

Twisting the $N = 2$ string

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The most general homogeneous monodromy conditions in $N = 2$ string theory are classified in terms of the conjugacy classes of the global symmetry group $U(1,1) \otimes \mathbb{Z}_2$. For classes which generate a discrete subgroup Γ , the corresponding target space backgrounds $\mathbb{C}^{1,1}/\Gamma$ include half spaces, complex orbifolds, and tori. We propose a generalization of the intercept formula to matrix-valued twists, but find massless physical states only for $\Gamma = \mathbb{1}$ (untwisted) and $\Gamma = \mathbb{Z}_2$ (in the manner of Mathur and Mukhi), as well as for Γ being a parabolic element of $U(1,1)$. In particular, the 16 \mathbb{Z}_2 -twisted sectors of the $N = 2$ string are investigated, and the corresponding ground states are identified via bosonization and BRST cohomology. We find enough room for an extended multiplet of “spacetime” supersymmetry, with the number of supersymmetries being dependent on global “spacetime” topology. However, world-sheet locality for the chiral vertex operators does not permit interactions among all massless “spacetime” fermions.

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I. INTRODUCTION

Since the discovery of $N = 2$ supersymmetric critical strings in 1976 [1,2], their status has undergone several fundamental changes. Initially constructed in *two* spacetime dimensions, they were lately recognized as the strings naturally living in a *four*-dimensional spacetime of apparently nonphysical (4,0) or (2,2) signature [3,4]. The interacting $N = 2$ string theory was shown to be closely related to *self-dual* four-dimensional field theories, and it was even conjectured to be the “master theory” for all integrable models [5]. Recently, some arguments were presented that the $N = 2$ string should also be able to support *target* (“spacetime”) self-dual supersymmetry, by relating it to the $N = 4$ fermionic string theory [6]. The appropriate framework for extended self-dual supersymmetry and supergravity in 2+2 dimensions was developed in Ref. [7]. Very recently, $N = 2$ strings reappeared in a quite different context of *universal* string theory, including the conventional $N = 0$ and $N = 1$ strings as particular vacua [8].

Despite all these amazing developments, the underlying symmetries and the physical spectrum of the naively “simple” (compared to the others) $N = 2$ string theory remain to be poorly understood. It is known from the calculation of the $N = 2$ string partition function on a torus

that there exists just a single massless spacetime boson in the spectrum of the *untwisted* $N = 2$ string moving in the flat (2+2)-dimensional background $\mathbb{R}^{2,2}$ [5]. The study of the related $N = 2$ Becchi-Rouet-Stora-Tyutin (BRST) cohomology with at least some continuous spacetime momenta was recently initiated in Ref. [9]. In addition, there are elements of the BRST cohomology corresponding to *discrete* states at vanishing momentum, $k = 0$. They can be most easily identified and investigated when using a compactified background $\mathbb{T}^{2,2}$ instead of $\mathbb{R}^{2,2}$ [10,11], but discrete states are not going to be the subject of this paper. In order to derive the *complete* spectrum of the critical $N = 2$ string in a flat background, and to address, in particular, the issue of “spacetime” fermionic physical states (with continuous momenta), one must investigate the most general monodromy conditions for the $N = 2$ string and their associated BRST cohomologies.

We are going to investigate in this paper *all* possible homogeneous twistings of the $N = 2$ string, and find those leading to consistent solutions. We distinguish a hierarchy of three different types of twists: namely, with increasing generality (i) those flipping only signs in the (bosonic) monodromies, (ii) those creating arbitrary phases around world-sheet cycles, and (iii) those mixing different string coordinates, leading to noncompact monodromies in a diagonal or Jordan normal basis. Clearly, the first type comprises the simplest (and, we believe, the most important) generalizations of the naive Neveu-Schwarz- and Raymond-type boundary conditions, and we are going to analyze them first. As for the second type of monodromies, the role of the *spectral flow* present in the $N = 2$ superconformal algebra has to be understood. Finally, the third monodromy type implies rather unusual topologies of the string target space, which may explain why, to our knowledge, it has not been explored in

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the past. We are particularly interested in backgrounds which allow *massless* physical states, in order to analyze the underlying effective field theory. This does not yet mean that the other backgrounds are inconsistent. Restricting ourselves to twisted $N = 2$ strings with massless ground states, we find only three possibilities: (i) the untwisted string as studied, e.g., by Ooguri and Vafa [5], (ii) implementing the previously known Mathur-Mukhi twist [12], and (iii) a peculiar background corresponding to parabolic elements of $U(1,1)$. We proceed with the BRST cohomology analysis only in the first two cases, in order to establish restrictions on interactions of the Mathur-Mukhi twisted states.

The paper is organized as follows. In Sec. II, we start with a general discussion of boundary conditions in string theory, formulate the framework for our subsequent investigation of general $N = 2$ string boundary conditions, and set up our notation. In Sec. III we recapitulate all the known local and global, continuous and discrete symmetries of the Brink-Schwarz $N = 2$ string action, and list four types of \mathbb{Z}_2 twist symmetries. This allows us to identify 16 possible \mathbb{Z}_2 monodromy patterns (called *sectors*) in the case of a closed $N = 2$ string. In Sec. IV we use bosonization techniques to introduce spin and twist fields. These fields are needed to specify the vertex operators creating the ground states in the 16 twisted sectors. Next, we calculate the critical intercepts for all these sectors, and exhibit the possible *massless* ground states. Section V is devoted to a discussion of the spectral flow in $N = 2$ string theory. We generalize the boundary conditions to twists by arbitrary phases and give, again, the relevant formulae for the ground state energy. The most general monodromy conditions are the subject of Sec. VI where we classify all possibilities in terms of the conjugacy classes of $U(1,1)$, and propose the most general intercept formula. We find a somewhat peculiar new massless background, related to the parabolic conjugacy class of $SU(1,1)$. Section VII deals with the BRST cohomology and interactions in $N = 2$ string theory. First, we check the BRST-invariance of the massless candidate (\mathbb{Z}_2 -twisted) ground states, and, second, investigate their possible interactions from the locality requirement for the vertex operator algebra. Our conclusions are summarized in Sec. VIII. Two Appendices comprise auxiliary information about the local transformation laws of the $N = 2$ string fields (Appendix A) and the $N = 2$ string BRST charge (Appendix B).

II. STRING BOUNDARY CONDITIONS

When varying a string action, some boundary terms appear. The action principle implies that certain constraints are to be added to the string equations of motion, in order to eliminate the boundary terms.

Varying the gauge-fixed bosonic string action¹ [13]

$$S_0 = \frac{1}{2\pi} \int_0^T d\tau \int_0^\pi d\sigma \eta^{\alpha\beta} \partial_\alpha X \cdot \partial_\beta X \quad (2.1)$$

yields not only the equations of motion ($\square X = 0$), but also the constraints²

$$\text{open : } X'|_{\sigma=0,\pi} = 0, \quad \dot{X}'|_{\tau=0,T} = 0; \quad (2.2)$$

$$\text{closed : } X' \text{ periodic, } \dot{X}'|_{\tau=0,T} = 0.$$

According to the minimal action principle, \dot{X} is supposed to vanish for the initial and final string states, while the “stringy” condition on X' is relevant. More general boundary conditions than the above are possible, but correspond to identifications in the target space and, hence, change global topology.

In the case of the $N = 1$ fermionic gauge-fixed string action [13]

$$S_1 = \frac{1}{2\pi} \int d\tau d\sigma \left\{ \partial_\alpha X \cdot \partial^\alpha X + \frac{i}{2} \bar{\Psi} \cdot \rho^\alpha \partial_\alpha \Psi \right\} \\ = \frac{1}{\pi} \int d^2\xi \{ \partial_+ X \cdot \partial_- X + \Psi_- \cdot \partial_+ \Psi_- + \Psi_+ \cdot \partial_- \Psi_+ \}, \quad (2.3)$$

where light-cone coordinates $\xi_\pm = \tau \pm \sigma$ as well as real Majorana-Weyl fermion fields Ψ_\pm have been introduced on the world-sheet, one gets, in addition to Eq. (2.2), the fermionic constraints

$$\text{open : } \Psi_+ = \pm \Psi_-|_{\sigma=0,\pi} = 0, \quad (2.4)$$

$$\text{closed : } \Psi_\pm \text{ periodic or antiperiodic.}$$

The two different possibilities in choosing the signs in Eq. (2.4) are relevant, and they lead to the known distinction between the Ramond (R) and Neveu-Schwarz (NS) sectors of the $N = 1$ string theory [13]. In principle, we are free to choose different boundary conditions for different target space components of Ψ_\pm . This would, however, destroy global target space symmetries, such as spacetime Lorentz invariance.

In the case of the $N = 2$ string, there are many more choices for the boundary conditions since both bosonic and fermionic string world-sheet fields ($N = 2$ superconformal “matter”) become *complex*. The gauge-fixed $N = 2$ string action has the form ($Z^\pm = X \pm iY$, $\Psi^\pm = \Psi^1 \pm i\Psi^2$)

$$S_2 = \frac{1}{\pi} \int d^2\xi \{ \partial_+ Z^+ \cdot \partial_- Z^- + \partial_- Z^+ \cdot \partial_+ Z^- \\ + \Psi_+^+ \cdot \partial_- \Psi_+^- + \Psi_-^+ \cdot \partial_+ \Psi_-^- + \Psi_+^- \cdot \partial_- \Psi_+^+ + \\ + \Psi_-^- \cdot \partial_+ \Psi_-^+ \}. \quad (2.5)$$

Zero modes of the Z fields are normally identified with the coordinates of embedding “spacetime” in which the

¹We suppress target space indices. The dots stand for their contractions with a flat metric $\eta_{\mu\nu}$.

²The prime and dot over a function mean the differentiation with respect to σ and τ , respectively.

$N = 2$ string propagates. Both the dimension and the signature of this “spacetime” are fixed quantum mechanically, when we require conformal anomaly cancellation (or nilpotency of the BRST charge) in the absence of additional propagating degrees of freedom (like Liouville modes). Critical $N = 2$ strings are known to live in two *complex* dimensions [3] (see Refs. [14,15] for the recent reviews about the $N = 2$ strings). The (2,2) spacetime signature of $N = 2$ critical string theory is apparently nonphysical. Consequently, there seems to be no compelling reason to insist on a direct physical interpretation of the $N = 2$ string target space at all, or assume it to be smooth everywhere. Let us therefore allow as much freedom as possible at this point, and do not even assume that our flat spacetime is globally a manifold.

Of course, we do not have total freedom in the choice of boundary conditions. They have to be compatible with the symmetries of the action. More precisely, we demand the action density to be single valued on the world sheet. Being moved around along a closed path, it should come back to its original value. The corresponding monodromies of the world-sheet fields must therefore conspire to produce a symmetry of the action (density). In order to take into account the constraints which accompany the gauge-fixed action (2.5), we shall investigate the monodromies for the full gauge-invariant action. The monodromies have to respect the local world-sheet symmetries as well as the global target space symmetries. Classifying allowed boundary conditions (and thus possible flat backgrounds) requires an analysis of all (global and local) symmetries. Apart from this, we do not constrain our world-sheet fields at all.

Stated differently, at the outset we allow a *multivaluedness* of the $N = 2$ string coordinates Z and Ψ . This implies that generic monodromy factors may appear in the boundary conditions, and each matter field may live in a quite general twisted complex bundle. Given a Riemann surface Σ to represent the $N = 2$ string world sheet after its “Euclideanization,” it is the allowed choice of the *global monodromies*, i.e., “phases” picked up by moving the matter fields around the cycles comprising a homology basis on Σ , that will be in question of our discussion in the following sections.

A monodromy matrix U generates a subgroup Γ of the symmetry group G of the string action. The twisted string target space is therefore obtained from the untwisted one through modding out by Γ . On the bosonic coordinates Z some subgroup $G_0 \subset G$ is realized, and the relevant monodromy creates some $\Gamma_0 \subset G_0$. Clearly, the *bosonic* background \mathcal{B} is simply the quotient space $\mathbb{C}^{1,1}/\Gamma_0$. If one likes to retain a (2+2)-dimensional manifold for \mathcal{B} locally, Γ_0 is required to be discrete but not necessarily finite. Typical examples are orbifolds from $\Gamma_0 = \mathbb{Z}_n, n \geq 2$, or tori from $\Gamma_0 = \mathbb{Z}$. The simplest cases are \mathbb{Z}_2 orbifolds, i.e., $\mathcal{B} = \mathbb{C}^{1,1}/\mathbb{Z}_2$. For this reason we shall discuss them first, in the two following sections. We will return to the general situation in Secs. V and VI. This means that fields of *integral* spin are allowed to pick up signs, i.e., be *double-valued* on Σ , just as for fields of *half-integral* spin. The complete monodromy behavior will be fixed from the signs picked up by the components

of Z , in addition to an overall sign between Z and Ψ related to the NS-R distinction.

A natural example of the twisted boundary conditions is the Mathur-Mukhi choice considered in Ref. [12],

$$Z^\pm(\pi) = Z^\mp(0), \quad \Psi^\pm(\pi) = \eta\Psi^\mp(0), \quad (2.6)$$

uniformly with respect to all the (suppressed) target space indices, with $\eta = \pm 1$. This allows us to choose different signs for real and imaginary parts of the fields, while keeping $X = \text{Re}Z^\pm$ periodic. The choice of boundary conditions in Eq. (2.6) is obviously consistent with the variation of the action (2.5).

The most conservative (Ooguri-Vafa) choice [5] of the $N = 2$ string boundary conditions in the form

$$Z^\pm(\pi) = Z^\pm(0), \quad \Psi^\pm(\pi) = \eta\Psi^\pm(0), \quad (2.7)$$

is the only one which allows us to keep the single valuedness of integral spin fields, and $\mathbb{C}^{1,1}$ as the consistent (2+2)-dimensional background spacetime for $N = 2$ string propagation. This choice only deals with untwisted line bundles and their square roots (spin bundles) to define fermions, just like for the $N = 1$ string. The two possible signs $\eta = \pm 1$ in Eq. (2.7) are common for all the world-sheet spinors, and correspond to the usual NS-R distinction familiar from the $N = 1$ case.

The two boundary conditions presented so far are blind to the spacetime indices of Z and Ψ and thus compatible with naive real spacetime Lorentz symmetry $O(1,1)$. This is also true for an additional overall sign flip in Eqs. (2.6) and (2.7), which doubles the number of such cases to four.³ It is conceivable that we might finally need to sum over all backgrounds or, equivalently, over all spin structures in the $N = 2$ string partition function. The only compelling reason to do so might possibly come from modular invariance, since the twists imply drastic consequences for the $N = 2$ moduli space. To address this issue, one needs a better understanding of $N = 2$ moduli space, which is a rather involved problem.

Before going any further, we want to briefly discuss our notation. Target space indices (internal and Lorentz) always appear as superscripts, world-sheet indices usually as subscripts. In real components

$$Z^{\mu\pm} = Z^{\mu 2} \pm iZ^{\mu 3}, \quad \Psi^{\mu\pm} = \Psi^{\mu 2} \pm i\Psi^{\mu 3}, \quad (2.8)$$

we have the fields $Z^{\mu i}(\xi)$ and $\Psi^{\mu i}(\xi)$, with generic monodromy conditions of the form

$$Z^{\mu i}(\pi) = M^{\mu i}_{\nu j} Z^{\nu j}(0), \quad (2.9)$$

and similarly for the Ψ 's. The lower-case Greek indices $\mu, \nu = 0, 1$ refer to a two-dimensional Minkowski space of signature $(-, +)$, while the lower case Latin indices $i, j = 2, 3$ refer to the real components of the complex fields. To avoid confusing the same numerical values of

³For a genus- h Riemann surface Σ , there are 4^{2h} possible boundary conditions of this type for the Z fields alone.

