

Yang-Mills solutions on de Sitter space of any dimension

Olaf Lechtenfeld and Gönül Ünal

*Institut für Theoretische Physik and Riemann Center for Geometry and Physics,
Leibniz Universität Hannover, Appelstraße 2, 30167 Hannover, Germany*



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For gauge groups $SO(n+1)$, $SU(m+1)$, and $Sp(\ell+1)$, we construct equivariant Yang-Mills solutions on de Sitter space in $n+1$, $2(m+1)$, and $4(\ell+1)$ spacetime dimensions. The latter is conformally mapped to a finite cylinder over a coset space realizing an appropriate unit sphere. The equivariance condition reduces the Yang-Mills system to an analog Newtonian particle in one or two dimensions subject to a time-dependent friction and a particular potential. We analyze some properties of the solutions such as their action and energy and display all analytic ones. Beyond dS_4 , all such configurations have finite energy but infinite action.

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I. INTRODUCTION AND SUMMARY

Analytic solutions of the Yang-Mills equations are not easy to come by in any spacetime manifold of dimension larger than two. Yet, they play a central role in the analysis of semiclassical behavior, vacuum structure, and admissible string backgrounds, to name a few. Imposing a sufficient amount of symmetry on Yang-Mills solutions, however, reduces the coupled system of nonlinear partial differential equations to ordinary (albeit cubic) matrix differential equations, which is amenable to solving explicitly. The simplest situation is that of maximally symmetric spacetimes and the gauge group being equal to its maximal compact symmetry subgroup G . Restricting our attention to G -equivariant Yang-Mills configurations then further simplifies the matrix equations of motion to scalar equations, which can be interpreted as Newtonian dynamics of a particle in some Euclidean space with a particular quartic potential. The latter system sometimes allows for consistent one-dimensional double-well subsystems, yielding special explicit solutions.

Much is known about analytic Yang-Mills solutions in Minkowski space $\mathbb{R}^{1,n}$ [1]. Typically, however, these are singular or have infinite energy, unless a Higgs field is added. The situation is better on de Sitter space in various spacetime dimensions $n+1$ because equal time slices may be chosen to be compact and the metric regulates the action integral at early and late times. The closed slicing of dS_{n+1} describes it as a cosh-cylinder over the n -sphere S^n , with a

G -action on S^n . Representing the unit sphere as a coset space G/H via

$$S^n = \frac{SO(n+1)}{SO(n)} \quad \text{or} \quad S^{2m+1} = \frac{SU(m+1)}{SU(m)} \\ \text{or} \quad S^{4\ell+3} = \frac{Sp(\ell+1)}{Sp(\ell)} \quad (1.1)$$

suggests looking for Yang-Mills solutions with gauge groups $SO(n+1)$, $SU(m+1)$, or $Sp(\ell+1)$ on the spaces dS_{n+1} , dS_{2m+2} , or $dS_{4\ell+4}$, respectively. Indeed, classical finite-energy and finite-action $SU(2)$ pure Yang-Mills fields have been found on dS_4 [2,3], employing the isomorphy of S^3 to $SU(2)$ and a conformal map from dS_4 to $\mathcal{I} \times S^3$, where \mathcal{I} is a finite time interval of length π .

It is thus natural to extend this analysis to higher-dimensional de Sitter spaces in the orthogonal, unitary, and symplectic incarnation based on (1.1). This is the subject of the present paper. After a geometric description of de Sitter space and the various spherical cosets in the following section, we perform the equivariant reduction of the Yang-Mills equations on the orthogonal, unitary, and symplectic cosets, arriving at a one- or two-dimensional Newtonian system with a particular quartic potential and a peculiar friction term. The latter arises away from four spacetime dimensions due to the noninvariance of the Yang-Mills equations under the conformal map

$$dS_{n+1} \rightarrow \mathcal{I} \times S^n \quad \text{with} \quad \mathcal{I} = \left(-\frac{\pi}{2}, \frac{\pi}{2} \right). \quad (1.2)$$

For all three cases, we compute the equations of motion, Newtonian potential, action, and energy of the analog-particle system and write down the Yang-Mills fields in terms of the particle trajectory. The final section

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discusses properties of generic and special solutions. As expected, above four spacetime dimensions, all solutions have infinite action but finite energy. The friction is detrimental below four dimensions as it renders the field strengths singular at the temporal boundary. However, purely color-magnetic solutions avoid the friction, and there exists an analytic one in every dimension. Corresponding to the unique local maximum of the Newtonian potential, it is unstable and given by the canonical H -connection.

In critical spacetime dimension four, one may use either $SU(2)$ or $\frac{SO(4)}{SO(3)}$ to describe the three-sphere. The absence of the friction term leads to a family of Abelian and of non-Abelian solutions, where the latter are given in terms of elliptic functions which solve the mechanical double-well problem. We reproduce the known $SU(2)$ results [where the color-magnetic solution is *half* the canonical $SU(2)$ connection] [4–6] and complement them with corresponding $SO(4)$ solutions, which, however, turn out to be equivalent.

We have thus constructed new explicit finite-energy (but infinite-action) Yang-Mills configurations on de Sitter space of any dimension. The story may be repeated for other coset representations, such as $S^6 = \frac{G_2}{SU(3)}$. Finally, our spacetime background is nondynamical; we do not consider its gravitational backreaction. The result of this effect is known for the four-dimensional Yang-Mills solutions (see, e.g., Sec. VIII of [7] and references therein) and may be generalized to higher dimensions.

II. DESCRIPTION OF DE SITTER SPACE

A. de Sitter space dS_{n+1} as a cylinder over S^n

It is well known that $(n+1)$ -dimensional de Sitter space dS_{n+1} can be embedded into $(n+2)$ -dimensional Minkowski space $\mathbb{R}^{n+1,1}$ as

$$\delta_{\alpha\beta}y^\alpha y^\beta - (y^{n+2})^2 = R^2 \quad \text{where } \alpha, \beta = 1, \dots, n+1, \quad (2.1)$$

with the metric

$$ds^2 = dy^\alpha dy^\beta - dy^{n+2} dy^{n+2}. \quad (2.2)$$

Topologically, de Sitter space dS_{n+1} is a cylinder over S^n , which is easily seen from the parametrization $\{\tau, \omega^\alpha\}$,

$$\begin{aligned} y^\alpha &= R\omega^\alpha \cosh \tau & \text{and} & & y^{n+2} &= R \sinh \tau \\ & & \text{with} & & \omega^\alpha \omega^\alpha &= 1, \end{aligned} \quad (2.3)$$

where $-\infty < \tau < \infty$ and ω^α embed the unit n -sphere into \mathbb{R}^{n+1} . An explicit form of the ω^α can be found in terms of the global coordinates $\{\varphi, \theta_1, \dots, \theta_{n-1}\}$ as

$$\begin{aligned} \omega^1 &= \rho_{n-1} \sin \varphi, & \omega^2 &= \rho_{n-1} \cos \varphi, \\ \omega^k &= \rho_{n-k+1} \cos \theta_{k-2} & \text{for } 3 \leq k \leq n+1 \end{aligned} \quad (2.4)$$

where

$$\begin{aligned} \rho_\ell &= \prod_{m=1}^{\ell} \sin \theta_{n-m}, & \rho_0 &= 1, \\ 0 \leq \varphi &\leq 2\pi, & 0 \leq \theta_1, \dots, \theta_{n-1} &< \pi. \end{aligned} \quad (2.5)$$

With the help of this parametrization, the flat metric (2.2) induces, on de Sitter space, the metric

$$ds^2 = R^2(-d\tau^2 + \cosh^2 \tau d\Omega_n^2) \quad (2.6)$$

where $d\Omega_n^2$ is the metric on the unit n -sphere. From the coordinates $\{\tau, \varphi, \theta_1, \dots, \theta_{n-1}\}$, we pass to conformal coordinates $\{t, \varphi, \theta_a\}$ by the time reparametrization [8]

$$t = \arctan(\sinh \tau) = 2 \arctan\left(\tanh \frac{\tau}{2}\right)$$

$$\text{with } \frac{d\tau}{dt} = \cosh \tau = \frac{1}{\cos t}, \quad (2.7)$$

so that

$$t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) = \mathcal{I} \quad (2.8)$$

ranges over the finite interval \mathcal{I} , and the metric becomes

$$ds^2 = \frac{R^2}{\cos^2 t}(-dt^2 + d\Omega_n^2). \quad (2.9)$$

We see that de Sitter space dS_{n+1} is conformally equivalent to a cylinder $\mathcal{I} \times S^n$ over the n -sphere, with a conformal factor of $\frac{R^2}{\cos^2 t}$.

B. S^n as coset $SO(n+1)/SO(n)$

In this subsection, we briefly summarize the well-known geometric properties of S^n realized as the coset space $\frac{SO(n+1)}{SO(n)}$. We denote by $\{I_A\}$, with $A = 1, \dots, \frac{n(n+1)}{2}$, the $SO(n+1)$ generators satisfying

$$[I_A, I_B] = f_{AB}^C I_C. \quad (2.10)$$

The structure constants f_{AB}^C give rise to the Cartan-Killing metric g on the Lie algebra $so(n+1)$,

$$g_{AB} = f_{AD}^C f_{CB}^D = -\text{tr}_{\text{adj}}(I_A I_B) = \kappa_n \delta_{AB}, \quad (2.11)$$

where the generators are taken in the adjoint representation and κ_n is some constant depending on their normalization.

