover, $\mathcal{F}\mathcal{F}^{-1} = 1$.

(ii) $\xi^\beta \overline{D}_\xi^\alpha \mathcal{F} = \mathcal{F}D_x^\beta x^\alpha$ and $x^\alpha D_x^\beta \mathcal{F}^{-1} = \mathcal{F}^{-1}\overline{D}_\xi^\alpha \xi^\beta$ for all $\alpha, \beta \in \mathbb{N}_0^n$.

(iii) $\mathcal{F} : L_1(\mathbb{R}^n) \to C_0(\mathbb{R}^n)$ and $[\mathcal{F}u](\xi) = \int e^{-ix\xi}u(x)dx$ (Riemann-Lebesgue Lemma).

(iv) $\mathcal{F} : L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n)$ and $\|\mathcal{F}u\|_{L_2(\mathbb{R}^n)} = \|u\|_{L_2(\mathbb{R}^n)}$ (Plancherel's Theorem).

(v) $\mathcal{F}\delta = 1$.

Note that integration over the covariables always refers to the measure $đ\xi$. Thus, no constants $(2\pi)^n$ appear in the equations above.
Let $(\phi_j)_{j\in\mathbb{N}_0}$ be a Littlewood-Paley decomposition of unity. By $\Phi_j := \phi_j(D) := \mathcal{F}^{-1}\phi_j(\cdot)\mathcal{F}$, we denote the associated Fourier multiplier on $\mathcal{S}'(\mathbb{R}^n)$. Note that $\Phi_j : \mathcal{S}'(\mathbb{R}^n) \to L_p(\mathbb{R}^n)$ is a regularizing pseudodifferential operator. For $s \in \mathbb{R}$ and $p \in [1, \infty]$, we define the Besov spaces and Bessel potential spaces:

$$B_p^s(\mathbb{R}^n) := \{u \in \mathcal{S}'(\mathbb{R}^n) : \|u\|_{B_p^s(\mathbb{R}^n)} < \infty\} \;\; \text{with} \;\; \|u\|^p_{B_p^s(\mathbb{R}^n)} := \sum 2^{sjp}\|\Phi_j u\|^p_{L_p(\mathbb{R}^n)}$$
$$H_p^s(\mathbb{R}^n) := \{u \in \mathcal{S}'(\mathbb{R}^n) : \|u\|_{H_p^s(\mathbb{R}^n)} < \infty\} \;\; \text{with} \;\; \|u\|^p_{H_p^s(\mathbb{R}^n)} := \left\|\sum 4^{sj}|\Phi_j u|^2\right\|^p_{L_p(\mathbb{R}^n)}.$$

These spaces are special cases of the function spaces treated in [45], denoted as $B_p^s(\mathbb{R}^n) = B_{p,p}^s(\mathbb{R}^n)$ and $H_p^s(\mathbb{R}^n) = F_{p,2}^s(\mathbb{R}^n)$. The topological spaces are well-defined, i.e., different choices of Littlewood-Paley decomposition of unity give rise to equivalent norms. According to [45], these spaces have the lifting property, i.e., for all $m \in \mathbb{R}$ the operator $\langle D\rangle^m$ is bounded from the space with parameter s to those with $s - m$. The definition of Littlewood-Paley decomposition implies that $H_p^0(\mathbb{R}^n) = L_p(\mathbb{R}^n)$. Therefore, $\|\langle D\rangle^s u\|_{L_p(\mathbb{R}^n)}$ is an equivalent norm on $H_p^s(\mathbb{R}^n)$, often used to define these spaces. It is well-known that for $s \in \mathbb{N}_0$ these spaces coincide with the Sobolev space $W_p^s(\mathbb{R}^n)$. The spaces introduced above have the following properties:

Theorem 2.3. *Let $1 < p \leq \infty$ and $s \in \mathbb{R}$. The following results hold:*

- *(Multiplier): Let $\psi \in B_\infty^\tau(\mathbb{R}^n)$, for some $\tau > 0$. Then ψ is a pointwise multiplication operator on $H_p^s(\mathbb{R}^n)$ and $B_p^s(\mathbb{R}^n)$ for all $|s| < \tau$. More precisely a constant $C > 0$ exists such that*

$$\|\psi u\|_{H_p^s(\mathbb{R}^n)} \leq C\|\psi\|_{B_\infty^\tau(\mathbb{R}^n)}\|u\|_{H_p^s(\mathbb{R}^n)} \;\; and \;\; \|\psi u\|_{B_p^s(\mathbb{R}^n)} \leq C\|\psi\|_{B_\infty^\tau(\mathbb{R}^n)}\|u\|_{B_p^s(\mathbb{R}^n)}.$$

- *(Dual): Let $1/p+1/q=0$. The dual of the Besov and Bessel potential spaces are:*
$$(H_p^s(\mathbb{R}^n))' = H_q^{-s}(\mathbb{R}^n) \text{ and } (B_p^s(\mathbb{R}^n))' = B_q^{-s}(\mathbb{R}^n).$$
- *(Embeddings): For all $\varepsilon>0$ the following embeddings hold.*
$$B_p^{s-\varepsilon}(\mathbb{R}^n) \hookrightarrow H_p^s(\mathbb{R}^n) \hookrightarrow B_p^{s+\varepsilon}(\mathbb{R}^n).$$
- *(Interpolation): Let $s=\theta s_0+(1-\theta)s_1$ for some $\theta\in[0,1]$. Then*
  - *(i)* $[H_p^{s_0}(\mathbb{R}^n), H_p^{s_1}(\mathbb{R}^n)]_{\theta,p} = B_p^s(\mathbb{R}^n)$.
  - *(ii)* $[H_p^{s_0}(\mathbb{R}^n), H_p^{s_1}(\mathbb{R}^n)]_{\theta} = H_p^s(\mathbb{R}^n)$.
  - *(iii)* $[B_p^{s_0}(\mathbb{R}^n), B_p^{s_1}(\mathbb{R}^n)]_{\theta,p} = B_p^s(\mathbb{R}^n)$.
  - *(iv)* $[B_p^{s_0}(\mathbb{R}^n), B_p^{s_1}(\mathbb{R}^n)]_{\theta} = B_p^s(\mathbb{R}^n)$.
- *(Trace): Let $\gamma_0 u(x') = u(x',0)$ for all $u\in\mathcal{D}(\mathbb{R}^n)$. If $s>1/p$, this operator extends to an element of $\mathcal{B}(H_p^s(\mathbb{R}^n), B^{s-1/p}(\mathbb{R}^{n-1}))$.*

Proof. All of these results can be found in [45]. □

We recall the definition of weighted Bessel potential spaces for $\mathbf{s}\in\mathbb{R}^2$ and $p\in(1,\infty)$:

$$H_p^{\mathbf{s}}(\mathbb{R}^n) := \{u\in\mathcal{S}'(\mathbb{R}^n) : \|u\|_{H_p^s(\mathbb{R})} < \infty\} \text{ with } \|u\|_{H_p^{\mathbf{s}}(\mathbb{R}^n)} := \|\mathcal{F}\langle\xi\rangle^{s_1}\mathcal{F}^{-1}\langle x\rangle^{s_2}u\|_{L_p(\mathbb{R}^n)}.$$

These spaces are Banach spaces with norm $\|\cdot\|_{H_p^{\mathbf{s}}(\mathbb{R}^n)}$. It is well-known that the Schwartz space and the space of tempered distributions can be expressed via the inductive limes and the projective limes, respectively:

$$\mathcal{S}(\mathbb{R}^n) = \bigcap_{\mathbf{s}\in\mathbb{R}^2} H_p^{\mathbf{s}}(\mathbb{R}^n) = \bigcap_{\mathbf{s}\in\mathbb{N}^2} H_p^{\mathbf{s}}(\mathbb{R}^n) \text{ and } \mathcal{S}'(\mathbb{R}^n) = \bigcup_{\mathbf{s}\in\mathbb{R}^2} H_p^{\mathbf{s}}(\mathbb{R}^n) = \bigcup_{\mathbf{s}\in\mathbb{N}^2} H_p^{\mathbf{s}}(\mathbb{R}^n).$$

In the following, we need spaces that consist of sequences of functions: Let E and F be Banach spaces and Γ be a countable index set. We say that a symmetric relation $\bowtie$ on Γ has finite width $N\in\mathbb{N}$, if

$$\sup_{l\in\Gamma} |\{k\in\Gamma : k\bowtie l\}| = N.$$

Definition 2.4. Let $\mathbb{A} : l_\infty(E)\to l_\infty(F)$ and $\bowtie$ be a symmetric relation of width $N\in\mathbb{N}$. We say that $\mathbb{A}$ has band structure, if it is of the form $(\mathbb{A}(u_l)_{l\in\Gamma})_k = \sum_{k\bowtie l} A_{kl}u_l$, where $A_{kl}\in\mathcal{B}(E,F)$ is a uniformly bounded family of operators.

Such operators naturally occur in the localisation process, where the index set labels the open covering. The indices are related if the intersection of the corresponding open sets is not empty. This relation is symmetric and has finite width for a suitable chosen open covering.

Lemma 2.5 (Band structure operator)**.** *Let $\mathbb{A} : l_\infty(E) \to l_\infty(F)$ have band structure. Let N be the width of the symmetric relation and $C = \sup_{k,l\in\Gamma} \|A_{kl}\|_{E,F}$. Then, $\mathbb{A} \in \mathcal{B}(l_p(E), l_p(F))$ and $\|\mathbb{A}\| \le CN^{\frac{p+1}{p}}$.*

Proof. We estimate the norm using the following computation:

$$\begin{aligned} \|\mathbb{A}(u_l)_{l\in\Gamma}\|_{l_p(F)}^p &= \sum_{k\in\Gamma} \|(\mathbb{A}(u_l)_{l\in\Gamma})_k\|_F^p = \sum_{k\in\Gamma} \left\| \sum_{k\bowtie l} A_{kl} u_l \right\|_F^p \le \sum_{k\in\Gamma} \left(\sum_{k\bowtie l} \|A_{kl}\|_{\mathcal{B}(E,F)} \|u_l\|_E \right)^p \\ &\le \sum_{k\in\Gamma} \left(NC \sup_{k\bowtie l} \|u_l\|_E \right)^p = (NC)^p \sum_{k\in\Gamma} \sup_{k\bowtie l} \|u_l\|_E^p \le (NC)^p \sum_{k\in\Gamma} \sum_{k\bowtie l} \|u_l\|_E^p \\ &\le (NC)^p \sum_{l\in\Gamma} \sum_{l\bowtie k} \|u_l\|_E^p = N(NC)^p \sum_{l\in\Gamma} \|u_l\|_E^p = N(NC)^p \|(u_l)_{l\in\Gamma}\|_{l_p(E)}^p. \end{aligned}$$

Here, we used the symmetry of the relation to interchange the summation. □

For the treatment of differential operators, the natural choice for E and F are Bessel potential or Besov spaces. In this case, we write $\mathbb{H}_p^s(\mathbb{R}^n) := l_p(\Gamma, H_p^s(\mathbb{R}^n))$ and $\mathbb{B}_p^s(\mathbb{R}^n) := l_p(\Gamma, B_p^s(\mathbb{R}^n))$. Moreover, we define $\mathbb{L}_p(\mathbb{R}^n) := l_p(\Gamma, L_p(\mathbb{R}^n))$. We do not refer to the index Γ set in the notation because it should be clear from the context. It is well-known that the spaces described above behave well under interpolation, see for instance [8, Theorem 5.1.2]. In our notation, the theorem reads as follows:

$$\begin{aligned} [\mathbb{H}_p^{s_0}(\mathbb{R}^n), \mathbb{H}_p^{s_1}(\mathbb{R}^n)]_\theta &= \mathbb{H}_p^s(\mathbb{R}^n), \\ [\mathbb{H}_p^{s_0}(\mathbb{R}^n), \mathbb{H}_p^{s_1}(\mathbb{R}^n)]_{\theta,p} &= \mathbb{B}_p^s(\mathbb{R}^n), \\ [\mathbb{B}_p^{s_0}(\mathbb{R}^n), \mathbb{B}_p^{s_1}(\mathbb{R}^n)]_\theta &= \mathbb{B}_p^s(\mathbb{R}^n), \text{ and} \\ [\mathbb{B}_p^{s_0}(\mathbb{R}^n), \mathbb{B}_p^{s_1}(\mathbb{R}^n)]_{\theta,p} &= \mathbb{B}_p^s(\mathbb{R}^n), \end{aligned}$$

where $\theta \in (0,1)$, $s = \theta s_0 + (1-\theta)s_1$, and $1 < p < \infty$.

2.2 Function Spaces on Euclidean Half Space

In this section, we summarize the relevant results for spaces of functions on euclidean half space, i.e., $\mathbb{R}_+^n := \{x \in \mathbb{R}^n : x_n \ge 0\}$. The majority of the results follows from the existence of a bounded extension operator and Section 2.1. We use Hamilton's definition of an extension operator, given in [22]. The advantage of his definition, over the one by Seeley in [39], is that explicit formulas for the dual operator are available. For more details, we refer to [4]. We define $\mathcal{S}(\mathbb{R}_+^n) := r^+\mathcal{S}(\mathbb{R}^n)$, where r^+ is the restriction to the closed set $\mathbb{R}_+^n$.

Lemma 2.6. *A function $h \in C^\infty((0,\infty), \mathbb{R})$ exists that has the following properties:*

$$\int_0^\infty t^s |h(t)|\, dt < \infty, \;\; (-1)^k \int_0^\infty t^k h(t)\, dt = 1, \;\; \textit{and} \;\; h(1/t) = -th(t),$$

for all $s \in \mathbb{R}$, $k \in \mathbb{Z}$, and $t > 0$.

For the existence of such a function, we refer to [4, Lemma 1.1.1]. Let u belong to $C_b(\mathbb{R}^n_+)$ or $C_b(\mathbb{R}^n)$. Then, for all $x \in \mathbb{R}^n$, we define:

$$[\varepsilon^k u](x) = (-1)^k \int_0^\infty t^k h(t) u(x', -tx_n)\, dt.$$

We further define an operator E that acts on $C_b(\mathbb{R}^n_+)$ as follows:

$$[Eu](x) := \begin{cases} u(x', x_n) & \text{if } \; x_n \geq 0, \\ \varepsilon^0 u(x', x_n) & \text{if } \; x_n < 0. \end{cases}$$

We are interested in the mapping properties of the latter operator. To this end, we observe:

(i) $x_n^{l'} D^l_{x_n} (x')^\alpha D^\beta_{x'} [\varepsilon^k u] = \varepsilon^{k+l-l'} [x_n^{l'} D^l_{x_n} (x')^\alpha D^\beta_{x'} u]$ for all $l, l' \in \mathbb{N}_0$, $k \in \mathbb{Z}$, $\alpha, \beta \in \mathbb{N}_0^{n-1}$.

(ii) $\|\varepsilon^k u\|_{L_p(\mathbb{R}^n_-)} \leq C \|u\|_{L_p(\mathbb{R}^n_+)}$, with a constant $C = C(k)$ for $k \in \mathbb{Z}$ and $1 \leq p \leq \infty$.

(iii) $[\varepsilon^k u](x', x_n) \to u(x', 0)$ as $x_n \searrow 0$ for all $k \in \mathbb{Z}$.

In particular, E is bounded from $H^{\mathbf{s}}_p(\mathbb{R}^n_+) \cap \mathcal{S}(\mathbb{R}^n_+)$ to $H^{\mathbf{s}}_p(\mathbb{R}^n)$ for all $\mathbf{s} \in \mathbb{N}_0^2$ and $1 < p < \infty$. Therefore, E is bounded from $\mathcal{S}(\mathbb{R}^n_+)$ to $\mathcal{S}(\mathbb{R}^n)$. Moreover, E is bounded from $C^k_b(\mathbb{R}^n_+)$ to $C^k_b(\mathbb{R}^n)$ and thus bounded from $B^s_\infty(\mathbb{R}^n_+)$ to $B^s_\infty(\mathbb{R}^n)$ for all $s > 0$. We define $\mathcal{S}_0(\mathbb{R}^n_+)$ to be the subspace of functions in $\mathcal{S}(\mathbb{R}^n_+)$ which vanish with all their derivatives at the boundary. Thus, the extension by zero, denoted as e^+, is a bounded operator from $\mathcal{S}_0(\mathbb{R}^n_+)$ to $\mathcal{S}(\mathbb{R}^n)$. The operator $Ru := r^+(u - \varepsilon^0 u)$ is bounded from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}_0(\mathbb{R}^n_+)$. We define two pairings:

$$\begin{aligned} \langle u, \phi \rangle_{\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)} &:= \int_{\mathbb{R}^n} u(x)\phi(x) dx \;\; \text{and} \\ \langle u, \phi \rangle_{\mathcal{S}(\mathbb{R}^n_+) \times \mathcal{S}_0(\mathbb{R}^n_+)} &:= \langle Eu, e^+\phi \rangle_{\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)} = \int_{\mathbb{R}^n_+} u(x)\phi(x) dx. \end{aligned}$$

Lemma 2.7. *The following identities hold:*

$$\begin{aligned} \langle Eu, \phi \rangle_{\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)} &= \langle u, R\phi \rangle_{\mathcal{S}(\mathbb{R}^n_+) \times \mathcal{S}_0(\mathbb{R}^n_+)} \\ \langle r^+ u, \phi \rangle_{\mathcal{S}(\mathbb{R}^n_+) \times \mathcal{S}_0(\mathbb{R}^n_+)} &= \langle u, e^+\phi \rangle_{\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)} \end{aligned}$$

Proof. The following computation is the essential step for the proof:

$$\begin{aligned} \int_{-\infty}^0 [\varepsilon^0 u](x', x_n) \phi(x', x_n)\, dx_n &= \int_{-\infty}^0 \int_0^\infty h(t) u(x', -tx_n) \phi(x', x_n)\, dt dx_n \\ &= \int_0^\infty \int_0^\infty h(t)/t u(x', y_n) \phi(x', -y_n/t)\, dt dy_n \\ &= \int_0^\infty \int_\infty^0 -h(1/s)/s u(x', y_n) \phi(x', -s y_n)\, ds dy_n \end{aligned}$$

$$= \int_0^\infty \int_\infty^0 h(s)u(x', y_n)\phi(x', -sy_n)\, dsdy_n$$
$$= -\int_0^\infty u(x', x_n)[\varepsilon^0\phi](x', x_n)dx_n$$

We obtain the first identity from the computation below:

$$\begin{aligned}\langle Eu, \phi\rangle_{\mathcal{S}(\mathbb{R}^n)\times\mathcal{S}(\mathbb{R}^n)} &= \int\int_0^\infty u(x)\phi(x)\, dx_n dx' + \int\int_{-\infty}^0 [\varepsilon^0 u](x', x_n)\phi(x', x_n)\, dx_n dx' \\ &= \int\int_0^\infty u(x)(\phi(x) - [\varepsilon^0\phi](x))\, dx_n dx' \\ &= \int_{\mathbb{R}^n_+} u(x)[R\phi](x)\, dx = \langle u, R\phi\rangle_{\mathcal{S}(\mathbb{R}^n_+)\times\mathcal{S}_0(\mathbb{R}^n_+)}.\end{aligned}$$

The second identity is obvious. □

We define $\mathcal{S}'(\mathbb{R}^n_+) := r^+\mathcal{S}'(\mathbb{R}^n)$. Here, r^+ is the restriction of distributions to the interior of $\mathbb{R}^n_+$. The test functions, with support in the interior of $\mathbb{R}^n_+$, form a dense subspace of $\mathcal{S}_0(\mathbb{R}^n_+)$. Thus, a unique pairing $\langle\cdot,\cdot\rangle_{\mathcal{S}'(\mathbb{R}^n_+)\times\mathcal{S}_0(\mathbb{R}^n_+)}$ exists which extends $\langle\cdot,\cdot\rangle_{\mathcal{S}(\mathbb{R}^n_+)\times\mathcal{S}_0(\mathbb{R}^n_+)}$. We define an extension operator R^* on $\mathcal{S}'(\mathbb{R}^n_+)$ which, according to Lemma 2.7, coincides with E on the dense set $\mathcal{S}(\mathbb{R}^n_+)$. For consistency, we call this operator E. We observe that $r^+Eu = u$ for all $u \in \mathcal{S}'(\mathbb{R}^n_+)$. Hence, $p^+ := Er^+$ and $p^- := 1 - Er^+$ are complementary projections on $\mathcal{S}'(\mathbb{R}^n)$ which give rise to the following direct sum decomposition:

$$\mathcal{S}'(\mathbb{R}^n) = p^+\mathcal{S}'(\mathbb{R}^n) \oplus p^-\mathcal{S}'(\mathbb{R}^n) = E\mathcal{S}'(\mathbb{R}^n_+) \oplus \{u \in \mathcal{S}'(\mathbb{R}^n) : \operatorname{supp} u \subset \mathbb{R}^n_-\}.$$

What we are primarily interested in are the subspaces $H^{\mathbf{s}}_p(\mathbb{R}^n_+)$ and $B^s_p(\mathbb{R}^n_+)$ of $\mathcal{S}'(\mathbb{R}^n_+)$ which are also defined via restriction. We observe that the restriction of E to $H^{\mathbf{s}}_p(\mathbb{R}^n_+)$ or $B^s_p(\mathbb{R}^n_+)$ is a bounded extension operator. We define $H^{\mathbf{s}}_{p;0}(\mathbb{R}^n_+)$ and $B^s_{p;0}(\mathbb{R}^n_+)$ as the closure of $\mathcal{S}_0(\mathbb{R}^n_+)$, with respect to the induced norm.

Theorem 2.8. *Let $1 < p < \infty$ and $s \in \mathbb{R}$. The following results hold:*

- *(Multiplier): Let $\psi \in B^\tau_\infty(\mathbb{R}^n_+)$, for some $\tau > 0$. Then, ψ is a pointwise multiplication operator on $H^s_p(\mathbb{R}^n_+)$ and $B^s_p(\mathbb{R}^n_+)$ for all $|s| < \tau$. More precisely, a constant $C > 0$ exists such that*

$$\|\psi u\|_{H^s_p(\mathbb{R}^n_+)} \le C\|\psi\|_{B^\tau_\infty(\mathbb{R}^n_+)}\|u\|_{H^s_p(\mathbb{R}^n_+)} \;\; \text{and} \;\; \|\psi u\|_{B^s_p(\mathbb{R}^n_+)} \le C\|\psi\|_{B^\tau_\infty(\mathbb{R}^n_+)}\|u\|_{B^s_p(\mathbb{R}^n_+)}.$$

- *(Dual): Let $1/p + 1/q = 0$. The dual of the Besov and Bessel potential spaces are:*

$$(H^s_p(\mathbb{R}^n_+))' = H^{-s}_{q;0}(\mathbb{R}^n_+) \;\; \text{and} \;\; (B^s_p(\mathbb{R}^n))' = B^{-s}_{q;0}(\mathbb{R}^n).$$

- *(Embeddings): For all $\varepsilon > 0$ the following embeddings hold.*

$$B^{s-\varepsilon}_p(\mathbb{R}^n_+) \hookrightarrow H^s_p(\mathbb{R}^n_+) \hookrightarrow B^{s+\varepsilon}_p(\mathbb{R}^n_+).$$

- *(Interpolation): Let $s = \theta s_0 + (1-\theta)s_1$ for some $\theta \in [0,1]$. Then*
    - *(i)* $[H_p^{s_0}(\mathbb{R}^n_+), H_p^{s_1}(\mathbb{R}^n_+)]_{\theta,p} = B_p^s(\mathbb{R}^n_+)$.
    - *(ii)* $[H_p^{s_0}(\mathbb{R}^n_+), H_p^{s_1}(\mathbb{R}^n_+)]_{\theta} = H_p^s(\mathbb{R}^n_+)$.
    - *(iii)* $[H_{p;0}^{s_0}(\mathbb{R}^n_+), H_{p;0}^{s_1}(\mathbb{R}^n_+)]_{\theta,p} = B_{p;0}^s(\mathbb{R}^n_+)$.
    - *(iv)* $[H_{p;0}^{s_0}(\mathbb{R}^n_+), H_{p;0}^{s_1}(\mathbb{R}^n_+)]_{\theta} = H_{p;0}^s(\mathbb{R}^n_+)$.
    - *(v)* $[B_p^{s_0}(\mathbb{R}^n_+), B_p^{s_1}(\mathbb{R}^n_+)]_{\theta,p} = B_p^s(\mathbb{R}^n_+)$.
    - *(vi)* $[B_p^{s_0}(\mathbb{R}^n_+), B_p^{s_1}(\mathbb{R}^n_+)]_{\theta} = B_p^s(\mathbb{R}^n_+)$.
- *(Trace): Let $\gamma_0^+ := \gamma_0 E$. This operator is well-defined and bounded from $H_p^s(\mathbb{R}^n_+)$ to $B^{s-1/p}(\mathbb{R}^{n-1})$, for $s > 1/p$.*

Proof. For the multiplier result, we observe that $\psi u = r^+ E\psi E u$. We thus obtain:

$$\|\psi u\|_{H_p^s(\mathbb{R}^n_+)} \le \|E\psi E u\|_{H_p^s(\mathbb{R}^n)} \le C\|E\psi\|_{B^\tau_\infty(\mathbb{R}^n)}\|Eu\|_{H_p^s(\mathbb{R}^n)} \le C\|\psi\|_{B^\tau_\infty(\mathbb{R}^n_+)}\|u\|_{H_p^s(\mathbb{R}^n_+)}.$$

The result on duality follows from the direct sum decomposition which these spaces inherit from the tempered distributions. We now prove the embedding result. To this end, we fix $u \in B_p^{s-\varepsilon}(\mathbb{R}^n_+)$ and $\tilde{u} \in B_p^{s-\varepsilon}(\mathbb{R}^n)$ such that $u = r^+\tilde{u}$. Then:

$$\|u\|_{H_p^s(\mathbb{R}^n_+)} \le \|\tilde{u}\|_{H_p^s(\mathbb{R}^n)} \le \|\tilde{u}\|_{B_p^{s+\varepsilon}(\mathbb{R}^n_+)}.$$

We obtain the first embedding by forming the infimum. The second embedding can be obtained by similar arguments. In the case of $H_p^s(\mathbb{R}^n_+)$ and $B_p^s(\mathbb{R}^n_+)$, the interpolation results follow from the fact that r^+ is a common retraction. The result for $H_{p;0}^s$ is obtained by duality. The trace is well-defined: For all $u \in \mathcal{S}(\mathbb{R}^n_+)$ we have $[\gamma_0^+ u](x') = \lim_{\varepsilon\to 0} u(x', \varepsilon)$. The trace is bounded as a composition of bounded operators. □

We write γ_0 instead of γ_0^+. From the context, it should be clear which operator we refer to.
We define $\mathbb{H}_p^s(\mathbb{R}^n_+) = l_p(\Gamma, H_p^s(\mathbb{R}^n_+))$, $\mathbb{H}_{p;0}^s(\mathbb{R}^n_+) = l_p(\Gamma, H_{p;0}^s(\mathbb{R}^n_+))$ and $\mathbb{B}_p^s(\mathbb{R}^n_+) = l_p(\Gamma, B_p^s(\mathbb{R}^n_+))$. Following the same arguments used in the last section, the interpolation results hold:

$$\begin{aligned}
[\mathbb{H}_p^{s_0}(\mathbb{R}^n_+), \mathbb{H}_p^{s_1}(\mathbb{R}^n_+)]_\theta &= \mathbb{H}_p^s(\mathbb{R}^n_+),\\
[\mathbb{H}_p^{s_0}(\mathbb{R}^n_+), \mathbb{H}_p^{s_1}(\mathbb{R}^n_+)]_{\theta,p} &= \mathbb{B}_p^s(\mathbb{R}^n_+),\\
[\mathbb{H}_{p;0}^{s_0}(\mathbb{R}^n_+), \mathbb{H}_{p;0}^{s_1}(\mathbb{R}^n_+)]_\theta &= \mathbb{H}_{p;0}^s(\mathbb{R}^n_+),\\
[\mathbb{H}_{p;0}^{s_0}(\mathbb{R}^n_+), \mathbb{H}_{p;0}^{s_1}(\mathbb{R}^n_+)]_{\theta,p} &= \mathbb{B}_{p;0}^s(\mathbb{R}^n_+),\\
[\mathbb{B}_p^{s_0}(\mathbb{R}^n_+), \mathbb{B}_p^{s_1}(\mathbb{R}^n_+)]_\theta &= \mathbb{B}_p^s(\mathbb{R}^n_+), \text{ and}\\
[\mathbb{B}_p^{s_0}(\mathbb{R}^n_+), \mathbb{B}_p^{s_1}(\mathbb{R}^n_+)]_{\theta,p} &= \mathbb{B}_p^s(\mathbb{R}^n_+).
\end{aligned}$$

Here, $\theta \in (0,1)$, $s = \theta s_0 + (1-\theta)s_1$, and $1 < p < \infty$. Furthermore, we need a well-known fact from the theory of distribution:

Lemma 2.9 (Jump relation)**.** *Let* $u \in \mathcal{S}(\mathbb{R}^n_+)$*. Then:*

$$D_n e^+ u = -i\gamma_0^*\gamma_0 u + e^+ D_n u \quad and$$
$$D_n^2 e^+ u = -\gamma_1^*\gamma_0 u + \gamma_0^*\gamma_1 u + e^+ D_n^2 u.$$

Proof. Observe that $e^+u = \Theta E u$, where Θ denotes the Heaviside function, hence:

$$D_n e^+ u = -i\delta E u + \Theta D_n E u = -i\delta \otimes \gamma_0 u + e^+ D_n u = -i\gamma_0^*\gamma_0 u + e^+ D_n.$$

The computation above relies on the fact that $\delta E u$ only depends on the values of Eu with $x_n = 0$, as well as $D_n E = E D_n$ on $\mathbb{R}^n_+$. We recall that $\gamma_1 = -\gamma_0\partial_n = -i\gamma_0 D_n$ which implies $\gamma_1^* = iD_n\gamma_0^*$. Iterative use of the identity above completes the proof. □

2.3 Function Spaces on Manifolds

For the results of this section, we follow [6] and [17].

Definition 2.10. A Riemannian manifold (X, g) without boundary has bounded geometry, if the injectivity radius is positive and all covariant derivatives of the curvature R are bounded, i.e.,:

$$\|\nabla^k R\|_{L_\infty(X)} \leq \infty \ \text{ for all } \ k \in \mathbb{N}_0.$$

Here, ∇ is the Levi-Civita connection.

We are primarily interested in Bessel potential spaces which generalize Sobolev spaces. The latter are defined as all functions which have L_p-bounded covariant derivatives up to a given order. For more details on these spaces, we refer to [7]. Robert Strichartz introduced the Bessel potential spaces as $H^s_p(X) := (1-\Delta_g)^{-s/2} L_p(X)$, see [42]. Additionally, we need Besov spaces because they naturally arise if we restrict functions to hypersurfaces. Both types of spaces can be described locally, using normal coordinates. The preferred point of view is the local description. For more details, we refer to [46, Chapter 7]. By definition: Let Γ be an index set for a uniform locally finite cover of X by normal coordinate charts U_l, with associate coordinates $\kappa_l : U_l \to V_l \subset \mathbb{R}^n$. Let $(\psi_l)_{l\in\Gamma}$ be a partition of unity subordinate to the cover. Given $\mathcal{T} := \{\Gamma, (U_l)_{l\in\Gamma}, (V_l)_{l\in\Gamma}, (\kappa_l)_{l\in\Gamma}, (\psi_l)_{l\in\Gamma}\}$, we define the following space:

$$H^{s,\mathcal{T}}_p(X) := \left\{ u \in \mathcal{D}'(X) : \|u\|_{H^s_p(X)} := \left(\sum_{l\in\Gamma} \|\kappa_{l,*}\psi_l u\|^p_{H^s_p(\mathbb{R}^n)} \right)^{1/p} < \infty \right\}, \tag{2.1}$$

where all functions $\kappa_l^*\psi_l u$ are extended by zero outside of V_l. We define Besov spaces in a similar fashion:

$$B^{s,\mathcal{T}}_p(X) := \left\{ u \in \mathcal{D}'(X) : \|u\|_{B^s_p(X)} := \left(\sum_{l\in\Gamma} \|\kappa_{l,*}\psi_l u\|^p_{B^s_p(\mathbb{R}^n)} \right)^{1/p} < \infty \right\}. \tag{2.2}$$

Different choices of $\mathcal{T}$ give rise to equivalent norms. In 2.1, each of these norms is equivalent to that of $H^s_p(X)$. In the following, we assume that $\mathcal{T}$ has been chosen. Thus, we drop it from the notation. We define a localization operator L by $u \mapsto (\kappa_{l,*}\psi_l u)_{l\in\Gamma}$. The operator is obviously linear and bounded from $H^s_p(X)$ to $\mathbb{H}^s_p(\mathbb{R}^n)$ and from $B^s_p(X)$ to $\mathbb{B}^s_p(\mathbb{R}^n)$. In fact, a function on X belongs to $H^s_p(X)$ or $B^s_p(X)$ if and only if $Lu \in \mathbb{H}^s_p(\mathbb{R}^n)$ or $Lu \in \mathbb{B}^s_p(\mathbb{R}^n)$, respectively. Moreover, $\|u\|_{H^s_p(X)} = \|Lu\|_{\mathbb{H}^s_p(\mathbb{R}^n)}$ and $\|u\|_{B^s_p(X)} = \|Lu\|_{\mathbb{B}^s_p(\mathbb{R}^n)}$. For each $l \in \Gamma$, we fix a bump function $\chi_l \in C_0^\infty(U_l)$ such that $\chi_l = 1$ on $\operatorname{supp}\psi_l$. We write $\chi_{l,*} := \kappa_{l,*}\chi_l$ and define a patching operator $P : (u_l)_{l\in\Gamma} \mapsto \sum_{l\in\Gamma}\kappa_l^*\chi_{l,*}u_l$. We define the relation $k \bowtie l :\Leftrightarrow \operatorname{supp}\chi_l \cap \operatorname{supp}\chi_k \neq \emptyset$ which is symmetric and has finite width because the cover of the manifold is uniform locally finite. The operator $\mathbb{D} = LP$ has band structure and is given by the family $D_{kl} = \kappa_{k,*}\psi_k\kappa_l^*\chi_{l,*} = \kappa_{k,*}\kappa_l^*\psi_{k,*}\chi_{l,*}$, where $\psi_{k,*} := \kappa_{l,*}\psi_k|_{U_l}$. The coordinate changes $\kappa_{k,*}\kappa_l^*$ belong to $\mathcal{B}(H^s_p(\mathbb{R}^n))\cap\mathcal{B}(B^s_p(\mathbb{R}^n))$ and are uniformly bounded with respect to the indices $l, k \in \Gamma$ because the geometry is bounded. The multiplication operator $\psi_{k,*}\chi_{l,*}$ is similarly bounded. Thus, Lemma 2.5 implies that $LP \in \mathcal{B}(\mathbb{H}^s_p(\mathbb{R}^n))\cap\mathcal{B}(\mathbb{B}^s_p(\mathbb{R}^n))$. Therefore, $P : \mathbb{H}^s_p(\mathbb{R}^n) \to H^s_p(X)$ and $P : \mathbb{B}^s_p(\mathbb{R}^n) \to B^s_p(X)$ are bounded operators. Note that $PL = 1$. Thus, P is a retraction from $\mathbb{H}^s_p(\mathbb{R}^n)$ to $H^s_p(M)$ with common coretraction L. In particular, the following interpolation results hold. Let $\theta \in (0,1)$, $s = \theta s_0 + (1-\theta)s_1$ and $1 < p < \infty$, then:

$$\begin{aligned}
[H^{s_1}_p(X), H^{s_2}_p(X)]_\theta &= H^s_p(X),\\
[H^{s_1}_p(X), H^{s_2}_p(X)]_{\theta,p} &= B^s_p(X),\\
[B^{s_1}_p(X), B^{s_2}_p(X)]_\theta &= B^s_p(X) \ \text{ and}\\
[B^{s_1}_p(X), B^{s_2}_p(X)]_{\theta,p} &= B^s_p(X).
\end{aligned}$$

From now on, we consider manifolds with boundaries. We assume that the boundary is a bounded hypersurface:

Definition 2.11. Let (X, g) be a Riemannian manifold without boundary and bounded geometry and Y be a hypersurface with (outward) unit normal field ν. We say that Y is a bounded hypersurface if the following conditions are satisfied:

(i) Y is a closed subset of X.

(ii) The second fundamental form $\mathbf{II}$ of Y in X and all its covariant derivatives along Y are bounded, i.e.,

$$\|(\nabla^Y)^k\mathbf{II}\|_{L_\infty(N)} < \infty \ \text{ for all } \ k \in \mathbb{N}_0.$$

(iii) There is a $\delta > 0$ such that $\exp^\perp : N \times (-\delta, \delta) \to M$ is injective.

Here, $\exp^\perp(p, x_n) := \exp^M_p(-\nu x_n)$.

If Y is a bounded hypersurface, then $(Y, g|_Y)$ is a manifold without boundary and bounded geometry, see [6] for the proof. It was observed in [17] that a choice of coordinates adapted to such a hypersurface exists which is compatible to the normal coordinates. The

Fermi coordinates are given by $\exp^{\perp}$ close to the boundary and by exp away from the boundary. With respect to Fermi coordinates, we define the spaces in (2.1). These spaces are equivalent to those defined by normal coordinates. The Fermi coordinates are used in the proof of the following result, see [17]:

Proposition 2.12. *Let (X, g) be a manifold without boundary and bounded geometry and Y a bounded hypersurface. For all $s > 1/p$ there is a surjective, bounded and linear map $\gamma_0 : H_p^s(X) \to B_p^{s-1/p}(Y)$ that coincides, with restriction to Y for smooth functions.*

The notation of a bounded hypersurfaces allows us to define manifolds with boundary and bounded geometry:

Definition 2.13. A Riemannian manifold (X_+, g_+) with smooth boundary ∂X_+ has bounded geometry if there is a Riemannian manifold (X, g) without boundary and bounded geometry. These manifolds are related as follows:

(i) X_+ and X have the same dimension. (If nothing else is mentioned $\dim X = n$.)

(ii) There is an isometric embedding $(X_+, g_+) \hookrightarrow (X, g)$.

(iii) The boundary ∂X_+ is a bounded hypersurface in X.

Theorem [6, Theorem 2.10], given below, proves that Definition 2.13 is equivalent to the definition provided by Thomas Schick in [34]. Schick's definition does not require a surrounding manifold of bounded geometry and is thus intrinsic.

Theorem 2.14. *Let (X_+, g) be a manifold with boundary such that the following assumptions hold.*

(N) There is a $r_\partial > 0$ such that the following map is a diffeomorphism onto its image.

$$\partial X_+ \times [0, r_\partial) \to X, (x', x_n) \mapsto \exp^{\perp}(x', x_n).$$

(I) There is a $r_{inj} > 0$ such that for all $r \leq r_{inj}$ and all $x \in X_+ \backslash U_r(\partial X_+)$ the exponential map $\exp_x : B_r(0) \subset T_xX \to X$ defines a diffeomorphism onto its image.