μ and i , we have taken a slightly unusual range for the lower-case Latin indices, so that we have

$$\underline{a} \equiv \{\mu i\} = (02, 03, 12, 13). \quad (2.10)$$

Sometimes, we write shorthand Z^μ for $(Z^{\mu 2}, Z^{\mu 3})$ or $(Z^{\mu +}, Z^{\mu -})$, as well as Z^i for (Z^{0i}, Z^{1i}) or Z^\pm for $(Z^{0\pm}, Z^{1\pm})$, and similarly for Ψ , suppressing irrelevant indices in an obvious fashion. It may also be convenient to choose spacetime light-cone coordinates $Z^{\uparrow\downarrow} = Z^0 \uparrow\downarrow Z^1$, where $\uparrow\downarrow$ is just another set of \pm signs (index i is suppressed here.)

In the $N = 2$ superconformal gauge, the fields (and ghosts) of the $N = 2$ string on the Euclidean world sheet become free, so that they can be decomposed into their holomorphic and antiholomorphic parts, as is usual in two-dimensional conformal field theory. For definiteness, we investigate only *closed* $N = 2$ strings in this paper. If not mentioned explicitly, we will consider only the right movers in the following. As usual, lower case letters will be used to denote the chiral parts of the matter fields, except for the Z fields (see Appendix B for more details).

III. TWISTING THE $N = 2$ STRING

The world-sheet action of $N = 2$ fermionic string theory reads⁴ [16]

$$S = \frac{1}{\pi} \int d^2 \xi e \left\{ \frac{1}{2} h^{\alpha\beta} \partial_\alpha Z^+ \cdot \partial_\beta Z^- + \frac{i}{2} \bar{\Psi}^+ \cdot \rho^\alpha \overleftrightarrow{D}_\alpha \Psi^- \right. \\ \left. + A_\alpha \bar{\Psi}^+ \cdot \rho^\alpha \Psi^- (\partial_\alpha Z^+ + \bar{\Psi}^+ \chi_\alpha^-) \cdot \bar{\chi}_\beta^+ \rho^\alpha \rho^\beta \Psi^- \right. \\ \left. + (\partial_\alpha Z^- + \bar{\chi}_\alpha^+ \Psi^-) \cdot \bar{\Psi}^+ \rho^\beta \rho^\alpha \chi_\beta^- \right\}, \quad (3.1)$$

where the $N = 2$ world-sheet supergravity multiplet, comprising a real zweibein $e_\alpha^a(\xi)$ with the metric $h_{\alpha\beta} = \eta_{ab} e_\alpha^a e_\beta^b$, a complex gravitino field $\chi_\alpha^\pm(\xi)$, and a real $U(1)$ gauge field $A_\alpha(\xi)$, has been introduced.

The Brink-Schwarz action (3.1) is known to be invariant under the following *local* world-sheet symmetries: reparametrization invariance, Lorentz invariance, $N = 2$ supersymmetry, phase and chiral $U(1)$ gauge invariances, and Weyl and super-Weyl invariances.

The explicit form of all the local transformation laws in components can be found in Appendix A (see Ref. [17] for the $N = 2$ superspace description of the $N = 2$ string

action and its symmetries). In particular, the local $U(1)$ symmetry acts nontrivially on the fields Ψ^\pm , χ_α^\pm and A_α only.

The action (3.1) also possesses *global continuous* invariances associated with target space symmetries: namely, global translation invariance $\mathbb{C}^{1,1}$, global $U(1)$ symmetry acting on the internal indices of the matter fields Z^i and Ψ^i , with complex eigenstates $Z^\pm = Z^2 \pm iZ^3$ and $\Psi^\pm = \Psi^2 \pm i\Psi^3$, global $O(1,1)$ Lorentz symmetry acting on the spacetime indices of the matter fields Z^μ and Ψ^μ , with light-cone eigenstates $Z^{\uparrow\downarrow} = Z^0 \uparrow\downarrow Z^1$ and $\Psi^{\uparrow\downarrow} = \Psi^0 \uparrow\downarrow \Psi^1$. The intersection of the two symmetries is a \mathbb{Z}_2 symmetry generated by the total sign flip PT of all matter fields, where P and T denote the usual parity and time reversal transformations, respectively.

The full global continuous symmetry of the action is, however, still larger, since the $O(1,1)$ symmetry can be extended to $U(1,1) = [U(1) \otimes SU(1,1)]/\mathbb{Z}_2$, where the factors $U(1)$ and \mathbb{Z}_2 coincide with the ones just mentioned. More precisely, we may split $O(1,1) = SO(1,1) \otimes \mathbb{Z}'_2$, with \mathbb{Z}'_2 generated by the parity P , and add complex boost and rotation generators to create $SU(1,1)$. The \mathbb{Z}'_2 is then contained in the product with the $U(1)$ symmetry already present.

In $2 + 2$ dimensions, the natural spacetime Lorentz symmetry for the $N = 2$ string is $SO(2,2) = [SU(1,1) \otimes SU(1,1)]/\mathbb{Z}_2$ [15]. The interaction terms in the action (3.1), however, break down one of the two $SU(1,1)$ factors to $U(1) \otimes \mathbb{Z}''_2$, with \mathbb{Z}''_2 representing the Mathur-Mukhi twist (Sec. II). It has been conjectured [6] that the total global Lorentz symmetry of the $N = 2$ string in the given formulation might actually be the remnant of a “hidden” $SO(2,2)$ symmetry, which presumably exhibits itself in the equivalent $N = 4$ supersymmetric formulation of the same string theory.

Finally, the $N = 2$ string action (3.1) is invariant under the four different \mathbb{Z}_2 twists of the fields:

$$\begin{aligned} \mathbb{Z}_2 : Z &\rightarrow -Z \text{ and the same for } \Psi, \\ \mathbb{Z}'_2 : Z^0 &\rightarrow Z^0, Z^1 \rightarrow -Z^1 \text{ and the same for } \Psi^{0,1}, \\ \mathbb{Z}''_2 : Z^\pm &\rightarrow Z^\mp \text{ and the same for } \Psi^\pm, \\ \mathbb{Z}'''_2 : Z &\rightarrow Z \text{ and } \Psi \rightarrow -\Psi, \end{aligned} \quad (3.2)$$

by adjusting the behavior of the $N = 2$ supergravity fields appropriately (see below). These *global discrete* symmetries are imbedded in the continuous groups discussed above. In particular, $\mathbb{Z}_2 = O(1,1) \cap U(1) = SU(1,1) \cap U(1)$ inverts (PT) , whereas \mathbb{Z}'_2 is a “large” transformation (P) in $O(1,1)$. The twist $\mathbb{Z}''_2 \subset SO(2,2)$ is the only one which is not contained in $U(1,1)$. The \mathbb{Z}'''_2 twist is part of the local $U(1)$ and does not affect Z ; it apparently resembles the usual NS \leftrightarrow R twist in the $N = 1$ string theory. In the special case of \mathbb{Z}_2 monodromies we are therefore going to use the same labels (NS and R) to distinguish these two cases, even though their actual meaning is quite different in the $N = 2$ string theory. Generally speaking, the NS (R) sector is characterized by $\Psi^{\mu i}$ and $Z^{\mu i}$ having opposite (identical) monodromy signs. In summary, the total ho-

⁴We use a purely imaginary (Majorana) representation for two-dimensional Dirac matrices ρ^α , $\alpha = 0, 1$: $\rho^0 = \sigma_2$, $\rho^1 = i\sigma_1$, $\rho^3 = \rho^0 \rho^1 = \sigma_3$, where σ_i , $i = 1, 2, 3$ are Pauli matrices, and $\{\rho^\alpha, \rho^\beta\} = -2\eta^{\alpha\beta}$, with $\eta = \text{diag}(-, +)$. The Majorana spinors $\Psi = (\Psi_-, \Psi_+)$ decompose into their Weyl parts Ψ_\pm . Dirac conjugation (denoted by an overbar) $\bar{\Psi} \equiv \Psi^T \rho^0$ does *not* include complex conjugation (denoted by a superscript $-$). Complex conjugation is explicit in our formulas, and it always acts first.

mogeneous *global* symmetry group of the $N = 2$ string is $G_0 = U(1,1) \otimes \mathbb{Z}_2''$, which is at the same time the maximal global monodromy group for the matter fields Z or Ψ , separately. The *relative* boundary conditions between Z and Ψ can only arise from *local* symmetries, so that Z and Ψ monodromies are rigidly related after gauge fixing.

It is not difficult to deduce the consequences of Eq. (3.2) for the $N = 2$ supergravity fields. First, one easily sees from inspecting the gravitino couplings in the action (3.1) that the complex gravitino field χ_α^\pm should transform under each twisting in exactly the *same* way as the complex matter field Ψ^\pm , uniformly for each α value. Under \mathbb{Z}_2'' , for example,

$$(\partial_\alpha Z^+ + \bar{\Psi}^+ \chi_\alpha^-) \xrightarrow{\text{twist}} (\partial_\alpha Z^- + \bar{\Psi}^- \tilde{\chi}_\alpha^-), \tag{3.3}$$

$$\bar{\chi}_\beta^+ \rho^\alpha \rho^\beta \Psi^- \xrightarrow{\text{twist}} \bar{\tilde{\chi}}_\beta^+ \rho^\alpha \rho^\beta \Psi^+ = \bar{\Psi}^+ \rho^\beta \rho^\alpha \tilde{\chi}_\beta^+,$$

where the tilde over a field means its \mathbb{Z}_2'' twisting. Similarly,

$$A_\alpha \bar{\Psi}^+ \cdot \rho^\alpha \Psi^- \xrightarrow{\text{twist}} \tilde{A}_\alpha \bar{\Psi}^- \cdot \rho^\alpha \Psi^+ = -\tilde{A}_\alpha \bar{\Psi}^+ \cdot \rho^\alpha \Psi^-. \tag{3.4}$$

The minus on the right-hand side (RHS) of Eq. (3.4) is important, since it implies

$$A_\alpha \xrightarrow{\text{twist}} -A_\alpha \tag{3.5}$$

under the \mathbb{Z}_2'' twist.

For the *untwisted* boundary conditions one gets

$$Z(\pi) = Z(0), \quad A_\alpha(\pi) = A_\alpha(0), \tag{3.6}$$

$$\Psi(\pi) = \eta \Psi(0), \quad \chi_\alpha(\pi) = \eta \chi_\alpha(0),$$

where the signs $\eta = \pm 1$ in the second line are in correspondence with each other and denote additional optional \mathbb{Z}_2''' twist, i.e., the NS-R option. For the \mathbb{Z}_2'' *twisted* boundary conditions one has instead

$$Z^+(\pi) = Z^-(0), \quad A_\alpha(\pi) = -A_\alpha(0), \tag{3.7}$$

$$\Psi^+(\pi) = \eta \Psi^-(0), \quad \chi_\alpha^+(\pi) = \eta \chi_\alpha^-(0).$$

Equation (3.7) implies, in particular, that the (Abelian) first Chern class $c = \frac{1}{2\pi} \int_\Sigma F$, $F \equiv dA$, associated with the $U(1)$ gauge field A on a Riemann surface Σ , has to vanish for such twisted boundary conditions.

The monodromy group $U(1,1) \otimes \mathbb{Z}_2''$ does not have room for all 16 independent sign choices of the four $Z^{\mu i}$ (or $\Psi^{\mu i}$) components. From the possibilities listed in Eq. (3.2) it is clear that in each \mathbb{Z}_2 twisting there must always be an *even* number of minuses among the four components⁵ (μi), in order for the compensating monodromy of the supergravity fields χ_α^\pm and A_α to exist. For instance, flipping only *one* component of Ψ , say Ψ^{02} , is inconsistent with the $N = 2$ string action, as can easily be seen from the term

$$A_\alpha \bar{\Psi}^+ \cdot \rho^\alpha \Psi^- = 2i A_\alpha (\bar{\Psi}^{02} \rho^\alpha \Psi^{03} - \bar{\Psi}^{12} \rho^\alpha \Psi^{13}). \tag{3.8}$$

We have checked the compatibility of any even twisting with all the local symmetries of the $N = 2$ string action (3.1), listed in Appendix A. The procedure is straightforward, and it simply determines the behavior of the local symmetry parameters under the twisting. The (superconformal) gauge fixing in $N = 2$ string theory results in the $N = 2$ superconformal algebra in terms of the currents associated with the residual symmetries. The related ghost structure and the BRST charge are well known [12,18,19], and the results are collected in Appendix B. Naturally, the boundary conditions of the currents and, hence, their moding, depend on the twisting, e.g., Eq. (3.7) implies an antiperiodic $U(1)$ current J .

An Abelian subgroup of the full symmetry group is $U(1) \otimes SO(1,1) \otimes U(1)_{\text{gauged}}$, where the first two factors represent global symmetries. Let (q, s, e) be the corresponding charges of various fields (including ghosts) with respect to $U(1)$, $SO(1,1)$, and $U(1)_{\text{gauged}}$, respectively. Their complete list is compiled in Table I. The rest of the $N = 2$ supergravity fields, (e_α^a, A_α) , the reparametrization ghosts (b, c) and the $U(1)$ ghosts (\tilde{b}, \tilde{c}) are all inert with respect to $U(1) \otimes SO(1,1) \otimes U(1)_{\text{gauged}}$.

The monodromy conditions for all the \mathbb{Z}_2 -twisted sectors of the closed $N = 2$ string are collected in Table II, where pluses and minuses mean periodicity and antiperiodicity, respectively. There are $2^4 = 16$ sectors in total, due to the \mathbb{Z}_2^4 twist group.

TABLE I. The charges (q, s, e) and conformal dimensions h of the world-sheet fields with respect to $U(1)$, $SO(1,1)$, $U(1)_{\text{gauged}}$ and Vir , respectively. Blanks appear where the fields are not eigenstates.

Field	Z^\pm	$Z^{\uparrow\downarrow}$	$Z^{\uparrow\downarrow\pm}$	Ψ^\pm	$\Psi^{\uparrow\downarrow}$	$\Psi^{\uparrow\downarrow\pm}$	t^\pm	$t^{\uparrow\downarrow}$	S^\pm	$S^{\uparrow\downarrow}$	χ_α^\pm	β^\pm	γ^\pm
q	± 1		± 1	± 1		± 1	$\pm \frac{1}{2}$		$\pm \frac{1}{2}$		0	0	0
s		$\uparrow\downarrow 1$	$\uparrow\downarrow 1$		$\uparrow\downarrow 1$	$\uparrow\downarrow 1$		$\uparrow\downarrow \frac{1}{2}$		$\uparrow\downarrow \frac{1}{2}$	0	0	0
e	0	0	0	± 1		± 1	0	0	$\pm \frac{1}{2}$		± 1	± 1	± 1
h	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$		$\frac{3}{2}$	$-\frac{1}{2}$

⁵Let us call these twistings *even*.

TABLE II. The \mathbb{Z}_2^4 monodromy conditions for the (twisted) sectors of the $N = 2$ closed string (right movers only). The last rows give conformal dimensions and local U(1) charges of the associated zero-momentum ground states (Sec. IV).