Let us decompose

$$so(n+1) = so(n) \oplus \mathfrak{m} \quad (2.12)$$

so that \mathfrak{m} is the orthogonal complement of the Lie subalgebra $so(n)$ inside $so(n+1)$. Then, the set of $SO(n+1)$ generators splits into two subsets,

$$\{I_A\} = \{I_a\} \cup \{I_i\} \quad \text{where } I_a \in \mathfrak{m} \quad \text{and} \quad I_i \in so(n) \quad (2.13)$$

with the index ranges $a = 1, \dots, n$ and $i = n+1, \dots, \frac{n(n+1)}{2}$. Since this coset is a symmetric space, $f_{ab}{}^c = 0$, and the commutation relations (2.10) decompose as

$$[I_i, I_j] = f_{ij}{}^k I_k, \quad [I_i, I_a] = f_{ia}{}^b I_b, \quad [I_a, I_b] = f_{ab}{}^i I_i. \quad (2.14)$$

The components of the Cartan-Killing metric (2.11) read

$$g_{ij} = f_{ik}{}^l f_{lj}{}^k + f_{ia}{}^b f_{bj}{}^a = \kappa_n \delta_{ij}, \\ g_{ab} = 2f_{ad}{}^i f_{ib}{}^d = \kappa_n \delta_{ab}, \quad g_{ia} = 0. \quad (2.15)$$

We note that $S^1 = SO(2)$ is special since the index i takes no values; hence, most terms are absent.

C. S^{2m+1} as coset $SU(m+1)/SU(m)$

Odd-dimensional spheres S^{2m+1} can also be realized as (nonsymmetric) coset spaces $\frac{SU(m+1)}{SU(m)}$. Let $\{\tilde{I}_A\}$ with $A = 1, \dots, m(m+2)$ be the generators of $SU(m+1)$ subject to

$$[\tilde{I}_A, \tilde{I}_B] = \tilde{f}_{AB}{}^C \tilde{I}_C. \quad (2.16)$$

Analogous to the previous subsection, we decompose

$$su(m+1) = su(m) \oplus \tilde{\mathfrak{m}} \quad (2.17)$$

and divide

$$\{\tilde{I}_A\} = \{\tilde{I}_a\} \cup \{\tilde{I}_i\} \quad \text{where } \tilde{I}_a \in \tilde{\mathfrak{m}} \quad \text{and} \quad \tilde{I}_i \in su(m) \quad (2.18)$$

with $a = 1, \dots, 2m+1$ and $i = 2m+2, \dots, m(m+2)$. In this case, the commutation relations (2.16) read

$$[\tilde{I}_i, \tilde{I}_j] = \tilde{f}_{ij}{}^k \tilde{I}_k, \quad [\tilde{I}_i, \tilde{I}_a] = \tilde{f}_{ia}{}^b \tilde{I}_b, \\ [\tilde{I}_a, \tilde{I}_b] = \tilde{f}_{ab}{}^i \tilde{I}_i + \tilde{f}_{ab}{}^c \tilde{I}_c. \quad (2.19)$$

The normalization of the Cartan-Killing metric

$$\tilde{g}_{AB} = \tilde{f}_{AD}{}^C \tilde{f}_{CB}{}^D = -\text{tr}_{\text{adj}}(\tilde{I}_A \tilde{I}_B) = \tilde{\kappa}_m \delta_{AB} \quad (2.20)$$

implies that

$$\tilde{g}_{ij} = \tilde{f}_{ik}{}^l \tilde{f}_{lj}{}^k + \tilde{f}_{ia}{}^b \tilde{f}_{bj}{}^a = \tilde{\kappa}_m \delta_{ij}, \\ \tilde{g}_{ab} = 2\tilde{f}_{ad}{}^i \tilde{f}_{ib}{}^d + \tilde{f}_{ad}{}^c \tilde{f}_{cb}{}^d = \tilde{\kappa}_m \delta_{ab}, \quad \tilde{g}_{ia} = 0. \quad (2.21)$$

We remark that $S^3 = SU(2)$ is particular since the index i takes no values, so many terms are vanishing.

D. $S^{4\ell+3}$ as coset $Sp(\ell+1)/Sp(\ell)$

Finally, there exists a symplectic coset realization of $S^{4\ell+3}$ as $\frac{Sp(\ell+1)}{Sp(\ell)}$. The $Sp(\ell+1)$ generators \hat{I}_A with $A = 1, \dots, (\ell+1)(2\ell+3)$ obey

$$[\hat{I}_A, \hat{I}_B] = \hat{f}_{AB}{}^C \hat{I}_C. \quad (2.22)$$

Splitting

$$sp(\ell+1) = sp(\ell) \oplus \hat{\mathfrak{m}} \quad \text{and} \\ \{\hat{I}_a\} = \{\hat{I}_a\} \cup \{\hat{I}_i\} \quad \text{where } \hat{I}_a \in \hat{\mathfrak{m}} \quad \text{and} \quad \hat{I}_i \in sp(\ell) \quad (2.23)$$

with $a = 1, \dots, 4\ell+3$ and $i = 4\ell+4, \dots, (\ell+1)(2\ell+3)$, the commutation relations (2.22) decompose as in (2.19). Analogously, relations (2.20) and (2.21) hold, with tildes replaced by hats and a normalization constant $\hat{\kappa}_\ell$.

E. Connections on spheres

To define a connection on S^n , we introduce orthonormal frames as follows: Let $\{\check{\nabla}_A\}$ be the left-invariant vector fields on $SO(n+1)$ satisfying the same commutation relations as the generators I_A , and let $\{\check{e}^A\}$ be the left-invariant one-forms on $SO(n+1)$ dual to the vector fields $\check{\nabla}_A$. The $SO(n+1)$ group multiplication induces a natural map

$$\alpha: SO(n+1) \rightarrow S^n \equiv \frac{SO(n+1)}{SO(n)} \quad \text{with} \quad g \mapsto g \cdot SO(n). \quad (2.24)$$

On any open subset $U \subset S^n$, one can invert this map by $\beta: U \rightarrow SO(n+1)$ with $\alpha \circ \beta = \text{id}$. In other words, β is a local section of a principal bundle $SO(n+1) \rightarrow S^n$. This allows us to pull back the one-forms \check{e}^A to left-invariant one-forms $e^A = \beta^* \check{e}^A$ on S^n . Splitting $\{e^A\} = \{e^a\} \cup \{e^i\}$, the set $\{e^a\}$ forms a basis on S^n , and the remaining one-forms e^i are dependent, i.e., $e^i = e^i{}_a e^a$ with real functions $e^i{}_a$. Using the group action, we can extend e^a from U to everywhere on S^n .

The normalization of the e^A is related to that of the structure constants via the Maurer-Cartan equations

$$de^A + \frac{1}{2} f_{BC}{}^A e^B \wedge e^C = 0, \quad (2.25)$$

and we choose it in such a way that

$$\delta_{ab}e^ae^b = d\Omega_n^2 \quad (2.26)$$

is the metric on the unit sphere, i.e., on S^n with radius one. This fixes the proportionality constant \varkappa_n in (2.11), as is derived in the Appendix. The metric (2.9) on the cylinder then takes the conformally flat form

$$ds^2 = \frac{R^2}{\cos^2 t}(-dt^2 + \delta_{ab}e^ae^b). \quad (2.27)$$

The connection one-forms $\omega^a{}_b$ on S^n may be found from Cartan's structure equations

$$de^a + \omega^a{}_b \wedge e^b = 0 \quad (2.28)$$

by comparing with the Maurer-Cartan equations:

$$\begin{aligned} 0 &= de^a + f_{bi}{}^ae^b \wedge e^i = de^a + f_{bi}{}^ae^i \wedge e^b \\ &\Rightarrow \omega^a{}_b = \omega^a{}_{cb}e^c = f_{ib}{}^ae^i e^c. \end{aligned} \quad (2.29)$$

The connection on the nonsymmetric coset for the odd-dimensional sphere S^{2m+1} can be obtained analogously. Left-invariant one-forms \check{e}^a on $SU(m+1)$ give rise to left-invariant one-forms \tilde{e}^a on S^{2m+1} , which we normalize (fixing $\tilde{\varkappa}_m$) in such a way that the metric on the unit sphere reads

$$d\Omega_{2m+1}^2 = \delta_{ab}\tilde{e}^a\tilde{e}^b \Rightarrow ds^2 = \frac{R^2}{\cos^2 t}(-dt^2 + \delta_{ab}\tilde{e}^a\tilde{e}^b). \quad (2.30)$$

From the Maurer-Cartan equations and $\tilde{e}^i = \tilde{e}^i{}_a\tilde{e}^a$, we read off the connection one-forms $\tilde{\omega}^a{}_b$ on S^{2m+1} :

$$\begin{aligned} 0 &= d\tilde{e}^a + \tilde{f}_{bi}{}^a\tilde{e}^b \wedge \tilde{e}^i + \frac{1}{2}\tilde{f}_{bc}{}^a\tilde{e}^b \wedge \tilde{e}^c \\ &\Rightarrow \tilde{\omega}^a{}_b = \tilde{\omega}^a{}_{cb}\tilde{e}^c = \left(\tilde{f}_{ib}{}^a\tilde{e}^i + \frac{1}{2}\tilde{f}_{cb}{}^a \right)\tilde{e}^c. \end{aligned} \quad (2.31)$$

The connection on the symplectic coset for $S^{4\ell+3}$ takes the same form, with left-invariant one-forms \check{e}^A on $Sp(\ell+1)$ descending to \hat{e}^a on $S^{4\ell+3}$. Again we normalize by fixing $\hat{\varkappa}_\ell$ so that

$$d\Omega_{4\ell+3}^2 = \delta_{ab}\hat{e}^a\hat{e}^b \Rightarrow ds^2 = \frac{R^2}{\cos^2 t}(-dt^2 + \delta_{ab}\hat{e}^a\hat{e}^b). \quad (2.32)$$

The connection one-forms $\hat{\omega}^a{}_b$ on $S^{4\ell+3}$ are obtained as in (2.29).

