On simple zeros of the Riemann zeta-function

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Zusammenfassung

Wir untersuchen die Verteilung einfacher Nullstellen der Riemannschen Zetafunktion.

Sei $L = \log T$ und $H \leq T$. Wir berechnen auf eine neue Art und Weise (alten Ideen von Atkinson und neuen Ideen von Jutila und Motohashi folgend) das quadratische Moment des Produktes von $F(s) = \zeta(s) + \frac{1}{L}\zeta'(s)$ und eines gewissen Dirichletpolynoms $A(s) = \sum_{n\leq M} \frac{a(n)}{n^s}$ der Länge $M = T^\theta$ mit $\theta < \frac{3}{8}$ in Nähe der kritischen Geraden: Ist R eine positive Konstante, $a = \frac{1}{2} - \frac{R}{L}$ und $a(n) = \mu(n)n^{a-\frac{1}{2}}\left(1 - \frac{\log n}{\log M}\right)$, so gilt

$$\begin{aligned}\int_T^{T+H} |AF(a+it)|^2 \, dt \quad &= \quad H\left(\frac{1}{2} + \frac{\theta}{6}\left(1 - R - \frac{1}{2R}\right) - \frac{1}{2R\theta}\left(1 + \frac{1}{R} + \frac{1}{2R^2}\right)\right. \\ &\quad \left. + e^{2R}\left(\frac{\theta}{12R} + \frac{1}{4R^3\theta}\right) + o(1)\right) + O\left(T^{\frac{1}{3}+\varepsilon}M^{\frac{4}{3}}\right).\end{aligned}$$

Hierin ist der Hauptterm wohlbekannt, aber der Fehlerterm wesentlich kleiner als bei anderen Ansätzen (z.B. $O\left(T^{\frac{1}{2}+\varepsilon}M\right)$). Bei einer bestimmten Wahl von R ergibt sich mit **Levinsons Methode**, daß ein positiver Anteil aller Zetanullstellen mit Imaginärteilen in $[T, T+H]$ auf der kritischen Geraden liegt und einfach ist, sofern $H \geq T^{0.591}$ (und eine bessere, aber kompliziertere Wahl von $A(s)$ erlaubt sogar $H \geq T^{0.552}$)!

Für noch kürzere Intervalle finden wir mit der **Methode von Conrey, Ghosh und Gonek**

$$\sum_{T<\gamma\leq T+H} \zeta'(\varrho) = \frac{HL^2}{4\pi} + O\left(HL + T^{\frac{1}{2}+\varepsilon}\right),$$

wobei über die nichtreellen Zetanullstellen $\varrho = \beta + i\gamma$ summiert wird. Also liegt in jedem Intervall $[T, T + T^{\frac{1}{2}+\varepsilon}]$ der Imaginärteil einer einfachen Nullstelle von $\zeta(s)$ bzw.

$$\sharp\left\{\varrho \ : \ T < \gamma \leq T + H, \ \zeta'(\varrho) \neq 0\right\} \gg HT^{-\frac{1}{2}-\varepsilon}.$$

Zusammen mit einer Dichteabschätzung für die Zetanullstellen mit Realteil $> \frac{1}{2}$ von Balasubramanian ergibt sich eine nichttriviale Einschränkung für die Realteile: Z.B. finden wir einfache Zetanullstellen $\varrho = \beta + i\gamma$ mit $T < \gamma \leq T + T^{0.55}$ und $\frac{1}{2} \leq \beta \leq \frac{41}{42} + \varepsilon$, wozu unser Ergebnis mit Levinsons Methode nicht fähig ist.

Schlagwörter: Riemannsche Zetafunktion, quadratisches Moment, Levinsons Methode.

Abstract

We investigate the distribution of simple zeros of the Riemann zeta-function.

Let $H \leq T$ and $L = \log T$. We calculate in a new way (following old ideas of Atkinson and new ideas of Jutila and Motohashi) the mean square of the product of $F(s) = \zeta(s) + \frac{1}{L}\zeta'(s)$ and a certain Dirichlet polynomial $A(s) = \sum_{n \leq M} \frac{a(n)}{n^s}$ of length $M = T^\theta$ with $\theta < \frac{3}{8}$ near the critical line: if R is a positive constant, $a = \frac{1}{2} - \frac{R}{L}$ and $a(n) = \mu(n) n^{a-\frac{1}{2}} \left(1 - \frac{\log n}{\log M}\right)$, then

$$\int_T^{T+H} |AF(a+it)|^2 \, dt = H\left(\frac{1}{2} + \frac{\theta}{6}\left(1 - R - \frac{1}{2R}\right) - \frac{1}{2R\theta}\left(1 + \frac{1}{R} + \frac{1}{2R^2}\right) + e^{2R}\left(\frac{\theta}{12R} + \frac{1}{4R^3\theta}\right) + o(1)\right) + O\left(T^{\frac{1}{3}+\varepsilon} M^{\frac{4}{3}}\right).$$

The main term is well known, but the error term is much smaller than the one obtained by other approaches (e.g. $O\left(T^{\frac{1}{2}+\varepsilon} M\right)$). It follows from **Levinson's method**, with an appropriate choice of R, that a positive proportion of the zeros of the zeta-function with imaginary parts in $[T, T+H]$ lie on the critical line and are simple, when $H \geq T^{0.591}$ (and by an optimal but more complicated choice of $A(s)$ even when $H \geq T^{0.552}$)!

For shorter intervals we find with the **Method of Conrey, Ghosh and Gonek**

$$\sum_{T < \gamma \leq T+H} \zeta'(\varrho) = \frac{HL^2}{4\pi} + O\left(HL + T^{\frac{1}{2}+\varepsilon}\right),$$

where the sum is taken over the nontrivial zeros $\varrho = \beta + i\gamma$ of $\zeta(s)$. So every interval $[T, T + T^{\frac{1}{2}+\varepsilon}]$ contains the imaginary part of a simple zero of $\zeta(s)$! Hence

$$\sharp \left\{\varrho \ : \ T < \gamma \leq T+H, \zeta'(\varrho) \neq 0\right\} \gg HT^{-\frac{1}{2}-\varepsilon}.$$

With a density result of Balasubramanian we get even a nontrivial restriction for the real parts: e.g. at the limit of our results with Levinson's method we find simple zeros $\varrho = \beta + i\gamma$ of the zeta-function with $T < \gamma \leq T + T^{0.55}$ and $\frac{1}{2} \leq \beta \leq \frac{41}{42} + \varepsilon$.

Keywords: Riemann zeta-function, mean square, Levinson's method.

* * *

The used notation is traditional as in the classical book of Titchmarsh [40]. So we write for example (k, l) for the **greatest common divisor** and $[k, l]$ for the **least common multiple** of the integers k, l. Every non standard writing or new definitions are given where they first occur; new defined notions are bold faced (as above). The symbol $\bullet$ marks the end of a proof. Implicit constants in $O(\ )$-terms may always depend on ε.

Basic analytical facts (e.g. the properties of the Gamma-function) are stated without citation; most of them can be found in [4]. Chapter 1 gives a brief introduction to the theory of the Riemann zeta-function; for more details see [40] and [26].

Contents

Chapter 1

Introduction

Let $s = \sigma + it, i = \sqrt{-1}$. Then the **Riemann zeta-function** is defined by

$$(1.1) \qquad \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad (\sigma > 1).$$

Since Euler's discovery (1737) of the analytic version of the unique prime factorization of the positive integers

$$(1.2) \qquad \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \in \mathbb{P}} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + ...\right) = \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^s}\right)^{-1} \qquad (\sigma > 1)$$

one has a close connection between the zeta-function and multiplicative number theory: because of the simple pole of $\zeta(s)$ in $s = 1$ the product does not converge. This means that there are infinitely many primes!
This fact is well known since Euclid's elementary proof, but the analytic access encodes much more arithmetic information as Riemann [35] showed: investigating $\zeta(s)$ as a function of a complex variable s (Euler deals only with real s), he discovered an analytic continuation to $\sigma > 0$ except for a single pole in $s = 1$ and a certain symmetry around $s = \frac{1}{2}$, the **functional equation**

$$(1.3) \qquad \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)$$

(a "real" version was conjectured and partially proved by Euler), such that $\zeta(s)$ is defined in the whole complex plane. From the Euler product (1.2) we deduce that there are no zeros in the halfplane $\sigma > 1$. Since the Gamma-function has no zeros at all, but poles at $s = 0, -1, -2, ...$, the functional equation (1.3) implies the existence of the **trivial zeros** of the zeta-function

$$\zeta(-2n) = 0 \qquad \text{for} \quad n \in \mathbb{N},$$

but no others in $\sigma < 0$. Since the poles of the Gamma-function are all simple the trivial zeros are also simple. **Nontrivial zeros** $\varrho = \beta + i\gamma$ can only occur in the **critical strip**

$0 \leq \sigma \leq 1$, but not on the real axis, which easily follows from the identity

$$\left(2^{1-s} - 1\right) \zeta(s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}$$

valid for $\sigma > 0$ (see [40], formula (2.12.4)). Since the zeta-function is real on the real axis we have by the reflection principle a further functional equation

$$(1.4) \qquad \zeta(\overline{s}) = \overline{\zeta(s)}.$$

Hence the nontrivial zeros lie symmetrically to the real axis and the **critical line** $\sigma = \frac{1}{2}$. There are infinitely many nontrivial zeros: define $N(T)$ as the number of zeros $\varrho = \beta + i\gamma$ with $0 \leq \beta \leq 1$, $0 < \gamma \leq T$ (counting multiplicities). Riemann conjectured and von Mangoldt proved the **Riemann-von Mangoldt-formula**

$$(1.5) \qquad N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T).$$

Riemann calculated the "first" zeros $\frac{1}{2} + i14.13...$, $\frac{1}{2} + i21.02...$, ... and stated the famous and yet unproved **Riemann hypothesis** that all zeros in the critical strip have real part $\frac{1}{2}$ or equivalently

$$\zeta(s) \neq 0 \quad \text{for} \quad \sigma > \frac{1}{2}.$$

The importance of the Riemann hypothesis lies in its connection with the distribution of primes: Riemann states an analogue to the **explicit formula**

$$(1.6) \qquad \Psi(x) := \sum_{p^k \leq x} \log p = x - \sum_{\varrho} \frac{x^\varrho}{\varrho} - \frac{\zeta'}{\zeta}(0) - \frac{1}{2} \log\left(1 - \frac{1}{x^2}\right),$$

where the summation is taken over the prime powers (in the case of an integer x the corresponding terms in the sums have to be halved); he works with the prime counting function $\pi(x) := \sum_{p \leq x} 1$, but for analytic reasons we prefer $\Psi(x)$. Hadamard and de la Vallée-Poussin showed that there are no zeros of $\zeta(s)$ "too close to $\sigma = 1$" (depending on t), but up to now no strip in $0 < \sigma < 1$ without zeros is known! Following Riemann's ideas and with new discoveries in complex analysis at hand they were able to prove (independently) the **prime number theorem** (1896)

$$\Psi(x) \sim x$$

or equivalently $\pi(x) \sim \frac{x}{\log x}$. The presently best known error term is

$$(1.7) \qquad \Psi(x) - x \ll x \exp\left(-C(\log x)^{\frac{3}{5}}(\log\log x)^{-\frac{1}{5}}\right)$$

due to Vinogradov and Korobov (1958). One may suggest by (1.6) and can show in fact for the error term in the prime number theorem

$$(1.8) \qquad \Psi(x) - x \ll x^\theta (\log x)^2 \qquad \Longleftrightarrow \qquad \zeta(s) \neq 0 \quad \text{in} \quad \sigma > \theta.$$

So by the symmetry of the nontrivial zeros the Riemann hypothesis states that the primes are distributed as uniformly as possible!

Many computations were done to find a counter example to the Riemann hypothesis: e.g. van de Lune, te Riele and Winter (1986) localized the first 1 500 000 001 zeros without exception on the critical line; moreover they all turned out to be simple! By observations like this it is conjectured, that all or at least almost all zeros of the zeta-function are simple. But, if $m(\varrho)$ denotes the multiplicity of the zero ϱ, it is only known that

$$m(\varrho) \ll \log |\gamma|,$$

which follows from the Riemann-von Mangoldt-formula (1.5).

Not only the vertical distribution of the zeros has arithmetical consequences, also their multiplicities: Cramér [11] showed, assuming the Riemann hypothesis,

$$\frac{1}{\log X} \int_1^X \left(\frac{\Psi(x) - x}{x} \right)^2 dx \sim \sum_{\varrho} \left| \frac{m(\varrho)}{\varrho} \right|^2 ,$$

where the sum is taken over distinct zeros; this mean value is minimal iff all zeros are simple. That would mean that the error term in the prime number theorem is on average much smaller than one may suggest by (1.8). This and further relations were elaborated by Mueller [33].

Another arithmetical correspondence combines Riemann's hypothesis and the simplicity of all zeros of $\zeta(s)$: Mertens conjectured for the Moebius transform of the coefficients of the Dirichlet series of $\frac{1}{\zeta(s)}$

$$\sum_{n \leq x} \mu(n) \ll x^{\frac{1}{2}} \tag{1.9}$$

with an implicit constant ≤ 1, where $\mu(n) = (-1)^r$ if n is squarefree and the product of r different primes, or otherwise $\mu(n) = 0$. This **Mertens hypothesis** was disproved by Odlyzko and te Riele (1983), but it is still open, whether (1.9) holds for some implicit constant > 1. This would imply Riemann's hypothesis and additionally that all zeros are simple (see [23])!

But what is known about the distribution of nontrivial zeros and their multiplicities? Hardy (1914) investigated the function

$$Z(t) := \pi^{-\frac{it}{2}} \frac{\Gamma\left(\frac{1}{4} + \frac{it}{2}\right)}{\left|\Gamma\left(\frac{1}{4} + \frac{it}{2}\right)\right|} \zeta\left(\frac{1}{2} + it\right),$$

which is real for real t. Since $Z(t)$ has exactly the same real zeros as $\zeta\left(\frac{1}{2} + it\right)$ it is possible to localize by the mean value theorem odd order zeros of the zeta-function on the critical line. In that way Hardy was the first to show that there are infinitely many zeros on the critical line. With the new idea of a "mollifier" (see also Chapter 2) Selberg was able to find even a positive proportion of all zeros: if $N_0(T)$ denotes the number of zeros ϱ of $\zeta(s)$ on the critical line with $0 < \gamma \le T$, he found

$$\liminf_{T\to\infty} \frac{N_0(T+H) - N_0(T)}{N(T+H) - N(T)} > 0$$

for $H \ge T^{\frac{1}{2}+\varepsilon}$. Karatsuba improved this result to $H \ge T^{\frac{27}{82}+\varepsilon}$ by technical refinement. The proportion is very small, about 10^{-6} as Min calculated; a later refinement by Žuravlev, using ideas of Siegel, gives after all $\frac{2}{21}$ if $H = T$ (cf. [26], p.36). However, the localized zeros are not necessarily simple!

Littlewood (1924) investigated the distribution of the zeros using an integrated version of the argument principle (see [40], §9.9): let $f(s)$ be regular in and upon the boundary of the rectangle $\mathcal{R}$ with vertices $a,\ a+iT,\ b+iT,\ b$ and not zero on $\sigma = b$, and let $\nu(\sigma, T)$ denote the number of zeros of $f(s)$ inside the rectangle with real part $> \sigma$ including those with imaginary part $= T$ but not $= 0$. Then **Littlewood's Lemma** states that

$$\int_{\partial\mathcal{R}} \log f(s)\, ds = -2\pi i \int_a^b \nu(\sigma, T)\, d\sigma.$$

Now let $N(\sigma, T)$ denote the number of zeros counted by $N(T)$, but only those with real part $> \sigma$. Then Littlewood found

$$N(\sigma, T) \ll T = o(N(T))$$

for every fixed $\sigma > \frac{1}{2}$. So "most" of the zeros lie arbitrarily "close to" the critical line!

Speiser [36] showed (discussing the Riemann surface of the zeta-function) that Riemann's hypothesis is equivalent to the nonvanishing of $\zeta'(s)$ in the strip $0 < \sigma < \frac{1}{2}$.
On the critical line we have another relation between the zeros of $\zeta(s)$ and $\zeta'(s)$: we rewrite the functional equation (1.3) as

$$\zeta(s) = \chi(s)\zeta(1-s), \tag{1.10}$$

where

$$\chi(s) := \frac{(2\pi)^s}{2\Gamma(s)\cos\frac{\pi s}{2}}$$

is a meromorphic function with only real zeros and poles. By Stirling's formula

$$\chi(s) = \left(\frac{t}{2\pi}\right)^{\frac{1}{2}-\sigma} \exp\left(i\left(\frac{\pi}{4} - t\log\frac{t}{2\pi e}\right)\right)\left(1 + O\left(\frac{1}{t}\right)\right) \qquad (t \ge 1). \tag{1.11}$$

Differentiation of (1.10) gives

$$\zeta'(s) = \chi'(s)\zeta(1-s) - \chi(s)\zeta'(1-s). \tag{1.12}$$

This shows by the reflection principle that every zero of the derivative of the zeta-function on the critical line is a multiple zero of the zeta-function itself:

$$\zeta'\left(\frac{1}{2} + it\right) = 0 \quad \Rightarrow \quad \zeta\left(\frac{1}{2} + it\right) = 0. \tag{1.13}$$

Thus the nonvanishing of $\zeta'(s)$ in $0 < \sigma \leq \frac{1}{2}$ would imply Riemann's hypothesis and that all zeros of the zeta-function are simple!

Levinson and Montgomery [30] gave a quantitative version of Speiser's surprising result applying the argument principle to the logarithmic derivative of (1.10) (for details of the following see the notes of §10 by Heath-Brown in [40]): if $\varrho' = \beta' + i\gamma'$ denotes the zeros of $\zeta'(s)$, then

$$\begin{aligned} (1.14) \qquad & \sharp\left\{\varrho' \,:\, -1 < \beta' < \frac{1}{2},\ T < \gamma' \leq T + H\right\} \\ &= \sharp\left\{\varrho \,:\, 0 < \beta < \frac{1}{2},\ T < \gamma \leq T + H\right\} + O(\log(T+H)). \end{aligned}$$

So there are as many nontrivial zeros of the zeta-function as of its derivative in the left half of the critical strip (apart from a small hypothetical error).
This plays an extremely important role in Chapter 2, so we give now a

Sketch of the proof of (1.14). Let T_1 and T_2 with $T_1 < T_2$ such that neither $\zeta(s)$ nor $\zeta'(s)$ vanishes for $t = T_j, -1 \leq \sigma \leq \frac{1}{2}$ (this may be assumed since the zeros of a non constant meromorphic function have no limit point). We observe the change in argument in $\frac{\zeta'}{\zeta}(s)$ around the rectangle $\mathcal{R}$ with vertices $\frac{1}{2} - \delta + iT_1,\ \frac{1}{2} - \delta + iT_2, -1 + iT_2$ and $-1 + iT_1$ for a small positive δ. Logarithmic differentiation of the functional equation (1.10) gives

$$(1.15) \qquad \frac{\zeta'}{\zeta}(s) = \frac{\chi'}{\chi}(s) - \frac{\zeta'}{\zeta}(1-s),$$

where by (1.11) or Stirling's formula

$$(1.16) \qquad \frac{\chi'}{\chi}(s) = -\log\frac{t}{2\pi} + O\left(\frac{1}{t}\right) \qquad (t \geq 1).$$

In particular we have

$$\frac{\zeta'}{\zeta}(-1+it) = -\log\frac{t}{2\pi} + O(1)$$

(since $\frac{\zeta'}{\zeta}(2+it)$ is given by logarithmic differentiation of the Euler product (1.2) as an absolutely convergent Dirichlet series). Thus on $\sigma = -1$ the change in argument in $\frac{\zeta'}{\zeta}(s)$ is $\ll 1$. For the other vertical edge we first get by (1.15) and (1.16)

$$(1.17) \qquad \mathrm{Re}\,\frac{\zeta'}{\zeta}\left(\frac{1}{2} + it\right) \leq -1 \qquad (t \neq \gamma)$$

for t large enough (since for $\sigma = \frac{1}{2}$ the symmetry between s and $1-s$ is just complex conjugation). In the neighbourhood of a zero $\varrho = \frac{1}{2} + i\gamma$ of multiplicity $m(\varrho)$ we have obviously

$$(1.18) \qquad \frac{\zeta'}{\zeta}(s) = \frac{m(\varrho)}{s-\varrho} + c + O(|s-\varrho|).$$

With $s = \frac{1}{2} + it \to \varrho$ follows Re $c \leq -1$ by (1.17), and with $s = \frac{1}{2} - \delta + it$ we get

$$\mathrm{Re}\ \frac{\zeta'}{\zeta}(s) = -\frac{m(\varrho)\delta}{|s-\varrho|^2} + \mathrm{Re}\ c + O(|s-\varrho|) \leq -\frac{1}{2}$$

for $|s - \varrho|$ small enough. Hence we have for a sufficiently small δ (that depends only on T_2)

$$\mathrm{Re}\ \frac{\zeta'}{\zeta}\left(\frac{1}{2} - \delta + it\right) \leq -\frac{1}{2}$$

for $T_1 \leq t \leq T_2$. It follows immediately that arg $\frac{\zeta'}{\zeta}(s)$ varies only by $\ll 1$ along this vertical line. Moreover it is well known that $\zeta(s)$ and $\zeta'(s)$ are both $\ll t^C$ for $\sigma \geq -2$ with a certain constant C (by the Phragmén-Lindelöf-principle). By the functional equation (1.10), (1.11) and (1.16) we also have

$$\begin{aligned} \zeta(-1+iT_j) &\gg T_j^{\frac{3}{2}}, \\ \zeta'(-1+iT_j) &\gg T_j^{\frac{3}{2}} \log T_j. \end{aligned}$$

Thus we find that $\arg \zeta(s)$ and $\arg \zeta'(s)$ both vary by $\ll \log T_2$ along the horizontal edges of $\mathcal{R}$ by use of Jensen's formula in a standard way (see §9.4 in [40]). So the total change of argument in $\frac{\zeta'}{\zeta}(s)$ around the rectangle $\mathcal{R}$ is $\ll \log T_2$. Now with the argument principle follows immediately (1.14) (with (1.5) the restricted T_1 resp. T_2 can obviously be replaced by arbitrary T resp. $T + H$ with an error $\ll \log(T + H)$).•

A similar "argument" holds for $\frac{\zeta'}{\zeta}(s) - s$. Therefore we have instead of (1.14)

$$\begin{aligned} &\sharp\left\{s \ :\ \frac{\zeta'}{\zeta}(s) = s,\ -1 < \sigma < \frac{1}{2},\ T < t \leq T + H\right\} \\ &= \ \sharp\left\{\varrho \ :\ 0 < \beta < \frac{1}{2},\ T < \gamma \leq T + H\right\} + O(\log(T + H)), \end{aligned}$$

too. So one could localize almost all nontrivial zeros of the zeta-function on the critical line by showing that the logarithmic derivative of the zeta function has no fixed points in the left half of the critical strip! A first approach of function iteration to Riemann's hypothesis was discussed by Hinkkanen [19]. But (because of Newton approximation) he prefers to iterate $\varepsilon\frac{\xi}{\xi'}(s) - s$, where $\xi(s) := s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$.

Moreover, Levinson and Montgomery find also

$$\sum_{0<\gamma'\leq T} \left(\beta' - \frac{1}{2}\right) \sim \frac{T}{2\pi} \log L,$$

so that frequently $\beta' > \frac{1}{2}$, and obviously the zeros of $\zeta'(s)$ are asymmetrically distributed (what one may expect by Speiser's result)!