No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
\mathbb{Z}_2''	R	NS	R	NS	R	NS	R	NS	R	NS	R	NS	R	NS	R	NS
Z^{02}	+	+	+	+	-	-	-	-	+	+	+	+	-	-	-	-
Z^{03}	+	+	+	+	-	-	-	-	-	-	-	-	+	+	+	+
Z^{12}	+	+	-	-	+	+	-	-	+	+	-	-	+	+	-	-
Z^{13}	+	+	-	-	+	+	-	-	-	-	+	+	-	-	+	+
Ψ^{02}	+	-	+	-	-	+	-	+	+	-	+	-	-	+	-	+
Ψ^{03}	+	-	+	-	-	+	-	+	-	+	-	+	+	-	+	-
Ψ^{12}	+	-	-	+	+	-	-	+	+	-	-	+	+	-	-	+
Ψ^{13}	+	-	-	+	+	-	-	+	-	+	+	-	-	+	+	-
χ^2	+	-	+	-	+	-	+	-	+	-	+	-	+	-	+	-
χ^3	+	-	+	-	+	-	+	-	-	+	-	+	-	+	-	+
A_α	+	+	+	+	+	+	+	+	-	-	-	-	-	-	-	-
h	0	0	0	$\frac{1}{4}$	0	$\frac{1}{4}$	0	$\frac{1}{2}$	0	0	0	0	0	0	0	0
e	0	0	$\pm\frac{1}{2}$	$\pm\frac{1}{2}$	$\pm\frac{1}{2}$	$\pm\frac{1}{2}$	0	0	-	-	-	-	-	-	-	-

IV. BOSONIZATION, SPIN FIELDS, AND TWIST FIELDS

The gauge-fixed action (2.5) is accompanied by a ghost action. It is well known [14,15] that the ghost systems appropriate for the $N = 2$ string are the reparametrization ghosts (b, c) , an anticommuting pair of free world-sheet fermions with conformal dimensions $(2, -1)$, the $N = 2$ supersymmetry ghosts (β^i, γ^i) or (β^\pm, γ^\pm) , two commuting pairs of free world-sheet fermions with conformal dimensions $(\frac{3}{2}, -\frac{1}{2})$, and the U(1) ghosts (\tilde{b}, \tilde{c}) , an anticommuting pair of free world-sheet fermions with conformal dimensions $(1, 0)$.

Since the superconformal ghosts (β^i, γ^i) carry a space-time index i associated with the target space where the $N = 2$ string lives, the spacetime properties of the physical $N = 2$ string states (to be determined by the BRST cohomology) may depend on their ghost structure as well. This would clearly be quite different from the conventional cases of the bosonic and $N = 1$ supersymmetric strings. To settle the framework for identifying the \mathbb{Z}_2 -twisted BRST-invariant states, we have to build up the ground states for all sectors (given in Table II) out of the single $N = 2$ super-SL(2,C) invariant vacuum $|0\rangle$. This can be done by introducing spin and \mathbb{Z}_2 twist fields for the matter and ghost fields, and using bosonization [20–22]. The spin fields twist world-sheet fermions, while the twist fields twist world-sheet bosons. From now on we employ planar complex coordinates (z, \bar{z}) for the Euclidean world sheet.

The chiral fermionic fields $\psi^{\mu i}(z)$ can be bosonized as⁶

$$\begin{aligned} \psi^{0i} &\cong \frac{1}{\sqrt{2}}(e^{\uparrow\phi^i} + e^{\downarrow\phi^i}), \\ \psi^{1i} &\cong \frac{1}{\sqrt{2}}(e^{\uparrow\phi^i} - e^{\downarrow\phi^i}), \end{aligned} \tag{4.1}$$

where \uparrow and \downarrow simply stand for $+$ and $-$. In Eq. (4.1) two real scalar bosons $\phi^i(z)$ have been introduced, with the operator product expansion (OPE):

$$\phi^i(z)\phi^j(w) \sim \delta^{ij} \ln(z-w). \tag{4.2}$$

Using the light-cone combinations

$$\frac{1}{\sqrt{2}}\psi^{\uparrow\downarrow i} \cong e^{\uparrow\downarrow\phi^i}, \tag{4.3}$$

we can construct the *spin* fields $S^{i,\uparrow\downarrow}(z)$ with helicity index $\uparrow\downarrow$ as

$$S^{i,\uparrow\downarrow} \cong e^{\uparrow\downarrow\phi^i/2}. \tag{4.4}$$

We choose the notation $\uparrow\downarrow$ for the spacetime light-cone coordinates in order to distinguish them from the complex internal ones $2 \pm i3$. The spin fields $S^{i,\uparrow\downarrow}$ twist the fermions $\psi^{\mu i}$ with respect to the index μ , which corresponds to the action of \mathbb{Z}_2' and the $(+ - + -)$ -type boundary conditions in Table II.

The convenient application of the bosonization procedure actually depends on the monodromy conditions under consideration. If the twist is applied with respect to the index i , which corresponds to the action of \mathbb{Z}_2 and the $(+ + - -)$ -type boundary conditions in Table II, different bosonization rules have to be introduced as [22]

$$\psi^{\mu 2} \cong \frac{i^\mu}{\sqrt{2}}(e^{\phi^\mu} + e^{-\phi^\mu}), \quad \psi^{\mu 3} \cong \frac{i^\mu}{i\sqrt{2}}(e^{\phi^\mu} - e^{-\phi^\mu}), \tag{4.5}$$

⁶Normal ordering is always suppressed, as well as cocycle operators [21].

with the OPE as

$$\phi^\mu(z)\phi^\nu(w) \sim \delta^{\mu\nu} \ln(z-w). \quad (4.6)$$

The associated $U(1)$ and $U(1)_{\text{gauged}}$ eigenfields are

$$\frac{1}{\sqrt{2}}\psi^{\mu\pm} \cong i^\mu e^{\pm\phi^\mu}, \quad S^{\mu,\pm} \cong e^{\pm\phi^\mu/2}. \quad (4.7)$$

The (q, s, e) charges of any spin field can be found in Table I. We can also formally define a ‘‘little’’ spin field $S^{\mu i}$ which simply sign flips the boundary condition of the individual fermion $\psi^{\mu i}$ and has conformal dimension $h = \frac{1}{16}$.

The reparametrization ghosts (b, c) are bosonized as [21]

$$b \cong e^{-\sigma}, \quad c \cong e^\sigma, \quad \text{with } \sigma(z)\sigma(w) \sim \ln(z-w). \quad (4.8)$$

Similarly, for the $U(1)$ ghosts (\tilde{b}, \tilde{c}) one has

$$\tilde{b} \cong e^{-\tilde{\sigma}}, \quad \tilde{c} \cong e^{\tilde{\sigma}}, \quad \text{with } \tilde{\sigma}(z)\tilde{\sigma}(w) \sim \ln(z-w). \quad (4.9)$$

As for the superconformal ghosts (β^i, γ^i) , the bosonization rules of Refs. [20,21] imply

$$\begin{aligned} \beta^i(z) &\cong e^{-\varphi^i} \partial \xi^i \cong e^{-\varphi^i + \theta^i} \partial \theta^i, \\ \gamma^i(z) &\cong \eta^i e^{\varphi^i} \cong e^{-\theta^i + \varphi^i}, \end{aligned} \quad (4.10)$$

$$\begin{aligned} \varphi^i(z)\varphi^j(w) &\sim -\delta^{ij} \ln(z-w), \\ \eta^i(z)\xi^j(w) &\sim \delta^{ij} \frac{1}{z-w}, \end{aligned}$$

where the auxiliary (η^i, ξ^i) conformal system of spin $(1, 0)$ has also been bosonized as

$$\eta^i \cong e^{-\theta^i}, \quad \xi^i \cong e^{\theta^i}, \quad (4.11)$$

with $\theta^i(z)\theta^j(w) \sim \delta^{ij} \ln(z-w)$.

The solitons $e^{\pm\varphi^i}$ are outside the monomial field algebra of (β^i, γ^i) ; however, one finds that

$$e^{\varphi^i} \cong \delta(\beta^i), \quad e^{-\varphi^i} \cong \delta(\gamma^i). \quad (4.12)$$

Alternatively, one can bosonize the complex linear combinations $\beta^\pm = \beta^2 \pm i\beta^3$ and $\gamma^\pm = \gamma^2 \pm i\gamma^3$:

$$\begin{aligned} \beta^\pm &\cong e^{-\varphi^\mp} \partial \xi^\pm, \quad \gamma^\pm \cong \eta^\pm e^{\varphi^\pm}, \\ \delta(\beta^\pm) &\cong e^{\varphi^\mp}, \quad \delta(\gamma^\pm) \cong e^{-\varphi^\pm}, \\ \varphi^\pm(z)\varphi^\pm(w) &\sim -\ln(z-w), \\ \eta^\pm(z)\xi^\mp(w) &\sim \frac{2}{z-w}, \end{aligned} \quad (4.13)$$

where η^\pm and ξ^\pm are not just linear combinations of the fields appearing in Eq. (4.11).

Here and in what follows we use the standard results [20,21] for free chiral bosons $\rho \in \{\sigma, \tilde{\sigma}, \varphi^i, \theta^i\}$:

$$h[e^{q\rho}] = \frac{\epsilon}{2} q(q - \tilde{Q}), \quad \rho(z)\rho(w) \sim \epsilon \ln(z-w), \quad \epsilon = \pm 1,$$

$$\tilde{Q}_\sigma = 3, \quad \tilde{Q}_{\tilde{\sigma}} = 1, \quad \tilde{Q}_{\varphi^i} = -2, \quad \tilde{Q}_{\theta^i} = 1, \quad (4.14)$$

where \tilde{Q} is a background charge, and the factor ϵ takes into account statistics. It follows that the spin fields $S^{i,\uparrow\downarrow}$ twisting ψ^i and χ^i , as well as the spin fields $S^{\mu,\pm}$ twisting ψ^μ , have conformal dimensions equal to $+\frac{1}{8}$. Similarly, the fields $e^{-\varphi^i/2}$ twisting β^i and γ^i as well as the fields $e^{-\varphi^\pm/2}$ twisting β^\pm and γ^\pm have conformal dimension equal to $+\frac{3}{8}$.

To describe the possible $N = 2$ string ground states corresponding to all the sectors listed in Table II, we start from the formal vacuum state $|0\rangle$ in the untwisted sector (2), which satisfies the standard constraints⁷ ($n \in \mathbb{Z}, r \in \mathbb{Z} + \frac{1}{2}$)

$$\begin{aligned} \alpha_n^a |0\rangle &= 0 \quad n \geq 0, \quad \psi_r^a |0\rangle = 0 \quad r \geq 1/2, \\ b_n |0\rangle &= 0 \quad n \geq -1, \quad c_n |0\rangle = 0 \quad n \geq 2, \end{aligned} \quad (4.15)$$

$$\begin{aligned} \beta_r^i |0\rangle &= 0 \quad r \geq -1/2, \quad \gamma_r^i |0\rangle = 0 \quad r \geq 3/2, \\ \tilde{b}_n |0\rangle &= 0 \quad n \geq 0, \quad \tilde{c}_n |0\rangle = 0 \quad n \geq 1. \end{aligned}$$

The conformal dimension of the state $|0\rangle$ is $h = 0$, and all its charges vanish, by definition. Like in the $N = 1$ string, the BRST cohomology comes in an infinite number of copies labeled by picture numbers. More precisely, there are *two* picture-changing operators for the $N = 2$ string: namely, $X^i(z) = \{Q_{\text{BRST}}, \xi^i(z)\}$. On top of this, there is a fourfold degeneracy represented by

$$e^\sigma |0\rangle, \quad e^{2\sigma} |0\rangle, \quad e^{\sigma+\tilde{\sigma}} |0\rangle, \quad e^{2\sigma+\tilde{\sigma}} |0\rangle \quad (4.16)$$

in the canonical ghost sectors.⁸ We pick the first member of this list here. Concerning the superconformal ghost pictures, we introduce the notation

$$\begin{aligned} |k; q_2, q_3\rangle &\cong \exp\left(\frac{i}{2}(k^+ \cdot Z^- + k^- \cdot Z^+)\right) \\ &\times e^{q_2\varphi^2 + q_3\varphi^3 + \sigma} |0\rangle \end{aligned} \quad (4.17)$$

and concentrate on the canonical values $q_i \in \{-1, -\frac{1}{2}\}$. In the following, $\mathcal{H}_{\pm\pm\pm\pm}^{\text{NS,R}}$ denote the 16 sectors of the complete Fock space of states, with the four signs corresponding to the signs for $Z^{\mu i}$ appearing in each column of Table II.

The reference state $|0\rangle$ is not BRST invariant. Our representative of the true $\mathcal{H}_{++++}^{\text{NS}}$ ground state with momentum k is given by $|k, -1, -1\rangle$, which has $q = s = e = 0$ and $h = \frac{1}{2}k^+ \cdot k^-$, as can easily be computed from Eq. (4.14). BRST invariance requires $h = 0$, which translates into the masslessness condition $k^+ \cdot k^- = 0$. In summary, one finds a massless neutral physical space-time scalar boson.

Among the candidates for the $\mathcal{H}_{++++}^{\text{R}}$ ground state

⁷The mode expansions of the fields and currents for this case are given in Appendix B.

⁸If the $U(1)$ gauge field is twisted, there is no \tilde{c} zero mode. In this case, only a twofold degeneracy arises, and we shall need $q_{\tilde{\sigma}} = \frac{1}{2}$.

there are four states of the type $S^{2,\uparrow}S^{3,\uparrow\downarrow}|k; -\frac{1}{2}, -\frac{1}{2}\rangle$. Requiring the ground state to be a $U(1)_{\text{gauged}}$ singlet leaves us with two possibilities:

$$S^{2,\uparrow}S^{3,\uparrow}|k; -\frac{1}{2}, -\frac{1}{2}\rangle \text{ and } S^{2,\downarrow}S^{3,\downarrow}|k; -\frac{1}{2}, -\frac{1}{2}\rangle, \quad (4.18)$$

which have all charges vanishing and $h = 0$, provided $k^+ \cdot k^- = 0$ again. In contrast to the NS sector, we identify a neutral massless physical vector.⁹

One may argue, however, that spectral flow moves us from the \uparrow component of this vector to the $\mathcal{H}_{++++}^{\text{NS}}$ ground state scalar and further to the \downarrow component, effectively identifying all three degrees of freedom (in the untwisted theory). In Sec. VII we will test for their BRST invariance.

To actually describe the twisted sectors, we need \mathbb{Z}_2 twist fields $t^{\mu i}(z)$, whose role is to twist the boundary conditions for the bosonic Z fields [23]. The twist fields generically act as

$$t^{\mu i}(z)\partial Z^{\nu j}(w) \sim \frac{\eta^{\mu\nu}\delta^{ij}}{\sqrt{z-w}}\tilde{t}(w), \quad (4.19)$$

where \tilde{t} is another (regular) twist field. These twist fields act trivially on ψ 's. The conformal dimension of the twist field, $h[t^{\mu i}] = \frac{1}{16}$, and the OPE

$$t^{\mu i}(z)t^{\mu j}(w) \sim -\frac{\eta^{\mu\nu}\delta^{ij}}{(z-w)^{1/8}} \quad (4.20)$$

were calculated in Ref. [24]. The results of Ref. [24] imply conformal dimensions equal to $+\frac{1}{8}$ for the twist fields $t^i \equiv t^{0i}t^{1i}$ (no sum) and $t^\mu \equiv t^{\mu 2}t^{\mu 3}$ (no sum), while the dimension of $t \equiv t^{02}t^{03}t^{12}t^{13}$ is equal to $+\frac{1}{4}$. This is consistent with the change in ground state energy due to a general $U(1)$ twist, as will be seen in Eq. (5.3) for $a = \frac{1}{2}$. We assume that all twist fields are Virasoro primaries. They carry global $U(1)$ and $SO(1,1)$ charges but are neutral under $U(1)_{\text{gauged}}$ (see Table I).