(B) For every $k \geq 0$, we have

$$\|\nabla^k R^X\|_{L_\infty(X_+)} < \infty \text{ and } \|(\nabla^{\partial X_+})^k \mathbf{II}\|_{L_\infty(\partial X_+)} < \infty.$$

Then, (M_+, g) is a manifold with boundary and bounded geometry.

As a next step, we define Bessel potential spaces on manifolds with boundary and bounded geometry as $H_p^s(X_+) := r_X^+ H_p^s(X)$. We endow these spaces with the image norm of r_X^+, where r_X^+ denotes the restriction in the sense of distribution to the open set X_+. We define L_+ by $u \mapsto r^+ L\tilde{u}$ with $r_X^+\tilde{u} = u$. Here, r^+ denotes the restriction of functions over the half space, applied to each component of the sequence. The operator is well-defined. It is linear and bounded as a map from $H^s_p(X_+)$ to $\mathbb{H}^s_p(\mathbb{R}^n_+)$. We define $P_+ := r^+_X PE : \mathbb{H}^s_p(\mathbb{R}^n_+) \to H^s_p(X_+)$, where E is the extension operator on functions over the half space, applied to each component of the sequence. We see that $P_+L_+ = 1$ on $H^s_p(X_+)$ in the calculation below:

$$P_+L_+u = r^+_X PEr^+L\tilde{u} = r^+_X PL\tilde{u} = r^+_X\tilde{u} = u.$$

We make use of the fact that the image of $1-Er^+$ consists of sequences of functions which have support in $\mathbb{R}^n_-$. In particular, the image of $P(1-Er^+)$ consists of functions in X which have support in $X\backslash X_+$. Thus, $E_X := PEL_+ : H^s_p(X_+) \to H^s_p(X)$ is a bounded extension operator. Note that this is a common method to construct the extension operator. We can use the operator to define the trace $\gamma^+_0 := \gamma_0 E_X : H^s_p(X_+) \to B^{s-1/p}_p(\partial X_+)$. Proposition 2.12 implies that the defined operator is bounded, for $s > 1/p$. In order to obtain a more readable notation, we write $r^+ = r^+_X$, $E = E_X$, $L = L_+$, and $P = P_+$. We define $H^s_{p;0}(X_+)$ as the closure of $C^\infty_0(X_+)$ with respect to the $H^s_p(X_+)$ norm. We observe that $H^s_{p;0}(X_+) \hookrightarrow H^s_p(X_+)$ for $s \geq 0$. We also obtain interpolation results since r^+ is a retraction with coretraction E. Let $\theta \in (0,1)$, $s = \theta s_0 + (1-\theta)s_1$ and $1 < p < \infty$, then:

$$[H^{s_1}_p(X_+), H^{s_2}_p(X_+)]_\theta = H^s_p(X_+), \tag{2.3}$$

$$[H^{s_1}_{p;0}(X_+), H^{s_2}_{p;0}(X_+)]_\theta = H^s_{p;0}(X_+), \tag{2.4}$$

$$[H^{s_1}_p(X_+), H^{s_2}_p(X_+)]_{\theta,p} = B^s_p(X_+), \tag{2.5}$$

$$[B^{s_1}_p(X_+), B^{s_2}_p(X_+)]_\theta = B^s_p(X_+) \text{ and} \tag{2.6}$$

$$[B^{s_1}_p(X_+), B^{s_2}_p(X_+)]_{\theta,p} = B^s_p(X_+). \tag{2.7}$$

3 Bounded H^∞-Calculus and Maximal Regularity

Alan McIntosh introduced the concept of a bounded H^∞-calculus in [29], first for operators on Hilbert space and later in [11] for operators on Banach space. For a more recent reference, we follow the lecture notes [27] provided by Peer Kunstmann and Lutz Weis. In particular, the perturbation results for this class of operators are important for the proof of the main result which, at least locally, is a perturbation of the case with smooth coefficients. We provide the relevant perturbation results in Section 3.2 after introducing the basic definitions in Section 3.1. Furthermore, in Section 3.3, we include some well-known results about how these operators relate to those with maximal regularity and those with bounded imaginary powers.

3.1 Definition of Bounded H^∞-calculus

Let E be a complex Banach space. Let $\mathcal{D}(A)$ be a subspace of E. A linear operator $A : \mathcal{D}(A) \to F$ is closed if its graph $\{(x, Ax) : x \in \mathcal{D}(A)\}$ is a closed subspace of $E \times F$. It is called densely defined if $D(A)$ is a dense subspace of E. The resolvent set, denoted as $\rho(A)$, is defined as all $\lambda \in \mathbb{C}$ such that $\lambda - A$ has a bounded inverse. The complement of the resolvent set is called the spectrum of A and is denoted by $\sigma(A)$. We define the sector of angle $\vartheta \in (0, \pi)$ as the following subset of the complex plane:

$$\Sigma_\vartheta := \{z \in \mathbb{C}\backslash\{0\} : |\arg z| < \vartheta\}.$$

Definition 3.1. A closed densely defined operator A is sectorial of angle $\vartheta \in (0, \pi)$, if

$$\sigma(A) \subset \overline{\Sigma}_\vartheta \text{ and } \{\lambda(\lambda - A)^{-1} : \theta \le \arg(\lambda) \le \pi\} \text{ is bounded for all } \vartheta < \theta \le \pi.$$

The infimum over all ϑ, for which A is sectorial, is called the spectral angle of A and is denoted by $\vartheta(A)$. We write $\mathcal{S}(E)$ for the set of all sectorial operators which are injective and have dense range. We denote by $H^\infty(\Sigma_\vartheta)$ all bounded holomorphic functions on the sector Σ_θ. It is well-known that the following subspace is dense with respect to the normal topology, i.e., uniform convergence on compact sets:

$$H_0^\infty(\Sigma_\vartheta) := \left\{ f \in H^\infty(\Sigma_\vartheta) : \exists\, C, \varepsilon > 0 : \ |f(\lambda)| \le C|\lambda(1+\lambda)^{-2}|^\varepsilon \right\}.$$

According to the decay properties of functions in the subspace above and of sectorial operators, the following integral is defined for all $\vartheta(A) < \theta < \vartheta$:

$$\Phi_A(f) := \frac{1}{2\pi i} \int_{\partial\Sigma_\theta} f(\lambda)(\lambda - A)^{-1}\, d\lambda. \tag{3.1}$$

It is well-known that Φ_A is an algebra morphism, formulated in the following theorem. For the proof, we refer to [27, Theorem 9.2].

Theorem 3.2. *Let E be a Banach space and $A \in \mathcal{S}(E)$ be a sectorial operator of angle $\vartheta \in (0, \pi)$. Let $\vartheta(A) < \theta < \vartheta$. Then, $\Phi_A : H_0^\infty(\Sigma_\vartheta) \to \mathcal{B}(E)$, defined by Equation* (3.1)*, is a linear and multiplicative map with the following properties:*

(i) Let $f_n, f \in H^\infty(\Sigma_\vartheta)$ *be uniformly bounded and* $f_n(\lambda) \to f(\lambda)$ *for* $\lambda \in \Sigma_\vartheta$*. Then, for all* $g \in H_0^\infty(\Sigma_\vartheta)$

$$\lim_{n\to\infty} \Phi_A(f_n g) = \Phi_A(fg) \ \ in \ \ \mathcal{B}(E).$$

(ii) If $f(\lambda) = \lambda(\mu_1 - \lambda)^{-1}(\mu_2 - \lambda)^{-1}$ *with* $\mu_1, \mu_2 \notin \overline{\Sigma}_\theta$*, then,*

$$\Phi_A(f) = A(A - \mu_1)^{-1}(A - \mu_2)^{-1}.$$

(iii) A $C > 0$ *exists such that* $\|\Phi_A(f)\| \le C \int_{\partial\Sigma_\theta} |f(\lambda)| \frac{d\lambda}{|\lambda|}$.

We observe that Φ_A is an unbounded operator and consider its closure. To this end, we provide the following definition:

Definition 3.3. Let $A \in \mathcal{S}(E)$ and $\vartheta > \vartheta(A)$. For $f \in H_0^\infty(\Sigma_\vartheta)$, we define

$$\|f\|_A := \|f\|_{L_\infty(\Sigma_\vartheta)} + \|\Phi_A(f)\|_{\mathcal{B}(E)}.$$

We further define $H_A^\infty(\Sigma_\vartheta)$ to be the class of functions $f \in H^\infty(\Sigma_\vartheta)$, for which a sequence $f_n \in H_0^\infty(\Sigma_\vartheta)$ exists with $f_n(\lambda) \to f(\lambda)$ for all $\lambda \in \Sigma_\vartheta$ and $\sup\{\|f\|_A : n \in \mathbb{N}\} < \infty$.

For functions in this class, the calculus defined in Theorem 3.2 has a unique extension. The exact result is provided in Theorem 3.4. For the proof, see [27, Theorem 9.6] for the existence and [27, Remark 9.7] for the uniqueness.

Theorem 3.4. *Let* $A \in \mathcal{S}(E)$ *and* $\vartheta > \vartheta(A)$*. Then, an extension* $\overline{\Phi}_A : H_A^\infty(\Sigma_\vartheta) \to \mathcal{B}(E)$ *of* Φ_A *exists with the following properties:*

(i) $\overline{\Phi}_A$ *is linear and multiplicative.*

(ii) $\tau_\mu := (\mu - \cdot)^{-1} \in H_A^\infty(\Sigma_\theta)$ *and* $\overline{\Phi}_A(\tau_\mu) = (\mu - A)^{-1}$ *if* $\mu \notin \overline{\Sigma}_\vartheta$.

(iii) If $f_n \in H_A^\infty(\Sigma_\vartheta)$ *and* $f \in H^\infty(\Sigma_\vartheta)$ *with* $f_n(\lambda) \to f(\lambda)$ *for all* $\lambda \in \Sigma_\vartheta$ *and* $\|f_n\|_{\Phi_A} \le C$*, then* $f \in H_A^\infty(\Sigma_\vartheta)$ *and* $\lim \overline{\Phi}_A(f_n)x = \overline{\Phi}_A(f)x$ *for all* $x \in E$ *and* $\|\overline{\Phi}_A(f)\| \le C$.

The extension is unique, i.e., if $\Psi : H_A^\infty(\Sigma_\vartheta) \to B(E)$ *satisfies (i)-(iii) then* $\Psi = \overline{\Phi}_A$.

We follow the common notation and write $f(A) := \overline{\Phi}_A f$. In general, $H_A^\infty(\Sigma_\vartheta)$ is a proper subspace of $H^\infty(\Sigma_\vartheta)$. What we are interested in is the situation in which these spaces coincide. Therefore, we define:

Definition 3.5 (Bounded H^∞-calculus)**.** Let $A \in \mathcal{S}(E)$ and $\vartheta > \vartheta(A)$. We say that A has a bounded $H^\infty(\Sigma_\vartheta)$-calculus if $H_A^\infty(\Sigma_\vartheta) = H^\infty(\Sigma_\vartheta)$. The infimum over all ϑ for which A has a bounded $H^\infty(\Sigma_\vartheta)$-calculus is denoted as $\vartheta_\infty(A)$.

According to closed graph theorem, A has a bounded H^∞-calculus if uniform estimates (3.1) exist. More formally:

Remark 3.6. *Let $A \in \mathcal{S}(E)$ and $\vartheta > \vartheta(A)$. Then, A has a bounded $H^\infty(\Sigma_\vartheta)$-calculus if and only if a $C > 0$ exists such that $\|f(A)\|_{\mathcal{B}(E)} \leq C\|f\|_{L_\infty(\Sigma_\vartheta)}$ for all $f \in H_0^\infty(\Sigma_\vartheta)$.*

For later argumentation, we need the following lemma:

Lemma 3.7. *Let E be a Banach space, $\mathcal{D}(A)$ a dense subspace, and $A : \mathcal{D}(A) \to E$ a linear operator. Assume that there a another Banach space F and operators L, P, B, and B' satisfying the following criteria exist:*

(i) *$L \in \mathcal{B}(E, F)$ and $P \in \mathcal{B}(F, E)$ such that $PL = 1$ on E.*

(ii) *$B, B' \in \mathcal{S}(F)$ have a bounded $H^\infty(\Sigma_\vartheta)$-calculus.*

(iii) *$L : \mathcal{D}(A) \to \mathcal{D}(B)$ and $P : \mathcal{D}(B') \to \mathcal{D}(A)$.*

(iv) *$LA = BL$ on $\mathcal{D}(A)$ and $AP = PB'$ on $\mathcal{D}(B)$.*

Then, A has a bounded $H^\infty(\Sigma_\vartheta)$-calculus.

Proof. We first verify that A is a sectorial operator. Let $\vartheta > \omega := \max\{\vartheta(B), \vartheta(B')\}$ and $\lambda \in \Sigma_\vartheta^c$. By assumption, $\lambda \in \rho(B) \cap \rho(B')$. We observe that $\lambda - A$ has a left inverse $P(\lambda - B')^{-1}L$ and a right inverse $P(\lambda - B)^{-1}L$. Moreover, the following estimate holds:

$$\|\lambda(\lambda - A)^{-1}\|_{\mathcal{B}(E)} = \|P\lambda(\lambda - B)^{-1}L\|_{\mathcal{B}(E)} \leq \|P\|_{\mathcal{B}(F,E)}\|\lambda(\lambda - B)^{-1}\|_{\mathcal{B}(F)}\|L\|_{\mathcal{B}(E,F)}$$

In particular, $\rho(A) \neq \emptyset$. Thus, A is closed operator. Moreover, we have shown that $A \in \mathcal{S}(E)$ and $\vartheta(A) \geq \omega$. Now, let $\vartheta > \theta > \omega_\infty := \max\{\vartheta_\infty(B), \vartheta_\infty(B')\}$ and $f \in H_0^\infty(\Sigma_\vartheta)$. Then:

$$\begin{aligned}
\|f(A)\|_{\mathcal{B}(E)} &= \left\| \int_{\partial\Sigma_\theta} f(\lambda)(\lambda - A)^{-1}\, d\lambda \right\|_{\mathcal{B}(E)} \\
&\leq \|P\|_{\mathcal{B}(F,E)} \left\| \int_{\partial\Sigma_\theta} f(\lambda)(\lambda - B)^{-1}\, d\lambda \right\|_{\mathcal{B}(F)} \|L\|_{\mathcal{B}(E,F)} \\
&\leq C\|P\|_{\mathcal{B}(F,E)}\|L\|_{\mathcal{B}(E,F)}\|f\|_{L_\infty(\Sigma_\theta)}.
\end{aligned}$$

Here, the constant $C > 0$ is the smallest which satisfies $\|f(B)\|_{\mathcal{B}(F)} < C\|f\|_{L_\infty(\Sigma_\vartheta)}$. We apply Remark 3.6 to finish the proof. □

3.2 Perturbation

This section addresses the question under which condition $A + B$ has a bounded $H^\infty(\Sigma_\vartheta)$-calculus in E, given that A has. First of all, we need the sum to define a sectorial operator. It is well-known that the sum is sectorial if B is A-bounded, i.e., B is a closed operator with $\mathcal{D}(B) \supset \mathcal{D}(A)$ and

$$\|Bx\|_E \leq C\|Ax\|_E \text{ for all } x \in \mathcal{D}(A).$$

According to the counterexample by Alan McIntosh and Atsushi Yagi in [30], the above condition cannot be sufficient for a bounded H^∞-calculus. We thus need additional assumptions. For example, the result below holds:

Theorem 3.8. *Let $A \in \mathcal{S}(E)$ have a bounded $H^\infty(\Sigma_\vartheta)$-calculus in E and $0 \in \rho(A)$. Let $\gamma \in (0,1)$ and suppose that B is a linear operator in E, satisfying $\mathcal{D}(B) \supset \mathcal{D}(A)$ and*

$$\|Bu\|_E \leq C\|A^{1-\gamma}u\|_E \text{ for all } u \in \mathcal{D}(A).$$

Then, $\nu + A + B$ has a bounded $H^\infty(\Sigma_\vartheta)$-calculus in E for $\nu \geq 0$ sufficiently large.

For the proof, we refer to [27, Proposition 13.1]. From now on, we assume that E is a uniformly convex Banach space which is true for all subspaces and quotient spaces of L_p-spaces with $1 < p < \infty$. Furthermore, we need a result for small perturbations:

Theorem 3.9. *Let E be a Banach space with the UMD property and $A \in \mathcal{S}(E)$ have a bounded $H^\infty(\Sigma_\vartheta)$-calculus and $0 \in \rho(A)$. Let B be a linear operator in E such that $\mathcal{D}(A) \subset \mathcal{D}(B)$ and an $\varepsilon > 0$ exist such that*

$$\|Bu\|_E \leq \varepsilon\|Au\|_E \text{ for all } u \in \mathcal{D}(A).$$

Suppose further that $\gamma \in (0,1)$ and a constant $C > 0$ exists such that

$$B(\mathcal{D}(A^{1+\gamma})) \subset \mathcal{D}(A^\gamma) \text{ and } \|A^\gamma Bx\|_E \leq C\|A^{1+\gamma}x\|_E \text{ for } x \in \mathcal{D}(A^{1+\delta}).$$

Then, $A+B$ has a bounded $H^\infty(\Sigma_\vartheta)$-calculus in E, provided ε is sufficiently small. Moreover, a constant $C_{A+B} := C_{A+B}(C_A, \varepsilon, C)$ exists such that

$$\|f(A+B)\|_{\mathcal{B}(E)} \leq C_{A+B}\|f\|_{H^\infty(\Sigma_\theta)}.$$

Here, C_A is the best constant that satisfies the above estimate with $B = 0$. For the proof, we refer to [12]. The size of ε depends on the constant C and the $\mathcal{R}$-bound of the resolvent of A over the sector Σ_ϑ, see also [27]. There, A is assumed to be $\mathcal{R}$-sectional which is true for operators with an $H^\infty(\Sigma_\vartheta)$-calculus in UMD spaces.
The term UMD is an abbreviation for „unconditional martingale differences“. For a precise definition of the UMD property, we refer to [25]. For this thesis, the most primitive examples of UMD spaces are sufficient: Any Hilbert space has the UMD property. Moreover, if $(\Omega, \mathcal{B}, \mu)$ is a sigma finite measure space, $1 < p < \infty$ and E has the UMD property, then, $L_p(\Omega, \mu; E)$ also has. In particular, $L_p(\mathbb{R}^n_+)$ and $\mathbb{L}_p(\mathbb{R}^n_+) = l_p(\Gamma; L_p(\mathbb{R}^n_+))$ have the UMD property.

3.3 Bounded Imaginary Powers and Maximal Regularity

The functional calculus has a unique extension to slowly growing holomorphic functions. We write $\varrho(\lambda) = \lambda\langle\lambda\rangle^{-2}$. For $\gamma > 0$, we define $H^\infty_\gamma(\Sigma_\vartheta)$ as the space of all holomorphic function on the sector that satisfy the following estimate:

$$\|f\|_{H^\infty_\gamma(\Sigma_\theta)} := \sup\{|\varrho(\lambda)|^\gamma |f(\lambda)| : \lambda \in \Sigma_\vartheta\} < \infty.$$

For each $f \in H^\infty_\gamma(\Sigma_\vartheta)$ and $A \in \mathcal{S}(E)$ which allows a bounded $H^\infty(\Sigma_\vartheta)$-calculus, we define an unbounded operator $f(A)$. We fix an even integer $k > \gamma$ and define:

$$\mathcal{D}(f(A)) := \{u \in E : [\varrho^k f](A)u \in \mathcal{D}(A^k) \cap R(A^k)\} \to E,\, u \mapsto \rho^{-k}(A)[\varrho^k f](A)u.$$

Here, $\mathcal{D}(A^k) \cap R(A^k)$ is the domain of $\varrho^{-k}(A)$. It is well-known that the operator defined above is closable. We make no notational distinction for the closure. For more details, see the appendix in [27]. The extension of the calculus is sufficient to treat complex powers, i.e., $\lambda^z \in H^\infty_{\Re z}(\Sigma_\vartheta)$ for all $\vartheta \in (0, \pi)$. In particular, imaginary powers are defined:

Definition 3.10. Let $A \in \mathcal{S}(E)$. We say A has bounded imaginary powers if, for all $t \in \mathbb{R}$, the operator $A^{it} \in \mathcal{B}(E)$ and constants $C, \vartheta > 0$ exists such that the following estimate holds:

$$\|A^{it}\|_{\mathcal{B}(E)} \leq Ce^{\vartheta|t|}.$$

The infimum over all ϑ is called the power angle, we denote it by $\vartheta_p(A)$.

Every operator $A \in \mathcal{S}(E)$ which has a bounded $H^\infty(\Sigma_\vartheta)$-calculus has bounded imaginary powers because $\exp(-\vartheta|t|)z^{it} \in H^\infty(\Sigma_\vartheta)$. Moreover, $\vartheta_p \leq \vartheta_\infty$. For an operator which has bounded imaginary powers, the domain of fractional power can be described via complex interpolation. For the proof, see [47, Theorem 1.15.2]:

Theorem 3.11. *If $A \in \mathcal{S}(E)$ has bounded imaginary powers and $0 \in \rho(A)$, then*

$$\mathcal{D}(A^\gamma) = [E, \mathcal{D}(A)]_\gamma \ \textit{ for all } \ \gamma \in (0, 1).$$

Under suitable conditions, operators which have bounded imaginary powers, also have maximal L_q-regularity, see Theorem 3.12. We only consider maximal L_q-regularity. Thus, dropping the specifier L_q is reasonable. We recall the definition: Let $-A$ be the generator of an analytic semigroup on a Banach space E. The solution to the Cauchy problem $\dot{y} + Ay = f$ with data $f \in L_q([0, T); E)$ and $y_0 = 0$ is provided by the „variation of constants formula“:

$$y(t) = \int_0^t T_{t-s} f(s)\, ds.$$

We say that A has maximal regularity on $[0, T)$, if y is differentiable almost everywhere, takes values in $\mathcal{D}(A)$, and the following estimate holds:

$$\|\dot{y}\|_{L_q([0,T);E)} + \|Ay\|_{L_q([0,T);E)} \leq C\|f\|_{L_q([0,T);E)}.$$

Theorem 3.12 (Dore & Venni)**.** *Let E be a Banach space with the UMD property and let $A \in \mathcal{S}(E)$ have bounded imaginary powers with $\vartheta_p(A) < \pi/2$. Then, A has maximal regularity.*

For the proof, we refer to [14]. Maximal regularity has several applications, for an overview and more details, see [27]. A first result, Theorem 3.13, by Philippe Clément and Shuanhu Li in [10] proves the short time existence of solutions to quasi-linear parabolic equations such as:

$$\dot{u}(t) + A(u(t))u(t) = f(t, u(t)) \text{ and } u(t_0) = u_0, \tag{3.2}$$

in $L_q(0, T; E_0)$ for some $1 < q < \infty$, for some finite T, and $\mathcal{D}(A(u(t))) = E_1$.

Theorem 3.13 (Clément & Li)**.** *We assume that $A(u_0)$ has maximal regularity and a neighbourhood U of u_0 exists in $E_q = [E_1, E_0]_{1/q,q}$ such that for all $u, u' \in U$:*

(i) $\|A(u) - A(u')\|_{\mathcal{B}(E_1;E_0)} \leq C\|u - u'\|_{E_q}$.

(ii) $\|f(t, u) - f(t', u')\|_{E_0} \leq C(\|u - u'\|_{E_q} + |t - t'|)$.

Then, a $T_* > 0$ exists such that the Equation (3.2) *has a unique solution in:*

$$L_q(0, T_*; E_1) \cap H_q^1(0, T_*; E_0).$$

According to [3, Theorem III.4.10.2], $u \in L_q(0, T_*; E_1) \cap H_q^1(0, T_*; E_0)$ implies that $u \in C([0, T_*]; E_q)$. One of the applications of the theory developed in this thesis is the short time existence of a solution to the porous medium equation. For the proof, we rely on the theorem above.

4 Boutet de Monvel's Calculus

In [9], Louis Boutet de Monvel introduced an algebra of operators which contains classical boundary value problems and parametrises, if they exists. Several monographs and research papers exist on this topic, most notably [20], [32], and [35]. The mentioned works all assume that the underlying pseudodifferential operators have symbols in $S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. For our purpose, we need operators based on symbols in the class $S^m_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$. Some results in the previous literature dealt with these operators, for instance [21]. The purpose of this section is to extend the basic results on Boutet de Monvel's calculus to the class of operators based on symbols in $S^m_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ with $0 \leq \delta < 1$. The proofs given in [20], [32], or [35] only need minor modifications. Adjusting the definitions and proofs in Boutet de Monvel's calculus allows us to fix the notation and conventions. For two reasons, we focus on operators that act on the half space with uniformly estimated symbols: First, it is sufficient for later use and second, localisation of symbol classes and transporting the results to manifolds is a well-known process.

4.1 Pseudodifferential Operators

Section 4.1 summarises the relevant definitions and results on pseudodifferential operators. We refer to [23] and [28] for more details. Operator-valued symbols, defined below, are important for later use. We follow [35] for the notation in this context. By σ_λ, we denote a strongly continuous group action on a Banach space E. If not specified differently this action is the trivial one, if $E = \mathbb{C}$, and is scaling action, if E is a function space on the (half-)line. The latter is defined as $[\sigma_{p,\lambda} f](x) = \lambda^{1/p} f(\lambda)$.

Definition 4.1. Let E, F be Banach spaces with strongly continuous group actions σ on E and $\tilde{\sigma}$ and F. A function $p \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; \mathcal{B}(E,F))$ is an operator-valued symbol of order $m \in \mathbb{R}$ and Hörmander type $(1,\delta)$, with $0 \leq \delta < 1$, if for any indices $\alpha, \beta, \gamma \in \mathbb{N}_0^n$ a constant $C = C_{\alpha,\beta,\gamma}$ exists such that

$$\|\tilde{\sigma}^{-1}_{\langle \xi' \rangle} D^\alpha_\xi D^\beta_x D^\gamma_y p(x,y,\xi) \sigma_{\langle \xi \rangle}\|_{\mathcal{B}(E,F)} \leq C \langle \xi \rangle^{m - |\alpha| + \delta(|\beta| + |\gamma|)} \tag{4.1}$$

We write $p \in S^m_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; (E,\sigma), (F,\tilde{\sigma}))$.

The expression $\langle \xi' \rangle := (1 + |\xi'|^2)^{1/2}$ is a standard notation in the context of pseudodifferential operators and is used throughout this thesis. For a shorter notation, the group actions mentioned above are dropped. To further simplify the notation, we write $\sigma_p = \sigma_{p,\langle \xi' \rangle}$. It is well-known that these symbol spaces are Fréchet spaces, endowed with the topology of best constant in (4.1). The definition of operator-valued symbols extends to projective and inductive limits. Let $E_1 \hookrightarrow E_2 \hookrightarrow \dots$ and $F_1 \hookleftarrow F_2 \hookleftarrow \dots$ be sequences of Banach spaces with the same group action, then:

$$S^m_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; E, \text{proj-lim}_k F_k) := \text{proj-lim}\, S^m_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; E, F_k),$$
$$S^m_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; \text{ind-lim}_k E_k, F) := \text{proj-lim}_k\, S^m_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; E_k, F), \quad \text{and}$$

$$S^m_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; \text{ind-lim}_k X_k, \text{proj-lim}_l Y_l) := \text{proj-lim}\, S^m_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; E_k, F_l).$$

In particular, $S^m_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; \mathcal{S}'(\mathbb{R}_+), \mathbb{C})$, $S^m_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; \mathbb{C}, \mathcal{S}(\mathbb{R}_+))$, and $S^m_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; \mathcal{S}'(\mathbb{R}_+), \mathcal{S}(\mathbb{R}_+))$ are defined and play a major role in Boutet de Monvel's calculus. For each symbol, we define an operator on Schwartz functions:

$$\text{op}(p) : \mathcal{S}(\mathbb{R}^n; E) \to \mathcal{S}(\mathbb{R}^n; F), \quad u \mapsto \int\int e^{i(x-y)\xi} p(x, y, \xi) u(y) dy đ\xi,$$

with the notation $đ\xi = (2\pi)^{-n} d\xi$. It is well-known that the mapping $p \mapsto \text{op}(p)$ is linear and bounded. In general, this mapping is not injective. However, it can be made injective, if we restrict the set of variables that p depends on. The most common restrictions only allow x-dependence or y-dependence. Then, p is the left or right symbol, respectively. Furthermore, we need symbols with (x', y_n)-dependency. Restricting the dependency of symbols is a continuous operation. We assume that symbols only depend on x, if not mentioned otherwise. Therefore, each pseudodifferential operator has a unique symbol. We denote operators by capital letters and symbols by small letters, i.e., $P = \text{op}(p)$. It is well-known that pseudodifferential operators form an algebra. In particular, the composition of two pseudodifferential operators is again a pseudodifferential operator. More formally:

Theorem 4.2. *There is a bounded bilinear map:*

$$\# : S^m_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n; E, E') \times S^{m'}_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n; E', F) \to S^{m+m'}_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n; E, F), \quad (p, q) \mapsto p\#q$$

given by the property that $\text{op}(p)\,\text{op}(q) = \text{op}(p\#q)$*. Moreover,*

$$p\#q \sim \sum_\alpha \frac{1}{\alpha!} [\partial^\alpha_x p](x, \xi) [D^\alpha_\xi q](x, \xi). \tag{4.2}$$

An explicit formula for $p\#q$ as an oscillatory integral that depends on the symbols p and q is available, see for instance [28]. Equation (4.2) gives the asymptotic expansion of the composed symbol, in the sense of the following definition:

Definition 4.3. Let $(m_j)_{j\in\mathbb{N}_0}$ be a monotonously decreasing sequence converging to $-\infty$. Let $p \in S^{m_0}(\mathbb{R}^n \times \mathbb{R}^n; E, F)$ and $p_j \in S^{m_j}(\mathbb{R}^n \times \mathbb{R}^n; E, F)$. We write

$$p \sim \sum_{j=0}^{\infty} p_j :\Leftrightarrow \forall N \in \mathbb{N} : \quad p - \sum_{j=0}^{N-1} p_j \in S^N_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n; E, F). \tag{4.3}$$

The symbol p is the asymptotic sum of $(p_j)_{j\in\mathbb{N}_0}$ which is unique modulo smoothing symbols, i.e., if p and q satisfy (4.3) then $p - q \in S^{-\infty}(\mathbb{R}^n \times \mathbb{R}^n; E, F)$. A key property of the symbol spaces defined above is that they are closed under asymptotic summation. This means: Given a sequence $(p_j)_{j\in\mathbb{N}_0}$ as in Definition 4.3, we can always find a $p \in S^{m_0}_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n; E, F)$ such that (4.3) holds. For a proof, we refer to [23]. Therefore, the symbol spaces are closed under parametrix construction. The exact statement is:

Theorem 4.4. *Let $p \in S^m_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ and $p(x,\xi)$ be invertible and $|p^{-1}(x,\xi)| \leq C\langle\xi\rangle^{-m}$ for $|\xi| \geq R \geq 0$. Then, a $p^{-\#} \in S^{-m}_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ exists such that $p\#p^{\#} = 1 + r$ and $p^{-\#}\#p = 1+l$ with $r, l \in S^{-\infty}(\mathbb{R}^n \times \mathbb{R}^n)$. Moreover, $|p^{-\#}|_* \leq C(\omega, |p|_*)$, $|r|_* \leq C(\omega, |p|_*)$ and $|l|_* \leq C(\omega, |p|_*)$.*

Above, we pointed out that every operator whose symbol only depends on x can also be expressed in terms of a symbol which depends on (x', y_n). These symbols are related by an asymptotic formula:

Lemma 4.5. *Let $p \in S^m_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$, then a $\tilde{p} \in S^m_{1,\delta}(\mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}^n)$ exists such that*

$$Pu = \int\int e^{ix'\xi'} \tilde{p}(x', y_n, \xi) \acute{u}(\xi', y_n)\, dy_n d\xi'.$$

Here, $\acute{u}(\xi', y_n) := [\mathcal{F}_{x' \mapsto \xi'} u](\xi', y_n)$. Moreover,

$$\tilde{p}(x', y_n, \xi) \sim \sum_{k \in \mathbb{N}_0} \frac{1}{k!} [\partial^k_{x_n} D^k_{\xi_n} p](x', y_n, \xi).$$

For the proof, we refer to [28]. To close this section, we recall the well-known mapping properties of pseudodifferential operators. For the proof, we refer to [1, Theorem 3.3 and 3.4]:

Theorem 4.6. *Let $p \in S^m_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n; E, F)$, with $0 \leq \delta < 1$. The group actions are assumed to be isometric. Then the following results hold.*

(a) If E and F are Hilbert spaces, then

$$P \in \mathcal{B}(H^s_p(\mathbb{R}^n, E), H^{s-m}_p(\mathbb{R}^n, F)) \ \text{ and } \ \|P\| \leq C|p|_*.$$

(b) If E and F are UMD spaces, then

$$P \in \mathcal{B}(B^s_p(\mathbb{R}^n, E), B^{s-m}_p(\mathbb{R}^n, F)) \ \text{ and } \ \|P\| \leq C|p|_*.$$

4.2 Wiener-Hopf Calculus

The simplest instance of Boutet de Monvel's calculus is the one dimensional case with constant coefficients. Here, the operators are Fourier multipliers of a certain type which are known as Wiener-Hopf operators. We recall some well-known facts about these operators, for more details, see [9], [20], and [32]. In order to describe Wiener-Hopf operators, we need additional notation.

Definition 4.7. We write $\mathcal{H}'_d$ for all polynomials of degree less than d and $\mathcal{H} := \cup\mathcal{H}_d$.

1. $\mathcal{H}^+ := \{\mathcal{F}(e^+u) : u \in \mathcal{S}(\mathbb{R}_+)\}$,
2. $\mathcal{H}^-_{-1} := \{\mathcal{F}(e^-u) : u \in \mathcal{S}(\mathbb{R}_-)\}$,

3. $\mathcal{H}^- := \mathcal{H}^-_{-1} \oplus \mathcal{H}'$ and

4. $\mathcal{H} := \mathcal{H}^+ \oplus \mathcal{H}^-$.

Well-known is that $\mathcal{H}$, $\mathcal{H}^+$, $\mathcal{H}^-$, and $\mathcal{H}^-_{-1}$ are algebras with respect to pointwise multiplication. Projections $h^\pm : \mathcal{H} \to \mathcal{H}^\pm$ exist. On $\mathcal{H}^+ \oplus \mathcal{H}^-_{-1} \subset L_2$, these projections are orthogonal and given by $h^\pm = \mathcal{F} e^\pm r^\pm \mathcal{F}^{-1}$. Moreover, we define the plus integral:

$$\int^+ : \mathcal{H} \to \mathbb{C},\ u \mapsto \int^+ u(\xi)\, đ\xi := \gamma_0 r^+ \mathcal{F}^{-1} u.$$

The plus integral only depends on h^+u, i.e., it is zero on $\mathcal{H}^-$. According to the dominate convergence theorem, if u is integrable, then the plus integral coincides with the usual integral, justifying the notation. The Cauchy Integral Theorem implies that for $u \in \mathcal{H}^+ \cap L_1$ the plus integral vanishes. In particular, the plus integral of pu for $u \in \mathcal{H}^+$ and $p \in \mathcal{H}$ only depends on h^-p. For more details, we refer to [32]. For Boutet de Monvel's calculus, the following definition is crucial:

Definition 4.8. We define a Wiener-Hopf operator:

$$\mathbf{a} = \begin{pmatrix} \mathbf{p} + \mathbf{g} & \mathbf{k} \\ \mathbf{t} & \mathbf{s} \end{pmatrix} : \begin{matrix} \mathcal{H}^+ \otimes E \\ \oplus \\ F \end{matrix} \to \begin{matrix} \mathcal{H}^+ \otimes E' \\ \oplus \\ F' \end{matrix}$$

The components are defined as follows:

(a) To $p \in \mathcal{H}_m \otimes \hom(E, E')$, we associate the operator $\mathbf{p}u := h^+(pu)$.

(b) To $g \in \mathcal{H}^+ \hat{\otimes} \mathcal{H}^-_d \otimes \hom(E, E')$, we associate an operator $\mathbf{g}u := \int^+ g(\cdot, \eta) u(\eta) đ\eta$.

(c) To $k \in \mathcal{H}^+ \otimes \hom(F, E')$, we associate the operator $\mathbf{k}\phi := k\phi$.

(d) To $t \in \mathcal{H}^-_d \otimes \hom(E, E')$, we associate the operator $\mathbf{t}u := \int^+ t(\eta) u(\eta) đ\eta$.

(e) To $s \in \hom(F, F')$, we associate the operator $\mathbf{s}\phi := s\phi$.

We call $m \in \mathbb{Z}$ the order and $d \in \mathbb{N}_0$ the class of the operator and write $\mathcal{WH}^{m,d}(E, F; E', F')$ for the space of all the Wiener-Hopf operators.

The class of Wiener-Hopf operators is closed under summation, given that the vector spaces match. What is less obvious is the fact that they are also closed under composition:

Theorem 4.9 (Composition)**.** *Let* $\mathbf{a} \in \mathcal{WH}^{m,d}(E, F, E', F')$ *and* $\mathbf{a}' \in \mathcal{WH}^{m',d'}(E', F', E'', F'')$. *Then,* $\mathbf{a}'' := \mathbf{a}\mathbf{a}' \in \mathcal{WH}^{m'',d''}(E, F, E'', F'')$. *The order is* $m'' = m + m'$ *and the class is* $d'' = \max(m' + d, d')$. *The components are given in the following list:*

1. $\mathbf{p}'' - \mathbf{g}''_1 = \mathbf{p}\mathbf{p}'$, *with* $p'' = pp'$ *and*

$$g''_1(\xi, \eta) := h^+_\xi h^-_\eta \frac{([h^-p](\xi) - [h^-p](\eta))\,([h^+p'](\xi) - [h^+p'](\eta))}{i(\xi - \eta)}.$$

2. $\mathbf{g}_2'' = \mathbf{p}\mathbf{g}'$, *with* $g_2''(\xi,\eta) := h_\xi^+ p(\xi) g'(\xi,\eta)$.

3. $\mathbf{g}_3'' = \mathbf{g}\mathbf{p}'$, *with* $g_3''(\xi,\eta) := h_\eta^- g(\xi,\eta) p'(\eta)$.

4. $\mathbf{g}_4'' = \mathbf{g}\mathbf{g}'$, *with* $g_4''(\xi,\eta) := \int^+ g(\xi,\zeta) g'(\zeta,\eta)\, đ\zeta$.

5. $\mathbf{g}_5'' = \mathbf{k}\mathbf{t}'$, *with* $g_5''(\xi,\eta) := k(\xi) t'(\eta)$.

6. $\mathbf{k}_1'' = \mathbf{p}\mathbf{k}'$, *with* $k_1'' := h^+(pk')$.