Chapter 2

Levinson's Method

The correspondence (1.14) between the distributions of the zeros of $\zeta(s)$ and its derivative is the starting point of Levinson's method. Let $H \leq T$ and $L = \log T$. If we write N for the left hand side of (1.14), it follows by the symmetry of the zeros of $\zeta(s)$ that

$$N_0(T+H) - N_0(T) = N(T+H) - N(T) - 2N + O(L). \tag{2.1}$$

So applying Littlewood's Lemma to $\zeta'(1-s)$ one may hope to get a good estimate of $N_0(T+H) - N_0(T)$. But with regard to (1.12) and (1.16) it is convenient to replace $\zeta'(1-s)$ by the approximation

$$F(s) := \zeta(s) + \frac{1}{L}\zeta'(s)$$

(see [5]), multiplied with a suitable Dirichlet polynomial $A(s)$ to "mollify" the wild behaviour of $\zeta'(1-s)$ in the critical strip. We take

$$A(s) := \sum_{n \leq M} \frac{a(n)}{n^s}$$

with $M = T^\theta$ and coefficients

$$a(n) := \mu(n) n^{a-\frac{1}{2}} \left(1 - \frac{\log n}{\log M}\right), \tag{2.2}$$

where a is a certain constant chosen later (then the product AF approximates $\frac{\zeta'}{\zeta}(1-s)$ and logarithmic derivatives have "small" order; see for that the theorem on the logarithmic derivative in [34], §IX.5). Since $AF(s)$ can only have more zeros than $F(s)$, this leads to

$$\begin{aligned} 2\pi\left(\frac{1}{2} - a\right) N &\leq \int_T^{T+H} \log |AF(a+it)|\, dt + O(L) \\ &\leq \frac{H}{2} \log \frac{1}{H} \int_T^{T+H} |AF(a+it)|^2\, dt + O(L) \end{aligned}$$

(we write here and in the sequel $AF(s)$ instead of $A(s)F(s)$). If one takes $a := \frac{1}{2} - \frac{R}{L}$ with a positive constant R chosen later, then by (2.1)

$$\frac{N_0(T+H) - N_0(T)}{N(T+H) - N(T)} \geq 1 - \frac{1}{R} \log \frac{1}{H} \int_T^{T+H} |AF(a+it)|^2\, dt + o(1). \tag{2.3}$$

Using this with $M = T^{\frac{1}{2}-\varepsilon}, H = TL^{-20}$ and $R = 1.3$ Levinson [29] localized more than one third of the nontrivial zeros of the zeta-function on the critical line, which is much more than Selberg's approach gives!

Moreover, Heath-Brown [17] and Selberg (unpublished) discovered that those localized zeros are all simple: let $N_0^{(r)}(T)$ denote the number of zeros $\varrho = \frac{1}{2} + i\gamma$ of multiplicity r with $0 < \gamma \leq T$. By (1.13) we have

$$\sharp\left\{\varrho' = \frac{1}{2} + i\gamma' \ : \ T < \gamma' \leq T + H\right\} = \sum_{r=2}^{\infty}(r-1)\left(N_0^{(r)}(T+H) - N_0^{(r)}(T)\right).$$

Hence we get instead of (2.1)

$$\begin{aligned} &N_0^{(1)}(T+H) - N_0^{(1)}(T) - \sum_{r=3}^{\infty}(r-2)\left(N_0^{(r)}(T+H) - N_0^{(r)}(T)\right) \\ = \ & N(T+H) - N(T) - 2\left(N + \sharp\left\{\varrho' = \frac{1}{2} + i\gamma' \ : \ T < \gamma' \leq T + H\right\}\right) + O(L). \end{aligned}$$

So we may replace (2.3) even by

$$\frac{N_0^{(1)}(T+H) - N_0^{(1)}(T)}{N(T+H) - N(T)} \geq 1 - \frac{1}{R}\log\frac{1}{H}\int_T^{T+H} |AF(a+it)|^2 \, dt + o(1). \tag{2.4}$$

By optimizing the technique Levinson himself and others improved the proportion $\frac{1}{3}$ sligthly (see [5]), but more recognizable is Conrey's new approach [6] to Levinson's method using Kloosterman sums. By that he was able to choose a mollifier of length $M = T^{\frac{4}{7}-\varepsilon}$ to show that more than two fifths of the zeros are simple and on the critical line! The use of longer mollifiers leads to larger proportions (as one can easily deduce from the asymptotic mean value in Theorem 2.1). But in general there exists no asymptotic formula for the mean square of $A\zeta(s)$, where $A(s)$ is an arbitrary Dirichlet polynomial of length T^θ with $\theta > 1$ (perhaps not even for $\theta \geq \frac{1}{2}$) as Balasubramanian, Conrey and Heath-Brown [3] showed. Farmer [13] observed that in the special case of Dirichlet polynomials given by (2.2) one could expect to have asymptotic formulas valid for all $\theta > 0$ ($\theta = \infty$ **-conjecture**). If it is possible to take mollifiers of infinite length, then almost all zeros lie on the critical line and are simple!

Levinson evaluated the integral in (2.4) using the **approximative functional equation**

$$\zeta(s) = \sum_{n\leq x}\frac{1}{n^s} + \chi(s)\sum_{n\leq y}\frac{1}{n^{1-s}} + O\left(x^{-\sigma} + |t|^{\frac{1}{2}-\sigma-1}y^{\sigma-1}\right),$$

valid for $0 < \sigma < 1$ and $2\pi xy = t$, which produces an error term $O\left(T^{\frac{1}{2}+\varepsilon}M\right)$.
To avoid errors of this order, we will calculate the integral in a different way combining ideas and methods of Atkinson, Jutila and Motohashi. Of course, we get the same main term as Levinson. So this leads not to an improvement of his result concerning the long interval, but with our error term we obtain better results for **short intervals** $[T, T+H]$ (i.e. $\limsup_{T\to\infty}\frac{\log H}{\log T} < 1$):

Theorem 2.1 *For $\theta < \frac{3}{8}$ we have*

$$\int_T^{T+H} |AF(a+it)|^2\, dt = H\left(\frac{1}{2}+\frac{\theta}{6}\left(1-R-\frac{1}{2R}\right)-\frac{1}{2R\theta}\left(1+\frac{1}{R}+\frac{1}{2R^2}\right)\right.$$
$$\left. +e^{2R}\left(\frac{\theta}{12R}+\frac{1}{4R^3\theta}\right)+o(1)\right)+O\left(T^{\frac{1}{3}+\varepsilon}M^{\frac{4}{3}}\right).$$

By (2.4) we find (after some computation) with the choice $R = 2.495$ and $\theta = 0.193$:

Corollary 2.2 *For $H \geq T^{0.591}$ a positive proportion of the zeros of the zeta-function with imaginary parts in $[T, T+H]$ lie on the critical line and are simple!*

Levinson's approach leads only to positive proportions in intervals of length $H \geq T^{0.693}$. But in both cases we observe the phenomenon that only short mollifiers give positive proportions in short intervals!

One can work with more complicated coefficients than (2.2) and with $\zeta(s) + \frac{\lambda}{L}\zeta'(s)$ instead of $F(s)$, where λ is optimally chosen later. Then one obtains even a positive proportion using mollifiers with $\theta = 0.164$ (as Farmer [14] showed). This leads to

$$\liminf_{T\to\infty} \frac{N_0^{(1)}(T+H) - N_0^{(1)}(T)}{N(T+H) - N(T)} > 0$$

whenever $H \geq T^{0.552}$!

Now we give the

2.1 Proof of Theorem 2.1

Before we start we give a sketch of the proof. Let

$$\int_T^{T+H} |AF(a+it)|^2\, dt = I(T,H) + E(T,H), \tag{2.5}$$

where $I(T,H)$ denotes the main term and $E(T,H)$ the error term. We are following Atkinson's approach to the mean square of the zeta-function on the critical line (or Motohashi's generalization to the mean square of the zeta-function multiplied with a Dirichlet polynomial): we split the integrand in the range where it is given as a product of Dirichlet series into its diagonal and nondiagonal terms (§2.1.1). In the nondiagonal terms we isolate certain parts that, as all diagonal terms, give contributions to the main part. Since these terms are analytic on the line of integration, we can calculate $I(T,H)$ very easily only by integrating certain functions involving the Gamma-function and evaluating the zeta-function near its pole (§2.1.2). Unfortunately, we need an analytic continuation of the remaining nondiagonal terms, which produces the error term, to the line of integration. After a certain transformation (§2.1.3), we are able to find such an analytic continuation by use of a Voronoi-type-formula (§2.1.6). With a special averaging technique due to Jutila (§2.1.8) we bound $E(T,H)$ by estimates of exponential integrals (§2.1.9).

2.1.1 Decomposition of the integrand

We have to integrate

$$
\begin{aligned}
(2.6)\qquad |AF(a+it)|^2 &= AF(z)AF(2a-z)\\
&= A\zeta(z)A\zeta(2a-z)+\frac{1}{L}\Big(A\zeta'(z)A\zeta(2a-z)\\
&\quad +A\zeta(z)A\zeta'(2a-z)\Big)+\frac{1}{L^2}A\zeta(z)A\zeta(2a-z)
\end{aligned}
$$

with $z=a+it$ along $T<t\le T+H$. First we try to find a more suitable expression of this: for $b,c\in \mathbb{N}_0 := \mathbb{N}\cup\{0\}$ and $\mathrm{Re}\ u, \mathrm{Re}\ v>1$ we have (writing $\sum_{k,l}$ for the double sum $\sum_k\sum_l$)

$$
\begin{aligned}
A\zeta^{(b)}(u)A\zeta^{(c)}(v) &= (-1)^{b+c}\sum_{k,l\le M}\sum_{m,n}\frac{a(k)a(l)}{(mk)^u(nl)^v}(\log m)^b(\log n)^c\\
&= \sum_{k,l\le M}\sum_{h=1}^{\infty}\frac{a(k)a(l)}{(h[k,l])^{u+v}}\left(\log\frac{k}{h[k,l]}\right)^b\left(\log\frac{l}{h[k,l]}\right)^c\\
&\quad +f(u,b;v,c;1)+f(v,c;u,b;1),
\end{aligned}
$$

where

$$
\begin{aligned}
&f(u,b;v,c;w(k,l))\\
&:= \sum_{m,n}\sum_{\substack{k,l\le M\\ m\equiv 0 \bmod k\\ m+n\equiv 0 \bmod l}}\frac{a(k)a(l)}{m^u(m+n)^v}w(k,l)\left(\log\frac{k}{m}\right)^b\left(\log\frac{l}{m+n}\right)^c
\end{aligned}
$$

with an arbitrary function $w(k,l)$. Define

$$
\begin{aligned}
S_0(u,v) &= \sum_{k,l\le M}\frac{a(k)a(l)}{[k,l]^{u+v}},\\
S_1(u,v) &= \sum_{k,l\le M}\frac{a(k)a(l)}{[k,l]^{u+v}}\log\frac{[k,l]}{k},\\
S_2(u,v) &= \sum_{k,l\le M}\frac{a(k)a(l)}{[k,l]^{u+v}}\log\frac{[k,l]}{k}\log\frac{[k,l]}{l}.
\end{aligned}
$$

Then we have

$$
(2.7)\qquad A\zeta(u)A\zeta(v)=\zeta(u+v)S_0(u,v)+f(u,0;v,0;1)+f(v,0;u,0;1).
$$

This gives an analytic continuation of $f(u,b;v,c;w(k,l))$ (as in [1] or [21], §15.2): since the Dirichlet polynomial $A(s)$ may be omitted from the following considerations, we observe with integration by parts

$$
\begin{aligned}
\sum_{n=1}^{\infty}(m+n)^{-v} &= \int_1^{\infty}(m+x)^{-v}\,d[x]\\
&= \frac{m^{1-v}}{v-1}-\frac{1}{2}m^{-v}+O\left(|v|^2 m^{-\mathrm{Re}\,v-1}\right).
\end{aligned}
$$

Hence we have for $k = l = 1$ in $f(u, 0; v, 0; 1)$

$$\begin{aligned}\sum_{m,n} m^{-u}(m+n)^{-v} &= \frac{1}{v-1}\sum_{m=1}^{\infty} m^{1-u-v} - \frac{1}{2}\sum_{m=1}^{\infty} m^{-u-v} \\ &\quad + O\left(|v|^2 \sum_{m=1}^{\infty} m^{-\mathrm{Re}\,(u+v)-1}\right),\end{aligned}$$

and therefore

$$\sum_{m,n} m^{-u}(m+n)^{-v} - \frac{1}{v-1}\zeta(u+v-1) + \frac{1}{2}\zeta(u+v)$$

is regular for $\mathrm{Re}\,(u+v) > 0$. So (2.7) holds by analytic continuation for u, v with $\mathrm{Re}\, u, \mathrm{Re}\, v \geq 0$ unless $u, v = 1$ or $u + v = 1, 2$. Analogously we find for the same range

$$\begin{aligned}(2.8)\qquad A\zeta'(u)A\zeta(v) &= \zeta'(u+v)S_0(u,v) - \zeta(u+v)S_1(u,v) \\ &\quad + f(u,1;v,0;1) + f(v,1;u,0;1), \\ (2.9)\qquad A\zeta'(u)A\zeta'(v) &= \zeta''(u+v)S_0(u,v) - 2\zeta'(u+v)S_1(u,v) + \zeta(u+v)S_2(u,v) \\ &\quad + f(u,1;v,1;1) + f(v,1;u,1;1).\end{aligned}$$

Now we transform f: with the well known formula (see [40], §2.4)

$$(2.10)\qquad \Gamma(s)N^{-s} = \int_0^{\infty} z^{s-1}e^{-Nz}\,dz \qquad\qquad (\sigma > 0)$$

we obtain

$$\begin{aligned}&f(u,0;v,0;1) \\ &= \frac{1}{\Gamma(u)\Gamma(v)}\sum_{k,l\leq M} a(k)a(l)\int_0^{\infty}\int_0^{\infty} x^{u-1}y^{v-1} \sum_{\substack{m,n \\ m\equiv 0 \bmod k \\ m+n\equiv 0 \bmod l}} e^{-mx-(m+n)y}\,dx\,dy.\end{aligned}$$

With

$$\frac{1}{l}\sum_{f=1}^{l}\exp\left(2\pi i\frac{Nf}{l}\right) = \begin{cases} 1 & , \quad N \equiv 0 \bmod l \\ 0 & , \quad N \not\equiv 0 \bmod l \end{cases}$$

(what we shall frequently use) we find

$$\begin{aligned}&\sum_{\substack{m,n \\ m\equiv 0 \bmod k \\ m+n\equiv 0 \bmod l}} e^{-mx-(m+n)y} \\ &= \sum_{\substack{m=1 \\ m\equiv 0 \bmod k}}^{\infty} e^{-m(x+y)}\sum_{n=1}^{\infty} e^{-ny}\frac{1}{l}\sum_{f=1}^{l}\exp\left(2\pi i\frac{(m+n)f}{l}\right) \\ &= \frac{1}{l}\sum_{f=1}^{l}\left(\exp\left(k(x+y) - 2\pi i\frac{kf}{l}\right) - 1\right)^{-1}\left(\exp\left(y - 2\pi i\frac{f}{l}\right) - 1\right)^{-1}.\end{aligned}$$

Hence with

$$\delta(f) := \begin{cases} 1 & , \quad kf \equiv 0 \bmod l \\ 0 & , \quad kf \not\equiv 0 \bmod l \end{cases}$$

it follows for $\mathrm{Re}\, u > 0, \mathrm{Re}\, v > 1$ and $\mathrm{Re}\, (u+v) > 2$ that

$$\begin{aligned} (2.11)\quad & f(u,0;v,0;1) \\ &= \frac{1}{\Gamma(u)\Gamma(v)} \sum_{k,l \le M} a(k)a(l) \Bigg\{ \frac{1}{l} \sum_{\substack{f=1 \\ \delta(f)=1}}^{l} \int_0^\infty y^{v-1} \left(\exp\left(y - 2\pi i \frac{f}{l} \right) - 1 \right)^{-1} \\ &\quad \times \int_0^\infty \frac{x^{u-1}}{k(x+y)}\, dx\, dy \quad + \quad \frac{1}{l} \sum_{f=1}^{l} \int_0^\infty y^{v-1} \left(\exp\left(y - 2\pi i \frac{f}{l} \right) - 1 \right)^{-1} \\ &\quad \times \int_0^\infty x^{u-1} \left(\left(\exp\left(k(x+y) - 2\pi i \frac{kf}{l} \right) - 1 \right)^{-1} - \frac{\delta(f)}{k(x+y)} \right) dx\, dy \Bigg\} \\ &=: \; F_1(u,0;v,0;1) \; + \; F_2(u,0;v,0;1). \end{aligned}$$

By the well known formula (see [22], formula (2.11))

$$(2.12) \qquad \int_0^\infty \frac{x^{u-1}}{x+y}\, dx \quad = \quad y^{u-1}\Gamma(u)\Gamma(1-u) \qquad (0 < \mathrm{Re}\, u < 1),$$

valid for $y > 0$, we obtain

$$\begin{aligned} & F_1(u,0;v,0;1) \\ &= \frac{\Gamma(1-u)}{\Gamma(v)} \sum_{k,l \le M} \frac{a(k)a(l)}{kl} \int_0^\infty y^{u+v-2} \sum_{\substack{f=1 \\ kf \equiv 0 \bmod l}}^{l} \left(\exp\left(y - 2\pi i \frac{f}{l} \right) - 1 \right)^{-1} dy. \end{aligned}$$

Since $kf \equiv 0 \bmod l$ iff $f \equiv 0 \bmod \frac{l}{(k,l)}$ the inner sum equals

$$\begin{aligned} \sum_{g=1}^{(k,l)} \sum_{N=1}^{\infty} \exp\left(-N \left(y - 2\pi i \frac{g}{(k,l)} \right) \right) \quad &= \quad (k,l) \sum_{\substack{N=1 \\ N \equiv 0 \bmod (k,l)}}^{\infty} e^{-Ny} \\ &= \quad (k,l) \left(e^{y(k,l)} - 1 \right)^{-1}. \end{aligned}$$

Thus we get, substituting $y(k,l) \mapsto y$,

$$\begin{aligned} & F_1(u,0;v,0;1) \\ &= \frac{\Gamma(1-u)}{\Gamma(v)} \sum_{k,l \le M} a(k)a(l) \frac{(k,l)^{2-u-v}}{kl} \int_0^\infty y^{u+v-2} \left(e^y - 1 \right)^{-1} dy. \end{aligned}$$

Define now

$$\begin{aligned} K_0(u,v) \quad &= \quad \sum_{k,l \le M} a(k)a(l) \frac{(k,l)^{2-u-v}}{kl}, \\ K_1(u,v) \quad &= \quad \sum_{k,l \le M} a(k)a(l) \frac{(k,l)^{2-u-v}}{kl} \log \frac{k}{(k,l)}, \\ K_2(u,v) \quad &= \quad \sum_{k,l \le M} a(k)a(l) \frac{(k,l)^{2-u-v}}{kl} \log \frac{k}{(k,l)} \log \frac{l}{(k,l)}. \end{aligned}$$

Then with

$$\Gamma\zeta(s) = \int_0^\infty z^{s-1}\,(e^z-1)^{-1}\,dz$$

(which is at first only defined for $\sigma > 1$, but leads to an analytic continuation of $\zeta(s)$ throughout the whole complex plane except for $s = 1$; see (2.10) and [40], §2.4) we obtain

$$F_1(u,0;v,0;1) = K_0(u,v)\frac{\Gamma(1-u)}{\Gamma(v)}\Gamma\zeta(u+v-1),$$

which gives an analytic continuation to $0 < \text{Re }u, \text{Re }v < 1$. If $F_2(u,0;v,0;1)$ also denotes the analytic continuation of the function F_2 given by (2.11) to a domain in the critical strip that includes the line of integration $\text{Re }u = \text{Re }v = a$, that we will give later (see §2.1.7), we have in the same range

$$f(u,0;v,0;1) = K_0(u,v)\frac{\Gamma(1-u)}{\Gamma(v)}\Gamma\zeta(u+v-1) + F_2(u,0;v,0;1). \tag{2.13}$$

Furthermore, one gets

$$\begin{aligned} f(u,1;v,0;1) &= \sum_{k,l\le M} a(k)a(l) \sum_{\substack{m,n \\ m\equiv 0 \bmod k \\ m+n\equiv 0 \bmod l}} \frac{\log k - \log m}{m^u(m+n)^v} \\ &= f(u,0;v,0;\log k) + \frac{\partial}{\partial u} f(u,0;v,0;1) \end{aligned}$$

with

$$\begin{aligned} &f(u,0;v,0;\log k) \\ &\quad = \frac{\Gamma(1-u)}{\Gamma(v)}\Gamma\zeta(u+v-1)\sum_{k,l\le M} a(k)a(l)\frac{(k,l)^{2-u-v}}{kl}\log k \\ &\qquad + F_2(u,0;v,0;\log k) \end{aligned}$$

and

$$\begin{aligned} &\frac{\partial}{\partial u} f(u,0;v,0;1) \\ &\quad = K_0(u,v)\left\{\frac{\Gamma(1-u)}{\Gamma(v)}(\Gamma'\zeta+\Gamma\zeta')(u+v-1) - \frac{\Gamma'(1-u)}{\Gamma(v)}\Gamma\zeta(u+v-1)\right\} \\ &\qquad + \frac{\Gamma(1-u)}{\Gamma(v)}\Gamma\zeta(u+v-1)\sum_{k,l\le M} a(k)a(l)\frac{(k,l)^{2-u-v}}{kl}\log\frac{1}{(k,l)} \\ &\qquad + \frac{\partial}{\partial u}F_2(u,0;v,0;1). \end{aligned}$$

Altogether we find

$$\begin{aligned} &f(u,1;v,0;1) \\ &\quad = K_1(u,v)\frac{\Gamma(1-u)}{\Gamma(v)}\Gamma\zeta(u+v-1) \\ &\qquad + K_0(u,v)\left\{\frac{\Gamma(1-u)}{\Gamma(v)}(\Gamma'\zeta+\Gamma\zeta')(u+v-1) - \frac{\Gamma'(1-u)}{\Gamma(v)}\Gamma\zeta(u+v-1)\right\} \\ &\qquad + F_2(u,0;v,0;\log k) + \frac{\partial}{\partial u}F_2(u,0;v,0;1). \end{aligned} \tag{2.14}$$

Finally, we have

$$\begin{aligned}
&f(u,1;v,1;1)\\
&= \sum_{k,l\le M} a(k)a(l) \sum_{\substack{m,n\\ m\equiv 0 \bmod k\\ m+n\equiv 0 \bmod l}} \frac{(\log k-\log m)(\log l-\log(m+n))}{m^u(m+n)^v}\\
&= f(u,0;v,0;\log k\log l)+\frac{\partial}{\partial u}f(u,0;v,0;\log l)+\frac{\partial}{\partial v}f(u,0;v,0;\log k)\\
&\quad +\frac{\partial}{\partial u}\frac{\partial}{\partial v}f(u,0;v,0;1)
\end{aligned}$$

with

$$\begin{aligned}
&f(u,0;v,0;\log k\log l)\\
&= \frac{\Gamma(1-u)}{\Gamma(v)}\Gamma\zeta(u+v-1)\sum_{k,l\le M} a(k)a(l)\frac{(k,l)^{2-u-v}}{kl}\log k\log l\\
&\quad +F_2(u,0;v,0;\log k\log l),
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial}{\partial u}f(u,0;v,0;\log l)\\
&= \left(\frac{\Gamma(1-u)}{\Gamma(v)}(\Gamma'\zeta+\Gamma\zeta')(u+v-1)-\frac{\Gamma'(1-u)}{\Gamma(v)}\Gamma\zeta(u+v-1)\right)\\
&\quad \times\sum_{k,l\le M} a(k)a(l)\frac{(k,l)^{2-u-v}}{kl}\log l\\
&\quad +\frac{\Gamma(1-u)}{\Gamma(v)}\Gamma\zeta(u+v-1)\sum_{k,l\le M} a(k)a(l)\frac{(k,l)^{2-u-v}}{kl}\log\frac{1}{(k,l)}\log l\\
&\quad +\frac{\partial}{\partial u}F_2(u,0;v,0;\log l),
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial}{\partial v}f(u,0;v,0;\log k)\\
&= \left(\frac{\Gamma(1-u)}{\Gamma(v)}(\Gamma'\zeta+\Gamma\zeta')(u+v-1)-\frac{\Gamma'}{\Gamma}(v)\frac{\Gamma(1-u)}{\Gamma(v)}\Gamma\zeta(u+v-1)\right)\\
&\quad \times\sum_{k,l\le M} a(k)a(l)\frac{(k,l)^{2-u-v}}{kl}\log k\\
&\quad +\frac{\Gamma(1-u)}{\Gamma(v)}\Gamma\zeta(u+v-1)\sum_{k,l\le M} a(k)a(l)\frac{(k,l)^{2-u-v}}{kl}\log\frac{1}{(k,l)}\log k\\
&\quad +\frac{\partial}{\partial v}F_2(u,0;v,0;\log k)
\end{aligned}$$

and

$$
\begin{aligned}
&\frac{\partial}{\partial u}\frac{\partial}{\partial v} f(u,0;v,0;1)\\
&= K_0(u,v)\left\{\frac{\Gamma'}{\Gamma}(v)\frac{\Gamma'(1-u)}{\Gamma(v)}\Gamma\zeta(u+v-1)-\left(\frac{\Gamma'}{\Gamma}(v)\frac{\Gamma(1-u)}{\Gamma(v)}+\frac{\Gamma'(1-u)}{\Gamma(v)}\right)\right.\\
&\quad\left.\times(\Gamma'\zeta+\Gamma\zeta')(u+v-1)+\frac{\Gamma(1-u)}{\Gamma(v)}(\Gamma''\zeta+2\Gamma'\zeta'+\Gamma\zeta'')(u+v-1)\right\}\\
&\quad+\left\{-\left(\frac{\Gamma'(1-u)}{\Gamma(v)}+\frac{\Gamma'}{\Gamma}(v)\frac{\Gamma(1-u)}{\Gamma(v)}\right)\Gamma\zeta(u+v-1)\right.\\
&\quad\left.+2\frac{\Gamma(1-u)}{\Gamma(v)}(\Gamma'\zeta+\Gamma\zeta')(u+v-1)\right\}\sum_{k,l\le M}a(k)a(l)\frac{(k,l)^{2-u-v}}{kl}\log\frac{1}{(k,l)}\\
&\quad+\frac{\Gamma(1-u)}{\Gamma(v)}\Gamma\zeta(u+v-1)\sum_{k,l\le M}a(k)a(l)\frac{(k,l)^{2-u-v}}{kl}\left(\log\frac{1}{(k,l)}\right)^2\\
&\quad+\frac{\partial}{\partial u}\frac{\partial}{\partial v}F_2(u,0;v,0;1).
\end{aligned}
$$