Using the spin and twist fields just introduced allows us to construct the ground states for the \mathbb{Z}_2 twisted sectors

of the $N = 2$ string (Table II). For example, for the $\mathcal{H}_{+-+}^{\text{NS}}$ sector one finds the doublet

$$t^1 S^{1,\pm}|k; -1, -1\rangle, \quad (4.21)$$

where the twist field t^1 has been used to twist the Z^{1i} , while the spin field S^1 was needed to twist the ψ^{1i} . The superconformal ghosts (β^i, γ^i) are untwisted. The conformal dimensions of the fields contributing to this state add up to $\frac{1}{4}$, and its local $U(1)$ charge equals $\pm\frac{1}{2}$. These properties will be in conflict with BRST invariance.

The candidate ground state for the $\mathcal{H}_{+--+}^{\text{NS}}$ sector takes the form [22]

$$t^3 S^{3,\uparrow\downarrow} e^{\tilde{\sigma}/2}|k; -1, -\frac{1}{2}\rangle. \quad (4.22)$$

Since the superconformal ghosts (β^3, γ^3) (and, in fact, χ^3) have to be twisted too, one needs the $g_3 = -\frac{1}{2}$ picture in Eq. (4.22). Finally, the appearance of another "spin" field $e^{\tilde{\sigma}/2}$ in Eq. (4.22) is due to the twist of the $U(1)$ gauge field A_α (see Table II).¹⁰

With conformal dimensions $h[e^{\tilde{\sigma}/2}] = -\frac{1}{8}$ and $h[t^3] = +\frac{1}{8}$, the conformal dimension of the ground state in the $\mathcal{H}_{+--+}^{\text{NS}}$ sector adds up to 0, as given in Table II. Note that the local $U(1)$ charge e is not defined here, since the $U(1)$ current $J(z)$ is half-integral moded. This destroys the spectral flow and also explains the blank slots in the last line of Table II. BRST invariance will further constrain this state (Sec. VII).

A different method to calculate conformal dimensions h (or critical intercepts) associated with these ground states consists of collecting the corresponding contributions to the intercept from the periodic (P) or antiperiodic (A) world-sheet bosons and fermions in each sector separately. The standard results [13] are displayed in Table III. Our results for the $N = 2$ string critical intercepts ($-h$) are given in the last row of Table II.

Completing the list of the candidate ground state for all the twisted sectors listed in Table II, we find (multiple sign choices are correlated)

$$\begin{aligned} \mathcal{H}_{++++}^{\text{NS,R}} &: |k; -1, -1\rangle, \quad S^{2,\uparrow\downarrow}S^{3,\uparrow\downarrow}|k; -\frac{1}{2}, -\frac{1}{2}\rangle \\ \mathcal{H}_{+-+}^{\text{NS,R}} &: t^1 S^{1,\pm}|k; -1, -1\rangle, \quad t^1 S^{0,\pm}|k; -\frac{1}{2}, -\frac{1}{2}\rangle \\ \mathcal{H}_{-+-}^{\text{NS,R}} &: t^0 S^{0,\pm}|k; -1, -1\rangle, \quad t^0 S^{1,\pm}|k; -\frac{1}{2}, -\frac{1}{2}\rangle \\ \mathcal{H}_{----}^{\text{NS,R}} &: t^0 t^1 S^{0,\pm} S^{1,\mp}|k; -1, -1\rangle, \quad t^0 t^1 |k; -\frac{1}{2}, -\frac{1}{2}\rangle \end{aligned}$$

whereas

$$\begin{aligned} \mathcal{H}_{+-+}^{\text{NS,R}} &: t^3 S^{3,\uparrow\downarrow} e^{\tilde{\sigma}/2}|k; -1, -\frac{1}{2}\rangle, \quad t^3 S^{2,\uparrow\downarrow} e^{\tilde{\sigma}/2}|k; -\frac{1}{2}, -1\rangle \\ \mathcal{H}_{+--+}^{\text{NS,R}} &: t^{03} t^{12} S^{03} S^{12} e^{\tilde{\sigma}/2}|k; -1, -\frac{1}{2}\rangle, \quad t^{03} t^{12} S^{02} S^{13} e^{\tilde{\sigma}/2}|k; -\frac{1}{2}, -1\rangle \end{aligned}$$

⁹The existence of this state in the spectrum of the $N = 2$ string was noticed in Ref. [22].

¹⁰This also follows from the structure of the BRST charge (Appendix B), where the $U(1)$ ghost field \tilde{c} multiplies the $U(1)$ current $J(z)$.

TABLE III. Contributions to (minus) the intercept of matter and ghost fields for periodic (P or R) and antiperiodic (A or NS) boundary conditions.

Sector	Complex boson	Complex fermion	(b, c)	(\tilde{b}, \tilde{c})	(β^i, γ^i)
P or R	$-\frac{1}{12}$	$+\frac{1}{12}$	$+\frac{1}{12}$	$+\frac{1}{12}$	$-\frac{1}{12}$
A or NS	$+\frac{1}{24}$	$-\frac{1}{24}$	$-\frac{1}{24}$	$-\frac{1}{24}$	$+\frac{1}{24}$

and quite similar for $\mathcal{H}_{-+-+}^{NS,R}$ and $\mathcal{H}_{-++-}^{NS,R}$. In all cases, the momenta will have to be adjusted for BRST invariance.

To compute the values of h for zero momentum in all sectors, it is useful to collect

$$\begin{aligned}
 h[|0; -1, -1\rangle] &= 0, \quad h[|0; -\frac{1}{2}, -\frac{1}{2}\rangle] = -\frac{1}{4}, \\
 h[|0; -1, -\frac{1}{2}\rangle] &= -\frac{1}{8}, \quad h[|0; -\frac{1}{2}, -1\rangle] = -\frac{1}{8}.
 \end{aligned}
 \tag{4.23}$$

With the values given above and in Table I we obtain massless states in all sectors but (4), (6), and (8) of Table II. The local $U(1)$ charges are only defined in the first eight sectors. Neutrality is impossible in sectors (3)–(6) of Table II.

The spacetime interpretation of the \mathbb{Z}_2 twists considered so far is quite clear. Depending on how many of the three twists \mathbb{Z}_2 , \mathbb{Z}'_2 , and \mathbb{Z}''_2 for Z we are permitting, the target space will be an orbifold

$$\mathcal{B} = \mathbb{C}^{1,1}/\mathbb{Z}_2 \text{ or } \mathbb{C}^{1,1}/\mathbb{Z}_2^2 \text{ or } \mathbb{C}^{1,1}/\mathbb{Z}_2^3,
 \tag{4.24}$$

which represents a cone or a half-space, a quarter-, or an eighth-space in $\mathbb{C}^{1,1}$, respectively. The first eight sectors of Table II contain spacetime bosons, whereas the ground states of sectors (9)–(16) are good candidates for spacetime fermions.

V. SPECTRAL FLOW

We are now going to consider more general monodromies parametrized by arbitrary phases:

$$\begin{aligned}
 Z^{\mu\pm}(\pi) &= +e^{\pm 2\pi i a^\mu} Z^{\mu\pm}(0), \\
 \Psi^{\mu\pm}(\pi) &= -e^{\pm 2\pi i a^\mu} e^{\pm 2\pi i \nu} \Psi^{\mu\pm}(0), \\
 \chi_\alpha^\pm(\pi) &= -e^{\pm 2\pi i \nu} \chi_\alpha^\pm(0),
 \end{aligned}
 \tag{5.1}$$

where the angles a^μ and ν are valued in $S^1 = \mathbb{R}/\mathbb{Z}$, and the signs are correlated. The special values $a^\mu = 0$ and $a^\mu = \frac{1}{2}$ correspond to the cases considered in Sec. III, namely, sectors (1)–(8). Similarly, the values $\nu = 0$ and $\nu = \frac{1}{2}$ correspond to the NS and R sectors, respectively, in our notation. Stated differently, the complex vectors Z^\pm and Ψ^\pm are subject to $U(1) \otimes U(1)$ boundary conditions, with a relative angle ν between Z and Ψ . We call these monodromies *complex elliptic*, for reasons to be explained in the following section. Clearly, a “good” target space arises only in case a^μ are rational, which leads to

$\mathcal{B} = \mathbb{C}^{1,1}/(\mathbb{Z}_n \otimes \mathbb{Z}_m)$, i.e., a product of cones.

It is worthy to notice here that the \mathbb{Z}'_2 -twisted (complex elliptic) boundary conditions of the type

$$\begin{aligned}
 Z^{\mu\pm}(\pi) &= +e^{\pm 2\pi i a^\mu} Z^{\mu\mp}(0), \\
 \Psi^{\mu\pm}(\pi) &= -e^{\pm 2\pi i a^\mu} e^{\pm 2\pi i \nu} \Psi^{\mu\mp}(0), \\
 \chi_\alpha^\pm(\pi) &= -e^{\pm 2\pi i \nu} \chi_\alpha^\mp(0),
 \end{aligned}
 \tag{5.2}$$

do not give anything new, since the phases in Eq. (5.2) can easily be removed up to signs by rescaling the fields [12], leaving just the sectors (9)–(16) of Table II.

The angle ν in Eq. (5.1) parametrizes rotations between the NS and R sectors, and it can be identified with the parameter of the *spectral flow* in the $N = 2$ superconformal algebra [5,25]. To understand this specific feature of the $N = 2$ string, one has to investigate the $N = 2$ supermoduli space. It is not hard to see that different choices of ν in Eq. (5.1) are simply related by shifts in the complex moduli of the $U(1)$ gauge field A [5]. Since we are instructed to finally integrate over all moduli, any amplitude has to be averaged over ν eventually. It seems to follow that monodromy sectors related by spectral flow should be identified, as happens in *globally* $N = 2$ superconformal field theory. This is not really true, for the following reason. For a given n -string amplitude, the $U(1)$ moduli are represented by nontrivial Wilson lines. Hence, we may shift the fermionic monodromies for each homology cycle individually, in particular for the cycles around the punctures. However, there are $n - 1$ *independent* homology cycles for a tree-level n -string amplitude. Consequently, exactly *one* of the fermionic boundary conditions of the different asymptotic string states cannot be altered by the spectral flow. More specifically, we can rotate all *but one* ν parameters to zero. Of course, the amplitude will have to vanish unless the “last” ν was zero from the beginning. However, the vanishing of an amplitude does not mean that any of its external states does not exist. In the example of a two-point function, we should indeed identify the NS-NS with the R-R propagator (and all interpolations); however, the NS-R correlation function (and all others with a relative twist) are genuinely different, albeit zero. In summary, the spectral flow does relate some string amplitudes, but does *not* identify NS with R states physically. Instead, we have a one-parameter family of distinct states labeled by $\nu \in [0, 1]$.

To calculate the ground state dimensions for the generalized boundary conditions in Eq. (5.1), one needs to sum up the relevant contributions from the $N = 2$ string fields. The basic formulas for the vacuum energy of a twisted complex boson and a twisted complex fermion are [26,27]

$$\begin{aligned}
 h_B &= 2 \sum_{n \in \mathbb{Z} + a} \frac{n}{2} = \sum_{n \in \mathbb{Z}} (n + a)^{-s} \Big|_{s=-1} = \zeta(-1, a) \\
 &= -\frac{1}{12} + \frac{1}{2} a(1 - a) =: f(a), \\
 h_F &= +\frac{1}{12} - \frac{1}{2} (a + \nu + \frac{1}{2})(1 - a - \nu - \frac{1}{2}) \\
 &= -f(a + \nu + \frac{1}{2}),
 \end{aligned}
 \tag{5.3}$$

TABLE IV. Contributions to the vacuum energy. The left column corresponds to the generalized boundary conditions in Eq. (5.1). The right column gives the result for the discrete \mathbb{Z}'_2 -twisted cases (9)–(16) of Table II. $\nu \in [-\frac{1}{2}, \frac{1}{2}]$ and $a^\mu \in [-\frac{1}{2} - \nu, \frac{1}{2} - \nu]$.

Twists	a^0	a^1	$Z^\pm \rightarrow Z^\mp$ and $\psi^\pm \rightarrow \psi^\mp$
	$\pm \pm$	$\pm \pm$	
Z	$\pm \pm$	$\mp \mp$	$\pm \mp \pm \mp$
ψ	$\pm \pm$	$\mp \mp$	$\pm \mp \pm \mp$
b, c	$f(a^0) + f(a^1)$	$f(a^0) + f(a^1)$	$f(0) + f(\frac{1}{2})$
β, γ	$-f(a^0 + \nu + \frac{1}{2}) - f(a^1 + \nu + \frac{1}{2})$	$-f(a^0 + \nu + \frac{1}{2}) - f(a^1 + \nu + \frac{1}{2})$	$-f(0) - f(\frac{1}{2})$
\bar{b}, \bar{c}	$-f(0)$	$-f(0)$	$-f(0)$
	$2f(\nu + \frac{1}{2})$	$2f(\nu + \frac{1}{2})$	$f(0) + f(\frac{1}{2})$
	$-f(0)$	$-f(0)$	$-f(\frac{1}{2})$
$h = h^0 + h^1$	$h^\mu = \begin{cases} a^\mu(\nu + \frac{1}{2}) & \text{for } a^\mu \in [0, \frac{1}{2} - \nu] \\ a^\mu(\nu - \frac{1}{2}) & \text{for } a^\mu \in [-\frac{1}{2} - \nu, 0] \end{cases}$		0

where $a \in [0, 1]$ in the first line and $a + \nu + \frac{1}{2} \in [0, 1]$ in the second one.¹¹ This result forces the conformal dimensions of general complex twist fields and spin fields to be equal to $h[t] = \frac{1}{2}a(1-a)$ and $h[S] = \frac{1}{2}(a+\nu)^2$, respectively. In adding the individual contributions of the different fields, care has to be exercised in order to make sure that the periodic argument of the function f is always taken from the interval $[0, 1]$, where Eq. (5.3) applies. In general, one has to create a list of case distinctions [12]. The results of counting are summarized in Table IV. The terms quadratic in a^μ cancel among bosons and fermions due to

$$f(a) - f(a + \nu + \frac{1}{2}) = a(\nu + \frac{1}{2}) + (\nu^2 - \frac{1}{4}). \quad (5.4)$$

The $U(1)_{\text{gauged}}$ charge for the ground states in question is given by $|e| = (|a^0| + |a^1|)$, where it is defined. It follows that, for a given ν , h is a symmetric, periodic, and piecewise linear function of a^0 and a^1 , whose derivatives jump on the lines $a^\mu = 0$ as well as $a^\mu = \frac{1}{2} - \nu$ and $a^\mu = -\frac{1}{2} - \nu$. Moreover, h is positive everywhere except at $a^0 = a^1 = 0$ and for $\nu = \pm \frac{1}{2}$, where it vanishes identically.