7. $\mathbf{k}_2'' = \mathbf{g}\mathbf{k}'$, *with* $k_2''(\xi) := \int^+ g(\xi,\eta) k'(\eta)\, đ\xi$.

8. $\mathbf{k}_3'' = \mathbf{k}\mathbf{s}'$, *with* $k_3'' := ks'$.

9. $\mathbf{t}_1'' = \mathbf{t}\mathbf{p}'$, *with* $t_1'' := h^-(tp')$.

10. $\mathbf{t}_2'' = \mathbf{t}\mathbf{g}'$, *with* $t_2'' := \int^+ t(\eta) g'(\eta,\xi)\, đ\eta$.

11. $\mathbf{t}_3'' = \mathbf{s}\mathbf{t}'$, *with* $t_3'' := st'$.

12. $\mathbf{s}_1'' = \mathbf{t}\mathbf{k}'$, *with* $s_1'' := \int^+ t(\eta) k'(\eta)\, đ\eta$.

13. $\mathbf{s}_2'' = \mathbf{s}\mathbf{s}'$, *with* $s_2'' := ss'$.

For the proof, we refer to Boutet de Monvel's original work [9, Theorem 1.12]. Moreover, the calculus is closed under inversion:

Theorem 4.10. *If* $\mathbf{a} \in \mathcal{WH}^{m,d}$ *is invertible, then* $\mathbf{a}^{-1} \in \mathcal{WH}^{-m,\max\{d-m,0\}}$.

A proof can be found in [9, Proposition 1.15]. Additionally, the following result is true, see for instance [32].

Theorem 4.11. *The following map is injective, linear, and bounded:*

$$\begin{pmatrix} \mathcal{H}^m \times \mathcal{H}^+ \hat{\otimes} \mathcal{H}^d_- & \mathcal{H}^+ \\ \mathcal{H}^d_- & \mathbb{C} \end{pmatrix} \ni a \mapsto \mathbf{a} \in \mathcal{B}(\mathcal{H}^+ \otimes E \oplus F; \mathcal{H}^+ \otimes E \oplus F)$$

4.3 Potential, Trace, and Singular Green Operators

In this section, we introduce three types of symbols which are part of the definition of Boutet de Monvel's calculus, see Definition 4.12. The action normal to the boundary of the associated operators can be interpreted as an operator-valued pseudodifferential operator. This interpretation is particularly useful to provide the composition rule for Boutet de Monvel operators.

Definition 4.12. Let $m \in \mathbb{R}$, $1 < p, q < \infty$, $1 = 1/p + 1/q$, and $d \in \mathbb{N}_0$. All functions below may be matrix valued.

- A function $k \in C^\infty(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R})$ belongs to the space $\mathcal{K}^m_{1,\delta}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$ of potential symbols of order m and Hörmander type $(1, \delta)$, if:

$$k_{[0]}(x', \xi'; \xi_n) := [\sigma_q k](x', \xi'; \xi_n) = \langle \xi' \rangle^{1/q} k(x', \xi'; \langle \xi' \rangle \xi_n) \in S^m_{1,\delta}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}) \hat{\otimes} \mathcal{H}^+_{\xi_n}.$$

- A function $t \in C^\infty(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R})$ belongs to the space $\mathcal{T}^{m,d}_{1,\delta}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$ of trace symbols of order m, class d, and Hörmander type $(1, \delta)$, if:

$$t_{[0]}(x', \xi'; \xi_n) := [\sigma_p t](x', \xi'; \xi_n) = \langle \xi' \rangle^{1/p} t(x', \xi'; \langle \xi' \rangle \xi_n) \in S^m_{1,\delta}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}) \hat{\otimes} \mathcal{H}^-_{d-1}.$$

- A function $g \in C^\infty(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R})$ belongs to the space $\mathcal{G}^{m,d}_{1,\delta}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$ of singular Green symbols of order m, class d, and Hörmander type $(1, \delta)$, if:

$$g_{[0]}(x', \xi'; \xi_n, \eta_n) := \langle \xi' \rangle g(x', \xi'; \langle \xi' \rangle \xi_n, \langle \xi' \rangle \eta_n) \in S^m_{1,\delta}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}) \hat{\otimes} \mathcal{H}^+_{\xi_n} \hat{\otimes} \mathcal{H}^-_{d-1,\eta_n}.$$

Note that $g_{[0]} = \sigma_q \sigma_p g$, if the group actions are applied to ξ_n and η_n, respectively.

The spaces $\mathcal{K}^m_{1,0}$, $\mathcal{T}^m_{1,0}$, and $\mathcal{G}^m_{1,0}$ are denoted as $S^{m-1/q}_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathcal{H}^+)$, $S^{m-1/p}_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathcal{H}^-_{d-1})$, and $S^{m-1}_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathcal{H}^+ \hat{\otimes} \mathcal{H}^-_{d-1})$ by Gerd Grubb in [20] . Stephan Rempel and Bert-Wolfgang Schulze denote them in [32] as $\mathfrak{K}^{m-1/q}(\mathbb{R}^{n-1} \times \mathbb{R}^n)$, $\mathfrak{T}^{m-1/p,d}(\mathbb{R}^{n-1} \times \mathbb{R}^n)$, and $\mathfrak{B}^{m-1,d}(\mathbb{R}^{n-1} \times \mathbb{R}^{n+1})$. The previously defined spaces are Fréchet spaces with the obvious topologies. For fixed (x', ξ'), the symbols above define Wiener-Hopf operators. Via Fourier transform, we obtain an action in the direction normal to the boundary:

$$\mathbf{k}(x', \xi') := r^+ \mathcal{F}^{-1}_{\xi_n \to x_n} \mathbf{k}(x', \xi', \xi_n) : \mathbb{C} \to \mathcal{S}(\mathbb{R}_+), \tag{4.4}$$
$$\mathbf{t}(x', \xi') := \mathbf{t}(x', \xi'; \xi_n) \mathcal{F}_{y_n \to \xi_n} e^+ : \mathcal{S}(\mathbb{R}_+) \to \mathbb{C} \text{ and} \tag{4.5}$$
$$\mathbf{g}(x', \xi') := r^+ \mathcal{F}^{-1}_{\xi_n \to x_n} \mathbf{g}(x', \xi'; \xi_n, \eta_n) \mathcal{F}_{y_n \to \eta_n} e^+ : \mathcal{S}(\mathbb{R}_+) \to \mathcal{S}(\mathbb{R}_+). \tag{4.6}$$

First of all, we focus on operators of class zero. We define the class of symbol-kernels as:

$$\begin{aligned} \tilde{k}(x', \xi'; x_n) &:= r^+ \mathcal{F}^{-1}_{\xi_n \to x_n} k(x', \xi'; \xi_n), \\ \tilde{t}(x', \xi'; x_n) &:= r^+ \overline{\mathcal{F}}^{-1}_{\xi_n \to x_n} t(x', \xi'; \xi_n), \quad \text{and} \\ \tilde{g}(x', \xi'; x_n, y_n) &:= r^+ \mathcal{F}^{-1}_{\xi_n \to x_n} \overline{\mathcal{F}}^{-1}_{\eta_n \to y_n} g(x', \xi'; \xi_n, \eta_n). \end{aligned}$$

The action in the direction normal to the boundary on $u \in \mathcal{S}(\mathbb{R}_+)$, respectively $\phi \in \mathbb{C}$, can thus be written as:

$$\begin{aligned} [\mathbf{k}(x', \xi')\phi](x_n) &= \tilde{k}(x', \xi'; x_n)\phi, \\ \mathbf{t}(x', \xi')u &= \int_0^\infty \tilde{t}(x', \xi'; y_n) u(y_n)\, dy_n, \quad \text{and} \\ [\mathbf{g}(x', \xi')u](x_n) &= \int_0^\infty \tilde{g}(x', \xi'; x_n, y_n) u(y_n)\, dy_n. \end{aligned}$$

We denote the space of all potential, trace, and singular Green symbol-kernels as:

$$\widetilde{\mathcal{K}}^m_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}),\;\; \widetilde{\mathcal{T}}^m_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}),\;\text{ resp. }\; \widetilde{\mathcal{G}}^m_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}).$$

The previous mentioned spaces inherit a tensor product structure from scaled symbol-kernels, i.e.:

$$\begin{aligned}\tilde{k}_{[0]} &:= \sigma_p^{-1}\tilde{k} \in S^m_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})\hat{\otimes}_\pi\mathcal{S}(\mathbb{R}_+),\\ \tilde{t}_{[0]} &:= \sigma_q^{-1}\tilde{t} \in S^m_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})\hat{\otimes}_\pi\mathcal{S}(\mathbb{R}_+),\;\text{ and}\\ \tilde{g}_{[0]} &:= \sigma_p^{-1}\sigma_p^{-1}\tilde{g} \in S^m_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})\hat{\otimes}_\pi\mathcal{S}(\mathbb{R}^2_{++}).\end{aligned}$$

This furnishes a natural topology on the symbol kernel spaces. With these topologies, the map taking a symbol to its symbol-kernel is bounded. This is a direct consequence of the interaction of scaling and Fourier transform. We observe that $\mathbf{t}(x',\xi')$ and $\mathbf{g}(x',\xi')$ extend to $\mathcal{S}'(\mathbb{R}_+)$, because they are integral operators with a kernel in $\mathcal{S}(\mathbb{R}_+)$. It is useful to have a more explicit description of the topology of symbol spaces and symbol-kernel spaces available. To this end, we need some properties of the scaling operator.

Lemma 4.13. *Let $[\sigma_p f](x_n) := \langle\xi'\rangle^{1/p} f(\langle\xi\rangle x_n)$, then the following results hold:*

(i) $\int f(x_n)[\sigma_p g](x_n)\,dx_n = \int[\sigma_q^{-1}f](x_n)g(x_n)\,dx_n$.

(ii) $\sigma_p\mathcal{F}_n = \mathcal{F}_n\sigma_q^{-1}$ *and* $\sigma_p\mathcal{F}_n^{-1} = \mathcal{F}_n^{-1}\sigma_q^{-1}$.

(iii) $\|\sigma_p f\|_{L_1} = \langle\xi'\rangle^{-1/q}\|f\|_{L_1}$, $\|\sigma_p f\|_{L_p} = \|f\|_{L_p}$ *and* $\|\sigma_p f\|_{L_\infty} = \langle\xi'\rangle^{1/p}\|f\|_{L_\infty}$.

(iv) $\|\sigma_p^{-1} f\|_{L_1} = \langle\xi'\rangle^{1/q}\|f\|_{L_1}$, $\|\sigma_p f\|_{L_p} = \|f\|_{L_p}$ *and* $\|\sigma_p^{-1} f\|_{L_\infty} = \langle\xi'\rangle^{1/p}\|f\|_{L_\infty}$.

(v) $\xi_n^{l'}D^l_{\xi_n}\sigma_p = \langle\xi'\rangle^{l-l'}\sigma_p\xi_n^{l'}D^l_{\xi_n}$ *and* $\xi_n^{l'}D^l_{\xi_n}\sigma_p^{-1} = \langle\xi'\rangle^{-l+l'}\sigma_p^{-1}\xi_n^{l'}D^l_{\xi_n}$.

(vi) $D^\alpha_{\xi'}D^\beta_{x'}\sigma_p = \sigma_p\sum s_{k,\alpha'}(\xi')\xi_n^k D^k_n D^{\alpha'}_{\xi'}D^{\beta'}_{x'}$ *with* $s_{k,\alpha'}(\xi')\in S^{-|\alpha|+|\alpha'|}_{1,0}(\mathbb{R}^{n-1})$.

The sum is taken over all $\alpha'\le\alpha$ and $k\in\{0,\dots,|\alpha-\alpha'|\}$. Moreover, $s_{0,|\alpha|}=1$.

Proof. The results (i), (ii), (iii), and (iv) can be obtained by change of variables. The chain rule implies (v). In combination with induction over $|\alpha|$, we obtain (vi). □

Now, we provide estimates for the symbols and symbol-kernels. The arguments are similar for potential, trace, and singular Green symbols. We thus focus on potential symbols. By definition, a smooth function k belongs to $\mathcal{K}^m_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$, if and only if $|k_{[0]}|^{\tilde{p}}_{\alpha,\beta,l,l'}\le\infty$ for all multi indices $\alpha,\beta\in\mathbb{N}_0^{n-1}$ and $1\le\tilde{p}\le\infty$. Here, $|\cdot|^{\tilde{p}}_{\alpha,\beta,l,l'} := |\cdot|_{\alpha,\beta}\hat{\otimes}_\pi|\cdot|^{\tilde{p}}_{l,l'}$. Moreover, $|\cdot|_{\alpha,\beta}$ denotes the seminorms in $S^m_{1,\delta}(\mathbb{R}^n\times\mathbb{R}^n)$ and $|\cdot|^{\tilde{p}}_{l,l'}$ the seminorms in $\mathcal{H}^+$. More explicitly:

$$\|[h^+\xi_n^{l'}D^l_{\xi_n}D^\alpha_{\xi'}D^\beta_{x'}k_{[0]}](x',\xi',\cdot)\|_{L_{\tilde{p}}(\mathbb{R})}\le C\langle\xi'\rangle^{m-|\alpha|+\delta|\delta|}.$$

Lemma 4.13 implies that the above and the following estimate are equivalent.

$$\|[h^+\xi_n^{l'}D_{\xi_n}^{l}D_{\xi'}^{\alpha}D_{x'}^{\beta}k](x',\xi',\cdot)\|_{L_{\tilde{p}}(\mathbb{R})} \leq C\langle\xi'\rangle^{m+m_{\tilde{p}}-|\alpha|+\delta|\delta|-l+l'}.$$

The factor $m_{\tilde{p}}$ follows from Estimate (iii) in Lemma 4.13. The factor is $1/p$, 0, or $-1/q$ for $\tilde{p}=1$, $\tilde{p}=q$, or $\tilde{p}=\infty$, respectively. By definition of the topology, the following estimate holds for the symbol-kernels:

$$\|x_n^l D_{x_n}^{l'}D_{\xi'}^{\alpha}D_{x'}^{\beta}\tilde{k}_{[0]}\|_{L_{\tilde{p}}(\mathbb{R}_+)} \leq C\langle\xi'\rangle^{m-|\alpha|+\delta|\beta|}.$$

Concerning Lemma 4.13 and $\tilde{k}_{[0]}=\sigma_p^{-1}\tilde{k}$, the next inequality is equivalent to the one above.

$$\|x_n^l D_{x_n}^{l'}D_{\xi'}^{\alpha}D_{x'}^{\beta}\tilde{k}\|_{L_{\tilde{p}}(\mathbb{R}_+)} \leq C\langle\xi'\rangle^{m-m_{\tilde{p}}-|\alpha|+\delta|\beta|-l+l'}.$$

Here, $m_{\tilde{p}}$ is now $-1/q$, 0, or $1/p$ if $\tilde{p}$ is $1,p$, or ∞, respectively. In the following lemma, we provide the results for the remaining symbols and symbol-kernels:

Lemma 4.14. *Let $l,l',l'',l'''\in\mathbb{N}_0$ and $\alpha,\beta\in\mathbb{N}_0^{n-1}$. By $C>0$, we denote a constant which may depend on the multi-indices used in the estimate.*

(i) A smooth function k belongs to $\in\mathcal{K}_{1,\delta}^{m}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$, if and only if it satisfies one (and hence all) of the following family of estimates:

$$\|[h^+\xi_n^{l'}D_{\xi_n}^{l}D_{\xi'}^{\alpha}D_{x'}^{\beta}k](x',\xi',\cdot)\|_{L_1(\mathbb{R})} \leq C\langle\xi'\rangle^{m+1/p-|\alpha|+\delta|\beta|-l+l'}.$$
$$\|[h^+\xi_n^{l'}D_{\xi_n}^{l}D_{\xi'}^{\alpha}D_{x'}^{\beta}k](x',\xi',\cdot)\|_{L_q(\mathbb{R})} \leq C\langle\xi'\rangle^{m-|\alpha|+\delta|\beta|-l+l'}.$$
$$\|[h^+\xi_n^{l'}D_{\xi_n}^{l}D_{\xi'}^{\alpha}D_{x'}^{\beta}k](x',\xi',\cdot)\|_{L_\infty(\mathbb{R})} \leq C\langle\xi'\rangle^{m-1/q-|\alpha|+\delta|\beta|-l+l'}.$$

(ii) A smooth function t belongs to $\mathcal{T}_{1,\delta}^{m,0}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$, if and only if it satisfies one (and hence all) of the following family of estimates:

$$\|[h^-\xi_n^{l'}D_{\xi_n}^{l}D_{\xi'}^{\alpha}D_{x'}^{\beta}t](x',\xi',\cdot)\|_{L_1(\mathbb{R})} \leq C\langle\xi'\rangle^{m+1/q-|\alpha|+\delta|\beta|-l+l'}$$
$$\|[h^-\xi_n^{l'}D_{\xi_n}^{l}D_{\xi'}^{\alpha}D_{x'}^{\beta}t](x',\xi',\cdot)\|_{L_p(\mathbb{R})} \leq C\langle\xi'\rangle^{m-|\alpha|+\delta|\beta|-l+l'}$$
$$\|[h^-\xi_n^{l'}D_{\xi_n}^{l}D_{\xi'}^{\alpha}D_{x'}^{\beta}t](x',\xi',\cdot)\|_{L_\infty(\mathbb{R})} \leq C\langle\xi'\rangle^{m-1/q-|\alpha|+\delta|\beta|-l+l'}$$

(iii) A smooth function g belongs to $\mathcal{G}_{1,\delta}^{m,0}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$, if and only if it satisfies one (and hence all) of the following family of estimates:

$$\|[h_{\xi_n}^+h_{\eta_n}^-\xi_n^{l'}D_{\xi_n}^{l}\eta_n^{l'''}D_{\eta_n}^{l''}D_{\xi'}^{\alpha}D_{x'}^{\beta}g](x',\xi',\cdot,\cdot)\|_{L_1(\mathbb{R}^2)} \leq C\langle\xi'\rangle^{m+1-|\alpha|+\delta|\beta|-l+l'-l''+l'''}$$
$$\|[h_{\xi_n}^+h_{\eta_n}^-\xi_n^{l'}D_{\xi_n}^{l}\eta_n^{l'''}D_{\eta_n}^{l''}D_{\xi'}^{\alpha}D_{x'}^{\beta}g](x',\xi',\cdot,\cdot)\|_{L_2(\mathbb{R}^2)} \leq C\langle\xi'\rangle^{m-|\alpha|+\delta|\beta|-l+l'-l''+l'''}$$
$$\|[h_{\xi_n}^+h_{\eta_n}^-\xi_n^{l'}D_{\xi_n}^{l}\eta_n^{l'''}D_{\eta_n}^{l''}D_{\xi'}^{\alpha}D_{x'}^{\beta}g](x',\xi',\cdot,\cdot)\|_{L_\infty(\mathbb{R}^2)} \leq C\langle\xi'\rangle^{m-1-|\alpha|+\delta|\beta|-l+l'-l''+l'''}$$

(iv) A smooth function $\tilde{k}$ belongs to $\tilde{\mathcal{K}}^m_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$, if and only if it is supported in $\mathbb{R}_+$ and satisfies one (and hence all) of the following family of estimates:

$$\|[x_n^l D_{x_n}^{l'} D_{\xi'}^{\alpha} D_{x'}^{\beta}\tilde{k}](x',\cdot,\xi')\|_{L_1(\mathbb{R}_+)} \le C\langle\xi'\rangle^{m-1/q-|\alpha|+\delta|\beta|-l+l'}$$
$$\|[x_n^l D_{x_n}^{l'} D_{\xi'}^{\alpha} D_{x'}^{\beta}\tilde{k}](x',\cdot,\xi')\|_{L_p(\mathbb{R}_+)} \le C\langle\xi'\rangle^{m-|\alpha|+\delta|\beta|-l+l'}$$
$$\|[x_n^l D_{x_n}^{l'} D_{\xi'}^{\alpha} D_{x'}^{\beta}\tilde{k}](x',\cdot,\xi')\|_{L_\infty(\mathbb{R}_+)} \le C\langle\xi'\rangle^{m+1/p-|\alpha|+\delta|\beta|-l+l'}$$

(v) A smooth function $\tilde{t}$ belongs to $\tilde{\mathcal{T}}^{m,0}_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$, if and only if it is supported in $\mathbb{R}_+$ and satisfies one (and hence all) of the following family of estimates:

$$\|[x_n^l D_{x_n}^{l'} D_{\xi'}^{\alpha} D_{x'}^{\beta}\tilde{t}](x',\cdot,\xi')\|_{L_1(\mathbb{R}_+)} \le C\langle\xi'\rangle^{m-1/p-|\alpha|+\delta|\beta|-l+l'}$$
$$\|[x_n^l D_{x_n}^{l'} D_{\xi'}^{\alpha} D_{x'}^{\beta}\tilde{t}](x',\cdot,\xi')\|_{L_q(\mathbb{R}_+)} \le C\langle\xi'\rangle^{m-|\alpha|+\delta|\beta|-l+l'}$$
$$\|[x_n^l D_{x_n}^{l'} D_{\xi'}^{\alpha} D_{x'}^{\beta}\tilde{t}](x',\cdot,\xi')\|_{L_\infty(\mathbb{R}_+)} \le C\langle\xi'\rangle^{m+1/q-|\alpha|+\delta|\beta|-l+l'}$$

(vi) A smooth function $\tilde{g}$ belongs to $\tilde{\mathcal{G}}^{m,0}_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$, if and only if it is supported in $\mathbb{R}^2_{++}$ and satisfies one (and hence all) of the following family of estimates:

$$\|x_n^l D_{x_n,+}^{l'} y_n^{l''} D_{y_n,+}^{l'''} D_{\xi'}^{\alpha} D_{x'}^{\beta}\tilde{g}](x',\xi',\cdot,\cdot)\|_{L_1(\mathbb{R}^2_{++})} \le C\langle\xi'\rangle^{m-1-|\alpha|+\delta|\beta|-l+l'-l''+l'''}$$
$$\|x_n^l D_{x_n}^{l'} y_n^{l''} D_{y_n}^{l'''} D_{\xi'}^{\alpha} D_{x'}^{\beta}\tilde{g}](x',\xi',\cdot,\cdot)\|_{L_2(\mathbb{R}^2_{++})} \le C\langle\xi'\rangle^{m-|\alpha|+\delta|\beta|-l+l'-l''+l'''}$$
$$\|x_n^l D_{x_n}^{l'} y_n^{l''} D_{y_n}^{l'''} D_{\xi'}^{\alpha} D_{x'}^{\beta}\tilde{g}](x',\xi',\cdot,\cdot)\|_{L_\infty(\mathbb{R}^2_{++})} \le C\langle\xi'\rangle^{m+1-|\alpha|+\delta|\beta|-l+l'-l''+l'''}$$

For all $N\in\mathbb{N}_0$, we define $|k|_N$ as the infimum over all constants such that the estimates (i) in Lemma 4.14 hold for all multi-indices with $|\alpha|,|\beta|,l,l'\le N$. From the discussion above, it is clear that the previously defined seminorms are equivalent. Similarly, we define sets of seminorms on $\mathcal{T}^{m,0}_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$, $\mathcal{G}^{m,0}_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$, $\widetilde{\mathcal{K}}^m_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$, $\widetilde{\mathcal{T}}^m_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$, and $\widetilde{\mathcal{G}}^m_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$.

It is well-known for symbols with Hömander type $(1,0)$ that $\mathbf{k}(x',\xi')$, $\mathbf{t}(x',\xi')$ and $\mathbf{g}(x',\xi')$ are operator-valued symbols. This result extends to general Hömander type. In fact, the proof given in [35, Theorems 3.7 and 3.9] can be generalised with obvious replacements. For completeness, we include the proof.

Theorem 4.15. *The following maps are linear, bounded, and bijective:*

1. $\widetilde{\mathcal{K}}^m_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})\ni k\mapsto\mathbf{k}\in S^m_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};\mathbb{C},\mathcal{S}(\mathbb{R}_+))$
2. $\widetilde{\mathcal{T}}^m_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})\ni t\mapsto\mathbf{t}\in S^m_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};\mathcal{S}'(\mathbb{R}_+),\mathbb{C})$
3. $\widetilde{\mathcal{G}}^m_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})\ni g\mapsto\mathbf{g}\in S^m_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};\mathcal{S}'(\mathbb{R}_+),\mathcal{S}(\mathbb{R}_+))$

Proof. Note that $\sigma_q^{-1} D_{\xi'}^{\alpha} D_{x'}^{\beta} \mathbf{k}\phi = \sigma_p^{-1}[D_{\xi'}^{\alpha} D_{x'}^{\beta} \tilde{k}]\phi = D_{\xi'}^{\alpha} D_{x'}^{\beta} \tilde{k}_{[0]}\phi$. Which implies:

$$\|D_{x_n}^{l_1} x_n^{l_2} \sigma_q^{-1}[D_{\xi'}^{\alpha} D_{x'}^{\beta} \mathbf{k}\phi]\|_{L_p(\mathbb{R}_+)} \leq |k|_* \langle \xi' \rangle^{m-|\alpha|+\delta|\beta|}.$$

The Bessel potential norm is equivalent to the Sobolev norm for integers. Thus, the above estimate implies the following.

$$\|\sigma_p^{-1} D_{\xi'}^{\alpha} D_{x'}^{\beta} \mathbf{k}\|_{\mathcal{B}(\mathbb{C}, H_p^{\mathbf{s}}(\mathbb{R}_+))} \leq C|k|_* \langle \xi' \rangle^{m-|\alpha|+\delta|\beta|} \text{ for all } \mathbf{s} \in \mathbf{N}_0^2.$$

This proves $\mathbf{k} \in S_{1,\delta}^m(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathbb{C}, \mathcal{S}(\mathbb{R}_+))$ and the boundedness of the map. The linearity and injectivity are obvious. What remains to be proven is surjectivity. For a given $\mathbf{k} \in S_{1,\delta}^m(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathbb{C}, \mathcal{S}(\mathbb{R}_+))$, we define $\tilde{k}(x', \xi'; x_n) := [\mathbf{k}(x', \xi')1](x_n)$. For all multi-indices $\alpha, \beta \in \mathbb{N}_0$ and $(l_1, l_2) \leq \mathbf{s} \in \mathbb{N}_0^2$, we obtain the following estimate:

$$\begin{aligned}
\|D_{x_n}^{l_1} x_n^{l_2} D_{\xi'}^{\alpha} D_{x'}^{\beta} \tilde{k}(x', \xi'; x_n)\|_{L_p(\mathbb{R}_+)} &= \langle \xi' \rangle^{l_1 - l_2} \|D_{x_n}^{l_1} x_n^{l_2} [\sigma_p^{-1} D_{\xi'}^{\alpha} D_{x'}^{\beta} \mathbf{k}(x', \xi')1](x_n)\|_{L_p(\mathbb{R}_+)} \\
&\leq C \langle \xi' \rangle^{l_1 - l_2} \|\sigma_p^{-1} D_{\xi'}^{\alpha} D_{x'}^{\beta} \mathbf{k}(x', \xi')\|_{\mathcal{B}(\mathbb{C}, H_p^{\mathbf{s}}(\mathbb{R}_+))} \\
&\leq C \langle \xi' \rangle^{m-|\alpha|+\delta|\beta|+l_1-l_2}.
\end{aligned}$$

According to Lemma 4.14, the function $\tilde{k}$ belongs to $\widetilde{\mathcal{K}}_{1,\delta}^m(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$. By linearity, the operator associated to $\tilde{k}$ coincides with $\mathbf{k}$.
Now, we look at trace symbol-kernels. By $\langle \cdot, \cdot \rangle$, we denote the pairing on $\mathcal{S}_0(\mathbb{R}_+) \times \mathcal{S}'(\mathbb{R}_+)$.

$$\begin{aligned}
|D_{\xi'}^{\alpha} D_{x'}^{\beta} \mathbf{t}(x', \xi') \sigma_p u| &= |\langle D_{\xi'}^{\alpha} D_{x'}^{\beta} \tilde{t}(x', \xi'; x_n), [\sigma_p u](x_n) \rangle| = |\langle \sigma_q^{-1} D_{\xi'}^{\alpha} D_{x'}^{\beta} \tilde{t}(x', \xi'; x_n), u(x_n) \rangle| \\
&\leq \|\sigma_q^{-1} D_{\xi'}^{\alpha} D_{x'}^{\beta} \tilde{t}(x', \xi'; x_n)\|_{H_p^{\mathbf{s}}(\mathbb{R}_+)} \|u\|_{H_{p;0}^{-\mathbf{s}}(\mathbb{R}_+)}.
\end{aligned}$$

We obtain $\|\sigma_q^{-1} D_{\xi'}^{\alpha} D_{x'}^{\beta} \tilde{t}(x', \xi'; x_n)\|_{H_p^{\mathbf{s}}(\mathbb{R}_+)} \leq |t_*| \langle \xi' \rangle^{m-|\alpha|+\delta|\beta|}$ based on similar arguments as for potential-kernels. The combination of previous results yields:

$$\|D_{\xi'}^{\alpha} D_{x'}^{\beta} \mathbf{t}(x', \xi') \sigma_p\|_{\mathcal{B}(H_{p;0}^{-1}(\mathbb{R}_+), \mathbb{C})} \leq \|\sigma_q^{-\mathbf{s}} D_{\xi'}^{\alpha} D_{x'}^{\beta} \tilde{t}(x', \xi'; x_n)\|_{H_p^{\mathbf{s}}(\mathbb{R}_+)} \leq |t_*| \langle \xi' \rangle^{m-|\alpha|+\delta|\beta|}.$$

This implies that $\mathbf{t} \in S_{1,\delta}^m(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathcal{S}'(\mathbb{R}_+), \mathbb{C})$ and the boundedness $\tilde{t} \mapsto \mathbf{t}$. Again, linearity and injectivity are clear. In the following, we consider surjectivity. For a given $\mathbf{t} \in S_{1,\delta}^m(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathcal{S}'(\mathbb{R}_+), \mathbb{C})$, we fixed (x', ξ'). Then, $\mathbf{t}(x', \xi')$ belongs to $(\mathcal{S}'(\mathbb{R}_+))'$. We identify $(\mathcal{S}'(\mathbb{R}_+))'$ with $\mathcal{S}(\mathbb{R}_+)$ and define $\tilde{t}(x', \xi'; x_n) := [\mathbf{t}(x', \xi')](x_n)$.

$$\begin{aligned}
\|D_{x_n}^{l_1} x_n^{l_2} D_{\xi'}^{\alpha} D_{x'}^{\alpha} \tilde{t}(x', \xi'; x_n)\|_{L_q(\mathbb{R}_+)} &= \langle \xi' \rangle^{l_1 - l_2} \|D_{x_n}^{l_1} x_n^{l_2} [\sigma_q^{-1} D_{\xi'}^{\alpha} D_{x'}^{\beta} \mathbf{t}(x', \xi')](x_n)\|_{L_q(\mathbb{R}_+)} \\
&\leq \langle \xi' \rangle^{l_1 - l_2} \|D_{\xi'}^{\alpha} D_{x'}^{\beta} \mathbf{t}(x', \xi') \sigma_p\|_{\mathcal{B}(H_{p;0}^{-\mathbf{s}}(\mathbb{R}_+), \mathbb{C})} \\
&\leq \langle \xi' \rangle^{m-|\alpha|+\delta|\beta|-l_1+l_2}.
\end{aligned}$$

According to Lemma 4.14, the function $\tilde{t}$ belongs to $\widetilde{T}_{1,\delta}^m(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$. The associated operator coincides with $\mathbf{t}$. For a singular Green operator, we observed that the L_∞ estimates in (vi) of Lemma 4.14 do not depend on the $1 < p < \infty$ used for scaling. Thus, we can assume that $p = 2$. In this case, the result is [35, Lemma 3.8.]. □

Now, we drop the zero class assumption. We consider trace symbols of class d which have no zero class part. Their form is $t_{[0]} \in S^m_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n)\hat{\otimes}\mathcal{H}'_{d-1}$ and this implies that symbols $s_{j,[0]} \in S^m_{1,\delta}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$ exist for $j \in \{0, \ldots, d-1\}$ such that:

$$t_{[0]}(x', \xi'; \xi_n) = \sum_{j=0}^{d-1} s_{j,[0]}(x', \xi')\xi_n^j.$$

We define $s_j(x', \xi') := i^j s_{j,[0]}(x', \xi')\langle\xi'\rangle^{-1/p-j} \in S^{m-1/p-j}_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^{n-1})$. Then,

$$t(x', \xi'; \xi_n) = [\sigma_p^{-1} t_{[0]}](x', \xi'; \xi_n) = \langle\xi'\rangle^{-1/p} \sum_{j=0}^{d-1} s_{j,[0]}(x', \xi')\langle\xi'\rangle^{-j}\xi_n^j = \sum_{j=0}^{d-1} s_j(x', \xi')(-i\xi_n)^j.$$

We include the factor i^j to obtain $\mathbf{t}(x', \xi') = \sum s_j(x', \xi')\gamma_j$ from the following computation:

$$\int^+ (-i\xi_n)^j [\mathcal{F}_n e^+ u](\xi_n)\, đ\xi_n = \gamma_0 r^+ \mathcal{F}_n^{-1}(-i\xi_n)^j \mathcal{F}_n e^+ u = \gamma_0(-\partial_{x_n})^j u = \gamma_j u.$$

Next, we consider singular Green operators. Let $g_{[0]} \in S^m_{1,\delta}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})\hat{\otimes}\mathcal{H}^+ \otimes \mathcal{H}'_{d-1}$, i.e., it is a polynomial in η_n of degree $d-1$, with coefficients $c_j \in S^m_{1,\delta}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})\hat{\otimes}\mathcal{H}^+$. We define $k_j(x', \xi'; \xi_n) := i^j \langle\xi'\rangle^{-1-j} c_j(x', \xi'; \xi_n/\langle\xi'\rangle)$. Note that $k_j \in \mathcal{K}^{m-j-1/p}_{1,\delta}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$ because $k_{j,[0]}(x', \xi'; \xi_n) = i^j \langle\xi'\rangle^{-j-1/p} c_j(x', \xi'; \xi_n) \in S^{m-j-1/p}_{1,\delta}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})\hat{\otimes}\mathcal{H}^+$.

$$g(x', \xi'; \xi_n, \eta_n) = \sum_{j=0}^{d-1} \langle\xi'\rangle^{-1} c_j(x', \xi'; \xi_n/\langle\xi'\rangle)(\eta_n/\langle\xi'\rangle)^j = \sum_{j=0}^{d-1} k_j(x', \xi'; \xi_n)(-i\eta_n)^j.$$

With the result above we obtain the following equality:

$$\begin{aligned}
[\mathbf{g}(x', \xi')u](x_n) &= r^+ \mathcal{F}^{-1}_{\xi_n \mapsto x_n} \int^+ g(x', \xi'; \xi_n, \eta_n)[\mathcal{F}_n e^+ u](\eta_n)\, đ\eta_n \\
&= \sum_{j=0}^{d-1} r^+ \mathcal{F}^{-1}_{\xi_n \mapsto x_n} k_j(x', \xi'; \xi_n) \int^+ (-i\eta_n)^j [\mathcal{F}_n e^+ u](\eta_n)\, đ\eta_n \\
&= \left[\sum_{j=0}^{d-1} \mathbf{k}_j(x', \xi')\gamma_j u\right](x_n).
\end{aligned}$$

A generic symbol of class d is a direct sum of a symbol of class zero and class d with no zero class part, according to the definition $\mathcal{H}^-_{d-1} := \mathcal{H}'_{d-1} \oplus \mathcal{H}^-_{-1}$. The previous discussion and Theorem 4.15 imply the result below.

Corollary 4.16 (operator valued symbols)**.** *Let* $\mathbf{s} \in \mathbb{R}^2$ *with* $s_1 > d - 1/p$. *The following maps are linear and bounded.*

(i) $\mathcal{K}^m_{1,\delta}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}) \ni k \mapsto \mathbf{k} \in S^m_{1,\delta}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathbb{C}, \mathcal{S}(\mathbb{R}_+))$

(ii) $\mathcal{T}^{m,0}_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})\ni t\mapsto \mathbf{t}\in S^m_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};\mathcal{S}'(\mathbb{R}_+),\mathbb{C})$

(iii) $\mathcal{G}^{m,0}_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})\ni g\mapsto \mathbf{g}\in S^m_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};\mathcal{S}'(\mathbb{R}_+),\mathcal{S}(\mathbb{R}_+))$

(iv) $\mathcal{T}^{m,d}_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})\ni t\mapsto \mathbf{t}\in S^m_{1,\delta}(\mathbb{R}^{n-1}\times\times\mathbb{R}^{n-1};H^{\mathbf{s}}_p(\mathbb{R}_+),\mathbb{C})$

(v) $\mathcal{G}^{m,d}_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})\ni g\mapsto \mathbf{g}\in S^m_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};H^{\mathbf{s}}_p(\mathbb{R}_+),\mathcal{S}(\mathbb{R}_+))$

Moreover, the maps (i), (ii), *and* (iii) *are bijections. The maps* (iv) *and* (v) *are injective and their images are operators of the following form:*

$$\mathbf{t}(x',\xi')=\sum s_j(x',\xi')\gamma_j+\mathbf{t}'(x',\xi') \quad \textit{and} \tag{4.7}$$

$$\mathbf{g}(x',\xi')=\sum \mathbf{k}_j(x',\xi')\gamma_j+\mathbf{g}'(x',\xi'), \tag{4.8}$$

where $j\in\{0,\dots,d-1\}$, $s_j\in S^{m-j-1/p}_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$, $k_j\in\mathcal{K}^{m-j-1/p}_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$, $t'\in\mathcal{T}^{m,0}_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$, *and* $g'\in\mathcal{G}^{m,0}_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$.

Proof. The results in (i), (ii), and (iii) directly follow from Theorem (4.15) and the fact that the Fourier transform is a bounded linear bijection from symbol spaces to the symbol-kernel spaces. If the class is not zero, then the symbol is a direct sum of a symbols for class zero and of class d with no class zero part. Thus, the discussion above finishes the proof. □

We restrict the operator-valued symbols to the form of (4.7) or (4.8). Thus, we obtain a bijection between symbols and operator-valued symbols. Theorem 4.15 and Corollary 4.16 provide two (three in the case of class zero) interchangeable concepts: The operator-valued symbols denoted by bold letters, the symbols denoted by plain letters, and the symbol-kernel denoted by letters with a tilde. From now on, we use these concepts interchangeably. For example, let $\mathbf{k}(x',\xi')$ be an operator-valued potential symbol, we write $k(x',\xi';\xi_n)$ for the symbol, instead of let $k(x',\xi';\xi_n)$ denote the symbol of $\mathbf{k}(x',\xi')$. We denote the associated pseudodifferential operator by capital letters, i.e., $K=\operatorname{op}\mathbf{k}$.

