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Sigma models for genuinely non-geometric backgrounds

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Abstract: The existence of genuinely non-geometric backgrounds, i.e. ones without geometric dual, is an important question in string theory. In this paper we examine this question from a sigma model perspective. First we construct a particular class of Courant algebroids as protobialgebroids with all types of geometric and non-geometric fluxes. For such structures we apply the mathematical result that any Courant algebroid gives rise to a 3D topological sigma model of the AKSZ type and we discuss the corresponding 2D field theories. It is found that these models are always geometric, even when both 2-form and 2-vector fields are neither vanishing nor inverse of one another. Taking a further step, we suggest an extended class of 3D sigma models, whose world volume is embedded in phase space, which allow for genuinely non-geometric backgrounds. Adopting the doubled formalism such models can be related to double field theory, albeit from a world sheet perspective.

Keywords: Sigma Models, Differential and Algebraic Geometry, String Duality, Flux compactifications

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1 Introduction

It is by now well established that useful string vacua, in particular those where potentials that can stabilize the moduli are generated, contain background fluxes. These fluxes come in several types, in particular standard ones such as NSNS flux, torsion or geometric flux and RR fluxes, but also non-standard types such as non-geometric fluxes in the NSNS [1] and RR [2] sectors. The latter can be described with techniques from the differential geometry of Lie and Courant algebroids [3–8] which are also used in Hitchin's generalized complex geometry [9]. They often appear in (generalized) T- or S-duals of standard geometries [10–13]. However, in most studies up to now these non-geometric string backgrounds are not truly new string vacua, even referring to ideal cases where the string equations of motion are/could be solved. This is so because the very essence of dualities is that the physics is the same on each side, and thus vacua which are related by dualities are just different

ways to describe the same physics. This can change only by having at hand vacua which are genuinely non-geometric, which means there is no duality transformation that can map the vacuum to a known, geometric one. In this sense, cases known as the Q-background and the R-background are not truly non-geometric from a string-theoretic point of view.

The main question that we would like to address in this paper is how genuinely non-geometric models could be described. This is arguably the most essential question in the study of generalized flux compactifications, which however has been posed and addressed only fragmentarily. One direction which was followed relies on the construction of exact conformal field theories based on asymmetric orbifolds [14]. Asymmetric orbifolds are backgrounds where left- and right-moving string coordinates see different geometries; for this reason such theories are indeed genuinely non-geometric string solutions and they can contain all types of fluxes [15–19]. Other approaches on the same problem include work on heterotic string vacua with non-geometric fluxes [20], where it is argued that geometric and non-geometric compactifications are equally typical, the study of cases where not only the internal geometry but also the external one is multi-valued and thus non-geometric [21], as well as a classification of the U-duality orbits of gaugings of (half-)maximal supergravities [22].

Our approach is different than the above ones and complements previous work in the string theory literature [3–8]. The starting point is Courant algebroids (CAs), which are structures introduced in refs. [23, 24] that provide a systematization of the properties of the Courant bracket introduced in ref. [25]. The authors of [23, 24] construct CAs as Lie bialgebroids, which is a special case of a more general construction performed in refs. [26, 27] using the notion of protobialgebroids (PBAs). The latter are structures that incorporate 3-index twists, corresponding to (some of) the NSNS fluxes that appear in string theory. In this paper we construct a class of PBAs, putting on firmer grounds and generalizing our previous work [8]. We consider PBAs following the spirit of twisting the generalized tangent bundle $\mathrm{TM} \oplus \mathrm{T}^\star\mathrm{M}$ of a d-torus by 2-form, 2-vector and (1,1)-tensor deformations. We choose a representative paradigm of (1,1) deformations that leads to nilmanifolds, which are also termed "twisted tori" in physics; this is a natural way to go beyond the toroidal case. This approach directly suggests how brackets, morphisms and generalized 3-forms should be defined in the class of PBAs we study. Given that any PBA gives rise to a CA, we then proceed in the construction of the latter and discuss an illustrative example of the class. The twist approach that we follow yields non-standard CAs.

Having constructed the desired structures over twisted tori, it is desirable for physics to study sigma models that correspond to them. An important mathematical result ([28]) states that given a CA one can construct a topological sigma model of the type introduced in the seminal work of Alexandrov, Kontsevich, Schwarz and Zaboronsky (AKSZ) [29] (see ref. [30] for a useful review). Applying this result, we construct in very explicit terms the sigma model corresponding to the class of CAs with all types of twists. Considering that the 3D topological theory has a 2D boundary, we derive consistency conditions for the action including all twists and deformations, thus generalizing previous results. Moreover, we show that in certain limits these conditions reproduce known results, for example integrability conditions for Dirac structures, such as the ones found in refs. [31–33] and additional ones

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derived in ref. [8]. Additionally, we discuss the corresponding 2D theories, add dynamics, and discuss in explicit detail a particular example with all types of fluxes.

After presenting models with all kinds of twists and 2-form and 2-vector deformations that are neither vanishing nor inverse of one another, we discuss whether such cases can be considered non-geometric in the sense of string theory. In this discussion one has to invoke T-duality in order to differentiate between two different kinds of 3-vector R flux, the one obtained by standard differential geometric methods (being a derivation of a non-Poisson 2-vector β, or equivalently the Schouten bracket of β with itself) and the one obtained by generalized T-duality. As already noticed in ref. [4] the former does not deserve to be called non-geometric, since it can be associated to a 2D sigma model with a standard target space. However, it is the second kind of R-flux that appears in string theory after T-duality. This means that a generalization of the formalism we apply in sections 2 and 3 is necessary in order to account for the R-flux originating in T-duality, and in order to examine the possibility of genuinely non-geometric backgrounds.

The above discussion leads us to propose an extension of the sigma models with standard target space to ones with phase space as target. This is also motivated by a similar approach adopted in ref. [6]. We choose to work with reference to the doubled formalism of string theory [34, 35], which parallels the first order formalism on phase space and introduces a set of coordinates $\tilde{X}_a$ in addition to the standard coordinates X^a, corresponding to the winding modes of closed strings. From a target space viewpoint, this has led to the development of double field theory (DFT) [36–41], which is currently under close scrutiny (see refs. [42–44] for reviews and a complete list of references). On the other hand, our approach has a 3D/2D perspective and does not use any results from DFT.[1] Once more it is assumed that the 3D manifold has a 2D boundary, and the consistency between the equations of motion and the boundary conditions uncovers an extended set of relations that have to be satisfied. We comment on their relation to the flux formulation of DFT [50–53] (see also [54–57]). Finally we write down in very explicit terms a 3D sigma model with doubled target where (i) all types of fluxes appear, (ii) 2-form and 2-vector deformations are neither vanishing nor inverse of one another and (iii) the R-flux is *not* of the type that can be reduced to a *standard* 2D sigma model. These properties suggest that this example is a nontrivial toy model of a genuinely non-geometric background.

2 Courant algebroids as protobialgebroids

In this section we define and construct Courant algebroids that accommodate 3-index twists of any type, which will be identified with geometric and non-geometric fluxes that appear in string theory. We call such structures "Courant-Roytenberg" algebroids, since they were introduced in refs. [26, 27]. Our approach in the presentation is to provide some basic definitions first, then apply them to construct a class of cases interesting for string theory, and finally to present in detail an explicit example. This approach will be followed in the following sections too.

[1] A new CFT approach of DFT was considered recently in refs. [45, 46]. Previous work along this line includes refs. [10, 47–49].

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2.1 Definitions of bialgebroids

Let us first define the notion of a protobialgebroid (PBA) and then discuss some particular limits. Note that PBAs are usually defined using supermanifolds [26, 27], but here we will use a more conservative, "bosonic" definition, which is more handy for applications in string theory (see however ref. [58]).

Definition 2.1 *Consider two dual vector bundles $(L, L^\star)$ over a manifold M, equipped with the following data:*

- *Skew-symmetric brackets $[\cdot,\cdot]_L$ on L and $[\cdot,\cdot]_{L^\star}$ on $L^\star$.*
- *Bundle morphisms (anchors) $\rho : L \to TM$ and $\rho_\star : L^\star \to TM$.*
- *Generalized 3-forms $\phi \in \Gamma(\wedge^3 L^\star)$ and $\psi \in \Gamma(\wedge^3 L)$.*

This structure is a ***protobialgebroid*** *provided that for $X, Y, Z \in \Gamma(L)$ and $\eta, \xi, \omega \in \Gamma(L^\star)$ the following properties hold:*

1. $[X, fY]_L = f[X,Y]_L + (\rho(X)f)Y$ *and* $[\eta, f\xi]_{L^\star} = f[\eta,\xi]_{L^\star} + (\rho_\star(\eta)f)\xi$, $f \in C^\infty(M)$,
2. $\rho([X,Y]_L) = [\rho(X), \rho(Y)]_{Lie} + \rho_\star\phi(X,Y,\cdot)$ *and* $\rho_\star([\eta,\xi]_{L^\star}) = [\rho_\star(\eta), \rho_\star(\xi)]_{Lie} + \rho\psi(\eta,\xi,\cdot)$,
3. $[[X,Y]_L, Z]_L + c.p. = \mathrm{d}_{L^\star}\phi(X,Y,Z) + \phi(\mathrm{d}_{L^\star}X, Y, Z) + \phi(X, \mathrm{d}_{L^\star}Y, Z) + \phi(X, Y, \mathrm{d}_{L^\star}Z)$,
   $[[\eta,\xi]_{L^\star}, \omega]_{L^\star} + c.p. = \mathrm{d}_L\psi(\eta,\xi,\omega) + \psi(\mathrm{d}_L\eta, \xi, \omega) + \psi(\eta, \mathrm{d}_L\xi, \omega) + \psi(\eta, \xi, \mathrm{d}_L\omega)$.
4. $\mathrm{d}_L\phi = 0$ *and* $\mathrm{d}_{L^\star}\psi = 0$.

Although this definition does not appear as such in the literature,[2] it is just the appropriate generalization of the definition 3.8.3 for a quasi-Lie bialgebroid in [26]. The four enumerated properties in definition 2.1 are generalizations of the familiar properties of the tangent bundle. They are lifted to the general vector bundles L and $L^\star$ with the aid of the maps ρ and $\rho_\star$, called anchors. More precisely, the first property is just the Leibniz rule for each bundle. Recall that the tangent bundle, whose sections are ordinary vector fields, is equipped with the standard Lie bracket of vector fields that satisfies the Leibniz rule

$$[X, fY]_{\mathrm{Lie}} = f[X,Y]_{\mathrm{Lie}} + (Xf)Y \,, \tag{2.1}$$

when $X, Y \in \Gamma(\mathrm{TM})$. It is then evident that property 1 for each bundle is the direct generalization of this rule. The second property is a twisted version of ρ and $\rho_\star$ being homomorphisms; ρ is a ϕ-homomorphism and $\rho_\star$ a ψ-homomorphism. The third property is a twisted version of the Jacobi identity. Finally, the fourth property states that the 3-objects ϕ and ψ are closed with respect to the corresponding derivations on each vector bundle. These derivations are in turn the direct generalizations of the standard exterior derivative on the tangent bundle, which acts on p-forms raising their degree by one. In

[2] It is briefly mentioned in ref. [59] though.

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particular, they are simply defined as maps $\mathrm{d}_L : \Gamma(\wedge^p L^\star) \to \Gamma(\wedge^{p+1} L^\star)$ and $\mathrm{d}_{L^\star} : \Gamma(\wedge^p L) \to \Gamma(\wedge^{p+1} L)$, acting as follows [60]:

$$\begin{aligned}\mathrm{d}_L\omega(X_1,\ldots,X_{p+1}) &= \sum_{i=1}^{p+1}(-1)^{i+1}\rho(X_i)\omega(X_1,\ldots,\hat{X}_i,\ldots,X_{p+1}) + \\ &+ \sum_{i<j}(-1)^{i+j}\omega([X_i,X_j]_L,X_1,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_{p+1})\,, \quad (2.2)\\ \mathrm{d}_{L^\star}\Omega(\eta_1,\ldots,\eta_{p+1}) &= \sum_{i=1}^{p+1}(-1)^{i+1}\rho_\star(\eta_i)\Omega(\eta_1,\ldots,\hat{\eta}_i,\ldots,\eta_{p+1}) + \\ &+ \sum_{i<j}(-1)^{i+j}\Omega([\eta_i,\eta_j]_{L^\star},\eta_1,\ldots,\hat{\eta}_i,\ldots,\hat{\eta}_j,\ldots,\eta_{p+1})\,, \quad (2.3)\end{aligned}$$

for arbitrary generalized p-forms $\omega \in \Gamma(\wedge^p L^\star)$ and $\Omega \in \Gamma(\wedge^p L)$. The property 4 is essentially the set of Bianchi identities for the structure.