The ground state energy equally follows from computing the $N = 2$ superconformal algebra in terms of the $N = 2$ string BRST currents. The relevant central extension terms $A(m)$ in the Virasoro subalgebra, having the form

$$[L_m^{\text{tot}}, L_n^{\text{tot}}] = (m-n)L_{m+n}^{\text{tot}} + A(m)\delta_{m+n,0}, \quad (5.5)$$

are at most *cubic* in m .¹² The total m^3 contribution to $A(m)$, being independent of the twisting phases, vanishes in the critical dimension. The coefficients of the terms linear in $2m$ sum to minus the critical intercept; they are collected in Table III. See Refs. [12,29] for the explicit

calculations in this approach.

The conclusions we can draw for the $N = 2$ string from the spectral flow analysis are the following. First, the corresponding R and NS sectors of Table II are *not* to be identified although they are related by spectral flow. This differs from one of the conclusions of Ref. [5]. In fact, one has a one-parameter family of distinct states here. Second, the conformal dimension h of the ground state *changes* under the spectral flow unless $a^0 = a^1 = 0$. If we accept only *massless gauge-singlet* states as physical¹³ [19,28], the two angles a^μ in Eq. (5.1) are required to vanish. Although the Ramond ground state is always massless, we would generally consider it to be unphysical, from a continuity argument. Hence, only states without $U(1) \otimes U(1)$ twist and, in particular, without \mathbb{Z}_2 or \mathbb{Z}'_2 twist survive, leaving only the (Ooguri-Vafa) states, i.e., sectors (1) and (2), from the first 8 sectors of Table II. Third, this discussion does not apply to the \mathbb{Z}''_2 -twisted states, where all sectors (9)–(16) remain. Their invariance under spectral flow is in line with the twisting of the gauge field, which eliminates all $U(1)$ moduli. Each of these discrete sectors may be generated from, say, sector (9) by the action of \mathbb{Z}_2 , \mathbb{Z}'_2 , and \mathbb{Z}''_2 twists in (3.2).

All this gives reasons to identify the ground states in the untwisted (Ooguri-Vafa) sectors (1) and (2) as physical spacetime *bosons*, whereas the ground states in the (Mathur-Mukhi) sectors (9)–(16) should be physical spacetime *fermions*. Demanding the ground states of all occurring sectors to be *massless*, only \mathbb{Z}''_2 and \mathbb{Z}'''_2 twists are permitted, selecting sectors (1),(2) and (9),(10) at most. Since each sector contributes only one or two physical degrees of freedom, \mathbb{Z}_2 twists leave room for a $2_B \oplus 2_F$ spacetime supermultiplet in the complex half-space $\mathcal{B} = \mathbb{C}^{1,1}/\mathbb{Z}''_2$. These observations partially support the idea of space-time extended supersymmetry in

¹¹Note that this expression is not periodic.

¹²For instance, $A(m) = \frac{1}{12}(1 - 3\tilde{Q}^2)m^3 + 2h_F m$ for an anti-commuting (b, c) -type system with background charge \tilde{Q} .

¹³This is naively “obvious” in the “double” light-cone gauge for the $N = 2$ string, where all the excitations of Z 's and Ψ 's disappear.

$N = 2$ string theory, put forward in Ref. [6]. However, nontrivial target space topology seems to be needed to realize it. In principle, the spectral flow parameter ν may take values different from 0 or $\frac{1}{2}$. The corresponding vertex operators, however, have little chance of leading to a local operator algebra (see Sec. VII).

VI. GENERAL MONODROMY CONDITIONS

We are now in a position to discuss the most general monodromy conditions for the $N = 2$ string. Let us concentrate on the Z fields, and arrange them into a complex doublet as

$$Z = \begin{pmatrix} Z^{0+} \\ Z^{1+} \end{pmatrix}, \quad Z^\dagger = (Z^{0-}, Z^{1-}), \tag{6.1}$$

where $Z^0 = Z^{02} + iZ^{03}$, $Z^1 = Z^{12} + iZ^{13}$. The fields ψ and χ_α can be treated similarly. The fields Z enter the $N = 2$ string action via the kinetic term, $\partial_\pm Z^\dagger \cdot \eta \cdot \partial_\mp Z$, where $\eta = \text{diag}(-, +)$ is the two-dimensional complex target space metric. The kinetic term is obviously invariant under the unitary transformations

$$Z \rightarrow UZ, \quad Z^\dagger \rightarrow Z^\dagger U^\dagger, \quad \text{with } U^\dagger \eta U = \eta. \tag{6.2}$$

As remarked earlier, these $U(1,1)$ transformations, together with similar compensating rotations of Ψ and χ , actually constitute the global continuous symmetry group of the *full* action (3.1).

We shall now consider arbitrary $U(1,1)$ transformations (6.2) as possible monodromies ($z \rightarrow e^{2\pi i z}$). The determinantal phase factor may be split off trivially. This is just the $U(1)$ factor corresponding to $a^0 + a^1$, which was ruled out in the last section. The remaining $SU(1,1)$ subgroup can be realized as

$$U = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix}, \quad |\alpha|^2 - |\beta|^2 = 1. \tag{6.3}$$

The corresponding $\mathfrak{su}(1,1)$ Lie algebra generators satisfy the relations

$$[L_1, L_2] = -L_3, \quad [L_2, L_3] = +L_1, \quad [L_3, L_1] = +L_2, \tag{6.4}$$

where L_1 and L_2 are noncompact and Hermitian, and L_3 is compact and anti-Hermitian. In our basis (6.1), L_3 turns out to be diagonal:

$$L_1 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad L_2 = \frac{1}{2} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},$$

$$L_3 = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{6.5}$$

Of course, the matrix representation depends on the choice of basis. For a different choice, the form of the Hermitian metric η changes, and it dictates the structure of the $SU(1,1)$ monodromy matrix via Eq. (6.2). In general, the only restrictions on the metric are $\eta^\dagger = \eta$

and $\det \eta = -1$, so that

$$\eta = \begin{pmatrix} \eta_0 + \eta_3 & \eta_1 - i\eta_2 \\ \eta_1 + i\eta_2 & \eta_0 - \eta_3 \end{pmatrix} \equiv \eta_\mu \sigma^\mu, \quad \eta_\mu \eta^\mu = 1, \tag{6.6}$$

where $\sigma^\mu = (\mathbf{1}, \boldsymbol{\sigma})$, and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are Pauli matrices. Each value for η determines a basis for the target space components of the fields. Different bases are related by linear field redefinitions:

$$Z \rightarrow \tilde{Z} = MZ, \quad Z^\dagger \rightarrow \tilde{Z}^\dagger = Z^\dagger M^\dagger, \tag{6.7}$$

forming the linear matrix group $GL(2, \mathbb{C})$. The redefined field \tilde{Z} has the monodromy

$$\tilde{Z} \equiv MZ \rightarrow MUZ = (MUM^{-1})MZ = (MUM^{-1})\tilde{Z} \equiv \tilde{U}\tilde{Z}, \tag{6.8}$$

so that Eq. (6.7) induces

$$U \rightarrow \tilde{U} = MUM^{-1}. \tag{6.9}$$

At the same time, the action is unchanged,

$$\tilde{Z}^\dagger \tilde{\eta} \tilde{Z} = Z^\dagger \eta Z, \tag{6.10}$$

so that the new metric takes the form

$$\tilde{\eta} = M^{-1\dagger} \eta M^{-1}. \tag{6.11}$$

Any two metrics (6.6) are related this way. Apparently, the determinantal factor of M merely leads to a complex rescaling, which is rather trivial. We therefore take $M \in SL(2, \mathbb{C})$. When $M \in SU(1,1)$, Eq. (6.11) becomes $\tilde{\eta} = \eta$, and, hence, \tilde{U} is conjugate to U in $U(1,1)$. However, when $M \notin SU(1,1)$, one gets $\tilde{\eta} \neq \eta$, which implies a change in the metric eigenvalues.

Given a fixed basis and metric η , we must consider two monodromies as equivalent if they are related by a global symmetry transformation $M \in U(1,1)$. Therefore, if we are interested in *all* inequivalent monodromies U of the $N = 2$ string, we need to consider all the *conjugacy classes* of¹⁴ $U(1,1) = [U(1) \otimes SU(1,1)]/Z_2$. These are labeled uniquely¹⁵ by the determinant $\det U$, corresponding to the $U(1)$ factor,¹⁶ and the normalized trace $(\det U)^{-1/2} \text{tr} U$, corresponding to the trace in $SU(1,1)$. The absolute value of the normalized trace can be either less than 2, equal to 2, or greater than 2, which distinguishes the so-called *elliptic*, *parabolic*, and *hyperbolic* cases, respectively.

For a generic $SU(1,1)$ monodromy of the form

$$\begin{pmatrix} Z^0 \\ Z^1 \end{pmatrix} (\sigma + \pi) = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} \begin{pmatrix} Z^0 \\ Z^1 \end{pmatrix} (\sigma), \tag{6.12}$$

¹⁴Plus the Z_2'' twist.

¹⁵Except if $|\text{tr} U| = 2$.

¹⁶We can actually take $(\det U)^2$ due to the sign ambiguity from Z_2 .

substituting the mode expansion

$$\begin{pmatrix} Z^0 \\ Z^1 \end{pmatrix}(\sigma) = \sum_n e^{2in\sigma} K(\sigma) \begin{pmatrix} a_n^0 \\ a_n^1 \end{pmatrix} \quad (6.13)$$

yields the equation

$$K(\sigma + \pi) = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} K(\sigma). \quad (6.14)$$

With $K(0) = \mathbb{1}$ we have $U = K(\pi)$. Introducing the $\text{su}(1,1)$ generators as

$$\begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} = \exp \left[\sum_{j=1}^3 \theta_j L_j \right], \quad (6.15)$$

we find the solution in the form

$$K(\sigma) = \exp \left[\frac{\sigma}{\pi} \sum_{j=1}^3 \theta_j L_j \right]. \quad (6.16)$$

Let us be more precise for the basis (6.1), i.e., when $\eta = \text{diag}(-, +)$. If just the compact generator appears, $\theta_j = 2\pi\theta\delta_{j3}$, one gets

$$K_{\text{ell}}(\sigma) = \begin{pmatrix} e^{i\theta\sigma} & 0 \\ 0 & e^{-i\theta\sigma} \end{pmatrix}, \quad (6.17)$$

which obviously corresponds to the elliptic case ($|\text{tr}U| < 2$). Indeed, setting $\theta = a^0 - a^1$ and multiplying $U_{\text{ell}} = K_{\text{ell}}(\pi)$ with the $U(1)$ phase $e^{i\pi(a^0+a^1)}$ just reproduce the complex elliptic monodromy of Eq. (5.1). As was already mentioned, the target space is a tensor product of two cones for the case of rational angles.

Turning on a noncompact generator, e.g., $\theta_j = 2\pi\theta\delta_{j1}$, leads to the prime example of a hyperbolic class:

$$K_{\text{hyp}}(\sigma) = \begin{pmatrix} \cosh \theta\sigma & \sinh \theta\sigma \\ \sinh \theta\sigma & \cosh \theta\sigma \end{pmatrix}. \quad (6.18)$$

Here, we find $|\text{tr}U| > 2$, and $U \in \text{SO}(1,1)$ is the real Lorentz symmetry already discussed in Sec. III. The mode expansion becomes complex and matrix valued:

$$\begin{pmatrix} Z^0 \\ Z^1 \end{pmatrix}(\sigma) = \sum_n \exp \left[2i \begin{pmatrix} n & -i\theta/2 \\ -i\theta/2 & n \end{pmatrix} \sigma \right] \begin{pmatrix} a_n^0 \\ a_n^1 \end{pmatrix}. \quad (6.19)$$

The alternative is to switch to a basis where L_1 is diagonal. This is achieved by taking

$$M = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \tilde{L}_1 = ML_1M^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\tilde{\eta} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad (6.20)$$

which simply brings us to the familiar light-cone basis,

$\tilde{Z} = \frac{1}{\sqrt{2}} \begin{pmatrix} Z^\uparrow \\ Z^\downarrow \end{pmatrix}$. Here, of course, we just get

$$\tilde{K}_{\text{hyp}}(\sigma) = \begin{pmatrix} e^{\theta\sigma} & 0 \\ 0 & e^{-\theta\sigma} \end{pmatrix}, \quad (6.21)$$

but the plane wave modes remain complex. The target space interpretation is also more transparent in this basis. The identifications

$$Z^\uparrow \cong e^{\theta\pi} Z^\uparrow, \quad Z^\downarrow \cong e^{-\theta\pi} Z^\downarrow \quad (6.22)$$

create two similar tori (with real modulus θ). Hence, the background \mathcal{B} is just their tensor product. One should remark that in this situation an additional \mathbb{Z}'_2 twist does not allow us to set $\theta = 0$ by rescaling Z , as was the case in Eq. (5.2). Here, it leads to yet another target space.

Rather unusual is the parabolic situation ($|\text{tr}U| = 2$) which is generated by either one of the nilpotent combinations

$$N_\pm = L_3 \pm L_2 = \frac{1}{2} \begin{pmatrix} i & \pm i \\ \mp i & -i \end{pmatrix} \quad (6.23)$$

and yields

$$K_{\text{par}}(\sigma) = \begin{pmatrix} 1 + i\theta\sigma & \pm i\theta\sigma \\ \mp i\theta\sigma & 1 - i\theta\sigma \end{pmatrix}. \quad (6.24)$$

In the light-cone basis, we have

$$\begin{aligned} \tilde{N}_+ &= \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix}, \quad \tilde{N}_- = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}, \\ \tilde{K}_+ &= \begin{pmatrix} 1 & 0 \\ 2i\theta\sigma & 1 \end{pmatrix}, \quad \tilde{K}_- = \begin{pmatrix} 1 & 2i\theta\sigma \\ 0 & 1 \end{pmatrix}. \end{aligned} \quad (6.25)$$

The background \mathcal{B} in this case is remarkable. From \tilde{K}_- , for example, we read off the identifications

$$Z^\uparrow \cong Z^\uparrow + 2\pi\theta(iZ^\downarrow), \quad (iZ^\downarrow \cong (iZ^\downarrow), \quad (6.26)$$

which is topologically a complex cone. The entire $iZ^\downarrow = 0$ axis has to be identified to a single point through which pass (almost) all straight lines.¹⁷ The metric $\tilde{\eta} = -\sigma_1$ on this cone degenerates for two different real sections: either $Z^{\uparrow\downarrow} = \uparrow\downarrow Z^{\uparrow\downarrow}$ or $Z^{\downarrow\pm} = \pm iZ^{\uparrow\pm}$ leads to a vanishing of $|Z|^2 = Z^{\uparrow-}Z^{\downarrow+} + Z^{\downarrow-}Z^{\uparrow+}$. This leaves room for null planes, just like in $\mathbb{C}^{1,1}$. It is also exceptional that $|\text{tr}U| = 2$ labels not one but *three* conjugacy classes: the identity class as well as the two parabolic ones generated by N_+ and N_- .

One should add that any $\text{SU}(1,1)$ monodromy matrix (6.15) can be factorized into a product of three matrices, with each being in one of the standard forms displayed above. However, there is always a basis in which it just takes the standard form corresponding to its (elliptic,

¹⁷If this is not done, one ends up with a non-Hausdorff space.

hyperbolic, or parabolic) nature.