III. EQUIVARIANT REDUCTION OF THE YANG-MILLS EQUATIONS

A. Yang-Mills equations on dS_{n+1}

Since the Yang-Mills equations transform in a simple fashion under a conformal rescaling of the metric, solutions on de Sitter space may be obtained from transformed Yang-Mills equations on the cylinder $\mathcal{I} \times S^n$ with the flat metric

$$\begin{aligned} d\bar{s}^2 &= -dt^2 + \delta_{ab}\bar{e}^a\bar{e}^b = -e^0e^0 + \delta_{ab}\bar{e}^a\bar{e}^b \quad \text{with} \\ e^0 &= dt \quad \text{and} \quad \bar{e}^a = \begin{cases} e^a \\ \tilde{e}^a \\ \hat{e}^a \end{cases}, \end{aligned} \quad (3.1)$$

depending on the case (2.27), (2.30), or (2.32), respectively. Therefore, we may raise and lower spacetime indices $\mu \in \{0, a\}$ with the Minkowski metric $(\eta_{\mu\nu}) = \text{diag}(-1, 1, \dots, 1)$. Structure constants with all indices down are completely antisymmetric. In this subsection, we treat the three cases in parallel, denoting the connection coefficients by $\bar{\omega}$.

We expand the gauge potential \mathcal{A} and gauge field \mathcal{F} in terms of the one-form basis $\{\bar{e}^0, \bar{e}^a\}$,

$$\mathcal{A} = \mathcal{A}_0\bar{e}^0 + \mathcal{A}_a\bar{e}^a \quad \text{and} \quad \mathcal{F} = \mathcal{F}_{0a}\bar{e}^0 \wedge \bar{e}^a + \frac{1}{2}\mathcal{F}_{ab}\bar{e}^a \wedge \bar{e}^b, \quad (3.2)$$

and take the temporal gauge $\mathcal{A}_0 = \mathcal{A}_t = 0$. Then, the Yang-Mills equations on $\mathcal{I} \times S^n$ read

$$\begin{aligned} \nabla_a\mathcal{F}^{a0} + \bar{\omega}_{ac}^a\mathcal{F}^{c0} + \bar{\omega}_{a0}^0\mathcal{F}^{a0} + \bar{\omega}_{ab}^0\mathcal{F}^{ab} + [\mathcal{A}_a, \mathcal{F}^{a0}] &= 0 \quad \text{and} \\ \nabla_0\mathcal{F}^{0b} + \nabla_a\mathcal{F}^{ab} + \bar{\omega}_{00}^0\mathcal{F}^{0b} + \bar{\omega}_{a0}^a\mathcal{F}^{0b} + \bar{\omega}_{0a}^0\mathcal{F}^{ab} \\ + \bar{\omega}_{ca}^c\mathcal{F}^{ab} + \bar{\omega}_{ca}^b\mathcal{F}^{ca} + [\mathcal{A}_a, \mathcal{F}^{ab}] &= 0, \end{aligned} \quad (3.3)$$

where $\nabla_0 = \frac{d}{dt}$ and ∇_a are the vector fields satisfying $\bar{e}^a(\nabla_b) = \delta_b^a$. We employ the flat metric $d\bar{s}^2$ on the cylinder, so Latin indices may be raised and lowered freely. The only effect of passing from $d\bar{s}^2$ to ds^2 is the occurrence of additional connection coefficients

$$\begin{aligned} \bar{\omega}_{00}^0 &= \tan t, \quad \bar{\omega}_{aa}^0 = \bar{\omega}_{0a}^a = n \tan t \quad (\text{sum over } a), \\ \bar{\omega}_{0b}^0 &= \bar{\omega}_{0b}^a = \bar{\omega}_{ab}^0 = 0 \end{aligned} \quad (3.4)$$

due to the conformal factor $\frac{R^2}{\cos^2 t}$. Inserting these into (3.3), the Yang-Mills equations take the form

$$\begin{aligned} \nabla_a\mathcal{F}^{a0} + \bar{\omega}_{ac}^a\mathcal{F}^{c0} + [\mathcal{A}_a, \mathcal{F}^{a0}] &= 0, \\ \nabla_0\mathcal{F}^{0b} + \nabla_a\mathcal{F}^{ab} + (n+1)\tan t\mathcal{F}^{0b} + \bar{\omega}_{ca}^c\mathcal{F}^{ab} + \bar{\omega}_{ca}^b\mathcal{F}^{ca} \\ + [\mathcal{A}_a, \mathcal{F}^{ab}] &= 0. \end{aligned} \quad (3.5)$$

Our aim is to find $\text{SO}(n+1)$, $\text{SU}(m+1)$, or $\text{Sp}(\ell+1)$ equivariant solutions to these equations.

B. Equivariant reduction on $\frac{\text{SO}(n+1)}{\text{SO}(n)}$

At this point we restrict ourselves to the canonical choice of $\text{SO}(n+1)$ for the gauge group, so the gauge potential \mathcal{A} and the field strength \mathcal{F} live in the adjoint representation of $\text{so}(n+1)$. Expanding in the generators $\{I_a, I_i\}$ and the one-forms $\{e^b\}$,

$$\mathcal{A} = I_i X^i_b e^b + I_a X^a_b e^b, \quad (3.6)$$

introduces a set $\{X^i_b, X^a_b\}$ of functions on $\mathcal{I} \times S^n$. In order to find explicit solutions to Eqs. (3.5), we impose $\text{SO}(n+1)$ equivariance on the gauge potential \mathcal{A} , which enforces

$$X^i_b = e^i_b \quad \text{and} \quad X^a_b = X^a_b(t) \quad \text{with} \quad f_{ia}{}^c X^a_b = f_{ib}{}^a X^c_a. \quad (3.7)$$

The first equation turns the first term in (3.6) into the canonical $\text{SO}(n)$ connection $I_i e^i$, while the last equation is more succinctly written in terms of n matrix functions $X_a(t) \in \mathfrak{m}$,

$$[I_i, X_a] = f_{ia}{}^b X_b \quad \text{for} \quad X_b := I_a X^a_b. \quad (3.8)$$

Hence, the equivariant gauge potential takes the form [9–12]

$$\mathcal{A} = I_i e^i + X_a(t) e^a \Leftrightarrow \mathcal{A}_a = I_i e^i_a + X_a(t). \quad (3.9)$$

The curvature of this connection is readily computed with the help of (3.8), and the equivariant gauge field reads

$$\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = \dot{X}_b e^0 \wedge e^b - \frac{1}{2} (f_{bc}{}^i I_i - [X_b, X_c]) e^b \wedge e^c \quad (3.10)$$

with the components

$$\mathcal{F}_{0b} = \dot{X}_b \quad \text{and} \quad \mathcal{F}_{ab} = -f_{ab}{}^i I_i + [X_a, X_b]. \quad (3.11)$$

Here, the overdot denotes a derivative with respect to t , i.e., $\dot{X}_a := \frac{dX_a}{dt}$.

Making use of (3.9) and (2.29), the first Yang-Mills equation in (3.5) reduces to

$$[X_a, \dot{X}^a] = 0, \quad (3.12)$$

while the second equation becomes

$$-\ddot{X}_a - (n-3) \tan t \dot{X}_a + \frac{1}{2} \varkappa_n X_a + [X_b, [X_b, X_a]] = 0, \quad (3.13)$$

where we have exploited the Jacobi identity and the normalization $f_{ba}{}^j f_{jb}{}^c = \frac{1}{2} \varkappa_n \delta_a{}^c$ of the Killing metric (2.15). We still have to solve the equivariance condition (3.8). Due to the decomposition

$$\text{adj}(\text{so}(n+1)) \rightarrow \text{adj}(\text{so}(n)) \oplus \mathfrak{n}, \quad (3.14)$$

with \mathfrak{n} denoting the $\text{so}(n)$ vector representation, there is only one free parameter in the general solution

$$X_a(t) = \phi(t) I_a, \quad (3.15)$$

where $\phi(t)$ is a real function of t . With this result, Eq. (3.12) is automatically satisfied, and the ordinary matrix differential equation (3.13) reduces to

$$\ddot{\phi} + (n-3) \tan t \dot{\phi} - \frac{1}{2} \varkappa_n (1 - \phi^2) \phi = 0. \quad (3.16)$$

Any solution $\phi(t)$ gives rise to a Yang-Mills field

$$\mathcal{F}_{0b} = \dot{\phi} I_b \quad \text{and} \quad \mathcal{F}_{ab} = (\phi^2 - 1) f_{ab}{}^i I_i. \quad (3.17)$$

Alternatively, one may evaluate the action functional on dS_{n+1} ,

$$\begin{aligned} S &= \frac{1}{4} \int_{dS_{n+1}} \text{tr}(\mathcal{F} \wedge * \mathcal{F}) \\ &= \frac{1}{8} \int_{\mathcal{I} \times S^n} \text{tr}(\mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu}) \left(\frac{R}{\cos t} \right)^{n+1} e^0 \wedge e^1 \wedge \dots \wedge e^n, \end{aligned} \quad (3.18)$$

where the factor $\left(\frac{R}{\cos t}\right)^{n+1}$ arises from the square root of the determinant of the metric (2.27). The trace is taken in the adjoint representation. Substituting the components (3.11) of the equivariant gauge field with $\mathcal{F}^{\mu\nu} = \left(\frac{R^2}{\cos^2 t}\right)^{-2} \mathcal{F}_{\mu\nu}$, we obtain ($\text{tr} = \text{tr}_{\text{adj}}$)