Given the general structure of a PBA, there are few special cases, depending on the presence or absence of the generalized 3-forms ϕ and ψ. They are collected in the following table:

ϕ	ψ	Structure
$\neq 0$	$\neq 0$	Protobialgebroid
$\neq 0$	$= 0$	Quasi-Lie bialgebroid
$= 0$	$\neq 0$	Lie quasibialgebroid
$= 0$	$= 0$	Lie bialgebroid

It is known from refs. [23, 24] that a Lie bialgebroid (LBA) gives rise to a Courant algebroid (CA) with vector bundle $E = L \oplus L^\star$, which will be defined below. More generally any PBA gives rise to a CA, as shown in refs. [26, 27].

2.2 Protobialgebroids made explicit

Let us construct a class of PBAs, based on nilmanifolds of step 2. Such manifolds are called "twisted tori" in the physics literature and they can be described as fibrations of toroidal fibers over toroidal bases, whose tangent bundle can be obtained from the tangent bundle of standard tori by an appropriate deformation of degree $(1,1)$. Additionally we consider deformations by elements of degree $(0,2)$ and $(2,0)$, as explained below. This structure was already partially constructed in ref. [8], where it was called "a QLBA with anchor on L instead of TM". The issue of anchors that did not project the vector bundles on TM will here be corrected using the more general protobialgebroid structure. Another difference is that in ref. [8] one had to invoke the Courant bracket as the bracket on the CA to do the computations, while now we are going to extract the consistent form of the bracket just from the deformation data, without a priori reference to the resulting CA. In particular, since the CA is not the standard one, the bracket on it is not simply the Courant bracket,

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but a more involved one, as explained in refs. [23, 24]. Finally, the associated CA was not constructed explicitly in ref. [8], and we will complete this construction later in this section.

2.2.1 Protobialgebroids over twisted tori

Consider M to be any d-dimensional step 2 nilmanifold with tangent and cotangent bundles spanned by

$$\theta_i = e_i^a(x)\partial_a \,, \tag{2.4}$$

$$e^i = e_a^i(x)\mathrm{d}x^a, \,, \tag{2.5}$$

where early indices $a, b, \ldots$ are curved (world indices) and late indices $i, j, \ldots$ are flat (freely falling frame indices), and the frame components $e_a^i(x)$ are such that the Maurer-Cartan equations hold. We choose to work on a (generally curved) nilmanifold because it is a direct but nontrivial generalization of a flat torus. In particular, the tangent and cotangent bundles of any nilmanifold can be obtained from the toroidal ones by means of a $(1,1)$ deformation $h(x) = \frac{1}{2}h_a^b(x)\mathrm{d}x^a \wedge \partial_b$:

$$\mathrm{TM} = e^{h(x)}\mathrm{T}T^d \,. \tag{2.6}$$

In the case of step 2 nilmanifolds the deformation can always be written as

$$h(x) = f_{ab}^c x^a \mathrm{d}x^b \wedge \partial_c \,, \tag{2.7}$$

for appropriate parameters f_{ab}^c corresponding to the structure constants of the associated nilpotent Lie algebra. One can simply jump to the toroidal case by setting f's to zero. Moreover, the choice of step 2, which amounts to the structure constant relation

$$f_{ab}^c f_{de}^b = 0 \,, \quad \text{no summation in } b \,, \tag{2.8}$$

is made for simplicity, but the general step case can also be addressed with the same methods. In the general case the deformation $h(x)$ is a more involved polynomial expression, and higher order relations among the structure constants hold (see ref. [61] for details). The basis 1-vectors and 1-forms are dual,

$$\theta_i(e^j) = \delta_i^j \,, \tag{2.9}$$

reflecting the duality of TM and $\mathrm{T}^\star\mathrm{M}$ as vector bundles. Additionally, the 1-forms satisfy the usual Maurer-Cartan equations.

Let us endow the twisted torus with a (not necessarily closed) 2-form and a (not necessarily Poisson) 2-vector:

$$B = \frac{1}{2}B_{ij}e^i \wedge e^j \,, \tag{2.10}$$

$$\beta = \frac{1}{2}\beta^{ij}\theta_i \wedge \theta_j \,. \tag{2.11}$$

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Using B and β as deformations the tangent and cotangent bundles can be twisted accordingly:[3]

$$L_B := e^B \mathrm{TM} = \mathrm{span}(\{\theta_i + B_{ij} e^j\}) \,, \tag{2.12}$$

$$L^\star_\beta := e^\beta \mathrm{T^\star M} = \mathrm{span}(\{e^i + \beta^{ij} \theta_j\}) \,. \tag{2.13}$$

Note that unlike TM and $\mathrm{T^\star M}$, which are dual due to the pairing (2.9), L_B and $L^\star_\beta$ are not mutually dual.[4] In ref. [8] we performed an independent change of basis on each bundle, such that duality is achieved. This is actually equivalent to performing an overall $e^B e^\beta$ twist[5] on $\mathrm{TM} \oplus \mathrm{T^\star M}$. We simply get

$$L_{B\beta} = e^B e^\beta \mathrm{TM} = \mathrm{span}(\{\Theta_i = \theta_i + B_{ij} e^j\}) \,, \tag{2.14}$$

$$L^\star_{B\beta} = e^B e^\beta \mathrm{T^\star M} = \mathrm{span}(\{E^i = e^i + \beta^{ik} B_{kj} e^j + \beta^{ij} \theta_j\}) \,. \tag{2.15}$$

We will never assume any constraining relation for $B\beta$; unlike what is usually assumed in the literature, here this combination is in general neither vanishing nor unity. The pair that appears in the definition of the protobialgebroids that we consider is then $(L, L^\star) = (L_{B\beta}, L^\star_{B\beta})$.

According to the definition, we should specify elements $\phi \in \Gamma(\wedge^3 L^\star_{B\beta})$ and $\psi \in \Gamma(\wedge^3 L_{B\beta})$. In the spirit of twisting the tangent and cotangent bundle data, we consider arbitrary elements $H \in \Gamma(\wedge^3 \mathrm{T^\star M})$ and $R \in \Gamma(\wedge^3 \mathrm{TM})$ and twist them to give

$$\begin{aligned}
\phi &= \frac{1}{6} \phi_{ijk} E^i \wedge E^j \wedge E^k \\
&= \frac{1}{6} \big((1+\beta B)^i_\rho (1+\beta B)^j_\sigma (1+\beta B)^k_\tau \phi_{ijk} e^\rho \wedge e^\sigma \wedge e^\tau \\
&\quad + 3 (1+\beta B)^i_\rho (1+\beta B)^j_\sigma \beta^{kl} \phi_{ijk} e^\rho \wedge e^\sigma \wedge \theta_l \\
&\quad + 3 (1+\beta B)^i_\rho \beta^{jl} \beta^{km} \phi_{ijk} e^\rho \wedge \theta_l \wedge \theta_m \\
&\quad + \beta^{il} \beta^{jm} \beta^{kn} \phi_{ijk} \theta_l \wedge \theta_m \wedge \theta_n \big) \,,
\end{aligned} \tag{2.16}$$

$$\begin{aligned}
\psi &= \frac{1}{6} \psi^{ijk} \Theta_i \wedge \Theta_j \wedge \Theta_k \\
&= \frac{1}{6} \big(\psi^{ijk} \theta_i \wedge \theta_j \wedge \theta_k \\
&\quad + 3 B_{kn} \psi^{ijk} \theta_i \wedge \theta_j \wedge e^n \\
&\quad + 3 B_{jm} B_{kn} \psi^{ijk} \theta_i \wedge e^m \wedge e^n \\
&\quad + B_{il} B_{jm} B_{kn} \psi^{ijk} e^l \wedge e^m \wedge e^n \big) \,.
\end{aligned} \tag{2.17}$$

Eqs. (2.16) and (2.17) exhibit that in the presence of the twists (ϕ and ψ) and the deformations (B and β) there are all types of fluxes turned on, as they were identified e.g. in ref. [8] in a less systematic way. Note that in some particular cases H and R can be

[3] These twisted bundles can equivalently be understood as graphs of B and β respectively, namely $L_B = \{X + B(X, \cdot); X \in \mathrm{TM}\}$ and $L^\star_\beta = \{\eta + \beta(\eta, \cdot); \eta \in \mathrm{T^\star M}\}$.

[4] Because the twist is not an element of $\mathrm{O}(d, d)$.

[5] The order of the twists, first with β and then with B, counts. A twist of the form $e^\beta e^B$ would lead to another path, which is however equivalent for our purposes.

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identified with the derivations of B and β respectively. However, it will not always be the case in this paper that these identifications are made. This will be explicitly stated when assumed.

Next we consider the bundle morphisms

$$\rho: \ L_{B\beta} \to \mathrm{TM}\,, \quad \rho(X) = e^{-\beta}e^{-B}X\,, \tag{2.18}$$

$$\rho_\star : L^\star_{B\beta} \to \mathrm{TM}\,, \quad \rho_\star(\eta) = \beta(e^{-\beta}e^{-B}\eta, \cdot)\ . \tag{2.19}$$

Here and in the following we use the symbol β also for the map $\beta : \mathrm{T}^\star\mathrm{M} \to \mathrm{TM}$ (often denoted as $\beta^\sharp$ in the literature). These are the candidates for anchors, being the twisted versions of the corresponding anchors on TM (unit map) and $\mathrm{T}^\star\mathrm{M}$ (β-morphism).

Now we have to define skew-symmetric closed brackets on each of the two vector bundles. Our strategy is once more to consider the corresponding brackets on TM and $\mathrm{T}^\star\mathrm{M}$ and twist them appropriately. Let us use the notation $X, Y \in \Gamma(L_{B\beta})$ and $\eta, \xi \in \Gamma(L^\star_{B\beta})$. Elements of TM are written as $\tilde{X} := e^{-\beta}e^{-B}X$, and elements of $\mathrm{T}^\star\mathrm{M}$ as $\tilde{\eta} := e^{-\beta}e^{-B}\eta$. The bracket on TM is the standard Lie bracket of vector fields, while the bracket on $\mathrm{T}^\star\mathrm{M}$ is

$$[\tilde{\eta}, \tilde{\xi}]_{\mathrm{K}} = \mathcal{L}_{\beta(\tilde{\eta},\cdot)}\tilde{\xi} - \mathcal{L}_{\beta(\tilde{\xi},\cdot)}\tilde{\eta} - \mathrm{d}\big(\beta(\tilde{\eta}, \tilde{\xi})\big)\,, \quad \tilde{\eta}, \tilde{\xi} \in \mathrm{T}^\star\mathrm{M}\,, \tag{2.20}$$

d being the standard de Rham differential. In the Poisson case this is the standard Koszul bracket of 1-forms. Then we consider the $e^B e^\beta$ twist of those brackets and write the Ansätze:

$$[X, Y]_{L_{B\beta}} = e^B e^\beta [e^{-\beta}e^{-B}X, e^{-\beta}e^{-B}Y]_{\mathrm{Lie}} + V\,, \tag{2.21}$$

$$[\eta, \xi]_{L^\star_{B\beta}} = e^B e^\beta [e^{-\beta}e^{-B}\eta, e^{-\beta}e^{-B}\xi]_K + W\,, \tag{2.22}$$

where $V \in L_{B\beta}$ and $W \in L^\star_{B\beta}$ are associated to the twists and they should be determined by consistency with the definition 2.1. In particular, for the bracket (2.21), the second requirement of the definition, combined with the anchors defined above, gives

$$\begin{aligned}
& \rho([X, Y]_{L_{B\beta}}) = [\rho(X), \rho(Y)]_{\mathrm{Lie}} + \rho_\star\phi(X, Y, \cdot) \\
\Leftrightarrow\ & \rho(e^B e^\beta [e^{-\beta}e^{-B}X, e^{-\beta}e^{-B}Y]_{\mathrm{Lie}}) + \rho(V) = [e^{-\beta}e^{-B}X, e^{-\beta}e^{-B}Y]_{\mathrm{Lie}} + \rho_\star\phi(X, Y, \cdot) \\
\Leftrightarrow\ & [e^{-\beta}e^{-B}X, e^{-\beta}e^{-B}Y]_{\mathrm{Lie}} + \rho(V) = [e^{-\beta}e^{-B}X, e^{-\beta}e^{-B}Y]_{\mathrm{Lie}} + \rho_\star\phi(X, Y, \cdot) \\
\Leftrightarrow\ & \rho(V) = \rho_\star\phi(X, Y, \cdot) \\
\Leftrightarrow\ & V = e^B e^\beta \beta(e^{-\beta}e^{-B}(\phi(X, Y, \cdot)), \cdot)\ .
\end{aligned} \tag{2.23}$$