The ground state energies calculated for the complex elliptic case in Sec. V are all based on Eq. (5.3). This equation needs to be generalized to the other cases (hyperbolic or parabolic). The complex elliptic case corresponds to

$$K(\sigma) = \begin{pmatrix} e^{2ia^0\sigma} & 0 \\ 0 & e^{2ia^1\sigma} \end{pmatrix} = \exp \left[2i \begin{pmatrix} a^0 & 0 \\ 0 & a^1 \end{pmatrix} \sigma \right] \\ = : \exp[2i\sigma\mathcal{A}_{\text{real}}], \quad (6.27)$$

and leads to the value $f(a^0) + f(a^1)$ in Table III, with $f(a) = -\frac{1}{12} + \frac{1}{2}a(1-a)$. The natural attempt is to view the result as arising from the trace

$$2f(\mathcal{A}) = -\frac{1}{6} + \frac{1}{2}\text{tr}[\mathcal{A}(\mathbb{1} - \mathcal{A})], \quad (6.28)$$

with the matrix $\mathcal{A} = \mathcal{A}_{\text{real}}$ of real angles taken from a ‘‘fundamental domain’’ since, again, this expression is not periodic. For a general $U \in \text{U}(1,1)$, one should factor off the $\text{U}(1)$ phase $e^{i\theta_0/2}$ and use the $\text{su}(1,1)$ algebra parametrization (6.15). Inserting

$$\mathcal{A}(\theta) = \frac{1}{2\pi i} \left[\theta_0 \frac{i}{2} \mathbb{1} + \sum_{j=1}^3 \theta_j L_j \right] \quad (6.29)$$

into the *basis-independent* formula (6.28) we arrive at

$$2f(\theta) = -\frac{1}{6} + \frac{1}{16\pi^2} \theta_0(4\pi - \theta_0) + \frac{1}{16\pi^2} (\theta_1^2 + \theta_2^2 - \theta_3^2), \quad (6.30)$$

for $\theta_0 \in [0, 4\pi]$ and $\theta_3 \in [|\theta_0 - 2\pi| - 2\pi, -|\theta_0 - 2\pi| + 2\pi]$. These restrictions arise from $a^\mu \in [0, 1]$ and the relations $\theta_0 = 2\pi(a^0 + a^1)$ and $\theta_3 = 2\pi(a^0 - a^1)$. The quadratic form in θ_j is nothing but the invariant length-squared of a boost or rotation vector associated to the action of $\text{so}(2,1) \cong \text{su}(1,1)$ in $(2+1)$ -dimensional Minkowski space. Time-, space-, and lightlike vectors are seen to belong to elliptic, hyperbolic, and parabolic conjugacy classes, respectively. Remarkably, the ground state energy takes the untwisted value only for $\theta_0 = 0 \pmod{4\pi}$ and a *lightlike* vector $\{\theta_i\}$, i.e., in the identity and parabolic classes, with possible \mathbb{Z}_2'' twist. A direct calculation of the L_0 eigenvalue of the ground state confirms this. We already know from Table IV that *elliptic* $\text{SU}(1,1)$ twists change the conformal dimension of the ground state, since for $\theta_0 = \theta_1 = \theta_2 = 0$ and $|\nu| \leq \frac{1}{2}$ we found

$$h = \begin{cases} |\theta_3|/4\pi & \text{if } |\theta_3| \leq 4\pi(\nu + \frac{1}{2}), \\ \nu + \frac{1}{2} & \text{if } |\theta_3| \geq 4\pi(\nu + \frac{1}{2}). \end{cases} \quad (6.31)$$

For hyperbolic twists, however, the ‘‘angles’’ θ_1 and θ_2 are no longer periodic, and the naive summation of contributions to the ground-state energy based on Eq. (6.30) gives a vanishing result. Looking back to the lesson learned in Sec. V, we should nevertheless expect a vanishing ground state energy only for $\nu = \frac{1}{2}$ and for at most \mathbb{Z}_2'' and/or parabolic twists.

VII. BRST COHOMOLOGY AND INTERACTIONS

We have assumed in Sec. IV that the spin fields $S(S^i$ or $S^\mu)$, as well as the twist fields $t(t^i$ or $t^\mu)$ are all primary and have conformal dimensions $h = \frac{1}{8}$. This does not mean, however, that they are superconformal primaries. They are *not*, in fact, as follows from examining the OPE’s of these fields with the $N = 2$ superconformal algebra generators $G^\pm = \psi^\pm \cdot \partial Z^\mp$ and $J = \frac{1}{4}\psi^- \cdot \psi^+$. In particular, the OPE $G^\pm(z)(St)(w)$ gives rise to the new field $(\Gamma^\pm S) \cdot \tilde{t}^\mp$ of dimensions $h = \frac{3}{4}$ and $\bar{h} = \frac{1}{4}$, where spacetime γ matrices $\Gamma^{\mu\pm} = \Gamma^{\mu 2} \pm i\Gamma^{\mu 3}$ have been introduced.¹⁸

Depending on the spin field, the γ matrices act either on the $\text{U}(1)$ index as $(\Gamma^{\mu i})^{\pm, \pm}$, or on the $\text{O}(1,1)$ index as $(\Gamma^{\mu i})^{\uparrow\downarrow, \uparrow\downarrow}$.

Our task now is to consider an action of the BRST charge in the form [11]

$$Q = Q_1 + Q_2 + Q_3, \quad Q_i = \oint j_i(z), \quad \oint \equiv \oint_0 \frac{dz}{2\pi i}, \\ j_1 = cT_{\text{tot}} + bc\partial c, \\ j_2 = \frac{1}{2}(\gamma^+ G^- + \gamma^- G^+ + 2\tilde{c}J), \\ j_3 = -b\gamma^+ \gamma^- - \tilde{b}(\gamma^+ \partial \gamma^- - \gamma^- \partial \gamma^+) \\ + \frac{1}{4}\tilde{c}(\beta^+ \gamma^- - \beta^- \gamma^+), \quad (7.1)$$

on the candidate ground states or the corresponding vertex operators creating these states.

First, the BRST operator of Eq. (7.1) annihilates the (untwisted) NS ground state $|k\rangle_{++++}^{\text{NS}} = |k; -1, -1\rangle$ when $k^+ \cdot k^- = 0$. The same is also true for the (untwisted) R ground states $|k\rangle_{++++}^{\text{R}} = S^{2,\uparrow\downarrow} S^{3,\uparrow\downarrow} |k; -\frac{1}{2}, -\frac{1}{2}\rangle$, provided that $k^{\uparrow 2} = 0 = k^{\downarrow 3}$ or $k^{\downarrow 2} = 0 = k^{\uparrow 3}$, for the upper or lower choice of sign, respectively. These momentum constraints imply $k^+ \cdot k^- = 0$ but are more stringent.

Second, let us investigate sectors (3) and (4) of Table II, where the combined twist $(St)^1$ or $S^0 t^1$ acting on $|k\rangle_{++++}^{\text{NS}}$ yields $|k\rangle_{++--}^{\text{NS}}$ or $|k\rangle_{+-+--}$, respectively. To this end, consider the OPE’s of the BRST currents $j_i(z)$ with the NS vertex operator of Eq. (4.21), expressed in complex notation:

$$V_{(4)}(w) = \bar{u}c\delta(\gamma^+)\delta(\gamma^-)(St)^1 \\ \times \exp(\frac{i}{2}(k^+ \cdot Z^- + k^- \cdot Z^+))(w), \quad (7.2)$$

which creates the $\mathcal{H}_{++--}^{\text{NS}}$ ground state having the spinor wave function $u(k)$. Using the bosonization formulas (4.13) and the relevant OPE’s:

¹⁸According to Eqs. (4.19) and (4.20), the twist field \tilde{t} has dimensions $h = \frac{5}{8}$ and $\bar{h} = \frac{1}{8}$.

$$\begin{aligned}
\gamma^\pm(z)\delta(\gamma^\pm)(w) &\sim (z-w)\eta^\pm(w), \\
\beta^\pm(z)\delta(\gamma^\mp)(w) &\sim \frac{1}{z-w}\partial\xi^\pm, \\
\psi^{\mu\pm}(z)S^\nu(w) &\sim \frac{\eta^{\mu\nu}}{\sqrt{z-w}}\Gamma^{\mu\pm}S^\nu(w), \\
\partial Z^{\mu\pm}(z)t^\nu(w) &\sim \frac{\eta^{\mu\nu}}{\sqrt{z-w}}\tilde{t}^{\mu\pm}(w),
\end{aligned} \tag{7.3}$$

we find

$$j_1(z)V_{(4)}(w) \sim \frac{h}{z-w}(\partial c)V_{(4)}(w), \tag{7.4a}$$

$$\begin{aligned}
j_2(z)V_{(4)}(w) &\sim \frac{1/2}{\sqrt{z-w}}\bar{u}c[\eta^+\delta(\gamma^-)(-ik^{1+})\Gamma^{1-} \\
&\quad + (+\leftrightarrow -)] \\
&\quad \times (St)^1 \exp\left[\frac{i}{2}(k^+ \cdot Z^- + k^- \cdot Z^+)\right](w) \\
&\quad + \frac{e/2}{z-w}\tilde{c}V_{(4)}(w),
\end{aligned} \tag{7.4b}$$

$$j_3(z)V_{(4)}(w) \sim \frac{1}{2}\tilde{c}\left[\frac{1-1}{z-w}\right]V_{(4)}(w) \sim \text{regular}, \tag{7.4c}$$

where Eq. (7.4c) follows due to a cancellation among the terms produced from $\gamma^-\beta^+ = 2\partial\varphi^-$ and $\gamma^+\beta^- = 2\partial\varphi^+$. Equation (7.4) implies that the \mathbb{Z}'_2 -twisted NS ground state is BRST closed provided it satisfies the three conditions

$$h = \frac{1}{4} + \frac{1}{2}k^+ \cdot k^- = 0, \quad \bar{u}k^{1\pm}\Gamma^{1\mp} = 0, \quad e (= \pm\frac{1}{2}) = 0, \tag{7.5}$$

where e is the *local* $U(1)$ charge of the state. Apparently, $V_{(4)}$ cannot be annihilated by the BRST charge.

Next, we do the same calculation for the \mathcal{H}_{+-}^R sector, where the relevant vertex operator has the form

$$\begin{aligned}
V_{(3)}(w) &= \bar{u}ce^{-\varphi^+/2}e^{-\varphi^-/2}S^0t^1 \\
&\quad \times \exp\left[\frac{i}{2}(k^+ \cdot Z^- + k^- \cdot Z^+)\right](w).
\end{aligned} \tag{7.6}$$

It follows that

$$j_1(z)V_{(3)}(w) \sim \frac{h}{z-w}(\partial c)V_{(3)}(w), \tag{7.7a}$$

$$\begin{aligned}
j_2(z)V_{(3)}(w) &\sim \frac{e/2}{z-w}\tilde{c}V_{(4)}(w) \\
&\quad + \text{terms of order } \frac{1}{z-w} \text{ and } \frac{1}{\sqrt{z-w}},
\end{aligned} \tag{7.7b}$$

$$j_3(z)V_{(3)}(w) \sim \frac{1}{2}\tilde{c}\left[\frac{\frac{1}{2}-\frac{1}{2}}{z-w}\right]V_{(4)}(w) \sim \text{regular}. \tag{7.7c}$$

Although now $h = \frac{1}{2}k^+ \cdot k^-$ vanishes for a massless excitation, again the vertex operator (7.6) is not BRST invariant, in particular because $e = \pm\frac{1}{2} \neq 0$.

Third, we come to the $(---)$ -type boundary conditions. For the NS vertex operator

$$\begin{aligned}
V_{(8)}(w) &= \bar{u}_0\bar{u}_1c\delta(\gamma^+)\delta(\gamma^-)(St)^0(St)^1 \\
&\quad \times \exp\left[\frac{i}{2}(k^+ \cdot Z^- + k^- \cdot Z^+)\right](w)
\end{aligned} \tag{7.8}$$

we get

$$\begin{aligned}
j_1(z)V_{(8)}(w) &\sim \frac{h}{z-w}(\partial c)V_{(8)}(w), \\
j_2(z)V_{(8)}(w) &\sim \frac{1/2}{\sqrt{z-w}}\bar{u}_0\bar{u}_1c[\eta^+\delta(\gamma^-)i(k^{0+}\Gamma^{0-}S^0S^1 \\
&\quad - k^{1+}\Gamma^{1-}S^1S^0) + (+\leftrightarrow -)]t^0t^1 \\
&\quad \times \exp\left[\frac{i}{2}(k^+ \cdot Z^- + k^- \cdot Z^+)\right](w) \\
&\quad + \frac{e/2}{z-w}\tilde{c}V_{(8)}(w),
\end{aligned} \tag{7.9a}$$

$$j_3(z)V_{(8)}(w) \sim \text{regular}. \tag{7.9b}$$

The BRST invariance requires

$$h = \frac{1}{2} + \frac{1}{2}k^+ \cdot k^- = 0,$$

$$\bar{u}_\mu k^{\mu\pm}\Gamma^{\mu\mp} = 0 \text{ (no sum)}, \quad e = 0. \tag{7.10}$$

This implies $k^+ \cdot k^- = -1$ and $k^{\mu\pm} = 0$, which cannot be achieved.

More easily, the \mathcal{H}_{-+-} sector with

$$\begin{aligned}
V_{(7)}(w) &= ce^{-\varphi^+/2}e^{-\varphi^-/2}t^0t^1 \\
&\quad \times \exp\left[\frac{i}{2}(k^+ \cdot Z^- + k^- \cdot Z^+)\right](w)
\end{aligned} \tag{7.11}$$

yields

$$j_1(z)V_{(7)}(w) \sim \frac{h}{z-w}(\partial c)V_{(7)}(w), \tag{7.12a}$$

$$\begin{aligned}
j_2(z)V_{(7)}(w) &\sim \frac{1/2}{\sqrt{z-w}}c[\eta^+e^{\varphi^+/2}e^{-\varphi^-/2}(-ik^+) \cdot \psi^- \\
&\quad + (+\leftrightarrow -)]t^0t^1 \exp\left[\frac{i}{2}(k^+ \cdot Z^- + k^- \cdot Z^+)\right] \\
&\quad \times (w) + \frac{e/2}{z-w}\tilde{c}V_{(7)}(w),
\end{aligned} \tag{7.12b}$$

$$j_3(z)V_{(7)}(w) \sim \tilde{c}\left[\frac{\frac{1}{2}-\frac{1}{2}}{z-w}\right]V_{(7)}(w) \sim \text{regular}, \tag{7.12c}$$

which is regular if $k^\pm = 0$ since $h = \frac{1}{2}k^+ \cdot k^-$ and $e = 0$ here. We do not consider this as a spacetime field degree of freedom.