4.4 Transmission Property

This section introduces a class of symbols with the so-called transmission property. The associated operators map the space of Schwartz functions into itself. The restricted class of pseudodifferential operators is still large enough to contain the symbols of differential operators and their parametrices, if they exists. We show that some well-known relations between potential-, trace-, and pseudodifferential symbol's continue to be true for a general Hörmander type. Moreover, the action in the direction normal to the boundary can be interpreted as an operator-valued symbol.

Definition 4.17. A symbol $p\in S^m_{1,\delta}(\mathbb{R}^n\times\mathbb{R}^n)$ has the transmission property (at $x_n=0$), provided that for all $l\in\mathbb{N}_0$ the following relation holds:

$$[\partial^l_{x_n}p](x',\xi',0,\langle\xi'\rangle\xi_n)\in S^m_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})\hat{\otimes}\mathcal{H}.$$

We write $p \in \mathcal{P}^m_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$ and $p \in \mathcal{CP}^m_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$, if p does not depend on x_n.

The definition is equivalent to the uniform transmission property with respect to $\mathbb{R}^n_+$ and $\mathbb{R}^n_-$, in [21].

Notation 4.18 (Principal symbol)**.** Let $\mathcal{X}^m_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$ stand for $\mathcal{P}^m_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$, $\mathcal{G}^{m,d}_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$, $\mathcal{K}^m_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$, $\mathcal{T}^{m,d}_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$, or $S^m_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$. Then,

$$x \approx y :\Leftrightarrow x - y \in \mathcal{X}^{m-(1-\delta)}_{1,\delta}(\mathbb{R}^n\times\mathbb{R}^n)$$

is an equivalence relation on $\mathcal{X}^m_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$. The equivalence class is the principal symbol of x. We make no notational distinction between classes and representatives, i.e., if $x \approx x_0$ we say that x_0 is a principal symbol of x.

Naturally, x is also a principal symbol of x_0. Thus, the notation is only useful if an explicit description of x_0 is available. It is common to speak of the (instead of a) principal symbol for classical operators because a conical choice of the representative, as the homogeneous part of highest order, exists. If an operator is constructed from a classical operator, we also speak of the principal symbol.
Now, we consider the action in the direction normal the boundary. First, we consider operators on the whole euclidean space. For a given $(x',\xi') \in \mathbb{R}^{n-1}\times\mathbb{R}^{n-1}$, the function $p(x',\xi';\cdot,\cdot)$ is a symbol in $S^m_{1,\delta}(\mathbb{R}\times\mathbb{R})$. We denote the associated pseudodifferential operator as $\mathbf{p}(x',\xi')$. Then:

Lemma 4.19 (Operator-valued symbol)**.** *The following map is linear and bounded:*

$$\mathcal{P}^m_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}) \ni p \mapsto \mathbf{p} \in S^m_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};\mathcal{S}(\mathbb{R}),\mathcal{S}(\mathbb{R})).$$

Proof. We prove that, for any $\alpha,\beta \in \mathbb{N}_0$ and $\mathbf{s} \in \mathbb{N}_0^2$, the operator below is linear, bounded, and satisfies symbol estimates, which is sufficient to prove the lemma.

$$\sigma_p^{-1} D^\alpha_{\xi'} D^\beta_{x'} \mathbf{p}(x',\xi')\sigma_p : H^{\mathbf{s}}_p(\mathbb{R}) \to H^{\mathbf{s}+(m,0)}_p(\mathbb{R}).$$

We can absorb $\alpha,\beta \in \mathbb{N}_0$ into the order of p. Thus, we assume that $\alpha = \beta = 0$. A straightforward calculation shows that $\mathbf{q}(x',\xi') := \sigma_p^{-1}\mathbf{p}(x',\xi')\sigma_p$ is a pseudodifferential operator with symbol $q(x',\xi';x_n,\xi_n) = p(x',\xi';\langle\xi'\rangle^{-1}x_n,\langle\xi'\rangle\xi_n)$. The symbol seminorms of $q(x',\xi';\cdot,\cdot)$ are related to those of p, as follows:

$$\begin{aligned}
|D^l_{\xi_n} D^{l'}_{x_n} q(x',\xi';x_n,\xi_n)| &= \langle\xi'\rangle^{l-l'} |[D^l_{\xi_n} D^{l'}_{x_n} p](x',\xi';x_n/\langle\xi'\rangle,\langle\xi'\rangle\xi_n)| \\
&\le |p|_* \langle\xi'\rangle^{l-l'} \langle\xi',\langle\xi'\rangle\xi_n\rangle^{m-l+\delta l'} \le |p|_* \langle\xi'\rangle^{m-(1-\delta)l'} \langle\xi_n\rangle^{m-l+\delta l'} \\
&\le |p|_* \langle\xi'\rangle^{m} \langle\xi_n\rangle^{m-l+\delta l'}.
\end{aligned} \tag{4.9}$$

The well-known mapping properties for pseudodifferential operators imply that the operator $\sigma_p^{-1} D^\alpha_{\xi'} D^\beta_{x'} \mathbf{p}(x',\xi')\sigma_p$ is indeed bounded and satisfies the symbol estimate. □

Next, we consider operators on the euclidean half space. We are interested in the truncated operator, i.e., $\mathbf{p}_+(x',\xi') := r^+\mathbf{p}(x',\xi')e^+$. We need the following result to investigate this operator. The result is interesting in its own right.

Proposition 4.20. *Let $\mathbf{k}(x',\xi') := r^+\mathbf{p}(x',\xi')\gamma_0^*$. This is a potential operator and the map:*

$$\mathcal{P}^m_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}) \ni p \mapsto k \in \mathcal{K}^{m+1/q}_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$$

is linear, bounded, and surjective. Moreover, $k(x',\xi',\xi_n) \approx h^+p'(x',\xi';0,\xi_n)$, where p' is a principal symbol of p.

Proof. Note that $\gamma_0^*\phi = \phi\otimes\delta$. We first assume that $p \in \mathcal{CP}^m_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$. Thus, the action in the direction normal to the boundary is a Fourier multiplier. Then:

$$r^+\mathbf{p}(x',\xi')\delta = r^+\mathcal{F}_n^{-1}p(x',\xi';\xi_n)\mathcal{F}_n\delta = r^+\mathcal{F}_n^{-1}h^+p(x',\xi';\xi_n)1.$$

The formula shows the symbol of the potential operator: $k(x',\xi';\xi_n) = h^+p(x',\xi';0,\xi_n)$. Now, we verify that $k \in \mathcal{K}^{m+1/q}_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$. The transmission property implies the following:

$$k_{[0]}(x',\xi';\xi_n) = \sigma_q h^+p(x',\xi';0,\xi_n) = \langle\xi'\rangle^{1/q}h^+p_{[0]}(x',\xi';\xi_n) \in S^{m+1/q}_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})\hat{\otimes}_\pi\mathcal{H}^+.$$

In general, for $p \in \mathcal{P}^m_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$, we can assume that p is given in (x',y_n)-form because $\mathbf{k}$ only depends on $\mathbf{p}$. Using Taylor expansion, as in 4.13 with $M=1$, implies:

$$\mathbf{p}(x',\xi') = \mathbf{p}_0^R(x',\xi') + \mathbf{p}_1^R(x',\xi')x_n.$$

Clearly, $x_n\delta = 0$. Thus, $\mathbf{k}(x',\xi')$ only depends on $\mathbf{p}_0^R(x',\xi') \in \mathcal{CP}^m_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$. Obviously, $k(x',\xi';\xi_n) = h^+_{\xi_n}p_0^R(x',\xi';\xi_n) \approx h^+_{\xi_n}p'(x',\xi';0,\xi')$. We have to show surjectivity of the map to finish the proof. For a given symbol-kernel $\tilde{k} \in \widetilde{K}^{m+1/q}_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$, we define:

$$q(x',\xi';\xi_n) := \mathcal{F}^{-1}_{\xi_n\to\xi_n}[E\tilde{k}](x',\xi';x_n).$$

Here, E is the extension operator introduced in Section 2. The boundedness of E and Riemann-Lebesgue's lemma imply the following estimate:

$$|\xi_n^{l'}D^l_{\xi_n}D^\alpha_{\xi'}D^\beta_{x'}q(x',\xi',\xi_n)| \le \|D^{l'}_{x_n}x_n^lD^\alpha_{\xi'}D^\beta_{x'}\tilde{k}(x',\xi';\cdot)\|_{L_1(\mathbf{R}_+)} \le |k|_*\langle\xi'\rangle^{m-|\alpha|+\delta|\beta|-l+l'}.$$

The last estimate is (iv) of Lemma 4.14. In particular, for every $N_1,N_2 \in \mathbb{N}_0$ with $N_1+N_2 = N$, the following estimate holds:

$$\langle\xi'\rangle^{2N_1}|\xi_n|^{2N_2}|D^l_{\xi_n}D^\alpha_{\xi'}D^\beta_{x'}q(x',\xi',\xi_n)| \le |k|_*\langle\xi'\rangle^{m-|\alpha|+\delta|\beta|-l}\langle\xi'\rangle^{2N}.$$

$\langle\xi\rangle^{2N}$ is a linear combination of $\langle\xi'\rangle^{2N_1}|\xi_n|^{2N_2}$. We thus obtain the following estimate:

$$|D^{\alpha'}_{\xi'}D^{\beta'}_{x'}D^l_{\xi_n}p(x',\xi',\xi_n)| \le C|k|_*\langle\xi'\rangle^{m-|\alpha|+\delta|\beta|-l}(\langle\xi'\rangle\langle\xi\rangle^{-1})^{2N} \text{ for all } N\in\mathbb{N}_0.$$

The estimate above implies that $p \in S^m_{1,\delta}(\mathbb{R}^{n-1} \times \mathbb{R}^n)$: For positive exponents, we can replace $\langle \xi' \rangle$ by $\langle \xi \rangle$ because the second term dominates the first. If the exponent is negative, we choose $N \in \mathbb{N}_0$ such that $2N$ exeedes the absolute value of the exponent. Therefore, we can use the estimate above to make the same replacement. The rapid decay in ξ_n for fixed ξ' implies the transmission property for q. We need to verify that the potential operator provided by the lemma coincides with the potential operator at the beginning of the construction:

$$[r^+ \mathbf{q}(x', \xi')\delta \otimes 1](x_n) = r^+ \mathcal{F}^{-1}_{\xi_n \to x_n} q(x', \xi', \xi_n) = r^+ E \tilde{k}(x', \xi'; x_n) = \tilde{k}(x', \xi'; x_n).$$

Thus, linearity implies $\mathbf{k}(x', \xi') = r^+ \mathbf{q}(x', \xi')\gamma_0^*$. □

Proposition 4.20 is well-known for $\delta = 0$. In fact, the proof above is a minor modification of the proof in [18].

Corollary 4.21. *The following map is linear and bounded:*

$$\mathcal{P}^m_{1,\delta}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}) \ni p \mapsto \mathbf{p}_+ \in S^m_{1,\delta}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathcal{S}(\mathbb{R}_+), \mathcal{S}(\mathbb{R}_+)).$$

Proof. We use the notation from the proof of Lemma 4.19. Note that the group action commutes with extension and restriction. Thus, it is sufficient to show that for all $N \in \mathbb{N}_0$:

$$\|\sigma_p^{-1} \mathbf{p}_+(x', \xi')\sigma_p\|_{\mathcal{L}(H^{\mathbf{s}}_p(\mathbb{R}_+), H^{\mathbf{s}-(m,0)}_p(\mathbb{R}_+))} \leq C|p|_* \langle \xi' \rangle^m \text{ for } 1/p - 1 < s_1 \leq 1/p + (1-\delta)N.$$

We proceed with induction. The induction hypotheses is that the estimate above holds for all $p \in \mathcal{P}^m_{1,\delta}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$ and $\mathbf{s} \in \mathbb{N}_0^2$, if s_1 belongs to the interval given above. The induction start is a direct consequence of Lemma 4.19, based on the well-known fact that $e^+ \in \mathcal{L}(H^{\mathbf{s}}(\mathbb{R}_+), H^{\mathbf{s}}(\mathbb{R}))$ for $1/p - 1 < s_1 < 1/p$. For the induction step, we use the equivalence of the following norms:

$$\| \cdot \|_{H^{\mathbf{s}+(1,0)}_p(\mathbb{R}_+)} \simeq \| \cdot \|_{H^{\mathbf{s}}_p(\mathbb{R}_+)} + \|D_{x_n} \cdot \|_{H^{\mathbf{s}}_p(\mathbb{R}_+)}.$$

We recall the jump relation $D_{x_n} e^+ u = e^+ D_{x_n} u - iu(0) \otimes \delta$ which implies the following identity:

$$\begin{aligned} D_{x_n} \sigma_p^{-1}[\mathbf{p}_+(x', \xi')\sigma_p u] =& \langle \xi' \rangle^{-1} \sigma_p^{-1} [D_{x_n} \mathbf{p}]_+(x', \xi')\sigma_p u + \sigma_p^{-1} \mathbf{p}_+(x', \xi')\sigma_p D_{x_n} u \\ &+ \langle \xi' \rangle^{-1/q} \sigma_p^{-1} r^+ \mathbf{p}(x', \xi')[u(0) \otimes \delta]. \end{aligned}$$

We use the fact that $D_{x_n} \sigma_p^{\pm 1} = \langle \xi' \rangle^{\pm 1} \sigma_p^{\pm 1} D_{x_n}$ and $[\sigma_p u](0) = \langle \xi \rangle^{1/p} u(0)$. Now, we separately estimate the operators on the right hand side of the equation above. The induction hypotheses imply the following estimate:

$$\|\sigma_p^{-1} \mathbf{p}_+(x', \xi')\sigma_p D_{x_n} u\|_{H^{\mathbf{s}-(m,0)}_p(\mathbb{R}_+)} \leq C|p|_* \langle \xi' \rangle^m \|D_{x_n} u\|_{H^{\mathbf{s}}_p(\mathbb{R}_+)} \leq C|p|_* \langle \xi' \rangle^m \|u\|_{H^{\mathbf{s}+(1,0)}_p(\mathbb{R}_+)}.$$

The symbol of $D_{x_n} \mathbf{p}$ belongs to $\mathcal{P}^{m+\delta}_{1,\delta}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$. Thus, the induction hypotheses imply:

$$\|\sigma_p^{-1} [D_{x_n} \mathbf{p}]_+ \sigma_p u\|_{H^{\mathbf{s}-(m,0)}(\mathbb{R}^n_+)} \leq C|p|_* \langle \xi' \rangle^{m+\delta} \|u\|_{H^{\mathbf{s}+(\delta,0)}(\mathbb{R}^n_+)}.$$

We use the induction hypotheses for $\mathbf{s}+(\delta,0)$ which explains that in each step of the induction the interval for s_1 increases only by $(1-\delta)$. Proposition 4.20 implies that a potential operator $\mathbf{k}\in S^{m+1/q}_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1};\mathbb{C};S(\mathbb{R}_+))$ exists such that $r^+\mathbf{p}(x',\xi')[u(0)\otimes\delta]=\mathbf{k}(x',\xi')\gamma_0 u$. Thus, the following estimate holds:

$$\|\sigma_p^{-1}\mathbf{k}(x',\xi')\gamma_0 u\|_{H^{\mathbf{s}-(m,0)}(\mathbb{R}_+)}\leq C|p|_*\langle\xi'\rangle^{m+1/q}|\gamma_0 u|\leq C|p|_*\langle\xi'\rangle^{m+1/q}\|u\|_{H^{\mathbf{s}}(\mathbb{R}_+)}.$$

These three estimates for the operators provide the induction step. □

Every potential operator can be constructed from a pseudodifferential operator with transmission property. The situation is similar for trace operators of class zero.

Proposition 4.22. *Let $\mathbf{t}(x',\xi'):=\gamma_0\mathbf{p}_+(x',\xi')$. This defines a trace operator. The map*

$$\mathcal{P}^m_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})\ni p\mapsto t\in\mathcal{T}^{m+1/p,0}_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$$

is linear, bounded, and surjective. Moreover, $t(x',\xi';\xi_n)=h^-_{-1}p(x',\xi';0,\xi_n)\approx h^-_{-1}p'(x',\xi';0,\xi_n)$, where p' is a the principal of p.

Proof. First, we assume $p\in\mathcal{CP}^m_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$. The integral $\int^+ pu$ only depends on the h^-_{-1} projection of p. We obtain the following identity:

$$\mathbf{t}(x',\xi')=\gamma_0 r^+\mathcal{F}^{-1}p(x',\xi';0,\cdot)\mathcal{F}e^+u=\int^+ p(x',\xi';0,\cdot)\mathcal{F}e^+u=\int^+ h^-_{-1}p(x',\xi';0,\cdot)\mathcal{F}e^+u.$$

The formula shows the symbol of the trace operator: $t(x',\xi';\xi_n):=[h^-_{-1}p](x',\xi';0,\xi_n)$. The transmission property implies that $t_{[0]}(x',\xi';\xi_n)=\langle\xi'\rangle^{1/p}h^-_{-1}p_{[0]}(x',\xi';\xi_n)$ belongs to $S^{m+1/p}_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})\otimes\mathcal{H}^-_{-1}$. Now, if $p\in\mathcal{P}^m_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$, we use expansion (4.12) with $M=1$, i.e.,

$$\mathbf{p}(x',\xi')=\mathbf{p}_0(x',\xi')+x_n\mathbf{p}_1(x',\xi'). \tag{4.10}$$

Since $\gamma x_n=0$, only $\mathbf{p}_0(x',\xi')\in\mathcal{CP}^m_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$ contributes to the trace operator. For a given $\tilde t\in\widetilde{\mathcal{T}}^{m+1/p,0}_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$, we define $q(x',\xi';\xi_n):=\overline{\mathcal{F}}E\tilde t(x',\xi',\xi_n)$. Note that $q\in\mathcal{CP}^m_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$, following the same argumentation as in the proof of Proposition 4.20. The derivation

$$\begin{aligned}\gamma_0\mathbf{q}_+(x',\xi')&=\int^+[\overline{\mathcal{F}}E\tilde t](x',\xi';\xi_n)[\mathcal{F}e^+u](\xi_n)\,đ\xi_n=\int[\overline{\mathcal{F}}E\tilde t](x',\xi';\xi_n)[\mathcal{F}e^+u](\xi_n)\,đ\xi_n\\&=\int[E\tilde t](x',\xi';x_n)[e^+u](x_n)\,dx_n=\int_{\mathbb{R}_+}\tilde t(x',\xi;x_n)u(x_n)\,dx_n=\mathbf{t}(x',\xi')u\end{aligned}$$

proves surjectivity. □

Proposition 4.20 and 4.22 imply the well-known duality of potential and trace operators:

Corollary 4.23. *The pointwise dual of a trace operator based on σ_p of order m and type 0 is a potential operator based on σ_q of order m and vice versa.*

Proof. Let $t \in \mathcal{T}^{m,0}_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$. Then, Proposition 4.22 ensures that a $p \in \mathcal{P}^{m-1/p}_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$ exists such that $\mathbf{t}(x',\xi') = \gamma_0\mathbf{p}_+(x',\xi')$. Thus, $t^* = r^+\mathbf{p}^*_+(x',\xi')\gamma^*_0$. Well-known is that $p^* \in \mathcal{P}^{m-1/p}_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$. Proposition 4.20 implies that $r^+\mathbf{p}^*_+(x',\xi')\gamma^*_0$ is a potential operator of order m, if p and q are interchanged. □

Proposition 4.20 and 4.22 provide a possibility to derive results for potential and trace operators, following from the corresponding result for pseudodifferential operators. For instance, the mapping properties are derived in this manner, see Section 4.6. Once these results are established, the corresponding result for a singular Green operator $G = KT$ follows. Then, the result is also true for generic singular Green operators of type zero, according to the following observation on the structure of these operators: Let $g_{[0]} \in S^m_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})\hat{\otimes}\mathcal{H}^+\hat{\otimes}\mathcal{H}^-_{-1}$ be given. According to Theorem 2.1, two sequences $(k_{[0],j})_{j\in\mathbb{N}_0} \in l_1(\mathbb{N}_0; S^m_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})\hat{\otimes}\mathcal{H}^+)$ and $(t_{[0],j})_{j\in\mathbb{N}_0} \in c_0(\mathbb{N}_0;\mathcal{H}^-_{-1})$ exist such that

$$g_{[0]}(x',\xi';\xi_n,\eta_n) = \sum_{j=0}^{\infty} k_{[0],j}(x',\xi';\xi_n)t_{[0],j}(x',\xi';\eta_n).$$

We identify $\mathcal{H}^-_{-1}$ with $S^0_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})\hat{\otimes}\mathcal{H}^-_{-1}$, by $v \mapsto 1\otimes v$. We define $k_j := \sigma_q^{-1}k_{[0],j}$ and $t_j := \sigma_p^{-1}t_{[0],j}$ which belong to $\mathcal{K}^m_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$ and $\mathcal{T}^{0,0}_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$. Thus, each singular Green operator of type zero is a convergent sum of the following form:

$$g(x',\xi';\xi_n,\eta_n) = \sum_{j=0}^{\infty} k_j(x',\xi';\xi_n)t_j(x',\xi';\eta_n) \text{ and } \mathbf{g}(x',\xi') = \sum_{j=0}^{\infty}\mathbf{k}_j(x',\xi')\mathbf{t}_j(x',\xi').$$

The symbol-classes defined in this and previous sections are closed under asymptotic summation:

Theorem 4.24. *Let $(m_l)_{l\in\mathbb{N}_0}$ be a monotonously decreasing sequence that converges to $-\infty$. Then:*

(a) Given $p_l \in \mathcal{P}^{m_l}_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$, a $p \in \mathcal{P}^{m_0}_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$ exists such that

$$p \sim \sum p_l.$$

(b) Given $k_l \in \mathcal{K}^{m_l}_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$, a $k \in \mathcal{K}^{m_0}_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$ exists such that

$$k \sim \sum k_l.$$

(c) Given $t_l \in \mathcal{T}^{m_l,d}_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$, a $t \in \mathcal{T}^{m_0,d}_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$ exists such that

$$t \sim \sum t_l.$$

(d) Given $g_l \in \mathcal{G}^{m_l,d}_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$*, a* $g \in \mathcal{G}^{m_0,d}_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$ *exists such that*

$$g \sim \sum g_l.$$

Proof. The discussion about the structure of singular Green operators, Proposition 4.20, and Proposition 4.22 imply that the proof of (a) is sufficient. By definition: $\mathcal{P}^m_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}) \subset S^m_{1,\delta}(\mathbb{R}^n\times\mathbb{R}^n)$, thus $p \in S^{m_0}_{1,\delta}(\mathbb{R}^n\times\mathbb{R}^n)$ exists. We have to verify that p satisfies the transmission property. The proof is similar to the proof in [32, 2.2.2.2. Proposition 1]. □

The idea of Boutet de Monvel's calculus is to group symbols in a matrix in order to form an algebra under composition. The matrix consists of a pseudodifferential symbol which has the transmission property, a singular Green symbol, a potential symbol, a trace symbol, and a pseudodifferential symbol (on the boundary). The Boutet de Monvel algebra is:

$$\mathcal{BM}^{m,d}_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}) := \begin{pmatrix} \mathcal{P}^m_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}) + G^{m-1,d}_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}) & \mathcal{K}^m_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}) \\ \mathcal{T}^{m,d}_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}) & S^m_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}) \end{pmatrix}.$$

According to Corollaries 4.16 and 4.21, the following map is linear and bounded:

$$\mathcal{BM}^{m,d}_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}) \ni b \mapsto \mathbf{b} := \begin{pmatrix} \mathbf{p}_+ + \mathbf{g} & \mathbf{k} \\ \mathbf{t} & s \end{pmatrix} \in S^m_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}; \mathcal{S}(\mathbb{R}^n_+)\oplus\mathcal{S}(\mathbb{R}^{n-1})).$$

The class of symbols $\mathcal{BM}^{m,d}_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$ is closed under addition, pointwise multiplication, and under asymptotic summation.

4.5 Composition

In this section, we verify that the Boutet de Monvel operators form an algebra:

Theorem 4.25. *A bilinear and bounded map*

$$\mathcal{BM}^{m,d}_{1,\delta} \times \mathcal{BM}^{m',d'}_{1,\delta} \ni (b,b') \mapsto b'' \in \mathcal{BM}^{m+m',\max\{m'+d,d'\}}_{1,\delta}$$

exists such that $B'' = BB'$*. Moreover,* $p'' \approx pp'$ *and* $b''_{[0]}(x',\xi') \approx b_{[0]}(x',\xi') \circ b'_{[0]}(x',\xi')$*. Here,* $b_{[0]}(x',\xi') \circ b'_{[0]}(x',\xi')$ *denotes the symbol of the Wiener-Hopf operator* $\mathbf{b}_{[0]}(x',\xi')\mathbf{b}'_{[0]}(x',\xi')$*, see Theorem 4.9 for explicit formulas.*

The asymptotic expansion of b'' is consistent with the asymptotic expansion for $\delta = 0$, see for instance [20]. In order to keep the notation simple, we only provide the principal symbol which is sufficient for our purpose. We follow the standard argument, i.e., prove the theorem for symbols, with no x_n dependence, and then use Taylor expansion and remainder estimates to generalise. We pointed out in the last section that the action of a Boutet de Monvel operator in the direction normal to the boundary can be interpreted as an operator-valued pseudodifferential operator. In particular, Theorem 4.2 implies:

$$\mathbf{b}''(x',\xi') = \mathbf{b}(x',\xi')\#\mathbf{b}'(x',\xi') \sim \sum D^\alpha_{\xi'}\mathbf{b}(x',\xi')\partial^\alpha_{x'}\mathbf{b}'(x',\xi'). \tag{4.11}$$

Thus, an investigation of the composition in the normal direction is sufficient.
Now, we assume that the symbols do not depend on x_n. Then, the operators on the right hand side of (4.11) are Wiener-Hopf operators, suitably composed with Fourier transforms. Therefore, Theorem 4.9 provides the composed symbols. We have proven the following result:

Lemma 4.26. *A bilinear and bounded map*

$$\mathcal{CBM}^{m,d}_{1,\delta} \times \mathcal{CBM}^{m,d}_{1,\delta} \ni (b,b') \mapsto b'' \in \mathcal{CBM}^{m'',d''}_{1,\delta}$$

exists such that $\mathbf{b}\mathbf{b}' = \mathbf{b}''$. *Moreover,* $b''_{[0]} \approx b_{[0]} \circ b'_{[0]}$.

In this section, we assume that the symbols of the pseudodifferential operators with transmission property are supported close to the boundary. This is no significant restriction because each symbol can be decomposed into two parts. One part has the support property and the other part vanishes to infinite order on the boundary. The latter part gives rise to negligible operators when composed with boundary symbols. Now, we drop the assumption that the symbol does not depend on x_n. This only affects compositions containing pseudodifferential operators with the transmission property. Let $p \in \mathcal{P}^m_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$ and p^R be its (x',y_n)-form. We define:

$$\begin{aligned}
p_j(x',\xi';\xi_n) &:= (j!)^{-1}[\partial^j_{x_n}p](x',\xi';0,\xi_n) \in \mathcal{CP}^{m+j\delta}_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}),\\
p_M(x',\xi';x_n,\xi_n) &:= \frac{1}{(M-1)!}\int_0^1 (1-s)^{M-1}\partial^M_{x_n}p(x',\xi';sx_n,\xi_n)\,ds \in \mathcal{P}^{m+\delta M}_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}),\\
p^R_j(x',\xi';\xi_n) &:= (j!)^{-1}[\partial^j_{y_n}p^R](x',\xi';0,\xi_n) \in \mathcal{CP}^{m+j\delta}_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}), \text{ and}\\
p^R_M(x',\xi';y_n,\xi_n) &:= \frac{1}{(M-1)!}\int_0^1 (1-s)^{M-1}\partial^M_{x_n}p^R(x',\xi';sy_n,\xi_n)\,ds \in \mathcal{P}^{m+\delta M}_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}).
\end{aligned}$$

We observe that dependence of p_j, p_M, p^R_j, and p^R_M on p is linear and bounded. The previously defined terms are the coefficients in the Taylor expansion of p (resp. p^R) at zero with respect to the x_n (resp. y_n) variable. Therefore:

$$p(x',\xi';x_n,\xi_n) = \sum_{j=0}^{M-1} x_n^j p_j(x',\xi';0,\xi_n) + x_n^M p_M(x',\xi';x_n,\xi_n) \text{ and} \tag{4.12}$$

$$p^R(x',\xi';y_n,\xi_n) = \sum_{j=0}^{M-1} y_n^j p^R_j(x',\xi';0,\xi_n) + y_n^M p^R_M(x',\xi';y_n,\xi_n). \tag{4.13}$$

In particular, we obtain two expansions for the operator valued symbol $\mathbf{p}(x',\xi')$:

$$\mathbf{p}(x',\xi') = \sum_{j<M} x_n^j \mathbf{p}_j(x',\xi') + x_n^M \mathbf{p}_M(x',\xi') = \sum_{j<M} \mathbf{p}_j^R(x',\xi') x_n^j + \mathbf{p}_M^R(x',\xi') x_n^M.$$

At first glance, the expansions do not look very promising. The order of the operators increase with j and x_n^j is not a uniformly bounded pseudodifferential operator. The key observation is that x_n^j has a regularising effect on boundary operators. More precisely, $\mathbf{k}_j := x_n^j \mathbf{k}$, $\mathbf{t}_j := \mathbf{t} x_n^j$, and $\mathbf{g}_{j,l} := x_n^j \mathbf{g} x_n^l$ are potential, trace, and singular Green operators with symbols $k_j = \overline{D}_{\xi_n}^j k$, $t_j = D_{\xi_n}^j t$, and $g_{j,l} = \overline{D}_{\xi_n}^j D_{\eta_n}^l g$, respectively. Symbol-kernel representations and basic results on the Fourier transform are used in the proof. Now, we focus on the composition $\mathbf{p}_+(x',\xi')\mathbf{k}(x',\xi')$:

Theorem 4.27. *Let $\mathbf{k}''(x',\xi') := \mathbf{p}_+(x',\xi')\mathbf{k}'(x',\xi')$. Then, $\mathbf{k}''$ is a potential operator of order $m+m'$ and the following map is bilinear and bounded:*

$$\mathcal{P}_{1,\delta}^m(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}) \times \mathcal{K}_{1,\delta}^{m'}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}) \ni (p,k') \mapsto k'' \in \mathcal{K}_{1,\delta}^{m+m'}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}).$$

Moreover, $\mathbf{k}''(x',\xi') \sim \sum \mathbf{p}_{j,+}^R(x',\xi')\mathbf{k}_j'(x',\xi')$ and $k''(x',\xi',\xi_n) \approx h^+ p(x',\xi';0,\xi_n)k'(x',\xi';\xi_n)$.

Proof. Given the expansion (4.13) for p, we obtain a corresponding expansion for k'':

$$\mathbf{k}''(x',\xi') := \mathbf{p}_+(x',\xi')\mathbf{k}'(x',\xi') = \sum_{j<M} \mathbf{p}_{j,+}^R(x',\xi') x_n^j \mathbf{k}'(x',\xi') + \mathbf{p}_{M,+}^R(x',\xi') x_n^M \mathbf{k}(x',\xi').$$

According to Lemma 4.26, $\mathbf{k}_j'' := \mathbf{p}_{j,+}^R(x',\xi') x_n^j \mathbf{k}'(x',\xi')$ is a potential operator of order $m+m'-(1-\delta)j$. Therefore, we only need to consider $\mathbf{k}_M'' := \mathbf{p}_{M,+}^R(x',\xi') x_n^M \mathbf{k}(x',\xi')$. We claim the following estimates hold for all $l, l', M \in \mathbb{N}_0$ that satisfy $M \geq [m-|\alpha|+\delta|\beta|]_+ + l'$:

$$\| x_n^l D_{x_n}^{l'} D_{\xi'}^\alpha D_{x'}^\beta \tilde{k}_M''(x',\xi';\cdot)\|_{L_p(\mathbb{R}_+)} \leq C|p|_*|k|_* \langle\xi'\rangle^{m+m'-(1-\delta)M-|\alpha|+\delta|\beta|-l+l'}. \tag{4.14}$$

In the following argumentation, we point out that these estimates are sufficient to prove $k'' \in \mathcal{K}_{1,\delta}^{m+m'}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$. We show that $\tilde{k}''$ satisfies the family of Estimates (iv) in Lemma 4.14. We fix indices α,β,l,l' and choose a decomposition with M large enough for (4.14) to hold. In particular, $\tilde{k}''$ satisfies the estimate for the chosen indices.
Proof of the claimed estimate: We observe that $\tilde{k}_M''(x',\xi';\cdot) = \mathbf{p}_{M,+}(x',\xi') x_n^M \tilde{k}'(x',\xi';\cdot)$. Thus, the scaled version $\tilde{k}_{M,[0]}''$ is:

$$\tilde{k}_{M,[0]}'' = \sigma_p^{-1}\left(\mathbf{p}_+(x',\xi') x_n^M \tilde{k}(x',\xi';\cdot)\right) = \langle\xi'\rangle^{-M}\sigma_p^{-1}\mathbf{p}_+(x',\xi')\sigma_p x_n^M \tilde{k}_{[0]}(x',\xi';\cdot).$$

We can absorb the derivative $D_{\xi'}^\alpha D_{x'}^\beta$ into the order of the operators. Therefore, we can assume $\alpha = \beta = 0$. We recall the following equation, derived in the proof of Corollary 4.21:

$$D_{x_n}\sigma_p^{-1}[\mathbf{p}_+(x',\xi')\sigma_p u] = \langle\xi'\rangle^{-1}\sigma_p^{-1}[D_{x_n}\mathbf{p}]_+(x',\xi')\sigma_p u + \sigma_{\langle\xi'\rangle}^{-1}\mathbf{p}_+(x',\xi')\sigma_p D_{x_n} u$$

$$+\langle\xi'\rangle^{-1/p}\sigma_p^{-1}r^+\mathbf{p}(x',\xi')[u(0)\otimes\delta].$$

The last term is zero because $u(x_n)=x_n\tilde{k}(x',\xi',x_n)$.

$$x_n\sigma_p^{-1}[\mathbf{p}_+(x',\xi')\sigma_p u]=\langle\xi'\rangle\sigma_p^{-1}[\overline{D}_{\xi_n}\mathbf{p}]_+\sigma_p u+\sigma_p^{-1}\mathbf{p}_+(x',\xi')\sigma_p[x_n u].$$

Given $l'<M$, by a repeated use of the above relations, we obtain that the scaled symbol kernel is a linear combination of terms of the form:

$$\langle\xi'\rangle^{-M+l_1-l_1'}\sigma_{\langle\xi'\rangle}^{-1}[D_{\xi_n}^{l_1}D_{x_n}^{l_1'}\mathbf{p}]_+(x',\xi')\sigma_{\langle\xi'\rangle}x_n^{M+l_2-l_2'}D_{x_n}^{l_3'}\tilde{k}_{[0]}'(x',\xi';\cdot).$$

The estimates for the symbol kernels and Corollary 4.21 imply that the scaled symbol kernel $\tilde{k}_M''$ satisfies the following estimate under the assumption that $l'<M$:

$$\|x_n^l D_{x_n}^{l'}\tilde{k}_{M,[0]}''(x',\xi';x_n)\|_{L_2(\mathbb{R}_+)}\leq C|p|_*|k|_*\langle\xi'\rangle^{m+m'-M(1-\delta)}.$$

The claimed estimates are the unscaled version of the estimates above. □

Next, we use the facts that trace operators are formal adjoints of potential operators and that singular Green operators can be decomposed into products of trace and potential operators in order to provide the following result:

Corollary 4.28. *Let* $\mathbf{t}''(x',\xi'):=\mathbf{t}(x',\xi')\mathbf{p}_+'(x',\xi')$, $\mathbf{g_1''}(x',\xi'):=\mathbf{p}_+(x',\xi')\mathbf{g}'(x',\xi')$, *and* $\mathbf{g_2'}(x',\xi'):=\mathbf{g}(x',\xi')\mathbf{p}_+{}'(x',\xi')$. *The following maps are bilinear and bounded:*

1. $\mathcal{T}_{1,\delta}^{m,d}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})\times\mathcal{P}_{1,\delta}^{m'}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})\ni(t,p')\mapsto t''\in\mathcal{G}_{1,\delta}^{m'+m,d'}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$.
2. $\mathcal{P}_{1,\delta}^{m}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})\times\mathcal{G}_{1,\delta}^{m',d'}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})\ni(p,g')\mapsto g_1''\in\mathcal{G}_{1,\delta}^{m',d'}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$.
3. $\mathcal{G}_{1,\delta}^{m,d}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})\times\mathcal{P}_{1,\delta}^{m'}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})\ni(g,p')\mapsto g_2''\in\mathcal{G}_{1,\delta}^{m+m',0}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$.