Thus

$$
\begin{aligned}
(2.15)\qquad &f(u,1;v,1;1)\\
&= K_2(u,v)\frac{\Gamma(1-u)}{\Gamma(v)}\Gamma\zeta(u+v-1)\\
&\quad+K_1(u,v)\left\{2\frac{\Gamma(1-u)}{\Gamma(v)}(\Gamma'\zeta+\Gamma\zeta')(u+v-1)\right.\\
&\quad\left.-\left(\frac{\Gamma'}{\Gamma}(v)\frac{\Gamma(1-u)}{\Gamma(v)}+\frac{\Gamma'(1-u)}{\Gamma(v)}\right)\Gamma\zeta(u+v-1)\right\}\\
&\quad+K_0(u,v)\left\{\frac{\Gamma'}{\Gamma}(v)\frac{\Gamma'(1-u)}{\Gamma(v)}\Gamma\zeta(u+v-1)\right.\\
&\quad-\left(\frac{\Gamma'(1-u)}{\Gamma(v)}+\frac{\Gamma'}{\Gamma}(v)\frac{\Gamma(1-u)}{\Gamma(v)}\right)(\Gamma'\zeta+\Gamma\zeta')(u+v-1)\\
&\quad\left.+\frac{\Gamma(1-u)}{\Gamma(v)}(\Gamma''\zeta+2\Gamma'\zeta'+\Gamma\zeta'')(u+v-1)\right\}\\
&\quad+F_2(u,0;v,0;\log k\log l)+\frac{\partial}{\partial u}F_2(u,0;v,0;\log l)\\
&\quad+\frac{\partial}{\partial v}F_2(u,0;v,0;\log k)+\frac{\partial}{\partial u}\frac{\partial}{\partial v}F_2(u,0;v,0;1).
\end{aligned}
$$

For $u+v=2a$ the sums $S_j(u,v)$ and $K_j(u,v)$ are independent of u and v, so let $S_j = S_j(z,2a-z)=S_j(2a-z,z)$ and $K_j=K_j(z,2a-z)=K_j(2a-z,z)$ for $j=0,1,2$. The error term $E(T,H)$ will arise from the integral over the functions F_2 and their derivatives with respect to u and v, evaluated at $u=z, v=2a-z$ and vice versa. Later we will bound those integrals in the same manner, so we shall denote (finite) sums of functions F_2 and their derivatives by $G(u,v)=G(z)$, respectively. Then we obtain with (2.13) in (2.7)

$$
\begin{aligned}
&A\zeta(z)A\zeta(2a-z)\\
&= S_0\zeta(2a)+K_0\left\{\frac{\Gamma(1-z)}{\Gamma(2a-z)}-\frac{\Gamma(1-2a+z)}{\Gamma(z)}\right\}\Gamma\zeta(2a-1)+G(z),
\end{aligned}
$$

and with (2.14) in (2.8)

$$
\begin{aligned}
&A\zeta'(z)A\zeta(2a-z)\\
&= S_0\zeta'(2a)-S_1\zeta(2a)\\
&\quad +K_0\left\{(\Gamma'\zeta+\Gamma\zeta')(2a-1)\left(\frac{\Gamma(1-z)}{\Gamma(2a-z)}+\frac{\Gamma(1-2a+z)}{\Gamma(z)}\right)\right.\\
&\quad \left.-\Gamma\zeta(2a-1)\left(\frac{\Gamma'(1-z)}{\Gamma(2a-z)}+\frac{\Gamma'(1-2a+z)}{\Gamma(z)}\right)\right\}\\
&\quad +K_1\left\{\frac{\Gamma(1-z)}{\Gamma(2a-z)}+\frac{\Gamma(1-2a+z)}{\Gamma(z)}\right\}\Gamma\zeta(2a-1)+G(z)\\
&= A\zeta(z)A\zeta'(2a-z)
\end{aligned}
$$

(after exchanging $u=z$ and $v=2a-z$ above). Finally, we get with (2.15) in (2.9)

$$
\begin{aligned}
&A\zeta'(z)A\zeta'(2a-z)\\
&= S_0\zeta''(2a)-2S_1\zeta'(2a)+S_2\zeta(2a)+K_0\left\{\left(\frac{\Gamma(1-z)}{\Gamma(2a-z)}+\frac{\Gamma(1-2a+z)}{\Gamma(z)}\right)\right.\\
&\quad\times(\Gamma''\zeta+2\Gamma'\zeta'+\Gamma\zeta'')(2a-1)-\left(\frac{\Gamma'}{\Gamma}(2a-z)\frac{\Gamma(1-z)}{\Gamma(2a-z)}\right.\\
&\quad\left.+\frac{\Gamma'}{\Gamma}(z)\frac{\Gamma(1-2a+z)}{\Gamma(z)}+\frac{\Gamma'(1-z)}{\Gamma(2a-z)}+\frac{\Gamma'(1-2a+z)}{\Gamma(z)}\right)(\Gamma'\zeta+\Gamma\zeta')(2a-1)\\
&\quad\left.+\left(\frac{\Gamma'}{\Gamma}(z)\frac{\Gamma'(1-2a+z)}{\Gamma(z)}+\frac{\Gamma'}{\Gamma}(2a-z)\frac{\Gamma'(1-z)}{\Gamma(2a-z)}\right)\Gamma\zeta(2a-1)\right\}\\
&\quad+K_1\left\{2\left(\frac{\Gamma(1-z)}{\Gamma(2a-z)}+\frac{\Gamma(1-2a+z)}{\Gamma(z)}\right)(\Gamma'\zeta+\Gamma\zeta')(2a-1)\right.\\
&\quad-\left(\frac{\Gamma'}{\Gamma}(2a-z)\frac{\Gamma(1-z)}{\Gamma(2a-z)}+\frac{\Gamma'}{\Gamma}(z)\frac{\Gamma(1-2a+z)}{\Gamma(z)}+\frac{\Gamma'(1-2a+z)}{\Gamma(z)}\right.\\
&\quad\left.\left.+\frac{\Gamma'(1-z)}{\Gamma(2a-z)}\right)\Gamma\zeta(2a-1)\right\}+K_2\left\{\frac{\Gamma(1-z)}{\Gamma(2a-z)}+\frac{\Gamma(1-2a+z)}{\Gamma(z)}\right\}\Gamma\zeta(2a-1)\\
&\quad+G(z).
\end{aligned}
$$

Collecting these expressions one obtains for the main term of (2.6)

$$
\begin{aligned}
(2.16)\quad &AF(z)AF(2a-z)\\
&= S_0\left\{\zeta(2a)+\frac{2}{L}\zeta'(2a)+\frac{1}{L^2}\zeta''(2a)\right\}-\frac{2S_1}{L}\left\{\zeta(2a)+\frac{1}{L}\zeta'(2a)\right\}\\
&\quad+\frac{S_2}{L^2}\zeta(2a)+K_0\left\{\left(\frac{\Gamma(1-z)}{\Gamma(2a-z)}-\frac{\Gamma(1-2a+z)}{\Gamma(z)}\right)\Gamma\zeta(2a-1)+\right.
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{2}{L}\left\{(\Gamma'\zeta+\Gamma\zeta')(2a-1)\left(\frac{\Gamma(1-z)}{\Gamma(2a-z)}+\frac{\Gamma(1-2a+z)}{\Gamma(z)}\right)\right.\\
&\left.- \Gamma\zeta(2a-1)\left(\frac{\Gamma'(1-z)}{\Gamma(2a-z)}+\frac{\Gamma'(1-2a+z)}{\Gamma(z)}\right)\right\}\\
&+ \frac{1}{L^2}\left\{\left(\frac{\Gamma(1-z)}{\Gamma(2a-z)}+\frac{\Gamma(1-2a+z)}{\Gamma(z)}\right)(\Gamma''\zeta+2\Gamma'\zeta'+\Gamma\zeta'')(2a-1)\right.\\
&- \left(\frac{\Gamma'}{\Gamma}(2a-z)\frac{\Gamma(1-z)}{\Gamma(2a-z)}+\frac{\Gamma'}{\Gamma}(z)\frac{\Gamma(1-2a+z)}{\Gamma(z)}\right.\\
&\left.+ \frac{\Gamma'(1-z)}{\Gamma(2a-z)}+\frac{\Gamma'(1-2a+z)}{\Gamma(z)}\right)(\Gamma'\zeta+\Gamma\zeta')(2a-1)\\
&\left.+ \left(\frac{\Gamma'}{\Gamma}(z)\frac{\Gamma'(1-2a+z)}{\Gamma(z)}+\frac{\Gamma'}{\Gamma}(2a-z)\frac{\Gamma'(1-z)}{\Gamma(2a-z)}\right)\Gamma\zeta(2a-1)\right\}\Bigg\}\\
&+\frac{K_1}{L}\left\{2\left(\frac{\Gamma(1-z)}{\Gamma(2a-z)}+\frac{\Gamma(1-2a+z)}{\Gamma(z)}\right)\Gamma\zeta(2a-1)+\frac{1}{L}\left(2\left(\frac{\Gamma(1-z)}{\Gamma(2a-z)}\right.\right.\right.\\
&\left.+ \frac{\Gamma(1-2a+z)}{\Gamma(z)}\right)(\Gamma'\zeta+\Gamma\zeta')(2a-1)-\left(\frac{\Gamma'}{\Gamma}(2a-z)\frac{\Gamma(1-z)}{\Gamma(2a-z)}\right.\\
&\left.\left.\left.+ \frac{\Gamma'}{\Gamma}(z)\frac{\Gamma(1-2a+z)}{\Gamma(z)}+\frac{\Gamma'(1-2a+z)}{\Gamma(z)}+\frac{\Gamma'(1-z)}{\Gamma(2a-z)}\right)\Gamma\zeta(2a-1)\right)\right\}\\
&+\frac{K_2}{L^2}\left\{\frac{\Gamma(1-z)}{\Gamma(2a-z)}+\frac{\Gamma(1-2a+z)}{\Gamma(z)}\right\}\Gamma\zeta(2a-1)+G(z).
\end{aligned}
$$

2.1.2 The main term

Before we integrate (2.16), we first calculate the factors that do not depend on z: from the functional equation (1.10) or

$$\Gamma\zeta(s) = \frac{(2\pi)^s}{2\cos\frac{\pi s}{2}}\,\zeta(1-s)$$

and its derivatives

$$(\Gamma'\zeta+\Gamma\zeta')(s) = \frac{(2\pi)^s}{2\cos\frac{\pi s}{2}}\left\{-\zeta'(1-s)+\zeta(1-s)\left(\log 2\pi+\frac{\pi}{2}\tan\frac{\pi s}{2}\right)\right\}$$

and

$$
\begin{aligned}
&(\Gamma''\zeta+2\Gamma'\zeta'+\Gamma\zeta'')(s)\\
&= \frac{(2\pi)^s}{2\cos\frac{\pi s}{2}}\left\{\zeta''(1-s)-2\zeta'(1-s)\left(\log 2\pi+\frac{\pi}{2}\tan\frac{\pi s}{2}\right)\right.\\
&\left.\quad+\zeta(1-s)\left(\left(\log 2\pi+\frac{\pi}{2}\tan\frac{\pi s}{2}\right)^2+\left(\frac{\pi}{2\cos\frac{\pi s}{2}}\right)^2\right)\right\},
\end{aligned}
$$

one easily gets

$$\Gamma\zeta(2a-1) = \frac{(2\pi)^{2a-1}}{2\cos\frac{\pi(2a-1)}{2}}\,\zeta(2-2a), \tag{2.17}$$

(2.18) $$(\Gamma'\zeta+\Gamma\zeta')(2a-1) = \frac{(2\pi)^{2a-1}}{2\cos\frac{\pi(2a-1)}{2}}\left\{-\zeta'(2-2a)+\zeta(2-2a)\left(\log 2\pi+\frac{\pi}{2}\tan\frac{\pi(2a-1)}{2}\right)\right\}$$

and

(2.19) $$\begin{aligned}(\Gamma''\zeta+2\Gamma'\zeta'+\Gamma\zeta'')(2a-1) &= \frac{(2\pi)^{2a-1}}{2\cos\frac{\pi(2a-1)}{2}}\left\{\zeta''(2-2a)-2\zeta'(2-2a)\left(\log 2\pi+\frac{\pi}{2}\tan\frac{\pi(2a-1)}{2}\right)\right.\\ &\left.+\zeta(2-2a)\left(\left(\log 2\pi+\frac{\pi}{2}\tan\frac{\pi(2a-1)}{2}\right)^2+\left(\frac{\pi}{2\cos\frac{\pi(2a-1)}{2}}\right)^2\right)\right\}.\end{aligned}$$

With Stirling's formula we find

$$\frac{\Gamma(1-z)}{\Gamma(2a-z)} = \exp\left(2a-1+(1-2a)\log(1-z)+O\left(\frac{1}{|z|}\right)\right).$$

Therefore we get

$$\begin{aligned}\int_{a+iT}^{a+i(T+H)}\frac{\Gamma(1-z)}{\Gamma(2a-z)}\,dz &= \left(e^{2a-1}+O\left(\frac{1}{T}\right)\right)\int_{a+iT}^{a+i(T+H)}(1-z)^{1-2a}\,dz\\ &= (-i)^{2-2a}\left(\frac{T}{e}\right)^{1-2a}H+O(HT^{-2a})\end{aligned}$$

and analogously

$$\int_{a-i(T+H)}^{a-iT}\frac{\Gamma(1-z)}{\Gamma(2a-z)}\,dz = i^{2-2a}\left(\frac{T}{e}\right)^{1-2a}H+O(HT^{-2a}).$$

Via $z\mapsto 2a-z$ we have

$$\int_{a\pm iT}^{a\pm i(T+H)}\frac{\Gamma(1-z)}{\Gamma(2a-z)}\,dz = \int_{a\mp i(T+H)}^{a\mp iT}\frac{\Gamma(1-2a+z)}{\Gamma(z)}\,dz,$$

so

$$\begin{aligned}&\frac{1}{2i}\left\{\int_{a-i(T+H)}^{a-iT}+\int_{a+iT}^{a+i(T+H)}\right\}\left(\frac{\Gamma(1-z)}{\Gamma(2a-z)}+\frac{\Gamma(1-2a+z)}{\Gamma(z)}\right)dz\\ &= 2\cos\left(\frac{\pi(1-2a)}{2}\right)\left(\frac{T}{e}\right)^{1-2a}H+O(HT^{-2a}).\end{aligned}$$

In a similar manner

$$\begin{aligned}&\frac{1}{2i}\left\{\int_{a-i(T+H)}^{a-iT}+\int_{a+iT}^{a+i(T+H)}\right\}\left(\frac{\Gamma'(1-z)}{\Gamma(2a-z)}+\frac{\Gamma'(1-2a+z)}{\Gamma(z)}\right)dz\\ &= 2\cos\left(\frac{\pi(1-2a)}{2}\right)\left(\frac{T}{e}\right)^{1-2a}HL+O(HLT^{-2a})\end{aligned}$$

and

$$\frac{1}{2i}\left\{\int_{a-i(T+H)}^{a-iT}+\int_{a+iT}^{a+i(T+H)}\right\}$$
$$\left(\frac{\Gamma'}{\Gamma}(2a-z)\frac{\Gamma'(1-z)}{\Gamma(2a-z)}+\frac{\Gamma'}{\Gamma}(z)\frac{\Gamma'(1-2a+z)}{\Gamma(z)}\right)dz$$
$$= 2\cos\left(\frac{\pi(1-2a)}{2}\right)\left(\frac{T}{e}\right)^{1-2a}HL^2+O(HL^2T^{-2a}),$$

respectively. With these formulas we are able to calculate the main part of

$$\int_T^{T+H}|AF(a+it)|^2\,dt$$
$$= \frac{1}{2i}\left\{\int_{a-i(T+H)}^{a-iT}+\int_{a+iT}^{a+i(T+H)}\right\}AF(z)AF(2a-z)\,dz$$

as follows: because of the simple pole of $\zeta(s)$ in $s=1$ we have for $k\in\mathbb{N}_0$

$$\zeta^{(k)}(s)=(-1)^k\frac{k!}{(s-1)^{k+1}}+O(1)\qquad (s\to 1), \tag{2.20}$$

so integrating the equation (2.16) we get with (2.17), (2.18) and (2.19) for the main part

$$\sim H\Bigg(S_0\left\{\zeta(2a)+\frac{2}{L}\zeta'(2a)+\frac{1}{L^2}\zeta''(2a)\right\}$$
$$-\frac{2S_1}{L}\left\{\zeta(2a)+\frac{1}{L}\zeta'(2a)\right\}+\frac{S_2}{L^2}\zeta(2a)$$
$$+\left(\frac{T}{2\pi e}\right)^{1-2a}\left(\frac{K_0}{L^2}\zeta''(2-2a)-\frac{2K_1}{L^2}\zeta'(2-2a)+\frac{K_2}{L^2}\zeta(2-2a)\right)\Bigg)$$
$$\sim H\Bigg(S_0\left\{\frac{1}{2a-1}-\frac{2}{L(2a-1)^2}+\frac{2}{L^2(2a-1)^3}\right\}$$
$$-\frac{2S_1}{L}\left\{\frac{1}{2a-1}-\frac{1}{L(2a-1)^2}\right\}+\frac{S_2}{L^2(2a-1)}$$
$$+\left(\frac{T}{2\pi e}\right)^{1-2a}\left(\frac{2K_0}{L^2(1-2a)^3}+\frac{2K_1}{L^2(1-2a)^2}+\frac{K_2}{L^2(1-2a)}\right)\Bigg).$$

Of course we have for $H=T$ the same main term as Levinson. With his further calculations (see [29], §11-14)

$$S_0\sim\frac{1}{\theta L}-\frac{1}{2}+a+\frac{\theta L}{3}\left(\frac{1}{2}-a\right)^2,\quad S_1\sim-\frac{1}{2}+\frac{\theta L}{3}\left(\frac{1}{2}-a\right),\quad S_2\sim\frac{\theta L}{3},$$
$$K_0\sim\frac{1}{\theta L}+\frac{1}{2}-a+\frac{\theta L}{3}\left(\frac{1}{2}-a\right)^2,\quad K_1\sim-\frac{1}{2}-\frac{\theta L}{3}\left(\frac{1}{2}-a\right),\quad K_2\sim\frac{\theta L}{3}$$

we obtain

$$I(T,H)\sim H\left(\frac{1}{2}+\frac{\theta}{6}\left(1-R-\frac{1}{2R}\right)-\frac{1}{2R\theta}\left(1+\frac{1}{R}+\frac{1}{2R^2}\right)+e^{2R}\left(\frac{\theta}{12R}+\frac{1}{4R^3\theta}\right)\right). \tag{2.21}$$

2.1.3 Transformation of the error term

For $G(u,v)$, i.e. for

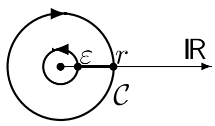
$$\begin{aligned} & F_2(u,0;v,0;w(k,l)) \\ = & \frac{1}{\Gamma(u)\Gamma(v)} \sum_{k,l\le M} a(k)a(l)w(k,l)\frac{1}{l}\sum_{f=1}^{l}\int_0^\infty y^{v-1}\left(\exp\left(y-2\pi i\frac{f}{l}\right)-1\right)^{-1} \\ & \times \int_0^\infty x^{u-1}\left(\left(\exp\left(k(x+y)-2\pi i\frac{kf}{l}\right)-1\right)^{-1}-\frac{\delta(f)}{k(x+y)}\right)dx\,dy \end{aligned}$$

and its derivatives with respect to u, v we need an analytic continuation to a domain that includes $\operatorname{Re} u = \operatorname{Re} v = a$. Obviously it is sufficient to find one for

$$\begin{aligned} \varphi(u,v) := & \frac{1}{\Gamma(u)\Gamma(v)l}\sum_{f=1}^{l}\int_0^\infty y^{v-1}\left(\exp\left(y-2\pi i\frac{f}{l}\right)-1\right)^{-1} \\ & \times\int_0^\infty x^{u-1}\left(\left(\exp\left(k(x+y)-2\pi i\frac{kf}{l}\right)-1\right)^{-1}-\frac{\delta(f)}{k(x+y)}\right)dx\,dy \end{aligned}$$

with fixed k, l. Following the ideas of Atkinson [1] and Motohashi [32] we first transform $\varphi(u,v)$ into a more suitable expression. Therefore we rewrite the integrals as Hankel integrals (as Riemann [35] did when proving the functional equation (1.3), see for that [40], §2.4 and [4], §III.5.6.4).