$$\begin{aligned} S &= \frac{1}{8} \int \left(\frac{R}{\cos t} \right)^{n-3} \text{tr}(-2\mathcal{F}_{0a} \mathcal{F}_{0a} + \mathcal{F}_{ab} \mathcal{F}_{ab}) \\ &\quad \times e^0 \wedge e^1 \wedge \dots \wedge e^n \\ &= \frac{1}{8} \int dt d^n \Omega_n \left(\frac{R}{\cos t} \right)^{n-3} (-2\dot{\phi}^2 \text{tr}(I_a I_a) \\ &\quad + (\phi^2 - 1)^2 f_{bc}{}^i f_{bc}{}^j \text{tr}(I_i I_j)) \\ &= \frac{1}{8} \varkappa_n \text{vol}(S^n) \int_{-\pi/2}^{\pi/2} dt \left(\frac{R}{\cos t} \right)^{n-3} \\ &\quad \times (2n\dot{\phi}^2 - (\phi^2 - 1)^2 f_{bc}{}^i f_{bc}{}^i) \\ &= \frac{1}{2} n \varkappa_n \text{vol}(S^n) \int_{-\pi/2}^{\pi/2} dt \left(\frac{R}{\cos t} \right)^{n-3} \\ &\quad \times \left(\frac{1}{2} \dot{\phi}^2 - \frac{1}{8} \varkappa_n (\phi^2 - 1)^2 \right), \end{aligned} \quad (3.19)$$

where we have used $f_{bc}^i f_{bc}^i = \frac{1}{2} \varkappa_n \delta_{bb} = \frac{1}{2} \varkappa_n n$ from (2.15). It is straightforward to see that the variation with respect to ϕ reproduces the equation of motion (3.16). To summarize, $\text{SO}(n+1)$ equivariance turns the Yang-Mills equations to the Newtonian dynamics of a particle on \mathbb{R} with time-dependent friction $\sim(n-3) \tan t$ and subject to the double-well potential

$$V(\phi) = \frac{1}{8} \varkappa_n (\phi^2 - 1)^2. \quad (3.20)$$

Finally, we compute the energy \mathcal{E} of our classical Yang-Mills configurations on dS_{n+1} , dual to de Sitter time τ . Mapping again to $\mathcal{I} \times S^n$ and fixing $t(\tau)$, the result is

$$\begin{aligned} \mathcal{E} &= -\frac{1}{4} \left(\frac{R}{\cos t} \right)^{n-4} \int_{S^n} \text{d}^n \Omega_n \text{tr} \left(\mathcal{F}_{0a} \mathcal{F}_{0a} + \frac{1}{2} \mathcal{F}_{ab} \mathcal{F}_{ab} \right) \\ &= \frac{1}{2} n \varkappa_n \text{vol}(S^n) (R \cosh \tau)^{n-4} \left(\frac{1}{2} \dot{\phi}^2 + \frac{1}{8} \varkappa_n (1 - \phi^2)^2 \right) \Big|_{t=t(\tau)}. \end{aligned} \quad (3.21)$$

For the Abelian case of $n=1$, the potential vanishes because $\varkappa_1 = 0$. The general solution to (3.16) is

$$\phi(t) = c \tan t + d \Rightarrow \mathcal{F}_{01} = \frac{c}{\cos^2 t} I_1, \quad (3.22)$$

which yields a singular action integral except for the trivial constant solution.

C. Equivariant reduction on $\frac{\text{SU}(m+1)}{\text{SU}(m)}$

In this subsection, we turn our attention to $\text{SU}(m+1)$ -equivariant gauge fields on odd-dimensional spheres and fix the gauge group to be $\text{SU}(m+1)$. We pass from the orthogonal to the unitary coset by putting tildes over most symbols. The $\text{SU}(m+1)$ equivariant connection takes the form

$$\mathcal{A} = \tilde{I}_i \tilde{e}^i + \tilde{X}_a(t) \tilde{e}^a \Leftrightarrow \mathcal{A}_a = \tilde{I}_i \tilde{e}^i{}_a + \tilde{X}_a(t), \quad (3.23)$$

where the $2m+1$ matrix functions $\tilde{X}_a(t)$ are subject to the equivariance constraint

$$[\tilde{I}_i, \tilde{X}_a] = \tilde{f}_{ia}{}^b \tilde{X}_b. \quad (3.24)$$

The connection (3.23) gives rise to the curvature

$$\mathcal{F} = \dot{\tilde{X}}_b \tilde{e}^0 \wedge \tilde{e}^b - \frac{1}{2} (\tilde{f}_{bc}{}^i \tilde{I}_i + \tilde{f}_{bc}{}^a \tilde{X}_a - [\tilde{X}_b, \tilde{X}_c]) \tilde{e}^b \wedge \tilde{e}^c \quad (3.25)$$

with the components

$$\mathcal{F}_{0b} = \dot{\tilde{X}}_b \quad \text{and} \quad \mathcal{F}_{ab} = -\tilde{f}_{ab}{}^i \tilde{I}_i - \tilde{f}_{ab}{}^c \tilde{X}_c + [\tilde{X}_a, \tilde{X}_b]. \quad (3.26)$$

While the first Yang-Mills equation in (3.5) yields

$$[\tilde{X}_a, \dot{\tilde{X}}^a] = 0 \quad (3.27)$$

the second equation descends to

$$\begin{aligned} -\ddot{\tilde{X}}_a - 2(m-1) \tan t \dot{\tilde{X}}_a + \left(\tilde{f}_{da}{}^i \tilde{f}_{id}{}^b - \frac{1}{2} \tilde{f}_{dc}{}^a \tilde{f}_{dc}{}^b \right) \tilde{X}_b \\ + \frac{3}{2} \tilde{f}_{bc}{}^a [\tilde{X}_b, \tilde{X}_c] + [\tilde{X}_b, [\tilde{X}_b, \tilde{X}_a]] = 0, \end{aligned} \quad (3.28)$$

again with the help of the Jacobi identity on the structure constants. In contrast to the previous subsection, our coset is no longer symmetric, which is manifested by $\tilde{f}_{ab}{}^c \neq 0$. So to evaluate the coefficient of \tilde{X}_b in (3.28), we need more information on the $\text{SU}(m+1)$ structure constants. Since the coset $\frac{\text{SU}(m+1)}{\text{SU}(m)}$ is an α -Sasakian manifold (with $\alpha = -\frac{1}{2} \tilde{\varkappa}_m$ in our normalization), the contact one-form defines a preferred direction, which we associate with $a=1$, and we may single out the corresponding \mathfrak{m} generator as special, thus subdividing

$$\{\tilde{I}_a\} = \{\tilde{I}_1\} \cup \{\tilde{I}_{a'}\} \quad \text{with} \quad a' = 2, \dots, 2m+1. \quad (3.29)$$

The structure constants can be chosen such that

$$\tilde{f}_{a'b'}{}^i \neq 0 \quad \text{and} \quad \tilde{f}_{a'b'}{}^1 \neq 0 \quad \text{but} \quad \tilde{f}_{ib'}{}^1 = \tilde{f}_{a'b'}{}^{c'} = 0. \quad (3.30)$$

The middle equation in (2.21) then splits into three parts,

$$\begin{aligned} \tilde{f}_{c'd'}{}^1 \tilde{f}_{c'd'}{}^1 = \tilde{\varkappa}_m \Rightarrow \tilde{f}_{c'1}{}^{a'} \tilde{f}_{c'1}{}^{b'} = \frac{1}{2m} \tilde{\varkappa}_m \delta^{a'b'} \quad \text{and} \\ \tilde{f}_{c'i}{}^{a'} \tilde{f}_{c'i}{}^{b'} = \frac{m-1}{2m} \tilde{\varkappa}_m \delta^{a'b'}. \end{aligned} \quad (3.31)$$

As a consequence, Eq. (3.28) simplifies to two different forms,

$$\begin{aligned} -\ddot{\tilde{X}}_{a'} - 2(m-1) \tan t \dot{\tilde{X}}_{a'} + \frac{m-2}{2m} \tilde{\varkappa}_m \tilde{X}_{a'} + 3\tilde{f}_{1b'}{}^{a'} [\tilde{X}_1, \tilde{X}_{b'}] \\ + [\tilde{X}_{b'}, [\tilde{X}_{b'}, \tilde{X}_{a'}]] + [\tilde{X}_1, [\tilde{X}_1, \tilde{X}_{a'}]] = 0, \\ -\ddot{\tilde{X}}_1 - 2(m-1) \tan t \dot{\tilde{X}}_1 - \frac{1}{2} \tilde{\varkappa}_m \tilde{X}_1 + \frac{3}{2} \tilde{f}_{b'c'}{}^1 [\tilde{X}_{b'}, \tilde{X}_{c'}] \\ + [\tilde{X}_{b'}, [\tilde{X}_{b'}, \tilde{X}_1]] = 0. \end{aligned} \quad (3.32)$$

This is consistent with the decomposition

$$\text{adj}(su(m+1)) \rightarrow \text{adj}(su(m)) \oplus \mathfrak{m} \oplus \bar{\mathfrak{m}} \oplus \mathbf{1}, \quad (3.33)$$

which implies that the general solution to the equivariance constraint (3.24) contains two free parameters: one for the fundamental $\text{su}(m)$ representation \mathbf{m} and its complex conjugate $\bar{\mathbf{m}}$ (\mathcal{A} is anti-Hermitian), indexed by a' , and one for the singlet $\mathbf{1}$, indexed by 1. Hence, the matrix functions $X_a(t)$ take the form¹

$$\tilde{X}_{a'}(t) = \varphi(t)\tilde{I}_{a'} \quad \text{and} \quad \tilde{X}_1(t) = \sqrt{2m}\psi(t)\tilde{I}_1 \quad (3.34)$$

where $\varphi(t)$ and $\psi(t)$ are real functions of t . This form automatically fulfills (3.12), and the two ordinary matrix differential equations (3.32) become

$$\begin{aligned} -\ddot{\varphi} - 2(m-1)\tan t\dot{\varphi} + \frac{m-2}{2m}\tilde{x}_m\varphi + \frac{3}{\sqrt{2m}}\tilde{x}_m\varphi\psi - \frac{1}{2}\tilde{x}_m\varphi^3 - \tilde{x}_m\varphi\psi^2 &= 0, \\ -\ddot{\psi} - 2(m-1)\tan t\dot{\psi} - \frac{1}{2}\tilde{x}_m\psi + \frac{3/2}{\sqrt{2m}}\tilde{x}_m\varphi^2 - \tilde{x}_m\psi\varphi^2 &= 0. \end{aligned} \quad (3.35)$$