Proof. We consider $\mathbf{t}''$ and initially assume that $d=0$.

$$\mathbf{t}''(x',\xi')=\mathbf{t}(x',\xi')\mathbf{p}_+'(x',\xi')=((\mathbf{p}_+')^*(x',\xi')\mathbf{t}^*(x',\xi'))^*.$$

Theorem 4.27 implies that $\mathbf{p}_+^*(x',\xi')\mathbf{t}^*(x',\xi')$ is a potential operator of order $m+m'$. Thus, according to Corollary 4.23, $\mathbf{t}'(x',\xi')$ is a trace operator of the same order. If the class is not zero, we choose $M\geq d-1$ in the expansion (4.12) of p. We notice that $\gamma_j x_n^M=0$ for $j\leq d-1=M$. Therefore, $\mathbf{t}(x',\xi')x_n^M\mathbf{p}_M'(x',\xi')$ has class zero. The other terms in the composition are independent of x_n and can thus be handled by Lemma 4.26.
We mentioned previously that every singular Green operator obeys a decomposition of the following form:

$$\mathbf{g}(x',\xi')=\sum_{j=0}^{d}\mathbf{k}_j(x',\xi')\gamma_j+\sum_{l=0}^{\infty}\mathbf{k}_l(x',\xi')\mathbf{t}_l(x',\xi').$$

We apply Theorem 4.27 to the decomposition in order to obtain the result for $\mathbf{g}_1''$. Observe that the types of g_1'' and g' coincide. Now, we consider $\mathbf{g}_2''$. Initially, we assume that the class of g is zero. Therefore, the result follows from duality. Next, we consider singular Green symbols without zero class part. According to the decomposition above, we can assume that $g = k_j\gamma_j$. The result follows from:

$$\mathbf{g}_2''(x',\xi') = \mathbf{k}_j(x',\xi')\gamma_j\mathbf{p}_+'(x',\xi') = \left((\mathbf{p}_+')^*(x',\xi')\gamma_j^*\mathbf{k}_j^*(x',\xi')\right)^*.$$

Note that the class of g_2'' is always zero. □

In the remainder of this section, we analyse the composition of two pseudodifferential operators with the transmission property. This composition has two terms. The first term is a truncated pseudodifferential operator with the symbol of the composed operator on the entire space. The second term, commonly referred to as the leftover term, is a singular Green operator:

$$\mathbf{g}''(x',\xi') := [\mathbf{g}''(p,p')](x',\xi') := (\mathbf{p}(x',\xi')\mathbf{p}'(x',\xi'))_+ - \mathbf{p}_+(x',\xi')\mathbf{p}_+'(x',\xi').$$

[19] provides the case $\delta = 0$. The argumentation below is similar. Using Taylor expansion, $\mathbf{p}(x',\xi') = \sum \mathbf{p}_i(x',\xi')x_n^i$ and $\mathbf{p}'(x',\xi') = \sum x_n^j\mathbf{p}_j'(x',\xi')$, where $p_j, p_j' \in \mathcal{CP}_{1,\delta}^{m+j\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$ for $i,j<M$ and $p_M, p_M' \in \mathcal{P}_{1,\delta}^{m+M\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$. Then, by bilinearity:

$$\mathbf{g}''(\mathbf{p}(x',\xi'),\mathbf{p}'(x',\xi')) = \sum_{0\le i,j\le M}\mathbf{g}''(\mathbf{p}_i(x',\xi')x_n^i, x_n^j\mathbf{p}_j'(x',\xi')) =: \sum_{0\le i,j\le M}\mathbf{g}_{ij}''(\mathbf{p}(x',\xi'),\mathbf{p}'(x',\xi')).$$

We first consider $\mathbf{g}_{ij}''(x',\xi')$ for $i,j<M$. We write $p_{i,i'} := D_{\xi_n}^{i-i'}p_i \in \mathcal{CP}_{1,\delta}^{m-(1-\delta)(i+i')}$ and $p_{j,j'}' := D_{\xi_n}^{j-j'}p_j' \in \mathcal{CP}_{1,\delta}^{m-(1-\delta)(j+j')}$ and observe that:

$$\mathbf{g}_{ij}''(x',\xi') = \sum C_{i,i',j,j'}x_n^{i'}\mathbf{g}(\mathbf{p}_{i,i'}(x',\xi'),\mathbf{p}_{j,j'}'(x',\xi'))x_n^{j'}.$$

We apply Lemma 4.26 to the right hand side of the equation above. Moreover, we use the fact that multiplication by x_n from the left or right decreases the order of singular Green operators which implies that $g_{ij}'' \in \mathcal{G}_{1,\delta}^{m+m'-(1-\delta)(i+j),0}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$. For the remaining terms we need the following technical result. We write J for the reflection along the boundary.

Lemma 4.29. *Let $N,l,l',l'',l''' \in \mathbb{N}_0$ and $\alpha,\beta\in\mathbb{N}_0^n$. For $p\in S_{1,\delta}^{m+\delta N}(\mathbb{R}^n\times\mathbb{R}^n)$, we define $\mathbf{g}_N^+(x',\xi') := r^+\mathbf{p}(x',\xi')x_n^Ne^-J$ and $\mathbf{g}_N^-(x',\xi') := Jr^-x_n^N\mathbf{p}(x',\xi')e^+$. Then,*

$$\|x_n^lD_{x_n}^{l'}y_n^{l''}D_{y_n}^{l'''}D_{\xi'}^{\alpha}D_{x'}^{\beta}\tilde{g}_{N,[0]}^{\pm}(x',\xi';\cdot,\cdot)\|_{L^2(\mathbb{R}_{++}^2)} \le C|p|_*\langle\xi'\rangle^{m-|\alpha|+\delta|\beta|-N},$$

if either $p\in\mathcal{CP}_{1,\delta}^{m+\delta N}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$ or $m-|\alpha|+\delta|\beta|-(1-\delta)N-l+l'-l''+l''' < -1/2$.

Proof. We only prove the results for g_N^+, the results for g_N^- can be derived analogously. A straightforward computation reduces the results to the case where all multi-indices vanish. We thus assume that $l = l' = l'' = l''' = 0$ and $\alpha = \beta = 0$. To simplify the computation, we define the following operator:

$$\mathbf{g}^\bullet(x', \xi') := \sum_{j=0}^{N} \binom{N}{j} x_n^{N-j} \mathbf{g}_0^+(x', \xi') x_n^j.$$

For later use, we express $\mathbf{g}^\bullet(x', \xi')$ in terms of $\mathbf{p}(x', \xi')$:

$$\begin{aligned}
[\mathbf{g}^\bullet(x', \xi')u](x_n) &= r^+ \int e^{i(x_n - y_n)\xi_n} \sum_{j=0}^{N} \binom{N}{j} x_n^{N-j} p(x', \xi'; x_n, \xi_n)(-y_n)^j [e^- u](-y_n)\, dy_n đ\xi_n \\
&= r^+ \int e^{i(x_n - y_n)\xi_n} (x_n - y_n)^N p(x', \xi'; x_n, \xi_n)[e^- u](-y_n)\, dy_n đ\xi_n \\
&= r^+ \int e^{i(x_n - y_n)\xi_n} [D_{\xi_n}^N p](x', \xi'; x_n, \xi_n)[e^- u](-y_n)\, dy_n đ\xi_n \\
&= [r^+ [D_{\xi_n}^N \mathbf{p}](x', \xi') e^- J u](x_n).
\end{aligned}$$

The estimates for the kernel of $\mathbf{g}_N^+(x', \xi')$ follow from the corresponding estimates for $\mathbf{g}^\bullet(x', \xi')$, since both are supported on the first quadrant and $\tilde{g}^\bullet(x', \xi; x_n, y_n) = (x_n + y_n)^N \tilde{g}_0^+(x', \xi; x_n, y_n)$.

$$\mathbf{g}_{[0]}^\bullet(x', \xi') = \sigma_p^{-1} \mathbf{g}^\bullet(x', \xi') \sigma_p = r^+ \sigma_p^{-1} [D_{\xi_n}^N \mathbf{p}](x', \xi') \sigma_p e^- J = r^+ \mathbf{q}(x', \xi') e^- J,$$

where $q \in S_{1,\delta}^{m-(1-\delta)N}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$ and is related to p by the following identity:

$$q(x', \xi'; x_n, \xi_n) = [D_{\xi_n}^N p](x', \xi'; x_n / \langle \xi' \rangle, \langle \xi' \rangle \xi_n).$$

As a pseudodifferential operator, $\mathbf{q}(x', \xi')$ has a singular integration kernel $\mathcal{F}_{\xi \mapsto z_n}^{-1} q(x', \xi'; x_n, \xi_n)|_{z_n = x_n - y_n}$. Via a change of variables $-y_n \rightsquigarrow y_n$ we obtain $\tilde{g}_{[0]}^\bullet$.

$$\begin{aligned}
\|\tilde{g}_{[0]}^\bullet(x', \xi'; \cdot, \cdot)\|_{L_2(\mathbb{R}_{++}^2)}^2 &= \int_0^\infty \int_0^\infty \left| \mathcal{F}_{\xi_n \mapsto z_n}^{-1} q(x', \xi'; x_n, \xi_n)|_{z_n = x_n + y_n} \right|^2 dy_n dx_n \\
&\le \int_0^\infty \langle x_n \rangle^{-2} \int_{x_n}^\infty \langle z_n \rangle^2 \left| \mathcal{F}_{\xi_n \mapsto z_n} q(x', \xi'; x_n, \xi_n) \right|^2 dz_n dx_n \\
&\le C^2 \sup_{x_n} \int \left| h^+ \langle D_{\xi_n} \rangle^2 q(x', \xi'; x_n, \xi_n) \right|^2 đ\xi_n.
\end{aligned}$$

If $p \in \mathcal{CP}_{1,\delta}^{m+\delta N}(\mathbb{R}^n \times \mathbb{R}^n)$, the right hand side can be estimated by the square of $C|p|_* \langle \xi' \rangle^{m-(1-\delta)j}$. If $p \notin \mathcal{CP}_{1,\delta}^{m+\delta N}(\mathbb{R}^n \times \mathbb{R}^n)$, we use the estimates of q derived in the proof of Lemma 4.19 to obtain:

$$\begin{aligned}
\|\tilde{g}_{[0]}^\bullet(x', \xi'; \cdot, \cdot)\|_{L_2(\mathbb{R}_{++}^2)}^2 &\le C^2 |p|_*^2 \langle \xi' \rangle^{2(m-(1-\delta)N)} \int \langle \xi_n \rangle^{2(m-(1-\delta)N)} đ\xi_n \\
&\le C |p|_*^2 \langle \xi' \rangle^{2(m-(1-\delta)N)}.
\end{aligned}$$

The last integral converges under the condition $m - (1-\delta)N < -1/2$. □

For M sufficiently large $x_n^M \mathbf{p}'(x',\xi')e^+u \in L_2(\mathbb{R})$ for all $u \in e^+\mathcal{S}(\mathbb{R}^n_+)$, since all distributions of a given order on the boundary eventually belong to the kernel of x_n^M. Therefore:

$$\mathbf{g}''_{i,M}(x',\xi') = r^+\mathbf{p}_i x_n^i(1-e^+r^+)x_n^M\mathbf{p}'_M e^+ = r^+\mathbf{p}_i x_n^i e^- J J r^- x_n^M \mathbf{p}'_M e^+ = \mathbf{g}_i^+(\mathbf{p}_i)\mathbf{g}_M^-(\mathbf{p}'_M).$$

Thus, Lemmata 4.26 and 4.29 imply that the symbol-kernel of $\mathbf{g}_{jM}$ satisfy the following estimate, if $m-|\alpha|+\delta|\beta|-(1-\delta)M-l+l'-l''+l'''<-1/2$:

$$\|x_n^l D_{x_n}^{l'} y_n^{l''} D_{y_n}^{l'''} D_{\xi'}^{\alpha} D_{x'}^{\beta} \tilde{g}''_{iM,[0]}(x',\xi';\cdot,\cdot)\|_{L_2(\mathbb{R}^2_{++})} \leq \langle\xi'\rangle^{m+m'-(1-\delta)(i+M)}.$$

The estimate also holds for the symbol-kernel of $\mathbf{g}''_{MM}$, if the restriction on the indices above also holds for m' instead of m. We observe that $\mathbf{g}''_{Mj} = (\mathbf{g}''_{jM})^*$. Thus, the estimate above also holds for the symbol-kernel of $\mathbf{g}''_{Mj}$. Now, we show that the symbol-kernel of $\mathbf{g}''$ satisfies the Estimates (vi) in Lemma 4.14. Therefore, $\mathbf{g}''$ is a singular Green operator of order $m+m'$. We fix the indices and choose decompositions for $\mathbf{p}$ and $\mathbf{p}'$ with an M large enough for the constraints above to be satisfied. Therefore, we obtain Estimates (vi) for the chosen indices.

4.6 Mapping Properties

In Section 4.6, we extend Boutet de Monvel operators to the scale of Banach space $H^s_p(\mathbb{R}^n_+) \oplus B^s_p(\mathbb{R}^{n-1})$. The proof given in [18] for $\delta = 0$ essentially extends to $0 \leq \delta < 1$. The main result is:

Theorem 4.30. *The map below is linear and bounded for all $s > d + 1/p - 1$.*

$$\mathcal{BM}^{m,d}_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}) \ni b \mapsto B \in \mathcal{B}\left(H^s_p(\mathbb{R}^n_+) \oplus B^s_p(\mathbb{R}^{n-1}), H^{s-m}_p(\mathbb{R}^n_+) \oplus B^{s-m}_p(\mathbb{R}^{n-1})\right).$$

We partition the proof into lemmata which provide the result for the components of the matrix. Initially, we consider potential operators:

Lemma 4.31. *The following map is linear and bounded for all $s \in \mathbb{R}$.*

$$\mathcal{K}^m_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}) \ni k \mapsto K \in \mathcal{B}(B^s_p(\mathbb{R}^{n-1}), H^{s-m}_p(\mathbb{R}^n_+))$$

Proof. Proposition 4.20 ensures that a pseudodifferential operator P with symbol $p \in \mathcal{P}^{m-1/q}_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$ exists such that $K = r^+P\gamma_0^*$. Moreover, the map

$$\mathcal{K}^m_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}) \ni k \mapsto p \in \mathcal{P}^{m-1/q}_{1,\delta}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$$

is linear and bounded. According to the trace theorem, the map below is linear and bounded:

$$\gamma_0^* : B^s_p(\mathbb{R}^{n-1}) \to H^{s-1/q}_p(\mathbb{R}^n) \text{ for all } s < 0.$$

Well-known is that $P : H^{s-1/q}_p(\mathbb{R}^n) \to H^{s-m}_p(\mathbb{R}^n)$ because P is a pseudodifferential operator of order $m-1/q$. Therefore, the lemma holds for $s<0$. We extend the result to all $s \in \mathbb{R}$ via order reduction. □

Next, we consider truncated pseudodifferential operators with transmission property:

Lemma 4.32. *The following map is linear and bounded for all $s > 1/p - 1$.*

$$\mathcal{P}^m_{1,\delta}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}) \ni p \mapsto P_+ \in \mathcal{B}(H^s_p(\mathbb{R}^n_+), H^{s-m}_p(\mathbb{R}^n_+)).$$

Proof. We provide an inductive proof. The induction hypotheses is that the lemma holds for $1/p-1 < s < (1-\delta)k+1/p$. Well-known is $H^s_{p,0}(\mathbb{R}^n_+) = H^s_p(\mathbb{R}^n_+)$ for $1/p-1 < s < 1/p$. In particular, $e^+ : H^s_{p;0}(\mathbb{R}^n_+) \to H^s_p(\mathbb{R}^n)$ is a bounded linear map. Therefore, the base case $k = 0$ is implied by the mapping properties of pseudodifferential operators. For the induction step, we use the equivalence of the norms below:

$$\| \cdot \| H^{\sigma+1}_p(\mathbb{R}^n_+) \simeq \sum_{|\alpha|\leq 1} \|D^\alpha_x \cdot \|_{H^\sigma_p(\mathbb{R}^n_+)}, \quad \text{where either } \ \sigma = s \ \text{ or } \ \sigma = s - m.$$

Here, the most complex term on the right hand side includes the derivative in the direction normal to the boundary. This term is decomposed via the jump relation:

$$D_{x_n} P_+ u = P_+ D_{x_n} u + [D_{x_n}, P]_+ u - i r^+ P \gamma_0^* \gamma_0 u. \tag{4.15}$$

The three terms on the right hand side are estimated separately. For the first term, we use the induction hypotheses:

$$\|P_+ D_{x_n} u\|_{H^{s-m}_p(\mathbb{R}^n_+)} \leq C|p|_* \|D_{x_n} u\|_{H^s_p(\mathbb{R}^n_+)} \leq C|p|_* \|u\|_{H^{s+1}_p(\mathbb{R}^n_+)}.$$

The symbol of the second term is $D_{x_n} p \in \mathcal{P}^{m+\delta}_{1,\delta}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$. Therefore, we can apply the induction hypotheses with $s + \delta$ which implies:

$$\|[D_{x_n}, P]_+ u\|_{H^{s-m}_p(\mathbb{R}^n_+)} \leq \|[D_{x_n}, P]_+ u\|_{H^{s+\delta}_p(\mathbb{R}^n_+)} \leq C|p|_* \|u\|_{H^{s+1}_p(\mathbb{R}^n_+)}.$$

In the argument above we rely on the fact that the result holds for $s+\delta$. Thus, the length of the interval increases by $1-\delta$ for each step of the induction. Lemma 4.31 and the trace Theorem imply:

$$\|r^+ P \gamma_0^* \gamma_0 u\|_{H^{s-m}_p(\mathbb{R}^n_+)} \leq C|p|_* \|\gamma_0 u\|_{B^{s+1/q}_p(\mathbb{R}^{n-1})} \leq C|p|_* \|u\|_{H^{s+1}_p(\mathbb{R}^n_+)}.$$

If $D^\alpha_x \neq D_{x_n}$, then equation (4.15) holds without the third term. □

Now, we consider trace operators:

Lemma 4.33. *The following maps are linear and bounded:*

(i) $\mathcal{T}^{m,0}_{1,\delta}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}) \ni t \mapsto T \in \mathcal{L}(H^s_p(\mathbb{R}_+), B^{s-m}_p(\mathbb{R}^n_+))$, *for* $s \in \mathbb{R}$.

(ii) $\mathcal{T}^{m,d}_{1,\delta}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}) \ni t \mapsto T \in \mathcal{L}(H^s_p(\mathbb{R}_+), B^{s-m}_p(\mathbb{R}^n_+))$, *for* $s > d + 1/p$.

Proof. We decompose T as $\sum_{j=0}^{d-1} S_j \gamma_j + T'$. The maps

$$\mathcal{T}_{1,\delta}^{m,d}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}) \ni t \mapsto s_j \in S_{1,\delta}^{m-j-1/p}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}) \text{ and}$$
$$\mathcal{T}_{1,\delta}^{m,d}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}) \ni t \mapsto t' \in \mathcal{T}_{1,\delta}^{m,0}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$$

are linear and bounded. According to the trace theorem, the lemma holds for $\sum_{j=0}^{d-1} S_j \gamma_j$. Thus, it is sufficient to prove (i). For each $t \in \mathcal{T}_{1,\delta}^{m,0}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$ a $p \in \mathcal{P}_{1,\delta}^{m-1/p}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$ exists such that $T = \gamma_0 P_+$. We initially assume that $s > m$. Therefore, the result holds according to the trace theorem and mapping properties of pseudodifferential operators. The results extend to all $s \in \mathbb{R}$ via order reduction. □

Finally we consider singular Green operators:

Lemma 4.34. *The following maps are linear and bounded:*

(i) $\mathcal{G}_{1,\delta}^{m,0}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}) \ni g \mapsto G \in \mathcal{L}(H_p^s(\mathbb{R}^n_+), H_p^{s-m}(\mathbb{R}^n_+))$, *for* $s \in \mathbb{R}$.

(ii) $\mathcal{G}_{1,\delta}^{m,d}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}) \ni g \mapsto G \in \mathcal{L}(H_p^s(\mathbb{R}^n_+), H_p^{s-m}(\mathbb{R}^n_+))$, *for* $s > d + 1/p$.

Proof. We decompose G as $\sum_{j=0}^{d-1} K_j \gamma_j + G'$. The maps

$$\mathcal{G}_{1,\delta}^{m,d}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}) \ni g \mapsto k_j \in \mathcal{K}_{1,\delta}^{m-j-1/p}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}) \text{ and}$$
$$\mathcal{G}_{1,\delta}^{m,d}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}) \ni g \mapsto g' \in \mathcal{G}_{1,\delta}^{m,0}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$$

are linear and bounded. According to the trace theorem and Lemma 4.31, the lemma holds for the sum. Therefore, it is sufficient to prove (i). Since the class is zero, a decomposition $G = \sum_{j=0}^{\infty} K_j T_j$ exists which is absolutely convergent with regards to seminorms. Therefore, the result follows from Lemma 4.31 and 4.33. □

5 Bounded H^∞-Calculus for a Degenerate Boundary Value Problem

For the proof of Theorem 1.2, a suitable description of the resolvent $(A_T - \lambda)^{-1}$ is mandatory. We explain the key idea of how this description is derived in a simple example, where $A = -\Delta$, $T = \gamma_0$, and $\nu = 1$. Here, the benefit is that we can point out the main ideas. However, the majority of abstract arguments can be replaced by explicit computations.
In the article [2], Shmuel Agmon proved a priori estimates for solutions of the following boundary value problem with spectral parameter:

$$\begin{cases} (1-\Delta-\lambda)_+ u &= f \;\text{ on }\; \mathbb{R}^n_+ \\ \gamma_0 u &= \phi \;\text{ on }\; \mathbb{R}^{n-1} \end{cases}. \tag{5.1}$$

Let $\lambda = \mu^2 e^{i\theta}$. The author observed that, given a solution u of (5.1), the function $\tilde{u} := u \otimes e_\mu$ with $e_\mu(z) = e^{i\mu z}$ solves the elliptic boundary problem below:

$$\begin{cases} (1-\Delta+e^{i(\pi+\theta)}D_z^2)_+ \tilde{u} &= \tilde{f} \;\text{ on }\; \mathbb{R}^{n+1}_+ \\ \gamma_0 \tilde{u} &= \tilde{\phi} \;\text{ on }\; \mathbb{R}^n. \end{cases} \tag{5.2}$$

The right hand side consists of $\tilde{f} := f \otimes e_\mu$ and $\tilde{\phi} = \phi \otimes e_\mu$. The a priori estimates were already established for the elliptic boundary value problem (5.2). For our purposes, a priori estimates are not sufficient. However, the basic idea can be extended to provide a relation between the inverse of (5.2) and (5.1). The following three operators are of interest:

$$\begin{aligned} Q_\theta &:= r^+ \mathcal{F}^{-1}(\langle\xi\rangle^2 + e^{i(\pi+\theta)}\zeta^2)^{-1}\mathcal{F}e^+, \\ K_\theta &:= r^+ \mathcal{F}'^{-1} e^{-\kappa_\theta(\xi',\zeta)x_n}\mathcal{F}', \;\text{ and} \\ G_\theta &:= -K_\theta\gamma_0 Q_\theta. \end{aligned}$$

Here, $\kappa_\theta(\xi',\zeta)$ is the root of the polynomial $a_\theta := \langle\xi\rangle^2 + e^{i(\pi+\theta)}\zeta^2$, with positve real part. The identities $A_\theta Q_\theta = 1$, $A_\theta K_\theta = 0$, $\gamma_0 K_\theta = 1$, and $\gamma_0(Q_\theta + G_\theta) = 0$ can be verified in a quick calculation. Therefore:

$$\begin{pmatrix} A_{\theta,+} \\ \gamma_0 \end{pmatrix}^{-1} = \begin{pmatrix} Q_{\theta,+} + G_\theta & K_\theta \end{pmatrix}. \tag{5.3}$$

The operators belong to Boutet de Monvel's calculus. We denote the symbols by lower case letters. Now, we apply (5.3) to the tensor product $\tilde{f} = f \otimes e_\mu$ and $\tilde{\phi} = \phi \otimes e_\mu$, respectively. For instance:

$$\begin{aligned} [Q_{\theta,+}\tilde{f}](x,z) &= r^+ \int e^{ix\xi + iz\zeta} q_\theta(\xi,\zeta)[\mathcal{F}e^+ f](\xi)\delta(\zeta-\mu)\, d\zeta đ\xi \\ &= e^{iz\mu} r^+ \int e^{ix\xi} q_\theta(\xi,\mu)[\mathcal{F}e^+ f](\xi)\, đ\xi \end{aligned}$$

$$=: e^{iz\mu}[Q_{\theta,\mu,+}f](x)$$

Here, $Q_{\theta,\mu}$ has the structure of a truncated pseudodifferential operator. The symbols of the operators Q_θ and $Q_{\theta,\mu}$ are related by restriction: $q_{\theta,\mu} = q_\theta|_{\zeta=\mu}$. We obtain $G_{\theta,\mu}$, $K_{\theta,\mu}$ and $A_{\theta,\mu,+}$ from G_θ, K_θ, and $A_{\theta,+}$ with a similar argumentation. We further observe that:

$$A_{\theta,+}[u \otimes e_\mu] = [A_{\theta,\mu,+}u] \otimes e_\mu = [(A-\lambda)_+u] \otimes e_\mu.$$

Using the previous relations, we verify that the function $u := (Q_{\theta,\mu,+} + G_{\theta,\mu})f + K_{\theta,\mu}\phi$ solves Problem (5.1) for given f and ϕ:

$$\begin{aligned}
[(A-\lambda)_+u] \otimes e_\mu = A_{\theta,+}[u \otimes e_\mu] &= A_{\theta,+}[((Q_{\theta,\mu,+} + G_{\theta,\mu})f + K_{\theta,\mu}\phi) \otimes e_\mu] \\
&= A_\theta(Q_{\theta,+} + G_\theta)(f \otimes e_\mu) + A_\theta K_\theta(\phi \otimes e_\mu)] \overset{(5.3)}{=} f \otimes e_\mu. \\
[\gamma_0 u] \otimes e_\mu &= \gamma_0(Q_\theta + G_\theta)(f \otimes e_\mu) + \gamma_0 K_\theta(\phi \otimes e_\mu)] \overset{(5.3)}{=} \phi \otimes e_\mu.
\end{aligned}$$

Therefore, the inverse of the parameter-dependent problem can be constructed for the inverse of the associated extended problem. For $\lambda = e^{i\theta}\mu^2$:

$$\begin{pmatrix} (A-\lambda)_+ \\ \gamma_0 \end{pmatrix}^{-1} = \begin{pmatrix} Q_{\theta,\mu,+} + G_{\theta,\mu} & K_{\theta,\mu} \end{pmatrix}.$$

What we are especially interested in is the left entry on the right hand side. Here, we observe:

$$(Q_{\theta,\mu,+} + G_{\theta,\mu})L_p(\mathbb{R}^n_+) \subset \mathcal{D}(A_T) := \{u \in L_p(\mathbb{R}^n_+) : A_+u \in L_p(\mathbb{R}^n_+), Tu = 0\}.$$

Therefore, we obtain an explicit formula for the resolvent on the ray $\lambda = e^{i\theta}\mu^2$:

$$(A_T - \lambda)^{-1} = Q_{\theta,\mu,+} + G_{\theta,\mu}.$$

Thus, the example encourages us to initially solve the extended problem for A and T:

$$\begin{aligned}
(A + e^{i(\pi+\theta)}D_z^2)_+\tilde{u} &= \tilde{f} \\
T\tilde{u} &= \tilde{\phi}.
\end{aligned}$$

In general, no explicit formulas for the inverse of the above problem exist. However, in Section 5.2, we verify that a parametrix exists. We can replace the inverse by a parametrix. However, the replacement generates an error term. To estimate the error term, we need to analyse the dependence on the parameters θ, μ and thus on λ of the operators above. The dependence on θ for $0 < \vartheta \le |\theta| \le \pi$ is not essential. In fact, we obtain uniform estimates on operator norms that only depend on ϑ. However, the dependence on μ is essential and thus discussed in Section 5.1.

5.1 The Spectral Parameter as a Co-variable

In this section, we consider pseudodifferential operators P with symbols that depend on covariables $(\xi, \zeta) \in \mathbb{R}^n \times \mathbb{R}$. We assume that the symbols do not depend on the variable z which corresponds to the covariables ζ. We write $S^m_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^{n+1}; E, F)$ for the space of such symbols. By restricting $\zeta = \mu$, we obtain a symbol p_μ in $S^m_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n; E, F)$, since $\langle \xi \rangle \leq \langle \xi, \mu \rangle \leq C_\mu \langle \xi \rangle$. With the same argumentation used in the example in the last section, we observe that $P[u \otimes e_\mu] = [P_\mu u] \otimes e_\mu$. The formal computation in this example is very common in the theory of pseudodifferential operators. Typically, the computation is applied to all co-variables in order to verify that each pseudodifferential operator has a unique symbol. A rigorous computation is based on the theory of oscillatory integrals, see [28] for more details. According to the computation below, the restriction $\zeta = \mu$ behaves well under composition:

$$\begin{aligned}(\mathrm{op}(p\#p')_\mu u) \otimes e_\mu &= \mathrm{op}(p\#p')(u \otimes e_\mu) = \mathrm{op}(p) \circ \mathrm{op}(p')(u \otimes e_\mu) = \mathrm{op}(p)(\mathrm{op}(p'_\mu)u) \otimes e_\mu) \\ &= (\mathrm{op}(p_\mu) \circ \mathrm{op}(p'_\mu)u) \otimes e_\mu) = (\mathrm{op}(p_\mu\#p'_\mu)u) \otimes e_\mu.\end{aligned}$$

Moreover, let p be elliptic with parametrix $p^{-\#}$ and remainder r. Then, the above equation implies:

$$[p^{-\#}]_\mu \# p_\mu = (p^{-\#}\#p)_\mu = (1+r)_\mu = 1 + r_\mu.$$

Therefore, p_μ is elliptic with parametrix $p_\mu^{-\#}$ and remainder r_μ. The restriction is of interest because it connects two types of expansions for pseudodifferential operators. First, the expansions with respect to decreasing symbol order. This type of expansion is typical in the calculus of pseudodifferential operators. Second, the expansions with respect to decay in the spectral parameter. We initially consider the case $\delta = 0$:

Lemma 5.1. *Let $0 \leq \sigma \leq m$ and $p \in S^{-m}_{1,0}(\mathbb{R}^n \times \mathbb{R}^{n+1}; E, F)$ with isometric group action on E and F. Then $p_\mu \in S^{-\sigma}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; E, F)$ and $|p_\mu|_* \leq |p|_* \langle \mu \rangle^{-m+\sigma}$.*

Proof. Note that for all $A \in \mathcal{B}(E, F)$ and $s, t \in \mathbb{R}$, the equality $\|A\|_{\mathcal{B}(E,F)} = \|\sigma_t A \sigma_s\|_{\mathcal{B}(E,F)}$ holds, since the group actions are isometric. We fix the multi indices $\alpha, \beta \in \mathbb{N}_0^n$. Moreover, we indicate the order of the seminorm under consideration, by a superscript. Then:

$$\begin{aligned}|p_\mu|^{-\sigma}_{\alpha,\beta} &= \sup_{x,\xi\in\mathbb{R}^n} \langle \xi \rangle^{\sigma+|\alpha|} \|\sigma^{-1}_{\langle \xi' \rangle} D^\alpha_\xi D^\beta_x p_\mu(x,\xi) \sigma_{\langle \xi' \rangle}\|_{\mathcal{B}(E,F)} \\ &= \langle \mu \rangle^{-m+\sigma} \sup_{x,\xi\in\mathbb{R}^n} \langle \mu \rangle^{m-\sigma} \langle \xi \rangle^{\sigma+|\alpha|} \|D^\alpha_\xi D^\beta_x p(x,\xi,\mu)\|_{\mathcal{B}(E,F)} \\ &\leq \langle \mu \rangle^{-m+\sigma} \sup_{x,\xi\in\mathbb{R}^n} \langle \xi, \mu \rangle^{m+|\alpha|} \|D^\alpha_\xi D^\beta_x p(x,\xi,\mu)\|_{\mathcal{B}(E,F)} \\ &\leq \langle \mu \rangle^{-m+\sigma} \sup_{x,\xi\in\mathbb{R}^n,\,\zeta\in\mathbb{R}} \langle \xi, \zeta \rangle^{|\alpha|+m} \|\sigma^{-1}_{\langle \xi', \zeta \rangle} D^\alpha_\xi D^\beta_x p(x,\xi,\zeta) \sigma_{\langle \xi', \zeta \rangle}\|_{\mathcal{B}(E,F)} \\ &= \langle \mu \rangle^{-m+\sigma} |p|^{-m}_{\alpha,\beta}.\end{aligned}$$

The computation above holds for all multi-indices. Thus, $|p_\mu|^{-\sigma}_* \leq |p|^{-m}_* \langle \mu \rangle^{-m+\sigma}$. □

For now, let E and F be Hilbert spaces. The lemma above and the mapping properties of pseudodifferential operators then imply that for $p \in S^{-m}_{1,0}(\mathbb{R}^n \times \mathbb{R}^{n+1}; E, F)$ the following holds:

$$P_\mu \in \mathcal{B}(L_p(\mathbb{R}^n; E), L_p(\mathbb{R}^n; F)) \text{ and } \|P_\mu\| \leq C|p|_* \langle \mu \rangle^{-m}.$$

We aim to prove that the same result holds for $0 \leq \delta < 1$. Let Λ^m be the order reduction operator with symbol $\langle \xi, \zeta \rangle^m \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^{n+1})$. We observe that $P_\mu = \Lambda^{-m}_\mu \Lambda^m_\mu P_\mu = \Lambda^{-m}_\mu (\Lambda^m P)_\mu$. We can thus assume that $m = 0$. For the following argumentation, we need the Schur's test:

Lemma 5.2 (Schur's Test)**.** *Let $\tilde{p} : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathcal{B}(E, F)$ be an integral kernel such that:*

$$\sup_{y \in \mathbb{R}^{n+1}} \int |\tilde{p}(x,y)|_{\mathcal{B}(E,F)}\, dx \leq M_1 \quad \textit{and}$$

$$\sup_{x \in \mathbb{R}^{n+1}} \int |\tilde{p}(x,y)|_{\mathcal{B}(E,F)}\, dy \leq M_2.$$

Then, the integral operator P belongs to $\mathcal{B}(L_p(\mathbb{R}^{n+1}; E), L_p(\mathbb{R}^{n+1}; F))$ and $\|P\| \leq M_1^{1/q} M_2^{1/p}$.

The assumption that the symbol is constant in the space direction corresponding to the co-variable ζ, allows use to interpret P as an operator on $\mathbb{R}^n \times \mathbb{S}_T$. Here, $\mathbb{S}_T$ denotes a circle of circumference T. In order to verify that this point of view is valid, we identify functions on $\mathbb{R}^n \times \mathbb{S}_T$ with T periodic functions and verify that P preserves periodicity.

$$\begin{aligned}
[Pu](x, z+T) :&= \int e^{i(x-y)\xi + i([z+T]-w)\zeta} p(x,\xi,\zeta) u(y,w)\, dy dw đ\xi đ\zeta \\
&= \int e^{i(x-y)\xi + i(z-[w-T])\zeta} p(x,\xi,\zeta) u(y,w)\, dy dw đ\xi đ\zeta \\
&= \int e^{i(x-y)\xi + i(z-\tilde{w})\zeta} p(x,\xi,\zeta) u(y,\tilde{w}+T)\, dy d\tilde{w} đ\xi đ\zeta \\
&= \int e^{i(x-y)\xi + i(z-\tilde{w})\zeta} p(x,\xi,\zeta) u(y,\tilde{w})\, dy d\tilde{w} đ\xi đ\zeta \\
&= [Pu](x,z)
\end{aligned}$$

As a next step, we consider the mapping properties of P in this identification.