Let $f(z)$ be a meromorphic function without poles on the nonnegative real axis. Consider the integral $\int_{\mathcal{C}} f(z)z^{w-1}\,dz$ with $0<w<1$ and the contour $\mathcal{C}$ that starts at $r>0$, proceeds along the positive real axis to a small $\varepsilon>0$, describes a circle of radius ε counterclockwise around the origin, returns along the real axis back to r and describes a circle of radius r clockwise around the origin. $\operatorname{Im}\log z$ varies on the small circle from 0 to 2π. With $z^{w-1}=e^{(w-1)\log z}$ we obtain



$$\begin{aligned} & \int_{\mathcal{C}} f(z)z^{w-1}\,dz \\ = & \left\{\int_r^\varepsilon+\int_{|z|=\varepsilon}-\int_{|z|=r}\right\}f(z)z^{w-1}\,dz+\int_\varepsilon^r f\left(ze^{2\pi i}\right)\left(ze^{2\pi i}\right)^{w-1}dz \\ = & \left(e^{2\pi iw}-1\right)\int_\varepsilon^r f(z)z^{w-1}\,dz+\left\{\int_{|z|=\varepsilon}-\int_{|z|=r}\right\}f(z)z^{w-1}\,dz. \end{aligned}$$

Now let $r\to\infty$ and $\varepsilon\to 0$. Since $\lim_{z\to 0,\infty} zf(z)=0$ the integrals over the circles vanish, and we obtain, using the calculus of residues,

$$\left(e^{2\pi iw}-1\right)\int_0^\infty f(z)z^w\,dz=\int_{\mathcal{C}} f(z)z^w\,dz=-2\pi i\sum \operatorname{Res} f(z)z^w.$$

Hence

$$\varphi(u,v) = \frac{1}{\Gamma(u)\Gamma(v)\left(e^{2\pi iu}-1\right)\left(e^{2\pi iv}-1\right)l}\times$$

$$\times \sum_{f=1}^{l} \int_{\mathcal{C}} y^{v-1} \left(\exp\left(y - 2\pi i \frac{f}{l} \right) - 1 \right)^{-1}$$
$$\times \int_{\mathcal{C}} x^{u-1} \left(\left(\exp\left(k(x+y) - 2\pi i \frac{kf}{l} \right) - 1 \right)^{-1} - \frac{\delta(f)}{k(x+y)} \right) dx\, dy.$$

Note that

$$\left(\exp\left(k(x+y) - 2\pi i \frac{kf}{l} \right) - 1 \right)^{-1} - \frac{\delta(f)}{k(x+y)}$$

is regular apart from simple poles in

$$x = -y + \frac{2\pi i}{k} \left(n + (1-\delta(f)) \frac{kf}{l} \right) \qquad (n \in \mathbb{Z}, n \neq 0 \iff \delta(f) = 1)$$

(in the case $\delta(f) = 1$ the pole in $x = -y$ can be lifted) with residues

$$\frac{1}{k} \left(-y + \frac{2\pi i}{k} \left(n + (1-\delta(f)) \frac{kf}{l} \right) \right)^{u-1}.$$

Thus we obtain

$$\varphi(u,v) = \frac{(-1)^u 2\pi i}{\Gamma(u)\Gamma(v)\left(e^{2\pi i u} - 1\right) kl} \sum_{f=1}^{l} \int_0^\infty y^{v-1} \left(\exp\left(y - 2\pi i \frac{f}{l} \right) - 1 \right)^{-1}$$
$$\times \sum_{\substack{n=-\infty \\ n \neq 0 \iff \delta(f)=1}}^{\infty} \left(y - 2\pi i \left(\frac{n}{k} + (1-\delta(f)) \frac{f}{l} \right) \right)^{u-1} dy$$

(where the path of integration is replaced by the original one). With $\Gamma(u)\Gamma(1-u) = \frac{\pi}{\sin \pi u}$ and (2.12) we have

$$\left(y - 2\pi i \left(\frac{n}{k} + (1-\delta(f)) \frac{f}{l} \right) \right)^{u-1}$$
$$= \frac{\sin \pi u}{\pi} \int_0^\infty \frac{x^{u-1}\, dx}{x + y - 2\pi i \left(\frac{n}{k} + (1-\delta(f)) \frac{f}{l} \right)}.$$

We can interpret the reciprocal of the denominator of the integrand as an integral

$$\frac{1}{x + y - 2\pi i \left(\frac{n}{k} + (1-\delta(f)) \frac{f}{l} \right)}$$
$$= \int_0^\infty \exp\left(-z \left\{ x + y - 2\pi i \left(\frac{n}{k} + (1-\delta(f)) \frac{f}{l} \right) \right\} \right) dz.$$

Since we may obviously exchange here (and below) summation and integration, we get

$$\varphi(u,v) = \frac{1}{\Gamma(u)\Gamma(v)kl} \sum_{f=1}^{l} \sum_{\substack{n=-\infty \\ n \neq 0 \iff \delta(f)=1}}^{\infty} \int_0^\infty y^{v-1} \left(\exp\left(y - 2\pi i \frac{f}{l} \right) - 1 \right)^{-1}$$
$$\times \int_0^\infty x^{u-1} \int_0^\infty \exp\left(-z \left\{ x + y - 2\pi i \left(\frac{n}{k} + (1-\delta(f)) \frac{f}{l} \right) \right\} \right) dz\, dx\, dy.$$

Now we replace $\left(\exp\left(y - 2\pi i\frac{f}{l}\right) - 1\right)^{-1}$ by a geometric series. Then $\varphi(u,v)$ equals

$$\begin{aligned} & \frac{1}{\Gamma(u)\Gamma(v)kl}\sum_{f=1}^{l}\sum_{m=1}^{\infty}\sum_{\substack{n=-\infty\\ n\neq 0 \iff \delta(f)=1}}^{\infty}\int_0^\infty\int_0^\infty\int_0^\infty y^{v-1}x^{u-1} \\ & \times \exp\left(-z\left\{x+y-2\pi i\left(\frac{n}{k}+(1-\delta(f))\frac{f}{l}\right)\right\} - m\left(y-2\pi i\frac{f}{l}\right)\right)\,dz\,dx\,dy \\ = \; & \frac{1}{\Gamma(u)\Gamma(v)kl}\sum_{f=1}^{l}\sum_{m=1}^{\infty}\sum_{\substack{n=-\infty\\ n\neq 0 \iff \delta(f)=1}}^{\infty}\int_0^\infty\int_0^\infty x^{u-1}e^{-xz}\,dx \\ & \times \int_0^\infty y^{v-1}e^{-y(m+z)}\,dy\,\exp\left(2\pi i\left(z\left\{\frac{n}{k}+(1-\delta(f))\frac{f}{l}\right\}+m\frac{f}{l}\right)\right)\,dz \end{aligned}$$

(exchanging the order of integration is obviously allowed by Fubini's theorem). By (2.10) and the substitution $z = ym$ we obtain

$$\begin{aligned} \varphi(u,v) \;=\; & \frac{1}{kl}\sum_{f=1}^{l}\sum_{m=1}^{\infty} m^{1-u-v}\sum_{\substack{n=-\infty\\ n\neq 0 \iff \delta(f)=1}}^{\infty}\int_0^\infty y^{-u}(1+y)^{-v} \\ & \times \exp\left(2\pi i\left(ym\left\{\frac{n}{k}+(1-\delta(f))\frac{f}{l}\right\}+\frac{mf}{l}\right)\right)\,dy. \end{aligned}$$

Let $\kappa = \frac{k}{(k,l)}, \lambda = \frac{l}{(k,l)}$. Then

$$\begin{aligned} m\left(\frac{n}{k}+(1-\delta(f))\frac{f}{l}\right) &= \frac{m}{(k,l)}\cdot\frac{\lambda n+\kappa(1-\delta(f))f}{\kappa\lambda}\,, \\ \frac{mf}{l} &= \frac{m}{(k,l)}\cdot\frac{f}{\lambda}\,. \end{aligned}$$

Now we interpret $\lambda n + \kappa(1-\delta(f))f$ as a linear form in n and f. This linear form represents (exactly once) all multiples of λ but not zero in the case of $\delta(f) = 1$ (since $n \neq 0 \iff \delta(f) = 1$) and otherwise all integers $\neq 0$. Let $\overline{\kappa}$ denote the inverse of κ mod λ. Thus with $N := \lambda n + \kappa(1-\delta(f))f$ we have $f \equiv N\overline{\kappa} \bmod \lambda$. Let further $M = \frac{m}{(k,l)}$. Then we obtain

$$\begin{aligned} \varphi(u,v) \;=\; & \frac{(k,l)^{2-u-v}}{kl}\sum_{M=1}^{\infty} M^{1-u-v}\sum_{\substack{N=-\infty\\ N\neq 0}}^{\infty}\exp\left(2\pi i M\frac{N\overline{\kappa}}{\lambda}\right) \\ & \times \int_0^\infty y^{-u}(1+y)^{-v}\exp\left(2\pi i y\frac{MN}{\kappa\lambda}\right)\,dy. \end{aligned}$$

Define now $e(z) = \exp(2\pi i z)$,

$$\sigma_\alpha(n) = \sum_{d|n} d^\alpha$$

and further $d(n) = \sigma_0(n)$. Then we obtain by setting $n = MN$

$$\varphi(u,v) = \frac{(k,l)^{2-u-v}}{kl}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\sigma_{1-u-v}\left(|n|\right)\,e\left(n\frac{\overline{\kappa}}{\lambda}\right)\int_0^\infty y^{-u}(1+y)^{-v}\,e\left(\frac{ny}{\kappa\lambda}\right)\,dy.$$

Now

$$\sum_{n=-\infty}^{-1} \sigma_{1-u-v}(|n|)\, e\left(n\frac{\overline{\kappa}}{\lambda}\right) \int_0^\infty y^{-u}(1+y)^{-v}\, e\left(\frac{ny}{\kappa\lambda}\right) dy$$
$$= \sum_{n=1}^{\infty} \sigma_{1-u-v}(n)\, e\left(-n\frac{\overline{\kappa}}{\lambda}\right) \int_0^\infty y^{-u}(1+y)^{-v}\, e\left(-\frac{ny}{\kappa\lambda}\right) dy$$

is just the conjugate of

$$\sum_{n=1}^{\infty} \sigma_{1-\overline{u}-\overline{v}}(n)\, e\left(n\frac{\overline{\kappa}}{\lambda}\right) \int_0^\infty y^{-\overline{u}}(1+y)^{-\overline{v}}\, e\left(\frac{ny}{\kappa\lambda}\right) dy,$$

as well as $u = z, v = 2a - z$ are conjugates on Re $z = a$. Define

$$g(u,v;k,l) = \sum_{n=1}^{\infty} \sigma_{1-u-v}(n)\, e\left(n\frac{\overline{\kappa}}{\lambda}\right) \int_0^\infty y^{-u}(1+y)^{-v}\, e\left(\frac{ny}{\kappa\lambda}\right) dy. \tag{2.22}$$

Then we have found

$$F_2(u,0;v,0;w(k,l)) = \sum_{k,l\leq M} a(k)a(l)w(k,l)\frac{(k,l)^{2-u-v}}{kl}\left(g(u,v;k,l) + \overline{g(\overline{u},\overline{v};k,l)}\right) \tag{2.23}$$

for $u = z, v = 2a - z$ and vice versa. Hence we may estimate first $g(u,v) := g(u,v;k,l)$ for fixed k,l to bound $G(u,v)$ later.

2.1.4 Estermann's zeta-function

For the analytic continuation of $G(u,v)$ into the critical strip we need estimates of the arithmetical function

$$D_\alpha\left(X;\frac{\kappa}{\lambda}\right) := \sum_{n\leq X} \sigma_\alpha(n)\, e\left(n\frac{\kappa}{\lambda}\right)$$

(if X is an integer, the term in the sum corresponding to X has to be halved) for small positive α and coprime integers κ, λ and $\lambda \geq 1$; note that by (2.22) we have to replace κ by $\overline{\kappa}$ in our later applications. This function was first studied by Jutila [25] in the case $\alpha = 0$ and Kiuchi [27] for $-1 < \alpha \leq 0$, but not for $\alpha > 0$. Unfortunately we also need estimates of derivatives of $D_\alpha\left(X;\frac{\kappa}{\lambda}\right)$ with respect to α for small positive α. Because of Cauchy's integral-formula we study $D_\alpha\left(X;\frac{\kappa}{\lambda}\right)$ as a function of a complex variable α.
$D\left(X;\frac{\kappa}{\lambda}\right)$ is the Moebius transform of the coefficients of the **Estermann zeta-function**

$$\begin{aligned} E_\alpha\left(s;\frac{\kappa}{\lambda}\right) &:= \sum_n \sigma_\alpha(n)\, e\left(n\frac{\kappa}{\lambda}\right) n^{-s} \qquad (\sigma > 1) \\ &= \sum_{f=1}^{\lambda} \sum_{\substack{d \\ d\equiv f \bmod \lambda}} d^{\alpha-s} \sum_m m^{-s}\, e\left(fm\frac{\kappa}{\lambda}\right) \\ &= \sum_{f=1}^{\lambda} \zeta(s-\alpha;f,\lambda)\zeta\left(s;e\left(f\frac{\kappa}{\lambda}\right)\right), \end{aligned} \tag{2.24}$$

where

$$\zeta(s;g,\lambda) := \sum_{\substack{n \\ n\equiv g \bmod \lambda}} n^{-s}$$

and

$$\zeta\left(s;e\left(\frac{g}{\lambda}\right)\right) := \sum_{n} n^{-s}\, e\left(n\frac{g}{\lambda}\right)$$

are both defined for $\sigma > 1$ and $g \in \mathbb{Z}$. First we recall some analytic properties of $E_\alpha\left(s;\frac{\kappa}{\lambda}\right)$ due to Estermann [12]: $\zeta\left(s;e\left(\frac{f}{\lambda}\right)\right)$ is an entire function for $f \not\equiv 0 \bmod \lambda$, and otherwise analytic except for a simple pole in $s=1$, which always holds for $\zeta(s;f,\lambda)$. In what follows we restrict our investigations to $\alpha \neq 0$ (the case $\alpha = 0$ can be treated similarly). Since by (2.24)

$$E_\alpha\left(s;\frac{\kappa}{\lambda}\right) - \zeta(s)\zeta(s-\alpha;\lambda,\lambda) = \sum_{f=1}^{\lambda-1} \zeta(s-\alpha;f,\lambda)\zeta\left(s;e\left(f\frac{\kappa}{\lambda}\right)\right)$$

is an entire function, it follows that $E_\alpha\left(s;\frac{\kappa}{\lambda}\right)$ has the same main part in $s=1$ as

$$\zeta(s)\zeta(s-\alpha;\lambda,\lambda) = \lambda^{\alpha-s}\zeta(s)\zeta(s-\alpha).$$

Hence

$$E_\alpha\left(s;\frac{\kappa}{\lambda}\right) = \frac{\lambda^{\alpha-1}}{s-1}\zeta(1-\alpha) + O(1) \qquad (s\to 1). \tag{2.25}$$

By (2.24) also

$$\begin{aligned} & E_\alpha\left(s;\frac{\kappa}{\lambda}\right) - \zeta(s-\alpha;\lambda,\lambda)\sum_{f=1}^{\lambda}\zeta\left(s;e\left(f\frac{\kappa}{\lambda}\right)\right) \\ &= \sum_{f=1}^{\lambda-1}\left(\zeta(s-\alpha;f,\lambda) - \zeta(s-\alpha;\lambda,\lambda)\right)\zeta\left(s;e\left(f\frac{\kappa}{\lambda}\right)\right) \end{aligned}$$

is entire. Thus $E_\alpha\left(s;\frac{\kappa}{\lambda}\right)$ has the same main part in $s=1$ as

$$\zeta(s-\alpha;\lambda,\lambda)\sum_{f=1}^{\lambda}\zeta\left(s;e\left(f\frac{\kappa}{\lambda}\right)\right) = \lambda^{1+\alpha-2s}\zeta(s)\zeta(s-\alpha).$$

So we find

$$E_\alpha\left(s;\frac{\kappa}{\lambda}\right) = \frac{\lambda^{-\alpha-1}}{s-\alpha-1}\zeta(1+\alpha) + O(1) \qquad (s\to 1+\alpha). \tag{2.26}$$

Apart from these simple poles $E_\alpha\left(s;\frac{\kappa}{\lambda}\right)$ is a regular function in the whole complex plane: as for Riemann's zeta-function Estermann proved functional equations, namely

$$\begin{aligned} \zeta(s;g,\lambda) &= \frac{1}{2\pi i}\left(\frac{2\pi}{\lambda}\right)^s \Gamma(1-s) \\ &\quad \times\left\{e\left(\frac{s}{4}\right)\zeta\left(1-s;e\left(\frac{g}{\lambda}\right)\right) - e\left(-\frac{s}{4}\right)\zeta\left(1-s;e\left(-\frac{g}{\lambda}\right)\right)\right\} \end{aligned}$$

and

$$\zeta\left(s; e\left(\frac{g}{\lambda}\right)\right) = \frac{1}{i}\left(\frac{2\pi}{\lambda}\right)^{s-1}\Gamma(1-s) \times\left\{e\left(\frac{s}{4}\right)\zeta(1-s;-g,\lambda) - e\left(-\frac{s}{4}\right)\zeta(1-s;g,\lambda)\right\}.$$

From this and (2.24) we deduce the functional equation

$$\begin{aligned}(2.27)\quad & E_\alpha\left(s;\frac{\kappa}{\lambda}\right) \\ &= \frac{1}{\pi}\left(\frac{2\pi}{\lambda}\right)^{2s-1-\alpha}\Gamma(1-s)\Gamma(1+\alpha-s) \\ &\quad\times\left\{\cos\left(\frac{\pi\alpha}{2}\right)E_\alpha\left(1+\alpha-s;\frac{\overline{\kappa}}{\lambda}\right) - \cos\left(\pi s-\frac{\pi\alpha}{2}\right)E_\alpha\left(1+\alpha-s;-\frac{\overline{\kappa}}{\lambda}\right)\right\}.\end{aligned}$$

This gives an analytic continuation of $E_\alpha\left(s;\frac{\kappa}{\lambda}\right)$ to the whole complex plane.

From (2.27) and Stirling's formula we find with the Phragmén-Lindelöf-principle (i.e. the theorem of Phragmén-Lindelöf applied to $E_\alpha\left(s;\frac{\kappa}{\lambda}\right)$ or directly by the convexity properties of Dirichlet series with a functional equation, analogously to its application to the zeta-function; see [38], §II.1.6 or [40], §5)

$$(2.28)\quad E_\alpha\left(s;\frac{\kappa}{\lambda}\right) \ll (\lambda|t|)^{1+\varepsilon-\sigma} \qquad (\text{Re }\alpha-\varepsilon\le\sigma\le 1+\varepsilon,\quad |t|\ge 1).$$

2.1.5 Preparations: Estimates of exponential integrals and Perron's formula

We interrupt the proof of Theorem 2.1 to state some well known results: in the sequel we often have to bound exponential integrals with the help of the following lemmas (see [20], §5.1 and [21], §2.1):

Lemma 2.3 (First derivative test) *Let $F(t)$ be real and differentiable with monotonic $F'(t)\ge m>0$ and $G(t)$ monotonic with $|G(t)|\le G$ on $[c,d]$. Then*

$$\int_c^d G(t)e^{iF(t)}\,dt \ll \frac{G}{m}.$$

This is proved by partial integration of

$$\int e^{iF(t)}\,dt = \int \frac{1}{iF'(t)}\cdot iF'(t)e^{iF(t)}\,dt$$

and the monotonic conditions. With the same idea we find

Lemma 2.4 *Let $F(t)$ and $G(t)$ be real functions, $F(t)$ differentiable and $\frac{G}{F'}(t)$ monotonic with $\left|\frac{F'(t)}{G(t)}\right|\ge m>0$ for $t\in[c,d]$. Then*

$$\int_c^d G(t)e^{iF(t)}\,dt \ll \frac{1}{m}.$$

In a similar way one can show

Lemma 2.5 (Second derivative test) *Let $F(t)$ be real and twice differentiable with monotonic $F''(t) \geq m > 0$ and $G(t)$ monotonic with $|G(t)| \leq G$ on $[c,d]$. Then*

$$\int_c^d G(t) e^{iF(t)}\, dt \ll \frac{G}{\sqrt{m}}.$$

Moreover we need

Lemma 2.6 (Perron's formula) *Let $c > 0, T > 2$ and $2 \leq X \notin \mathbb{Z}$. If $\sum_{n=1}^{\infty} \frac{b(n)}{n^s}$ converges absolutely for $\sigma > c$, then we have*

$$\begin{aligned} \sum_{n \leq X} b(n) &= \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \sum_{n=1}^{\infty} \frac{b(n)}{n^s} \frac{X^s}{s}\, ds \\ &+ O\left(\frac{X^c}{T} \sum_{n=1}^{\infty} \frac{|b(n)|}{n^c} + \max_{\frac{3}{4}X \leq n \leq \frac{5}{4}X} |b(n)| \left(1 + \frac{X \log X}{T}\right)\right). \end{aligned}$$

The proof (see [38], §II.2.1) is an immediate consequence of the truncated version of the well known formula (also called Perron's formula)

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{y^s}{s}\, ds = \begin{cases} 0 & , \quad 0 < y < 1, \\ \frac{1}{2} & , \quad y = 1, \\ 1 & , \quad y > 1 \end{cases} .$$

2.1.6 A truncated Voronoi-type formula

Now we are ready to study $D_\alpha\left(X; \frac{\kappa}{\lambda}\right)$. Let $a = \text{Re}\ \alpha$. For technical reasons we first consider $a < 0$.

Theorem 2.7 *Let $\lambda \leq X,\ 1 \leq N \ll X$ and $\left|\alpha + \frac{1}{2}\right| < \frac{1}{2}$. Then we have*

$$\begin{aligned} D_\alpha\left(X; \frac{\kappa}{\lambda}\right) &= \frac{X^{1+\alpha}}{1+\alpha} \lambda^{-1-\alpha} \zeta(1+\alpha) + X \lambda^{\alpha-1} \zeta(1-\alpha) \\ &+ E_\alpha\left(0; \frac{\kappa}{\lambda}\right) + \Delta_\alpha\left(X; \frac{\kappa}{\lambda}\right) \end{aligned}$$

with

$$\begin{aligned} &\Delta_\alpha\left(X; \frac{\kappa}{\lambda}\right) \\ &= -X^{\frac{1}{2}+\frac{\alpha}{2}} \sum_{n \leq N} \sigma_\alpha(n)\, e\left(-n\frac{\overline{\kappa}}{\lambda}\right) n^{-\frac{1}{2}-\frac{\alpha}{2}} \left(\cos\left(\frac{\pi\alpha}{2}\right) Y_{1+\alpha}\left(\frac{4\pi (nX)^{\frac{1}{2}}}{\lambda}\right) \right. \\ &\left. \quad + \sin\left(\frac{\pi\alpha}{2}\right) J_{1+\alpha}\left(\frac{4\pi (nX)^{\frac{1}{2}}}{\lambda}\right)\right) + O\left(\lambda N^{-\frac{1}{2}} X^{\frac{1}{2}+\varepsilon}\right), \end{aligned}$$

where $J_v(z)$ and $Y_v(Z)$ are the Bessel functions (given below by (2.31) and (2.32)). Especially for real α we have

$$\Delta_\alpha\left(X;\frac{\kappa}{\lambda}\right) = \frac{1}{\pi\sqrt{2}}\lambda^{\frac12}X^{\frac14+\frac{\alpha}{2}}\sum_{n\le N}\sigma_\alpha(n)\,e\left(-n\frac{\overline{\kappa}}{\lambda}\right)n^{-\frac34-\frac{\alpha}{2}}\cos\left(\frac{4\pi(nX)^{\frac12}}{\lambda}-\frac{\pi}{4}\right) +O\left(\lambda N^{-\frac12}X^{\frac12+\varepsilon}\right).$$

The proof of this "complex" result is very similar to Jutila's "real" one or Kiuchi's generalization (the real case is stated for completeness to compare with Kiuchi's result).