For any solution $(\varphi(t), \psi(t))$, we gain a Yang-Mills configuration

$$\begin{aligned} \mathcal{F}_{0b'} &= \dot{\varphi}\tilde{I}_{b'} \quad \text{and} \quad \mathcal{F}_{a'b'} = (\varphi^2 - 1)\tilde{f}_{a'b'}^i\tilde{I}_i + (\varphi^2 - \sqrt{2m}\psi)\tilde{f}_{a'b'}^1\tilde{I}_1, \\ \mathcal{F}_{01} &= \sqrt{2m}\dot{\psi}\tilde{I}_1 \quad \text{and} \quad \mathcal{F}_{a'1} = \varphi(\sqrt{2m}\psi - 1)\tilde{f}_{a'1}^{b'}\tilde{I}_{b'}. \end{aligned} \quad (3.36)$$

Alternatively, the Yang-Mills action on dS_{2m+2} with the input (3.34) can be computed as ($\text{tr} = \text{tr}_{\text{adj}}$)

$$\begin{aligned} S &= \frac{1}{4} \int_{\mathcal{I} \times S^{2m+1}} \text{tr}(F \wedge *F) \\ &= \frac{1}{8} \int dt d^{2m+1} \Omega_{2m+1} \left(\frac{R}{\cos t} \right)^{2m-2} \{ (-2\dot{\varphi}^2 \text{tr}(\tilde{I}_{a'}\tilde{I}_{a'}) - 4m\dot{\psi}^2 \text{tr}(\tilde{I}_1\tilde{I}_1) + (1 - \varphi^2)^2 \tilde{f}_{b'c'}^i \tilde{f}_{b'c'}^j \text{tr}(\tilde{I}_i\tilde{I}_j) \\ &\quad + (\sqrt{2m}\psi - \varphi^2)^2 \tilde{f}_{b'c'}^1 \tilde{f}_{b'c'}^1 \text{tr}(\tilde{I}_1\tilde{I}_1) + 2\varphi^2(1 - \sqrt{2m}\psi)^2 \tilde{f}_{1c'}^{a'} \tilde{f}_{1c'}^{d'} \text{tr}(\tilde{I}_{a'}\tilde{I}_{d'}) \} \\ &\propto \int dt \left(\frac{R}{\cos t} \right)^{2m-2} \{ 4m\dot{\varphi}^2 + 4m\dot{\psi}^2 - (m-1)\tilde{x}_m(1 - \varphi^2)^2 - \tilde{x}_m(\sqrt{2m}\psi - \varphi^2)^2 - 2\tilde{x}_m\varphi^2(1 - \sqrt{2m}\psi)^2 \}. \end{aligned} \quad (3.37)$$

Variation with respect to φ and ψ properly yields the equations of motion (3.35). This time, we find a two-dimensional Newtonian dynamics with time-dependent friction. Dividing by $8m$, the external potential is extracted as

$$\begin{aligned} \tilde{x}_m^{-1}V(\varphi, \psi) &= \frac{m-1}{8m}(1 - \varphi^2)^2 + \frac{1}{4m}\varphi^2(1 - \sqrt{2m}\psi)^2 + \frac{1}{8m}(\sqrt{2m}\psi - \varphi^2)^2 \\ &= \frac{m-1}{8m} - \frac{m-2}{4m}\varphi^2 + \frac{1}{4}\psi^2 - \frac{3/2}{\sqrt{2m}}\varphi^2\psi + \frac{1}{8}\varphi^4 + \frac{1}{2}\varphi^2\psi^2. \end{aligned} \quad (3.38)$$

Contour plots for $m = 1$ and $m = 2$ are displayed in Fig. 1. Finally, the de Sitter energy reads

$$\mathcal{E} = m\tilde{x}_m \text{vol}(S^{2m+1})(R \cosh \tau)^{2m-3} \left(\frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}\dot{\psi}^2 + V(\varphi, \psi) \right) \Big|_{t=t(\tau)}. \quad (3.39)$$

D. Equivariant reduction on $\frac{\text{Sp}(\ell+1)}{\text{Sp}(\ell)}$

Finally, we look at equivariant solutions for the symplectic gauge group $\text{Sp}(\ell+1)$. Equations (3.23)–(3.27) of the previous subsection carry over with tildes exchanged for hats, and the matrix differential equations read

$$-\hat{X}_a - 4\ell \tan t \hat{X}_a + \left(\hat{f}_{da}^i \hat{f}_{id}^b - \frac{1}{2} \hat{f}_{dc}^a \hat{f}_{dc}^b \right) \hat{X}_b + \frac{3}{2} \hat{f}_{bc}^a [\hat{X}_b, \hat{X}_c] + [\hat{X}_b, [\hat{X}_b, \hat{X}_a]] = 0. \quad (3.40)$$

¹The prefactor $\sqrt{2m}$ is chosen for later convenience.

At this stage, we exploit the fact that $\frac{\text{Sp}(\ell+1)}{\text{Sp}(\ell)}$ is α -tri-Sasakian, which defines a special subalgebra in $\hat{\mathfrak{m}}$ that we span with the generators \hat{I}_1, \hat{I}_2 , and \hat{I}_3 , thus splitting

$$\{\hat{I}_a\} = \{\hat{I}_{a''}\} \cup \{\hat{I}_{a'}\} \quad \text{with} \quad a'' = 1, 2, 3 \quad \text{and} \quad a' = 4, \dots, 4\ell + 3. \quad (3.41)$$

The structure constants can be arranged such that

$$\hat{f}_{a'b'}^i \neq 0 \quad \text{and} \quad \hat{f}_{a'b'}^{c''} \neq 0 \quad \text{and} \quad \hat{f}_{12}^3 \neq 0 \quad \text{but} \quad \hat{f}_{ib'}^{c''} = \hat{f}_{ib''}^{c''} = \hat{f}_{a'b''}^{c''} = \hat{f}_{a'b'}^{c'} = 0. \quad (3.42)$$

We normalize to unit radius the $S^3 \simeq \text{Sp}(1)$ generated by $\{\hat{I}_{a''}\}$ by taking

$$d\Omega_3^2 = \delta_{a''b''} \hat{e}^{a''} \hat{e}^{b''} \Leftrightarrow \hat{f}_{a''d''}^{c''} \hat{f}_{c''b''}^{d''} = \hat{\chi}_0 \delta_{a''b''} \quad \text{with} \quad \hat{\chi}_0 = 8 \Leftrightarrow \hat{f}_{a''b''}^{c''} = 2\epsilon_{a''b''}^{c''}. \quad (3.43)$$

The middle equation in (2.21) with hats instead of tildes then determines the following partial sums:

$$\begin{aligned} \hat{f}_{c'd''}^{a''} \hat{f}_{c''d''}^{b''} &= \frac{2\hat{\chi}_\ell}{(\ell+2)} \delta^{a''b''}, & \hat{f}_{c'd'}^{a''} \hat{f}_{c'd'}^{b''} &= \frac{\ell\hat{\chi}_\ell}{(\ell+2)} \delta^{a''b''}, \\ \hat{f}_{c'd''}^{a'} \hat{f}_{c'd''}^{b'} &= \frac{3\hat{\chi}_\ell}{4(\ell+2)} \delta^{a'b'}, & \hat{f}_{c'i}^{a'} \hat{f}_{c'i}^{b'} &= \frac{(2\ell+1)\hat{\chi}_\ell}{4(\ell+2)} \delta^{a'b'}. \end{aligned} \quad (3.44)$$

After this splitting, the matrix differential equations (3.40) read

$$\begin{aligned} -\ddot{\hat{X}}_{a'} - 4\ell \tan t \dot{\hat{X}}_{a'} + \frac{(\ell-1)}{2(\ell+2)} \hat{\chi}_\ell \hat{X}_{a'} + 3\hat{f}_{b'c''}^{a'} [\hat{X}_{b'}, \hat{X}_{c''}] + [\hat{X}_{b'}, [\hat{X}_{b'}, \hat{X}_{a'}]] + [\hat{X}_{b''}, [\hat{X}_{b''}, \hat{X}_{a'}]] &= 0, \\ -\ddot{\hat{X}}_{a''} - 4\ell \tan t \dot{\hat{X}}_{a''} - \frac{1}{2} \hat{\chi}_\ell \hat{X}_{a''} + \frac{3}{2} \hat{f}_{bc}^{a''} [\hat{X}_b, \hat{X}_c] + [\hat{X}_b, [\hat{X}_b, \hat{X}_{a''}]] &= 0, \end{aligned} \quad (3.45)$$

where in the last two terms $\{b, c\} = \{b', c'\} \cup \{b'', c''\}$, of course. When we consider the decomposition

$$\text{adj}(sp(\ell+1)) \rightarrow (\text{adj}(sp(\ell)), \mathbf{1}) \oplus (\mathbf{2}\ell, \mathbf{2}) \oplus (\mathbf{1}, \text{adj}(sp(1))), \quad (3.46)$$

of $sp(\ell+1)$ irreps into $sp(\ell) \otimes sp(1)$ ones, we see that again there are two free parameters in the equivariance condition (3.24): one for the 4ℓ -dimensional fundamental representation indexed by a' , and one for the three-dimensional adjoint representation of $sp(1)$, indexed by a'' . Bearing this in mind, the equivariant form of $\hat{X}_a(t)$ is²

$$\hat{X}_{a'}(t) = \sqrt{3}\varphi(t)\hat{I}_{a'} \quad \text{and} \quad \hat{X}_{a''}(t) = \sqrt{4\ell}\psi(t)\hat{I}_{a''}. \quad (3.47)$$