The last new pattern comes from the discrete \mathcal{H}_{+-} sectors. It is now more convenient to express the BRST current in the real component fields, as given in Eq. (B14) and bosonized in Eq. (4.10). In particular, looking at \mathcal{H}_{+-}^{NS} with

$$V_{(10)}(w) = \bar{u} c e^{\tilde{\sigma}/2} e^{-\varphi^2} e^{-\varphi^3/2} S^3 t^3 e^{i(k^2 \cdot Z^2 + k^3 \cdot Z^3)}(w), \quad (7.13)$$

we obtain

$$j_1(z) V_{(10)}(w) \sim \frac{h}{z-w} (\partial c) V_{(10)}(w), \quad (7.14a)$$

$$j_2(z) V_{(10)}(w) \sim \frac{1/2}{z-w} \bar{u} c e^{\tilde{\sigma}/2} e^{-\varphi^2} e^{\varphi^3/2} \eta^3 (-i k_\mu^2) \times \Gamma^{\mu 3} S^3 t^3 e^{i(k^2 \cdot Z^2 + k^3 \cdot Z^3)}(w) + \frac{1}{\sqrt{z-w}} (\text{terms linear in } k^3), \quad (7.14b)$$

$$j_3(z) V_{(10)}(w) \sim \text{regular}, \quad (7.14c)$$

where $h = \frac{1}{2} k^i \cdot k^i$. Notice that the charge-measuring term $\tilde{c}J$ in j_2 does not lead to a singularity. To obtain a vanishing BRST commutator, we have to request not only $k^i \cdot k^i = 0$ and the Dirac equation

$$\begin{aligned} \bar{u} k_\mu^2 \Gamma^{\mu 3} &= -\frac{1}{2} \bar{u} (k^{\uparrow 2} \Gamma^{\downarrow 3} + k^{\downarrow 2} \Gamma^{\uparrow 3}) \\ &= -\frac{1}{2} k^{\uparrow 2} \bar{u}^\downarrow (\Gamma^{\downarrow 3})^{\uparrow, \downarrow} - \frac{1}{2} k^{\downarrow 2} \bar{u}^\uparrow (\Gamma^{\uparrow 3})^{\downarrow, \uparrow}, \end{aligned} \quad (7.15)$$

but also set k^3 to zero. The latter is not surprising, since we created the twisted state on the border of the half-space $\mathbb{C}^{1,1}/\mathbb{Z}_2''$ so that its transversal momentum gets trapped at $k^3 = -k^3$. Furthermore, the Dirac equation (7.15) reads

$$\bar{u}^\downarrow k^{\uparrow 2} = 0 \text{ and } \bar{u}^\uparrow k^{\downarrow 2} = 0, \quad (7.16)$$

so that one helicity is removed, e.g., by choosing $k^{\uparrow 2} = 0$ and $\bar{u}^\uparrow = 0$, i.e., taking only $S^{3, \uparrow}$ in Eq. (7.13). With these requirements, $V_{(10)}$ does create a BRST-invariant state with lightlike momentum $k^{\downarrow 2} \neq 0$. Similar calculations for \mathcal{H}_{+--+}^R as well as the remaining sectors (11)–(16) arrive at the same conclusion.

The analysis above allows us to make the statement that our candidate ground states in sectors (1), (2) and (9)–(16) are actually the *physical* states; i.e., they represent the BRST cohomology classes. Indeed, they are BRST closed, as we have explicitly showed, while they cannot be BRST exact (i.e., of the type $Q|*$) because there are no candidates for a star state with the correct ghost and picture numbers and conformal dimension, in the case of a ground state. Since the NS-R pairs of sectors (9)–(16) are related by simple coordinate relabeling, we do not consider them separately, but restrict ourselves to (9) and (10) from now on.

We have yet to show that our physical vertex operators form a *local* field algebra. To this end, we should investigate their mutual operator products. Equivalently, we are going to address *interactions* among the $N = 2$ string physical states (cf. Ref. [22]). Generally speaking, the required mutual locality of the vertex operators is expected to impose some constraints on the allowed interactions.

As representatives of the \mathbb{Z}_2'' -twisted $N = 2$ string

physical states, we choose the ground states in the sectors $\mathcal{H}_{++++}^{\text{NS}}, \mathcal{H}_{++++}^{\text{R}}, \mathcal{H}_{+--+}^{\text{NS}}$, and $\mathcal{H}_{+--+}^{\text{R}}$: namely,

$$\begin{aligned} \mathcal{H}_{++++}^{\text{NS}} : \Phi &= c e^{-\varphi^2} e^{-\varphi^3} e^{i(k^2 \cdot Z^2 + k^3 \cdot Z^3)}, \\ \mathcal{H}_{++++}^{\text{R}} : \Upsilon^{\uparrow \downarrow} &= c e^{-\varphi^2/2} e^{-\varphi^3/2} S^{2, \uparrow \downarrow} S^{3, \uparrow \downarrow} e^{i(k^2 \cdot Z^2 + k^3 \cdot Z^3)}, \\ \mathcal{H}_{+--+}^{\text{NS}} : \Xi^{\uparrow \downarrow} &= c e^{\tilde{\sigma}/2} e^{-\varphi^2} e^{-\varphi^3/2} S^{3, \uparrow \downarrow} t^3 e^{i k^2 \cdot Z^2}, \\ \mathcal{H}_{+--+}^{\text{R}} : \Lambda^{\uparrow \downarrow} &= c e^{\tilde{\sigma}/2} e^{-\varphi^2/2} e^{-\varphi^3} S^{2, \uparrow \downarrow} t^3 e^{i k^2 \cdot Z^2}, \end{aligned} \quad (7.17)$$

with $k^2 \cdot k^2 + k^3 \cdot k^3 = 0$ in the first case and $k^{\uparrow i} = 0$ or $k^{\downarrow i} = 0$ in the other three, corresponding to the chosen helicity.

On one hand, using the OPE structure of the bosonized fields among themselves, viz.,

$$\begin{aligned} e^{\tilde{\sigma}/2}(z) e^{\tilde{\sigma}/2}(w) &\sim (z-w)^{1/4} e^{\tilde{\sigma}}(w), \\ e^{-\varphi^i/2}(z) e^{-\varphi^i/2}(w) &\sim (z-w)^{-1/4} e^{-\varphi^i}(w), \end{aligned} \quad (7.18)$$

$$\begin{aligned} S^{i, \uparrow}(z) S^{i, \uparrow}(w) &\sim (z-w)^{1/4} e^{2\phi^i}(w), \\ S^{i, \uparrow}(z) S^{i, \downarrow}(w) &\sim (z-w)^{-1/4}, \end{aligned}$$

we find that interactions between the sectors $\mathcal{H}_{++++}^{\text{NS}}$ and $\mathcal{H}_{+--+}^{\text{NS,R}}$ seem to be forbidden, since the relevant OPE is *not local*:

$$\begin{aligned} e^{-\varphi^2 - \varphi^3} e^{i k^2 \cdot Z^2}(z) e^{-\varphi^2} e^{-\varphi^3/2} e^{i k'^2 \cdot Z^2}(w) \\ \sim (z-w)^{-3/2} (z-w)^{k^2 \cdot k'^2}, \end{aligned} \quad (7.19)$$

and quite similarly in the other cases.¹⁹ On the other hand, each of the three triples $(\Phi, \Upsilon^\uparrow, \Upsilon^\downarrow)$, $(\Upsilon^\uparrow, \Xi^\downarrow, \Lambda^\downarrow)$, and $(\Upsilon^\downarrow, \Xi^\uparrow, \Lambda^\uparrow)$ have *local* OPE's among themselves, as can easily be checked using Eq. (7.18) again. The non-localities do not yet mean that interactions between the triplets are impossible, since for closed strings the left-moving (chiral) fields still have to be combined with the right-moving ones to complete the full vertex operators. As the example of the nonsupersymmetric $O(16) \otimes O(16)$ string showed [30–32], square root singularities in the OPE's may disappear when the proper Gliozzi-Scherk-Olive (GSO) projection is applied for modular invariance. Hence, an *asymmetric* (with respect to the left- and right-moving degrees of freedom) GSO projection may allow us to keep more than two interacting spacetime fermions in the theory. To settle this question, one should study the “bosonized lattice” of Ref. [21], which in our case is a direct product of two (1,1)-dimensional (half-integral) Lorentzian lattices²⁰ for the right movers, and once more

¹⁹For consistency in the twisted string, we must put $k^3 = 0$ for its untwisted states as well, effectively reducing the theory to $(1+1)$ dimensions. See also Ref. [22].

²⁰The signature arises from the sign difference between the OPE's of Eqs. (4.2) and (4.10). The twist fields are irrelevant here, since the combination $t^3 e^{\tilde{\sigma}/2}$ is local with any vertex operator as long as $k^3 = 0$.

for the left movers. At the same time it is rather clear that the continuous spectral flow family $\nu \in [-\frac{1}{2}, \frac{1}{2}]$ has little chance to survive the final locality test, except for $\nu = 0$ and $\nu = \frac{1}{2}$, i.e., in the well-known NS and R sectors.

In order to test *full* locality, we need to look at (tree-level) amplitudes, i.e., correlation functions of vertex operators. The nonvanishing three-point amplitude for the ground state physical “scalar” in the $\mathcal{H}_{++++}^{\text{NS}}$ sector was constructed by Ooguri and Vafa [5]. Admitting also the $\mathcal{H}_{++++}^{\text{R}}$ sector, one easily computes, for example,

$$\langle \Phi_x \Upsilon_y^\dagger \Upsilon_z^\dagger \rangle = 1, \tag{7.20}$$

as expected from conformal invariance. For the one-boson–two-fermion amplitude we also find

$$\langle \Upsilon_x^\dagger \Xi_y^\dagger \Lambda_z^\dagger \rangle = \langle \Upsilon_x^\dagger \Xi_y^\dagger \Lambda_z^\dagger \rangle = 1, \tag{7.21}$$

which is encouraging. Ghost-number conservation does not permit other three-point functions.

The vanishing of the tree-level bosonic four-point function $\langle \Phi_x \Phi_y \Phi_z \Phi_w \rangle$ [5] is most easily verified in our approach by computing the equivalent (because of spectral flow) amplitude $\langle \Upsilon_x \Upsilon_y \Upsilon_z \Upsilon_w \rangle$ as follows ($\{x_i | i = 1, \dots, 4\} = \{x, y, z, w\}$):

$$\begin{aligned} \frac{1}{3}[\langle \Upsilon_x^\dagger \Upsilon_y^\dagger \Upsilon_z^\dagger \Upsilon_w^\dagger \rangle + \langle \Upsilon_x^\dagger \Upsilon_y^\dagger \Upsilon_z^\dagger \Upsilon_w^\dagger \rangle + \langle \Upsilon_x^\dagger \Upsilon_y^\dagger \Upsilon_z^\dagger \Upsilon_w^\dagger \rangle] &\sim \left\langle \prod_i e^{-\varphi/2}(x_i) \right\rangle^2 [\langle S_x^\dagger S_y^\dagger S_z^\dagger S_w^\dagger \rangle^2 + \langle S_x^\dagger S_y^\dagger S_z^\dagger S_w^\dagger \rangle^2 + \langle S_x^\dagger S_y^\dagger S_z^\dagger S_w^\dagger \rangle^2] \\ &= \prod_{i < j} x_{ij}^{-1} [(x-y)(z-w) - (x-z)(y-w) + (x-w)(y-z)] = 0, \end{aligned} \tag{7.22}$$

where we defined $x_{ij} \equiv x_i - x_j$, as usual. The crucial relative signs emerge from carefully taking into account the suppressed cocycle operators [21].

VIII. CONCLUSIONS

Our motivation for twisting the $N = 2$ string was driven by the search for more physical states in $N = 2$ string theory defined in $2 + 2$ dimensions. An arbitrary twisting implies a locally flat background \mathcal{B} for $N = 2$ string propagation, which is not just $\mathbb{C}^{1,1}$ but has non-trivial global topology. Backgrounds induced by twisting generically take the form $\mathcal{B} = \mathbb{C}^{1,1}/\Gamma_0$, where a discrete group Γ_0 is generated by elements of the *global* symmetry group which act on the bosonic coordinate fields $Z^{\mu i}$. For flat backgrounds, the global homogeneous symmetry group with nontrivial Z action is isomorphic to $U(1,1) \otimes \mathbb{Z}_2''$. Its action on the fermionic coordinates is identical. On top of this, there exists the spectral flow labeling the overall mismatch of bosonic and fermionic boundary conditions. It generates a $U(1)$ family of sectors interpolating between the NS- and R-like sectors of the $N = 2$ string. We have found four types of different \mathbb{Z}_2 twists leading to sixteen different sectors to consider. One of the twists originated from the spectral flow and does not alter the background, while two more are contained in the global $U(1,1)$ symmetry of the $N = 2$ string.

The quantized critical $N = 2$ string theory can be conveniently described in the $N = 2$ superconformal gauge by introducing ghosts for the gauge-fixed local symmetries and the corresponding BRST charge. Using bosonization techniques, we constructed the spin and twist fields which actually implement all the \mathbb{Z}_2 twists mentioned above. Next, we identified the ground states of the sixteen \mathbb{Z}_2 -twisted sectors of the $N = 2$ string,

as well as their conformal dimensions and local $U(1)$ charges. The “spacetime” interpretation of the twists is based on the orbifolds and semi-spaces in Eq. (4.24) as possible target spaces for $N = 2$ string propagation. Half of the ground states were recognized as candidates for spacetime bosons, whereas the \mathbb{Z}_2'' -twisted half suggested an interpretation as spacetime fermions. Not all of these states are, however, physical states identified as representatives of BRST cohomology classes.

The most general homogeneous monodromies of the $N = 2$ string were shown to be classified by the conjugacy classes of $U(1,1)$ and the \mathbb{Z}_2'' twist of complex conjugation. They naturally split into three different groups: elliptic, parabolic, and hyperbolic, only the first type having been considered in the past. We proposed the formula of Eq. (6.30) for the vacuum energy of the arbitrarily $U(1,1)$ -twisted ground state, by generalizing the result known for the elliptic case. Searching for *massless* ground states, we were able to restrict them to the (possibly \mathbb{Z}_2'' -twisted) identity and parabolic conjugacy classes. The latter gives a previously unknown massless background with interesting properties. The former leads back to $\mathbb{C}^{1,1}$ which yields four of the sixteen \mathbb{Z}_2 sectors upon twisting by \mathbb{Z}_2'' and/or spectral flow.

The presence of the spectral flow relating the R- and NS-type sectors of the $N = 2$ string does not mean that these are to be identified. The spectral flow modifies fermionic boundary conditions for each homology cycle on the world-sheet separately. For a tree-level amplitude, this leaves *one* combination of the external fermionic boundary conditions unchanged. More specifically, we can always choose the fermionic monodromy of one external state to be invariant under the spectral flow. In particular, we have a $U(1)$ family of two-point functions, indexed by the *relative* boson-fermion monodromy mismatch, most of which have to vanish. Hence, a rigid

U(1) label should be attached to any state. In this paper we have only considered the \mathbb{Z}_2 subset denoted by NS and R.

From the world-sheet point of view, the twistings considered in this paper have drastic consequences for the $N = 2$ supergravity fields as well. We are used to the fact that gravitini may be antiperiodic around world-sheet cycles, but, e.g., the Mathur-Mukhi twist also implies a *double-valued* Abelian gauge field [see Eq. (3.7)]. In other words, we are dealing with a double cover of the world sheet. Alternatively, one may put $A_\alpha \equiv 0$, which amounts to consistently truncating to $N = 1$ supergravity. It can be shown [33] that there actually exist only *two* different GSO projections of the $N = 2$ string, one corresponding to the untwisted (spacetime bosonic) theory, the other leading to the Mathur-Mukhi-twisted theory containing spacetime bosons *and* fermions.