Proof. Let $\varepsilon>0, N\in\mathbb{N}$ and the parameter T be given by

$$(\lambda T)^{2+\varepsilon}=4\pi^2X\left(N+\frac12\right).$$

Applying Perron's formula Lemma 2.6 we find

$$D_\alpha\left(X;\frac{\kappa}{\lambda}\right)=\frac{1}{2\pi i}\int_{1+\varepsilon-iT}^{1+\varepsilon+iT}E_\alpha\left(s;\frac{\kappa}{\lambda}\right)\frac{X^s}{s}\,ds+O\left(\lambda N^{-\frac{1}{2+\varepsilon}}X^{\frac{1+\varepsilon}{2+\varepsilon}+\varepsilon}\right),$$

where the bound of the error term arises from the well known fact $d(n)\ll n^\varepsilon$ (see [38], §I.5.3) and $|\sigma_\alpha(n)|\le d(n)$. Now we evaluate the integral above by integrating on the rectangular contour with vertices $1+\varepsilon\pm iT, a-\varepsilon\pm iT$: by (2.28) we have for the integrals along the horizontal paths

$$\int_{a-\varepsilon\pm iT}^{1+\varepsilon\pm iT}E_\alpha\left(s;\frac{\kappa}{\lambda}\right)\frac{X^s}{s}\,ds\ll\lambda N^{-\frac{1}{2+\varepsilon}}X^{\frac{1+\varepsilon}{2+\varepsilon}+\varepsilon}.$$

Applying the calculus of residues, (2.25) and (2.26) we have

$$D_\alpha\left(X;\frac{\kappa}{\lambda}\right) = \frac{1}{2\pi i}\int_{a-\varepsilon-iT}^{a-\varepsilon+iT}E_\alpha\left(s;\frac{\kappa}{\lambda}\right)\frac{X^s}{s}\,ds+E_\alpha\left(0;\frac{\kappa}{\lambda}\right)+X\lambda^{\alpha-1}\zeta(1-\alpha) +\frac{X^{1+\alpha}}{1+\alpha}\lambda^{-1-\alpha}\zeta(1+\alpha)+O\left(\lambda N^{-\frac{1}{2+\varepsilon}}X^{\frac{1+\varepsilon}{2+\varepsilon}+\varepsilon}\right).$$

Hence

$$\Delta_\alpha\left(X;\frac{\kappa}{\lambda}\right)=\frac{1}{2\pi i}\int_{a-\varepsilon-iT}^{a-\varepsilon+iT}E_\alpha\left(s;\frac{\kappa}{\lambda}\right)\frac{X^s}{s}\,ds+O\left(\lambda N^{-\frac{1}{2+\varepsilon}}X^{\frac{1+\varepsilon}{2+\varepsilon}+\varepsilon}\right).$$

Using the functional equation (2.27) it follows that

$$\Delta_\alpha\left(X;\frac{\kappa}{\lambda}\right) = \frac{1}{2\pi^2 i}\left(\frac{\lambda}{2\pi}\right)^{1+\alpha}\sum_{n=1}^{\infty}\sigma_\alpha(n)\,e\left(-n\frac{\overline{\kappa}}{\lambda}\right)n^{-1-\alpha}\left(\cos\left(\frac{\pi\alpha}{2}\right)I_1-\sin\left(\frac{\pi\alpha}{2}\right)I_2\right) +\frac{1}{\pi^2}\left(\frac{\lambda}{2\pi}\right)^{1+\alpha}\cos\left(\frac{\pi\alpha}{2}\right)\sum_{n=1}^{\infty}\sigma_\alpha(n)\sin\left(2\pi n\frac{\overline{\kappa}}{\lambda}\right)n^{-1-\alpha}I_3 +O\left(\lambda N^{-\frac{1}{2+\varepsilon}}X^{\frac{1+\varepsilon}{2+\varepsilon}+\varepsilon}\right), \tag{2.29}$$

where

$$\begin{aligned} I_1 &:= \int_{a-\varepsilon-iT}^{a-\varepsilon+iT} \Gamma(1-s)\Gamma(1+\alpha-s)(1-\cos\pi s)\left(\frac{4\pi^2nX}{\lambda^2}\right)^s \frac{ds}{s}, \\ I_2 &:= \int_{a-\varepsilon-iT}^{a-\varepsilon+iT} \Gamma(1-s)\Gamma(1+\alpha-s)\sin\pi s\left(\frac{4\pi^2nX}{\lambda^2}\right)^s \frac{ds}{s}, \\ I_3 &:= \int_{a-\varepsilon-iT}^{a-\varepsilon+iT} \Gamma(1-s)\Gamma(1+\alpha-s)\left(\frac{4\pi^2nX}{\lambda^2}\right)^s \frac{ds}{s}. \end{aligned}$$

With Stirling's formula we find

$$\Gamma(1-s)\Gamma(1+\alpha-s) \ll e^{-\pi t}t^{1+a-2\sigma} \qquad (t \geq 1).$$

It follows that

$$I_3 \ll n^{a-\varepsilon}\lambda^{-a}N^{\frac{2\varepsilon-a}{2+\varepsilon}}X^{a-\varepsilon+\frac{2\varepsilon-a}{2+\varepsilon}}$$

and therefore the contribution of the sum with I_3 to (2.29) is $\ll \lambda N^{\frac{2\varepsilon-a}{2+\varepsilon}}X^{a-\varepsilon+\frac{2\varepsilon-a}{2+\varepsilon}}$. For the other integrals we distinguish the cases $n \leq N$ and $n > N$. First $n > N$: let

$$\begin{aligned} I_1 &= \left\{\int_{a-\varepsilon+i}^{a-\varepsilon+iT} + \int_{a-\varepsilon-i}^{a-\varepsilon+i} + \int_{a-\varepsilon-iT}^{a-\varepsilon-i}\right\} \\ &\quad \Gamma(1-s)\Gamma(1+\alpha-s)(1-\cos\pi s)\left(\frac{4\pi^2nX}{\lambda^2}\right)^s \frac{ds}{s} \\ &=: I_{1,1} + I_{1,2} + I_{1,3} \end{aligned}$$

and $I_2 =: I_{2,1} + I_{2,2} + I_{2,3}$, respectively. By Stirling's formula we have

$$\begin{aligned} &\Gamma(1-s)\Gamma(1+\alpha-s)(1-\cos\pi s)\left(\frac{4\pi^2nX}{\lambda^2}\right)^s \frac{1}{s} \\ &= A(\sigma)\left(\frac{4\pi^2nX}{\lambda^2}\right)^\sigma t^{a-2\sigma}e^{iF(t)}\left(1+O\left(\frac{1}{t}\right)\right) \qquad (t \geq 1) \end{aligned}$$

with a bounded function $A(\sigma)$ and

$$F(t) := t\log\frac{4\pi^2nX}{\lambda^2} + (\mathrm{Im}\ \alpha - 2t)\log t + 2t.$$

With Lemma 2.4 we obtain

$$I_{1,1},\ I_{1,3} \ll n^{a-\varepsilon}\lambda^{-a}N^{\frac{2\varepsilon-a}{2+\varepsilon}}X^{a-\varepsilon+\frac{2\varepsilon-a}{2+\varepsilon}},$$

since

$$\begin{aligned} F'(t) &= \log\frac{4\pi^2nX}{\lambda^2t^2} + O\left(\frac{1}{t}\right) \qquad (1 \leq t \leq T) \\ &\geq (\varepsilon + o(1))\log(\lambda T) + O(1) \end{aligned}$$

is positive for T large enough. Easily we find

$$I_{1,2} \ll n^{a-\varepsilon}\lambda^{-2a}N^{\varepsilon}X^{a+\varepsilon}.$$

Thus the contribution of these terms with $n > N$ to (2.29) is $\ll \lambda N^{\frac{2\varepsilon-a}{2+\varepsilon}}X^{a-\varepsilon+\frac{2\varepsilon-a}{2+\varepsilon}}$. Analogously we get the same estimate for I_2. Summing up, we conclude from (2.29)

$$\begin{aligned}(2.30)\quad &\Delta_\alpha\left(X;\frac{\kappa}{\lambda}\right)\\ &= \frac{1}{2\pi^2 i}\left(\frac{\lambda}{2\pi}\right)^{1+\alpha}\sum_{n\le N}\sigma_\alpha(n)\,e\left(-n\frac{\overline{\kappa}}{\lambda}\right)n^{-1-\alpha}\left(\cos\left(\frac{\pi\alpha}{2}\right)I_1-\sin\left(\frac{\pi\alpha}{2}\right)I_2\right)\\ &\quad +O\left(\lambda N^{-\frac{1}{2+\varepsilon}}X^{\frac{1+\varepsilon}{2+\varepsilon}+\varepsilon}\right).\end{aligned}$$

Now we write

$$\begin{aligned}I_1 &= \left\{\int_{-i\infty}^{i\infty}-\int_{iT}^{i\infty}-\int_{-i\infty}^{-iT}-\int_{-iT}^{a-\varepsilon-iT}-\int_{a-\varepsilon+iT}^{iT}\right\}\\ &\qquad \Gamma(1-s)\Gamma(1+\alpha-s)(1-\cos\pi s)\left(\frac{4\pi^2 nX}{\lambda^2}\right)^s\frac{ds}{s}\\ &=: I_{1,4}-I_{1,5}-I_{1,6}-I_{1,7}-I_{1,8}\end{aligned}$$

and $I_2 =: I_{2,4}-I_{2,5}-I_{2,6}-I_{2,7}-I_{2,8}$, respectively. With Mellin's inversion formula (see [39], §1.29) and the functional equation of the Gamma function we get by $s\mapsto 1+\alpha-\frac{s}{2}$

$$\begin{aligned}I_{1,4} &= -\left(\frac{4\pi^2 nX}{\lambda^2}\right)^{1+\alpha}\left\{\int_{2+2a-i\infty}^{2+2a+i\infty}2^{s-1}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2}-\alpha-1\right)\right.\\ &\qquad\times\left(\frac{4\pi(nX)^{\frac12}}{\lambda}\right)^{-s}ds+\int_{2+2a-i\infty}^{2+2a+i\infty}2^{s}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2}-\alpha-1\right)\\ &\qquad\left.\times\cos\left(\frac{\pi s}{2}-\pi(1+\alpha)\right)\left(\frac{4\pi(nX)^{\frac12}}{\lambda}\right)^{-s}ds\right\}\\ &= -2\pi^2 i\left(\frac{2\pi(nX)^{\frac12}}{\lambda}\right)^{1+\alpha}\left(\frac{2}{\pi}K_{1+\alpha}\left(\frac{4\pi(nX)^{\frac12}}{\lambda}\right)+Y_{1+\alpha}\left(\frac{4\pi(nX)^{\frac12}}{\lambda}\right)\right)\end{aligned}$$

and in the same way

$$I_{2,4} = 2\pi^2 i\left(\frac{2\pi(nX)^{\frac12}}{\lambda}\right)^{1+\alpha}J_{1+\alpha}\left(\frac{4\pi(nX)^{\frac12}}{\lambda}\right)$$

with the well known representation of the Bessel functions as Mellin transforms

$$\begin{aligned}(2.31)\quad J_v(z) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}2^{s-v-1}\Gamma\left(\frac{s}{2}\right)\left(\Gamma\left(v-\frac{s}{2}+1\right)\right)^{-1}z^{v-s}\,ds,\\ K_v(z) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}2^{s-v-2}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2}-v\right)z^{v-s}\,ds,\\ (2.32)\quad Y_v(z) &= -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}2^{s-v-1}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2}-v\right)\cos\left(\frac{\pi s}{2}-v\pi\right)z^{v-s}\,ds\end{aligned}$$

valid for $2|v| \le c < |v| + \frac{3}{2}$ (see [39], §7.9). Since $n \le N$, we get, using again Lemma 2.4,

$$I_{1,5}, I_{1,6} \ll T^a \left(\log \left(\frac{N + \frac{1}{2}}{n} \right)^{-1} \right)^{-1} .$$

So their contribution to (2.30) is $\ll \lambda N^{\frac{a}{2+\varepsilon}} X^{\frac{a}{2+\varepsilon}}$. At last we bound trivially

$$I_{1,7},\ I_{1,8} \ll n^{a-\varepsilon} \lambda^{-a} N^{\frac{2\varepsilon - a}{2+\varepsilon}} X^{a-\varepsilon+\frac{2\varepsilon-a}{2+\varepsilon}}$$

and the contributions of these terms to (2.30) are $\ll \lambda N^{\frac{2\varepsilon-a}{2+\varepsilon}} X^{a-\varepsilon+\frac{2\varepsilon-a}{2+\varepsilon}}$. In a similar way we obtain the same estimate for I_2 as for I_1. Summing up, we may replace (2.30) by

$$\begin{aligned} &\Delta_\alpha \left(X; \frac{\kappa}{\lambda} \right) \\ &= -X^{\frac{1}{2}+\frac{\alpha}{2}} \sum_{n \le N} \sigma_\alpha(n)\, e\left(-n \frac{\overline{\kappa}}{\lambda} \right) n^{-\frac{1}{2}-\frac{\alpha}{2}} \left(\cos\left(\frac{\pi\alpha}{2} \right) \left\{ Y_{1+\alpha} \left(\frac{4\pi (nX)^{\frac{1}{2}}}{\lambda} \right) \right. \right. \\ &\quad \left. \left. + \frac{2}{\pi} K_{1+\alpha} \left(\frac{4\pi (nX)^{\frac{1}{2}}}{\lambda} \right) \right\} + \sin\left(\frac{\pi\alpha}{2} \right) J_{1+\alpha} \left(\frac{4\pi (nX)^{\frac{1}{2}}}{\lambda} \right) \right) \\ &\quad + O\left(\lambda N^{-\frac{1}{2+\varepsilon}} X^{\frac{1+\varepsilon}{2+\varepsilon}+\varepsilon} \right). \end{aligned}$$

Now we make use of the well known asymptotic formulas of the Bessel functions (see [28], p.87 and [41], §7)

$$J_v(z) = \frac{j(v)}{\sqrt{z}} \cos\left(z - \frac{\pi v}{2} - \frac{\pi}{4} \right) + O\left(|z|^{-\frac{3}{2}} \right), \tag{2.33}$$

$$K_v(z) = \frac{k(v)}{\sqrt{z}} \exp(-z) \left(1 + O(|z|^{-1}) \right),$$

$$Y_v(z) = \frac{y(v)}{\sqrt{z}} \sin\left(z - \frac{\pi v}{2} - \frac{\pi}{4} \right) + O\left(|z|^{-\frac{3}{2}} \right) \tag{2.34}$$

with bounded functions $j(v), k(v), y(v)$ independent on z and

$$j(v) = y(v) = \sqrt{\frac{2}{\pi}}, \tag{2.35}$$

for real v; note that the $O(\ )$-terms do not depend on v. Obviously $K_{1+\alpha}$ gives only a small contribution to the error term. In the case of real α we find

$$\begin{aligned} \Delta_\alpha \left(X; \frac{\kappa}{\lambda} \right) &= \frac{1}{\pi\sqrt{2}} \lambda^{\frac{1}{2}} X^{\frac{1}{4}+\frac{\alpha}{2}} \sum_{n \le N} \sigma_\alpha(n)\, e\left(-n \frac{\overline{\kappa}}{\lambda} \right) n^{-\frac{3}{4}-\frac{\alpha}{2}} \cos\left(\frac{4\pi (nX)^{\frac{1}{2}}}{\lambda} - \frac{\pi}{4} \right) \\ &\quad + O\left(\lambda N^{-\frac{1}{2+\varepsilon}} X^{\frac{1+\varepsilon}{2+\varepsilon}+\varepsilon} \right). \end{aligned}$$

Since $\varepsilon > 0$ can be chosen arbitrarily small the assertions of Theorem 2.7 follows.•

With the identity

$$\sigma_\alpha(n) = \sum_{d|n} d^\alpha = \sum_{d|n} \left(\frac{n}{d} \right)^\alpha = n^\alpha \sigma_{-\alpha}(n)$$

we will now reflect our formulas to the case of α with positive real part. After $\alpha \mapsto -\alpha$ we get with partial summation for $X \notin \mathbb{Z}$

$$\begin{aligned} D_\alpha\left(X;\frac{\kappa}{\lambda}\right) &= \sum_{n\le X}\sigma_{-\alpha}(n)\,e\left(n\frac{\kappa}{\lambda}\right)n^\alpha \\ &= x^\alpha D_{-\alpha}\left(x;\frac{\kappa}{\lambda}\right)\Big|_{x=1}^{X} - \alpha\int_1^X D_{-\alpha}\left(x;\frac{\kappa}{\lambda}\right)x^{\alpha-1}\,dx \\ &= \frac{X^{1+\alpha}}{1+\alpha}\lambda^{-1-\alpha}\zeta(1+\alpha) + X\lambda^{\alpha-1}\zeta(1-\alpha) + X^\alpha\Delta_{-\alpha}\left(X;\frac{\kappa}{\lambda}\right) \\ &\quad -\alpha\int_1^X x^{\alpha-1}\Delta_{-\alpha}\left(x;\frac{\kappa}{\lambda}\right)dx + O(1). \end{aligned}$$

We have

$$\begin{aligned} &\alpha\int_1^X x^{\alpha-1}\Delta_{-\alpha}\left(x;\frac{\kappa}{\lambda}\right)dx \\ &= -\alpha\sum_{n\le N}\sigma_\alpha(n)\,e\left(-n\frac{\overline{\kappa}}{\lambda}\right)n^{-\frac12-\frac{\alpha}{2}}\int_1^X x^{\frac{\alpha}{2}-\frac12}\left(\cos\left(\frac{\pi\alpha}{2}\right)Y_{1-\alpha}\left(\frac{4\pi(nx)^{\frac12}}{\lambda}\right)\right. \\ &\quad \left.-\sin\left(\frac{\pi\alpha}{2}\right)J_{1-\alpha}\left(\frac{4\pi(nx)^{\frac12}}{\lambda}\right)\right)dx + O\left(\lambda N^{-\frac12}X^{\frac12+a+\varepsilon}\right). \end{aligned}$$

Applying Lemma 2.4 by use of the asymptotic expansions (2.33) and (2.34), the integral is bounded by $\ll \lambda^{\frac32}n^{-\frac34}$, and so the sum is

$$\ll \lambda^{\frac32}\sum_{n\le N}n^{\frac{a}{2}+\varepsilon-\frac54},$$

since $\sigma_\alpha(n)\ll n^{a+\varepsilon}$. Thus the integral above is bounded by $\lambda^{\frac32}$ if $a<\frac12$. Finally we find our "new" Δ as

$$\begin{aligned} &X^\alpha\Delta_{-\alpha}\left(X;\frac{\kappa}{\lambda}\right) \\ &= -X^{\frac12+\frac{\alpha}{2}}\sum_{n\le N}\sigma_\alpha(n)\,e\left(-n\frac{\overline{\kappa}}{\lambda}\right)n^{-\frac12-\frac{\alpha}{2}}\left(\cos\left(\frac{\pi\alpha}{2}\right)Y_{1-\alpha}\left(\frac{4\pi(nX)^{\frac12}}{\lambda}\right)\right. \\ &\quad \left.-\sin\left(\frac{\pi\alpha}{2}\right)J_{1-\alpha}\left(\frac{4\pi(nX)^{\frac12}}{\lambda}\right)\right) + O\left(\lambda N^{-\frac12}X^{\frac12+a+\varepsilon}\right). \end{aligned}$$

So we have proved (using in the real case once more the asymptotic expansions with (2.35))

Corollary 2.8 *Let $\lambda\le X,\ 1\le N\ll X$ and $\left|\alpha-\frac14\right|<\frac14$. Then we have*

$$D_\alpha\left(X;\frac{\kappa}{\lambda}\right) = \frac{X^{1+\alpha}}{1+\alpha}\lambda^{-1-\alpha}\zeta(1+\alpha) + X\lambda^{\alpha-1}\zeta(1-\alpha) + \Delta_\alpha\left(X;\frac{\kappa}{\lambda}\right)$$

with

$$\begin{aligned}\Delta_\alpha\left(X;\frac{\kappa}{\lambda}\right) &= -X^{\frac12+\frac{\alpha}{2}}\sum_{n\le N}\sigma_\alpha(n)\,e\left(-n\frac{\overline{\kappa}}{\lambda}\right)n^{-\frac12-\frac{\alpha}{2}}\Bigg(\cos\left(\frac{\pi\alpha}{2}\right)Y_{1-\alpha}\left(\frac{4\pi(nX)^{\frac12}}{\lambda}\right)\\ &\quad -\sin\left(\frac{\pi\alpha}{2}\right)J_{1-\alpha}\left(\frac{4\pi(nX)^{\frac12}}{\lambda}\right)\Bigg)+O\left(\lambda N^{-\frac12}X^{\frac12+a+\varepsilon}+\lambda^{\frac32}\right),\end{aligned}$$

and for real α

$$\begin{aligned}\Delta_\alpha\left(X;\frac{\kappa}{\lambda}\right) &= \frac{1}{\pi\sqrt2}\lambda^{\frac12}X^{\frac14+\frac{\alpha}{2}}\sum_{n\le N}\sigma_\alpha(n)\,e\left(-n\frac{\overline{\kappa}}{\lambda}\right)n^{-\frac34-\frac{\alpha}{2}}\cos\left(\frac{4\pi(nX)^{\frac12}}{\lambda}-\frac{\pi}{4}\right)\\ &\quad +O\left(\lambda N^{-\frac12}X^{\frac12+a+\varepsilon}+\lambda^{\frac32}\right).\end{aligned}$$

We remark that it should be possible to replace $O\left(\lambda^{\frac32}\right)$ by $E_\alpha\left(0;\frac{\kappa}{\lambda}\right)$ explicitly, which is only $\ll\lambda\log\lambda$.

Putting $N=\lambda^{\frac{2}{3-2a}}X^{\frac{1-2a}{3-2a}}$, it follows immediately from the corollary that

$$\Delta_\alpha\left(X;\frac{\kappa}{\lambda}\right)\ll\lambda^{\frac{2-2a}{3-2a}}X^{\frac{1}{3-2a}+a+\varepsilon}+\lambda^{\frac32}, \tag{2.36}$$

but one might conjecture more; sometimes Δ_α must be smaller since

Theorem 2.9 *For* $1\le\lambda\le X$ *and* $\left|\alpha-\frac14\right|<\frac14$ *we have*

$$\int_{\frac{X}{2}}^{X}\left|\Delta_\alpha\left(x,\frac{\kappa}{\lambda}\right)\right|^2dx \ll \lambda X^{\frac32+a}+\lambda^2\left(X^{1+2a+\varepsilon}+X^{\frac54+\frac{a}{2}+\varepsilon}\right)+\lambda^3X,$$

and for real α

$$\begin{aligned}\int_{1}^{X}\left|\Delta_\alpha\left(x,\frac{\kappa}{\lambda}\right)\right|^2dx &= \lambda X^{\frac32+\alpha}\,\frac{\zeta\left(\frac32+\alpha\right)\zeta\left(\frac32-\alpha\right)\zeta^2\left(\frac32\right)}{(6+4\alpha)\pi^2\,\zeta(3)}\\ &\quad +O\left(\lambda^2\left(X^{1+2a+\varepsilon}+X^{\frac54+\frac{a}{2}+\varepsilon}\right)+\lambda^3X\right).\end{aligned}$$

Proof. First we consider the "real case" $\alpha=a$: integration of the truncated Voronoi-type-formula of Corollary 2.8 above with $N=X$ gives

$$\begin{aligned}&\int_{\frac{X}{2}}^{X}\left|\Delta_\alpha\left(x,\frac{\kappa}{\lambda}\right)\right|^2dx\\ &= \frac{\lambda}{4\pi^2}\sum_{m,n\le X}\sigma_\alpha(m)\sigma_\alpha(n)\,e\left((n-m)\frac{\overline{\kappa}}{\lambda}\right)(mn)^{-\frac34-\frac{\alpha}{2}}\\ &\quad\times\int_{\frac{X}{2}}^{X}x^{\frac12+\alpha}\cos\left(\frac{4\pi(mx)^{\frac12}}{\lambda}-\frac{\pi}{4}\right)\cos\left(\frac{4\pi(nx)^{\frac12}}{\lambda}-\frac{\pi}{4}\right)dx\\ &\quad+O\Bigg(\lambda^3X+\lambda^2X^{1+2a+\varepsilon}+\left(\lambda^{\frac32}X^{\frac14+\frac{3a}{2}+\varepsilon}+\lambda^2X^{\frac14+\frac{a}{2}}\right)\\ &\quad\times\int_{\frac{X}{2}}^{X}\left|\sum_{n\le X}\sigma_\alpha(n)\,e\left(n\frac{\overline{\kappa}}{\lambda}\right)n^{-\frac34-\frac{\alpha}{2}}x^{\frac14+\frac{\alpha}{2}}\cos\left(\frac{4\pi(nx)^{\frac12}}{\lambda}-\frac{\pi}{4}\right)\right|dx\Bigg).\end{aligned} \tag{2.37}$$

Since one easily calculates

$$\int x^{\mu} \cos^2\left(\nu x^{\frac{1}{2}}\right) dx = \frac{x^{1+\mu}}{1+\mu} + O\left(x^{\frac{1}{2}+\mu}\right) \qquad (\mu > -1)$$

we have

$$\sum_{n>X} \sigma_\alpha^2(n) n^{-\frac{3}{2}-\alpha} \int_{\frac{X}{2}}^{X} x^{\frac{1}{2}+\alpha} \cos^2\left(\frac{4\pi(nx)^{\frac{1}{2}}}{\lambda} - \frac{\pi}{4}\right) dx \ll X^{1+2a+\varepsilon}.$$

So the diagonal terms $m = n$ in the sum above deliver

$$\begin{aligned} &\frac{\lambda}{4\pi^2} \sum_{n\le X} \sigma_\alpha^2(n) n^{-\frac{3}{2}-\alpha} \int_{\frac{X}{2}}^{X} x^{\frac{1}{2}+\alpha} \cos^2\left(\frac{4\pi(mx)^{\frac{1}{2}}}{\lambda} - \frac{\pi}{4}\right) dx \\ = \;& \lambda X^{\frac{3}{2}+\alpha} \frac{1-2^{-\frac{3}{2}-\alpha}}{(6+4\alpha)\pi^2} \sum_{n=1}^{\infty} \sigma_\alpha^2(n) n^{-\frac{3}{2}-\alpha} + O\left(\lambda X^{1+2a+\varepsilon}\right). \end{aligned}$$