For the bracket (2.22) the analogous requirement is

$$\rho_\star([\eta, \xi]_{L^\star_{B\beta}}) = [\rho_\star(\eta), \rho_\star(\xi)]_{\mathrm{Lie}} + \rho\psi(\eta, \xi, \cdot)\ . \tag{2.24}$$

A similar computation leads to the result

$$\rho_\star(W) = \beta(e^{-\beta}e^{-B}W, \cdot) = e^{-\beta}e^{-B}\psi(\eta, \xi, \cdot) - \frac{1}{2}[\beta, \beta]_{\mathrm{S}}(e^{-\beta}e^{-B}\eta, e^{-\beta}e^{-B}\xi, \cdot)\,, \tag{2.25}$$

where $[\cdot, \cdot]_{\mathrm{S}}$ is the Schouten bracket. This equation should be solved for W in order to fully determine the bracket on $L^\star_{B\beta}$. Unlike the previous case this is not straightforward, since

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it depends on the invertibility of $\rho_\star$ (while ρ is always invertible in our approach). For invertible β, it is easy to solve for W and plug it in the Ansatz (2.22). However, the case of non-invertible β is more interesting for our purposes. A way to solve (2.25) is to assume that the right hand side is zero, namely

$$\psi(\eta,\xi,\cdot)=\frac{1}{2}e^B e^\beta[\beta,\beta]_{\rm S}(e^{-\beta}e^{-B}\eta,e^{-\beta}e^{-B}\xi,\cdot)\,, \tag{2.26}$$

and that $\beta^3\phi=0$. Then we set

$$W=\phi\big(e^Be^\beta\beta(e^{-\beta}e^{-B}\eta,\cdot),e^Be^\beta\beta(e^{-\beta}e^{-B}\xi,\cdot),\cdot\big)\ . \tag{2.27}$$

We will see later that these conditions are mild enough to assure that nontrivial cases indeed exist. According to the above, the brackets on the two vector bundles are determined to be[6]

$$[X,Y]_{L_{B\beta}}=e^Be^\beta\Big([e^{-\beta}e^{-B}X,e^{-\beta}e^{-B}Y]_{\rm Lie}+\beta(e^{-\beta}e^{-B}(\phi(X,Y,\cdot)),\cdot)\Big)\,, \tag{2.28}$$

$$[\eta,\xi]_{L^\star_{B\beta}}=e^Be^\beta[e^{-\beta}e^{-B}\eta,e^{-\beta}e^{-B}\xi]_{\rm K}+\phi\big(e^Be^\beta\beta(e^{-\beta}e^{-B}\eta,\cdot),e^Be^\beta\beta(e^{-\beta}e^{-B}\xi,\cdot),\cdot\big). \tag{2.29}$$

The skew-symmetry of the brackets (2.28) and (2.29) follows from the skew-symmetry of the Lie and Koszul brackets and the antisymmetry of ϕ. Closedness is also rather obvious. The big brackets in eq. (2.28) contain an element of TM. Then this element is acted upon with e^Be^β, yielding elements of $L_{B\beta}$, as required. Similarly, both terms in eq. (2.29) are elements of $L^\star_{B\beta}$. The brackets can be computed explicitly for the basis elements Θ_i and E^i; they yield the results

$$[\Theta_i,\Theta_j]_{L_{B\beta}}=(f^k_{ij}-\beta^{km}\phi_{mij})\Theta_k\,, \tag{2.30}$$

$$[E^i,E^j]_{L^\star_{B\beta}}=(\theta_k\beta^{ij}-2\beta^{il}f^j_{lk}+\beta^{il}\beta^{jm}\phi_{klm})E^k\ . \tag{2.31}$$

With the above elements ϕ and ψ, the brackets and the anchors, we have now collected all the input ingredients of a protobialgebroid, as required from the definition 2.1. In the appendix we collect the proofs of the properties 1-4 in this definition.

2.2.2 Explicit example

In order to exhibit that nontrivial cases with nonvanishing B and β and with $B\beta\neq 1$ indeed exist, let us consider as an example the 3D nilmanifold based on the Heisenberg algebra with single structure constant $f^3{}_{12}=1$. The full basis is

$$\theta_1=\partial_1\,,\qquad \theta_2=\partial_2+x^1\partial_3\,,\qquad \theta_3=\partial_3\,, \tag{2.32}$$

$$e^1=\mathrm{d}x^1\,,\qquad e^2=\mathrm{d}x^2\,,\qquad e^3=\mathrm{d}x^3-x^1\mathrm{d}x^2\ . \tag{2.33}$$

[6]Note that these brackets seem to contain only ϕ and not ψ explicitly. However, as it is clear from eq. (2.26), ψ is not zero and this is essential for the ψ-homomorphism equation (2.24) to hold, as it should for a protobialgebroid. However, ψ will appear explicitly when we construct the bracket of the corresponding Courant algebroid, where the two twisted homomorphism conditions are replaced by a single homomorphism condition for the anchor of the Courant algebroid.

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It can be checked that the manifold has a Poisson structure [62], given by the 2-vector

$$\theta_P = \mu\theta_1 \wedge \theta_3 + \nu\theta_2 \wedge \theta_3 \ . \tag{2.34}$$

Therefore, any non-Poisson 2-vector will necessarily include $\theta_1 \wedge \theta_2$. Here we consider such a 2-vector,

$$\beta = \sqrt{c}\theta_1 \wedge \theta_2 \, , \tag{2.35}$$

where c is a real constant. Its Schouten bracket gives:

$$[\beta, \beta]_S = 2R = 2c\theta_1 \wedge \theta_2 \wedge \theta_3 \, , \tag{2.36}$$

where for this example we identified the Schouten bracket with R (and $\mathrm{d}B$ with H below), which is not always the case. Notably, β being constant in the basis θ_i is enough to produce a nonvanishing 3-vector. Additionally we consider a 2-form proportional to the symplectic leaves of the manifold, which are $e^1 \wedge e^3$ and $e^2 \wedge e^3$. To be precise, we restrict the 2-form only on one leaf and take

$$B = Nx^1 e^2 \wedge e^3 \ . \tag{2.37}$$

This 2-form is not closed, giving

$$\mathrm{d}B = H = Ne^1 \wedge e^2 \wedge e^3 \ . \tag{2.38}$$

The twisted bases are given as

$$L_{B\beta} = \mathrm{span}(\{\Theta_i\} = \{\theta_1, \theta_2 + Nx^1 e^3, \theta_3 - Nx^1 e^2\}) \, , \tag{2.39}$$

$$L^\star_{B\beta} = \mathrm{span}(\{E^i\} = \{e^1 + \sqrt{c}Nx^1 e^3 + \sqrt{c}\theta_2, e^2 - \sqrt{c}\theta_1, e^3\}) \ . \tag{2.40}$$

The closed brackets among the basis elements $\{\Theta_i, E^i\}$ are found via eqs. (2.28) and (2.29). They are

$$[\Theta_1, \Theta_2]_{L_{B\beta}} = \Theta_3 \, , \qquad [\Theta_1, \Theta_3]_{L_{B\beta}} = -\sqrt{c}N\Theta_1 \, , \qquad [\Theta_2, \Theta_3]_{L_{B\beta}} = -\sqrt{c}N\Theta_2 \, , \tag{2.41}$$

$$[E^1, E^2]_{L^\star_{B\beta}} = cNE^3 \, , \quad [E^1, E^3]_{L^\star_{B\beta}} = \sqrt{c}E^1 \, , \qquad [E^2, E^3]_{L^\star_{B\beta}} = \sqrt{c}E^2 \ . \tag{2.42}$$

Note that these are different from the ones in ref. [8], because the brackets have changed.

We specify the anchors from eqs. (2.18) and (2.19):

$$\rho(\Theta_i) = \theta_i \, , \tag{2.43}$$

$$\rho_\star(E^i) = \beta^{ij}\theta_j \ . \tag{2.44}$$

Note that unlike ref. [8], the anchors are morphisms to the TM, as required.

Finally, the 3-elements are:

$$\phi = Ne^1 \wedge e^2 \wedge e^3 + \sqrt{c}N(e^2 \wedge e^3 \wedge \theta_2 + e^1 \wedge e^3 \wedge \theta_1) + cNe^3 \wedge \theta_1 \wedge \theta_2 \, , \tag{2.45}$$

$$\psi = c\theta_1 \wedge \theta_2 \wedge \theta_3 + cNx^1(\theta_3 \wedge \theta_1 \wedge e^3 + \theta_2 \wedge \theta_1 \wedge e^2) + c(Nx^1)^2\theta_1 \wedge e^2 \wedge e^3 \ . \tag{2.46}$$

It is simple to check that they satisfy the Bianchi identities $\mathrm{d}_{L_{B\beta}}\phi = 0$ and $\mathrm{d}_{L^\star_{B\beta}}\psi = 0$ respectively (see appendix).

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2.3 The induced Courant algebroid

We recall the definition of a Courant algebroid according to refs. [23, 24].

Definition 2.2 *A **Courant algebroid** is a quadruplet $(E, [\cdot,\cdot]_E, \langle\cdot,\cdot\rangle_E, a)$ of the following data:*

- *a vector bundle E over M,*
- *a skew-symmetric bracket on $\Gamma(E)$,*
- *a non-degenerate symmetric bilinear form on E,*
- *and an anchor map $a : E \to TM$,*

such that for $\mathfrak{X}_i \in \Gamma(E)$:

1. $[[\mathfrak{X}_1,\mathfrak{X}_2]_E,\mathfrak{X}_3]_E + c.p. = \mathcal{D}\mathcal{N}(\mathfrak{X}_1,\mathfrak{X}_2,\mathfrak{X}_3)\,, \quad 3\mathcal{N} = \langle[\mathfrak{X}_1,\mathfrak{X}_2]_E,\mathfrak{X}_3\rangle_E + c.p.\,,$
2. $a([\mathfrak{X}_1,\mathfrak{X}_2]_E) = [a(\mathfrak{X}_1), a(\mathfrak{X}_2)]_{Lie}\,,$
3. $[\mathfrak{X}_1, f\mathfrak{X}_2]_E = f[\mathfrak{X}_1,\mathfrak{X}_2]_E + (a(\mathfrak{X}_1)f)\mathfrak{X}_2 - \langle\mathfrak{X}_1,\mathfrak{X}_2\rangle_E\mathcal{D}f\,,\ f \in C^\infty(M)\,,$
4. $\langle\mathcal{D}f,\mathcal{D}g\rangle_E = 0\,,\ \ f,g \in C^\infty(M)\,,$
5. $a(\mathfrak{X})\langle\mathfrak{X}_1,\mathfrak{X}_2\rangle_E = \langle[\mathfrak{X},\mathfrak{X}_1]_E + \mathcal{D}\langle\mathfrak{X},\mathfrak{X}_1\rangle_E,\mathfrak{X}_2\rangle_E + \langle\mathfrak{X}_1,[\mathfrak{X},\mathfrak{X}_2]_E + \mathcal{D}\langle\mathfrak{X},\mathfrak{X}_2\rangle_E\rangle_E\,,$

where $\mathcal{D} : C^\infty(M) \to \Gamma(E)$ is a map such that $\langle\mathcal{D}f,\mathfrak{X}\rangle_E = \frac{1}{2}a(\mathfrak{X})f$.

According to Roytenberg there is a CA associated to any PBA [26, 27]. Its construction is rather simple. Recall that according to Liu-Weinstein-Xu the general bracket of a CA is not just the Courant bracket, but a more general expression [23, 24]. The Courant bracket only arises in the case where the CA is standard, i.e. $E = \mathrm{TM} \oplus \mathrm{T}^\star\mathrm{M}$ **and** the bracket on the cotangent bundle is taken to be zero (providing a trivial extension of the tangent bundle). In the case at hand the cotangent bundle is equipped with a non-trivial bracket, and the CA is non-standard. Therefore the correct bracket on the CA should be the $L_{B\beta}$ bracket plus the $L^\star_{B\beta}$ bracket with appropriate additional terms and twists.