Lemma 5.3. *Let $p \in S^0_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^{n+1}; E, F)$ and E, F be Hilbert spaces. Then, a constant $C > 0$ exists such that for all T:*

$$P \in \mathcal{B}(L_p(\mathbb{R}^n \times \mathbb{S}_T; E), L_p(\mathbb{R}^n \times \mathbb{S}_T; F)) \quad \textit{and} \quad \|P\| \leq C|p|_*.$$

Proof. We identify $u \in L_p(\mathbb{R}^n \times \mathbb{S}_T; E)$ with a T-periodic function and write

$$u = \sum_{j \in \mathbb{Z}} u_j \text{ with } u_j(x,z) := u|_{\mathbb{R}^n \times [-T/2, T/2]}(x, z - Tj).$$

Note that for every $j \in \mathbb{Z}$, u_j belongs to $L_p(\mathbb{R}^n \times \mathbb{R}; E)$ and $\|u_j\|_{L_p(\mathbb{R}^n\times\mathbb{R};E)} = \|u\|_{L_p(\mathbb{R}^n\times\mathbb{S}_T;E)}$. The pseudodifferential operator P is represented via the integral kernel $\tilde{p}(x, z, y, w)$:

$$\tilde{p}(x, z, y, w) = \int e^{i(x-y)\xi + i(z-w)\zeta} p(x, \xi, \zeta) \, đ\xi đ\zeta.$$

Since p is of order zero, we obtain the estimate

$$\|\tilde{p}(x, z, y, w)\|_{\mathcal{B}(E,F)} \leq C|p|_*(|x-y|^2 + |z-w|^2)^{-l/2}$$

for all even $l \in \mathbb{N}$ with $l > n+1$ with a suitable seminorm $|p|_*$ for p. If $|j| \geq 2$, $z \in [-T/2, T/2]$ and $w \in \operatorname{supp} u_j$, then $|z-w| \geq (|j|-1)T$. Therefore:

$$\begin{aligned}\|\tilde{p}(x, z, y, w)\|_{\mathcal{B}(E,F)} &\leq C|p|_*(|x-y|^2 + (|j|-1)^2T^2)^{-(n+2)/2} \\ &\leq C|p|_*((|j|-1)T)^{-(n+2)}\langle |x-y|/(|j|-1)T\rangle^{-(n+2)}.\end{aligned}$$

We write χ_j for the indicator function of $[-T/2 + jT, T/2 + jT]$. A quick computation shows that for $|j| \geq 2$ the following estimates hold:

$$\begin{aligned}&\int \chi_0(z)\|\tilde{p}(x, z, y, w)\|_{\mathcal{B}(E,F)}\chi_j(w) \, dwdy \leq C|p|_*T^{-1}(|j|-1)^{-2} \quad \text{and} \\ &\int \chi_0(z)\|\tilde{p}(x, z, y, w)\|_{\mathcal{B}(E,F)}\chi_j(w) \, dzdx \leq C|p|_*T^{-1}(|j|-1)^{-2}.\end{aligned}$$

For $|j| \geq 2$, an application of Schur's Test yields:

$$\|Pu_j\|_{L_p(\mathbb{R}^n\times\mathbb{S}_T;F)} = \|\chi_0 P \chi_j u_j\|_{L_p(\mathbb{R}^n\times\mathbb{R};F)} \leq C|p|_*T^{-1}(|j|-1)^{-2}\|u\|_{L_p(\mathbb{R}^n\times\mathcal{S}_T;E)}.$$

In particular the right hand side is summable. Therefore:

$$\begin{aligned}\|Pu\|_{L_p(\mathbb{R}^n\times\mathbb{S}_T;F)} = &\sum_{j\in\{-1,0,1\}} \|Pu_j\|_{L_p(\mathbb{R}^n\times\mathbb{S}_T;F)} + \sum_{|j|\geq 2} \|Pu_j\|_{L_p(\mathbb{R}^n\times\mathbb{S}_T;F)} \\ \leq &C\Big(3|p|_*\|u\|_{L_p(\mathbb{R}^n\times\mathbb{S}_T;E)} + 2T^{-1}\sum_{j\in\mathbb{N}} j^{-2}|p|_*\|u\|_{L_p(\mathbb{R}^n\times\mathbb{S}_T;E)}\Big) \\ \leq &C\max\{1, T^{-1}\}|p|_*\|u\|_{L_p(\mathbb{R}^n\times\mathbb{S}_T;E)}.\end{aligned}$$

Note, we need the assumption that E and F are Hilbert spaces in order to estimate the first three terms. The estimate is independent of T which is obvious for $T \geq 0$. However, we can prove that the bound also holds for $T < 1$. To this end, we choose $N \in \mathbb{N}$ such that $NT \geq 1$ and consider a T-periodic function as an NT-periodic function. Note that $\|u\|_{L_p(\mathbb{R}^n\times\mathbb{S}_{NT};E)} = N^{-1/p}\|u\|_{L_p(\mathbb{R}^n\times\mathbb{S}_T;E)}$. Therefore, the arguments above can be applied:

$$\begin{aligned}\|Pu\|_{L_p(\mathbb{R}^n\times\mathbb{S}_T;F)} &= N^{-1/p}\|Pu\|_{L_p(\mathbb{R}^n\times\mathbb{S}_{NT};F)} \leq C|p|_*N^{-1/p}\|u\|_{L_p(\mathbb{R}^n\times\mathbb{S}_{NT};E)} \\ &= C|p|_*\|u\|_{L_p(\mathbb{R}^n\times\mathbb{S}_T;E)}.\end{aligned}$$

Here, the constant C is the same as in the estimate for $T > 1$. □

We apply Lemma 5.3 with $T = 2\pi/\mu$ to the left hand side of $P(u \otimes e_\mu) = (P_\mu u) \otimes e_\mu$:

$$\|(P_\mu u) \otimes e_\mu\|_{L_p(\mathbb{R}^n \times \mathbb{S}_T; F)} = \|P(u \otimes e_\mu)\|_{L_p(\mathbb{R}^n \times \mathbb{S}_T; F)} \leq C|p|_* \|u \otimes e_\mu\|_{L_p(\mathbb{R}^n \times \mathbb{S}_T; E)}.$$

The estimate holds for all μ because the constant in Lemma 5.3 is independent of T. According to Fubini's Theorem, the norm of the tensor product is the product of the norms. Thus, $\|P_\mu u\|_{L_p(\mathbb{R}^n)} \leq C|p|_* \|u\|_{L_p(\mathbb{R}^n)}$. In sum, we have proven:

Theorem 5.4. *Let E and F be Hilbert spaces with isometric group action and $p \in S^{-m}_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^{n+1}; E, F)$. Then:*

$$P_\mu \in \mathcal{B}(L_p(\mathbb{R}^n; E), L_p(\mathbb{R}^n; F)) \text{ and } \|P_\mu\| \leq C|p|_* \langle \mu \rangle^{-m}.$$

The assumption that E and F are Hilbert spaces is too restrictive. We can allow E and F to be UMD spaces by the following arguments. We fix an $\varepsilon > 0$ and assume that $p \in S^{-2\varepsilon}_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^{n+1}; E, F)$. Then, the mapping properties imply that $P \in \mathcal{B}(B^{-\varepsilon}_p(\mathbb{R}^{n+1}; E), B^{\varepsilon}_p(\mathbb{R}^{n+1}; F))$. Applying the embedding results for the Besov spaces implies $P \in \mathcal{B}(L_p(\mathbb{R}^{n+1}; E), L_p(\mathbb{R}^{n+1}; F))$. With the arguments above we therefore obtain:

Theorem 5.5. *Let E and F be UMD spaces with isometric group action and $p \in S^{-m-\varepsilon}(\mathbb{R}^n \times \mathbb{R}^{n+1}; E, F)$. Then:*

$$P_\mu \in \mathcal{B}(L_p(\mathbb{R}^n; E), L_p(\mathbb{R}^n; F)) \text{ and } \|P_\mu\| \leq C|p|_* \langle \mu \rangle^{-m}.$$

We apply the result to Boutet de Monvel operators:

Corollary 5.6. *Let $m \geq 0$ and $\varepsilon > 0$.*

(i) Let $p \in \mathcal{P}^{-m}_{1,\delta}(\mathbb{R}^{n-1} \times \mathbb{R}^n)$. Then,

$$P_{\mu,+} \in \mathcal{B}(L_p(\mathbb{R}^n_+)) \text{ and } \|P_\mu\| \leq C|p|_* \langle \mu \rangle^{-m}.$$

(ii) Let $k \in \mathcal{K}^{-m-\varepsilon}_{1,\delta}(\mathbb{R}^{n-1} \times \mathbb{R}^n)$. Then,

$$K_\mu \in \mathcal{B}(L_p(\mathbb{R}^{n-1}); L_p(\mathbb{R}^n_+)) \text{ and } \|K_\mu\| \leq C|p|_* \langle \mu \rangle^{-m}.$$

(iii) Let $t \in \mathcal{T}^{-m-\varepsilon,0}_{1,\delta}(\mathbb{R}^{n-1} \times \mathbb{R}^n)$. Then,

$$T_\mu \in \mathcal{B}(L_p(\mathbb{R}^n_+); L_p(\mathbb{R}^{n-1})) \text{ and } \|T_\mu\| \leq C|p|_* \langle \mu \rangle^{-m}.$$

(iv) Let $g \in \mathcal{G}^{-m-\varepsilon,0}_{1,\delta}(\mathbb{R}^{n-1} \times \mathbb{R}^n)$. Then,

$$G_\mu \in \mathcal{B}(L_p(\mathbb{R}^n_+)) \text{ and } \|G_\mu\| \leq C|p|_* \langle \mu \rangle^{-m}.$$

Proof. For Result (i), we make use of the fact that the extension by zero is a bounded operator for L_p-functions and apply Theorem 5.4 with $E = F = \mathbb{C}$. For the other results, the spaces E and F are either $L_p(\mathbb{R}_+)$ or $\mathbb{C}$ which are UMD spaces. The group actions are σ_p in the case of $L_p(\mathbb{R}_+)$ and the trivial one in the case of $\mathbb{C}$. Both are clearly isometric. Therefore, the result for potential operators, trace operators, and singular Green operators follows from Theorem 5.5. □

5.2 The Parametrix Construction

Recall that $\mathcal{A}$ be an M-elliptic second order differential operator with smooth coefficients. By A we denote the operator $\mathcal{A}$ acting on $L_p(\mathbb{R}^n)$ which has symbol:

$$a(x,\xi) = \sum_{1\le i,j\le n} a_{ij}(x)\xi^i\xi^j + \sqrt{-1}\sum_{1\le i\le n} b_i(x)\xi^i + c(x). \tag{5.4}$$

Moreover, the principal part is a symmetric, uniformly strictly positive, and bounded bilinear form:

$$M^{-1}|\xi|^2 \le \sum_{1\le i,j\le n} a_{ij}(x)\xi_i\xi_j \le M|\xi|^2 \text{ for all } x,\xi\in\mathbb{R}^n.$$

In this section, we construct a parametrix to the extended problem:

$$\begin{pmatrix} A_{\theta,+} \\ T \end{pmatrix}^{-\#} = \begin{pmatrix} (A_\theta^{-\#})_+ + G_\theta^T & K_\theta^T \end{pmatrix}.$$

Here, $A_\theta := A + e^{i(\pi+\theta)}D_z^2$. In particular, we are interested in the operator G_θ^T. The construction of the parametrix splits into the construction of two parametrices. The first is the parametrix of the associated Dirichlet problem, see Section 5.2.1. The result is:

Lemma 5.7. *Let $0 < \vartheta \le |\theta| \le \pi$ and let A be an M-elliptic operator. Then,*

- $a_\theta^{-\#} \in \mathcal{P}_{1,0}^{-2}(\mathbb{R}^{n-1}\times\mathbb{R}^n)$,
- $g_\theta^D \in \mathcal{G}_{1,0}^{-2,0}(\mathbb{R}^{n-1}\times\mathbb{R}^n)$,
- $k_\theta^D \in \mathcal{K}_{1,0}^{-1/p}(\mathbb{R}^{n-1}\times\mathbb{R}^n)$, *and*
- $r_\theta^D \in \mathcal{BM}_{1,0}^{-\infty,0}(\mathbb{R}^{n-1}\times\mathbb{R}^n)$,

exist with $|a_\theta^{-\#}|_*, |g_\theta^D|_*, |k_\theta^D|_*, |r_\theta^D|_* \le C(|a|_*, M, \vartheta)$ such that the associated operators satisfy

$$\begin{pmatrix} A_{\theta,+} \\ \gamma_0 \end{pmatrix}\begin{pmatrix} A_\theta^{-\#} + G_\theta^D & K_\theta^D \end{pmatrix} = 1 + R_\theta^D.$$

Moreover, the principal symbols are:

$$\begin{aligned}
a_\theta^{-\#}(x,\xi';x_n,\xi_n) &\approx \frac{1}{a_{nn}(x)}\frac{1}{\kappa_\theta^+(x,\xi',\zeta)+i\xi_n}\frac{1}{\kappa_\theta^-(x,\xi',\zeta)-i\xi_n} \\
g_\theta^D(x',\xi',\zeta;\xi_n,\eta_n) &\approx \frac{-1}{a_{nn}(x')(\kappa_\theta^+(x',\xi',\zeta)+\kappa_\theta^-(x',\xi',\zeta))}\frac{1}{\kappa_\theta^+(x',\xi',\zeta)+i\xi_n}\frac{1}{\kappa_\theta^-(x',\xi',\zeta)-i\eta_n}. \\
k_\theta^D(x',\xi',\zeta;\xi_n) &\approx \frac{1}{\kappa^+(x',\xi',\zeta)+i\xi_n}.
\end{aligned}$$

The definition of a_{nn}, κ_θ^+ and κ_θ^- is provided in the next section. The parametrix is used to reduce the problem to the boundary. By assumption, the trace operator is $T = \varphi_1\gamma_1 + \varphi_0\gamma_0$, with $\varphi_1, \varphi_0 \geq 0$ and $\varphi_1 + \varphi_0 > 0$. We define the Dirichlet-to-Neumann operator as $\Pi_\theta := \gamma_1 K_\theta^D$. Furthermore, we observe:

$$\begin{pmatrix} A_{\theta,+} \\ T \end{pmatrix} \begin{pmatrix} (A_\theta^{-\#})_+ + G_\theta^D & K_\theta^D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \varphi_1\gamma_1((A_\theta^{-\#})_+ + G_\theta^D) & \varphi_1\Pi_\theta + \varphi_0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & \varphi_0 \end{pmatrix} R_\theta^D.$$

The first operator on the right hand side is a lower triangular matrix which has a parametrix if the diagonal entries have. The reduced problem is the construction of a parametrix to the operator $S_\theta := \varphi_1\Pi_\theta + \varphi_0$, a pseudodifferential operator on the boundary. This operator is, in general, not elliptic. However, the assumption on the operator T ensures the existence of a parametrix. The lemma below is proven in Section 5.2.2:

Lemma 5.8. *Let $0 < \vartheta \leq |\theta| \leq \pi$, A be a M-elliptic operator and $\varphi_1, \varphi_2 \geq 0$ be smooth functions such that $\varphi_1 + \varphi_0 > 0$. Then, symbols*

$$s_\theta^{-\#} \in S^0_{1,1/2}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}) \text{ and } r_\theta^S \in S^{-\infty}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$$

exist with $|s_\theta^{-\#}|_*, |r_\theta^S|_* \leq C(|a|_*, |t|_*, M, \vartheta)$, satisfying the following equation:

$$S_\theta S_\theta^{-\#} = 1 + R_\theta^S.$$

Moreover, $s^{-\#}\#\varphi_1 \in S^{-1}_{1,1/2}(\mathbb{R}^{n-1} \times \mathbb{R}^n)$ and the principal symbol is:

$$s_\theta^{-\#}(x', \xi', \zeta) \approx \frac{1}{\varphi_1(x')\kappa_\theta^+(x', \xi', \zeta) + \varphi_0(x')}.$$

Given Lemma 5.8, we obtain the parametrix of the triangular matrix above:

$$\begin{pmatrix} 1 & 0 \\ \varphi_1\gamma_1((A_\theta^{-\#})_+ + G_\theta^D) & S_\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -S_\theta^{-\#}\varphi_1\gamma_1((A_\theta^{-\#})_+ + G_\theta^D) & S_\theta^{-\#} \end{pmatrix} = 1 + \begin{pmatrix} 0 & 0 \\ -R_\theta^S\varphi_1\gamma_1((A_\theta^{-\#})_+ + G_\theta^D) & R_\theta^S \end{pmatrix}.$$

In particular, the parametrix of the extended problem can be defined as follows:

$$\begin{pmatrix} A_{\theta,+} \\ T \end{pmatrix}^{-\#} := \begin{pmatrix} A_{\theta,+}^{-\#} + G_\theta^T & K_\theta^T \end{pmatrix} := \begin{pmatrix} A_{\theta,+}^{-\#} + G_\theta^D & K_\theta^D \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -S_\theta^{-\#}\varphi_1\gamma_1((A_\theta^{-\#})_+ + G_\theta^D) & S_\theta^{-\#} \end{pmatrix}.$$

The singular Green operator is $G_\theta^T = G_\theta^D - K_\theta^D S_\theta^{-\#}\varphi_1\gamma_1((A_\theta^{-\#})_+ + G_\theta^D)$ and the potential operator is $K_\theta^T = K_\theta^D S_\theta^{-\#}$. The remainder term is:

$$R_\theta^T = \begin{pmatrix} 1 & 0 \\ -R_\theta^S\varphi_1\gamma_1((A_\theta^{-\#})_+ + G_\theta^D) & R_\theta^S \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & \varphi_0 \end{pmatrix} R_\theta^D \begin{pmatrix} 1 & 0 \\ -S_\theta^{-\#}\varphi_1\gamma_1((A_\theta^{-\#})_+ + G_\theta^D) & S_\theta^{-\#} \end{pmatrix}.$$

The order of the singular Green term in the parametrix of the extended problem is the same as in the Dirichlet case. This follows from the partial result of Lemma 5.8: The operator $S_\theta^{-\#}\varphi_1$ has order -1. Both, the symbol seminorms of the parametrix and the remainder term to the extended problem can be estimated by a constant. This constant depends on the seminorms of the symbol in Lemma 5.7 and 5.8. In summary:

Lemma 5.9. *Let $0 < \vartheta \leq |\theta| \leq \pi$, A be an M-elliptic operator and T be as in (1.2). Then, symbols*

- $a_\theta^{-\#} \in \mathcal{P}^{-2}_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^n)$,
- $g_\theta^T \in \mathcal{G}^{-2,0}_{1,\delta}(\mathbb{R}^{n-1} \times \mathbb{R}^n)$,
- $k_\theta^T \in \mathcal{K}^{-1/p}_{1,\delta}(\mathbb{R}^{n-1} \times \mathbb{R}^n)$ *and*
- $r_\theta^T \in \mathcal{BM}^{-\infty}_{1,\delta}(\mathbb{R}^{n-1} \times \mathbb{R}^n)$,

exist with $|a_\theta^{-\#}|_*, |g_\theta^T|_*, |k_\theta^T|_*, |r_\theta^T| \leq C(|a|_*, |t|_*, M, \vartheta)$ such that the following equation holds:

$$\begin{pmatrix} A_{\theta,+} \\ T \end{pmatrix} \begin{pmatrix} A_\theta^{-\#} \oplus G_\theta^T & K_\theta^T \end{pmatrix} = 1 + R_\theta^T.$$

The principal symbols of these operators are:

$$\begin{aligned}
a_\theta^{-\#}(x', \xi'; x_n, \xi_n) &\approx \frac{1}{a_{nn}(x)} \frac{1}{\kappa_\theta^+(x, \xi', \zeta) + i\xi_n} \frac{1}{\kappa_\theta^-(x, \xi', \zeta) - i\xi_n} \\
g_\theta^T(x', \xi', \zeta; \xi_n, \eta_n) &\approx s_\theta^T(x', \xi', \zeta) \frac{1}{\kappa^+(x', \xi', \zeta) + i\xi_n} \frac{1}{\kappa_\theta^-(x', \xi', \zeta) - i\eta_n}. \\
s_\theta^T(x', \xi', \zeta) &= \frac{1}{a_{nn}(x')} \left[\frac{\varphi_1(x')}{\varphi_1(x')\kappa_\theta^+(x', \xi', \zeta) + \varphi_0(x')} + \frac{1}{\kappa_\theta^+(x', \xi', \zeta) + \kappa_\theta^-(x', \xi', \zeta)} \right]. \\
k_\theta^T(x', \xi', \zeta; \xi_n) &\approx \frac{\varphi_1(x')}{\varphi_1(x')\kappa_\theta^+(x', \xi', \zeta) + \varphi_0(x')} \frac{1}{\kappa_\theta^+(x', \xi', \zeta) + i\xi_n}.
\end{aligned}$$

5.2.1 The Parametrix to the Dirichlet Problem

The construction of the parametrix to the Dirichlet problem is well-known. For the sake of completeness and to fix the notation, we include the construction. The extended operator $A_\theta := A + e^{i(\pi+\theta)} D_z^2$ has symbol $a_\theta(x, \xi, \zeta) = a(x, \xi) + e^{i(\pi+\theta)}\zeta^2$. The symbol is a polynomial of degree two and thus belongs to $\mathcal{P}^2_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^n)$. The symbol's seminorms can be estimated: $|a_\theta|_* \leq \max\{1, |a|_*\}$. Now, we verify that $a_\theta(x, \xi, \zeta)$ is an elliptic symbol. Therefore, we consider the principal part which is equivalent to setting $b_i(x) = c(x) = 0$.

$$\begin{aligned}
|a_\theta(x, \xi, \zeta)|^2 &= (a(x, \xi) + \cos(\pi + \theta)\zeta^2)^2 + \sin^2(\pi + \theta)\zeta^4 \\
&= a^2(x, \xi) + 2\cos(\pi + \theta) a(x, \xi)\zeta^2 + \zeta^4
\end{aligned}$$

$$\geq \min\{1, 1 + \cos(\pi + \theta)\}(a^2(x, \xi) + \zeta^4)$$
$$\geq \min\{1, 1 + \cos(\pi + \theta)\} \min\{1, M^{-1}\} 2^{-1}(|\xi, \zeta|^4).$$

For $\vartheta < |\theta| \leq \pi$, we obtain a constant $c = c(\vartheta, M) > 0$ such that $|a_\theta(x, \xi, \zeta)| \geq c|\xi, \zeta|^2$. Therefore, the following result is a special case of Theorem 4.4:

Lemma 5.10. *Let $0 < \vartheta \leq |\theta| \leq \pi$. A parametrix $a_\theta^{-\#} \in \mathcal{P}_{1,0}^{-2}(\mathbb{R}^{n-1} \times \mathbb{R}^n)$ and a remainder term $r_\theta \in \mathcal{P}_{1,0}^{-\infty}(\mathbb{R}^{n-1} \times \mathbb{R}^n)$ exist such that:*

$$a_\theta \# a_\theta^{-\#} = 1 - r_\theta.$$

Moreover, $|a_\theta^{-\#}|_*, |r_\theta|_* \leq C$ with a constant $C = C(|a|_*, \vartheta, M) > 0$.

Notation 5.11. The principal symbol of the parametrix is the point-wise inverse of the operator's principal symbol. To be precise, the principal symbol has to be multiplied by a zero excision function which depends on $|\xi, \zeta|$. We decided to drop the excision function from the notation to keep the layout readable. Two reasons justify this form of notation. First, treating the excision term is a well-known process. Second, we are interested in the case where ζ is large. If ζ is indeed large, the excision function equals one.

For later computations, a decomposition of the principal symbol of the parametrix is useful. The decomposition contains two parts, one in $\mathcal{H}^+$ and one in $\mathcal{H}_{-1}^-$. For a fixed (x, ξ', ζ), the principal symbol $a_\theta(x, \xi', \xi_n, \zeta)$ is a polynomial of degree two in ξ_n.

$$\begin{aligned} a_\theta(x, \xi', \xi_n, \zeta) &= \left(\sum_{1 \leq i,j < n} a_{ij}(x) \xi_i \xi_j + e^{i(\pi+\theta)} \zeta^2 \right) + 2 \left(\sum_{1 \leq i < n} a_{in}(x) \xi_i \right) \xi_n + a_{nn}(x) \xi_n^2 \\ &=: a_{nn}(x)(q_\theta(x, \xi', \zeta) + 2p(x, \xi')\xi_n + \xi_n^2). \end{aligned} \tag{5.5}$$

We label the two complex roots as $\pm i\kappa_\theta^\pm(x, \xi', \zeta)$. They are:

$$\kappa_\theta^\pm(x, \xi', \zeta) = \pm ip(x, \xi', \zeta) + \sqrt{q_\theta(x, \xi', \zeta) - p^2(x, \xi', \zeta)}.$$

We choose the negative real axis as the branch cut of the root. We observe:

$$|a_{nn}(x) \left(q_\theta(x, \xi', \zeta) - p^2(x, \xi') \right)| = |a_\theta(x, \xi', -p(x, \xi'), \zeta)| \geq c|\xi', -p(x, \xi'), \zeta|^2 \geq c|\xi', \zeta|^2.$$

Here, $c = c(M, \vartheta) > 0$ is the constant from the elliptic estimate. Therefore, the absolute value of the root is bounded from below by $\sqrt{c/M}|\xi', \zeta|$. Next, we consider the argument of the root. To this end, we observe:

$$a_{nn}(x) \left(q_\theta(x, \xi', 0) - p^2(x, \xi') \right) = a_\theta(x, \xi', -p(x, \xi'), 0) \geq M^{-1}|\xi', -p(x, \xi')|^2 \geq M^{-1}|\xi'|^2.$$

The argument of $e^{i(\pi+\theta)}\zeta^2$ belongs to the interval $[-\pi + \vartheta, \pi - \vartheta]$. According to the observation above, the argument of $q_\theta(x, \xi', \zeta) - p^2(x, \xi') = q_\theta(x, \xi', 0) - p^2(x, \xi') + e^{i(\pi+\theta)}\zeta^2$ belongs to the same interval. Therefore, the argument of the root belongs to interval

$[-\frac{\pi}{2}+\frac{\vartheta}{2}, \frac{\pi}{2}-\frac{\vartheta}{2}]$. The real part of the root coincides with the real part of $\kappa_\theta^\pm$. Thus, a constant $c = c(M, \vartheta) > 0$ exists such that the following estimate holds:

$$\Re\kappa_\theta^\pm(x, \xi', \zeta) \geq c|\xi', \zeta|. \tag{5.6}$$

A quick computation shows that $\kappa_\theta^\pm$ is homogeneous of degree one, i.e., $\kappa_\theta^\pm(x, t\xi', t\zeta) = t\kappa_\theta^\pm(x, \xi', \zeta)$. We are primarily interested in the situation where $x_n = 0$ and write $\kappa_\theta^\pm(x', \xi', \zeta) := \kappa_\theta^\pm((x', 0), \xi', \zeta)$. The homogeneity of $\kappa_\theta^\pm$ implies it is a symbol in $S^1_{1,0}(\mathbb{R}^{n-1}\times\mathbb{R}^n)$ which is elliptic according to (5.6). By decomposition into partial fractions, we obtain:

$$a_\theta^{-\#}(x, \xi, \zeta) \approx \frac{1}{a_{nn}(x)(\kappa_\theta^+(x, \xi', \zeta) + \kappa_\theta^-(x, \xi', \zeta))}\left[\frac{1}{\kappa_\theta^+(x, \xi', \zeta) + i\xi_n} + \frac{1}{\kappa_\theta^-(x, \xi', \zeta) - i\xi_n}\right].$$

The first summand belongs to $\mathcal{H}^+$ and the second to $\mathcal{H}^-_{-1}$ as a function of ξ_n. Similarly, we obtain a decomposition for the principal symbol of $A_\theta^{-\#}\partial_{x_n}$:

$$i\xi_n a_\theta^{-\#}(x, \xi, \zeta) \approx \frac{1}{a_{nn}(x')(\kappa_\theta^+(x, \xi', \zeta) + \kappa_\theta^-(x, \xi', \zeta))}\left[-\frac{\kappa_\theta^+(x, \xi', \zeta)}{\kappa_\theta^+(x, \xi', \zeta) + i\xi_n} + \frac{\kappa_\theta^-(x, \xi', \zeta)}{\kappa_\theta^-(x, \xi', \zeta) - i\xi_n}\right]$$

Now, we derive the potential operator in the parametrix to the extended Dirichlet problem. Form the jump relation, we obtain the following identity, see also [23, Chapter XX].

$$A_\theta e^+ = e^+ A_\theta + P^c\gamma \text{ with } P^c(\phi, \psi) := a_{nn}(x')(-2i\gamma_0^* p(x', D') - \gamma_1^*)\phi + \gamma_0^* a_{nn}(x')\psi.$$

We define two operators K_θ^0 and K_θ^1 such that $K_\theta^0\phi + K_\theta^1\psi := r^+ A_\theta^{-\#}\mathcal{P}^c(\phi, \psi)^t$, show that they are potential operators, and compute their principal symbol. To this end, we define the two auxiliary operators $K^j_{\theta,a} := r^+ A_\theta^{-\#}\gamma_j^*$ for $j = 0, 1$ which are potential operators of order $-1 + j - 1/p$ according to Proposition 4.20. Note that $\gamma_1^* = (-\gamma_0\partial_{x_n})^* = \partial_{x_n}\gamma_0^*$. In particular, we compute the principal symbols of the axillary operators by means of the decomposition derived above:

$$k^0_{\theta,a}(x', \xi, \zeta) = h^+_{\xi_n} a_\theta^{-\#}(x', \xi, \zeta) \approx \frac{1}{a_{nn}(x')(\kappa_\theta^+(x', \xi', \zeta) + \kappa_\theta^-(x', \xi', \zeta))}\ \frac{1}{\kappa_\theta^+(x', \xi', \zeta) + i\xi_n}.$$

$$k^1_{\theta,a}(x', \xi, \zeta) = h^+_{\xi_n} i\xi_n a_\theta^{-\#}(x', \xi, \zeta) \approx \frac{-\kappa_\theta^+(x', \xi', \zeta)}{a_{nn}(x')(\kappa_\theta^+(x', \xi', \zeta) + \kappa_\theta^-(x', \xi', \zeta))}\ \frac{1}{\kappa_\theta^+(x', \xi', \zeta) + i\xi_n}.$$

Thus, the operators $K_\theta^0 := (K^0_{\theta,a} a_{nn}(x')(-2ip(x', D')) - K^1_{\theta,a})$ and $K_\theta^1 = K^0_{\theta,a} a_{nn}(x')$ are potential operators according to Theorem 4.25. The principal symbols are:

$$k_\theta^0(x', \xi, \zeta) \approx \frac{\kappa_\theta^-(x', \xi', \zeta)}{\kappa_\theta^+(x', \xi', \zeta) + \kappa_\theta^-(x', \xi', \zeta)}\ \frac{1}{\kappa_\theta^+(x', \xi', \zeta) + i\xi_n}.$$

$$k_\theta^1(x', \xi, \zeta) \approx \frac{1}{\kappa_\theta^+(x', \xi', \zeta) + \kappa_\theta^-(x', \xi', \zeta)}\ \frac{1}{\kappa_\theta^+(x', \xi', \zeta) + i\xi_n}.$$

We used the fact that $-2ip(x', \xi') = \kappa_\theta^-(x', \xi', \zeta) - \kappa_\theta^+(x', \xi', \zeta)$ in the derivation of k_θ^0. The composed operators $C_\theta^{ij} := \gamma_i K_\theta^j$ for $i, j \in \{0, 1\}$ are pseudodifferential operators on the boundary. Their principal symbol can be derived from the symbol-kernel of the potential operators and the identity $\mathcal{F}^{-1}_{\xi_n \mapsto x_n}(\kappa_\theta^+ + i\xi_n)^{-1} = \theta(x_n)\exp(-\kappa_\theta^+ x_n)$:

$$
\begin{aligned}
c_\theta^{00}(x',\xi',\zeta) &= \gamma_0 \tilde{k}_\theta^0(x',\xi',\zeta) \approx \frac{\kappa_\theta^-(x',\xi',\zeta)}{\kappa_\theta^+(x',\xi',\zeta)+\kappa_\theta^-(x',\xi',\zeta)}.\\
c_\theta^{01}(x',\xi',\zeta) &= \gamma_0 \tilde{k}_\theta^1(x',\xi',\zeta) \approx \frac{1}{\kappa_\theta^+(x',\xi',\zeta)+\kappa_\theta^-(x',\xi',\zeta)}.\\
c_\theta^{10}(x',\xi',\zeta) &= \gamma_1 \tilde{k}_\theta^0(x',\xi',\zeta) \approx \frac{\kappa_\theta^-(x',\xi',\zeta)\kappa_\theta^+(x',\xi',\zeta)}{\kappa_\theta^+(x',\xi',\zeta)+\kappa_\theta^-(x',\xi',\zeta)}.\\
c_\theta^{11}(x',\xi',\zeta) &= \gamma_1 \tilde{k}_\theta^1(x',\xi',\zeta) \approx \frac{\kappa_\theta^+(x',\xi',\zeta)}{\kappa_\theta^+(x',\xi',\zeta)+\kappa_\theta^-(x',\xi',\zeta)}.
\end{aligned}
$$

In particular, C^{01} is an elliptic operator which allows us to define a pseudodifferential operator on the boundary $S_\theta := (C_\theta^{01})^{-\#}(1 - C_\theta^{00})$, with principal symbol:

$$s_\theta(x',\xi',\zeta) \approx \kappa^+(x',\xi',\zeta).$$

Moreover, we define a potential operator $K_\theta^D := K_\theta^0 + K_\theta^1 S_\theta$. According to the computation below, this operator is the entry in the parametrix to the extended Dirichlet problem:

$$
\begin{aligned}
\gamma_0 K_\theta^D &= \gamma_0 K_\theta^0 + \gamma_0 K_\theta^1 S_\theta = C_\theta^{00} + C_\theta^{01}(C_\theta^{01})^{-\#}(1 - C_\theta^{00}) \sim 1\\
A_{\theta,+} K_\theta^D &= A_{\theta,+} K_\theta^0 + A_{\theta,+} K_\theta^1 S_\theta \sim 0
\end{aligned}
$$

In the second line, we used $A_{\theta,+} r^+ A_\theta^{-\#} \sim r^+$ and the fact that the image of P_θ^c consists of distributions with support on the boundary. These distributions belong to the kernel of the restriction. In sum, the result is:

Lemma 5.12. *Given $0 < \vartheta \le |\theta| \le \pi$. A potential operator K_θ^D with symbol $k_\theta^D \in \mathcal{K}_{1,0}^{-1/p}(\mathbb{R}^{n-1} \times \mathbb{R}^n)$ exists such that*

$$A_{\theta,+} K_\theta^D \sim 0 \ \text{ and } \ \gamma_0 K_\theta^D \sim 1.$$

Moreover, $|k_\theta^D|_* \le C = C(M, \vartheta, |a|_*)$ and the principal symbol is:

$$k_\theta^D(x',\xi',\zeta) \approx \frac{1}{\kappa^+(x',\xi',\zeta) + i\xi_n}.$$

The seminorms of the remainder terms can be bounded by a constant, depending on the same parameters as in Lemma 5.10. The operators C_θ^{ij} are entries of the Calderón projector, for more details see [23, Chaper XX]. The Dirichlet-to-Neumann operator is defined as $\Pi_\theta := \gamma_1 K_\theta^D$. It is a pseudodifferential operator on the boundary of order 1 and $|\pi_\theta|_* \le C(|a_*|, M, \vartheta)$ according to Lemma 5.12. The principal symbol $\pi_\theta \approx \kappa_\theta^+$ can be computed with the second row of the Calderón projector. In particular, the Dirichlet-to-Neumann operator is elliptic. The singular Green operator in the parametrix of the extended problem is $G_\theta^D = -K_\theta^D \gamma_0 A_{\theta,+}^{-\#}$, has order -2, and satisfies

$$A_{\theta,+} G_\theta^D \sim 0 \ \text{ and } \ \gamma_0 G_\theta^D = -\gamma_0 K_\theta^D \gamma_0 A_{\theta,+}^{-\#} \sim -\gamma_0 A_{\theta,+}^{-\#}.$$

The principal symbol of the singular Green operator is the product of the two principal symbols of the potential operator K_θ^D and the trace operator $\gamma_0 A_{\theta,+}^{-\#}$. According to Proposition 4.22, the principal symbol of the trace operator can be computed with the decomposition above. Therefore:

$$g_\theta^D(x',\xi',\zeta;\xi_n,\eta_n) \approx \frac{-1}{a_{nn}(x')(\kappa_\theta^+(x',\xi',\zeta)+\kappa_\theta^-(x',\xi',\zeta))}\,\frac{1}{\kappa_\theta^+(x',\xi',\zeta)+i\xi_n}\,\frac{1}{\kappa_\theta^-(x',\xi',\zeta)-i\eta_n}.$$

Now, the construction of the parametrix to the extended problem is complete. At the beginning of this section, we indicated that we need the principal symbol of the trace operator $T_\theta^D := \gamma_1(A_{\theta,+}^{-\#}+G_\theta^D)$. The principal symbol of $-\partial_{x_n} A_\theta^{-\#}$ is $-i\xi_n a_\theta^{-1}(x',\xi',\zeta)$. We can thus compute the principal symbol of $T'_\theta := \gamma_1 A_{\theta,+}^{-\#}$ with Proposition 4.22 and the decomposition above:

$$t'_\theta(x',\xi',\zeta;\eta_n) \approx \frac{-1}{a_{nn}(x')(\kappa_\theta^+(x',\xi',\zeta)+\kappa_\theta^-(x',\xi',\zeta))}\,\frac{\kappa_\theta^-(x',\xi',\zeta)}{\kappa_\theta^-(x',\xi',\zeta)-i\eta_n}.$$

Furthermore, the principal symbol of the trace operator $T''_\theta = \gamma_1 G_\theta^D = -\gamma_1 K_\theta^D \gamma_0 A_{\theta,+}^{-\#}$ can be computed with Theorem 4.25:

$$t''_\theta(x',\xi',\zeta;\eta_n) \approx \frac{-1}{a_{nn}(x')(\kappa_\theta^+(x',\xi',\zeta)+\kappa_\theta^-(x',\xi',\zeta))}\,\frac{\kappa_\theta^+(x',\xi',\zeta)}{\kappa_\theta^-(x',\xi',\zeta)-i\eta_n}$$

Since T_θ^D is the sum of T'_θ and T''_θ, it has principal symbol:

$$t_\theta^D(x',\xi',\zeta;\eta_n) \approx \frac{-1}{a_{nn}(x')}\,\frac{1}{\kappa_\theta^-(x',\xi',\zeta)-i\eta_n}.$$

5.2.2 The parametrix on the boundary

This section proves Lemma 5.8 and compares the degenerate to the non-degenerate case. Furthermore, we point out the necessity for the Boutet de Monvel calculus with $0 \neq \delta$.

Theorem 5.13 (parametrix)**.** *Let $0 \le m' \le m$, $0 \le \delta \le 1$ and $p \in S^m_{1,0}(\mathbb{R}^{n-1}\times\mathbb{R}^n)$. Suppose constants $c>0$ and $\rho \ge 0$ exist such that for all $|\xi,\zeta| \ge \rho$ the following estimates hold:*

$$|p(x',\xi',\zeta)| \ge c\langle \xi,\zeta\rangle^{-m'} \quad \textit{and} \tag{5.7}$$
$$|\partial_x^\beta \partial_\xi^\alpha p(x',\xi',\zeta)||p(x',\xi',\zeta)^{-1}| \le C\langle\xi,\zeta\rangle^{-\rho|\alpha|+\delta|\beta|}. \tag{5.8}$$

Then, symbols $p^{-\#} \in S^{-m'}_{1,\delta}(\mathbb{R}^n\times\mathbb{R}^n)$ and $r \in S^{-\infty}(\mathbb{R}^n\times\mathbb{R}^n)$ exist with $|p^{-\#}|_*, |r|_* \le C(c,|p|_*)$ such that the associated operators satisfy $PP^{-\#} = 1-R$. The operator $P^{-\#}$ is also a left parametrix.

For the proof of the theorem, we refer to [28]. Given $0 < \vartheta \leq |\theta| \leq \pi$, the operator

$$S_\theta = \varphi_1 \Pi_\theta + \varphi_0$$

is a pseudodifferential operator on the boundary of order 1 and Hörmander type $(1, 0)$ with symbol $s_\theta(x', \xi', \zeta) := \varphi_1(x')\pi_\theta(x', \xi', \zeta) + \varphi_0(x')$. In the last subsection, we pointed out that the principal symbol of Π_θ is κ_θ^+. Moreover, $\Re\kappa_\theta^+(x', \xi', \zeta) \geq c|\xi, \zeta|$, with a constant $c = c(M, \vartheta)$. Thus, for sufficiently large $|\xi, \zeta|$, the estimate from below $\Re\pi_\theta(x', \xi', \zeta) \geq 1$ holds. According to Assumption (1.2), the function $\varphi_1, \varphi_0 \geq 0$ satisfies $\varphi_1 + \varphi_0 \geq c > 0$. Therefore, for sufficiently large $|\xi, \zeta|$, the following estimate holds:

$$|s_\theta(x', \xi', \zeta)| \geq \varphi_1(x')\Re\pi_\theta(x', \xi', \zeta) + \varphi_0(x') \geq \varphi_1(x') + \varphi_0(x') \geq c > 0. \tag{5.9}$$

The estimate is Assumption (5.7) with $m' = 0$. Furthermore, we observe:

$$D_\zeta^l D_{\xi'}^\alpha D_{x'}^\beta s_\theta(x', \xi', \zeta) = \sum D_{x'}^{\beta_1}\varphi_1(x')\pi_\theta(x', \xi', \zeta)\frac{D_\zeta^l D_{\xi'}^\alpha D_{x'}^{\beta_2}\pi_\theta(x', \xi', \zeta)}{\pi_\theta(x', \xi', \zeta)} + \delta_{0,|\alpha|+l} D_{x'}^{\beta_3}\varphi_0(x').$$

In the equation, we sum over all $\beta_1, \beta_2, \beta_3 \in \mathbb{N}_0^{n-1}$ with $\beta_1 + \beta_2 + \beta_3 = \beta$. Since $\pi_\theta \in S_{1,0}^1(\mathbb{R}^{n-1} \times \mathbb{R}^n)$ is an elliptic symbol, the following estimate holds:

$$\left|\frac{D_{x'}^\beta D_{\xi'}^\alpha D_\zeta^l \pi_\theta(x', \xi', \zeta)}{\pi_\theta(x', \xi', \zeta)}\right| \leq C\langle\xi', \zeta\rangle^{-|\alpha|-l}.$$

Thus, s_θ satisfies Assumption (5.8), if for all $\beta \in \mathbb{N}_0^{n-1}$, the estimate below holds:

$$\left|\frac{D_{x'}^\beta \varphi_1(x')\pi_\theta(x', \xi', \zeta)}{s_\theta(x', \xi', \zeta)}\right| \leq C\langle\xi', \zeta\rangle^{|\beta|/2}. \tag{5.10}$$

In the case of $|\beta| \geq 2$, the estimate holds, implied by $|D_{x'}^\beta\varphi_1(x')| \leq |t|_*$, $|\pi_\theta(x', \xi', \zeta)| \leq |\pi_\theta|_*\langle\xi', \zeta\rangle$, and Estimate 5.9. In the case of $|\beta| = 0$, Equation (5.9) implies Equation (5.10):

$$\left|\frac{\varphi_1(x')\pi_\theta(x', \xi', \zeta)}{s_\theta(x', \xi', \zeta)}\right| = \left|\frac{s_\theta(x', \xi', \zeta) - \varphi_0(x')}{s_\theta(x', \xi', \zeta)}\right| \leq C.$$

Now, we consider the case of $|\beta| = 1$. To this end, we need two estimates. By assumption, φ_1 is a non-negative real function. Therefore, the following estimate holds. For the proof we refer to [43, Lemma 5.3].

$$|D_{x'_j}\varphi_1(x')| \leq 1/2|\varphi_1(x')|^{1/2}\|D_{x'_j}^2\varphi\|_\infty^{1/2} \leq C\varphi_1(x')^{1/2} \text{ for } j \in \{1, \ldots, n-1\}.$$

The previous estimate implies the first of the two estimates:

$$|D_{x'_j}\varphi_1(x')\pi_\theta(x', \xi, \zeta)| \leq C|\varphi_1(x')|^{1/2}|\pi_\theta(x', \xi', \zeta)| \leq C|\varphi_1(x')\pi_\theta(x', \xi', \zeta)|^{1/2}\langle\xi', \zeta\rangle^{1/2}.$$

The second estimate is obtained from the fact that $\Re\pi_\theta(x',\xi',\zeta), \varphi_1(x'), \varphi_0(x') \geq 0$ and Equation (5.9):

$$\begin{aligned}|s_\theta(x',\xi',\zeta)|^2 &= (\varphi_1(x')\Re\pi_\theta(x',\xi',\zeta)+\varphi_0(x'))^2 + (\varphi_1(x')\Im\pi_\theta(x',\xi',\zeta))^2\\ &\geq (\varphi_1(x')\Re\pi_\theta(x',\xi',\zeta))^2 + (\varphi_0(x'))^2 + (\varphi_1(x')\Im\pi_\theta(x',\xi',\zeta))^2\\ &= |\varphi_1(x')\pi_\theta(x',\xi',\zeta)|^2 + |\varphi_0(x')|^2 \geq 1/2(|\varphi_1(x')\pi_\theta(x',\xi',\zeta)| + |\varphi_0(x')|)^2\\ &\geq 1/8(|\varphi_1(x')\pi_\theta(x',\xi',\zeta)|+c)^2.\end{aligned}$$

The function $g(y) = y(y^2+c)^{-1}$ is bounded. Thus, we obtain Equation (5.10) for $|\beta| = 1$ from the two estimates derived above:

$$\left|\frac{D_{x'_j}\varphi_1(x')\pi_\theta(x',\xi,\zeta)}{s_\theta(x',\xi,\zeta)}\right| \leq C\frac{|\varphi_1(x')\pi_\theta(x',\xi',\zeta)|^{1/2}}{|\varphi_1(x')\pi_\theta(x',\xi',\zeta)|+c}\langle\xi',\zeta\rangle^{1/2} \leq C\langle\xi',\zeta\rangle^{1/2}.$$

Therfore, s_θ satisfies Assumption (5.8). According to Theorem 5.13, symbols $s_\theta^{-\#} \in S^0_{1,1/2}(\mathbb{R}^{n-1}\times\mathbb{R}^n)$ and $r_\theta \in S^{-\infty}_{1,0}(\mathbb{R}^{n-1}\times\mathbb{R}^n)$ exist such that the associated operator satisfies $S_\theta S_\theta^{-\#} = 1 + R_\theta$. Moreover, the seminorms can be estimated by $|s_\theta^{-\#}|, |r_\theta| \leq C$, with a constant $C = C(|a|_*, |t|_*, M, \vartheta)$.