Otherwise, the nondiagonal terms $m \neq n$ contribute

$$\begin{aligned} = \;& \frac{\lambda}{\pi^2} \sum_{\substack{m,n \\ n<m\le X}} \sigma_\alpha(m)\sigma_\alpha(n)\, e\left((n-m)\frac{\overline{\kappa}}{\lambda}\right)(mn)^{-\frac{3}{4}-\frac{\alpha}{2}} \\ & \times \left\{ \int_{\frac{X}{2}}^{X} x^{\frac{1}{2}+\alpha} \cos\left(\frac{4\pi}{\lambda}\left(\sqrt{mx}-\sqrt{nx}\right)\right) dx \right. \\ & \left. + \int_{\frac{X}{2}}^{X} x^{\frac{1}{2}+\alpha} \sin\left(\frac{4\pi}{\lambda}\left(\sqrt{mx}+\sqrt{nx}\right)\right) dx \right\} \\ =: \;& S_1 + S_2. \end{aligned}$$

Once more with Lemma 2.4 we find

$$\int_{\frac{X}{2}}^{X} x^{\frac{1}{2}+\alpha} \sin\left(\frac{4\pi}{\lambda}\left(\sqrt{mx}+\sqrt{nx}\right)\right) dx \ll \frac{\lambda X^{1+a}}{\sqrt{m}+\sqrt{n}}.$$

Thus $S_2 \ll \lambda^2 X^{1+a}$. For S_1 we have to work a little bit more: analogously we get for $m > n$

$$\int_{\frac{X}{2}}^{X} x^{\frac{1}{2}+\alpha} \cos\left(\frac{4\pi}{\lambda}\left(\sqrt{mx}-\sqrt{nx}\right)\right) dx \ll \frac{\lambda X^{1+a}}{\sqrt{m}-\sqrt{n}}$$

and therefore

$$S_1 \;\ll\; \lambda^2 X^{1+a} \left\{ \sum_{\substack{m,n \\ m\le X, n\le \frac{m}{2}}} + \sum_{\substack{m,n \\ \frac{m}{2}\le n<m\le X}} \right\} \sigma_a(m)\sigma_a(n)(mn)^{-\frac{3}{4}-\frac{a}{2}}(\sqrt{m}-\sqrt{n})^{-1}.$$

We find that the first sum is

$$\ll \sum_{m\le X} \sigma_a(m) m^{-\frac{5}{4}-\frac{a}{2}} \sum_{n\le \frac{m}{2}} \sigma_a(n) n^{-\frac{3}{4}-\frac{a}{2}} \ll X^{a+\varepsilon},$$

and that the second sum is

$$\ll \sum_{m\le X}\sigma_a(m)m^{-1-a}\sum_{\frac{m}{2}<n<m}\sigma_a(n)(m-n)^{-1}\ll X^{a+\varepsilon}.$$

Hence altogether $S_1+S_2\ll\lambda^2X^{1+2a+\varepsilon}$. With the Cauchy-Schwarz-inequality we now estimate the integral in the error term of (2.37)

$$\int_{\frac{X}{2}}^{X}\left|\sum_{n\le X}\sigma_\alpha(n)\,e\left(-n\frac{\overline{\kappa}}{\lambda}\right)n^{-\frac34-\frac{\alpha}{2}}\cos\left(\frac{4\pi(nx)^{\frac12}}{\lambda}-\frac{\pi}{4}\right)\right|\,dx$$
$$\le\ X^{\frac12}\left(\int_{\frac{X}{2}}^{X}\left|\sum_{n\le X}\sigma_\alpha(n)\,e\left(-n\frac{\overline{\kappa}}{\lambda}\right)n^{-\frac34-\frac{\alpha}{2}}\cos\left(\frac{4\pi(nx)^{\frac12}}{\lambda}-\frac{\pi}{4}\right)\right|^2\,dx\right)^{\frac12}.$$

Rewriting the squared modulus as a product of conjugates we bound the diagonal and nondiagonal terms by $\ll\lambda^2X^{\frac54+\frac{a}{2}+\varepsilon}$ in the same way as above. Summing up, we have

$$(2.38)\qquad \int_{\frac{X}{2}}^{X}\left|\Delta_\alpha\left(x,\frac{\kappa}{\lambda}\right)\right|^2\,dx \;=\; \lambda X^{\frac32+\alpha}\frac{1-2^{-\frac32-\alpha}}{(6+4\alpha)\pi^2}\sum_{n=1}^{\infty}\sigma_\alpha^2(n)n^{-\frac32-\alpha}$$
$$+O\left(\lambda^2\left(X^{1+2a+\varepsilon}+X^{\frac54+\frac{a}{2}+\varepsilon}\right)+\lambda^3X\right).$$

It is well known (see [40], formula (1.3.3)) that

$$\sum_{n=1}^{\infty}\sigma_\alpha^2(n)n^{-\frac32-\alpha}=\frac{\zeta\left(\frac32+\alpha\right)\zeta\left(\frac32-\alpha\right)\zeta^2\left(\frac32\right)}{\zeta(3)}.$$

Then, applying formula (2.38) with $2^{1-k}X$ instead of X and adding up the results for $k\in\mathbb{N}$, the assertion for real α follows. In the case of $\alpha\notin\mathbb{R}$ we may argue analogously. Using once more the asymptotic expansions (2.33) and (2.34) we get the estimate of Theorem 2.9.•

From Theorem 2.9 the existence of some x with small $\Delta_\alpha\left(x;\frac{\kappa}{\lambda}\right)$ follows immediately: by the Cauchy-Schwarz-inequality we have for $0<a<\frac14$

$$(2.39)\qquad \min_{\frac{X}{2}\le x\le X}\left|\Delta_\alpha\left(x;\frac{\kappa}{\lambda}\right)\right| \;\ll\; X^{-\frac12}\left(\int_{\frac{X}{2}}^{X}\left|\Delta_\alpha\left(x,\frac{\kappa}{\lambda}\right)\right|^2\,dx\right)^{\frac12}$$
$$\ll\; \lambda^{\frac12}X^{\frac14+\frac{\alpha}{2}}+\lambda\left(X^{\frac18+\frac{\alpha}{4}+\varepsilon}+X^{\alpha+\varepsilon}\right)+\lambda^{\frac32}.$$

Obviously $\sigma_\alpha(n)$ is differentiable with respect to α, so let

$$\sigma_\alpha^{[m]}(n)=\left(\frac{\partial}{\partial\alpha}\right)^m\sigma_\alpha(n)\qquad(m\in\mathbb{N}_0)$$

and accordingly for related functions. We are now able to state formulas for the derivatives of $D_\alpha\left(x;\frac{\kappa}{\lambda}\right)$ with respect to α. Let

$$(2.40)\qquad \Theta_\alpha\left(X;\frac{\kappa}{\lambda}\right)=\frac{X^{1+\alpha}}{1+\alpha}\lambda^{-1-\alpha}\zeta(1+\alpha)+X\lambda^{\alpha-1}\zeta(1-\alpha).$$

This defines an analytic function in α. Moreover, $\Delta_\alpha\left(X;\frac{\kappa}{\lambda}\right) = D_\alpha\left(X;\frac{\kappa}{\lambda}\right) - \Theta_\alpha\left(X;\frac{\kappa}{\lambda}\right)$ is analytic. Hence we have

$$D_\alpha^{[m]}\left(X;\frac{\kappa}{\lambda}\right) = \Theta_\alpha^{[m]}\left(X;\frac{\kappa}{\lambda}\right) + \Delta_\alpha^{[m]}\left(X;\frac{\kappa}{\lambda}\right).$$

One easily calculates

$$\begin{aligned}(2.41)\qquad \Theta_\alpha^{[1]}\left(X;\frac{\kappa}{\lambda}\right) &= \frac{X^{1+\alpha}}{1+\alpha}\lambda^{-1-\alpha}\left\{\zeta(1+\alpha)\left(\log X - \frac{1}{1+\alpha} - \log\lambda\right) + \zeta'(1+\alpha)\right\}\\ &\quad + X\lambda^{\alpha-1}\left(\log\lambda\zeta(1-\alpha) - \zeta'(1-\alpha)\right)\end{aligned}$$

and

$$\begin{aligned}(2.42)\qquad &\Theta_\alpha^{[2]}\left(X;\frac{\kappa}{\lambda}\right)\\ &= \frac{X^{1+\alpha}}{1+\alpha}\lambda^{-1-\alpha}\left\{\left(\frac{2}{(1+\alpha)^2} + (\log X - \log\lambda)\left(\log X - \log\lambda - \frac{2}{1+\alpha}\right)\right)\right.\\ &\quad \left.\times\zeta(1+\alpha) + 2\zeta'(1+\alpha)\left(\log X - \log\lambda - \frac{1}{1+\alpha}\right) + \zeta''(1+\alpha)\right\}\\ &\quad + X\lambda^{\alpha-1}\left\{(\log\lambda)^2\zeta(1-\alpha) - 2\log\lambda\zeta'(1-\alpha) + \zeta''(1-\alpha)\right\}.\end{aligned}$$

It remains to give estimates of $\Delta_\alpha^{[m]}\left(X;\frac{\kappa}{\lambda}\right)$. With the Cauchy integral formula we find a generalization of (2.36) for $0 < a < \frac{1}{6}$, namely

$$\begin{aligned}(2.43)\qquad \Delta_\alpha^{[m]}\left(X;\frac{\kappa}{\lambda}\right) &= \frac{m!}{2\pi i}\int_{|\alpha-z|=\frac{a}{2}} \frac{\Delta_z\left(X;\frac{\kappa}{\lambda}\right)}{(\alpha-z)^{m+1}}\,dz\\ &\ll \frac{m!}{a^m}\max_{|\alpha-z|=\frac{a}{2}}\left|\Delta_z\left(X;\frac{\kappa}{\lambda}\right)\right|\\ &\ll \frac{m!}{a^m}\left(\lambda^{\frac{2-3a}{3-3a}}X^{\frac{1}{3-3a}+\frac{3a}{2}+\varepsilon} + \lambda^{\frac{3}{2}}\right).\end{aligned}$$

But as in the case $m = 0$ we find better estimates for certain values of X. Theorem 2.9 and the Cauchy-Schwarz-inequality imply

$$\begin{aligned}&\int_{\frac{X}{2}}^{X}\left|\Delta_\alpha^{[m]}\left(x;\frac{\kappa}{\lambda}\right)\right|^2 dx\\ &= \int_{\frac{X}{2}}^{X}\left|\frac{m!}{2\pi i}\int_{|\alpha-z|=\frac{a}{2}}\frac{\Delta_z\left(x;\frac{\kappa}{\lambda}\right)}{(\alpha-z)^{m+1}}\,dz\right|^2 dx\\ &\ll \left(\frac{m!}{a^m}\right)^2\left(\lambda X^{\frac{3}{2}+\frac{3a}{2}} + \lambda^2\left(X^{1+3a+\varepsilon} + X^{\frac{5}{4}+\frac{3a}{4}+\varepsilon}\right) + \lambda^3 X\right).\end{aligned}$$

Hence we may replace (2.39) by

$$(2.44)\qquad \min_{\frac{X}{2}\le x\le X}\left|\Delta_\alpha^{[m]}\left(x;\frac{\kappa}{\lambda}\right)\right| \ll \frac{m!}{a^m}\left(\lambda^{\frac{1}{2}}X^{\frac{1}{4}+\frac{3a}{4}} + \lambda\left(X^{\frac{1}{8}+\frac{3a}{8}+\varepsilon} + X^{\frac{3a}{2}+\varepsilon}\right) + \lambda^{\frac{3}{2}}\right)$$

for $0 < a < \frac{1}{6}$.

2.1.7 Analytic continuation

We need an analytic continuation of $g(u,v)$ to a domain in the critical strip that includes the line $\operatorname{Re} u = \operatorname{Re} v = a = \frac{1}{2} - \frac{R}{L}$. Let

$$h(u,v,x) = \int_0^\infty y^{-u}(1+y)^{-v}\, e\left(\frac{xy}{\kappa\lambda}\right) dy.$$

This defines an analytic function for $\operatorname{Re} u < 1$ and $0 < \operatorname{Re} v < 1$. By Cauchy's theorem we have also

$$h(u,v,x) = \int_0^{i\infty} y^{-u}(1+y)^{-v}\, e\left(\frac{xy}{\kappa\lambda}\right) dy, \tag{2.45}$$

which we shall use frequently in the sequel. Let $N \in \mathbb{N}$, then we have by Stieltjes integration for $\operatorname{Re} u < 0$

$$\begin{aligned}\sum_{n>N} \sigma_{1-u-v}(n)\, e\left(n\frac{\overline{\kappa}}{\lambda}\right) h(u,v,n) &= \int_{N+\frac{1}{2}}^\infty h(u,v,x)\, dD_{1-u-v}\left(x;\frac{\overline{\kappa}}{\lambda}\right)\\ &= -h\left(u,v,N+\frac{1}{2}\right)\Delta_{1-u-v}\left(N+\frac{1}{2};\frac{\overline{\kappa}}{\lambda}\right)\\ &\quad + \int_{N+\frac{1}{2}}^\infty h(u,v,x)\frac{\partial}{\partial x}\Theta_{1-u-v}\left(x;\frac{\overline{\kappa}}{\lambda}\right) dx\\ &\quad - \int_{N+\frac{1}{2}}^\infty \Delta_{1-u-v}\left(x;\frac{\overline{\kappa}}{\lambda}\right)\frac{\partial}{\partial x} h(u,v,x)\, dx.\end{aligned}$$

With (2.22) follows that

$$\begin{aligned} g(u,v) &= \sum_{n=1}^\infty h(u,v,n)\sigma_{1-u-v}(n)\, e\left(n\frac{\overline{\kappa}}{\lambda}\right)\\ &= \sum_{n\le N} \sigma_{1-u-v}(n)\, e\left(n\frac{\overline{\kappa}}{\lambda}\right) h(u,v,n)\\ &\quad -h\left(u,v,N+\frac{1}{2}\right)\Delta_{1-u-v}\left(N+\frac{1}{2};\frac{\overline{\kappa}}{\lambda}\right)\\ &\quad + \int_{N+\frac{1}{2}}^\infty h(u,v,x)\frac{\partial}{\partial x}\Theta_{1-u-v}\left(x;\frac{\overline{\kappa}}{\lambda}\right) dx\\ &\quad - \int_{N+\frac{1}{2}}^\infty \Delta_{1-u-v}\left(x;\frac{\overline{\kappa}}{\lambda}\right)\frac{\partial}{\partial x} h(u,v,x)\, dx\\ &=: \sum_{j=1}^4 g_j(u,v).\end{aligned} \tag{2.46}$$

Obviously the functions $g_1(u,v)$ and $g_2(u,v)$ are analytic in the same range as $h(u,v,x)$, in particular if $u,v \in \mathcal{D} := \left\{s\ :\ \frac{1}{2}-\varepsilon < \sigma < \frac{1}{2}\right\}$ by small positive ε and T large enough. For $g_3(u,v)$ we have to work a little bit more: by (2.40)

$$\frac{\partial}{\partial x}\Theta_{1-u-v}\left(x;\frac{\overline{\kappa}}{\lambda}\right) = x^{1-u-v}\lambda^{-2+u+v}\zeta(2-u-v) + \lambda^{-u-v}\zeta(u+v).$$

Thus we consider

$$g_3(u,v) = \int_{N+\frac{1}{2}}^{\infty} h(u,v,x)\left(x^{1-u-v}\lambda^{-2+u+v}\zeta(2-u-v) + \lambda^{-u-v}\,\zeta(u+v)\right) dx.$$

With (2.45) we get

$$\begin{aligned}
&\int_{N+\frac{1}{2}}^{\infty}\int_0^{\infty} y^{-u}(1+y)^{-v}\, e\left(\frac{xy}{\kappa\lambda}\right) \\
&\qquad \times \left(x^{1-u-v}\lambda^{-2+u+v}\zeta(2-u-v) + \lambda^{-u-v}\,\zeta(u+v)\right) dy\,dx \\
&= \int_{N+\frac{1}{2}}^{\infty} \left(x^{1-u-v}\lambda^{-2+u+v}\zeta(2-u-v) + \lambda^{-u-v}\,\zeta(u+v)\right) \\
&\qquad \times \int_0^{i\infty} y^{-u}(1+y)^{-v}\, e\left(\frac{xy}{\kappa\lambda}\right) dy\,dx.
\end{aligned}$$

Now it is easy to calculate

$$\begin{aligned}
&\int_{N+\frac{1}{2}}^{\infty}\int_0^{i\infty} y^{-u}(1+y)^{-v}\, e\left(\frac{xy}{\kappa\lambda}\right) dy\,dx \\
&= -\frac{\kappa\lambda}{2\pi i}\int_0^{\infty} y^{-u-1}(1+y)^{-v}\, e\left(\frac{\left(N+\frac{1}{2}\right)y}{\kappa\lambda}\right) dy.
\end{aligned}$$

Similarly we have

$$\begin{aligned}
&\int_{N+\frac{1}{2}}^{\infty}\int_0^{i\infty} x^{1-u-v} y^{-u}(1+y)^{-v}\, e\left(\frac{xy}{\kappa\lambda}\right) dy\,dx \\
&= -\left(N+\frac{1}{2}\right)^{1-u-v}\frac{\kappa\lambda}{2\pi i}\int_0^{i\infty} y^{-u-1}(1+y)^{-v}\, e\left(\frac{\left(N+\frac{1}{2}\right)y}{\kappa\lambda}\right) dy \\
&\qquad -(1-u-v)\frac{\kappa\lambda}{2\pi i}\int_{N+\frac{1}{2}}^{\infty} x^{-u-v}\int_0^{i\infty} y^{-u-1}(1+y)^{-v}\, e\left(\frac{xy}{\kappa\lambda}\right) dy\,dx.
\end{aligned}$$

Substituting $w = xy$, the second integral above equals

$$\begin{aligned}
&\int_0^{i\infty} e\left(\frac{w}{\kappa\lambda}\right) w^{-1-u}\int_{N+\frac{1}{2}}^{\infty} (x+w)^{-v}\,dx\,dw \\
&\quad = \frac{1}{1-v}\int_0^{i\infty} w^{-1-u}\left(N+\frac{1}{2}+w\right)^{1-v} e\left(\frac{w}{\kappa\lambda}\right) dw \\
&\quad = \frac{1}{1-v}\left(N+\frac{1}{2}\right)^{1-u-v}\int_0^{i\infty} y^{-u-1}(1+y)^{1-v}\, e\left(\frac{\left(N+\frac{1}{2}\right)y}{\kappa\lambda}\right) dy.
\end{aligned}$$

Summing up, we have once more with (2.45)

$$\begin{aligned}
&g_3(u,v) \qquad (2.47)\\
&= -\frac{\kappa\lambda}{2\pi i}\left\{\left(\lambda^{-u-v}\,\zeta(u+v) + \lambda^{-2+u+v}\zeta(2-u-v)\left(N+\frac{1}{2}\right)^{1-u-v}\right)\times\right.
\end{aligned}$$

$$\times \int_0^\infty y^{-u-1}(1+y)^{-v}\, e\left(\frac{\left(N+\frac{1}{2}\right)y}{\kappa\lambda}\right) dy$$

$$+\lambda^{-2+u+v}\zeta(2-u-v)\frac{1-u-v}{1-v}\left(N+\frac{1}{2}\right)^{1-u-v}$$

$$\times \int_0^\infty y^{-u-1}(1+y)^{1-v}\, e\left(\frac{\left(N+\frac{1}{2}\right)y}{\kappa\lambda}\right) dy\Bigg\}.$$

But these integrals are obviously uniformly convergent for $u, v \in \mathcal{D}$, so we have an analytic continuation of $g_3(u,v)$ into $\mathcal{D}$. Finally we consider

$$g_4(u,v) = -\int_{N+\frac{1}{2}}^\infty \Delta_{1-u-v}\left(x; \frac{\overline{\kappa}}{\lambda}\right) \frac{\partial}{\partial x} h(u,v,x)\, dx.$$

We have

$$\begin{aligned}
\frac{\partial}{\partial x} h(u,v,x)\, dx &= \frac{2\pi i}{\kappa\lambda}\int_0^{i\infty} y^{1-u}(1+y)^{-v}\, e\left(\frac{xy}{\kappa\lambda}\right) dy \\
&= \frac{2\pi i}{\kappa\lambda} x^{u-2}\int_0^{i\infty} w^{1-u}\left(1+\frac{w}{x}\right)^{-v} e\left(\frac{w}{\kappa\lambda}\right) dw \\
&\ll x^{\mathrm{Re}\, u-2}.
\end{aligned}$$

By (2.36) we get

$$\Delta_{1-u-v}\left(x; \frac{\overline{\kappa}}{\lambda}\right) \ll x^{\frac{1}{1+2\mathrm{Re}\,(u+v)}+1-\mathrm{Re}\,(u+v)+\varepsilon}.$$

Hence $g_4(u,v)$ is an analytic function if $u, v \in \mathcal{D}$. Therefore, we have found by use of the Voronoi-type-formula of §2.1.6 an analytic continuation of $g(u,v)$ to the domain $\mathcal{D}$ in the critical strip that includes the line Re $u =$ Re $v = a$.

2.1.8 Exponential averaging

The following technique is due to Jutila (see [24] or [21], §15.5) and often leads to better estimates than other ones (compare for example the error terms in the asymptotic mean square formulas of the product of the zeta-function with a Dirichlet polynomial in [3] and [32]). It bounds our error term by an average. Define

$$\int_0^T |AF(a+it)|^2\, dz = I(T) + E(T),$$

where $I(T)$ denotes the main term and $E(T)$ the error term. The mean square is obviously an increasing function for positive T. Thus we have for $0 \le V \le t$

$$E(V) \le E(t) + I(t) - I(V) = E(t) + I(V, t-V)$$

since by the definition (2.5) of the mean square in short intervals $I(T,H) = I(T+H) - I(T)$. So by (2.21)

$$E(V) \le E(t) + O(t-V).$$

With the formula

$$\text{(2.48)}\qquad \int_{-\infty}^{\infty} \exp\left(Ax - Bx^2\right) dx = \sqrt{\frac{\pi}{B}} \exp\left(\frac{A^2}{4B}\right) \qquad (\text{Re } B > 0)$$

(which follows by quadratic completion from the well known formula $\int_{-\infty}^{\infty} \exp\left(-x^2\right) dx = \sqrt{\pi}$) integration of the last inequality leads to

$$\frac{\sqrt{\pi}}{2} GE(V) \leq \int_0^{\infty} E(V+u) \exp\left(-\frac{u^2}{G^2}\right) du + O(G^2).$$

Note that $E(-u) = E(u)$ and $E(u) \ll u$ (by [29]). Thus we obtain

$$\text{(2.49)}\qquad E(T) \ll \max_{\frac{T}{2} \leq V \leq 2T} \min_{L \leq G \leq VL^{-1}} \left\{ G + \frac{1}{G} \left| \int_{-\infty}^{\infty} E(V+u) \exp\left(-\frac{u^2}{G^2}\right) du \right| \right\}.$$

Of course this also leads to an upper bound for the error term of the mean square for short intervals since $E(T,H) = E(T+H) - E(T)$ by (2.5).