It automatically satisfies the condition (3.27) and reduces the matrix differential equations (3.45) to two coupled differential equations,

$$\begin{aligned} -\ddot{\varphi} - 4\ell \tan t \dot{\varphi} + \frac{\ell-1}{2(\ell+2)} \hat{\chi}_\ell \varphi + \frac{9\sqrt{\ell}}{2(\ell+2)} \hat{\chi}_\ell \varphi \psi - \frac{3}{2} \hat{\chi}_\ell \varphi^3 - \frac{3\ell}{\ell+2} \hat{\chi}_\ell \varphi \psi^2 &= 0, \\ -\ddot{\psi} - 4\ell \tan t \dot{\psi} - \frac{1}{2} \hat{\chi}_\ell \psi + \frac{6\sqrt{\ell}}{\ell+2} \hat{\chi}_\ell \psi^2 + \frac{9\sqrt{\ell}}{4(\ell+2)} \hat{\chi}_\ell \varphi^2 - \frac{8\ell}{\ell+2} \hat{\chi}_\ell \psi^3 - \frac{3\ell}{\ell+2} \hat{\chi}_\ell \varphi \psi^2 &= 0. \end{aligned} \quad (3.48)$$

For any solution (φ, ψ) to these equations, we obtain a Yang-Mills configuration

$$\begin{aligned} \mathcal{F}_{0b'} &= \sqrt{3}\dot{\varphi}\hat{I}_{b'} \quad \text{and} \quad \mathcal{F}_{a'b'} = (3\varphi^2 - 1)\hat{f}_{a'b'}^i \hat{I}_i + (3\varphi^2 - \sqrt{4\ell}\psi)\hat{f}_{a'b'}^{a''} \hat{I}_{a''}, \\ \mathcal{F}_{0a''} &= \sqrt{4\ell}\dot{\psi}\hat{I}_{a''}, \quad \mathcal{F}_{a'b''} = \sqrt{3}\varphi(\sqrt{4\ell}\psi - 1)\hat{f}_{a'b''}^{c'} \hat{I}_{c'}, \quad \mathcal{F}_{a''b''} = \sqrt{4\ell}\psi(\sqrt{4\ell}\psi - 1)\hat{f}_{a''b''}^{c''} \hat{I}_{c''}. \end{aligned} \quad (3.49)$$

²The functions $\varphi(t)$ and $\psi(t)$ in this subsection are different from those in the previous one. We choose the prefactors for later convenience.

The action functional on de Sitter space $dS_{4\ell+4}$ with the ansatz (3.47) can be computed as

$$\begin{aligned}
 S &= \frac{1}{4} \int_{\mathcal{I} \times S^{4\ell+3}} \text{tr}(F \wedge *F) \\
 &= \frac{1}{8} \int dt d^{4\ell+3} \Omega_{4\ell+3} \left(\frac{R}{\cos t} \right)^{4\ell} \{ (-6\dot{\varphi}^2 \text{tr}(\hat{I}_{a'} \hat{I}_{a'}) - 8\ell \dot{\psi}^2 \text{tr}(\hat{I}_{a''} \hat{I}_{a''}) + (1 - 3\varphi^2)^2 \hat{f}_{b'c'}^i \hat{f}_{b'c'}^j \text{tr}(\hat{I}_i \hat{I}_j) \\
 &\quad + (\sqrt{4\ell}\psi - 3\varphi^2)^2 \hat{f}_{b'c'}^{a''} \hat{f}_{b'c'}^{b''} \text{tr}(\hat{I}_{a''} \hat{I}_{b''}) + 6\varphi^2 (1 - \sqrt{4\ell}\psi)^2 \hat{f}_{b''c'}^d \hat{f}_{b''c'}^d \text{tr}(\hat{I}_{a'} \hat{I}_{d'}) \\
 &\quad + 4\ell \psi^2 (1 - \sqrt{4\ell}\psi)^2 \hat{f}_{b''c''}^{a''} \hat{f}_{b''c''}^{d''} \text{tr}(\hat{I}_{a''} \hat{I}_{d''}) \} \\
 &\propto \int dt \left(\frac{R}{\cos t} \right)^{4\ell} \left\{ 24\ell \dot{\varphi}^2 + 24\ell \dot{\psi}^2 - \frac{\ell(2\ell+1)}{\ell+2} \hat{\chi}_\ell (1 - 3\varphi^2)^2 - \frac{3\ell}{\ell+2} \hat{\chi}_\ell (\sqrt{4\ell}\psi - 3\varphi^2)^2 \right. \\
 &\quad \left. - \frac{6\ell}{\ell+2} \hat{\chi}_\ell (3\varphi^2 + 4\psi^2) (1 - \sqrt{4\ell}\psi)^2 \right\} \quad (3.50)
 \end{aligned}$$

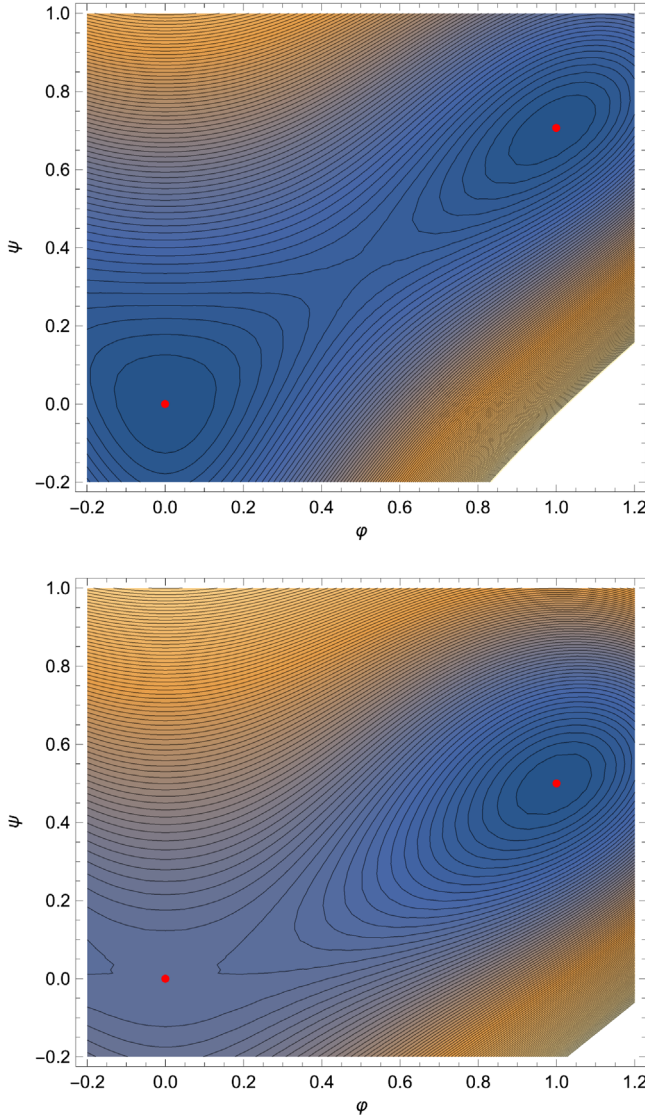


FIG. 1. Newtonian potential contours for SU(2) (top) and $\frac{\text{SU}(3)}{\text{SU}(2)}$ (bottom).

where again $\text{tr} = \text{tr}_{\text{adj}}$. It is possible to verify the equations of motions (3.48) by varying this action with respect to φ and ψ . Dividing by 48ℓ , the corresponding external potential is read off as

$$\begin{aligned}
 \frac{1}{\hat{\chi}_\ell} V(\varphi, \psi) &= \frac{2\ell+1}{48(\ell+2)} (1 - 3\varphi^2)^2 \\
 &\quad + \frac{1}{16(\ell+2)} (\sqrt{4\ell}\psi - 3\varphi^2)^2 \\
 &\quad + \frac{1}{8(\ell+2)} (3\varphi^2 + 4\psi^2) (1 - \sqrt{4\ell}\psi)^2 \\
 &= \frac{2\ell+1}{48(\ell+2)} - \frac{\ell-1}{4(\ell+2)} \varphi^2 + \frac{1}{4} \psi^2 \\
 &\quad - \frac{9\sqrt{\ell}}{4(\ell+2)} \varphi^2 \psi - \frac{2\sqrt{\ell}}{\ell+2} \psi^3 + \frac{3}{8} \varphi^4 \\
 &\quad + \frac{3\ell}{2(\ell+2)} \varphi^2 \psi^2 + \frac{2\ell}{\ell+2} \psi^4. \quad (3.51)
 \end{aligned}$$

Figure 2 shows contour plots for $\ell = 1$ and $\ell = 2$. The energy of these Yang-Mills configurations on de Sitter space $dS_{4\ell+4}$ is computed as

$$\begin{aligned}
 \mathcal{E} &= 6\ell \hat{\chi}_\ell \text{vol}(S^{4\ell+3}) (R \cosh \tau)^{4\ell-1} \\
 &\quad \times \left(\frac{1}{2} \dot{\varphi}^2 + \frac{1}{2} \dot{\psi}^2 + V(\varphi, \psi) \right) \Big|_{t=\tau}. \quad (3.52)
 \end{aligned}$$

IV. FINITE-ACTION YANG-MILLS SOLUTIONS ON dS_{n+1}

A. Conformal factor and friction

In the previous section, we have shown that equivariance with respect to the isometry group of the coset reduces the Yang-Mills equations on dS_{n+1} to Newtonian mechanics of a particle in a particular potential in one or two space dimensions with an additional friction term $\sim \tan t \dot{\phi}$ or $\sim \tan t(\dot{\varphi}, \dot{\psi})$, respectively. Away from $n = 3$, the general

