Physics Letters B 790 (2019) 191–196

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Supersymmetric many-body Euler–Calogero–Moser model

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ARTICLE INFO

Article history:
Received 13 December 2018
Received in revised form 10 January 2019
Accepted 11 January 2019
Available online 18 January 2019
Editor: M. Cvetič

Keywords:
Spin-Calogero models
$\mathcal{N}$-extended supersymmetry
Euler–Calogero–Moser system

ABSTRACT

We explicitly construct a supersymmetric $so(n)$ spin-Calogero model with an arbitrary even number $\mathcal{N}$ of supersymmetries. It features $\frac{1}{2}\mathcal{N}n(n+1)$ rather than $\mathcal{N}n$ fermionic coordinates and a very simple structure of the supercharges and the Hamiltonian. The latter, together with additional conserved currents, form an $osp(\mathcal{N}|2)$ superalgebra. We provide a superspace description for the simplest case, namely $\mathcal{N}=2$ supersymmetry. The reduction to an $\mathcal{N}$-extended supersymmetric goldfish model is also discussed.

1. Introduction

In recent years notable progress was achieved in the supersymmetrization of the bosonic matrix models [1–6]. It has been known for a long time that matrix models are an efficient tool of constructing conformally invariant systems (see e.g. [7] and refs. therein) For example, the Calogero model as well as its different extensions [8–12] are closely related to matrix models and can be obtained from them by a reduction procedure. The supersymmetrization of matrix models consists in replacing the bosonic matrix entries by superfields [1–5]. While this approach has been quite successful for $\mathcal{N}\leq 4$ extended supersymmetry, it seems to be less efficient or even inapplicable for $\mathcal{N}>4$ supersymmetric cases.[1] In contrast, the Hamiltonian approach has no serious restriction on the number of supersymmetries, due to the absence of auxiliary components.

The key feature of a supersymmetric extension of one-dimensional models within the Hamiltonian approach is the appearance of additional fermionic matrix degrees of freedom accompanying the standard $\mathcal{N}n$ fermions customarily required for an $\mathcal{N}$-extended supersymmetric system with n bosonic coordinates. Recently we implemented this feature to construct a supersymmetric extension of Hermitian matrix models which admits an arbitrary number of supersymmetries [6]. We also provided a supersymmetrization of the reduction procedure which yields an $\mathcal{N}$-extended n-particle supersymmetric Calogero model. The question we address in this paper is how to (if possible) repeat this supersymmetrization procedure for the real symmetric matrix model [8].

In the bosonic case, the free matrix model associated with real symmetric matrices (see e.g. [11]) results in a spin generalization of the n-particle Calogero–Moser model, which is also known as the Euler–Calogero–Moser (ECM) model [8,9] and described by the Hamiltonian

$$H=\frac{1}{2}\sum_{i=1}^{n}p_i^2+\frac{1}{2}\sum_{i\neq j}^{n}\frac{\ell_{ij}^2}{\left(x_i-x_j\right)^2}\,. \tag{1.1}$$

It depends on the coordinates $x_i(t)$ and momenta $p_i(t)$ of each particle as well as on the internal degrees of freedom encoded in the angular momenta $\ell_{ij}=-\ell_{ji}$. The coordinates and momenta satisfy the standard Poisson brackets

$$\left\{x_i,p_j\right\}=\delta_{ij}, \tag{1.2}$$

while the Poisson brackets of the angular momenta form the $so(n)$ algebra

$$\left\{\ell_{ij},\ell_{km}\right\}=\frac{1}{2}\left(\delta_{ik}\ell_{jm}+\delta_{jm}\ell_{ik}-\delta_{jk}\ell_{im}-\delta_{im}\ell_{jk}\right). \tag{1.3}$$

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[1] An up to now unique example of a matrix system with $\mathcal{N}=8$ supersymmetry has appeared in [5] in $\mathcal{N}=4$ superspace.

https://doi.org/10.1016/j.physletb.2019.01.018

The ECM model with the Hamiltonian (1.1) possesses conformal invariance. Indeed, if we define the conserved currents of the dilatation D and conformal boost K as

$$D=-\frac{1}{2}\sum_{i=1}^{n}x_i p_i+tH \quad \text{and} \quad K=\frac{1}{2}\sum_{i=1}^{n}x_i^2-t\sum_{i=1}^{n}x_i p_i+t^2H, \tag{1.4}$$

then it is easy to demonstrate that they generate the one-dimensional conformal algebra $so(1,2)$:

$$\{H,K\}=2D, \quad \{H,D\}=H, \quad \{K,D\}=-K. \tag{1.5}$$

The equations of motion which follow from the Hamiltonian (1.1),

$$\ddot{x}_i=2\sum_{k\neq i}\frac{\ell_{ik}^2}{(x_i-x_k)^3} \quad \text{and}$$
$$\dot{\ell}_{ij}=-\sum_{k\neq i,j}\ell_{ik}\ell_{kj}\left(\frac{1}{(x_i-x_k)^2}-\frac{1}{(x_k-x_j)^2}\right), \tag{1.6}$$

consistently reduce to (see e.g. [13,11,12])

$$\ddot{x}_i=2\sum_{j\neq i}\frac{\dot{x}_i\dot{x}_j}{x_i-x_j} \tag{1.7}$$

upon setting

$$\ell_{ij}=-\left(x_i-x_j\right)\sqrt{\dot{x}_i\dot{x}_j}. \tag{1.8}$$

This maximally superintegrable system is known as the goldfish model [14,15].

In what follows we will construct an $\mathcal{N}$-extended supersymmetric generalization of the Hamiltonian (1.1) and demonstrate an $Osp(\mathcal{N}|2)$ invariance of this $\mathcal{N}=2M$ supersymmetric ECM model. We also provide a superfield description for the simplest case of $\mathcal{N}=2$ supersymmetry. Finally, we will perform the supersymmetric version of the reduction (1.8), ending up with an $\mathcal{N}$-extended supersymmetric goldfish model.