According to these, the vector bundle we consider is $E = L_{B\beta} \oplus L^\star_{B\beta}$, with the bracket:

$$\begin{aligned}[X+\eta, Y+\xi]_E = {} & [X,Y]_{L_{B\beta}} + \mathcal{L}_X\xi - \mathcal{L}_Y\eta - \frac{1}{2}\mathrm{d}_{L_{B\beta}}(X(\xi) - Y(\eta)) \\ & + [\eta,\xi]_{L^\star_{B\beta}} + \mathcal{L}_\eta Y - \mathcal{L}_\xi X + \frac{1}{2}\mathrm{d}_{L^\star_{B\beta}}(X(\xi) - Y(\eta)) \\ & - \phi(X,Y,\cdot) - \psi(\eta,\xi,\cdot)\,, \end{aligned} \tag{2.47}$$

where the Lie derivatives are defined as

$$\mathcal{L}_X = \mathrm{d}_{L_{B\beta}}\iota_X + \iota_X\mathrm{d}_{L_{B\beta}} \quad \text{and} \quad \mathcal{L}_\eta = \mathrm{d}_{L^\star_{B\beta}}\iota_\eta + \iota_\eta\mathrm{d}_{L^\star_{B\beta}}\,. \tag{2.48}$$

The anchor is just the sum of the two anchors,

$$a(X+\eta) = \rho(X) + \rho_\star(\eta) = e^{-\beta}e^{-B}X + \beta(e^{-\beta}e^{-B}\eta)\,. \tag{2.49}$$

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The symmetric bilinear is the standard one,

$$\langle X+\eta, Y+\xi\rangle_E = \frac{1}{2}(X(\xi)+Y(\eta)) \ . \tag{2.50}$$

These are the data of the CA that corresponds to the PBA structure of the previous sections. Note also that

$$\mathcal{D} = \mathrm{d}_{L_{B\beta}} + \mathrm{d}_{L^\star_{B\beta}} \ . \tag{2.51}$$

It can be directly checked that the requirements 1-5 are satisfied. Notably, the anchor a of the CA is a homomorphism due to property 2 in the definition 2.2. Recall also that the maps ρ and $\rho_\star$ are not exact homomorphisms, as dictated by property 2 in the definition 2.1. This works as follows. Consider elements of E which lie entirely in $L_{B\beta}$, i.e. $\mathfrak{X} = X$ and $\eta = 0$. The bracket of E between such elements is

$$[X,Y]_E = [X,Y]_{L_{B\beta}} - \phi(X,Y,\cdot) \ . \tag{2.52}$$

Then we compute

$$\begin{aligned} a([X,Y]_E) &= \rho([X,Y]_{L_{B\beta}}) - \rho_\star\phi(X,Y,\cdot) \\ &= \big([\rho(X),\rho(Y)]_{\mathrm{Lie}} + \rho_\star\phi(X,Y,\cdot)\big) - \rho_\star\phi(X,Y,\cdot) \\ &= [a(X),a(Y)]_{\mathrm{Lie}} \, , \end{aligned} \tag{2.53}$$

as required. A similar computation holds for the dual case.

Although in this section we used only index-free notation, it is useful to introduce CA indices $I, J, \ldots$, ranging from 1 to $2d$. An arbitrary generalized vector is written as

$$\mathfrak{X} = (\mathfrak{X}_I) = (\mathfrak{X}^i, \mathfrak{X}_i) \in \Gamma(E) \, , \tag{2.54}$$

namely the index I splits into upper and lower indices according to $\mathfrak{X} = \mathfrak{X}^i\Theta_i + \mathfrak{X}_i E^i$.

3 The associated AKSZ sigma model

3.1 Topological sigma model, boundary terms and dynamics

Every Courant algebroid has an associated (topological) sigma model of the type described by Alexandrov, Kontsevich, Schwarz and Zaboronsky (AKSZ) in ref. [29]. This can be inferred e.g. by the discussion of Roytenberg in the paper [28]. A physicists-friendly review is [30] (see also the paper [63]). The master action contains fields with ghost number 0, 1, 2 and 3. Let us focus on the 0-ghost sector of the action:

$$S_{\Sigma_3}[X,A,F] = \int_{\Sigma_3}\left(F_a\wedge \mathrm{d}X^a + \frac{1}{2}\eta_{IJ}A^I\wedge\mathrm{d}A^J - P^a_I A^I\wedge F_a + \frac{1}{6}T_{IJK}A^I\wedge A^J\wedge A^K\right) . \tag{3.1}$$

The explanation for the ingredients of this action is the following. This is a membrane topological action in 3D. The indices I, J are Courant algebroid indices, while the index a is a curved index, as before. X^a are the world volume scalars on the membrane, or in other words the components of the map $X : \Sigma_3 \to \mathrm{M}$, M being the target spacetime. A^I is valued

in $\Omega^1(\Sigma_3, X^\star E)$, where $X^\star$ denotes the pull back with respect to the world volume scalar fields. Additionally, F_a is a world volume 2-form in $\Omega^2(\Sigma_3, X^\star \mathrm{T}^\star\mathrm{M})$. In the membrane model it plays the role of an auxiliary field that will be integrated out in the reduced string model. Moreover, η is the $\mathrm{O}(d,d)$ invariant metric, namely

$$\eta_{IJ} = \begin{pmatrix} 0 & \mathbb{1}_d \\ \mathbb{1}_d & 0 \end{pmatrix} , \tag{3.2}$$

and P_I^a is the anchor matrix defined through the relation

$$a(\mathfrak{X}_I) = P_I^a(X)\partial_a \, , \tag{3.3}$$

where $a : E \to \mathrm{TM}$ is the anchor of the CA. Finally, $T \in \Omega^3(\Sigma_3, X^\star E)$ is a generalized 3-form.

We assume that the manifold Σ_3 has a boundary, say $\partial\Sigma_3 := \Sigma_2$, since this is the relevant case for physical applications. The above action can be decorated with a general topological boundary term as in ref. [4] (see also [6, 30]):

$$S_{\partial\Sigma_3,\mathrm{top}} = \int_{\Sigma_2} \frac{1}{2}\mathcal{B}_{IJ}(X)A^I \wedge A^J \, . \tag{3.4}$$

More explicitly, with the splitting $A^I = (q^i, p_i)$,

$$\frac{1}{2}\mathcal{B}_{IJ}(X)A^I \wedge A^J = \frac{1}{2}\mathcal{B}_{ij}(X)q^i \wedge q^j + \frac{1}{2}\mathcal{B}^{ij}(X)p_i \wedge p_j + \frac{1}{2}\mathcal{B}^i_j(X)q^j \wedge p_i \, . \tag{3.5}$$

In the class of CAs we examine, all terms will play a role.

Additionally, in order to make contact with physics, dynamics should be added to the topological theory (thus breaking its topological nature). In this section our approach will be to study the 3D topological theory, then reduce it to the corresponding 2D field theory on the boundary and add dynamics at the level of this 2D theory. This is either done by simply adding a standard kinetic term

$$\int_{\Sigma_2} \frac{1}{2} g_{ij} e^i \wedge \star e^j \, , \tag{3.6}$$

or in certain cases a kinetic term formed with the inverse metric

$$\int_{\Sigma_2} \frac{1}{2} g^{ij} p_i \wedge \star p_j \, , \tag{3.7}$$

as in ref. [4]. The corresponding 2D theories are related to the dynamical sigma models discussed in refs. [64, 65].

A final comment has to do with the functional dependence of the quantities that appear in the above actions. In this section we assume that the various background field components $\mathcal{B}_{IJ}$, the anchor matrix P_I^a and the twist T solely depend on the scalar fields X^a. These assumptions will be lifted in section 4, where in the spirit of the first order formalism we will allow everything to depend both on X^a and the corresponding momenta.

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3.2 The AKSZ model for the Courant algebroid $E = L_{B\beta} \oplus L^{\star}_{B\beta}$

Let us now specialize to the class of Courant algebroids that we discuss in this paper. The ingredients of the topological membrane action (3.1) can be further specified. We hereby use the splitting $A^I = (q^i, p_i)$ referring to the basis (e^i, θ_i). According to eqs. (2.18)–(2.19), or more particularly eqs. (2.43)–(2.44), we immediately obtain the components P^a_I of the anchor matrix:

$$P^a_i = \mu e^a_i(X)\,, \tag{3.8}$$

$$P^{ai} = \nu\beta^{ij}(X)e^a_j(X)\ . \tag{3.9}$$

Note that we used the freedom to introduce parameters $\mu, \nu \in \{0, 1\}$, since the CA structure is rigid against trivialization of the anchors. These parameters are relevant in taking interesting limits, as will become clear later in this section.

Given the above ingredients the bulk action is

$$S^{(\phi,\psi)}_{\Sigma_3} = \int_{\Sigma_3}\Big(F_a\wedge \mathrm{d}X^a + kq^i\wedge \mathrm{d}p_i + k'p_i\wedge \mathrm{d}q^i - (\mu e^a_i q^i + \nu\beta^{ij}e^a_j p_i)\wedge F_a + f - \phi - \psi\Big)\,, \tag{3.10}$$

with f being the geometric flux

$$f = \frac{1}{2}f^k_{ij}q^i\wedge q^j\wedge p_k\,, \tag{3.11}$$

while, recalling that we work in the (e^i, θ_i) basis, ϕ and ψ are the twists given by the expansions

$$\begin{aligned}\phi = \frac{1}{6}\big(&(1+\beta B)^i_\rho(1+\beta B)^j_\sigma(1+\beta B)^k_\tau\phi_{ijk}q^\rho\wedge q^\sigma\wedge q^\tau\\ &+3(1+\beta B)^i_\rho(1+\beta B)^j_\sigma\beta^{kl}\phi_{ijk}q^\rho\wedge q^\sigma\wedge p_l\\ &+3(1+\beta B)^i_\rho\beta^{jl}\beta^{km}\phi_{ijk}q^\rho\wedge p_l\wedge p_m\\ &+\beta^{il}\beta^{jm}\beta^{kn}\phi_{ijk}p_l\wedge p_m\wedge p_n\big)\,,\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}\psi = \frac{1}{6}\big(&\psi^{ijk}p_i\wedge p_j\wedge p_k\\ &+3B_{kn}\psi^{ijk}p_i\wedge p_j\wedge q^n\\ &+3B_{jm}B_{kn}\psi^{ijk}p_i\wedge q^m\wedge q^n\\ &+B_{il}B_{jm}B_{kn}\psi^{ijk}q^l\wedge q^m\wedge q^n\big)\ .\end{aligned} \tag{3.13}$$

We used the fact that we are free to introduce two additional parameters k and k'. According to the general action (3.1) they have to satisfy

$$k + k' = 1\ . \tag{3.14}$$

The most symmetric choice is $k = k' = \frac{1}{2}$, and in absence of boundary one can always perform an integration by parts to change it to an arbitrary choice satisfying the condition (3.14). In the presence of a 2D boundary these parameters are used for interpolation between different limits.

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Now let us specify the boundary action. This contains all possible terms incorporating B, β and h deformations.[7] Accordingly, the boundary action is given by eqs. (3.4) and (3.5), with each set of components given as

$$\mathcal{B}_{ij} = B_{ij}\,, \quad \mathcal{B}^{ij} = \beta^{ij}\,, \quad \mathcal{B}_i^j = h_i^j\,. \tag{3.15}$$

The full action that we consider is then

$$S = S_{\Sigma_3}^{(\phi,\psi)} + S_{\partial\Sigma_3,\text{top}}^{(B,\beta,h)}\,. \tag{3.16}$$

This action comes with a set of consistency conditions. First, the boundary conditions should match with the equations of motion on the boundary. This implies that we have to vary the action with respect to X^a, q^i and p_i, set the variations to zero and determine appropriate boundary conditions. Performing this task we obtain

$$\begin{aligned}
\delta_{X^a} S|_{\Sigma_2} &= F_a + \frac{1}{2}\partial_a B_{jk} q^j \wedge q^k + \frac{1}{2}\partial_a \beta^{jk} p_j \wedge p_k + \frac{1}{2}\partial_a h_j^k q^j \wedge p_k = 0\,,\\
\delta_{q^i} S|_{\Sigma_2} &= -(k' p_i + B_{ij} q^j + \frac{1}{2} h_i^j p_j) = 0\,,\\
\delta_{p_i} S|_{\Sigma_2} &= -(k q^i + \beta^{ij} p_j - \frac{1}{2} h_j^i q^j) = 0\,.
\end{aligned} \tag{3.17}$$

These conditions are generalizations of the ones that appear e.g. in [30] and [33] and they can be solved in many ways, as we will explore below. The second consistency condition that has to be satisfied reads as

$$(\mu e_i^a(X) q^i + \nu \beta^{ij} e_j^a p_i) \wedge F_a = f - \phi - \psi \qquad \text{on } \Sigma_2\,. \tag{3.18}$$

Normally this condition follows from the classical master equation [30]. Alternatively it can be viewed as vanishing of the sector of the bulk action that does not reduce to the boundary via the field equations.