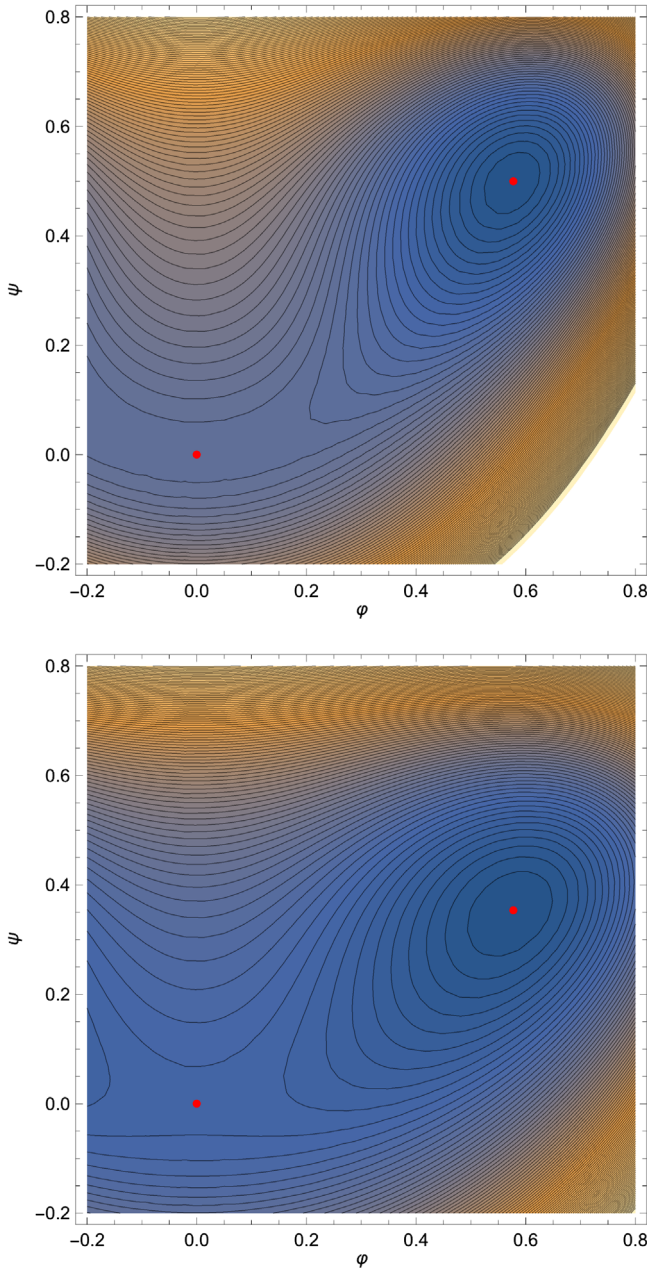


FIG. 2. Newtonian potential contours for $\frac{\text{Sp}(2)}{\text{Sp}(1)}$ (top) and $\frac{\text{Sp}(3)}{\text{Sp}(2)}$ (bottom).

solution to the nonlinear differential equations (3.16), (3.35), or (3.48) is not known, and the only analytic solutions are the constant ones, with the analog particle sitting at a local extremum of the potential. For nonconstant solutions, it is possible to figure out the behavior of ϕ [or (ϕ, ψ)] and of the action numerically and asymptotically near the boundary $t = \pm \frac{\pi}{2}$. One finds that

$$\phi\left(\frac{\pi}{2} - \epsilon\right) \stackrel{\epsilon \rightarrow 0}{\sim} \begin{cases} \epsilon^{-1} & \text{for } n = 1 \\ \log \epsilon & \text{for } n = 2, \\ \text{constant} + \epsilon^{n-2} & \text{for } n \geq 3 \end{cases} \quad (4.1)$$

and likewise for (ϕ, ψ) . Hence, the friction term leads to a blowing-up for $n < 3$ (negative friction) and to a freezing for $n > 3$ (positive friction). For $n > 3$, the friction is harmless, but the action (3.19) is infinite due to the conformal factor $\left(\frac{R}{\cos t}\right)^{n-3}$, unless we deal with just the vacuum solution. For $n < 3$ the conformal factor is benign but the blowing-up at the boundary $t = \pm \frac{\pi}{2}$ renders the action (3.19) divergent, unless ϕ or (ϕ, ψ) is constant. The $n = 3$ case is very special because the friction term always vanishes, and the particle solutions in a double-well potential are known explicitly in terms of elliptic functions. Here, both the action and energy are finite.

B. Vacuum and purely magnetic solutions

As is exemplified in Figs. 1 and 2, the vacuum solutions ($V = 0$) in the three cases are given by

$$\begin{aligned} \phi_{\text{vac}} = \pm 1 \quad \text{or} \quad (\phi, \psi)_{\text{vac}} = \left(\pm 1, \frac{1}{\sqrt{2m}}\right) \quad \text{or} \\ (\phi, \psi)_{\text{vac}} = \left(\pm \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{4\ell}}\right), \end{aligned} \quad (4.2)$$

and we do not consider them further because they correspond to the pure-gauge configuration

$$\mathcal{A} = \bar{I}_A \bar{e}^A \Rightarrow \mathcal{F} \equiv 0. \quad (4.3)$$

The only nontrivial analytic solution to (3.16), (3.35), or (3.48) in any dimension is the constant solution

$$\phi_{\text{mag}} = 0 \quad \text{or} \quad (\phi, \psi)_{\text{mag}} = (0, 0), \quad (4.4)$$

respectively. It corresponds to the analog particle sitting on a local maximum or on a saddle point (for $m \geq 2$) of the potential, respectively (see Figs. 1 and 2). Since $\mathcal{F}_{0b} = 0$, the ensuing gauge field is purely magnetic, with

$$\begin{aligned} \mathcal{A}_a = I_i e^i{}_a \Rightarrow \mathcal{F}_{ab} = -f_{ab}{}^i I_i, \\ \mathcal{A}_a = \tilde{I}_i \tilde{e}^i{}_a \Rightarrow \mathcal{F}_{a'b'} = -\tilde{f}_{a'b'}{}^i \tilde{I}_i \quad \text{and} \quad \mathcal{F}_{a''1} = 0, \\ \mathcal{A}_a = \hat{I}_i \hat{e}^i{}_a \Rightarrow \mathcal{F}_{a'b'} = -\hat{f}_{a'b'}{}^i \hat{I}_i \quad \text{and} \quad \mathcal{F}_{a''b''} = 0 = \mathcal{F}_{a''b''}. \end{aligned} \quad (4.5)$$

Note that \mathcal{F} is valued in the stabilizer subgroup. Its action (3.19), (3.37), or (3.50) becomes

$$\begin{aligned} S &= -\frac{1}{8} \frac{n}{2} \chi_n^2 \text{vol}(S^n) \int_{-\pi/2}^{\pi/2} dt \left(\frac{R}{\cos t}\right)^{n-3}, \\ S &= -\frac{1}{8} (m-1) \tilde{\chi}_m^2 \text{vol}(S^{2m+1}) \int_{-\pi/2}^{\pi/2} dt \left(\frac{R}{\cos t}\right)^{2m-2}, \\ S &= -\frac{1}{8} \frac{\ell(2\ell+1)}{\ell+2} \hat{\chi}_\ell^2 \text{vol}(S^{4\ell+3}) \int_{-\pi/2}^{\pi/2} dt \left(\frac{R}{\cos t}\right)^{4\ell}, \end{aligned} \quad (4.6)$$

respectively, which is finite only for $n \leq 3$ or $m \leq 1$. Finally, the de Sitter energy is given by

$$\begin{aligned}\mathcal{E} &= \frac{1}{8} \frac{n}{2} \chi_n^2 \text{vol}(S^n) (R \cosh \tau)^{n-4}, \\ \mathcal{E} &= \frac{1}{8} (m-1) \tilde{\chi}_m^2 \text{vol}(S^{2m+1}) (R \cosh \tau)^{2m-3}, \\ \mathcal{E} &= \frac{1}{8} \frac{\ell(2\ell+1)}{\ell+2} \tilde{\chi}_\ell^2 \text{vol}(S^{4\ell+3}) (R \cosh \tau)^{4\ell-1}.\end{aligned}\quad (4.7)$$

The exception is $S^3 = \text{SU}(2)$ (the case $m = 1$), where $(\varphi, \psi) = (0, 0)$ is again a vacuum solution. Since for $m = 1$ the index i takes no values, the field strength completely vanishes, as do the action and energy. However, as can be seen in the top half of Fig. 1, in this case, the potential admits two saddle points half-way between the vacuum solutions,

$$(\varphi, \psi)_{\text{mag}}^{m=1} = \left(\pm \frac{1}{2}, \frac{1}{2\sqrt{2}} \right), \quad (4.8)$$

which yield the configurations

$$\begin{aligned}\mathcal{A}_{a'} &= \pm \frac{1}{2} \tilde{I}_{a'} \quad \text{and} \quad \mathcal{A}_1 = \frac{1}{2} \tilde{I}_1 \Rightarrow \mathcal{A} = \frac{1}{2} \tilde{I}_a \tilde{e}^a \\ \Rightarrow \mathcal{F} &= -\frac{1}{8} \tilde{f}_{ab}{}^c \tilde{I}_c \tilde{e}^a \wedge \tilde{e}^b,\end{aligned}\quad (4.9)$$

i.e., just one-half of the canonical $\text{SU}(2)$ connection. For comparison, the orthogonal coset representation of the same space (the case $n = 3$) yields precisely the canonical $\text{SO}(3)$ connection inside $\text{SO}(4)$ as in (4.5).

With the values

$$\begin{aligned}\chi_2 = 2 &\Leftrightarrow f_{ab}{}^3 = \epsilon_{ab}, & \chi_3 = 4 &\Leftrightarrow f_{ab}{}^i = \epsilon_{ab}{}^{i-3}, \\ \tilde{\chi}_1 = 8 &\Leftrightarrow \tilde{f}_{ab}{}^c = 2\epsilon_{ab}{}^c,\end{aligned}\quad (4.10)$$

the values for the action and the energy of the magnetic Yang-Mills solutions on dS_3 and dS_4 (the orthogonal and the unitary coset) are displayed in the table below.