2. $\mathcal{N}$-extended supersymmetric Euler–Calogero–Moser model

2.1. Extended super Poincaré algebra

The bosonic ECM model (1.1) can be obtained from a free ensemble of real symmetric matrices. This feature is parallel to the descendence of the $su(n)$ spin-Calogero model [9] from the Hermitian matrix model (for details see [7]), for which a supersymmetrization has been constructed in [6]. In full analogy with that case, to construct $\mathcal{N}$ supercharges Q^a and $\overline{Q}_b$ generating an $\mathcal{N}=2M$ superalgebra

$$\left\{Q^a,\overline{Q}_b\right\}=-2\mathrm{i}\,\delta^a_b H \quad \text{and}$$
$$\left\{Q^a,Q^b\right\}=\left\{\overline{Q}_a,\overline{Q}_b\right\}=0 \quad \text{for} \quad a,b=1,2,\dots M, \tag{2.1}$$

one has to introduce two types of fermions:

- $\mathcal{N}\times n$ fermions ψ_i^a and $\bar{\psi}_{i\,a}=\left(\psi_i^a\right)^\dagger$ with $i=1,\dots,n$. These fermions can be combined with the bosonic coordinates $x_i(t)$ into $\mathcal{N}=2M$ supermultiplets.
- $\frac{1}{2}\mathcal{N}\times n(n-1)$ additional fermions $\rho_{ij}^a=\rho_{ji}^a$ and $\bar{\rho}_{ij\,a}=\left(\rho_{ij}^a\right)^\dagger$ subject to $\rho_{ii}^a=\bar{\rho}_{ii\,a}=0$ (no sum).

In total, we thus utilize $\frac{1}{2}\mathcal{N}n(n+1)$ fermions of type ψ and ρ, which we demand to obey the following Poisson brackets

$$\left\{\psi_i^a,\bar{\psi}_{j\,b}\right\}=-\mathrm{i}\delta^a_b\delta_{ij} \quad \text{and}$$
$$\left\{\rho_{ij}^a,\bar{\rho}_{km\,b}\right\}=-\frac{\mathrm{i}}{2}\delta^a_b\left(1-\delta_{ij}\right)\left(1-\delta_{km}\right)\left(\delta_{ik}\delta_{jm}+\delta_{im}\delta_{jk}\right). \tag{2.2}$$

Using these fermions one can construct the composite objects

$$\Pi_{ij}=-\Pi_{ji}$$
$$=-\mathrm{i}\Big[\left(\psi_i^a-\psi_j^a\right)\bar{\rho}_{ij\,a}+\left(\bar{\psi}_{i\,a}-\bar{\psi}_{j\,a}\right)\rho_{ij}^a$$
$$+\sum_{k=1}^{n}\left(\rho_{ik}^a\bar{\rho}_{kj\,a}-\rho_{jk}^a\bar{\rho}_{ki\,a}\right)\Big], \tag{2.3}$$

which satisfy the $so(n)$ Poisson brackets (1.3),

$$\left\{\Pi_{ij},\Pi_{km}\right\}=\frac{1}{2}\left(\delta_{ik}\Pi_{jm}+\delta_{jm}\Pi_{ik}-\delta_{jk}\Pi_{im}-\delta_{im}\Pi_{jk}\right), \tag{2.4}$$

and which Poisson-commute with the fermions ψ and ρ as follows,

$$\left\{\Pi_{ij},\psi_k^a\right\}=\left(\delta_{ik}-\delta_{jk}\right)\rho_{ij}^a,$$
$$\left\{\Pi_{ij},\rho_{km}^a\right\}=-\frac{1}{2}\left(1-\delta_{km}\right)\Big[\left(\delta_{ik}\delta_{jm}+\delta_{im}\delta_{jk}\right)\left(\psi_i^a-\psi_j^a\right) \tag{2.5}$$
$$+\left(\delta_{nm}\delta_{jk}+\delta_{kn}\delta_{jm}\right)\rho_{in}^a-\left(\delta_{nm}\delta_{ik}+\delta_{kn}\delta_{im}\right)\rho_{jn}^a\Big].$$

The key idea for constructing the supercharges $Q^a,\overline{Q}_a$ generating (2.1) is to "prolong" ℓ_{ij} to $\ell_{ij}+\Pi_{ij}$ in all expressions, leading to

$$Q^a=\sum_{i=1}^{n}p_i\psi_i^a-\sum_{i\neq j}^{n}\frac{\left(\ell_{ij}+\Pi_{ij}\right)\rho_{ij}^a}{x_i-x_j} \quad \text{and}$$
$$\overline{Q}_a=\sum_{i=1}^{n}p_i\bar{\psi}_{i\,a}-\sum_{i\neq j}^{n}\frac{\left(\ell_{ij}+\Pi_{ij}\right)\bar{\rho}_{ij\,a}}{x_i-x_j} \tag{2.6}$$

which, together with the Hamiltonian

$$H=\frac{1}{2}\sum_{i=1}^{n}p_i^2+\frac{1}{2}\sum_{i\neq j}^{n}\frac{\left(\ell_{ij}+\Pi_{ij}\right)^2}{\left(x_i-x_j\right)^2} \tag{2.7}$$

indeed obey the $\mathcal{N}=2M$ super Poincaré algebra (2.1) and thus describe an $\mathcal{N}=2M$ supersymmetric extension of the n-particle Euler–Calogero–Moser model. To confirm this fact it is most convenient to treat Π_{ij} as independent objects, which by themselves span the $so(n)$ algebra (2.4) and Poisson-commute with the fermions as in (2.5). Due to these properties, our construction is valid for an arbitrary number of supersymmetries, in a full analogy with the extended supersymmetric $su(n)$-spin Calogero model [6].

2.2. Superconformal invariance

The bosonic n-particle ECM model admits a dynamical conformal symmetry. Our $\mathcal{N}=2M$ supersymmetric extension with the supercharges (2.6) and Hamiltonian (2.7) possesses a dynamical superconformal symmetry. Indeed, starting from the conserved conformal boost current

$$K=\frac{1}{2}\sum_{i=1}^{n}x_i^2-t\sum_{i=1}^{n}x_i p_i+t^2H, \tag{2.8}$$

the remaining conserved currents can easily be obtained by successively Poisson-commuting the super Poincaré generators with K. In this way one finds the full list of conserved currents:

$$
\begin{aligned}
D &= -\frac{1}{2}\sum_{i=1}^{n} x_i p_i + tH, \\
J^a{}_b &= -\sum_{i=1}^{n} \psi_i^a \bar\psi_{i\,b} - \sum_{i\neq j}^{n} \rho_{ij}^a \bar\rho_{ij\,b}, \\
I^{ab} &= -\sum_{i=1}^{n} \psi_i^a \psi_i^b - \sum_{i\neq j}^{n} \rho_{ij}^a \rho_{ij}^b, \\
\bar I_{ab} &= \sum_{i=1}^{n} \bar\psi_{i\,a} \bar\psi_{i\,b} + \sum_{i\neq j}^{n} \bar\rho_{ij\,a} \bar\rho_{ij\,b}, \\
S^a &= \sum_{i=1}^{n} x_i \psi_i^a - tQ^a, \\
\overline{S}_a &= \sum_{i=1}^{n} x_i \bar\psi_{i\,a} - t\overline{Q}_a.
\end{aligned} \tag{2.9}
$$