Let us now explore some boundary conditions. First, we consider

$$\begin{aligned}
F_a|_{\Sigma_2} &= -\frac{1}{2}\partial_a B_{jk} q^j \wedge q^k - \frac{1}{2}\partial_a \beta^{jk} p_j \wedge p_k - \frac{1}{2}\partial_a h_j^k q^j \wedge p_k\,,\\
\delta q^i|_{\Sigma_2} &= 0\,,\\
(k q^i + \beta^{ij} p_j - \frac{1}{2} h_j^i q^j)|_{\Sigma_2} &= 0\,.
\end{aligned} \tag{3.19}$$

Notably, the mild condition $h_k^i h_j^k = 0$ allows us (for $k \neq 0$) to write

$$q^i = -\frac{1}{k}\chi_k^i \beta^{kj} p_j\,, \tag{3.20}$$

where we introduced shorthand notation $\chi = 1 + \frac{1}{2k}h$. A medium long calculation shows that (3.18) reduces to the bulk/boundary consistency condition

$$\mathcal{R}^{ijk} - \frac{1}{k}\mathcal{Q}_n^{[ij}\beta^{\underline{pk}]}\chi_p^n + \frac{1}{k^2}\mathcal{F}_{mn}^{[i}\beta^{\underline{pj}}\beta^{\underline{qk}]}\chi_p^m\chi_q^n - \frac{1}{k^3}\mathcal{H}_{lmn}\beta^{pi}\beta^{qj}\beta^{rk}\chi_p^l\chi_q^m\chi_r^n = 0\,, \tag{3.21}$$

[7] Although when we discuss specific examples we never add excess geometric flux on the twisted torus, in the general discussion such a possibility is retained.

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where we defined

$$\begin{aligned}
\mathcal{R}^{ijk} &= \psi^{ijk} - 3\nu\beta^{[il}\theta_l\beta^{jk]} + \beta^{li}\beta^{mj}\beta^{nk}\phi_{lmn}\,,\\
\mathcal{Q}^{ij}_k &= -3\mu\theta_k\beta^{ij} + 3\nu\beta^{[il}\theta_l h^{j]}_k + 3B_{lk}\psi^{ijl} + 3(1+\beta B)^l_k\beta^{mi}\beta^{nj}\phi_{lmn}\,,\\
\mathcal{F}^i_{jk} &= -3\mu\theta_{[j}h^i_{k]} - 3f^i_{jk} - 3\nu\beta^{il}\theta_l B_{jk} + 3B_{lj}B_{mk}\psi^{lmi} + 3(1+\beta B)^l_j(1+\beta B)^m_k\beta^{ni}\phi_{lmn}\,,\\
\mathcal{H}_{ijk} &= (1+\beta B)^l_i(1+\beta B)^m_j(1+\beta B)^n_k\phi_{lmn} - 3\mu\theta_{[i}B_{jk]} + B_{li}B_{mj}B_{nk}\psi^{lmn}\ .
\end{aligned} \tag{3.22}$$

This long expression reveals the rich structure of the type of models we consider. In certain limits the condition (3.21) simplifies drastically and reduces to known results, as we will discuss in the next section.

Second, consider the alternative boundary conditions

$$\begin{aligned}
F_a|_{\Sigma_2} &= -\frac{1}{2}\partial_a B_{jk}q^j\wedge q^k - \frac{1}{2}\partial_a\beta^{jk}p_j\wedge p_k - \frac{1}{2}\partial_a h^k_j q^j\wedge p_k\,,\\
(k'p_i + B_{ij}q^j + \frac{1}{2}h^j_i p_j)|_{\Sigma_2} &= 0\,,\\
\delta p_i|_{\Sigma_2} &= 0\ .
\end{aligned} \tag{3.23}$$

As before, for $k'\neq 0$ and defining $\chi' = 1 - \frac{1}{2k'}h$ we can write

$$p_i = -\frac{1}{k'}\chi'^k_i B_{kj}q^j\,, \tag{3.24}$$

which will now yield a consistency condition different from the previous case. The new calculation leads to

$$\mathcal{H}_{ijk} - \frac{1}{k'}\mathcal{F}^n_{[ij}B_{\underline{p}k]}\chi'^p_n + \frac{1}{k'^2}\mathcal{Q}^{mn}_{[i}B_{\underline{p}j}B_{\underline{q}k]}\chi'^p_m\chi'^q_n - \frac{1}{k'^3}\mathcal{R}^{lmn}B_{pi}B_{qj}B_{rk}\chi'^p_l\chi'^q_m\chi'^r_n = 0\,, \tag{3.25}$$

with the same definitions (3.22).

Finally, let us comment on the possibility of using the boundary conditions

$$\begin{aligned}
F_a|_{\Sigma_2} &= -\frac{1}{2}\partial_a B_{jk}q^j\wedge q^k - \frac{1}{2}\partial_a\beta^{jk}p_j\wedge p_k - \frac{1}{2}\partial_a h^k_j q^j\wedge p_k\,,\\
(k'p_i + B_{ij}q^j + \frac{1}{2}h^j_i p_j)|_{\Sigma_2} &= 0\,,\\
(kq^i + \beta^{ij}p_j - \frac{1}{2}h^i_j q^j)|_{\Sigma_2} &= 0\ .
\end{aligned} \tag{3.26}$$

Now we obtain that both (3.20) and (3.24) should hold. This leads to the equations

$$\begin{aligned}
(1 - \frac{1}{kk'}\chi' B\chi\beta)^j_i p_j &= 0\,,\\
(1 - \frac{1}{kk'}\chi\beta\chi' B)^i_j q^j &= 0
\end{aligned} \tag{3.27}$$

which in general force $p_i = q^i = 0$, which is way too restrictive.

As a final remark, it should be clear that the above sets of boundary conditions are just two illustrative cases and they do not exhaust the range of possibilities, since one can impose mixed boundary conditions too. We will encounter interesting cases of mixed boundary conditions later.

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Twisted Dirac structure	Bracket $[\cdot,\cdot]_T$	Condition
L_B	$[\cdot,\cdot]_H$	$\mathrm{d}B = H$
L^*_β	$[\cdot,\cdot]_R$	$\frac{1}{2}[\beta,\beta] = R$
L_B	$[\cdot,\cdot]_R$	$(\mathrm{d}B)_{ijk} - \frac{1}{3}B_{il}B_{jm}B_{kn}R^{lmn} = 0$
L^*_β	$[\cdot,\cdot]_H$	$([\beta,\beta])^{ijk} - \frac{2}{3}\beta^{il}\beta^{jm}\beta^{kn}H_{lmn} = 0$
L_B	$[\cdot,\cdot]_{HR}$	$\mathrm{d}B = H$ and $B_{il}B_{jm}B_{kn}R^{lmn} = 0$
L^*_β	$[\cdot,\cdot]_{HR}$	$\frac{1}{2}[\beta,\beta] = R$ and $\beta^{il}\beta^{jm}\beta^{kn}H_{lmn} = 0$

Table 1. Integrability conditions for the almost Dirac structures L_B and $L^\star_\beta$ with H or/and R twists.

3.3 Bulk/boundary versus integrability conditions for Dirac structures

Let us explore some limits of the bulk/boundary consistency conditions (3.21) and (3.25) and show that they reduce to previously obtained results. In particular we show that they are equivalent to the integrability conditions for twisted almost Dirac structures. Recall that a Dirac structure L is a subbundle of a CA E which satisfies the following two conditions:

$$\langle L, L\rangle_E = 0\,, \tag{3.28}$$

$$[L, L]_E \in L\,, \tag{3.29}$$

namely it is maximally isotropic and involutive with respect to the CA bracket [25]. An almost Dirac structure is just a maximal isotropic subbundle, i.e. the bundle before the second condition is imposed. Imposing the closure condition yields an integrability condition for L.

In ref. [8], the study of twisted almost Dirac structures led us to the results summarized in table 1 for the vector bundles $L_B = e^B\mathrm{TM}$ and $L^\star_\beta = e^\beta\mathrm{T}^\star\mathrm{M}$ with various choices of the CA bracket.[8]

The brackets that appear on the table are the Courant bracket twisted by H, R or both.[9] With the choice of bracket $[\cdot,\cdot]_T$, the integrability condition of the third column must hold. The conditions in the first and second row are of course standard. Additionally, the condition in the fourth row is also standard and it corresponds to the H-twisted Poisson sigma model [31, 32, 66]. This table can also be obtained in the context of the AKSZ sigma models and we now show how (see also [30, 33] for related discussions).

[8] Slight differences to [8] in factors and signs are due to change of conventions on one hand and different way of presentation on the other hand.

[9] One should be cautious about the differences with the twists ϕ and ψ. In ref. [8] it was assumed that the bracket twists are exactly $H = \mathrm{d}B$ and $R = \frac{1}{2}[\beta,\beta]_\mathrm{S}$, while in the present setting the twists ϕ and ψ are more general. We clarify this further below.

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Case 1: Dirac structure L_B. In this case we set $\beta = 0$ and $h = 0$, and we keep only a nonvanishing B. Additionally, we make the following choice of parameters:

$$k = 0\,, \quad k' = 1\,, \quad \mu = 1\,, \quad \nu = \{0, 1\}\ . \tag{3.30}$$

The relation (3.24) simply becomes

$$p_i = -B_{ij} q^j\ . \tag{3.31}$$

Then eq. (3.22) gives

$$\begin{aligned} \mathcal{R}^{ijk} &= \psi^{ijk}\,, \\ \mathcal{Q}^{ij}_k &= 3B_{lk}\psi^{ijl}\,, \\ \mathcal{F}^i_{jk} &= -3f^i_{jk} + 3B_{lj}B_{mk}\psi^{lmi}\,, \\ \mathcal{H}_{ijk} &= \phi_{ijk} - 3\theta_{[i}B_{jk]} + B_{li}B_{mj}B_{nk}\psi^{lmn}\ . \end{aligned} \tag{3.32}$$

The bulk/boundary consistency condition (3.25) reduces to the significantly simpler expression

$$\phi_{ijk} - 3\theta_{[i}B_{jk]} - 3f^l_{[ij}B_{k]l} = 0\,, \tag{3.33}$$

or equivalently

$$\phi - \mathrm{d}B = 0 \tag{3.34}$$

in index-free notation. In order to compare this condition with the ones given in table 1, we recall that these results were obtained by twisting the bracket on L_B with a 3-vector R and/or a 3-form H. Therefore it is useful to write the fluxes ϕ and ψ reduced to the boundary as

$$\phi + \psi = \frac{1}{6}(\phi_{ijk} + B_{li}B_{mj}B_{nk}\psi^{lmn})q^i \wedge q^j \wedge q^k + \frac{1}{6}\psi^{ijk}p_i \wedge p_j \wedge p_k\ . \tag{3.35}$$

The first line in table 1 corresponds to the case of $\phi_{ijk} = H_{ijk}$ and $\psi^{ijk} = R^{ijk} = 0$, whence the integrability condition (3.34) gives $dB = H$. Similarly, the third line corresponds to $\phi_{ijk} = B_{il}B_{jm}B_{kn}\psi^{lmn}$ and $\psi^{ijk} = R^{ijk}$, leading to the integrability condition $(dB)_{ijk} = \frac{1}{3}B_{il}B_{jm}B_{kn}R^{lmn}$. Finally, for the fifth line we have $\phi_{ijk} = H_{ijk}$ and $\psi^{ijk} = R^{ijk}$ thus giving $dB = H$ and $B_{il}B_{jm}B_{kn}R^{lmn} = 0$.

Case 2: Dirac structure $L^\star_\beta$. In this case we set $B = 0$ and $h = 0$, and we keep only a nonvanishing β. We choose the parameters

$$k = 1\,, \quad k' = 0\,, \quad \mu = 1\,, \quad \nu = 0\ . \tag{3.36}$$

The relation (3.20) becomes

$$q^i = -\beta^{ij}p_j\,, \tag{3.37}$$

and the definitions (3.22)

$$\mathcal{R}^{ijk} = \psi^{ijk} + \beta^{li}\beta^{mj}\beta^{nk}\phi_{lmn}\,,$$

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$$\begin{aligned}\mathcal{Q}^{ij}_k &= -3\theta_k\beta^{ij} + 3\beta^{li}\beta^{mj}\phi_{klm}\,,\\ \mathcal{F}^i_{jk} &= -3f^i_{jk} + 3\beta^{li}\phi_{jkl}\,,\\ \mathcal{H}_{ijk} &= \phi_{ijk}\ .\end{aligned} \tag{3.38}$$

Then the bulk/boundary consistency condition (3.21) reduces to

$$\psi^{ijk} - 3\beta^{[il}\theta_l\beta^{jk]} - 3f^i_{mn}\beta^{mj}\beta^{nk} = 0\,, \tag{3.39}$$

or equivalently

$$\psi - \frac{1}{2}[\beta,\beta]_{\mathrm{S}} = 0\ . \tag{3.40}$$

As before, this expression directly yields the integrability conditions for the almost Dirac structure $L^\star_\beta$ appearing in the second, fourth and sixth rows of table 1. To make this explicit we write the fluxes ϕ and ψ reduced to the boundary as

$$\phi + \psi = \frac{1}{6}\phi_{ijk}q^i\wedge q^j\wedge q^k + \frac{1}{6}(\psi^{ijk} + \beta^{li}\beta^{mj}\beta^{nk}\phi_{lmn})p_i\wedge p_j\wedge p_k\,, \tag{3.41}$$

Then the second row in table 1 corresponds to $\phi_{ijk} = 0$ and $\psi^{ijk} = R^{ijk}$, thus reducing (3.40) to $R = \frac{1}{2}[\beta,\beta]_{\mathrm{S}}$. Similarly, the fourth line in the table is obtained when $\phi_{ijk} = H_{ijk}$ and $\psi^{ijk} = -\beta^{li}\beta^{mj}\beta^{nk}\phi_{lmn}$, resulting in the integrability condition $\beta^{il}\beta^{jm}\beta^{kn}H_{lmn} = \frac{3}{2}([\beta,\beta]_{\mathrm{S}})^{ijk}$. For the sixth line $\phi_{ijk} = H_{ijk}$ and $\psi^{ijk} = R^{ijk}$ and the integrability condition (3.40) reduces to $R = \frac{1}{2}[\beta,\beta]_{\mathrm{S}}$ and $\beta^{li}\beta^{mj}\beta^{nk}H_{lmn} = 0$.