2.1.9 Bounding the error term

Let $\delta = \frac{1}{2} - a$. Since (2.46) guarantees an analytic continuation of $g(u,v)$ to a domain in the critical strip that includes Re $u =$ Re $v = a$, the derivatives with respect to u and v are regular in the same range as well. By (2.23) a typical summand of $G(u,v)$ is

$$\sum_{k,l \leq M} a(k)a(l)w(k,l) \frac{(k,l)^{2-u-v}}{kl} \sum_{n=1}^{\infty} \sigma_{2\delta}^{[m]}(n)\, e\left(n\frac{\overline{\kappa}}{\lambda}\right)$$
$$\times \int_0^{\infty} \frac{(\log y)^b (\log(1+y))^c}{y^u (1+y)^v}\, e\left(\frac{ny}{\kappa\lambda}\right) dy,$$

where the log-factors arise from differentiation of (2.22) with respect to u and v; note that $0 \leq b, c, m \leq 2$ (higher derivatives do not occur). Since for fixed k, l we now have to deal with the integral above with $u = z, v = 2a - z$ (the other case $u = 2a - z, v = z$ can be treated analogously) instead of $h(u,v;x)$ as in §2.1.7, we define

$$h(z,x) = \int_0^{\infty} \frac{(\log y)^b (\log(1+y))^c}{y^z (1+y)^{2a-z}}\, e\left(\frac{xy}{\kappa\lambda}\right) dy.$$

Differentiation of (2.46) at $u = z, v = 2a - z$ leads to

$$\begin{aligned}
\text{(2.50)}\qquad g(z) \;&:=\; \sum_n \sigma_{2\delta}^{[m]}(n)\, e\left(n\frac{\overline{\kappa}}{\lambda}\right) h(z,n) \\
&=\; \sum_{n \leq N} \sigma_{2\delta}^{[m]}(n)\, e\left(n\frac{\overline{\kappa}}{\lambda}\right) h(z,n) \\
&\quad -h\left(z, N+\frac{1}{2}\right) \Delta_{2\delta}^{[m]}\left(N+\frac{1}{2}; \frac{\overline{\kappa}}{\lambda}\right) \\
&\quad + \int_{N+\frac{1}{2}}^{\infty} h(z,x) \frac{\partial}{\partial x} \Theta_{2\delta}^{[m]}\left(x; \frac{\overline{\kappa}}{\lambda}\right) dx \;+
\end{aligned}$$

$$
\begin{aligned}
&-\int_{N+\frac{1}{2}}^{\infty} \Delta_{2\delta}^{[m]}\left(x; \frac{\overline{\kappa}}{\lambda}\right) \frac{\partial}{\partial x} h(z,x)\,dx \\
=: &\sum_{j=1}^{4} g_j(z).
\end{aligned}
$$

Now we take an exponential average over

$$
\int_{a-iT}^{a+iT} g(z)dz = \int_{a-iT}^{a+iT} \sum_{j=1}^{4} g_j(z)dz =: \sum_{j=1}^{4} E_j(T)
$$

(later we sum over k, l). Let $\frac{T}{2} \le V \le 2T$ and $L \le G \le VL^{-1}$. Up to now N is a free parameter, but with regard to $g_2(z)$ we may choose an N with $\kappa\lambda V \le N \le 2\kappa\lambda V$ such that

$$
\Delta_{2\delta}^{[m]}\left(N+\frac{1}{2}; \frac{\overline{\kappa}}{\lambda}\right) \ll L^2\left(\lambda^{\frac{1}{2}} N^{\frac{1}{4}} + \lambda N^{\frac{1}{8}+\varepsilon} + \lambda^{\frac{3}{2}}\right) \tag{2.51}
$$

by (2.44) (note that $T^{c\delta} \ll 1$ for every constant c, so we can frequently eliminate certain factors like $N^{2\delta}$). First we consider

$$
\begin{aligned}
&\int_{-\infty}^{\infty} E_1(V+u) \exp\left(-\frac{u^2}{G^2}\right) du \\
&= \sum_{n\le N} \sigma_{2\delta}^{[m]}(n)\, e\left(n\frac{\overline{\kappa}}{\lambda}\right) \int_{-\infty}^{\infty} \int_{a-i(V+u)}^{a+i(V+u)} h(z,n) \exp\left(-\frac{u^2}{G^2}\right) dz\,du
\end{aligned}
\tag{2.52}
$$

Let $l(y) = \log\left(1+\frac{1}{y}\right)$, then

$$
y^z(1+y)^{2a-z} = y^a(1+y)^a e^{-itl(y)}.
$$

Now with (2.48) we have

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \int_{a-i(V+u)}^{a+i(V+u)} h(z,n) \exp\left(-\frac{u^2}{G^2}\right) dz\,du \\
&= \int_{-\infty}^{\infty} \int_{a-i(V+u)}^{a+i(V+u)} \int_0^{\infty} \frac{(\log y)^b (\log(1+y))^c}{y^z(1+y)^{2a-z}}\, e\left(\frac{ny}{\kappa\lambda}\right) \exp\left(-\frac{u^2}{G^2}\right) dy\,dz\,du \\
&= \int_0^{\infty} \frac{(\log y)^b (\log(1+y))^c}{y^a(1+y)^a l(y)}\, e\left(\frac{ny}{\kappa\lambda}\right) \\
&\quad\times \int_{-\infty}^{\infty} \left(e^{i(V+u)l(y)} - e^{-i(V+u)l(y)}\right) \exp\left(-\frac{u^2}{G^2}\right) du\,dy \\
&= 2\sqrt{\pi} i G \int_0^{\infty} \frac{(\log y)^b (\log(1+y))^c}{y^a(1+y)^a l(y)}\, e\left(\frac{ny}{\kappa\lambda}\right) \exp\left(-\frac{G^2 l^2(y)}{4}\right) \sin\left(Vl(y)\right) dy
\end{aligned}
\tag{2.53}
$$

(changing the order of integration is obviously allowed by Fubini's theorem). Hence we have with (2.52)

$$\begin{aligned} & \frac{1}{2\sqrt{\pi} i G} \int_{-\infty}^{\infty} E_1(V+u) \exp\left(-\frac{u^2}{G^2}\right) du \\ = & \sum_{n \le N} \sigma_{2\delta}^{[m]}(n)\, e\left(n\frac{\overline{\kappa}}{\lambda}\right) \int_0^{\infty} p_1(n,y) q(y)\, dy \\ =: & \ P_1, \end{aligned}$$

where

$$q(y) := \frac{1}{y^{\frac{1}{2}}(1+y)^{\frac{1}{2}} l(y)} \exp\left(-\frac{G^2 l^2(y)}{4}\right)$$

and

$$p_1(x,y) := y^{\delta}(1+y)^{\delta}(\log y)^b (\log(1+y))^c\, e\left(\frac{xy}{\kappa\lambda}\right) \sin\left(V l(y)\right).$$

Later we also need

$$p_2(x,y) := y^{\delta}(1+y)^{\delta}(\log y)^b (\log(1+y))^c\, e\left(\frac{xy}{\kappa\lambda}\right) \cos\left(V l(y)\right).$$

Clearly

$$q(y) \ll \exp\left(-\frac{G^2}{20}\right) \qquad (0 < y \le 1). \tag{2.54}$$

Moreover we have

$$q'(y) = \frac{1}{y^{\frac{3}{2}}(1+y)^{\frac{3}{2}}} \left(\frac{G^2}{2} - \frac{2y+1}{2l(y)} + \frac{1}{l^2(y)}\right) \exp\left(-\frac{G^2 l^2(y)}{4}\right).$$

Hence

$$q'(y) \ll \begin{cases} \exp\left(-\frac{G^2}{20}\right) & , \quad 0 < y < 1 \\ G^2 y^{-3} \exp\left(-\frac{G^2}{4y^2}\right) & , \quad y \ge 1 \end{cases} \tag{2.55}$$

(this bound is the reason for the decomposition of the integrand above). Now let

$$\begin{aligned} P_1 &= \sum_{n \le N} \sigma_{2\delta}^{[m]}(n)\, e\left(n\frac{\overline{\kappa}}{\lambda}\right) \left\{\int_0^{GL^{-1}} + \int_{GL^{-1}}^{\infty}\right\} p_1(n,y) q(y)\, dy \\ &=: P_{1,1} + P_{1,2}. \end{aligned}$$

It is easy to show with the second mean value theorem that

$$\int_0^{GL^{-1}} p_1(n,y) q(y)\, dy \ll \exp\left(-\frac{L^2}{4}\right) (GL^{-1})^{1+\varepsilon}$$

(the exp-factor motivates splitting the integration at $y = GL^{-1}$). So we find with Corollary 2.8, (2.41) and (2.42) that

$$P_{1,1} \ll \kappa\lambda G V^{1+\varepsilon} \exp\left(-\frac{L^2}{4}\right).$$

To bound $P_{1,2}$ we define

$$r_j(x,y) = \int_{GL^{-1}}^{y} p_j(x,z)dz \qquad (j=1,2).$$

Then we have with Lemma 2.5 that

$$r_j(x,y) \ll y^{\frac{3}{2}+2\delta+\varepsilon} V^{-\frac{1}{2}}. \tag{2.56}$$

Moreover, we obtain with Lemma 2.3 for $x > \kappa\lambda V G^{-2} L^2$ that

$$r_j(x,y) \ll \frac{\kappa\lambda}{x} y^{2\delta+\varepsilon}. \tag{2.57}$$

By

$$\begin{aligned}\int_{GL^{-1}}^{\infty} p_1(n,y) q(y)\,dy &= \int_{GL^{-1}}^{\infty} q(y)\,dr_1(n,y)\\ &= -\int_{GL^{-1}}^{\infty} r_1(n,y) q'(y)\,dy\end{aligned}$$

we have with (2.55), (2.56) and (2.57)

$$\begin{aligned}P_{1,2} &= -\left\{\sum_{n\le\kappa\lambda VG^{-2}L^2} + \sum_{\kappa\lambda VG^{-2}L^2<n\le N}\right\} \sigma_{2\delta}^{[m]}(n)\, e\left(n\frac{\overline{\kappa}}{\lambda}\right) \int_{GL^{-1}}^{\infty} r_1(n,y)q'(y)\,dy\\ &\ll G^2 V^{-\frac{1}{2}} \sum_{n\le\kappa\lambda VG^{-2}L^2} \sigma_{2\delta}^{[m]}(n)\, e\left(n\frac{\overline{\kappa}}{\lambda}\right) \int_{GL^{-1}}^{\infty} y^{-\frac{3}{2}+2\delta+\varepsilon}\,dy\\ &\quad + G^2\kappa\lambda \sum_{\kappa\lambda VG^{-2}L^2<n\le N} \sigma_{2\delta}^{[m]}(n)\, e\left(n\frac{\overline{\kappa}}{\lambda}\right) \frac{1}{n} \int_{GL^{-1}}^{\infty} y^{-3+2\delta+\varepsilon}\,dy.\end{aligned}$$

By Corollary 2.8, (2.41), (2.42) and (2.43) the first term is bounded by $\ll \kappa V^{\frac{1}{2}+\varepsilon} G$. In order to bound the second one we use partial summation

$$\begin{aligned}&\sum_{\kappa\lambda VG^{-2}L^2<n\le N} \sigma_{2\delta}^{[m]}(n)\, e\left(n\frac{\overline{\kappa}}{\lambda}\right) \frac{1}{n}\\ &= \frac{1}{N} \sum_{\kappa\lambda VG^{-2}L^2<n\le N} \sigma_{2\delta}^{[m]}(n)\, e\left(n\frac{\overline{\kappa}}{\lambda}\right)\\ &\quad + \int_{\kappa\lambda VG^{-2}L^2}^{N} \sum_{\kappa\lambda VG^{-2}L^2<n\le x} \sigma_{2\delta}^{[m]}(n)\, e\left(n\frac{\overline{\kappa}}{\lambda}\right) \frac{dx}{x^2}\\ &\ll V^{\varepsilon}\lambda^{-1}.\end{aligned}$$

Since $P_{1,1}$ is negligible, we end up with

$$P_1 \ll \kappa V^{\varepsilon}\left(G^{-\frac{1}{2}} V^{\frac{1}{2}} + 1\right). \tag{2.58}$$

Now with (2.53)

$$\begin{aligned}&\frac{1}{2\sqrt{\pi} i G} \int_{-\infty}^{\infty} E_2(V+u) \exp\left(-\frac{u^2}{G^2}\right) du\\ &= -\Delta_{2\delta}^{[m]}\left(N+\frac{1}{2}\right) \int_0^{\infty} p_1\left(N+\frac{1}{2}, y\right) q(y)\,dy\\ &=: P_2.\end{aligned}$$

As above $\int_0^{GL^{-1}} p_1\left(N+\frac{1}{2},y\right) q(y)\,dy$ is negligible. The main contribution to P_2 comes from

$$\int_{GL^{-1}}^{\infty} p_1\left(N+\frac{1}{2},y\right) q(y)\,dy = -\int_{GL^{-1}}^{\infty} r_1\left(N+\frac{1}{2},y\right) q'(y)\,dy,$$

which is bounded by

$$G^2\frac{\kappa\lambda}{N}\int_{GL^{-1}}^{\infty} y^{2\delta+\varepsilon-3}\,dy \ll G^{\varepsilon}\frac{\kappa\lambda}{N}L^2,$$

using (2.55) and (2.57). Hence we get with (2.51)

$$P_2 \ll V^{\varepsilon}\left(\kappa^{\frac{1}{4}}\lambda^{\frac{3}{4}}V^{-\frac{3}{4}} + \kappa^{\frac{1}{8}}\lambda^{\frac{9}{8}}V^{-\frac{7}{8}} + \lambda^{\frac{3}{2}}V^{-1}\right). \tag{2.59}$$

We now consider with (2.47)

$$\begin{aligned} g_3(z) &= -\frac{\kappa\lambda}{2\pi i}\Bigg\{\left(Q_1(N,\lambda,\delta)\lambda^{2\delta-1} + Q_2(N,\lambda,\delta)\lambda^{-1-2\delta}\left(N+\frac{1}{2}\right)^{2\delta}\right) \\ &\quad\times\int_0^{\infty}\frac{(\log y)^b(\log(1+y))^c}{y^{1+z}(1+y)^{2a-z}}\,e\left(\frac{\left(N+\frac{1}{2}\right)y}{\kappa\lambda}\right)dy \\ &\quad+Q_3(N,\lambda,\delta)\lambda^{-1-2\delta}\left(N+\frac{1}{2}\right)^{2\delta} \\ &\quad\times\int_0^{\infty}\frac{(\log y)^b(\log(1+y))^c}{y^{1+z}(1+y)^{2a-z-1}}\,e\left(\frac{\left(N+\frac{1}{2}\right)y}{\kappa\lambda}\right)dy\Bigg\}, \end{aligned}$$

where the $Q_j(N,\lambda,\delta)$ are certain polynomials in $\log\left(N+\frac{1}{2}\right)$, $\log\lambda$ and $\frac{d}{ds}\zeta(s)$ evaluated at $s=1\pm 2\delta$. They are computed from (2.41) and (2.42) by differentiation with respect to u and v; note that $Q_j \ll N^{\varepsilon}$. Analogously to (2.53) we find

$$\begin{aligned} &\frac{\sqrt{\pi}}{G}\int_{-\infty}^{\infty} E_3(V+u)\exp\left(-\frac{u^2}{G^2}\right)du \\ &= -\kappa\lambda\Bigg\{\left(Q_1(N,\lambda,\delta)\lambda^{2\delta-1} + Q_2(N,\lambda,\delta)\lambda^{-1-2\delta}\left(N+\frac{1}{2}\right)^{2\delta}\right) \\ &\quad\times\int_0^{\infty}\frac{(\log y)^b(\log(1+y))^c}{y^{1+a}(1+y)^{a}l(y)}\,e\left(\frac{\left(N+\frac{1}{2}\right)y}{\kappa\lambda}\right)\exp\left(-\frac{u^2}{G^2}\right)\sin(Vl(y))\,dy \\ &\quad+Q_3\left(2\delta,\log\left(N+\frac{1}{2}\right)\log\lambda,\zeta(1+2\delta)\right)\lambda^{-1-2\delta}\left(N+\frac{1}{2}\right)^{2\delta} \\ &\quad\times\int_0^{\infty}\frac{(\log y)^b(\log(1+y))^c}{y^{1+a}(1+y)^{a-1}l(y)}\,e\left(\frac{\left(N+\frac{1}{2}\right)y}{\kappa\lambda}\right)\exp\left(-\frac{u^2}{G^2}\right)\sin(Vl(y))\,dy\Bigg\} \\ &=: P_3. \end{aligned}$$

Thus we have to bound

$$\int_0^\infty \frac{(\log y)^b(\log(1+y))^c}{y^{1+a}(1+y)^{a-\omega}l(y)}\, e\left(\frac{\left(N+\frac{1}{2}\right)y}{\kappa\lambda}\right)\exp\left(-\frac{G^2l^2(y)}{4}\right)\sin\left(Vl(y)\right)\,dy$$
$$= \left(\int_0^{GL^{-1}} + \int_{GL^{-1}}^\infty\right)\frac{(1+y)^\omega}{y}\, p_1\left(N+\frac{1}{2}, y\right) q(y)\,dy$$

for $\omega = 0$ and $\omega = 1$. Once more the short integrals are negligible. By the choice of N we get analogously to the estimate of $P_{1,2}$

$$\int_{GL^{-1}}^\infty \int_{GL^{-1}}^y \frac{(1+z)^\omega}{z}\, p_1\left(N+\frac{1}{2}, z\right)\, dz\, q'(y)\, dy \ll \frac{\kappa\lambda}{N} L^3 G^{\varepsilon+\omega-1}.$$

Hence

(2.60) $\quad P_3 \ll \kappa V^{\varepsilon-1}.$

Finally, we obtain by (2.53)

$$\int_{-\infty}^\infty E_4(V+u)\exp\left(-\frac{u^2}{G^2}\right)du$$
$$= -\int_{N+\frac{1}{2}}^\infty \Delta_{2\delta}^{[m]}\left(x;\frac{\overline{\kappa}}{\lambda}\right)\frac{\partial}{\partial x}\int_{-\infty}^\infty\int_{a-i(V+u)}^{a+i(V+u)}\int_0^\infty \frac{(\log y)^b(\log(1+y))^c}{y^z(1+y)^{2a-z}}$$
$$\times e\left(\frac{xy}{\kappa\lambda}\right)\exp\left(-\frac{u^2}{G^2}\right)dy\,dz\,du\,dx$$
$$= -2\sqrt{\pi} iG\int_{N+\frac{1}{2}}^\infty \Delta_{2\delta}^{[m]}\left(x;\frac{\overline{\kappa}}{\lambda}\right)\frac{\partial}{\partial x}\int_0^\infty \frac{(\log y)^b(\log(1+y))^c}{y^a(1+y)^a l(y)}$$
$$\times e\left(\frac{xy}{\kappa\lambda}\right)\exp\left(-\frac{G^2l^2(y)}{4}\right)\sin\left(Vl(y)\right)\,dy\,dx.$$

With $w = xy$ we find

$$\frac{\partial}{\partial x}\int_0^\infty \frac{(\log y)^b(\log(1+y))^c}{y^a(1+y)^a l(y)}\, e\left(\frac{xy}{\kappa\lambda}\right)\exp\left(-\frac{G^2l^2(y)}{4}\right)\sin\left(Vl(y)\right)\,dy$$
$$= \frac{\partial}{\partial x}\left\{x^{-2\delta}\int_0^\infty \frac{(\log\frac{w}{x})^b(\log(1+\frac{w}{x}))^c}{w^a(x+w)^a l(\frac{w}{x})}\right.$$
$$\left.\times e\left(\frac{w}{\kappa\lambda}\right)\exp\left(-\frac{G^2l^2(\frac{w}{x})}{4}\right)\sin\left(Vl\left(\frac{w}{x}\right)\right)\,dw\right\}$$
$$= \frac{1}{x}\int_0^\infty\int_0^\infty \frac{(\log y)^b(\log(1+y))^c}{y^a(1+y)^a l(y)}\, e\left(\frac{xy}{\kappa\lambda}\right)\exp\left(-\frac{G^2l^2(y)}{4}\right)$$
$$\times\left\{\left(2a-1-\frac{2a+2cy+G^2}{2(1+y)}-\frac{b}{\log y}-\frac{1}{(1+y)\log(1+y)}\right)\sin\left(Vl(y)\right)\right.$$
$$\left.+\frac{V\cos\left(Vl(y)\right)}{1+y}\right\}dy.$$

Hence we have

$$\begin{aligned}
&\frac{1}{2\sqrt{\pi}iG}\int_{-\infty}^{\infty} E_4(V+u)\exp\left(-\frac{u^2}{G^2}\right)du\\
&= \ -\int_{N+\frac{1}{2}}^{\infty}\frac{\Delta_{2\delta}^{[m]}\left(x;\frac{\kappa}{\lambda}\right)}{x}\int_0^{\infty}\frac{(\log y)^b(\log(1+y))^c}{y^a(1+y)^a l(y)}\, e\left(\frac{xy}{\kappa\lambda}\right)\exp\left(-\frac{G^2l^2(y)}{4}\right)\\
&\quad\times\left\{\left(2a-1-\frac{2a+2cy+G^2}{2(1+y)}-\frac{b}{\log y}-\frac{1}{(1+y)\log(1+y)}\right)\sin(Vl(y))\right.\\
&\quad\left.+\frac{V\cos(Vl(y))}{1+y}\right\}dy\,dx\\
&=: \ P_4.
\end{aligned}$$

Obviously it is sufficient to consider

$$\begin{aligned}
&\int_{N+\frac{1}{2}}^{\infty}\frac{\Delta_{2\delta}^{[m]}\left(x;\frac{\kappa}{\lambda}\right)}{x}\left\{\int_0^{\kappa\lambda Vx^{-1}}+\int_{\kappa\lambda Vx^{-1}}^{\infty}\right\}p_j(x,y)\,\frac{y^{\mu}}{(1+y)^{\nu}}\,q(y)\,dy\,dx\\
&=: \ P_{4,1}+P_{4,2}
\end{aligned}$$

for $\mu=\nu=0,1$ or $\mu=0,\nu=1$ by $j=1,2$. With (2.54) we bound

$$\int_0^{\kappa\lambda Vx^{-1}} p_j(x,y)\,\frac{y^{\mu}}{(1+y)^{\nu}}\,q(y)\,dy\,dx \le \frac{\kappa\lambda V}{x}\exp\left(-\frac{G^2}{20}\right).$$

Since by (2.43)

$$\int_{N+\frac{1}{2}}^{\infty}\frac{\Delta_{2\delta}^{[m]}\left(x;\frac{\kappa}{\lambda}\right)}{x^2}\,dx \ll \lambda^{\frac{2}{3}}N^{\varepsilon-\frac{2}{3}}+\lambda^{\frac{3}{2}}N^{\varepsilon-1},$$

$P_{4,1}$ is negligible. If $\mu=\nu=0,1$ we have as above

$$\begin{aligned}
(2.61)\qquad &\int_{\kappa\lambda Vx^{-1}}^{\infty} p_j(x,y)\,\frac{y^{\mu}}{(1+y)^{\nu}}\,q(y)\,dy\\
&= \ -\int_{\kappa\lambda Vx^{-1}}^{\infty}\int_{\kappa\lambda Vx^{-1}}^{y} p_j(x,z)\left(\frac{z}{1+z}\right)^{\mu}dz\,q'(y)\,dy.
\end{aligned}$$

With (2.55) and (2.57) this double integral is

$$\ll \frac{\kappa\lambda}{x}\left\{\exp\left(-\frac{G^2}{20}\right)+G^2\int_1^{\infty} y^{2\delta+\varepsilon-3}\exp\left(-\frac{G^2}{4y^2}\right)dy\right\}.$$

By the transformation $w=\frac{G}{y}$ one easily sees that

$$\int_1^{\infty} y^{-m}\exp\left(-\frac{G^2}{4y^2}\right)dy \ll G^{1-m}.$$

Hence

$$P_{4,2} \ll \kappa\lambda G^{\varepsilon}\int_{N+\frac{1}{2}}^{\infty}\frac{\Delta_{2\delta}^{[m]}\left(x;\frac{\kappa}{\lambda}\right)}{x^2}\,dx$$

if $\mu = \nu$. Now with the Cauchy-Schwarz-inequality and (2.44) we get

$$\begin{aligned}
&\int_{N+\frac{1}{2}}^{\infty} \frac{\Delta_{2\delta}^{[m]}\left(x;\frac{\overline{\kappa}}{\lambda}\right)}{x^2}\,dx \\
&= \sum_{n=0}^{\infty} \int_{2^n\left(N+\frac{1}{2}\right)}^{2^{n+1}\left(N+\frac{1}{2}\right)} \frac{\Delta_{2\delta}^{[m]}\left(x;\frac{\overline{\kappa}}{\lambda}\right)}{x^2}\,dx \\
&\leq \sum_{n=0}^{\infty} \left(\int_{2^n\left(N+\frac{1}{2}\right)}^{2^{n+1}\left(N+\frac{1}{2}\right)} \left|\Delta_{2\delta}^{[m]}\left(x;\frac{\overline{\kappa}}{\lambda}\right)\right|^2 dx \int_{2^n\left(N+\frac{1}{2}\right)}^{2^{n+1}\left(N+\frac{1}{2}\right)} \frac{dx}{x^4} \right)^{\frac{1}{2}} \\
&\ll N^{\varepsilon}\left(\lambda^{\frac{1}{2}}N^{-\frac{3}{4}} + \lambda N^{-\frac{7}{8}} + \lambda^{\frac{3}{2}}N^{-1}\right).
\end{aligned}$$

This leads to

$$P_{4,2} \ll V^{\varepsilon}\left(\kappa^{\frac{1}{4}}\lambda^{\frac{3}{4}}V^{-\frac{3}{4}} + \kappa^{\frac{1}{8}}\lambda^{\frac{9}{8}}V^{-\frac{7}{8}} + \lambda^{\frac{3}{2}}V^{-1}\right) \tag{2.62}$$

if $\mu = \nu$. Otherwise, when $\mu = 0$ and $\nu = 1$, we argue with

$$\int_{\kappa\lambda V x^{-1}}^{\infty} p_j(x,y)\,\frac{q(y)}{1+y}\,dy = -\int_{\kappa\lambda V x^{-1}}^{\infty} \int_{\kappa\lambda V x^{-1}}^{y} p_j(x,z)\,dz \left(\frac{q(y)}{1+y}\right)' dy$$

instead of (2.61). By a simlar estimate as above we obtain in that case the bound of (2.62) multiplied with G^{-1}. Thus

$$P_4 \ll V^{\varepsilon}\left(\kappa^{\frac{1}{4}}\lambda^{\frac{3}{4}}V^{-\frac{3}{4}} + \kappa^{\frac{1}{8}}\lambda^{\frac{9}{8}}V^{-\frac{7}{8}} + \lambda^{\frac{3}{2}}V^{-1}\right)(G + VG^{-1}); \tag{2.63}$$

note that this also bounds P_2.