Together with the supercharges $Q^a, \overline{Q}_a$ (2.6), the Hamiltonian H (2.7) and the conformal boost current K (2.8) they form an $osp(\mathcal{N}|2)$ superalgebra:

$$
\begin{aligned}
&\{H,K\} = 2D, \quad \{H,D\} = H, \quad \{K,D\} = -K, \\
&\{J^a{}_b, J^c{}_d\} = \mathrm{i}\big(\delta^c_b J^a{}_d - \delta^a_d J^c{}_b\big), \quad \{J^a{}_b, I^{cd}\} = \mathrm{i}\big(\delta^c_b I^{ad} - \delta^d_b I^{ac}\big), \\
&\{J^a{}_b, \bar I_{cd}\} = -\mathrm{i}\big(\delta^a_c \bar I_{bd} - \delta^a_d \bar I_{bc}\big), \\
&\{I^{ab}, \bar I_{cd}\} = \mathrm{i}\big(\delta^a_c J^b{}_d - \delta^a_d J^b{}_c - \delta^b_c J^a{}_d + \delta^b_d J^a{}_c\big), \\
&\{D, Q^a\} = -\frac{1}{2} Q^a, \quad \{D, \overline{Q}_a\} = -\frac{1}{2}\overline{Q}_a, \\
&\{D, S^a\} = \frac{1}{2} S^a, \quad \{D, \overline{S}_a\} = \frac{1}{2}\overline{S}_a, \\
&\{H, S^a\} = -Q^a, \quad \{H, \overline{S}_a\} = -\overline{Q}_a, \\
&\{K, Q^a\} = S^a, \quad \{K, \overline{Q}_a\} = \overline{S}_a, \\
&\{J^a{}_b, Q^c\} = \mathrm{i}\,\delta^c_b\, Q^a, \quad \{J^a{}_b, S^c\} = \mathrm{i}\,\delta^c_b\, S^a, \\
&\{J^a{}_b, \overline{Q}_c\} = -\mathrm{i}\,\delta^a_c\, \overline{Q}_b, \quad \{J^a{}_b, \overline{S}_c\} = -\mathrm{i}\,\delta^a_c\, \overline{S}_b, \\
&\{I^{ab}, \overline{Q}_c\} = -\mathrm{i}\big(\delta^a_c Q^b - \delta^b_c Q^a\big), \quad \{I^{ab}, \overline{S}_c\} = -\mathrm{i}\big(\delta^a_c S^b - \delta^b_c S^a\big), \\
&\{\bar I_{ab}, Q^c\} = \mathrm{i}\big(\delta^c_a \overline{Q}_b - \delta^c_b \overline{Q}_a\big), \quad \{\bar I_{ab}, S^c\} = \mathrm{i}\big(\delta^c_a \overline{S}_b - \delta^c_b \overline{S}_a\big), \\
&\{Q^a, \overline{Q}_b\} = -2\mathrm{i}\delta^a_b H, \quad \{S^a, \overline{S}_b\} = -2\mathrm{i}\delta^a_b K, \\
&\{Q^a, \overline{S}_b\} = 2\,\mathrm{i}\,\delta^a_b\, D + J^a{}_b, \quad \{S^a, \overline{Q}_b\} = 2\,\mathrm{i}\,\delta^a_b\, D - J^a{}_b, \\
&\{Q^a, S^b\} = I^{ab}, \quad \{\overline{Q}_a, \overline{S}_b\} = -\bar I_{ab}.
\end{aligned} \tag{2.10}
$$

A $u(M)$ subalgebra is generated by $J^a{}_b$ and extended to an $so(2M)$ subalgebra by adding I^{ab} and $\bar I_{ab}$.

3. $\mathcal{N}=2$ supersymmetric Euler–Calogero–Moser model in superspace

With the Hamiltonian description of an $\mathcal{N}$-extended supersymmetric ECM model at hand, it is quite instructive to construct the superfield description of the simplest case with $\mathcal{N}=2$ supersymmetry. Such a description may be useful for understanding the general structure of the given supersymmetric construction, especially the role played by the additional ρ-type fermions and the currents ℓ_{ij}.

To obtain a superspace representation of the $\mathcal{N}=2$ supersymmetric Euler–Calogero–Moser model, defined with $M=1$ by the supercharges $Q, \overline{Q}$ (2.6) and the Hamiltonian (2.7), one firstly has to solve two tasks:

- assemble the physical components $x_i, \psi_i, \bar\psi_i, \rho_{ij}$ and $\bar\rho_{ij}$ into appropriate $\mathcal{N}=2$ superfields,
- introduce auxiliary bosonic superfields $v_i, \bar v_i$ whose leading components realize ℓ_{ij} via bilinear combinations.

Let us start with the first task. From the structure of the supercharges $Q, \overline{Q}$ (2.6) it is clear that $\mathcal{N}=2$ supersymmetry transforms the coordinates x_i into the fermions $\psi_i, \bar\psi_i$. Thus, one must introduce n bosonic $\mathcal{N}=2$ superfields $\boldsymbol{x}_i$ with the following components,

$$
x_i = \boldsymbol{x}_i|, \quad \psi_i = -\mathrm{i}D\boldsymbol{x}_i|, \quad \bar\psi_i = -\mathrm{i}\overline{D}\boldsymbol{x}_i|, \quad A_i = \frac{1}{2}\left[\overline{D}, D\right]\boldsymbol{x}_i|. \tag{3.1}
$$

Here, $|$ denotes the $\theta=\bar\theta=0$ projection, while D and $\overline{D}$ are $\mathcal{N}=2$ covariant derivatives obeying the relations

$$
\left\{D, \overline{D}\right\} = 2\mathrm{i}\partial_t \quad \text{and} \quad \left\{D, D\right\} = \left\{\overline{D}, \overline{D}\right\} = 0. \tag{3.2}
$$