The pattern is already obvious. The choice of bracket corresponds to the choice of twist in the membrane action. The choice of Dirac structure deformation corresponds to the choice of boundary condition on the boundary string. The integrability condition corresponds to consistency of the boundary conditions with the bulk action. This dictionary is summarized as:

Courant algebroid	Sigma model
Bracket twist $[\cdot,\cdot]_T$	Bulk term $-\int_{\Sigma_3} T$
Dirac structure deformation $L_{\mathcal{B}}$	Boundary term $\int_{\partial\Sigma_3}\mathcal{B}$
Integrability condition for Dirac structure	Bulk/boundary consistency condition

3.4 2D sigma models with dynamics

Up to now we discussed the 3D topological field theory. For physical applications, notably for string theory, it is necessary to look at the corresponding 2D theory on the boundary and add dynamics to it. Let us first revisit the two cases of section 3.3 from this perspective.

For the first case of L_B, the F_a equation of motion yields

$$q^i = e^i_a \mathrm{d}X^a = e^i\ . \tag{3.42}$$

Using this to integrate out the auxiliary field and adding dynamics in the standard way, we obtain the familiar 2D field theory with Wess-Zumino term

$$S = \int_{\Sigma_2}\left(\frac{1}{2}g_{ij}e^i\wedge\star e^j + \frac{1}{2}B_{ij}e^i\wedge e^j\right) - \int_{\Sigma_3}\frac{1}{6}\phi_{ijk}e^i\wedge e^j\wedge e^k\ . \tag{3.43}$$

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In the second case of $L^\star_\beta$, the F_a equation again gives (3.42), and integrating out the auxiliary 2-form produces the action

$$S = \int_{\Sigma_2} \left(\frac{1}{2}\tilde{g}^{ij} p_i \wedge \star p_j + p_i \wedge e^i + \frac{1}{2}\beta^{ij} p_i \wedge p_j \right) - \int_{\Sigma_3} \frac{1}{6}\psi^{ijk} p_i \wedge \ p_j \wedge p_k \,, \tag{3.44}$$

where in the spirit of the first order formalism we added dynamics with the inverse metric[10] $\tilde{g}^{ij}$, exactly as in ref. [4]. Some remarks regarding the action (3.44) are in order. First, when the bracket is twisted only with a 3-form H, one has $\psi^{ijk} = -\beta^{li}\beta^{mj}\beta^{nk}H_{lmn}$ and this is precisely the H-twisted Poisson sigma model [66] on a nilmanifold. In that case one can write the standard kinetic term. Furthermore, if β is invertible and its inverse is equal to B, then the action (3.44) with $g = -B\tilde{g}^{-1}B$ is equivalent to (3.43).

3.5 Explicit sigma model with both B and β, and $B\beta \notin \{0, 1\}$

In section 3.3 we showed that the general formulae of section 3.2 reproduce known results in the limits $B = 0$ and $\beta = 0$ respectively. However, in general none of B and β is zero, and moreover they do not have to satisfy any relation of the sort $B\beta = 1$, as is sometimes assumed. The results of section 3.2 reflect such general cases. In the present section we want to show that these results are not empty, in the sense that there indeed exist nontrivial cases where the consistency conditions of the AKSZ sigma model can be satisfied.

In order to be very explicit, let us consider the toy example of section 2.2.2, where the nonvanishing components of B and β are $B_{23} = NX^1$ and $\beta^{12} = \sqrt{c}$. Therefore

$$B\beta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sqrt{c}NX^1 & 0 & 0 \end{pmatrix} , \tag{3.45}$$

which is neither vanishing nor unity. In very explicit terms, the sigma model is

$$\begin{aligned} S = \int_{\Sigma_3} \Big(& F_a \wedge \mathrm{d}X^a + \frac{1}{2} q^i \wedge \mathrm{d}p_i + \frac{1}{2} p_i \wedge \mathrm{d}q^i - (q^1 - \sqrt{c}p_2) \wedge F_1 - (q^2 + \sqrt{c}p_1) \wedge F_2 \\ & - (q^3 + X^1 q^2 + \sqrt{c}X^1 p_1) \wedge F_3 + q^1 \wedge q^2 \wedge p_3 - Nq^1 \wedge q^2 \wedge q^3 \\ & - \sqrt{c}N(q^2 \wedge q^3 \wedge p_2 + q^1 \wedge q^3 \wedge p_1) - cNq^3 \wedge p_1 \wedge p_2 - cp_1 \wedge p_2 \wedge p_3 \\ & - cNX^1(p_3 \wedge p_1 \wedge q^3 + p_2 \wedge p_1 \wedge q^2) - c(NX^1)^2 p_1 \wedge q^2 \wedge q^3 \Big) \\ & + \int_{\Sigma_2} \left(NX^1 q^2 \wedge q^3 + \sqrt{c}p_1 \wedge p_2 \right) , \end{aligned} \tag{3.46}$$

where the indices a and i run from 1 to 3, and we have made the choices $k = k' = \frac{1}{2}$ and $\mu = \nu = 1$. Proceeding with the variations, the δ_{X^a} ones directly lead to the boundary condition

$$F_1 = -Nq^2 \wedge q^3 \,, \quad F_2 = F_3 = 0 \,. \tag{3.47}$$

[10] More precisely this is *not* an inverse metric but the standard metric on the dual algebroid structure.

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The variations δ_{p_i} and δ_{q^i} lead to the following set of relations:

$$(\frac{1}{2}q^1+\sqrt{c}p_2)\delta p_1=0\,,\qquad (\frac{1}{2}q^2-\sqrt{c}p_1)\delta p_2=0\,,\qquad (\frac{1}{2}q^3)\delta p_3=0\,,$$
$$(\frac{1}{2}p_1)\delta q^1=0\,,\quad (\frac{1}{2}p_2+NX^1q^3)\delta q^2=0\,,\quad (\frac{1}{2}p_3-NX^1q^2)\delta q^3=0\,. \tag{3.48}$$

Additionally, taking into account (3.47), the bulk/boundary consistency condition is

$$\begin{aligned}&N(q^1-\sqrt{c}p_2)\wedge q^2\wedge q^3+q^1\wedge q^2\wedge p_3-Nq^1\wedge q^2\wedge q^3\\&-\sqrt{c}N(q^2\wedge q^3\wedge p_2+q^1\wedge q^3\wedge p_1)-cNq^3\wedge p_1\wedge p_2-cp_1\wedge p_2\wedge p_3\\&-cNX^1(p_3\wedge p_1\wedge q^3+p_2\wedge p_1\wedge q^2)-c(NX^1)^2p_1\wedge q^2\wedge q^3=0\,.\end{aligned} \tag{3.49}$$

Now we have to choose appropriate boundary conditions, consistent with eqs. (3.48) and (3.49). The choice corresponding to (3.23) is

$$\delta p_i=0\,,\quad p_1=0\,,\quad p_2=-2NX^1q^3\,,\quad p_3=2NX^1q^2\,. \tag{3.50}$$

It is observed that eq. (3.13) gives $\psi=0$. This is a legitimate possibility but it is not so interesting because it makes one of the twists vanish. On the other hand, the choice $\delta q^i=0$ of (3.19) is *not* consistent with (3.49) for $c\neq 0$. This indicates that mixed boundary conditions are appropriate in order to keep both ϕ and ψ nonvanishing. We can find such conditions by first noting that

- $\delta q^1\neq 0\quad\Rightarrow\quad p_1=0\quad\Rightarrow\quad\psi=0\,,$
- $\delta p_3\neq 0\quad\Rightarrow\quad q^3=0\quad\Rightarrow\quad\phi=0\,,$
- $\big(\delta p_1\neq 0\quad\text{and}\quad\delta q^2\neq 0\big)\quad\Rightarrow\quad\big(p_2\propto q^3\quad\text{and}\quad q^1\propto q^3\big)\quad\Rightarrow\quad\phi=0\,,$
- $\big(\delta p_2\neq 0\quad\text{and}\quad\delta q^3\neq 0\big)\quad\Rightarrow\quad\big(p_3\propto p_1\quad\text{and}\quad q^2\propto p_1\big)\quad\Rightarrow\quad\psi=0\,.$

This leads to the necessary requirements

$$\delta q^1=0\,,\quad\delta p_3=0\,,\quad\big(\delta p_1=0\quad\text{or}\quad\delta q^2=0\big)\,,\quad\big(\delta p_2=0\quad\text{or}\quad\delta q^3=0\big)\,. \tag{3.51}$$

Let us choose $\delta q^2=\delta p_2=0$ for the last two requirements. The remaining boundary conditions from (3.48) are

$$q^1=-2\sqrt{c}p_2\,,\quad p_3=2NX^1q^2\,,\quad\text{on }\Sigma_2\,. \tag{3.52}$$

In order to be able to solve the bulk/boundary consistency condition we choose additionally

$$q^3+\frac{1}{2}\sqrt{c}X^1p_1=0\quad\text{on }\Sigma_2\,. \tag{3.53}$$

Then we find that on the boundary

$$\phi=\frac{1}{2}cNX^1q^2\wedge p_1\wedge p_2\,, \tag{3.54}$$
$$\psi=2\sqrt{c}Nq^2\wedge q^3\wedge p_2\,, \tag{3.55}$$

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and it is checked that the condition (3.49) is satisfied. This shows that the boundary conditions that were chosen are consistent with the AKSZ action, while both twists ϕ and ψ and both deformations B and β are nonvanishing. Focusing on 2D, the corresponding action can be brought to the form

$$\int_{\Sigma_2} \left(\frac{1}{2} g_{ij} e^i \wedge \star e^j + \frac{1}{2} p_i \wedge e^i + N X^1 e^2 \wedge e^3 + \sqrt{c} p_1 \wedge p_2 - \sqrt{c} N X^1 p_1 \wedge e^3 \right) . \tag{3.56}$$

This is a nontrivial case from the general class of 2D field theories called Dirac sigma models, introduced and studied in refs. [64, 65].

4 Toward a sigma model description of double field theory

We would like to examine to what extend the approach we adopted up to now can be carried on to account for genuinely non-geometric cases. As mentioned in the introduction, non-geometric situations are better understood in the doubled formalism, where non-geometry is triggered by the presence of dual coordinates. In the doubled field theory these where implemented in an effective field theory on some doubled spacetime. Here we do not work in a target space field theory framework, but instead we formulate the appropriate sigma model. This is close in spirit to the inspiring attempt of ref. [6] to describe non-geometric backgrounds in the context of AKSZ sigma models. The authors used this approach to discuss quantization of non-geometric backgrounds and limited their description to the single presence of R flux. This case is however known to be T-dual to standard H flux backgrounds and as such it is *not* a genuinely non-geometric background. In the following we will extend and generalize the scope of AKSZ inspired sigma models to account for more general cases.

4.1 Sigma models with doubled target space

Let us recall a key point in the analysis of ref. [6]. Consider the sigma model (3.1) associated to the standard CA on a torus. Moreover let $T = R$ be the only generalized 3-form with R a constant 3-vector. This means that the 3D action is[11]

$$S_R[X, A, F] = \int_{\Sigma_3} \left(F_a \wedge \mathrm{d}X^a + q^a \wedge \mathrm{d}p_a - q^a \wedge F_a + \frac{1}{6} R^{abc} p_a \wedge p_b \wedge p_c \right) , \tag{4.1}$$

where we used only early Latin indices because for the moment we refer to the flat torus. Integrating out the 2-form F_a one obtains

$$S_R[X, A, F] = \int_{\Sigma_2} p_a \wedge \mathrm{d}X^a + \int_{\Sigma_3} \frac{1}{6} R^{abc} p_a \wedge p_b \wedge p_c \, , \tag{4.2}$$

with X^a-independent R^{abc} by assumption. The equation of motion for X^a is simply

$$\mathrm{d}p_a = 0 \, . \tag{4.3}$$

[11] As mentioned in ref. [6] a 2D kinetic term should be included for consistency with the equations of motion. We do not explicitly write it here because it will not play a crucial role in the argument.