The crucial check of our \mathbb{Z}_2 -twisted ground states as candidate physical states comes from the analysis of the BRST cohomology. We have found that only some of these states are physical, namely, one scalar, one vector, and two spinors. Moreover, the interactions among the would-be physical states were analyzed from the viewpoint of world-sheet locality, needed for an unambiguous definition of the conformal field theory correlation functions. It turned out that either three bosons or else, one boson and two fermions may coexist. The consequences can be found in Ref. [33].

Still, there is room for $N = 2$ spacetime supersymmetry to be present, although the numbers of *physical* boson and fermion degrees of freedom we obtained are not enough to support the *maximal* spacetime supersymmetry advocated in Ref. [6]. The BRST approach to gauge theories is well-known to be based on canonical (unitary) quantization. The notion of unitarity is, however, quite formal in $2+2$ dimensions. This is related to the fact that in the covariant Lagrangian description of self-dual (supersymmetric) field theories in $2+2$ dimensions one-half of the fields are usually the (propagating) Lagrange multipliers for the other half, and vice versa [7]. The BRST cohomology may have selected those half of the (super) self-dual states with *positive* norms. The remaining half represents an equal number of ghost states having *negative* norms, which are, however, needed for the covariant [with respect to $SO(2,2)$] description of $N = 2$ string theory. This effectively doubles the number of states in the game, and may open a way for the $N = 4$ spacetime supersymmetry of the (ghost-) extended theory. More studies are needed to resolve this issue.

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APPENDIX A: LOCAL SYMMETRIES OF THE BS ACTION

In this Appendix, we list the infinitesimal field transformation laws corresponding to the local two-dimensional symmetries of the Brink-Schwarz (BS) action (3.1).

(i) For reparametrization invariance:

$$\begin{aligned}\delta e_\alpha^a &= \xi^\beta \partial_\beta e_\alpha^a + e_\beta^a \partial_\alpha \xi^\beta, \\ \delta \chi_\alpha &= \xi^\beta \partial_\beta \chi_\alpha + \chi_\beta \partial_\alpha \xi^\beta, \\ \delta A_\alpha &= \xi^\beta \partial_\beta A_\alpha + A_\beta \partial_\alpha \xi^\beta, \\ \delta Z &= \xi^\alpha \partial_\alpha Z, \quad \delta \Psi = \xi^\alpha \partial_\alpha \Psi.\end{aligned}\tag{A1}$$

(ii) For Lorentz invariance:

$$\begin{aligned}\delta e_\alpha^a &= l \varepsilon^a{}_b e_\alpha^b, \\ \delta \chi_\alpha &= -\frac{1}{2} l \rho_3 \chi_\alpha, \\ \delta A_\alpha &= \delta Z = \delta \Psi = 0.\end{aligned}\tag{A2}$$

(iii) For $N = 2$ extended supersymmetry:

$$\begin{aligned}\delta e_\alpha^a &= -2i\bar{\varepsilon}\rho^a\chi_\alpha + \text{H.c.}, \\ \delta \chi_\alpha &= (\partial_\alpha + \frac{1}{2}\omega_\alpha\rho_3 - iA_\alpha)\varepsilon, \\ \delta A_\alpha &= \varepsilon^{\beta\gamma}\bar{\varepsilon}\rho_3\rho_\alpha(\partial_\beta + \frac{1}{2}\omega_\beta\rho_3 - iA_\beta)\chi_\gamma + \text{H.c.}, \\ \delta Z &= -2\bar{\varepsilon}\Psi, \\ \delta \Psi &= i\rho^\beta\varepsilon(\partial_\beta Z + 2\bar{\chi}_\beta\Psi).\end{aligned}\tag{A3}$$

(iv) For phase and chiral gauge invariances:

$$\begin{aligned}\delta e_\alpha^a &= \delta Z = 0, \\ \delta \chi_\alpha &= i\alpha\chi_\alpha - i\rho_3\hat{\alpha}\chi_\alpha, \\ \delta A_\alpha &= \partial_\alpha\alpha + \varepsilon_\alpha{}^\beta\partial_\beta\hat{\alpha}, \\ \delta \Psi &= i\alpha\Psi + i\hat{\alpha}\rho_3\Psi.\end{aligned}\tag{A4}$$

(v) For Weyl and super-Weyl invariances:

$$\begin{aligned}\delta e_\alpha^a &= \sigma e_\alpha^a, \\ \delta \chi_\alpha &= \frac{1}{2}\sigma\chi_\alpha + \rho_\alpha\eta, \\ \delta A_\alpha &= \bar{\chi}_\beta\rho_\alpha\rho^\beta\eta + \text{H.c.}, \\ \delta Z &= 0, \quad \delta \Psi = -\frac{1}{2}\sigma\Psi.\end{aligned}\tag{A5}$$

In the above, ζ , l , ε , α , $\hat{\alpha}$, σ and η are parameters of reparametrization, Lorentz, $N = 2$ extended supersymmetry, phase, chiral, Weyl, and $N = 2$ super-Weyl (superconformal) local transformations, respectively. We use the notation

$$\begin{aligned}\varepsilon^{\alpha\beta} &= e^{-1}\epsilon^{\alpha\beta}, \\ \omega_\alpha(e, \chi) &= \omega_\alpha(e) + [i\bar{\chi}_\beta\rho^\beta\chi_\alpha + \text{H.c.}],\end{aligned}\tag{A6}$$

where $\epsilon^{\alpha\beta}$ is the Levi-Civita symbol, and $\omega_\alpha(e)$ is the conventional gravitational connection in two dimensions.

APPENDIX B: $N = 2$ STRING BRST CHARGE

In this Appendix, the standard results needed to introduce the $N = 2$ string BRST charge²¹ are summarized.

The OPE's for the chiral parts of the matter fields representing the $N = 2$ string coordinates are

$$\begin{aligned} Z^{i\mu}(z)Z^{j\nu}(w) &\sim -\delta^{ij}\eta^{\mu\nu}\ln(z-w), \\ \psi^{i\mu}(z)\psi^{j\nu}(w) &\sim -\delta^{ij}\eta^{\mu\nu}\frac{1}{z-w}. \end{aligned} \quad (\text{B1})$$

The $N = 2$ currents associated with the superconformally gauge-fixed $N = 2$ string action take the form²²

$$\begin{aligned} T(z) &= -\frac{1}{2}\partial Z^+ \cdot \partial Z^- + \frac{1}{4}\psi^+ \cdot \partial\psi^- + \frac{1}{4}\psi^- \cdot \partial\psi^+ \\ &= -\frac{1}{2}(\partial Z^{i\mu}\partial Z_\mu^i - \psi^{i\mu}\partial\psi_\mu^i) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}, \\ G^\pm(z) &= \partial Z^\mp \cdot \psi^\pm = \partial Z^{i\mu}\psi_\mu^i \pm i\varepsilon^{ij}\partial Z^{i\mu}\psi_\mu^j \\ &= \sum_{n\in\mathbb{Z}} G_n^\pm z^{-n-3/2}, \end{aligned}$$

$$J(z) = \frac{1}{4}\psi^- \cdot \psi^+ = \frac{i}{4}\varepsilon^{ij}\psi^{i\mu}\psi_\mu^j = \sum_{n\in\mathbb{Z}} J_n z^{-n-1}, \quad (\text{B2})$$

where the mode expansions on the RHS of Eq. (B2) and all the equations below are valid for the untwisted R-type boundary conditions, for definiteness. For the twisted and/or NS-type boundary conditions, the moding on the RHS of Eq. (B2) and the mode expansions below have to be appropriately modified. The currents (B2) form the $N = 2$ superconformal algebra with central extension.

The reparametrization ghosts

$$b(z) = \sum_{n\in\mathbb{Z}} b_n z^{-n-2}, \quad c(z) = \sum_{n\in\mathbb{Z}} c_n z^{-n+1}, \quad (\text{B3})$$

satisfy the OPE

$$b(z)c(w) \sim \frac{1}{z-w}, \quad \{c_m, b_n\} = \delta_{m+n,0}. \quad (\text{B4})$$

The $N = 2$ superconformal ghosts

$$\begin{aligned} \beta^\pm(z) &= \sum_{n\in\mathbb{Z}} \beta_n^\pm z^{-n-3/2}, \\ \gamma^\pm(z) &= \sum_{n\in\mathbb{Z}} \gamma_n^\pm z^{-n+1/2}, \end{aligned} \quad (\text{B5})$$

satisfy the OPE

$$\begin{aligned} \beta^+(z)\gamma^-(w) &\sim -\frac{2}{z-w}, \\ \beta^-(z)\gamma^+(w) &\sim -\frac{2}{z-w}, \end{aligned} \quad (\text{B6})$$

which imply the (only nonvanishing) commutation relations

$$[\gamma_m^+, \beta_n^-] = [\gamma_m^-, \beta_n^+] = 2\delta_{m+n,0}. \quad (\text{B7})$$

Finally, the anticommuting U(1) ghosts

$$\tilde{b}(z) = \sum_{n\in\mathbb{Z}} \tilde{b}_n z^{-n-1}, \quad \tilde{c}(z) = \sum_{n\in\mathbb{Z}} \tilde{c}_n z^{-n}, \quad (\text{B8})$$

associated with the Abelian local invariance of $N = 2$ string theory, have the OPE

$$\tilde{b}(z)\tilde{c}(w) \sim \frac{1}{z-w}, \quad \{\tilde{c}_m, \tilde{b}_n\} = \delta_{m+n,0}. \quad (\text{B9})$$

The full BRST-invariant generators read

$$\begin{aligned} T_{\text{tot}} &= \{Q_{\text{BRST}}, b\} = T - 2b\partial c - (\partial b)c - \tilde{b}\partial\tilde{c} - \frac{3}{4}[\beta^-\partial\gamma^+ + \beta^+\partial\gamma^-] - \frac{1}{4}[\gamma^+\partial\beta^- + \gamma^-\partial\beta^+], \\ G_{\text{tot}}^\pm &= [Q_{\text{BRST}}, \beta^\pm] = G^\pm - 2b\gamma^\pm - 4\tilde{b}\partial\gamma^\pm - 2(\partial\tilde{b})\gamma^\pm + \frac{3}{2}\beta^\pm\partial c + (\partial\beta^\pm)c + \frac{1}{2}\beta^\pm\tilde{c}, \\ J_{\text{tot}} &= \{Q_{\text{BRST}}, \tilde{b}\} = J + \tilde{b}\partial c + (\partial\tilde{b})c + \frac{1}{4}[\beta^+\gamma^- - \beta^-\gamma^+]. \end{aligned} \quad (\text{B10})$$

They imply, in particular, the mode expansions²³

$$\begin{aligned} L_m^{\text{tot}} &= \{Q_{\text{BRST}}, b_m\} = L_m + (m-n)b_{m+n}c_{-n} - n\tilde{b}_{n+m}\tilde{c}_{-n} + \frac{1}{4}(m-2n)[\beta_{m+n}^-\gamma_{-n}^+ + \beta_{m+n}^+\gamma_{-n}^-] - A_0\delta_{m,0}, \\ G_m^{\pm, \text{tot}} &= [Q_{\text{BRST}}, \beta_m^\pm] \\ &= G_m^\pm - 2b_{m+n}\gamma_{-n}^\pm - 2(m-n)\tilde{b}_{m+n}\gamma_{-n}^\pm + (\frac{1}{2}n-m)\beta_{m+n}^\pm c_{-n} + \frac{1}{2}\beta_{m+n}^\pm\tilde{c}_{-n}, \\ J_m^{\text{tot}} &= \{Q_{\text{BRST}}, \tilde{b}_m\} = J_m - m\tilde{b}_{m+n}c_{-n} + \frac{1}{4}[\beta_{m+n}^+\gamma_{-n}^- - \beta_{m+n}^-\gamma_{-n}^+] - B_0\delta_{m,0}, \end{aligned} \quad (\text{B11})$$

where normal-ordering ambiguity constants A_0 and B_0 have been introduced.

The $N = 2$ BRST charge is given by

$$Q_{\text{BRST}} = \oint_0 \frac{dz}{2\pi i} j_{\text{BRST}}(z), \quad (\text{B12})$$

²¹We are grateful to Jan Bischoff for helping us to check some of the formulas listed below.

²²Normal ordering is applied whenever an ambiguity arises.

²³Summation over repeated indices is always understood.

with the BRST current having the form

$$j_{\text{BRST}}(z) = cT + bc\partial c + \tilde{b}c\partial\tilde{c} - \frac{3}{4}c[\beta^-\partial\gamma^+ + \beta^+\partial\gamma^-] - \frac{1}{4}c[\gamma^+\partial\beta^- + \gamma^-\partial\beta^+] + \frac{1}{2}[\gamma^-G^+ + \gamma^+G^-] \\ + \tilde{c}J - \gamma^-\gamma^+b + [\gamma^-\partial\gamma^+ - \gamma^+\partial\gamma^-]\tilde{b} + \frac{1}{4}\tilde{c}[\beta^+\gamma^- - \beta^-\gamma^+] + \frac{3}{8}\partial[c(\beta^+\gamma^- + \beta^-\gamma^+)]. \quad (\text{B13})$$

In terms of real fields,

$$j_{\text{BRST}}(z) = cT + bc\partial c + \tilde{b}c\partial\tilde{c} - \frac{3}{2}c[\beta^2\partial\gamma^2 + \beta^3\partial\gamma^3] - \frac{1}{2}c[\gamma^2\partial\beta^2 + \gamma^3\partial\beta^3] \\ + \gamma^2[\partial Z^2 \cdot \psi^2 + \partial Z^3 \cdot \psi^3] + \gamma^3[\partial Z^2 \cdot \psi^3 - \partial Z^3 \cdot \psi^2] + \frac{i}{2}\tilde{c}\psi^2 \cdot \psi^3 \\ - [\gamma^2\gamma^2 + \gamma^3\gamma^3]b + 2i[\gamma^2\partial\gamma^3 - \gamma^3\partial\gamma^2]\tilde{b} + \frac{i}{2}\tilde{c}[\gamma^2\beta^3 - \gamma^3\beta^2] + \frac{3}{4}\partial[c(\beta^2\gamma^2 + \beta^3\gamma^3)]. \quad (\text{B14})$$

The total derivative terms have been adjusted to make the current j_{BRST} BRST exact and turn it into a conformal primary of $h = 1$ and $e = 0$. It follows

$$Q_{\text{BRST}} = c_{-n}L_n + \frac{1}{2}\gamma_{-n}^-G_n^+ + \frac{1}{2}\gamma_{-n}^+G_n^- + \frac{1}{2}\tilde{c}_{-n}J_n \\ - \frac{1}{2}(m-n)c_{-m}c_{-n}b_{m+n} - \gamma_{-m}^-\gamma_{-n}^+b_{m+n} - (m-n)\gamma_{-m}^-\gamma_{-n}^+\tilde{b}_{m+n} \\ + nc_{-m}\tilde{c}_{-n}\tilde{b}_{m+n} + \frac{1}{4}(m-2n)c_{-m}[\beta_{m+n}^+\gamma_{-n}^- + \beta_{m+n}^-\gamma_{-n}^+] + \frac{1}{4}\tilde{c}_{-m}[\beta_{m+n}^+\gamma_{-n}^- - \beta_{m+n}^-\gamma_{-n}^+] - A_0c_0 - B_0\tilde{c}_0. \quad (\text{B15})$$

The BRST generators (B11) satisfy the $N = 2$ superconformal algebra without central extension, while the BRST charge $Q_{\text{BRST}} = 0$, when the complex target space dimension is two and the intercept takes its critical value under the given boundary conditions.

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