The fermions $\rho_{ij}, \bar\rho_{ij}$ are put into $n(n-1)$ fermionic superfields $\boldsymbol{\rho}_{ij}, \bar{\boldsymbol{\rho}}_{ij}$, symmetric and of zero diagonal in the indices i, j, i.e.

$$
\boldsymbol{\rho}_{ij} = \boldsymbol{\rho}_{ji}, \quad \bar{\boldsymbol{\rho}}_{ij} = \bar{\boldsymbol{\rho}}_{ji}, \qquad \boldsymbol{\rho}_{ii} = \bar{\boldsymbol{\rho}}_{ii} = 0 \quad (\text{no sum}). \tag{3.3}
$$

As $\mathcal{N}=2$ superfields the $\boldsymbol{\rho}_{ij}$ and $\bar{\boldsymbol{\rho}}_{ij}$ contain a lot of components. However, their leading components ρ_{ij} and $\bar\rho_{ij}$ transform under the $\mathcal{N}=2$ supersymmetry generated by Q and $\overline{Q}$ (2.6) as follows,

$$
\begin{aligned}
\delta_Q \rho_{ij} &\sim \mathrm{i}\bar\epsilon\left[\frac{\psi_i-\psi_j}{x_i-x_j}\rho_{ij} - \sum_{k\neq i,j}^{n}\frac{x_i-x_j}{(x_i-x_k)\left(x_j-x_k\right)}\rho_{ik}\rho_{jk}\right], \\
\delta_{\overline{Q}} \bar\rho_{ij} &\sim \mathrm{i}\epsilon\left[\frac{\bar\psi_i-\bar\psi_j}{x_i-x_j}\bar\rho_{ij} - \sum_{k\neq i,j}^{n}\frac{x_i-x_j}{(x_i-x_k)\left(x_j-x_k\right)}\bar\rho_{ik}\bar\rho_{jk}\right].
\end{aligned} \tag{3.4}
$$

To realize these transformations in superspace we are forced to impose the following nonlinear chirality conditions,

$$
\begin{aligned}
D\boldsymbol{\rho}_{ij} &= \mathrm{i}\left[\frac{\boldsymbol{\psi}_i-\boldsymbol{\psi}_j}{\boldsymbol{x}_i-\boldsymbol{x}_j}\boldsymbol{\rho}_{ij} - \sum_{k\neq i,j}^{n}\frac{\boldsymbol{x}_i-\boldsymbol{x}_j}{(\boldsymbol{x}_i-\boldsymbol{x}_k)\left(\boldsymbol{x}_j-\boldsymbol{x}_k\right)}\boldsymbol{\rho}_{ik}\boldsymbol{\rho}_{jk}\right], \\
\overline{D}\bar{\boldsymbol{\rho}}_{ij} &= \mathrm{i}\left[\frac{\bar{\boldsymbol{\psi}}_i-\bar{\boldsymbol{\psi}}_j}{\boldsymbol{x}_i-\boldsymbol{x}_j}\boldsymbol{\rho}_{ij} - \sum_{k\neq i,j}^{n}\frac{\boldsymbol{x}_i-\boldsymbol{x}_j}{(\boldsymbol{x}_i-\boldsymbol{x}_k)\left(\boldsymbol{x}_j-\boldsymbol{x}_k\right)}\bar{\boldsymbol{\rho}}_{ik}\bar{\boldsymbol{\rho}}_{jk}\right].
\end{aligned} \tag{3.5}
$$

These conditions leave in the superfields $\boldsymbol{\rho}_{ij}$ and $\bar{\boldsymbol{\rho}}_{ij}$ only the components

$$
\rho_{ij} = \boldsymbol{\rho}_{ij}|, \quad B_{ij} = \overline{D}\boldsymbol{\rho}_{ij}|, \qquad \bar\rho_{ij} = \bar{\boldsymbol{\rho}}_{ij}|, \quad \overline{B}_{ij} = D\bar{\boldsymbol{\rho}}_{ij}|\,. \tag{3.6}
$$

To get the correct Poisson brackets for $\psi_i, \bar\psi_i$ and $\rho_{ij}, \bar\rho_{ij}$ (2.2) after passing to the Hamiltonian formalism, the kinetic terms for these fermionic components must read

$$
\begin{aligned}
\mathcal{L}^{\psi}_{kin} &= \frac{\mathrm{i}}{2}\sum_{i=1}^{n}\left(\dot\psi_i\bar\psi_i - \psi_i\dot{\bar\psi}_i\right) \quad \text{and} \\
\mathcal{L}^{\rho}_{kin} &= \frac{\mathrm{i}}{2}\sum_{i,j}^{n}\left(\dot\rho_{ij}\bar\rho_{ij} - \rho_{ij}\dot{\bar\rho}_{ij}\right).
\end{aligned} \tag{3.7}
$$

Altogether, we arrive at the following superfield action for the purely $\mathcal{N}=2$ supersymmetric system with $l_{ij}=0$,

$$S_0=\int \mathrm{d}t\,\mathrm{d}^2\theta\Big[-\frac{1}{2}\sum_{i=1}^n \mathcal{D}\boldsymbol{x}_i\,\overline{\mathcal{D}}\boldsymbol{x}_i+\frac{1}{2}\sum_{i,j}^n \boldsymbol{\rho}_{ij}\bar{\boldsymbol{\rho}}_{ij}\Big],$$
$$\mathrm{d}^2\theta\equiv \mathcal{D}\overline{\mathcal{D}}. \tag{3.8}$$

Now we come to the second task: realize the ℓ_{ij} in terms of auxiliary semi-dynamical variables. As $so(n)$ generators the ℓ_{ij} possess the standard realization

$$\hat{\ell}_{ij}=\frac{\mathrm{i}}{2}\big(v_i\bar{v}_j-v_j\bar{v}_i\big) \tag{3.9}$$

in terms of $2n$ bosonic variables $v_i,\bar{v}_i$ subject to

$$\{v_i,\bar{v}_j\}=-\mathrm{i}\delta_{ij}. \tag{3.10}$$

To implement these new semi-dynamical variables $v_i,\bar{v}_i$ at the superfield level, we have to introduce $2n$ bosonic superfields $\boldsymbol{v}_i,\bar{\boldsymbol{v}}_i$. Additional information about these superfields again comes from the transformation of their first components under $\mathcal{N}=2$ supersymmetry. These transformations can be learned from the explicit structure of the supercharges Q, $\overline{Q}$ (2.6), with the ℓ_{ij} being replaced by their realization $\hat{\ell}_{ij}$ (3.9):

$$\delta_Q v_i\sim \mathrm{i}\bar{\epsilon}\sum_{j\neq i}^n \frac{\rho_{ij}v_j}{x_i-x_j}\quad\text{and}\quad \delta_{\overline{Q}}\bar{v}_i\sim \mathrm{i}\epsilon\sum_{j\neq i}^n\frac{\bar{\rho}_{ij}\bar{v}_j}{x_i-x_j}. \tag{3.11}$$