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This means that the 1-form p_a may be written locally as

$$p_a = \mathrm{d}\tilde{X}_a \,, \tag{4.4}$$

where $\tilde{X}_a \in C^\infty(\Sigma_3, X^\star \mathrm{T}^\star\mathrm{M})$. These $\tilde{X}_a$ are similar to the dual coordinates of DFT, which is the reason for our choice of notation. As suggested in ref. [6], in the sigma model they essentially correspond to an augmented embedding of the 2-dimensional boundary theory on Σ_2 in the full cotangent bundle of the target manifold M. In other words there are generalized (or doubled) target space coordinates $(\mathbb{X}^I) = (X^a, \tilde{X}_a)$ which correspond to the map $\mathbb{X} = (\mathbb{X}^I) : \Sigma_3 \to \mathrm{T}^\star\mathrm{M}$. Note that the appearance of the dual coordinates is very natural in this context, since they were suggested by the equations of motion of the sigma model.

An alternative way to think about the above doubling is in the spirit of the topological approach to T-duality [67, 68], which was explained via Courant algebroids in refs. [69–71]. In this approach there is a product manifold $\mathrm{M} \times \tilde{\mathrm{M}}$ of original and dual spaces and T-duality corresponds to an isomorphism of twisted K-theories [67, 68]. In [70] it was shown that this can be extended to an isomorphism between the corresponding CAs. Here we associate $\mathbb{X}^I$ to the product manifold $\mathrm{M} \times \tilde{\mathrm{M}}$. Presumably, the AKSZ sigma models for CAs over this extended target space correspond to the ones we will consider below. We plan to study this correspondence carefully in future work.

Once one considers the possibility of such generalized embeddings, it is natural to allow all the fields that appear in the model to depend both on X^a and $\tilde{X}_a$. In that case insisting on the formulation (3.1) for the sigma model is rather restrictive. From the viewpoint of physics, eq. (3.1) does not contain $d\tilde{X}_a$ at all, which should not be the case in general. Thus, returning to the general case, our proposal here is twofold. First, allow B, β, h, a and T to depend on both X^a and $\tilde{X}_a$. Second, introduce a second world volume 2-form $\tilde{F}^a \in \Omega^2(\Sigma_3, \mathbb{X}^\star\mathrm{TM})$; note that this is again an auxiliary world volume 2-form like F_a, with the difference of having a *vector* index instead. Then we write the 3-dimensional action

$$S_{\Sigma_3} = \int_{\Sigma_3} \left(F_a \wedge \mathrm{d}X^a + \tilde{F}^a \wedge \mathrm{d}\tilde{X}_a + \frac{1}{2}\eta_{IJ} A^I \wedge \mathrm{d}A^J - P^a_I A^I \wedge F_a - \tilde{P}_{aI} A^I \wedge \tilde{F}^a + \frac{1}{6} T_{IJK} A^I \wedge A^J \wedge A^K \right) . \tag{4.5}$$

In more compact notation, writing $P^J_I = (P^a_I, \tilde{P}_{aI})$ and $F^I = (F_a, \tilde{F}^a)$ for $F^I \in \Omega^2(\Sigma_3, \mathbb{X}^\star E)$, we get

$$S_{\Sigma_3} = \int_{\Sigma_3} \left(\delta_{IJ} F^I \wedge \mathrm{d}\mathbb{X}^J + \frac{1}{2}\eta_{IJ} A^I \wedge \mathrm{d}A^J - \delta_{JK} P^J_I A^I \wedge F^K + \frac{1}{6} T_{IJK} A^I \wedge A^J \wedge A^K \right) . \tag{4.6}$$

The boundary action is the same as before, namely

$$S_{\Sigma_2} = \int_{\Sigma_2} \frac{1}{2} \mathcal{B}_{IJ} A^I \wedge A^J \,, \tag{4.7}$$

with the difference that $\mathcal{B} = \mathcal{B}(X, \tilde{X})$. An important remark regards the object $\tilde{P}_{aI}$, which was absent before. These are the components of a map $\tilde{P} : E \to \mathrm{T}^\star\mathrm{M}$ that maps elements of the Courant algebroid to the cotangent bundle. Examples of such a map is the unit map on 1-forms and the map $B^\sharp : \mathrm{TM} \to \mathrm{T}^\star\mathrm{M}$ that acts simply as $B^\sharp(X_i) = B_{ij}\eta^j$.

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Our purpose now is to consider the analog of the construction we did for the CA $L_{B\beta} \oplus L^{\star}_{B\beta}$, bearing in mind that a complete mathematical characterization of the construction is due. The ingredients are similar to the standard case. We consider the twists ϕ and ψ, as given in eqs. (3.12) and (3.13), as well as the geometric twist f of the nilmanifold that appears in eq. (3.11). Then the action reads as

$$\begin{aligned} S = \int_{\Sigma_3} \Big(& F_a \wedge \mathrm{d}X^a + \tilde{F}^a \wedge \mathrm{d}\tilde{X}_a + k q^i \wedge \mathrm{d}p_i + k' p_i \wedge \mathrm{d}q^i \\ & - (\mu e^a_i q^i + \nu \beta^{ij} e^a_j p_i) \wedge F_a - (\mu' e^i_a p_i + \nu' B_{ij} e^j_a q^i) \wedge \tilde{F}^a + f - \phi - \psi \Big) \\ & + \int_{\Sigma_2} \left(\frac{1}{2} B_{ij} q^i \wedge q^j + \frac{1}{2} \beta^{ij} p_i \wedge p_j + \frac{1}{2} h^j_i q^i \wedge p_j \right) . \end{aligned} \tag{4.8}$$

For the map $\tilde{P}$ we took

$$\tilde{P}^i_a = \mu' e^i_a \quad \text{and} \quad \tilde{P}_{ai} = \nu' B_{ij} e^j_a \,, \tag{4.9}$$

which is the natural choice. As before, the parameters μ, ν, μ', ν' are valued in $\{0, 1\}$, which reflects the flexibility of trivializing the corresponding map or not. Once more, k and k' should satisfy $k + k' = 1$.

Next we determine the equations of motion on the boundary by varying with respect to $X^a, \tilde{X}_a, q^i$ and p_i. The only new equation is

$$\delta_{\tilde{X}_a} S|_{\Sigma_2} = \tilde{F}^a + \frac{1}{2} \tilde{\partial}^a B_{jk} q^j \wedge q^k + \frac{1}{2} \tilde{\partial}^a \beta^{jk} p_j \wedge p_k + \frac{1}{2} \tilde{\partial}^a h^k_j q^j \wedge p_k = 0 \,, \tag{4.10}$$

where $\tilde{\partial}^a = \partial / \partial \tilde{X}_a$. The other three equations are exactly as in (3.17). Additionally, the bulk/boundary condition that should hold reads as

$$(\mu e^a_i q^i + \nu \beta^{ij} e^a_j p_i) \wedge F_a + (\mu' e^i_a p_i + \nu' B_{ij} e^j_a q^i) \wedge \tilde{F}^a = f - \phi - \psi \qquad \text{on } \Sigma_2 \,. \tag{4.11}$$

This has to be consistent with the choice of boundary conditions that guarantee the equations of motion on the boundary.

Let us examine how the boundary conditions that were considered in section 3.2 are modified. First we consider the boundary conditions

$$\begin{aligned} F_a|_{\Sigma_2} &= -\frac{1}{2} \partial_a B_{jk} q^j \wedge q^k - \frac{1}{2} \partial_a \beta^{jk} p_j \wedge p_k - \frac{1}{2} \partial_a h^k_j q^j \wedge p_k \,, \\ \tilde{F}^a|_{\Sigma_2} &= -\frac{1}{2} \tilde{\partial}^a B_{jk} q^j \wedge q^k - \frac{1}{2} \tilde{\partial}^a \beta^{jk} p_j \wedge p_k - \frac{1}{2} \tilde{\partial}^a h^k_j q^j \wedge p_k \,, \\ \delta q^i|_{\Sigma_2} &= 0 \,, \\ (k q^i + \beta^{ij} p_j - \frac{1}{2} h^i_j q^j)|_{\Sigma_2} &= 0 \,. \end{aligned} \tag{4.12}$$

The bulk/boundary consistency condition (4.11) becomes formally identical to (3.21), namely

$$\mathcal{R}^{ijk} - \frac{1}{k} \mathcal{Q}^{[ij}_n \beta^{pk]} \chi^n_p + \frac{1}{k^2} \mathcal{F}^{[i}_{mn} \beta^{pj} \beta^{qk]} \chi^m_p \chi^n_q - \frac{1}{k^3} \mathcal{H}_{lmn} \beta^{pi} \beta^{qj} \beta^{rk} \chi^l_p \chi^m_q \chi^n_r = 0 \,,$$

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but with the upgraded definitions

$$\begin{aligned}
\mathcal{R}^{ijk} &= \psi^{ijk} - 3\nu\beta^{[il}\theta_l\beta^{jk]} + \beta^{li}\beta^{mj}\beta^{nk}\phi_{lmn} - 3\mu'\tilde{\theta}^{[i}\beta^{jk]}\,, \\
\mathcal{Q}^{ij}_k &= -3\mu\theta_k\beta^{ij} + 3\nu\beta^{[il}\theta_l h^{j]}_k + 3B_{lk}\psi^{ijl} + 3(1+\beta B)^l_k\beta^{mi}\beta^{nj}\phi_{lmn} + 3\mu'\tilde{\theta}^{[i}h^{j]}_k - 3\nu' B_{kl}\tilde{\theta}^l\beta^{ij}\,, \\
\mathcal{F}^i_{jk} &= -3\mu\theta_{[j}h^i_{k]} - 3f^i_{jk} - 3\nu\beta^{il}\theta_l B_{jk} + 3B_{lj}B_{mk}\psi^{lmi} + 3(1+\beta B)^l_j(1+\beta B)^m_k\beta^{ni}\phi_{lmn} \\
&\quad -3\mu'\tilde{\theta}^i B_{jk} - 3\nu' B_{[jl}\tilde{\theta}^l h^i_{k]}\,, \\
\mathcal{H}_{ijk} &= (1+\beta B)^l_i(1+\beta B)^m_j(1+\beta B)^n_k\phi_{lmn} - 3\mu\theta_{[i}B_{jk]} + B_{li}B_{mj}B_{nk}\psi^{lmn} - 3\nu' B_{[il}\tilde{\theta}^l B_{jk]}\,,
\end{aligned} \tag{4.13}$$

where we defined $\tilde{\theta}^i = e^i_a\tilde{\partial}^a$. Similarly, the boundary conditions

$$\begin{aligned}
F_a|_{\Sigma_2} &= \frac{1}{2}\partial_a B_{jk}q^j\wedge q^k + \frac{1}{2}\partial_a\beta^{jk}p_j\wedge p_k + \frac{1}{2}\partial_a h^k_j q^j\wedge p_k\,, \\
\tilde{F}^a|_{\Sigma_2} &= \frac{1}{2}\tilde{\partial}^a B_{jk}q^j\wedge q^k + \frac{1}{2}\tilde{\partial}^a\beta^{jk}p_j\wedge p_k + \frac{1}{2}\tilde{\partial}^a h^k_j q^j\wedge p_k\,, \\
(k'p_i - B_{ij}q^j - \frac{1}{2}h^j_i p_j)|_{\Sigma_2} &= 0\,, \\
\delta p_i|_{\Sigma_2} &= 0\,,
\end{aligned} \tag{4.14}$$

lead to the generalization of the alternative condition (3.25), namely

$$\mathcal{H}_{ijk} - \frac{1}{k'}\mathcal{F}^n_{[ij}B_{\underline{p}k]}\chi'^p_n + \frac{1}{k'^2}\mathcal{Q}^{mn}_{[i}B_{\underline{p}j}B_{\underline{q}k]}\chi'^p_m\chi'^q_n - \frac{1}{k'^3}\mathcal{R}^{lmn}B_{pi}B_{qj}B_{rk}\chi'^p_l\chi'^q_m\chi'^r_n = 0\,,$$

with the definitions (4.13).