Now, we compare the degenerate to the non-degenerate case. In the non-degenerate case, the parametrix $s_\theta^{-\#}$ belongs to $S^{-1}_{1,0}(\mathbb{R}^{n-1}\times\mathbb{R}^n)$. In this case, a Boutet de Monvel calculus with $\delta = 0$ is sufficient for further argumentation. By contrast, $\delta = 1/2$ in the degenerate case. According to the results of Section 4, $\delta = 1/2$ is not a significant drawback. However, the difference in the order of the parametrix is a serious issue. The parametrix has order 0 in the degenerate case and order -1 in the non-degenerate case.

Loosely speaking, the Parametrix of S_θ behaves poorly on the zeros of φ_1. An important observation is we regain the loss in the order, if we multiply the parametrix by φ_1. Using the fact that $\varphi_1 \sim s_\theta\#\pi_\theta^{-\#} - \varphi_0\pi_\theta^{-\#}$, we obtain:

$$s_\theta^{-\#}\#\varphi_1 \sim s_\theta^{-\#}(s_\theta\#\pi_\theta^{-\#} - \varphi_0\pi_\theta^{-\#}) \sim \pi_\theta^{-\#} - s_\theta^{-\#}\#\varphi_0\pi_\theta^{-\#}.$$

The right hand side is obviously of order -1. In sum, Lemma 5.8 holds.

5.3 The resolvent of A_T

The example at the beginning of Section 5 indicates that the resolvent is related to the inverse of the extended problem. In general, no explicit formulas for the inverse are available. However, a parametrix was explicitly constructed in Section 5.2. The relation of the parametrix to the extended problem to the resolvent, is similar to the relation of the inverse to the resolvent. An error term occurs, whenever the parametrix is used instead of the inverse. We can, however, control the error term with the result of Section 5.1. We thus obtain an identity for the resolvent, see Proposition 5.15. The identity is adequate for deriving the estimates in Remark 3.6, if the Banach space is $L_p(\mathbb{R}^n_+)$. For the proof of Proposition 5.15, we need:

Lemma 5.14. *Let $0 < \vartheta \leq \pi$, A be M-elliptic, and T be as in* (1.2). *Then, for all $N \in \mathbb{N}$, a constant $c = c(|a|_*, |t|_*, M, \vartheta)$ exists such that on each ray $\lambda = e^{i\theta}\mu^2$, with $\vartheta/2 < |\theta| \leq \pi$, for $|\lambda| \geq c$, the resolvent of A_T exists and the following estimate holds:*

$$\|(A_T - \lambda)^{-1} - (A_{\theta,\mu}^{-\#} + G_{\theta,\mu}^T)\| \leq C\langle\lambda\rangle^{-N}.$$

Here, the constant is $C = C(|a|_*, |t|_*, M, \vartheta)$.

Proof. Let $\begin{pmatrix} A_\theta^{-\#} + G_\theta^T & K_\theta^T \end{pmatrix}$ be the parametrix and R_θ^T the remainder to the extended problem, see Lemma 5.9. With the same argument used in the example at the beginning of Section 5, we obtain:

$$\begin{pmatrix} (A-\lambda)_+ \\ T \end{pmatrix} \begin{pmatrix} A_{\theta,\mu}^{-\#} + G_{\theta,\mu}^T & K_{\theta,\mu}^T \end{pmatrix} = 1 + R_{\theta,\mu}^T.$$

The application of Corollary 5.6 to the remainder $r_\theta^T \in \mathcal{BM}_{1,\delta}^{-\infty,0}(\mathbb{R}^{n-1} \times \mathbb{R}^n)$ implies that $R_{\theta,\mu}^T \in \mathcal{B}(L_p(\mathbb{R}_+^n) \oplus L_p(\mathbb{R}^{n-1}))$ and $\|R_{\theta,\mu}^T\| \leq C|r_\theta|_*\langle\mu\rangle^{-2N} = C|r_\theta|_*\langle\lambda\rangle^{-N}$, for all $N \in \mathbb{N}_0$. We define $c := c(|a|_*, |t|_*, M, \vartheta) := C|r_\theta|_*$, choosing the same seminorm as before. In particular, $\|R_{\theta,\mu}^T\| < 1$ for $|\lambda| > c$. Thus, the inverse of $1 + R_{\theta,\mu}^T$ exists and is provided by the Neumann series. Therefore:

$$\begin{pmatrix} (A-\lambda)_+ \\ T \end{pmatrix}^{-1} = \begin{pmatrix} A_{\theta,\mu}^{-\#} + G_{\theta,\mu}^T & K_{\theta,\mu}^T \end{pmatrix} (1 + R_{\theta,\mu}^T)^{-1} = \begin{pmatrix} A_{\theta,\mu}^{-\#} + G_{\theta,\mu}^T & K_{\theta,\mu}^T \end{pmatrix} + \mathcal{O}(\langle\lambda\rangle^{-N}).$$

The statement follows from the above equation. □

For later use, the representation of the resolvent in Lemma 5.14 is not sufficient. The main issue is that the construction of a parametrix is, in general, not a finite process. However, given any order $-m$, two symbols q_θ' and g_θ' can be computed in a finite number of steps such that $a_\theta^{-\#} - q_\theta' =: q_\theta'' \in \mathcal{P}_{1,\delta}^{-m}(\mathbb{R}^{n-1} \times \mathbb{R}^n)$ and $g_\theta^T - g_\theta' =: g_\theta'' \in \mathcal{G}_{1,\delta}^{-m,0}(\mathbb{R}^{n-1} \times \mathbb{R}^n)$. We choose $m = 2 + 3\varepsilon$ for some $\varepsilon > 0$. Thus, q_θ' and g_θ' are the principal symbols of $A_\theta^{-\#}$ and G_θ^T, respectively. An application of Corollary 5.6 yields:

$$\|Q_{\theta,\mu,+}''\|_{\mathcal{B}(L_p(\mathbb{R}_+^n))} \leq C|q_\theta''|_*\langle\mu\rangle^{-2-3\varepsilon} \quad \text{and} \quad \|G_{\theta,\mu}''\|_{\mathcal{B}(L_p(\mathbb{R}_+^n))} \leq C|g_\theta''|_*\langle\mu\rangle^{-2-2\varepsilon}.$$

Both norms decay like $\langle\mu\rangle^{-2-2\varepsilon} = \langle\lambda\rangle^{-1-\epsilon}$. Therefore:

$$(A_T - \lambda)^{-1} = Q_{\theta,\mu,+}' + G_{\theta,\mu}' + \mathcal{O}(\langle\lambda\rangle^{-1-\varepsilon}). \tag{5.11}$$

In particular, the remainder term is integrable on the ray $\lambda = e^{i\theta}\mu^2$. The Identity (5.11) only holds for sufficiently large λ. In fact, for small λ, the resolvent may not exist. Therefore, we shift the operator A_T to guarantee the existence of the resolvent.

Proposition 5.15 (Structure of the Resolvent)**.** *Let A be M-elliptic and T be as in* (1.2). *For a given $0 < \vartheta \leq \pi$, a constant $0 \leq \nu = \nu(|a|_*, |t|_*, M, \vartheta)$ exists such that $\sigma(A_T + \nu) \subset \Sigma_{\vartheta/2}$. Moreover, a constant $0 \leq c = c(|a|_*, |t|_*, M, \vartheta)$ exists such that on* *each ray $\lambda = e^{\pm i\theta}\mu^2$, with $\vartheta < |\theta| \leq \pi$, for all $|\mu| \geq c$, the following equation for the resolvent holds:*

$$(A_T + \nu - \lambda)^{-1} = Q'_{\theta,\mu,+} + G'_{\theta,\mu} + \mathcal{O}(\langle\lambda\rangle^{-1-\varepsilon}), \quad \textit{for some } 0 < \varepsilon. \tag{5.12}$$

Here, $Q'_{\theta,+}$ and G'_θ are the operators associated to the principal symbols of $A_\theta^{-\#}$ and G_θ^T, respectively.

Proof. We choose $c = c(|a|_*, |t|_*, M, \vartheta)$ as in Lemma 5.14 with $N = 2$. In particular, the complement of the key-hole region $\Sigma_{\vartheta/2} \cup B_c$ belongs to $\rho(A_T)$. We choose $\nu = \nu(|a|_*, |t|_*, M, \vartheta)$ such that $\Sigma_{\vartheta/2} \cup B_c \subset \Sigma_{\vartheta/2} - \nu$. Therefore, $\Sigma^c_{\vartheta/2}$ belongs to $\rho(A_T + \nu)$. Corollary 5.6 and Lemma 5.14 imply $\|(A_T - \lambda)^{-1}\| \leq C\langle\lambda\rangle^{-1+\varepsilon}$ for all $\varepsilon > 0$ with a constant $C = C(|a|_*, |t|_*, M, \vartheta)$. After possibly enlarging $c = c(|a|_*, |t|_*, M, \vartheta)$, we can assume that $\|\nu(A_T - \lambda)^{-1}\| < 1$ for $|\lambda| \geq c$. Therefore:

$$\begin{aligned}(A_T + \nu - \lambda)^{-1} =& (A_T - \lambda)^{-1}(1 + \nu(A_T - \lambda)^{-1})^{-1} = (A_T - \lambda)^{-1}\sum_{k=0}^{\infty}\nu^k(A_T - \lambda)^{-k} \\ =& (A_T - \lambda)^{-1} + \mathcal{O}(\langle\lambda\rangle^{-2}) = Q'_{\theta,\mu,+} + G'_{\theta,\mu} + \mathcal{O}(\langle\lambda\rangle^{-1-\varepsilon}),\end{aligned}$$

which is Equation (5.12). Thus, the proof is complete. □

Note that $0 \in \rho(A_T + \nu)$ and the inverse is the sum of a truncated pseudodifferential operator and a singular Green operator of order -2. In particular, the mapping properties derived in Section 4.6 imply that $(A_T + \nu)^{-1} \in \mathcal{B}(H^s_p(\mathbb{R}^n_+); H^{s+2}_p(\mathbb{R}^n_+)))$ for $s > 1/p - 1$. Therefore, the following estimate holds for all $u \in \mathcal{D}(A_T)$:

$$\|u\|_{H^{2+s}(\mathbb{R}^n_+)} = \|(A_T + \nu)^{-1}(A_T + \nu)u\|_{H^{2+s}(\mathbb{R}^n_+)} \leq C\|(\nu + A)u\|_{H^s_p(\mathbb{R}^n_+)}. \tag{5.13}$$

5.4 Proof of the Auxiliary Result

This section proves Theorem 1.2. Let A_T satisfy the assumptions of the Theorem. For a given $0 < \vartheta < \pi$, we choose $\nu = \nu(|a|_*, |t|_*, M, \vartheta/2)$ according to Proposition 5.15. In particular, the resolvent set of $A_T + \nu$ contains the complement of the sector of angle $\vartheta/2$. Moreover, the resolvent belongs to $\mathcal{O}(\lambda^{-1+\varepsilon})$ for any $\varepsilon > 0$. We fix a $\vartheta/2 < \theta < \vartheta$. Therefore, for all $f \in H^\infty_0(\Sigma_\theta)$, the integral below is defined:

$$f(A_T) := \frac{i}{2\pi}\int_{\partial\Sigma_\theta} f(\lambda)(A_T + \nu - \lambda)^{-1}\, d\lambda.$$

According to Remark 3.6, the operator $A_T + \nu$ allows a bounded $H^\infty(\Sigma_\vartheta)$-calculus if and only if the following estimate holds for all $f \in H^\infty_0(\Sigma_\vartheta)$:

$$\left\|\frac{i}{2\pi}\int_{\partial\Sigma_\theta} f(\lambda)(A_T + \nu - \lambda)^{-1}\, d\lambda\right\|_{\mathcal{B}(L_p(\mathbb{R}^n_+))} \leq C\|f\|_{L_\infty(\Sigma_\vartheta)} \text{ for all } f \in H^\infty_0(\Sigma_\vartheta).$$

The boundary of the sector Σ_θ is the union of the two rays $\lambda = e^{\pm i\theta}\mu^2$. On each of these rays, Proposition 5.15 provides a description of the resolvent. We write $Q'_\lambda := Q'_{\pm\theta,\mu}$, where the sign depends on the ray $\lambda = e^{\pm i\theta}\mu^2$. Similarly, we define G'_λ. For a given $f \in H_0^\infty(\Sigma_\vartheta)$, the operator $-2\pi i f(A_T)$ is the sum of the following three operators:

$$
\begin{aligned}
I_1 &= \int_{\partial\Sigma_\theta} f(\lambda) Q'_{\lambda,+}\, d\lambda, \\
I_2 &= \int_{\partial\Sigma_\theta} f(\lambda) \left[(A_T + \nu - \lambda)^{-1} - (Q'_{\lambda,+} + G'_\lambda) \right] d\lambda, \quad \text{and} \\
I_3 &= \int_{\partial\Sigma_\theta} f(\lambda) G'_\lambda\, d\lambda.
\end{aligned}
$$

Thus, Theorem 1.2 holds, if constants $C_i = C_i(|a|_*, |t|_*, M, \vartheta)$ for $i = 1, 2, 3$ exist such that the estimates below hold:

$$
\|I_i\|_{\mathcal{B}(L_p(\mathbb{R}^n_+))} \le C_i \|f\|_{L_\infty(\Sigma_\vartheta)} \quad \text{for all} \quad f \in H_0^\infty(\Sigma_\vartheta).
$$

The first estimate is well-known, see [38]. For completeness, we include the proof.

Lemma 5.16. *A constant $C = C(|a|_*, M, \vartheta) > 0$ exists such that*

$$
\left\| \int_{\partial\Sigma_\theta} f(\lambda) Q'_{\lambda,+}\, d\lambda \right\|_{\mathcal{B}(L_p(\mathbb{R}^n_+))} \le C \|f\|_{L_\infty(\Sigma_\vartheta)}.
$$

Proof. The symbol seminorms of q_λ, with respect to order zero, belong to $\mathcal{O}(\langle\lambda\rangle^{-1})$. Therefore, the following integral defines a symbol of order zero:

$$
q_f(x,\xi) := \int_{\partial\Sigma_\theta} f(\lambda) q'_\lambda(x,\xi)\, d\lambda. \tag{5.14}
$$

Here, q'_λ is the parameter-dependent symbol of Q'_λ:

$$
q'_\lambda(x,\xi) = \left(\sum_{1\le i,j\le n} a_{ij}(x)\xi_i\xi_j - \lambda \right)^{-1}.
$$

The symbol satisfies the following estimates for all $\alpha, \beta \in \mathbb{N}_0^n$:

$$
|D_\xi^\alpha D_x^\beta q'_\lambda(x,\xi)| \le C\langle\xi\rangle^{-2-|\alpha|} \quad \text{and} \quad |D_\xi^\alpha D_x^\beta q'_\lambda(x,\xi)| \le C\langle\xi\rangle^{-|\alpha|}\langle\lambda\rangle^{-1}.
$$

For given $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{N}_0^n$, the function $\lambda \mapsto D_\xi^\alpha D_x^\beta q_\lambda(x,\xi)$ is holomorphic on the complement of $\Sigma_\vartheta \cap B_\rho$ with $\rho := 2M|\xi|^2$. Therefore, the function $\lambda \mapsto f(\lambda) D_\xi^\alpha D_x^\beta q_\lambda(x,\xi)$ is holomorphic. According to the estimate above and the assumption $f \in H_0^\infty(\Sigma_\vartheta)$, the function is $\mathcal{O}(\langle\lambda\rangle^{-1-\varepsilon})$ at infinity. Thus, we can differentiate under the integral and change the path of integration in Equation 5.14:

$$
D_x^\beta D_\xi^\alpha q_f(x,\xi) = \int_{\partial(\Sigma_\vartheta \cap B_\rho)} f(\lambda) D_x^\beta D_\xi^\alpha q'_\lambda(x,\xi)\, d\lambda.
$$

The length of the path of integration $\partial(\Sigma_\vartheta \cap B_\rho)$ is proportional to $\rho = 2M|\xi|^2$. The standard estimates for the integral on the right hand side is:

$$\begin{aligned} |D_x^\beta D_\xi^\alpha q_f(x,\xi)| &\le |\partial\Sigma_\vartheta \cap B_\rho| \|f\|_{L_\infty(\Sigma_\vartheta)} \sup_\lambda |D_x^\beta D_\xi^\alpha q_\lambda(x,\xi)| \\ &\le CM|q_\lambda|_* \|f\|_{L_\infty(\Sigma_\vartheta)} |\xi|^2 \langle\xi\rangle^{-2-|\alpha|} \\ &\le CM|q_\lambda|_* \|f\|_{L_\infty(\Sigma_\vartheta)} \langle\xi\rangle^{-|\alpha|}. \end{aligned}$$

Since $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{N}_0^n$ were chosen arbitrarily, the inequality implies that $q_f \in S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ and $|q_f|_* \le C\|f\|_{L_\infty(\Sigma_\vartheta)}$. According to the mapping properties of pseudodifferential operators, we obtain:

$$\|Q_f\|_{\mathcal{B}(L_p(\mathbb{R}^n))} \le C\|f\|_{L_\infty(\Sigma_\vartheta)}.$$

Here, the constant C equals $C(|a|_*, M, \vartheta)$. Therefore, $\|Q_{f,+}\|_{\mathcal{B}(L_p(\mathbb{R}^n_+))} \le C\|f\|_{L_\infty(\Sigma_\vartheta)}$. A change of the order of integration shows that $Q_{f,+}$ is the operator of interest. □

For the second estimate, we use the decay of $(A_T + \nu - \lambda)^{-1} - (Q'_{\lambda,+} + G'_\lambda)$ which is at least $|\lambda|^{-1-\varepsilon}$ for some $\varepsilon > 0$:

Lemma 5.17. *A constant $C = C(|a|_*, |t|_*, M, \vartheta) > 0$ exists such that*

$$\left\| \int_{\partial\Sigma_\theta} f(\lambda) \left((A_T - \lambda)^{-1} - Q'_{\lambda,+} + G'_\lambda \right) d\lambda \right\|_{\mathcal{B}(L_p(\mathbb{R}^n_+))} \le C\|f\|_{L_\infty(\Sigma_\vartheta)}.$$

Proof. According to Proposition 5.15, constants $c = c(|a|_*, |t|_*, M, \vartheta) \ge 0$ and $C = C(|a|_*, |t|_*, M, \vartheta) > 0$ exist such that for some $\varepsilon > 0$ and all $|\lambda| \ge c$ the estimate below holds:

$$\left\|(A_T + \nu - \lambda)^{-1} - (Q'_{\lambda,+} \oplus G'_\lambda)\right\|_{\mathcal{B}(L_p(\mathbb{R}^n_+))} \le C\langle\lambda\rangle^{-1-\varepsilon}.$$

Proposition 5.15 also implies $\partial\Sigma_\theta \subset \rho(A_T + \nu)$. Since the resolvent is continuous,

$$\sup\{\|(A_T - \lambda)^{-1}\|_{\mathcal{B}(L_p(\mathbb{R}^n_+))} : \lambda \in \Sigma_\theta \text{ and } |\lambda| \le c\} < \infty.$$

The operator $Q'_{\lambda,+} + G'_\lambda$ also continuously depends on λ. Therefore:

$$\sup\{\|Q'_{\lambda,+} + G'_\lambda\|_{\mathcal{B}(L_p(\mathbb{R}^n_+))} : \lambda \in \Sigma_\theta \text{ and } |\lambda| \le c\} < \infty.$$

In sum, the function $\lambda \mapsto \|(A_T + \nu - \lambda)^{-1} - (Q'_{\lambda,+} + G'_\lambda)\|$ is integrable on Σ_θ. Thus, we obtain the result with the standard estimate for the integral. □

Next, we provide the third estimate. To this end, we parametrise the boundary:

$$\int_{\partial\Sigma_\theta} f(\lambda) G'_\lambda \, d\lambda = \int_0^\infty 2\mu e^{i\theta} f(e^{i\theta}\mu^2) G'_{\theta,\mu} \, d\mu + \int_0^\infty 2\mu e^{-i\vartheta} f(e^{-i\theta}\mu^2) G'_{-\theta,\mu} \, d\mu. \qquad (5.15)$$

Both integrals can be estimated with the same arguments. Therefore, we can focus on the first Integral. To estimate the integral, we use the symbol-kernel representation of $G'_{\theta,\mu}$. According to Lemma 5.9, an $s_\theta^T \in S^{-1}_{1,1/2}(\mathbb{R}^{n-1}\times\mathbb{R}^n)$ exists such that the symbol-kernel of $G'_{\theta,\mu}$ is:

$$\tilde{g}'_\theta(x',\xi',\zeta;x_n,y_n) = s_\theta^T(x',\xi',\zeta)\exp(-\kappa_\theta^+(x',\xi',\zeta)x_n - \kappa_\theta^-(x',\xi',\zeta)y_n). \tag{5.16}$$

Moreover, we use the following result:

Lemma 5.18. *Let $\sigma \in S^1_{1,0}(\mathbb{R}^{n-1}\times\mathbb{R}^n)$ and $\Re\sigma(x',\xi',\zeta) \geq c|\xi',\zeta|$. Then, the map*

$$\mathbb{R}_+ \ni t \mapsto \exp(-\sigma(x',\xi',\zeta)t) \in S^0_{1,0}(\mathbb{R}^{n-1}\times\mathbb{R}^n)$$

is uniformly bounded. In fact, $\sup\{|\exp(-\sigma(x',\xi',\zeta)t)|_* : t\in\mathbb{R}_+\} \leq C = C(|\sigma|_*, c)$.

Proof. A simple induction over $|\alpha|+|\beta|+l = N$ shows that $D^\alpha_{\xi'}D^\beta_{x'}D^l_\zeta \exp(-\sigma(x',\xi',\zeta)t)$ is a linear combination over all $\alpha_1+\cdots+\alpha_k=\alpha$, $\beta_1+\cdots+\beta_k=\beta$, $l_1+\cdots+l_k$, and $k\leq N$. The terms in the linear combination have the following structure:

$$\left(D^{\alpha_1}_{\xi'}D^{\beta_1}_{x'}D^{l_1}_\zeta\sigma(x',\xi',\zeta)\cdots D^{\alpha_k}_{\xi'}D^{\beta_k}_{x'}D^{l_k}_\zeta\sigma(x',\xi',\zeta)\right)(-t)^k\exp(-\sigma(x',\xi',\zeta)t).$$

Furthermore, the assumption $\sigma \in S^1_{1,0}(\mathbb{R}^{n-1}\times\mathbb{R}^n)$ implies:

$$\left|D^{\alpha_1}_{\xi'}D^{\beta_1}_{x'}D^{l_1}_\zeta\sigma(x',\xi',\zeta)\cdots D^{\alpha_n}_{\xi'}D^{\beta_k}_{x'}D^{l_k}_\zeta\sigma(x',\xi',\zeta)\right| \leq \prod_{i=1}^k |\sigma|_*|\xi',\zeta|^{1-|\alpha_i|+l_i} = |\sigma|_*^k|\xi',\zeta|^{k-|\alpha|+l}.$$

Moreover, we use the fact that $s^k\exp(-s)$ is bounded on the positive real axis in order to obtain:

$$\left|(-t)^k\exp(-\sigma(x',\xi',\zeta)t)\right| = t^k\exp(-\Re\sigma(x',\xi',\zeta)t) \leq t^k\exp(-c|\xi',\zeta|t) \leq c^{-k}|\xi',\zeta|^{-k}C.$$

According to the last two estimates, all terms in the linear combination can be estimated by $C|\xi',\zeta|^{-|\alpha|+l}$. □

Lemma 5.19. *A constant $C = C(|a|_*, |t|_*, M, \vartheta)$ exists such that*

$$\left\|\int_{\partial\Sigma_\theta} f(\lambda)G'_\lambda\,d\lambda\right\|_{\mathcal{B}(L_p(\mathbb{R}^n_+))} \leq C\|f\|_{L_\infty(\Sigma_\vartheta)} \text{ for all } f\in H_0^\infty(\Sigma_\vartheta).$$

Proof. According to Equation (5.15), providing the estimate for the following operator is sufficient:

$$I^+ := 2^{-1}e^{-i\theta}\int_{\lambda=e^{i\theta}\mu^2} f(\lambda)G'_\lambda\,d\lambda = \int_0^\infty \mu f(\mu^2e^{i\theta})G'_{\theta,\mu}\,d\mu:$$

For the estimate, we use the explicit description of the symbol-kernel of G'_θ in Equation (5.16). In the equation $s^T_\theta \in S^{-1}_{1,1/2}(\mathbb{R}^{n-1}\times\mathbb{R}^n)$ and thus $\zeta s^T_\theta(x',\xi',\zeta)\in S^0_{1,1/2}(\mathbb{R}^{n-1}\times\mathbb{R}^n)$.

In Section 5.2, we observed that the roots $\kappa_\theta^\pm$ are strongly elliptic. Moreover, a constant $c = c(|a|_*, M, \vartheta) > 0$ exists such that:

$$\Re\kappa_\theta^\pm(x', \xi', \zeta) \geq 2c|\xi, \zeta|.$$

Therefore, $\sigma_\theta^\pm(x', \xi', \zeta) := \kappa_\theta^\pm(x', \xi', \zeta) - c\zeta$ satisfies the assumption of Lemma 5.18. Thus, the map below is uniformly bounded:

$$\mathbb{R}^2_{++} \ni (x_n, y_n) \mapsto h_\theta(x', \xi', \zeta; x_n, y_n) := \zeta e^{c\zeta(x_n+y_n)}\tilde{g}'_\theta(x', \xi', \zeta; x_n, y_n) \in S^0_{1,1/2}(\mathbb{R}^{n-1} \times \mathbb{R}^n).$$

Now, we analyze the action of $G_{\theta,\mu}$ in the direction transversal to the boundary. To this end, we define a family of operators that act on $\mathcal{S}(\mathbb{R}^{n-1})$:

$$[G'_{\theta,\mu}(x_n, y_n)v](x') := \int e^{ix'\xi'}\tilde{g}'_{\theta,\mu}(x', \xi'; x_n, y_n)\hat{v}(\xi')\, đ\xi'.$$

Similarly, we define $H_{\theta,\mu}(x_n, y_n)$. According to the definition of h_θ,

$$\mu e^{c\mu(x_n+y_n)}G'_{\theta,\mu}(x_n, y_n) = H_{\theta,\mu}(x_n, y_n).$$

For a given $(x_n, y_n) \in \mathbb{R}^2_{++}$, we apply Theorem 5.4 to the right hand side. Since the seminorms of h_θ are uniformly bounded with respect to $(x_n, y_n) \in \mathbb{R}^2_{++}$, we obtain:

$$\|\mu G'_{\theta,\mu}(x_n, y_n)v\|_{L_p(\mathbb{R}^{n-1})} \leq e^{-c\mu(x_n+y_n)}\|H_{\theta,\mu}v\|_{L_p(\mathbb{R}^{n-1})} \leq e^{-c\mu(x_n+y_n)}C\|v\|_{L_p(\mathbb{R}^{n-1})}.$$

Furthermore, if $u \in \mathcal{S}(\mathbb{R}^{n-1}) \otimes \mathcal{S}(\mathbb{R}_+)$ is a simple tensor, such as $u = v \otimes w$, then:

$$[I^+u](x', x_n) = \int_0^\infty \int_0^\infty f(\mu^2 e^{i\theta})[\mu G'_{\theta,\mu}(x_n, y_n)v](x')w(y_n)\, dy_n d\mu.$$

In order to provide the estimate for I^+, it is sufficient to consider simple tensors because they span a dense subset of $L_p(\mathbb{R}^n_+)$. Therefore:

$$\begin{aligned}
\|I^+u\|_{L_p(\mathbb{R}^n_+)} &\leq \|f\|_{L_\infty(\Sigma_\vartheta)} \left\| \int_0^\infty \int_0^\infty \|\mu G_{\theta,\mu}(x_n, y_n)v\|_{L_p(\mathbb{R}^{n-1})}|w(y_n)|\, dy_n d\mu \right\|_{L_p(\mathbb{R}_+)} \\
&\leq C\|f\|_{L_\infty(\Sigma_\vartheta)}\|v\|_{L_p(\mathbb{R}^{n-1})} \left\| \int_0^\infty \int_0^\infty \exp(-c\mu(x_n + y_n))|w(y_n)|\, dy_n d\mu \right\|_{L_p(\mathbb{R}_+)} \\
&\leq C\|f\|_{L_\infty(\Sigma_\vartheta)}\|v\|_{L_p(\mathbb{R}^{n-1})} \left\| \int_0^\infty \frac{|w(y_n)|}{x_n + y_n}\, dy_n \right\|_{L_p(\mathbb{R}_+)} \leq C\|v\|_{L_p(\mathbb{R}^{n-1})}\|w\|_{L_p(\mathbb{R}_+)} \\
&\leq C\|f\|_{L_\infty(\Sigma_\vartheta)}\|u\|_{L_p(\mathbb{R}^n_+)}.
\end{aligned}$$

Note that we used L_p-boundedness of the Hilbert transform in the derivation above. The estimate implies that $I^+ \in \mathcal{B}(L_p(\mathbb{R}^n_+))$ and $\|I^+\| \leq C\|f\|_{L_\infty(\Sigma_\vartheta)}$. Here, $C = C(|a|_*, M, |t|_*, \vartheta)$ is the constant in the estimate above. □

According to the argumentation at the beginning of this subsection, Lemma 5.16, 5.17, and 5.19 imply Theorem 1.2. Now, we prove Corollary 1.3:

Proof. By assumption, $a(x,\xi) = \sum a_{ij}\xi_i\xi_j$. Thus, $|a|_{\alpha,\beta} = 0$ if $|\beta| \neq 0$ or $|\alpha| > 2$. Obviously for $|\alpha| \in \{0,1,2\}$, the estimate $|D^\alpha \xi_j\xi_i| \leq 2\langle\xi\rangle^{2-|\alpha|}$ holds. Therefore, $|a|_* \leq C\sum|a_{ij}|$. Since all matrix norms are equivalent, we can replace the sum by the largest eigenvalue of $(a_{ij})_{1\leq i,j\leq n}$. The largest eigenvalue is bounded by M, according to the definition of M-ellipticity. Therefore, the constants in Theorem 1.2 only depend on M, $|t|_*$, and ϑ. □

5.5 Proof of the Main Result

In this subsection, we extend the results of the last subsection in two directions. First, we reduce the regularity assumption for the differential operator. The sufficient regularity assumption is the same as in the non-degenerate case, i.e., C^τ for the highest order coefficients and L_∞ for the remaining coefficients. Second, we replace the underlying euclidean half-space by manifolds with boundary and bounded geometry.

5.5.1 The Main Result for Euclidean Half Space

We reduce the regularity assumptions of the auxiliary result, i.e., we prove Theorem 1.1 for the case of $X_+ = \mathbb{R}^n_+$. In this case, the differential operator has the following form:

$$\mathcal{A} := \sum_{1\leq i,j\leq n} a_{ij}(x)D_iD_j + \sqrt{-1}\sum_{1\leq j\leq n} b_j(x)D_j + c(x),$$

with coefficients $a_{ij} \in C^\tau(\mathbb{R}^n)$ for some $\tau > 0$ and $b_j, c \in L_\infty(\mathbb{R}^n)$. We use the classical approach of freezing the coefficients. We only freeze the coefficients of the differential operator, *not* those of the boundary operator. The localisation scheme we use is inspired by [27]. The localization provides a family of operators. Each of these operators is a small perturbation of an operator with frozen coefficients. We prove that they allow a bounded $H^\infty(\Sigma_\theta)$-calculus in a uniform manner. By assembling the local operators, we can conclude that $A_T + \nu$ itself allows a bounded $H^\infty(\Sigma_\theta)$-calculus.
Now, we describe the localisation scheme. We choose a small $r > 0$. Later on, we clarify how r is choose. We define the cubes $Q = (-r,r)^n$ and $Q_l := Q+l$, with $l \in \Gamma := r(\mathbb{Z}\times\mathbb{N}_0)$. Note that $\mathbb{R}^n_+ \subset \cup_{l\in\Gamma}Q_l$. We choose a positive function $\psi \in C_0^\infty(Q)$ such that $\gamma_1\psi = 0$ and

$$\sum_{l\in\Gamma}\psi_l(x) = 1 \text{ for all } x \in \mathbb{R}^n_+, \text{ where } \psi_l(x) = \psi(x-l).$$

Moreover, we choose two positive functions $\chi, \chi' \in C_0^\infty(Q)$ such that $\chi' = 1$ on $\operatorname{supp}\psi$ and $\chi = 1$ on $\operatorname{supp}\chi'$. We define χ_l and χ'_l similar to ψ_l. For all $l \in \Gamma$, we define an unbounded operator A_l with domain $\mathcal{D}(A_l) = \mathcal{D}(A_T)$. The operator maps $u \in \mathcal{D}(A_l) \subset L_p(\mathbb{R}^n_+)$ to $r^+ \mathcal{A}_l e^+ \in L_p(\mathbb{R}^n_+)$, where $\mathcal{A}_l$ is the differential operator below:

$$\mathcal{A}_l = \sum_{|\alpha|=2} \left(a_\alpha(l) + \chi'_l(x)[a_\alpha(x) - a_\alpha(l)]\right) D^\alpha.$$

We define A'_T similar to A_T, with respect to the principal part of $\mathcal{A}$. Observe that $A_l \psi_l = A'_T \psi_l$. The main technical difficulty is to prove that each operator in the family $(A_l)_{l\in\Gamma}$ allows a bounded $H^\infty(\Sigma_\vartheta)$-calculus with uniform estimates. We provide the precise statement here and postpone the proof to the next subsection.

Lemma 5.20. *For a given $0 < \vartheta < \pi$, two constants $\nu = \nu(M, |t|_*, \vartheta) \geq 0$ and $r = r(M, |t|_*, \vartheta) > 0$ exist such that for all $l \in \Gamma$, the operator $A_l + \nu$ allows a bounded $H^\infty(\Sigma_\vartheta)$-calculus. Moreover, a constant $C := C(M, |t|_*, \vartheta) > 0$ exists such that*

$$\|f(A_l)\|_{\mathcal{B}(L_p(\mathbb{R}^n_+))} \leq C \|f\|_{L_\infty(\Sigma_\vartheta)} \quad \textit{for all } f \in H^\infty(\Sigma_\vartheta) \textit{ and } l \in \Gamma.$$

We define the localization operator L and the patching operator P as follows:

$$\begin{aligned} L &: L_p(\mathbb{R}^n_+) \to \mathbb{L}_p(\mathbb{R}^n_+), \quad u \mapsto (\psi_l u)_{l\in\Gamma}. \\ P &: \mathbb{L}_p(\mathbb{R}^n_+) \to L_p(\mathbb{R}^n_+), \quad (u_l)_{l\in\Gamma} \mapsto \sum_{l\in\Gamma} \chi_l u_l. \end{aligned}$$

Moreover, we define the operator $\mathbb{T} : \mathbb{H}^2_p(\mathbb{R}^n_+) \to \mathbb{B}^{1-1/p}_p(\mathbb{R}^{n-1})$, $(u_l)_{l\in\Gamma} \to (Tu_l)_{l\in\Gamma}$. For further argumentation, we collect some properties of the previously defined operators:

Lemma 5.21. *Let L, P, and $\mathbb{T}$ be as above and $s \geq 0$. Then, the following results hold:*

(i) $L \in \mathcal{B}(H^s_p(\mathbb{R}^n_+); \mathbb{H}^s_p(\mathbb{R}^n_+))$.

(ii) $P \in \mathcal{B}(\mathbb{H}^s_p(\mathbb{R}^n_+); H^s_p(\mathbb{R}^n_+))$.

(iii) $PL = 1$.

(iv) $L : H^2_p(\mathbb{R}^n_+) \cap \ker T \to \mathbb{H}^2_p \cap \ker \mathbb{T}$.

(v) $P : \mathbb{H}^2_p \cap \ker \mathbb{T} \to H^2_p(\mathbb{R}^n_+) \cap \ker T$.