This form of the transformations implies that, like $\boldsymbol{\rho}_{ij}$ and $\bar{\boldsymbol{\rho}}_{ij}$, also the superfields $\boldsymbol{v}_i$ and $\bar{\boldsymbol{v}}_i$ are subject to nonlinear chirality conditions,

$$\mathcal{D}\boldsymbol{v}_i=\mathrm{i}\sum_{j\neq i}^n\frac{\boldsymbol{\rho}_{ij}\boldsymbol{v}_j}{\boldsymbol{x}_i-\boldsymbol{x}_j}\quad\text{and}\quad \overline{\mathcal{D}}\bar{\boldsymbol{v}}_i=\mathrm{i}\sum_{j\neq i}^n\frac{\bar{\boldsymbol{\rho}}_{ij}\bar{\boldsymbol{v}}_j}{\boldsymbol{x}_i-\boldsymbol{x}_j}. \tag{3.12}$$

These conditions leave in the superfields $\boldsymbol{v}_i$ and $\bar{\boldsymbol{v}}_i$ only the components

$$v_i=\boldsymbol{v}_i|,\quad C_i=-\mathrm{i}\overline{\mathcal{D}}\boldsymbol{v}_i|,\qquad \bar{v}_i=\bar{\boldsymbol{v}}_i|,\quad \overline{C}_i=-\mathrm{i}\mathcal{D}\bar{\boldsymbol{v}}_i|. \tag{3.13}$$

Finally, to have the brackets (3.10), the kinetic terms for $v_i,\bar{v}_i$ must take the form

$$\mathcal{L}^v_{kin}=-\frac{\mathrm{i}}{2}\sum_{i=1}^n\big(\dot{v}_i\bar{v}_i-v_i\dot{\bar{v}}_i\big). \tag{3.14}$$

Therefore, the interaction part ($l_{ij}\neq 0$) of the superfield action reads

$$S_1=-\frac{1}{2}\int \mathrm{d}t\,\mathrm{d}^2\theta\,\sum_{i=1}^n \boldsymbol{v}_i\bar{\boldsymbol{v}}_i. \tag{3.15}$$

Combining everything, we conclude that the superfield action should have the form

$$S=S_0+S_1$$
$$=\int \mathrm{d}t\,\mathrm{d}^2\theta\Big[-\frac{1}{2}\sum_{i=1}^n\mathcal{D}\boldsymbol{x}_i\,\overline{\mathcal{D}}\boldsymbol{x}_i+\frac{1}{2}\sum_{i,j}^n\boldsymbol{\rho}_{ij}\bar{\boldsymbol{\rho}}_{ij}-\frac{1}{2}\sum_{i=1}^n\boldsymbol{v}_i\bar{\boldsymbol{v}}_i\Big], \tag{3.16}$$

where the superfields $\boldsymbol{\rho}_{ij},\bar{\boldsymbol{\rho}}_{ij},\boldsymbol{v}_i$ and $\bar{\boldsymbol{v}}_i$ are subject to the constraints (3.5) and (3.12), respectively.

Despite the extremely simple form of the superfield action (3.16), its component version looks quite complicated due to the nonlinear chirality constraints (3.5) and (3.12). We will write the corresponding component Lagrangian as the sum of a kinetic term $\mathcal{L}_{kin}$, auxiliary-field terms $\mathcal{L}^A_{aux},\mathcal{L}^B_{aux},\mathcal{L}^C_{aux}$ and a “matter” term $\mathcal{L}_{matter}$,

$$\mathcal{L}=\mathcal{L}_{kin}+\mathcal{L}^A_{aux}+\mathcal{L}^B_{aux}+\mathcal{L}^C_{aux}+\mathcal{L}_{matter}. \tag{3.17}$$