Space	Value of the action S	Value of the energy \mathcal{E}
dS_3	$-\frac{2 \cdot 2^2}{16} \text{vol}(S^2) \frac{2}{R} = -4\pi/R$	$2\pi(R \cosh \tau)^{-2}$
$\text{dS}_4^{\text{orth}}$	$-\frac{3 \cdot 4^2}{16} \text{vol}(S^3) \pi = -6\pi^3$	$6\pi^2(R \cosh \tau)^{-1}$
$\text{dS}_4^{\text{unit}}$	$-\frac{3 \cdot 8^2}{128} \text{vol}(S^3) \pi = -3\pi^3$	$3\pi^2(R \cosh \tau)^{-1}$

C. General equivariant dS_4 solutions

As already mentioned, for dS_4 all equivariant Yang-Mills solutions enjoy finite action and energy, for gauge group $\text{SO}(4)$ (case $n = 3$) as well as for gauge group $\text{SU}(2)$ (case $m = 1$). We have the $\text{SO}(4)$ - and $\text{SU}(2)$ -equivariant connections

$$\mathcal{A} = I_i e^i + \phi I_a e^a \quad \text{and} \quad \mathcal{A} = \varphi \tilde{I}_{a'} \tilde{e}^{a'} + \sqrt{2} \psi \tilde{I}_1 \tilde{e}^1 \quad (4.11)$$

which with (4.10) produce the fields

$$\begin{aligned}\mathcal{F} &= \dot{\phi} I_a e^0 \wedge e^a + \frac{1}{2} (\phi^2 - 1) \epsilon_{ab}{}^{i-3} I_i e^a \wedge e^b \quad \text{and} \\ \mathcal{F} &= \dot{\varphi} \tilde{I}_{a'} e^0 \wedge \tilde{e}^{a'} + \sqrt{2} \dot{\psi} \tilde{I}_1 e^0 \wedge \tilde{e}^1 \\ &\quad + (\varphi^2 - \sqrt{2} \psi) \epsilon_{a'b'}{}^1 \tilde{I}_1 \tilde{e}^{a'} \wedge \tilde{e}^{b'} \\ &\quad + 2\varphi (1 - \sqrt{2} \psi) \epsilon_{1a'b'} \tilde{I}_{b'} \tilde{e}^{a'} \wedge \tilde{e}^1,\end{aligned}\quad (4.12)$$

respectively. The Yang-Mills equations boil down to

$$\begin{aligned}\ddot{\phi} &= 2\phi - 2\phi^3 = -\frac{\partial V}{\partial \phi} \quad \text{with} \quad V = \frac{1}{2} (\phi^2 - 1) \quad \text{and} \\ \left\{ \begin{aligned} \ddot{\phi} &= -4\phi + 12\sqrt{2}\phi\psi - 4\phi^3 - 8\phi\psi^2 = -\frac{\partial V}{\partial \phi} \\ \ddot{\psi} &= -4\psi + 6\sqrt{2}\phi^2 - 8\psi\phi^2 = -\frac{\partial V}{\partial \psi} \end{aligned} \right\} \quad \text{with} \\ V &= 2\phi^2 (\sqrt{2}\psi - 1)^2 + (\sqrt{2}\psi - \phi^2)^2,\end{aligned}\quad (4.13)$$

respectively.

Let us first look at the unitary case. Besides the constant solutions discussed above, one finds two types of non-constant analytic solutions easily visible in the top contour plot of Fig. 1,

$$\begin{aligned}\text{Abelian : } &(\varphi, \psi) = (0, c \cos 2(t - t_0)) \quad \text{with} \quad c, t_0 \in \mathbb{R}, \\ \text{Non-Abelian : } &(\varphi, \psi) = \left(\frac{1}{2} (1 + \phi), \frac{1}{2\sqrt{2}} (1 + \phi) \right) \\ &\quad \text{with} \quad \dot{\phi} = 2\phi - 2\phi^3.\end{aligned}\quad (4.14)$$

Both types of solutions and the related Yang-Mills fields have been discussed in [2,3]. Interestingly, the ‘‘Newton equation’’ for $\phi(t)$ above is the same as in the orthogonal case, presented in the first line of (4.13). Its general solution is given in terms of elliptic functions, except for the bounce solution

$$\phi(t) = \sqrt{2} \text{sech}(\sqrt{2}(t - t_0)), \quad (4.15)$$

which approaches the purely magnetic solution $\phi = 0$ for $t \rightarrow \pm\infty$. For any solution $\phi(t)$, the $\text{su}(2)$ Yang-Mills potential and field take the form

$$\begin{aligned}\mathcal{A} &= \frac{1}{2} (1 + \phi) \tilde{I}_a \tilde{e}^a \quad \text{and} \\ \mathcal{F} &= \frac{1}{2} \dot{\phi} \tilde{I}_a e^0 \wedge \tilde{e}^a + \frac{1}{4} (\phi^2 - 1) \epsilon_{ab}{}^c \tilde{I}_c \tilde{e}^a \wedge \tilde{e}^b.\end{aligned}\quad (4.16)$$

Let us compare this to the orthogonal case. The $\text{so}(4)$ -valued configurations

$$\begin{aligned}\mathcal{A} &= (I_i e^i_a + \phi I_a) e^a \quad \text{and} \\ \mathcal{F} &= \dot{\phi} I_a e^0 \wedge e^a + \frac{1}{2} (\phi^2 - 1) \epsilon_{ab}{}^{i-3} I_i e^a \wedge e^b\end{aligned}\quad (4.17)$$

merely embed the $su(2)$ Yang-Mills solution into $so(4)$ and represent nothing new.

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APPENDIX: STRUCTURE CONSTANT NORMALIZATION FOR UNIT SPHERES

How are the structure constants to be normalized so that the corresponding coset manifold is a sphere of radius one? To answer this question, we embed the unit spheres as

$$S^n \hookrightarrow \mathbb{R}^{n+1}, \quad S^{2m+1} \hookrightarrow \mathbb{C}^{m+1}, \quad S^{4\ell+3} \hookrightarrow \mathbb{H}^{\ell+1} \quad (\text{A1})$$

where \mathbb{H} denotes the quaternionic number field. The three spaces \mathbb{R}^{n+1} , \mathbb{C}^{m+1} , and $\mathbb{H}^{\ell+1}$ carry the defining unitary representation of $SO(n+1)$, $SU(m+1)$, and $Sp(\ell+1)$, respectively. With elementary matrices $E_{\alpha\beta}$ defined by matrix elements

$$(E_{\alpha\beta})_{\gamma\delta} = \delta_{\alpha\gamma}\delta_{\beta\delta} \quad \text{with} \quad \alpha, \beta, \gamma, \delta = 1, 2, \dots, n+1$$

$$\text{or } m+1 \quad \text{or } \ell+1, \quad (\text{A2})$$

some of the one-parameter subgroups in all three cases have the form

$$U_{\alpha\beta}(t) = (E_{\alpha\alpha} + E_{\beta\beta}) \cos t + (E_{\alpha\beta} - E_{\beta\alpha}) \sin t$$

$$= \exp\{t(E_{\alpha\beta} - E_{\beta\alpha})\} \quad \text{with} \quad t \in [0, 2\pi). \quad (\text{A3})$$

For a suitable choice of the subgroup $SO(n)$, $SU(m)$, or $Sp(\ell)$, the circle $U_{\alpha\beta}(t)$ realizes a great circle in the coset space, i.e., the unit sphere, with an obvious circumference of 2π . In other words, we may identify $E_{\alpha\beta} - E_{\beta\alpha}$ with a generator \bar{I}_a of \mathfrak{m} in the defining representation.

To connect this to the structure constants, we compute the trace of the square of this generator in the defining as well as in the adjoint representation of $SO(n+1)$, $SU(m+1)$, or $Sp(\ell+1)$:

$$\text{tr}_{\text{def}}(\bar{I}_a^2) = \text{tr}((E_{\alpha\beta} - E_{\beta\alpha})^2) = -2 \quad \text{and}$$

$$\text{tr}_{\text{adj}}(\bar{I}_a \bar{I}_b) = \bar{f}_{aC}^D \bar{f}_{bD}^C = -\bar{g}_{ab} = -\bar{\kappa} \delta_{ab}. \quad (\text{A4})$$

On the other hand, the two traces are related by the dual Coxeter number h^\vee ,

$$\text{tr}_{\text{adj}}(XY) = \begin{cases} h^\vee \text{tr}_{\text{def}}(XY) & \text{for } X, Y \in so(n+1) \quad \text{with } h^\vee = n-1 \\ 2h^\vee \text{tr}_{\text{def}}(XY) & \text{for } X, Y \in su(m+1) \quad \text{with } h^\vee = m+1, \\ h^\vee \text{tr}_{\text{def}}(XY + YX) & \text{for } X, Y \in sp(\ell+1) \quad \text{with } h^\vee = \ell+2 \end{cases} \quad (\text{A5})$$

We conclude that (including even $n=2$)

$$\kappa_n = 2(n-1), \quad \tilde{\kappa}_m = 4(m+1), \quad \hat{\kappa}_\ell = 4(\ell+2) \quad (\text{A6})$$

for the orthogonal, unitary, and symplectic cases, respectively.

A rescaling of the structure constants causes the following changes,

$$\bar{f}_{..} \mapsto \frac{1}{\rho} \bar{f}_{..}, \quad \bar{I}_{..} \mapsto \frac{1}{\rho} \bar{I}_{..}, \quad \bar{e} \mapsto \rho \bar{e},$$

$$d\Omega^2 \mapsto \rho^2 d\Omega^2, \quad \bar{g}_{..} \mapsto \rho^2 \bar{g}_{..}, \quad \kappa \mapsto \rho^2 \kappa, \quad (\text{A7})$$

leading to a sphere of radius ρ .

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