Proof. The operators L and P are special cases of localisation and patching operators in Section 2.3. Therefore, *(i)*, *(ii)*, and *(iii)* hold. By assumption, $\gamma_1 \psi = 0$. Thus, $\gamma_1 \psi_l = 0$ for all $l \in \Gamma$. Therefore, $L : \ker T \to \ker \mathbb{T}$. Similarly, we obtain $P : \ker \mathbb{T} \to T$. □

We define operators $A_{lk} := \delta_{lk} A_l$ with domain $\mathcal{D}(A_{lk}) = H^2_p(\mathbb{R}^n_+) \cap \ker T$. Moreover, we define:

$$\mathbb{A} : \mathcal{D}(\mathbb{A}) := \mathbb{H}^2_p(\mathbb{R}^n_+) \cap \ker \mathbb{T} \subset \mathbb{L}_p(\mathbb{R}^n_+) \to \mathbb{L}_p(\mathbb{R}^n_+), \quad (u_k)_{k\in\Gamma} \mapsto \left(\sum_{k\in\Gamma} A_{lk} u_k \right)_{l\in\Gamma}.$$

Similarly, we define $\mathbb{B}$ and $\mathbb{D}$ for the following families of operators with index set $\Gamma \times \Gamma$:

$$B_{lk} := \delta_{lk} A_{low} + [\psi_l, A]\psi_k \text{ and } D_{lk} = \delta_{lk} A_{low} + \psi_l [A_k + A_{low}, \psi_k].$$

Here, A_{low} denotes $A_T - A'_T$. On $\Gamma \times \Gamma$, $l \bowtie k :\Leftrightarrow Q_l \cap Q_k \neq \emptyset$ defines a symmetric relation. For a given $l \in \Gamma$, the set $\Gamma_l := \{k \in \Gamma : k \bowtie l\}$ is finite. In particular, $B_{lk} = 0$ and $D_{lk} = 0$ if $k \notin \Gamma_l$. Therefore, all sums in the definition of $\mathbb{B}$ and $\mathbb{D}$ are finite. A quick computation shows that the operators $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{D}$ are related:

$$\begin{aligned} LA_T &= (\mathbb{A} + \mathbb{B})L \text{ on } H^2_p(\mathbb{R}^n_+) \cap \ker T \text{ and} \\ A_T P &= P(\mathbb{A} + \mathbb{D}) \text{ on } \mathbb{H}^2_p(\mathbb{R}^n_+) \cap \ker \mathbb{T}. \end{aligned}$$

For more details on the computation, see Section 5.5.3. We use the previously described localisation scheme to prove Theorem 1.1. According to Lemma 3.7, the following claim implies Theorem 1.1.

We claim. For given $0 < \vartheta \leq \pi$, two constants $\nu = \nu(M, |t|_*, \vartheta) \geq 0$ and $r = r(M, |t|_*, \vartheta) > 0$ exist such that both $\mathbb{A} + \mathbb{B} + \nu$ and $\mathbb{A} + \mathbb{D} + \nu$ allow a bounded $H^\infty(\Sigma_\theta)$-calculus in $\mathbb{L}_p(\mathbb{R}^n_+)$.

The claim can be proven in two steps. First:

Lemma 5.22. *For given $0 < \vartheta \leq \pi$, two constants $\nu = \nu(M, |t|_*, \vartheta) \geq 0$ and $r = r(M, |t|_*, \vartheta) > 0$ exist such that $\mathbb{A} + \nu$ allows a bounded $H^\infty(\Sigma_\theta)$-calculus in $\mathbb{L}_p(\mathbb{R}^n_+)$.*

Proof. For given $0 < \vartheta \leq \pi$, we choose $\nu, r > 0$, according to Lemma 5.20. Therefore, each operator in the family $A_l + \nu$ allows a bounded $H^\infty(\Sigma_\vartheta)$-calculus in $L_p(\mathbb{R}^n_+)$. Furthermore, the operator $\mathbb{F} := f(\mathbb{A} + \nu)$ is a matrix operator with entries $\mathbb{F}_{lk} = \delta_{lk} f(A_l + \nu)$. In particular, $\mathbb{F}$ has band structure of width 1 and the entries are uniformly estimated by $C\|f\|_{H^\infty(\Sigma_\theta)}$. Lemma 2.5 implies that $f(\mathbb{A} + \nu) \in \mathcal{B}(\mathbb{L}_p(\mathbb{R}^n_+))$ and $\|f(\mathbb{A} + \nu)\| \leq C\|f\|_{H^\infty(\Sigma_\vartheta)}$. The constant C is the same as in Lemma 5.20. □

As the second step, we verify that both $\mathbb{B}$ and $\mathbb{D}$ are lower order perturbations of $\mathbb{A} + \nu$ such as in Theorem 3.8 which implies:

Lemma 5.23. *For given $0 < \vartheta \leq \pi$, two constants $\nu = \nu(M, |t|_*, \vartheta) \geq 0$ and $r = r(M, |t|_*, \vartheta) > 0$ exist such that $\mathbb{A} + \mathbb{B} + \nu$ and $\mathbb{A} + \mathbb{D} + \nu$ allow a bounded $H^\infty(\Sigma_\vartheta)$-calculus in $\mathbb{L}_p(\mathbb{R}^n_+)$.*

Proof. We can assume that $0 \in \rho(\nu + \mathbb{A})$, otherwise we increase ν. Thus, for $0 < \gamma < 1$, the operator $(\nu + \mathbb{A})^{(1-\gamma)}$ is invertible. In particular, $\|\cdot\|_{\mathcal{D}((\nu+\mathbb{A})^{1-\gamma})}$ and $\|(\nu + \mathbb{A})^{1-\gamma} \cdot\|_{\mathbb{L}_p(\mathbb{R}^n_+)}$ are equivalent norms. According to Lemma 5.22, the operator $\nu + \mathbb{A}$ has a bounded $H^\infty(\Sigma_\theta)$-calculus in $\mathbb{L}_p(\mathbb{R}^n_+)$. Therefore, $\nu + \mathbb{A}$ has bounded imaginary powers. According to Theorem 3.11, the domain of $(\nu + \mathbb{A})^{1-\gamma}$ is:

$$\mathcal{D}((\nu + \mathbb{A})^{1-\gamma}) = [\mathbb{L}_p(\mathbb{R}^n_+), \mathcal{D}(\mathbb{A})]_{1-\gamma} \hookrightarrow [\mathbb{L}_p(\mathbb{R}^n_+), \mathbb{H}^2_p(\mathbb{R}^n_+))]_{1-\gamma} = \mathbb{H}^{2-2\gamma}_p(\mathbb{R}^n_+). \tag{5.17}$$

The results for $\mathbb{B}$ and $\mathbb{D}$ are proven in a similar manner. Thus, we can focus on $\mathbb{B}$. All operators B_{lk} are first order differential operators. Therefore, for each $\gamma < 1/2$, the following estimate holds:

$$\|B_{lk}u_k\|_{L_p(\mathbb{R}^n_+)} \leq C\|u_k\|_{H^1_p(\mathbb{R}^n_+)} \leq C\|u_k\|_{H^{2-2\gamma}_p(\mathbb{R}^n_+)}. \tag{5.18}$$

Here, the constant $C > 0$ depends on the L_∞ norm of the coefficients and is thus independent of k and l. The estimate above implies that $\mathbb{B}$ is a band structure operator and therefore bounded for $\mathbb{H}^{2-2\gamma}_p(\mathbb{R}^n_+)$ to $\mathbb{L}_p(\mathbb{R}^n_+)$. In sum:

$$\|\mathbb{B}(u_l)_{l\in\Gamma}\|_{\mathbb{L}_p(\mathbb{R}^n_+)} \leq \|(\mathbb{A}+\nu)^{1-\gamma}(u_l)_{l\in\Gamma}\|_{\mathbb{L}_p(\mathbb{R}^n_+)}.$$

Therefore, we can apply Theorem 3.8 which, in turn, proves the result. □

5.5.2 A Technical Lemma

Now, we consider the operator family $(A_l)_{l\in\Gamma}$. For each $l \in \Gamma$, the operator A_l is the L_p-realization of $\mathcal{A}_l = \mathcal{A}^c_l + \mathcal{A}^s_l$ with domain A_T. Here, the right hand side is defined as:

$$\mathcal{A}^c_l := \sum_{1\leq i,j\leq n} a_{ij}(l)D_iD_j \quad \text{and} \quad \mathcal{A}^s_l := \sum_{1\leq i,j\leq n} \chi'_l(x)[a_{ij}(x) - a_{ij}(l)]D_iD_j.$$

Based on $\mathcal{A}^c_l$ and $\mathcal{A}^s_l$, we define A^c_l and A^s_l similar to A_l. The coefficients of A^c_l are constant. In fact, we obtain A^c_l from A_T by freezing the coefficients of the principal part at $x = l$. According to the definition of M-ellipticity, the operator A^c_l inherits M-ellipticity from A_T. Therefore, Corollary 1.3 applies to A^c_l. Thus, we obtain:

Lemma 5.24. *For a given $0 < \vartheta \leq \pi$, a constant $\nu = \nu(M, |t|_*, \vartheta) \geq 0$ exists such that $A^c_l + \nu$ has a bounded $H^\infty(\Sigma_\vartheta)$-calculus in $L_p(\mathbb{R}^n_+)$. Moreover, a constant $C = C(|t|_*, M, \vartheta) > 0$ exists such that the following estimate holds for all $f \in H^\infty(\Sigma_\vartheta)$:*

$$\|f(A^c_l + \nu)\|_{\mathcal{B}(L_p(\mathbb{R}^n_+))} \leq C\|f\|_{L_\infty(\Sigma_\vartheta)}.$$

The constants in the lemma above are independent of l and r. Therefore, we have some freedom in the choice of $r > 0$. We can choose $r > 0$ such that A^s_l is a small perturbation of $A^c_l + \nu$ such as in Theorem 3.9. The application of the theorem proves Lemma 5.20. Now, we discuss in detail how we choose $r > 0$. The coefficients $a^s_{l,ij}(x) = \chi'_l(x)[a_{ij}(x) - a_{ij}(l)]$ of A^s_l are uniformly small such as in the following lemma:

Lemma 5.25. *Let $0 < \sigma \leq \tau < 1$. A constant $C > 0$ exists such that the following estimates hold:*

$$\|a^s_{l,ij}\|_\infty \leq C\|a_{ij}\|_{C^\tau(\mathbb{R}^n_+)}r^\tau \quad \textit{and} \quad \|a^s_{l,ij}\|_{C^\sigma(\mathbb{R}^n_+)} \leq C\|a_{ij}\|_{C^\tau(\mathbb{R}^n_+)}r^{\tau-\sigma}.$$

Proof. We recall that r is proportional to the diameter of the cube Q defined in the last subsection. We obtain the first estimate from the definition of the Hölder norm:

$$\begin{aligned}\|a^s_{l,ij}\|_\infty = \sup|\chi'_l(x)(a_{ij}(x)-a_{ij}(l))| \quad &\leq \sup\left\{\frac{|a_{ij}(x)-a_{ij}(l)|}{|x-l|^\tau}|x-l|^\tau : x\in\operatorname{supp}(\chi'_l)\right\}\\ \leq C\|a_{ij}\|_{C^\tau(\mathbb{R}^n)}r^\tau.\end{aligned}$$

By definition, $a^s_{l,ij}\in C^\tau(\mathbb{R}^n)$ is a product of Hölder continuous functions. Therefore, the Hölder seminorm of $a^s_{l,ij}$ can be estimated:

$$[a^s_{l,ij}]_{C^\tau(\mathbb{R}^n)} \leq [\chi'_l]_{C^\tau(\mathbb{R}^n)}\|a_{ij}(\cdot)-a_{ij}(l)\|_\infty + \|\chi'_l\|_\infty[a_{ij}]_{C^\tau(\mathbb{R}^n)} \leq C\|a_{ij}\|_{C^\tau(\mathbb{R}^n)}.$$

To estimate the C^σ seminorm, we separate the cases $|x-y|\geq r$ and $|x-y|<r$:

$$\begin{aligned}\sup_{|x-y|\geq r}\frac{|a^s_{l,ij}(x)-a^s_{l,ij}(y)|}{|x-y|^\sigma} &\leq 2\|a^s_{l,ij}\|_\infty r^{-\sigma} \leq C\|a_{ij}\|_{C^\tau(\mathbb{R}^n)}r^{\tau-\sigma} \quad\text{and}\\ \sup_{0<|x-y|<r}\frac{|a^s_{l,ij}(x)-a^s_{l,ij}(y)|}{|x-y|^\sigma} &\leq \sup_{0<|x-y|<r}\frac{|a^s_{l,ij}(x)-a^s_{l,ij}(y)|}{|x-y|^\tau}r^{\tau-\sigma} \leq C\|a_{ij}\|_{C^\tau(\mathbb{R}^n)}r^{\tau-\sigma}.\end{aligned}$$

The estimate for the C^σ norm of $a^s_{l,ij}$ is a direct consequence of the estimates above. □

Next, we verify that the lemma above implies the following estimate:

$$\|A^s_l u\|_{L_p(\mathbb{R}^n_+)} \leq Cr^\tau\|(A^c_l+\nu)u\|_{L_p(\mathbb{R}^n_+)} \quad\text{for all}\quad u\in H^2_p(\mathbb{R}^n_+)\cap\ker T. \tag{5.19}$$

We know that $C^\tau(\mathbb{R}^n_+)\hookrightarrow\mathcal{B}(H^s_p(\mathbb{R}^n_+))$ as a multiplication operator for $s\in[0,\tau]$. Therefore:

$$\|A^s_l u\|_{L_p(\mathbb{R}^n_+)} \leq \|a^s_{l,ij}\|_{C(\mathbb{R}^n_+)}\|u\|_{H^2_p(\mathbb{R}^n_+)} \leq Cr^\tau\|u\|_{H^2_p(\mathbb{R}^n_+)}.$$

Furthermore, on $H^2_p(\mathbb{R}^n_+)\cap\ker T$, the norm $\|(A^c_l+\nu)\cdot\|_{L_p(\mathbb{R}^n_+)}$ and the $H^2_p(\mathbb{R}^n_+)$ norm are equivalent because $(A^c_l+\nu)$ is invertible. Therefore, Equation (5.19) holds. Now, we compute the domain of $(A^c_l+\nu)^\gamma$ for $2\gamma<\min\{1/p,\tau\}$. According to Theorem 3.11, the domain is:

$$\mathcal{D}((A^c_l+\nu)^\gamma) = [L_p(\mathbb{R}^n_+), H^2_p(\mathbb{R}^n_+)\cap\ker T]_\gamma.$$

By interpolation, the embedding $H^2_{p;0}(\mathbb{R}^n_+)\hookrightarrow H^2_p(\mathbb{R}^n_+)\cap\ker T\hookrightarrow H^2_p(\mathbb{R}^n_+)$ implies:

$$H^{2\gamma}_{p;0}(\mathbb{R}^n_+)\hookrightarrow[L_p(\mathbb{R}^n_+), H^2_p(\mathbb{R}^n_+)\cap\ker T]_\gamma\hookrightarrow H^{2\gamma}(\mathbb{R}^n_+).$$

Therefore, $\mathcal{D}((A^c_l+\nu)^\gamma) = H^{2\gamma}_p(\mathbb{R}^n_+)$ because $H^{2\gamma}_p(\mathbb{R}^n_+) = H^{2\gamma}_{p;0}(\mathbb{R}^n_+)$ for $2\gamma<1/p$. Furthermore, the operator $(A^c_l+\nu)^\gamma$ is invertible. Thus, $\|(A^c_l+\nu)^\gamma\cdot\|_{L_p(\mathbb{R}^n_+)}$ and $\|\cdot\|_{H^{2\gamma}_p(\mathbb{R}^n_+)}$ are equivalent norms on $\mathcal{D}((A^c_l+\nu)^\gamma)$. We make us of Lemma 5.25 and $C^\sigma(\mathbb{R}^n_+)\hookrightarrow\mathcal{B}(H^s_p(\mathbb{R}^n_+))$ to obtain the following estimate:

$$\|(A^c_l+\nu)^\gamma A^s_l u\|_{L_p(\mathbb{R}^n_+)} \leq C\|A^s_l u\|_{H^{2\gamma}_p(\mathbb{R}^n_+)} \leq Cr^{\tau-2\gamma}\|u\|_{H^{2+2\gamma}_p(\mathbb{R}^n_+)}.$$

We can further estimate the right hand side with Estimate (5.13):

$$\|u\|_{H_p^{2+2\gamma}(\mathbb{R}_+^n)} \leq C\|(\nu + A_l^c)u\|_{H_p^{2\gamma}(\mathbb{R}_+^n)} \leq \|(\nu + A_l^c)^{1+\gamma}u\|_{L_p(\mathbb{R}_+^n)}.$$

In sum, the following estimate holds for all $u \in \mathcal{D}((\nu + A_l^c)^{1+\gamma})$:

$$\|(\nu + A_l^c)^\gamma A_l^s u\|_{L_p(\mathbb{R}_+^n)} \leq C r^{\tau-2\gamma}\|(\nu + A_l^c)^{1+\gamma}u\|_{L_p(\mathbb{R}_+^n)}. \tag{5.20}$$

The constants in Equation (5.19) and (5.20) are independent of l and r. Therefore, we can choose r such that Theorem 3.9 applies to $\nu + A_l^c + A_l^s$ and thus Lemma 5.20 holds.

5.5.3 The Main Result for Manifolds

Now, let (X_+, g) be a manifold with boundary and bounded geometry. We choose an atlas of Fermi coordinates $\kappa_l : U_l \subset X_+ \to V_l \subset \mathbb{R}_+^n$ with index set Γ such that $\sup_{l\in\Gamma} |\{k \in \Gamma : U_k \cap U_l \neq \emptyset\}| =: N < \infty$. We also choose a subordinated partition of unity $(\psi_l)_{l\in\Gamma}$ such that $\partial_\nu \psi_l = 0$ for all $l \in \Gamma$. Here, ν denotes an outward unit normal vector field on ∂X_+. For each ψ_l, we choose positive functions $\chi_l', \chi_l \in C_0^\infty(U_l)$ such that $\chi_l = 1$ on $\operatorname{supp}\psi_l$ and $\chi_l' = 1$ on $\operatorname{supp}\chi_l$. We denote $\chi_{l,*} = \kappa_{l,*}\chi_l \in C_0^\infty(V_l) \subset C_0^\infty(\mathbb{R}^n)$. Similarly, we define $\chi_{l,*}'$. Moreover, we write $\tilde{\kappa}_l(x') := \kappa_l(x', 0)$ for the induced chart on the boundary. Let $\mathcal{A}$ be a sufficiently regular M-elliptic second order differential operator on X and T be a boundary operator as in (1.2). For each $l \in \Gamma$, we define the following operators:

$$\mathcal{A}_l := -\Delta(1 - \chi_{l,*}') + \kappa_{l,*}\mathcal{A}\kappa_l^*\chi_{l,*}' \qquad \text{and} \qquad T_l := \gamma_0(1 - \chi_{l,*}') + \tilde{\kappa}_{l,*}T\kappa_l^*\chi_{l,*}'.$$

For each $l \in \Gamma$, the operator $\mathcal{A}_l$ is an M-elliptic second order differential operator on $\mathbb{R}^n$ which satisfies the regularity assumption in Section 5.5.1. Moreover, the norms of the coefficients of the local representations of $\mathcal{A}$ are bounded by M. Therefore, the norms of the coefficients are uniformly bounded with respect to $l \in \Gamma$. Moreover, the seminorms $|t_l|_*$ are uniformly bounded with respect to $l \in \Gamma$. We define:

$$A_l : \mathcal{D}(A_l) := \{u \in H_p^2(\mathbb{R}_+^n) : T_l u = 0\} \to L_p(\mathbb{R}_+^n),\ u \mapsto r^+\mathcal{A}_l e^+ u.$$

Each operator A_l satisfies the assumptions in Section 5.5.1. Therefore, we can apply Theorem 1.1 to A_l and obtain:

Lemma 5.26. *For a given $0 < \vartheta \leq \pi$, a constant $\nu = \nu(M, |t|_*, \vartheta) \geq 0$ exists such that for all $l \in \Gamma$, the operator $A_l + \nu$ allows a bounded $H^\infty(\Sigma_\vartheta)$-calculus. Moreover, a constant $C = C(M, |t|_*, \vartheta) > 0$ exists such that for all $l \in \Gamma$, the following estimate holds:*

$$\|f(A_l + \nu)\|_{\mathcal{L}(L_p(\mathbb{R}_+^n))} \leq C\|f\|_{L_\infty(\Sigma_\vartheta)} \ \text{for all}\ f \in H^\infty(\Sigma_\vartheta).$$

In Section 2.3, we defined the localization operator and the patching operator below:

$$L : L_p(X_+) \to \mathbb{L}_p(\mathbb{R}_+^n),\ \ u \mapsto (\kappa_{l,*}\psi_l u)_{l\in\mathbb{N}}.$$
$$P : \mathbb{L}_p(\mathbb{R}_+^n) \to L_p(X_+),\ \ (u_l)_{l\in\mathbb{N}} \to \sum_{l\in\mathcal{I}} \kappa_l^*\chi_{l,*}u_l.$$

When we introduced the two operators, we also proved the following results:

- $L \in \mathcal{B}(H^s_p(M_+); \mathbb{H}^s_p(\mathbb{R}^n_+))$,
- $P \in \mathcal{B}(\mathbb{H}^s_p(\mathbb{R}^n_+); H^s_p(M_+))$, and
- $PL = 1$.

Furthermore, we now define $\mathbb{T} : \mathbb{H}^2_p(\mathbb{R}^n_+) \to \mathbb{B}^{1-1/p}_p(\mathbb{R}^{n-1})$, $(u_l)_{l\in\mathcal{I}} \mapsto (T_l u_l)_{l\in\mathcal{I}}$. Next, we verify that the localization operator maps the kernel of T to the kernel of $\mathbb{T}$. To this end, we fix a $u \in \ker T$. For each component of the sequence $\mathbb{T}Lu$, the following calculation holds:

$$T_l \kappa_{l,*}\psi_l u = \tilde{\kappa}_{l,*} T \kappa_l^* \chi'_{l,*}\kappa_{l,*}\psi_l u = \tilde{\kappa}_{l,*} T \psi_l u = \tilde{\kappa}_{l,*}\psi_l T u + \tilde{\kappa}_{l,*}\gamma_0(\partial_\nu \psi_l) u = 0.$$

Here, we use the assumption $\partial_\nu \psi_l = 0$ for all $l \in \Gamma$. Now, we consider the patching operator. The assumption that $\gamma_0 \partial_\nu \chi_l = 0$ is locally given by $\gamma_0 \partial_n \chi_{l,*} = 0$. Therefore:

$$TP(u_l)_{l\in\Gamma} = T\sum_{l\in\Gamma} \tilde{\kappa}_l^* \chi_{l,*} u_l = \sum_{l\in\Gamma} \tilde{\kappa}_l^* T_l \chi_{l,*} u_l = \sum_{l\in\Gamma} \tilde{\kappa}_l^* \chi_{l,*} T_l u_l + \varphi_1 \tilde{\kappa}_l^* \gamma_0(\partial_n \chi_{l,*}) u_l = P\mathbb{T}(u_l)_{l\in\Gamma}.$$

Thus, the patching operator maps the kernel of $\mathbb{T}$ into the kernel of T. We define $\mathcal{D}(\mathbb{A}) := \mathbb{H}^2_p(\mathbb{R}^n_+) \cap \ker \mathbb{T}$. Note that $(u_l)_{l\in\Gamma} \in \mathcal{D}(\mathbb{A})$ implies that $u_l \in \mathcal{D}(A_l)$ for all $l \in \Gamma$. Therefore, the following definition is reasonable:

$$\mathbb{A} : \mathcal{D}(\mathbb{A}) := \mathbb{H}^2_p(\mathbb{R}^n_+) \cap \ker \mathbb{T} \subset \mathbb{L}_p(\mathbb{R}^n_+) \to \mathbb{L}_p(\mathbb{R}^n_+),\ (u_l)_{l\in\Gamma} \mapsto (A_l u_l)_{l\in\Gamma}.$$

The following result is proven similarly to Lemma 5.22: We only have to replace Lemma 5.20 by Lemma 5.26.

Lemma 5.27. *For a given $0 < \vartheta \le \pi$, a constant $\nu = \nu(M, \vartheta, |t_*|) \ge 0$ exists such that $\mathbb{A} + \nu$ allows a bounded $H^\infty(\Sigma_\vartheta)$-calculus in $\mathbb{L}_p(\mathbb{R}^n_+)$. Moreover, a constant $C = C(M, \vartheta, |t_*|) \ge 0$ exists such that:*

$$\|f(\mathbb{A}+\nu)\|_{\mathcal{B}(\mathbb{L}_p(\mathbb{R}^n_+))} \le C\|f\|_{L_\infty(\Sigma_\vartheta)} \ \text{for all}\ f \in H^\infty(\Sigma_\vartheta).$$

Next, we compute $\mathbb{B}L := LA_T - \mathbb{A}L$. For a given $l \in \Gamma$, we observe:

$$(LA_T)_l = \kappa_{l,*}A_T\psi_l + \kappa_{l,*}[\psi_l, A_T] = A_l\kappa_{l,*}\psi_l + \kappa_{l,*}[\psi_l, A_T] = (\mathbb{A}L)_l + \kappa_{l,*}[\psi_l, A_T].$$

We rewrite the last term with the help of the partition of unity:

$$\kappa_{l,*}[\psi_l, A_T] = \sum_{k\in\Gamma} \kappa_{l_*}[\psi_l, A_T]\chi'_{k,*}\kappa_k^*\kappa_{k,*}\psi_k = \sum_{k\in\Gamma} B_{lk}\kappa_{k,*}\psi_k, \ \text{ where }\ B_{lk} := \kappa_{l_*}[\psi_l, A_T]\chi'_{k,*}\kappa_k^*.$$

The operators B_{lk} are first order differential operators which have bounded coefficients that can be estimated independent of l and k. The operator $\mathbb{B} : \mathbb{H}^2_p(\mathbb{R}^n_+) \subset \mathbb{L}_p(\mathbb{R}^n_+) \to \mathbb{L}_p(\mathbb{R}^n_+)$ is represented by the infinite matrix (B_{lk}). Next, we consider the patching operator. We observe:

$$A_T P = \sum_{l\in\Gamma} A_T \kappa_l^* \chi_{l,*} = \sum_{l\in\Gamma} \kappa_l^* A_l \chi_{l,*} = P\mathbb{A} + \sum_{l\in\Gamma} \kappa_l^*[A_l, \chi_{l,*}].$$

Again, through an application of the partition of unity, we can write the last term as:

$$\sum_{l\in\Gamma} \kappa_l^*[A_l, \chi_{l,*}] = \sum_{l\in\Gamma} \psi_l \sum_{k\in\Gamma} \kappa_k^*[A_k, \chi_{k,*}] = \sum_{l\in\Gamma} \kappa_l^* \chi_{l,*} \sum_{k\in\Gamma} \kappa_{l,*}\psi_l \kappa_k^*[A_k, \chi_{k,*}] =: P\mathbb{D}.$$

The operator $\mathbb{D} : \mathbb{H}_p^2(\mathbb{R}_+^n) \subset \mathbb{L}_p(\mathbb{R}_+^n) \to \mathbb{L}_p(\mathbb{R}_+^n)$ is an infinite matrix with entries $D_{lk} := \kappa_{l,*}\psi_l\kappa_k^*[A_k, \chi_{k,*}]$. Note that the entries are first order differential operators which have bounded coefficients that can be estimated independent of l and k. The operators $\mathbb{A}, \mathbb{B}$, and $\mathbb{D}$ have the same properties as those in Section 5.5.1. Therefore, the proof of Lemma 5.23 also holds for the operators $\mathbb{A}, \mathbb{B}$, and $\mathbb{D}$:

Lemma 5.28. *For a given* $0 < \vartheta \leq \pi$, *a constant* $\nu = \nu(M, |t|_*, \vartheta) \geq 0$ *exists such that* $\mathbb{A} + \mathbb{B} + \nu$ *and* $\mathbb{A} + \mathbb{D} + \nu$ *allow a bounded* $H^\infty(\Sigma_\theta)$*-calculus in* $\mathbb{L}_p(X_+)$.

Following the same arguments as in Section 5.5.1, we obtain Theorem 1.1 by Lemmata 3.7 and 5.28.

6 The Porous Medium Equation

In this section, we illustrate the application of the theory developed in this thesis to nonlinear parabolic partial differential equations. A simple example for this type of equations is the porous medium equation below:

$$(PME)\begin{cases}\dot{v}-\Delta_g v^m = 0\\ Tv=\phi\\ v|_{t=0}=v_0\end{cases}. \tag{6.1}$$

This equation, arises for instance, in the description of the gas flow through a porous medium. We consider the case were the initial data $v_0 \in H^2_p(X_+)$ is a strictly positive real valued function and the boundary data is independent of time and compatible with the initial data, i.e., $\phi = Tv_0$. Under this assumption, we can provide the short time existence of a solution to Problem (6.1). More precisely:

Theorem 6.1. *Let $n/p + 2/q < 1$. Let v_0 and ϕ satisfy the assumption above. Then, a constant $t_* > 0$ exists such that the Problem* (6.1) *has a unique solution v in:*

$$v \in L_q(0, t_*; H^2_p(X_+) \cap \{Tv = Tv_0\}) \cap W^1_q(0, t_*; L_p(X_+)).$$

The proof we present is inspired by [36]. We define $u := v - v_0$ and consider the following equivalent parabolic problem:

$$\begin{cases}\dot{u}-\Delta_g (u+v_0)^m = 0\\ Tu=0\\ u|_{t=0}=0\end{cases}. \tag{6.2}$$

A quick computation shows that v solves (6.1) if and only if u solves (6.2). Therefore, we focus on Problem (6.2). Next, we write Problem (6.2) as an abstract parabolic problem. To this end, we need the following identity which can easily be verified in local coordinates:

$$\Delta_g(u+v_0)^m = m(u+v_0)^{m-1}\Delta_g u + m(m-1)(u+v_0)^{m-2}|\nabla(u+v_0)|^2_g + m((u+v_0))^{m-1}\Delta_g v_0.$$

The first term on the right hand side is the highest order term. Therefore, we define $A(u) := -m(u+v_0)^{m-1}\Delta_{g,T}$ and:

$$f(u) := m(m-1)(u+v_0)^{m-2}|\nabla(u+v_0)|^2_g + m((u+v_0))^{m-1}\Delta_{g,T} v_0.$$

According to the definitions above, Problem (6.2) is the abstract parabolic problem:

$$\dot{u} + A(u)u = f(u) \text{ and } u|_{t=0} = 0. \tag{6.3}$$

In the following, we verify that (6.3) satisfies the assumptions of Theorem 3.13. To this end, we define $E_0 = L_p(X_+)$ and $E_1 := H^2_p(X_+) \cap \ker T$. The trace space is defined as:

$$E_q := [E_1, E_0]_{1/q,q} \hookrightarrow [H^2_p(X_+), L_p(X_+)]_{1/q,q} = B^{2-2/q}_{p,q}(X_+) \hookrightarrow C^\tau(X_+). \tag{6.4}$$

Here, the last embedding only holds if $2-2/q-n/p > \tau > 0$. According to the assumptions of Theorem 6.1, the inequality holds. The operator $A(u_0) = -mv_0^{m-1}\Delta_{g,T}$ satisfies the assumptions of Theorem 1.1 because, by assumption, v_0 is strictly positive. Therefore, a suitable shift of $A(u_0)$ allows a bounded H^∞-calculus and thus $A(u_0)$ has maximal L_q-regularity. Maximal regularity is part of the assumptions of Theorem 3.13. Next, we consider the remaining assumptions of the theorem. To this end, we need the following result:

Lemma 6.2. *Let $v_0 \in C^\tau(X_+)$ with $\Re v_0 \geq \delta > 0$. We define:*

$$W := \{z \in \mathbb{C} : |z| < \|v_0\|_{C^\tau} + 3\delta/4, \;\; \Re z > \delta(1-3/4)\} \;.$$

A neighbourhood V of v_0 in $C^\tau(X_+)$ and a constant $C := C(\delta, \|v_0\|_{C^\tau(X_+)})$ exist such that for all $f \in H^\infty(W)$ and $u, u' \in V$ the following estimates hold:

$$\|f(u)\|_{C^\tau(X_+)} \leq C\|f\|_{L_\infty(W)} \;\; \text{and} \;\; \|f(u) - f(u')\|_{C^\tau(X_+)} \leq C\|f\|_{L_\infty(W)}\|u-u'\|_{C^\tau(X_+)}.$$

Proof. We choose $V := B(v_0, \delta/4)$. Since all functions in V are continuous, we obtain:

$$\operatorname{im} V := \cup_{u\in V} \operatorname{im} u \subset W'' := \{z \in \mathbb{C} : |z| < \|v_0\|_{C^\tau} + \delta/4, \;\; \Re z > \delta(1-1/4)\}.$$

Furthermore, we define $W' := \{z \in \mathbb{C} : |z| < \|v_0\|_{C^\tau} + \delta/2, \;\; \Re z > \delta(1-1/2)\}$. By definition, some distance between the boundary of W'' and the boundary of W' exists, i.e., $d(\partial W'', \partial W') \geq \delta/4$. Therefore, $|\eta - u(x)| \geq \delta/4$ for all $u \in V$, $\eta \in \partial W'$ and $x \in X_+$. It is well-known that such a lower bound implies that $(\eta-u)^{-1} \in C^\tau(X_+)$. Moreover, the following estimate holds:

$$\|(\eta-u)^{-1}\|_{C^\tau(X_+)} \leq 16/\delta^2\|\eta-u\|_{C^\tau(X_+)} \leq 16/\delta^2(2\|v_0\|_{C^\tau(X_+)} + 3\delta/4) =: S.$$

We can estimate the length of the boundary: $|\partial W'| \leq 2\pi(\|v_0\|_{C^\tau(X)} + \delta/2) := 2\pi L$. For all $u \in V$ and $x \in X_+$, we obtain the following identity from the Cauchy integral representation:

$$f(u(x)) = \frac{1}{2\pi i}\int_{\partial W'} f(\eta)(\eta - u(x))^{-1}\, d\eta.$$

Thus, we obtain the first estimate $\|f(u)\|_{C^\tau(X_+)} \leq LS\|f\|_{H^\infty(W)}$. For $u, u' \in V$, we use the resolvent identity to obtain:

$$f(u(x)) - f(u'(x)) = \frac{u'(x) - u(x)}{2\pi i}\int_{\partial W'} f(\eta)(\eta - u(x))^{-1}(\eta - u'(x))^{-1}\, d\eta$$

We can estimate the $C^\tau(X_+)$-norm of the integral as before. Therefore, the $C^\tau(X_+)$-norm of the left hand side can be estimated as stated in the lemma. □

According to the assumptions of Theorem 6.1 and Embedding (6.4), the function v_0 satisfies the assumptions of Lemma 6.2. We choose a neighbourhood V of v_0, according to Lemma 6.2. Additionally, we choose a neighbourhood U of zero in E_q such that the image of $U + v_0$ under the Embedding (6.4) belongs to V. For $i \in \{1, 2\}$, Lemma 6.2 applies to $f(z) := z^{m-i}$. Therefore:

$$\|(u+v_0)^{m-i}\|_{C^\tau(X_+)} \leq C \quad \text{for all} \quad u \in U \quad \text{and} \tag{6.5}$$

$$\|(u+v_0)^{m-i} - (u'+v_0)^{m-i}\|_{C^\tau(X_+)} \leq C\|u-u'\|_{E_q} \quad \text{for all} \quad u, u' \in U. \tag{6.6}$$

We recall $C^\tau(X_+) \hookrightarrow \mathcal{B}(E_0)$ as a multiplication operator. Thus, Estimate 6.6 implies

$$\|A(u) - A(u')\|_{\mathcal{B}(E_1;E_0)} \leq m\|(u+v_0)^{m-1} - (u'+v_0)^{m-1}\|_{\mathcal{B}(E_0)}\|\Delta_{g,T}\|_{\mathcal{B}(E_1;E_0)} \leq C\|u-u'\|_{E_q}$$

for all $u, u' \in U$. Therefore, Assumption (i) in Theorem 3.13 is satisfied. Next, we verify Assumption (ii). To this end, we define $h(u) = (u+v_0)^{m-2}|\nabla(u+v_0)|_g^2$ and observe:

$$\begin{aligned} h(u) - h(u') =& (u+v_0)^{m-2}|\nabla(u+v_0)|_g^2 - (u'+v_0)^{m-2}|\nabla(u'+v_0)|_g^2 \\ =& \left((u+v_0)^{m-2} - (u'+v_0)^{m-2}\right)|\nabla(u+v_0)|_g^2 \\ &+ (u'+v_0)^{m-2}\left(|\nabla(u+v_0)|_g^2 - |\nabla(u'+v_0)|_g^2\right) \\ =& \left((u+v_0)^{m-2} - (u'+v_0)^{m-2}\right)|\nabla(u+v_0)|_g^2 \\ &+ (u'+v_0)^{m-2}\langle\nabla(u-u'), \nabla(u+v_0)\rangle_g \\ &+ (u'+v_0)^{m-2}\langle\nabla(u'+v_0), \nabla(u-u')\rangle_g. \end{aligned}$$

The assumption $1 > n/p + 2/q$ and the Embedding (6.4) imply that $E_q \hookrightarrow C^1(X_+)$ and $E_q \hookrightarrow H_p^1(X_+)$. Thus, for all $u, u' \in E_q$, the following estimate holds:

$$\|\langle\nabla_g u, \nabla_g u'\rangle_g\|_{E_0} = \|\langle\nabla_g u, \nabla_g u'\rangle_g\|_{L_p(X_+)} \leq \|u\|_{C^1(X_+)}\|u'\|_{H_p^1(X_+)} \leq \|u\|_{E_q}\|u'\|_{E_q}.$$

Therefore, for $u, u' \in U$, we obtain:

$$\begin{aligned} \||\nabla(u+v_0)|^2\|_{E_0} &\leq C, \\ \|\langle\nabla(u-u'), \nabla(u-v_0)\rangle_g\|_{E_0} &\leq C\|u-u'\|_{E_q}, \quad \text{and} \\ \|\langle\nabla(u'-v_0), \nabla(u-u')\rangle_g\|_{E_0} &\leq C\|u-u'\|_{E_q}. \end{aligned}$$

The Estimates (6.5), (6.6), and those above imply $\|h(u) - h(u')\|_{E_0} \leq C\|u-u'\|_{E_q}$. We obtain $\|((u-v_0)^{m-1} - (u-v_0)^{m-1})\Delta_g v_0\|_{E_0} \leq C\|u-u'\|_{E_q}$ for all $u, u' \in U$, from the assumption $v_0 \in H_p^2(X_+)$ and Estimate (6.6). Thus, $\|f(u) - f(u')\|_{E_0} \leq C\|u-u'\|_{E_q}$ for all $u, u' \in U$. In other words, Assumption (ii) is satisfied. Therefore, Theorem 3.13 can be applied to Problem (6.3) which completes the proof of Theorem 6.1.

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