The explicit form of these terms is

$$\begin{aligned}
\mathcal{L}_{kin}&=\frac{1}{2}\sum_{i=1}^n\dot{x}_i\dot{x}_i+\frac{\mathrm{i}}{2}\sum_{i=1}^n\big(\dot{\psi}_i\bar{\psi}_i-\psi_i\dot{\bar{\psi}}_i\big)\\
&\quad+\frac{\mathrm{i}}{2}\sum_{i,j}^n\big(\dot{\rho}_{ij}\bar{\rho}_{ij}-\rho_{ij}\dot{\bar{\rho}}_{ij}\big)-\frac{\mathrm{i}}{2}\sum_{i=1}^n\big(\dot{v}_i\bar{v}_i-v_i\dot{\bar{v}}_i\big),\\
\mathcal{L}^A_{aux}&=\frac{1}{2}\sum_{i=1}^n A_iA_i-\sum_{j\neq i}^n\frac{A_i-A_j}{x_i-x_j}\rho_{ij}\bar{\rho}_{ij},\\
\mathcal{L}^B_{aux}&=\frac{1}{2}\sum_{i,j=1}^n B_{ij}\overline{B}_{ij}+\frac{\mathrm{i}}{2}\sum_{j\neq i}^n\Big[\frac{\psi_i-\psi_j}{x_i-x_j}B_{ij}\bar{\rho}_{ij}\\
&\quad+\frac{\bar{\psi}_i-\bar{\psi}_j}{x_i-x_j}\overline{B}_{ij}\rho_{ij}+\frac{B_{ij}v_j\bar{v}_i}{x_i-x_j}-\frac{\overline{B}_{ij}v_i\bar{v}_j}{x_i-x_j}\Big]\\
&\quad+\mathrm{i}\sum_{k\neq i,j}^n\frac{x_i-x_j}{(x_i-x_k)(x_j-x_k)}\Big[B_{ik}\rho_{jk}\bar{\rho}_{ij}+\overline{B}_{ik}\bar{\rho}_{jk}\rho_{ij}\Big],\\
\mathcal{L}^C_{aux}&=-\frac{1}{2}\sum_{i=1}^n C_i\overline{C}_i+\frac{1}{2}\sum_{j\neq i}^n\frac{1}{x_i-x_j}\Big[\rho_{ij}C_j\bar{v}_i-\bar{\rho}_{ij}\overline{C}_jv_i\Big],\\
\mathcal{L}_{matter}&=\frac{1}{2}\sum_{i\neq j,k}^n\frac{\rho_{ij}\bar{\rho}_{ik}}{(x_i-x_j)(x_i-x_k)}v_j\bar{v}_k\\
&\quad-\frac{1}{2}\sum_{j\neq i}^n\Big[\frac{\psi_i-\psi_j}{(x_i-x_j)^2}\bar{\rho}_{ij}v_i\bar{v}_j-\frac{\bar{\psi}_i-\bar{\psi}_j}{(x_i-x_j)^2}\rho_{ij}v_j\bar{v}_i\Big]\\
&\quad+\frac{1}{2}\sum_{j\neq i}^n\frac{(\psi_i-\psi_j)(\bar{\psi}_i-\bar{\psi}_j)}{(x_i-x_j)^2}\rho_{ij}\bar{\rho}_{ij}\\
&\quad+\frac{1}{2}\sum_{i,j\neq k,l}\frac{(x_i-x_j)^2}{(x_i-x_k)(x_j-x_k)(x_i-x_l)(x_j-x_l)}\\
&\quad\times\rho_{ik}\rho_{jk}\bar{\rho}_{il}\bar{\rho}_{jl}\\
&\quad+\sum_{i,j\neq k}^n\frac{1}{(x_i-x_k)(x_j-x_k)}\\
&\quad\times\Big[\frac{x_i-x_j}{x_j-x_k}\big(\psi_j-\psi_k\big)-\big(\psi_i-\psi_j\big)\Big]\bar{\rho}_{ik}\bar{\rho}_{jk}\rho_{ij}\\
&\quad+\sum_{i,j\neq k}^n\frac{1}{(x_i-x_k)(x_j-x_k)}\\
&\quad\times\Big[\frac{x_i-x_j}{x_j-x_k}\big(\bar{\psi}_j-\bar{\psi}_k\big)-\big(\bar{\psi}_i-\bar{\psi}_j\big)\Big]\rho_{ik}\rho_{jk}\bar{\rho}_{ij}.
\end{aligned} \tag{3.18}$$

To go on-shell we eliminate the auxiliary fields $A_i,B_{ij},\overline{B}_{ij},C_i,\overline{C}_i$ using their equations of motion,

$$A_i=2\sum_{j\neq i}^n\frac{\rho_{ij}\bar{\rho}_{ij}}{x_i-x_j},\qquad C_i=\sum_{j\neq i}^n\frac{\bar{\rho}_{ij}v_j}{x_i-x_j},\qquad \overline{C}_i=\sum_{j\neq i}^n\frac{\rho_{ij}\bar{v}_j}{x_i-x_j},$$

$$B_{ij} = \frac{\mathrm{i}}{2}\frac{v_i\bar{v}_j - v_j\bar{v}_i}{x_i - x_j} - \mathrm{i}\frac{(\bar{\psi}_i - \bar{\psi}_j)\,\rho_{ij}}{x_i - x_j} + \mathrm{i}\sum_{k\neq i,j}^{n}\frac{1}{x_i - x_j}\left(\frac{x_i - x_k}{x_k - x_j}\rho_{ik}\bar{\rho}_{jk} - \frac{x_j - x_k}{x_k - x_i}\rho_{jk}\bar{\rho}_{ik}\right),$$

$$\overline{B}_{ij} = \frac{\mathrm{i}}{2}\frac{v_i\bar{v}_j - v_j\bar{v}_i}{x_i - x_j} - \mathrm{i}\frac{(\psi_i - \psi_j)\,\bar{\rho}_{ij}}{x_i - x_j} - \mathrm{i}\sum_{k\neq i,j}^{n}\frac{1}{x_i - x_j}\left(\frac{x_i - x_k}{x_k - x_j}\rho_{jk}\bar{\rho}_{ik} - \frac{x_j - x_k}{x_k - x_i}\rho_{ik}\bar{\rho}_{jk}\right). \tag{3.19}$$

After the substitution of the auxiliary components by the expressions (3.19), a straightforward but slightly tedious calculation brings the Lagrangian (3.17) to the extremely simple form

$$\mathcal{L} = \frac{1}{2}\sum_{i=1}^{n}\dot{x}_i\dot{x}_i + \frac{\mathrm{i}}{2}\sum_{i=1}^{n}\left(\dot{\psi}_i\bar{\psi}_i - \psi_i\dot{\bar{\psi}}_i\right) + \frac{\mathrm{i}}{2}\sum_{i,j}^{n}\left(\dot{\rho}_{ij}\bar{\rho}_{ij} - \rho_{ij}\dot{\bar{\rho}}_{ij}\right) - \frac{\mathrm{i}}{2}\sum_{i=1}^{n}\left(\dot{v}_i\bar{v}_i - v_i\dot{\bar{v}}_i\right) - \sum_{i\neq j}^{n}\frac{(\hat{\ell}_{ij} + \Pi_{ij})^2}{2\,(x_i - x_j)^2}, \tag{3.20}$$

where Π_{ij} is still defined as in (2.3) for $a = 1$ and $\hat{\ell}_{ij}$ is expressed in terms of semi-dynamical variables as in (3.9). Thus, the superfield action (3.16), with the superfields $\boldsymbol{\rho}_{ij}, \bar{\boldsymbol{\rho}}_{ij}, \boldsymbol{v}_i$ and $\bar{\boldsymbol{v}}_i$ nonlinearly constrained by (3.5) and (3.12), indeed describes the $\mathcal{N} = 2$ supersymmetric Euler–Calogero–Moser model.