Now we finish the proof. We observe that $w(k,l) \ll (kl)^{\varepsilon}$ and $a(k) \ll 1$ by definition. With (2.58), (2.59), (2.60) and (2.63) we get

$$\begin{aligned}
&\frac{1}{G}\int_{-\infty}^{\infty} E(V+u)\exp\left(-\frac{u^2}{G^2}\right) du \\
&\ll \sum_{k,l\leq M} \frac{a(k)a(l)w(k,l)}{[kl]} \max_{\frac{T}{2}\leq V\leq 2T} \min_{L\leq G\leq VL^{-1}} \{|P_1| + |P_2| + |P_3| + |P_4|\} \\
&\ll \min_{L\leq G\leq \frac{T}{2L}} M^2\left(T^{\frac{1}{2}+\varepsilon}G^{-\frac{1}{2}} + T^{\varepsilon-\frac{3}{4}}G\right).
\end{aligned}$$

Since this is bounded by the first term if $G \ll T^{\frac{5}{6}}$, we have with (2.49)

$$E(T) \ll \min_{L\leq G\leq T^{\frac{5}{6}}} \left\{G + G^{-\frac{1}{2}}T^{\frac{1}{2}+\varepsilon}M^2\right\}.$$

Balancing with $G = T^{\frac{1}{3}+\varepsilon}M^{\frac{4}{3}}$ we find for $\theta < \frac{3}{8}$

$$E(T+H) \ll E(T) \ll T^{\frac{1}{3}+\varepsilon}M^{\frac{4}{3}}.$$

Together with (2.21) we have proved Theorem 2.1. •

Perhaps with refinements one can improve Theorem 2.1: for the error term in Atkinson's asymptotic mean square formula one can get $E(T) \ll T^{\frac{7}{22}+\varepsilon}$ using exponential sums (see [22], §2.7); if the same exponent would hold in the error term above, one could obviously find positive proportions for slightly smaller H. But in view of the corresponding Ω-result $E(T) = \Omega\left(T^{\frac{1}{4}}\right)$ (see [22], §3.2) the exponent must be $\geq \frac{1}{4}$.

* * *

A different remarkable method to localize simple zeros of the zeta-function is due to Montgomery [31]. Investigating the vertical distribution of zeros he found, assuming Riemann's hypothesis, that more than two thirds of the zeros are simple. Assuming in addition his yet unproved **pair correlation conjecture**

$$\sum_{\substack{0<\gamma_1,\gamma_2\leq T\\ \frac{2\pi\alpha}{L}\leq\gamma_1-\gamma_2\leq\frac{2\pi\beta}{L}}} 1 \sim \left(\int_\alpha^\beta \left(1-\left(\frac{\sin \pi u}{\pi u}\right)^2\right) du + 1_{[\alpha,\beta]}(0)\right)\frac{TL}{2\pi},$$

where $1_{[\alpha,\beta]}$ is the characteristic function of the interval $[\alpha,\beta]$, it even follows that almost all zeros are simple!

But since all these assumptions are speculative, we now investigate the method of Conrey, Ghosh and Gonek that yields some unconditional results on simple zeros of $\zeta(s)$...

Chapter 3

The Method of Conrey, Ghosh and Gonek

Conrey, Ghosh and Gonek [9] were able to prove the existence of infinitely many nontrivial simple zeros of the zeta-function in a much easier way than Levinson had done. But with the minor effort the information about the real parts of the localized zeros gets lost! Since $\zeta'(\varrho)$ does not vanish iff ϱ is a simple zero of $\zeta(s)$, the basic idea of Conrey, Ghosh and Gonek is to interpret $\sum_{\varrho} \zeta'(\varrho)$ as a sum of residues. Note that as a logarithmic derivative $\frac{\zeta'}{\zeta}(s)$ has only simple poles in the zeros (see the expansion (1.18)) and the unique pole of $\zeta(s)$. So by the calculus of residues one has

$$\sum_{1\leq\gamma\leq T} \zeta'(\varrho) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{\zeta'}{\zeta}\zeta'(s)\, ds,$$

where $\mathcal{C}$ is a path corresponding to the condition of summation. Conrey, Ghosh and Gonek calculated the integral as $\sim \frac{TL^2}{4\pi}$. Since this tends to infinity with $T \to \infty$ the series over $\zeta'(\varrho)$ diverges, too. But this means that there are infinitely many nontrivial simple zeros of the zeta-function!

Fujii [15] was able to sharpen the result of Conrey, Ghosh and Gonek to

$$\begin{aligned}\sum_{1\leq\gamma\leq T} \zeta'(\varrho) &= \frac{T}{4\pi}\left(\log\frac{T}{2\pi}\right)^2 + (\gamma-1)\frac{T}{2\pi}\log\frac{T}{2\pi} + (c_1-\gamma)\frac{T}{2\pi}\\ &\quad + O\left(T\exp\left(-c\sqrt{L}\right)\right),\end{aligned}$$

where (the Euler-Mascheroni-constant) γ and c_1 arise from the Laurent expansion

$$\zeta(s) = \frac{1}{s-1} + \gamma + c_1(s-1) + \ldots$$

and c is a certain positive constant.

We will slightly generalize the result of Conrey, Ghosh and Gonek to short intervals:

Theorem 3.1 *Let $T^{\frac{1}{2}+\varepsilon} \leq H \leq T$. Then we have*

$$\sum_{T<\gamma\leq T+H} \zeta'(\varrho) = \frac{HL^2}{4\pi} + O(HL).$$

We deduce immediately that the distance between two consecutive simple zeros is $\ll T^{\frac{1}{2}+\varepsilon}$. Let $N_s(T)$ count the number of nontrivial simple zeros $\varrho = \beta + i\gamma$ with $0 < \gamma \leq T$. Dividing the interval $[T, T+H]$ into $HT^{-\frac{1}{2}-\varepsilon}$ disjoint subintervals of length $T^{\frac{1}{2}+\varepsilon}$ we have

Corollary 3.2 *If $T^{\frac{1}{2}+\varepsilon} \leq H \leq T$, then*

$$N_s(T+H) - N_s(T) \gg HT^{-\frac{1}{2}-\varepsilon}.$$

For "small" H this trivial estimate yields more than our results of Chapter 2 give.

Assuming Riemann's hypothesis Gonek [16] was able to show that

$$\sum_{1\leq\gamma\leq T} |\zeta'(\varrho)|^2 \sim \frac{TL^4}{2\pi}, \tag{3.1}$$

which leads to $N_s(T) \gg T$ by the Cauchy-Schwarz-inequality. The **generalized Lindelöf hypothesis** states that

$$L\left(\frac{1}{2} + it, \chi\right) \ll (q(1+|t|))^{\varepsilon}$$

for every(!) Dirichlet L-function $L(s,\chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$ to a character $\chi \bmod q$. Assuming this and additionally Riemann's hypothesis Conrey, Ghosh and Gonek [10] even find (once more using a mollifier) a positive proportions of simple zeros:

$$\liminf_{T\to\infty} \frac{N_s(T)}{N(T)} \geq \frac{19}{27},$$

which is larger than what Montgomery's pair correlation approach delivers assuming Riemann's hypothesis. We expect the same for short intervals $[T, T+H]$ whenever $H \geq T^{\frac{1}{2}+\varepsilon}$. Note, that unconditional estimates of the left hand side of (3.1) do not lead to an improvement of Corollary 3.2 since the order of the zeta-function in the critical strip is not to be known as small as the Lindelöf hypothesis asserts.

As remarked above there is no immediate information about the real parts of the localized simple zeros. But with the density result of Balasubramanian [2]

$$N(\sigma, T+H) - N(\sigma, T) \ll H^{4\frac{1-\sigma}{3-2\sigma}} L^{100} \qquad (T^{\frac{27}{82}} \leq H \leq T)$$

we get a nontrivial restriction: since

$$HT^{-\frac{1}{2}} = H^{4\frac{1-\sigma}{3-2\sigma}} \qquad \Longleftrightarrow \qquad \sigma = \frac{3 + 2\frac{\log H}{L}}{2 + 4\frac{\log H}{L}}$$

we find simple zeros $\varrho = \beta + i\gamma$ of the zeta function with

$$T < \gamma \le T + H \qquad \text{and} \qquad \frac{1}{2} \le \beta \le \frac{3 + 2\frac{\log H}{L}}{2 + 4\frac{\log H}{L}} + \varepsilon.$$

If for example $H = T^{0.55}$, i.e. the limit of our results in Chapter 2, we have

$$T < \gamma \le T + T^{0.55} \qquad \text{and} \qquad \frac{1}{2} \le \beta \le \frac{41}{42} + \varepsilon.$$

Now we give the

3.1 Proof of Theorem 3.1

We start with a brief description. In the beginning we give some (local and global) estimates of the zeta-function and its derivatives ζ', $\frac{\zeta'}{\zeta}$ in the critical strip. That enables us to bound certain integrals which occur by interpretation of $\sum_\varrho \zeta'(\varrho)$ as a sum of residues (§3.1.1). With a certain exponential integral (§3.1.2) we can evaluate the remaining integral, which gives the main term (§3.1.3).

3.1.1 Interpretation as a sum of residues

It follows from the Riemann-von Mangoldt-formula (1.5) that the ordinates of the nontrivial zeros cannot lie too dense: for any given T_0 and fixed H we can find a T with $T_0 < T \le T_0+1$ and

$$\text{(3.2)} \qquad \min_\gamma |t - \gamma| \gg \frac{1}{L} \qquad \text{for} \quad t = T, T + H.$$

From the partial fraction of $\frac{\zeta'}{\zeta}(s)$ we get

$$\frac{\zeta'}{\zeta}(s) = \sum_{|t-\gamma|\le 1} \frac{1}{s - \varrho} + O(L) \qquad (-1 \le \sigma \le 2)$$

(see [40], Theorem 9.6(A)) and the Riemann-von Mangoldt-formula (1.5) once again implies

$$\text{(3.3)} \qquad \frac{\zeta'}{\zeta}(s) \ll L \qquad \text{for} \quad t = T, T + H$$

in the same strip.

We obtain a global estimate with the Phragmén - Lindelöf principle (analogous to (2.28) in §2.1.4; see once more [40], §5 or [38], §II.1.6): with the functional equations (1.10) and (1.11) one easily gets

$$\text{(3.4)} \qquad \zeta(s) \ll t^{\frac{1-\sigma}{2}+\varepsilon} \qquad \left(-\frac{1}{L} \le \sigma \le 1 + \frac{1}{L}\right).$$

But the real order of the zeta-function in the critical strip is conjectured to be much smaller: the **Lindelöf hypothesis** asserts that

$$\zeta(s) \ll t^{\varepsilon} \qquad \left(\sigma \ge \frac{1}{2}\right)$$

or equivalently $\zeta\left(\frac{1}{2}+it\right) \ll t^{\varepsilon}$. On the other hand Montgomery (1977) found

$$\max_{1\leq t\leq T} \log|\zeta(s)| \geq \frac{1}{20}\left(\sigma-\frac{1}{2}\right)^{\frac{1}{2}} \frac{L^{1-\sigma}}{(\log L)^{\sigma}}$$

for fixed $\sigma \in \left(\frac{1}{2},1\right)$. Moreover Tsang (1992) proved

$$\max_{T\leq t\leq 2T} \log|\zeta(s)| \gg \begin{cases} L^{\frac{1}{2}}(\log L)^{-\frac{1}{2}} & , \quad \frac{1}{2}\leq\sigma\leq\frac{1}{2}+\frac{1}{\log L} \\ \left(\sigma-\frac{1}{2}\right)^{-\frac{1}{2}} L^{1-\sigma}(\log L)^{-1} & , \quad \frac{1}{2}+\frac{1}{\log L}<\sigma\leq\frac{3}{5} \end{cases} .$$

By the way, the author [37] showed that this order is also satisfied on arbitrary rectifiable paths (with a positive distance to $\sigma=1$) instead of vertical lines. Hence, if we interpret $|\zeta(s)|$ as an analytical landscape over the critical strip, we see that there exist no "real valleys"!

We continue in our proof. With the Cauchy integral-formula we have

$$\zeta'(s) = \frac{1}{2\pi i}\int_{|s-z|=\frac{2}{L}} \frac{\zeta(z)}{(s-z)^2}\,dz.$$

Hence estimates of $\zeta(s)$ give estimates for $\zeta'(s)$. It follows from (3.4) that

$$\zeta(s),\ \zeta'(s) \ll t^{\frac{1}{2}+\varepsilon} \qquad \left(-\frac{1}{L}\leq\sigma\leq 1+\frac{1}{L}\right) \tag{3.5}$$

for $t \ll T$. So we have

$$\frac{\zeta'}{\zeta}\zeta'(s) \ll T^{\frac{1}{2}+\varepsilon} \qquad \left(-\frac{1}{L}\leq\sigma\leq 1+\frac{1}{L}\right) \tag{3.6}$$

whenever (3.3) holds.

Let $a = 1+\frac{1}{L}$. Now integrating on the rectangular contour with vertices $a+iT, a+i(T+H), 1-a+i(T+H), 1-a+iT$, we find by the calculus of residues

$$\begin{aligned} &\sum_{T<\gamma\leq T+H} \zeta'(\varrho) \\ &= \frac{1}{2\pi i}\left\{\int_{a+iT}^{a+i(T+H)} + \int_{a+i(T+H)}^{1-a+i(T+H)} + \int_{1-a+i(T+H)}^{1-a+iT} + \int_{1-a+iT}^{a+iT}\right\}\frac{\zeta'}{\zeta}\zeta'(s)\,ds \\ &=: \sum_{j=1}^{4}\mathcal{I}_j. \end{aligned}$$

We will see that the main contribution comes from $\mathcal{I}_3$ so we bound the other integrals first. On the line $\sigma=a$ we may expand the integrand in its absolutely convergent Dirichlet series: logarithmic differentiation of the Euler product (1.2) shows

$$\begin{aligned} \frac{\zeta'}{\zeta}(s) &= -\sum_{p,k}\frac{\log p}{p^{ks}} \\ &= -\sum_{m=1}^{\infty}\frac{\Lambda(m)}{m^s} \qquad \text{with} \quad \Lambda(m) = \begin{cases} \log p & , \quad m=p^k \\ 0 & , \quad \text{otherwise} \end{cases} , \end{aligned} \tag{3.7}$$

if we use von Mangoldt's function Λ. This leads to

$$\begin{aligned}\mathcal{I}_1 &= \sum_{m,n} \frac{\Lambda(m)\log n}{(mn)^a} \frac{1}{2\pi} \int_T^{T+H} \frac{dt}{(mn)^{it}} \\ &\ll \frac{\zeta'}{\zeta}\zeta'(a).\end{aligned}$$

From (2.20) we get $\mathcal{I}_1 \ll L^3$. With (3.6) follows immediately $\mathcal{I}_2, \mathcal{I}_4 \ll T^{\frac{1}{2}+\varepsilon}$. Hence

$$\sum_{T<\gamma\leq T+H} \zeta'(\varrho) = \mathcal{I}_3 + O\left(T^{\frac{1}{2}+\varepsilon}\right). \tag{3.8}$$

3.1.2 A certain exponential integral

We will make use of

Lemma 3.3 *Let a be fixed and A large enough. Then*

$$\frac{1}{2\pi}\int_A^B \exp\left(it\log\frac{t}{em}\right)\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}} dt$$
$$= \begin{cases} \left(\frac{m}{2\pi}\right)^a \exp\left(-im+\frac{\pi i}{4}\right) + O\left(A^{a-\frac{1}{2}}\right) & , \quad A < m \leq B \leq 2A \\ O\left(A^{a-\frac{1}{2}}\right) & , \quad m \leq A \text{ or } m > B \end{cases}.$$

The proof follows from the Taylor expansion of the integrand (see [16]). By (1.11) we have as an immediate application

Lemma 3.4 *Let $\sum_{n=1}^{\infty} \frac{a(n)}{n^s}$ converge uniformly for $\sigma > 1$ and absolutely for $\sigma > 1+\varepsilon$. Let $a(n) \ll n^{\varepsilon}$ and $a = 1 + (\log U)^{-1}$. Then*

$$\frac{1}{2\pi i}\int_{a+i}^{a+iU} \chi(1-s) r^s \sum_{n=1}^{\infty} \frac{a(n)}{n^s}\, ds = \sum_{n\leq \frac{rU}{2\pi}} a(n)\, e\left(-\frac{n}{r}\right) + O\left(U^{\frac{1}{2}+\varepsilon}\right).$$

3.1.3 The main term

It remains to evaluate I_3: via $s \mapsto 1-\overline{s}$ we find

$$\mathcal{I}_3 = -\frac{1}{2\pi i}\int_{a+iT}^{a+i(T+H)} \frac{\zeta'}{\zeta}\zeta'(1-\overline{s})\, ds.$$

By the reflection principle this is the conjugate of

$$-\frac{1}{2\pi i}\int_{a+iT}^{a+i(T+H)} \frac{\zeta'}{\zeta}\zeta'(1-s)\, ds.$$

With the functional equations (1.12) and (1.15) one easily gets

$$\begin{aligned}\frac{\zeta'}{\zeta}\zeta'(1-s) &= \left(\frac{\chi'}{\chi}(s) - \frac{\zeta'}{\zeta}(s)\right)\left(\chi'(1-s)\zeta(s) - \chi(1-s)\zeta'(s)\right) \\ &= -2\frac{\chi'}{\chi}\chi(1-s)\zeta'(s) + \chi(1-s)\frac{\zeta'}{\zeta}\zeta'(s) + \left(\frac{\chi'}{\chi}\right)^2 \chi(1-s)\zeta(s).\end{aligned}$$

Hence $\overline{\mathcal{I}_3}$ equals

$$\frac{1}{2\pi i}\int_{a+iT}^{a+i(T+H)}\left(2\frac{\chi'}{\chi}\chi(1-s)\zeta'(s)-\chi(1-s)\frac{\zeta'}{\zeta}\zeta'(s)-\left(\frac{\chi'}{\chi}\right)^2\chi(1-s)\zeta(s)\right)ds$$
$$=:\ 2\mathcal{F}_1+\mathcal{F}_2+\mathcal{F}_3.$$

We find with (1.16)

$$\begin{aligned}\mathcal{F}_1 &= -\frac{1}{2\pi i}\int_{a+iT}^{a+i(T+H)}\left(\log\frac{t}{2\pi}+O\left(\frac{1}{t}\right)\right)\chi(1-s)\zeta'(s)\,ds\\ &= -\int_T^{T+H}\left(\log\frac{t}{2\pi}+O\left(\frac{1}{T}\right)\right)d\left(\frac{1}{2\pi i}\int_{a+i}^{a+it}\chi(1-s)\zeta'(s)\,ds\right).\end{aligned}$$

With Lemma 3.4 the integrator equals

$$\frac{1}{2\pi i}\int_{a+i}^{a+it}\chi(1-s)\zeta'(s)\,ds=-\sum_{n\le\frac{t}{2\pi}}\log n+O\left(t^{\frac12+\varepsilon}\right).$$

Hence by partial summation and the Taylor expansion of the logarithm

$$\begin{aligned}\mathcal{F}_1 &= \left(L+O\left(\frac{H}{T}\right)\right)\sum_{\frac{T}{2\pi}<n\le\frac{T+H}{2\pi}}\log n+O\left(T^{\frac12+\varepsilon}\right)\\ &= \frac{HL^2}{2\pi}+O(HL+T^{\frac12+\varepsilon}).\end{aligned}$$

In a similar way we get

$$\mathcal{F}_2=-\sum_{\frac{T}{2\pi}<mn\le\frac{T+H}{2\pi}}\Lambda(m)\log n+O\left(T^{\frac12+\varepsilon}\right).$$

The sum above equals

$$\sum_{m\le\frac{T+H}{2\pi}}\Lambda(m)\sum_{\frac{T}{2\pi m}<n\le\frac{T+H}{2\pi m}}\log n=\sum_{m\le\frac{T+H}{2\pi}}\Lambda(m)\left(\frac{H}{2\pi m}\log\frac{T}{2\pi m}+O\left(\frac{H}{m}\right)\right).$$

by partial integration. Since $\Psi(x)=\sum_{m\le x}\Lambda(m)$ by (3.7) we are able to calculate with the prime number theorem (1.7) and summation by parts

$$\begin{aligned}\sum_{m\le\frac{T+H}{2\pi}}\frac{\Lambda(m)}{m} &= L+O(1),\\ \sum_{m\le\frac{T+H}{2\pi}}\frac{\Lambda(m)\log m}{m} &= \frac{L^2}{2}+O(L).\end{aligned}$$

This leads to

$$\mathcal{F}_2=-\frac{HL^2}{4\pi}+O(HL+T^{\frac12+\varepsilon}). \tag{3.9}$$

Further

$$\begin{aligned}\mathcal{F}_3 &= -\frac{1}{2\pi i}\int_{a+iT}^{a+i(T+H)}\left(\left(\log\frac{t}{2\pi}\right)^2+O\left(\frac{\log t}{t}\right)\right)\chi(1-s)\zeta(s)\,ds\\ &= -\int_T^{T+H}\left(\left(\log\frac{t}{2\pi}\right)^2+O\left(\frac{L}{T}\right)\right)d\left(\frac{1}{2\pi i}\int_{a+i}^{a+it}\chi(1-s)\zeta(s)\,ds\right).\end{aligned}$$

Once again with Lemma 3.4

$$\frac{1}{2\pi i}\int_{a+i}^{a+it}\chi(1-s)\zeta(s)\,ds=\sum_{n\le\frac{t}{2\pi}}1+O\left(t^{\frac{1}{2}+\varepsilon}\right).$$

We obtain

$$\mathcal{F}_3=-\frac{HL^2}{2\pi}+O\left(HL+T^{\frac{1}{2}+\varepsilon}\right).$$

Since the main terms are invariant under conjugation we have altogether

$$\mathcal{I}_3=\frac{HL^2}{4\pi}+O\left(HL+T^{\frac{1}{2}+\varepsilon}\right).$$

By (3.8) this leads to

$$\sum_{T<\gamma\le T+H}\zeta'(\varrho)=\frac{HL^2}{4\pi}+O\left(HL+T^{\frac{1}{2}+\varepsilon}\right)$$

for every T with the condition (3.2). To get this uniformly in T we allow an arbitrarily T at the expense of an error $\ll T^{\frac{1}{2}+\varepsilon}$ (by shifting the path of integration using (3.5)). So we have proved Theorem 3.1. •

* * *

Of course, the results of Chapter 3 could be transferred to Dirichlet L-functions, and by the work of Hilano [18] also those of Chapter 2. But unfortunately Levinson's method does not work for L-functions associated with holomorphic cusp forms, as Farmer [14] observed, and also the method of Conrey, Ghosh and Gonek fails in the special case of Ramanujan's τ-function. However, Conrey and Ghosh [7] developped yet another method to detect unconditionally simple zeros of Dirichlet series of a much more general class. They proved the existence of infinitely many nontrivial simple zeros of Ramanujan's τ-function. Unfortunately this approach does not guarantee a positive proportion of simple zeros, so there remains a lot to do...

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