To conclude, let us make a few comments:

- The nonlinear chirality conditions (3.5) can be slightly simplified by passing to different superfields

$$\boldsymbol{\xi}_{ij} \equiv \frac{\boldsymbol{\rho}_{ij}}{\boldsymbol{x}_i - \boldsymbol{x}_j}, \quad \bar{\boldsymbol{\xi}}_{ij} \equiv \frac{\bar{\boldsymbol{\rho}}_{ij}}{\boldsymbol{x}_i - \boldsymbol{x}_j} \quad \Rightarrow \quad D\boldsymbol{\xi}_{ij} + \mathrm{i}\sum_{k=1}^{n}\boldsymbol{\xi}_{ik}\boldsymbol{\xi}_{jk} = 0, \quad \overline{D}\bar{\boldsymbol{\xi}}_{ij} + \mathrm{i}\sum_{k=1}^{n}\bar{\boldsymbol{\xi}}_{ik}\bar{\boldsymbol{\xi}}_{jk} = 0.$$

  However, the Lagrangian, Hamiltonian and Poisson brackets will look more complicated in terms of $\boldsymbol{\xi}_{ij}$ and $\bar{\boldsymbol{\xi}}_{ij}$, despite the fact that the constraints for these new superfields do no longer involve the superfields $\boldsymbol{x}_i$.
- The auxiliary superfields $\boldsymbol{v}_i, \bar{\boldsymbol{v}}_i$ cannot be redefined in a similar manner. Thus, the nonlinear chirality constraints (3.12) are unavoidable.
- The superfield action (3.16) looks like a free action for all superfields involved. However, all interactions are hidden inside the nonlinear chirality constraints (3.5) and (3.12). This feature makes our construction quite different from most $\mathcal{N} = 2$ supersymmetric mechanics where the interactions are generated via superpotentials. We are curious whether our mechanism to turn on interactions may be applied elsewhere for constructing new interacting superfield models.

4. Supersymmetric goldfish model

To construct an $\mathcal{N} = 2M$ supersymmetric extension of the bosonic n-particle goldfish model (1.7) one has to impose a modified version of the constraints (1.8). It is not too hard to guess such constraints to be

$$\widetilde{G}_{ij} \equiv \ell_{ij} + (x_i - x_j)\sqrt{\dot{x}_i\dot{x}_j} + \Pi_{ij} \approx 0. \tag{4.1}$$

One may check that these constraints weakly commute with the Hamiltonian (2.7), with the supercharges (2.6) and with each other, hence they are first class.

To get the equations of motion, one has to evaluate the brackets of all component fields involved with the Hamiltonian (2.7) and then to impose the constraints (4.1). This results in the following equations of motion:

$$\dot{x}_i = p_i, \qquad \dot{p}_i = 2\sum_{j\neq i}^{n}\frac{p_i p_j}{x_i - x_j},$$

$$\dot{\psi}_i^a = 2\sum_{j\neq i}^{n}\frac{\sqrt{p_i p_j}}{x_i - x_j}\rho_{ij}^a, \qquad \dot{\bar{\psi}}_{i\,a} = 2\sum_{j\neq i}^{n}\frac{\sqrt{p_i p_j}}{x_i - x_j}\bar{\rho}_{ij\,a},$$

$$\dot{\rho}_{ij}^a = -\frac{\sqrt{p_i p_j}}{x_i - x_j}\left(\psi_i^a - \psi_j^a\right) + \sum_{k\neq i,j}^{n}\left[\frac{\sqrt{p_i p_k}}{x_i - x_k}\rho_{jk}^a + \frac{\sqrt{p_j p_k}}{x_j - x_k}\rho_{ik}^a - 2\delta_{ij}\frac{\sqrt{p_i p_k}}{x_i - x_k}\rho_{ik}^a\right],$$

$$\dot{\bar{\rho}}_{ij\,a} = -\frac{\sqrt{p_i p_j}}{x_i - x_j}\left(\bar{\psi}_{i\,a} - \bar{\psi}_{j\,a}\right) + \sum_{k\neq i,j}^{n}\left[\frac{\sqrt{p_i p_k}}{x_i - x_k}\bar{\rho}_{jk\,a} + \frac{\sqrt{p_j p_k}}{x_j - x_k}\bar{\rho}_{ik\,a} - 2\delta_{ij}\frac{\sqrt{p_i p_k}}{x_i - x_k}\bar{\rho}_{ik\,a}\right]. \tag{4.2}$$

The $\mathcal{N}$-extended supersymmetry transformations, generated by Poisson-commuting $\mathrm{i}\left(\bar{\epsilon}_a Q^a + \epsilon^a \overline{Q}_a\right)$ with all components fields and then by imposing the constraints (4.1), have the form

$$\delta x_i = \mathrm{i}\left(\bar{\epsilon}_a\psi_i^a + \epsilon^a\bar{\psi}_{i\,a}\right),$$

$$\delta p_i = 2\mathrm{i}\sum_{j\neq i}^{n}\frac{\sqrt{p_i p_j}}{x_i - x_j}\left(\bar{\epsilon}_a\rho_{ij}^a + \epsilon^a\bar{\rho}_{ij\,a}\right),$$

$$\delta\psi_i^a = 2\mathrm{i}\sum_{j\neq i}^{n}\frac{\rho_{ij}^a}{x_i - x_j}\left(\bar{\epsilon}_b\rho_{ij}^b + \epsilon^b\bar{\rho}_{ij\,b}\right) - \epsilon^a p_i,$$

$$\delta\bar{\psi}_{i\,a} = 2\mathrm{i}\sum_{j\neq i}^{n}\frac{\bar{\rho}_{ij\,a}}{x_i - x_j}\left(\bar{\epsilon}_b\rho_{ij}^b + \epsilon^b\bar{\rho}_{ij\,b}\right) - \bar{\epsilon}_a p_i,$$

$$\delta\rho_{ij}^a = -\epsilon^a\sqrt{p_i p_j} + \epsilon^a\delta_{ij}p_i - \mathrm{i}\frac{\psi_i^a - \psi_j^a}{x_i - x_j}\left(\bar{\epsilon}_b\rho_{ij}^b + \epsilon^b\bar{\rho}_{ij\,b}\right) + \mathrm{i}\sum_{k\neq i}^{n}\frac{\rho_{jk}^a}{x_i - x_k}\left(\bar{\epsilon}_b\rho_{ik}^b + \epsilon^b\bar{\rho}_{ik\,b}\right) + \mathrm{i}\sum_{k\neq j}^{n}\frac{\rho_{ik}^a}{x_j - x_k}\left(\bar{\epsilon}_b\rho_{jk}^b + \epsilon^b\bar{\rho}_{jk\,b}\right) - 2\mathrm{i}\delta_{ij}\sum_{k\neq i}^{n}\frac{\rho_{ik}^a}{x_i - x_k}\left(\bar{\epsilon}_b\rho_{ik}^b + \epsilon^b\bar{\rho}_{ik\,b}\right),$$

$$\delta\bar{\rho}_{ij\,a} = -\bar{\epsilon}_a\sqrt{p_i p_j} + \bar{\epsilon}_a\delta_{ij}p_i - \mathrm{i}\frac{\bar{\psi}_{i\,a} - \bar{\psi}_{j\,a}}{x_i - x_j}\left(\bar{\epsilon}_b\rho_{ij}^b + \epsilon^b\bar{\rho}_{ij\,b}\right) + \mathrm{i}\sum_{k\neq i}^{n}\frac{\bar{\rho}_{jk\,a}}{x_i - x_k}\left(\bar{\epsilon}_b\rho_{ik}^b + \epsilon^b\bar{\rho}_{ik\,b}\right) + \mathrm{i}\sum_{k\neq j}^{n}\frac{\bar{\rho}_{ik\,a}}{x_j - x_k}\left(\bar{\epsilon}_b\rho_{jk}^b + \epsilon^b\bar{\rho}_{jk\,b}\right) - 2\mathrm{i}\delta_{ij}\sum_{k\neq i}^{n}\frac{\bar{\rho}_{ik\,a}}{x_i - x_k}\left(\bar{\epsilon}_b\rho_{ik}^b + \epsilon^b\bar{\rho}_{ik\,b}\right). \tag{4.3}$$

One may verify that these transformations form the $\mathcal{N}=2$ superalgebra and leave the equations of motion (4.2) invariant.

After imposing the constraints (4.1), the Hamiltonian (2.7) and the supercharges (2.6) acquire the form

$$H_{red} = \frac{1}{2}\left(\sum p_i\right)^2 \quad \text{and}$$
$$\left(Q^a\right)_{red} = \sum_i p_i\psi_i^a + \frac{1}{2}\sum_{i\neq j}\sqrt{p_i p_j}\,\rho^a_{ij},$$
$$\left(\overline{Q}_a\right)_{red} = \sum_i p_i\bar\psi_{i\,a} + \frac{1}{2}\sum_{i\neq j}\sqrt{p_i p_j}\,\bar\rho_{ij\,a}. \tag{4.4}$$

It is clear that the correct equations of motion require a deformation of the basic Poisson brackets (1.2), (2.2), similarly to the purely bosonic case [12]. We plan to analyze the corresponding deformation of the Poisson brackets elsewhere.

5. Conclusion

We proposed a novel $\mathcal{N}$-extended supersymmetric $so(n)$ spin-Calogero model by a direct supersymmetrization of the bosonic Euler–Calogero–Moser system [8]. The constructed model contains

- n bosonic coordinates x_i which stem from the diagonal part of a real symmetric matrix,
- the off-shell elements of this symmetric matrix, which enter the supercharges and the Hamiltonian only through $so(n)$ currents ℓ_{ij},
- $\mathcal{N}n$ fermions ψ_i^a and $\bar\psi_{i\,a}$, which combine with the x_i to n supermultiplets,
- $\frac{1}{2}\mathcal{N}\times n(n-1)$ additional fermions $\rho^a_{ij}=\rho^a_{ji}$ and $\bar\rho_{ij\,a}=\left(\rho^a_{ij}\right)^\dagger$ for $i\neq j$.

The supercharges Q^a and $\overline{Q}_b$ and the Hamiltonian form an $\mathcal{N}$-extended Poincaré superalgebra and have the standard structure up to cubic in the fermions. Additional conserved currents enlarge this superalgebra to a dynamical $osp(\mathcal{N}|2)$ superconformal symmetry of the ECM model. Having performed the Hamiltonian reduction of the ECM model, we obtained the $\mathcal{N}$-supersymmetric goldfish system for n particles.

The structure of the $so(n)$ spin-Calogero supercharges (2.6) and Hamiltonian (2.7) is quite similar to the supercharges and the Hamiltonian of the extended supersymmetric $su(n)$ spin-Calogero model [6]. Indeed, the former can be obtained from the latter by restricting the $su(n)$ currents ℓ_{ij} to the $so(n)$ subalgebra, imposing antisymmetry in their indices, and likewise restricting the matrix fermions ρ^a_{ij} and $\bar\rho_{ij\,a}$ to be symmetric in their indices. Upon such a reduction, the composite object Π_{ij} also becomes antisymmetric in (i,j) and generates an $so(n)$ algebra. The first-class constraints $\ell_{ii}+\Pi_{ii}\approx 0$ present in the $su(n)$ spin-Calogero model [6] are then satisfied automatically, and the reduced supercharges and Hamiltonian will coincide with the supercharges (2.6) and Hamiltonian (2.7). However, the compatibility of this reduction with the extended supersymmetry is *not a priori* evident and has to be checked explicitly.

The superfield description of our model in the simplest case of $\mathcal{N}=2$ supersymmetry features

- coordinates x_i and fermions $\psi_i, \bar\psi_j$ forming standard unconstrained bosonic superfields of type $(1,2,1)$,
- fermionic symmetric matrices $\rho_{ij}, \bar\rho_{ij}$ (with vanishing diagonal), subject to nonlinear chirality constraints,
- $2n$ bosonic $\mathcal{N}=2$ semi-dynamical superfields $v_i, \bar v_i$ also obeying some nonlinear chirality constraints.

The superspace action contains only the standard kinetic terms for all superfields. It is only the nonlinear constraints which result in a rather complicated component action. However, after eliminating the auxiliary components via their equations of motion, the action acquires quite a simple form again, with an interaction quadratic and quartic in the fermions.

The presented $\mathcal{N}=2$ supersymmetric case is not too illuminating, because it can also be constructed without matrix fermions ρ_{ij} and $\bar\rho_{ij}$, in analogy with the $\mathcal{N}=2$ supersymmetric Calogero model [16,17]. One may discard the terms quadratic in ρ_{ij} and $\bar\rho_{ij}$ in the nonlinear chirality constraints (3.5). Thus, the generic superfield structure of the $\mathcal{N}$-extended ECM model becomes visible at $\mathcal{N}=4$ only. We are planning to address this elsewhere.

Acknowledgements

We are grateful to V. Gerdt and A. Khvedelidze for stimulating discussions. This work was partially supported by the Heisenberg–Landau program. The work of S.K. was partially supported by Russian Science Foundation grant 14-11-00598, the one of A.S. by RFBR grants 18-02-01046 and 18-52-05002 Arm-a. This article is based upon work from COST Action MP1405 QSPACE, supported by COST (European Cooperation in Science and Technology).

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