4.2 The pure R flux limit

Let us briefly revisit the pure R flux limit of ref. [6] with the results of section 4.1. Consider $B = h = 0$ and $\beta = \beta(\tilde{X})$ to be independent of X^a. Additionally, let us turn off the ϕ flux and geometric flux f just for the present example, namely $f = \phi = 0$. With these assumptions, eqs. (4.13) reduce to

$$\begin{aligned}
\mathcal{R}^{ijk} &= \psi^{ijk} - 3\tilde{\partial}^{[i}\beta^{jk]}\,, \\
\mathcal{Q} = \mathcal{F} = \mathcal{H} &= 0\,.
\end{aligned} \tag{4.15}$$

We choose the boundary conditions (4.12), in which case the bulk/boundary condition (3.21) reads as

$$\mathcal{R}^{ijk} = 0 \quad\Rightarrow\quad \psi^{ijk} = 3\tilde{\partial}^{[i}\beta^{jk]}\,. \tag{4.16}$$

In case β is linear in $\tilde{X}_a$, e.g. $\beta = \tilde{N}\epsilon^{ijk}\delta^a_k\tilde{X}_a p_i\wedge p_j$, we can identify ψ with the constant R flux, e.g. $R = \tilde{N}$. This is similar to the case considered in [6], where more details may be found.

4.3 Genuine non-geometry?

The main motivation for the formulation we propose in section 4.1 was to examine the possibility to construct genuinely non-geometric models, in the sense that was explained in the Introduction. This is essentially the message of the results in section 4.1, but in order

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to make sure that they are not empty and indeed contain nontrivial cases we construct here an explicit toy model.

Consider the sigma model on the twisted torus of section 3.5, upgraded to a model of the type (4.8) with the following background fields instead:

$$B = NX^1 q^2 \wedge q^3 \,, \quad \beta = \tilde{N}\tilde{X}_2 p_1 \wedge p_3 \,. \tag{4.17}$$

Both B and β are nonvanishing, and they satisfy

$$\beta B = \begin{pmatrix} 0 & -N\tilde{N}X^1\tilde{X}_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \,. \tag{4.18}$$

Moreover, in this case we identify ϕ and ψ with the corresponding derivations, namely

$$\phi = Nq^1 \wedge q^2 \wedge q^3 \,, \quad \psi = \tilde{N} p_1 \wedge p_2 \wedge p_3 \,. \tag{4.19}$$

Making the choices $k = k' = \frac{1}{2}$ and $\mu = \mu' = \nu = \nu' = 1$, the model is given as

$$\begin{aligned} S = \int_{\Sigma_3} \Big(& F_a \wedge \mathrm{d}X^a + \tilde{F}^a \wedge \mathrm{d}\tilde{X}_a + \frac{1}{2} p_i \wedge \mathrm{d}q^i + \frac{1}{2} q^i \wedge \mathrm{d}p_i \\ & - F_1 \wedge (q^1 - \tilde{N}\tilde{X}_2 p_3) - F_2 \wedge q^2 - F_3 \wedge (q^3 + X^1 q^2 + \tilde{N}\tilde{X}_2 p_1) \\ & - \tilde{F}^1 \wedge p_1 - \tilde{F}^2 \wedge (p_2 - X^1 p_3 - NX^1 q^3 - N(X^1)^2 q^2) - \tilde{F}^3 \wedge (p_3 + NX^1 q^2) \\ & + q^1 \wedge q^2 \wedge p_3 - \phi - \psi \Big) \\ + \int_{\Sigma_2} & \Big(NX^1 q^2 \wedge q^3 + \tilde{N}\tilde{X}_2 p_1 \wedge p_3 \Big) \,, \end{aligned} \tag{4.20}$$

where in the present case we find

$$\begin{aligned} \phi &= Nq^1 \wedge q^2 \wedge q^3 + N\tilde{N}\tilde{X}_2 (q^2 \wedge q^3 \wedge p_3 + q^2 \wedge q^1 \wedge p_1) + N(\tilde{N}\tilde{X}_2)^2 q^2 \wedge p_3 \wedge p_1 \,, \\ \psi &= \tilde{N} p_1 \wedge p_2 \wedge p_3 + N\tilde{N}X^1 (p_3 \wedge p_1 \wedge q^3 + p_2 \wedge p_1 \wedge q^2) + \tilde{N}(NX^1)^2 p_1 \wedge q^2 \wedge q^3 \,. \end{aligned} \tag{4.21}$$

The equations of motion for X^a and $\tilde{X}_a$ lead to the boundary conditions for F_a and $\tilde{F}^a$:

$$\begin{aligned} F_1 &= -Nq^2 \wedge q^3 \,, & F_2 &= F_3 = 0 \,, \\ \tilde{F}^2 &= -\tilde{N} p_1 \wedge p_3 \,, & \tilde{F}^1 &= \tilde{F}^3 = 0 \,. \end{aligned} \tag{4.22}$$

Additionally, the equations of motion for q^i and p_i lead to:

$$\begin{aligned} &(\tfrac{1}{2} q^1 + \tilde{N}\tilde{X}_2 p_3)\delta p_1 = 0 \,, \qquad (\tfrac{1}{2} q^2)\delta p_2 = 0 \,, \qquad (\tfrac{1}{2} q^3 - \tilde{N}\tilde{X}_2)\delta p_3 = 0 \,, \\ &(\tfrac{1}{2} p_1)\delta q^1 = 0 \,, \quad (\tfrac{1}{2} p_2 + NX^1 q^3)\delta q^2 = 0 \,, \quad (\tfrac{1}{2} p_3 - NX^1 q^2)\delta q^3 = 0 \,. \end{aligned} \tag{4.23}$$

An analysis similar to section 3.5 dictates the boundary conditions

$$\delta q^1 = 0 \,, \quad \delta p_2 = 0 \,, \quad \big(\delta p_3 = 0 \quad \text{or} \quad \delta q^2 = 0\big) \,, \quad \big(\delta p_1 = 0 \quad \text{or} \quad \delta q^3 = 0\big) \,. \tag{4.24}$$

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Out of the last two requirements we make the random choice $\delta p_3 = \delta q^3 = 0$. The remaining boundary conditions are

$$q^1 = -2\tilde{N}\tilde{X}_2 p_3\,, \quad p_2 = -2NX^1 q^3\,. \tag{4.25}$$

Imposing also that $X^1 p_1 - 2\tilde{X}_2 q^2 = 0$, we find that on the boundary

$$\begin{aligned} \phi &= -N\tilde{N}\tilde{X}_2 p_3 \wedge q^2 \wedge q^3\,, \\ \psi &= N\tilde{N}X^1 p_1 \wedge p_3 \wedge q^3\,, \end{aligned} \tag{4.26}$$

and that the bulk/boundary consistency condition

$$F_1 \wedge (q^1 - \tilde{N}\tilde{X}_2 p_3) + \tilde{F}^2 \wedge (p_2 - X^1 p_3 - NX^1 q^3 - N(X^1)^2 q^2) + q^1 \wedge q^2 \wedge p_3 - \phi - \psi = 0\,, \tag{4.27}$$

is satisfied. This means that the model is a nontrivial case where the twists ϕ and ψ, as well as the deformations B and β, are nonvanishing. Unlike the model with pure 3-vector flux, which is well known to be T-dual to standard geometric models, the present case cannot be T-dualized to a standard geometry. Thus it constitutes a genuine case of non-geometry.

The latter statement is corroborated by attempting to write down the corresponding 2D string model. This is not possible just in terms of X^a; instead $\tilde{X}_a$ necessarily appear, similarly to the pure R-flux models considered in refs. [4, 6] but in a significantly more complicated way. The topological sector of the corresponding model can be written as

$$\begin{aligned} &\int_{\Sigma_3} \Big(8N^2\tilde{N}\tilde{X}_1 \mathrm{e}^1 \wedge e^2 \wedge e^3 + 4N\tilde{N}^2 X^2 \tilde{X}_2 \mathrm{d}\tilde{X}_1 \wedge \mathrm{d}\tilde{X}_2 \wedge \mathrm{d}\tilde{X}_3 + N\tilde{N}\tilde{X}_2 e^1 \wedge e^2 \wedge \mathrm{d}\tilde{X}_1 + \\ &-2N\tilde{N}(\tilde{X}_2 e^2 - \tilde{X}_1 e^1) \wedge e^3 \wedge \mathrm{d}\tilde{X}_3 \Big) + \\ &+ \int_{\Sigma_2} \Big(-NX^1(1 + 4N\tilde{N}X^1\tilde{X}_1) e^2 \wedge e^3 + \tilde{N}(\tilde{X}_2 + NX^2(X^1)^2 + 2N\tilde{N}X^2\tilde{X}_2^2) \mathrm{d}\tilde{X}_1 \wedge \mathrm{d}\tilde{X}_3 + \\ &+ \frac{3}{2} N\tilde{N}X^1\tilde{X}_2 e^2 \wedge \mathrm{d}\tilde{X}_1 - 2N\tilde{N}X^1\tilde{X}_1(e^3 - X^2 e^1) \wedge \mathrm{d}\tilde{X}_3 \Big)\,, \end{aligned} \tag{4.28}$$

which supports the above remarks. Notice that for $\tilde{N} \to 0$, namely when the deformation β is turned off, we obtain

$$\int_{\Sigma_2} -NX^1 e^2 \wedge e^3\,, \tag{4.29}$$

while for $N \to 0$ (or $B = 0$) we obtain

$$\int_{\Sigma_2} \tilde{N}\tilde{X}_2 \mathrm{d}\tilde{X}_1 \wedge \mathrm{d}\tilde{X}_3\,, \tag{4.30}$$

as expected.

5 Conclusions

The extended nature of the fundamental degrees of freedom in string theory leads to duality symmetries, whose consequences are unconventional from a traditional field theory

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viewpoint. One of these consequences is our encounter with non-geometric string backgrounds. A central question in this line of research is whether such backgrounds are always equivalent (up to duality) to previously known geometric ones or there exist ones that are truly new. Recent developments, mainly in the context of DFT, suggest that duality orbits of flux configurations that do not intersect geometric regions indeed exist [22].

In this paper we addressed the problem of constructing sigma models that correspond to genuinely non-geometric backgrounds. This approach is inspired by previous work along these lines in the string theory literature [4, 6], which we extended and generalized. The underlying mathematical setting is that of Courant algebroids, which has recently found applications in the physics of string theory [4–8]. Here we constructed a general class of CAs with base manifolds being twisted tori. The choice of twisted tori is made for a number of reasons, in particular (i) they are the simplest nontrivial generalization of flat tori that retain parallelizability and they can be endowed with all kinds of generalized complex structures [72],[12] (ii) they naturally incorporate geometric fluxes, and (iii) they play a central role in flux compactifications, notably in Scherk-Schwarz reductions. We followed the approach of introducing the basic mathematical notions first, then applying them for general twisted tori of step 2, and finally examining in detail an illustrative example from the class.

In order to reach our main goal of constructing relevant sigma models, we resided on the result that every CA structure over a manifold M has an associated topological sigma model with M as target space [28]. For physical applications, it is natural to consider manifolds with boundary and add general topological boundary terms and also kinetic terms that break the topological nature of the model. Studying the corresponding membrane sigma models for the class of CAs we constructed, we found general bulk/boundary consistency conditions appearing in eqs. (3.21) and (3.25). These expressions generalize on one hand previously known integrability conditions for Dirac structures [31–33], and on the other hand allow for a systematic characterization of fluxes, extending expressions found in [4]. In certain limits, our expressions reproduce previous results; on the other hand we also studied in detail a case where both 2-form and 2-vector deformations coexist meaningfully without being inverse of one another.

However, in order to really account for cases that appear in string theory via generalized T-duality, the above sigma models cannot be the end of the story. This was already noticed in [4], and later in [6], where sigma models of an extended type were first suggested. These sigma models have the phase space of M as target space, instead of M itself. Inspired by this approach, we proposed a minimal systematic generalization of the previous sigma models that incorporates this doubled point of view. Analysing such models we found that the bulk/boundary consistency conditions take again the form appearing in eqs. (3.21) and (3.25), albeit with an upgraded set of definitions that characterize the fluxes of the model. Then we were able to write down an explicit example of a model which combines the following properties: (i) all types of generalized fluxes are present, (ii) it cannot be reduced to a 2D theory with standard target space and (iii) it cannot be dualized to a

[12]For example, several twisted tori admit a symplectic structure and their phase space can be completely characterized, see [61].

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standard geometric model. This makes this example, and any other constructed similarly, an excellent toy model for genuinely non-geometric backgrounds.

Finally, it is interesting to compare our results with the expressions for fluxes found in the context of DFT and its generalized Scherk-Schwarz dimensional reduction on twisted doubled tori [50–53]. For this comparison, it is not enough to look at eqs. (4.13), which contain less information than the corresponding ones from DFT. The relevant equations are instead the full conditions (3.21) and (3.25) with the definitions (4.13). These two equations give the "H" and "R" flux in the present formulation, which actually contain all terms appearing in DFT plus additional terms of higher order in the combinations of B and β. For the other two sets of fluxes the comparison is not yet possible, since we have not determined general expressions for mixed boundary conditions in this paper. However, we can conclude that our formulation encompasses results from DFT and it would be interesting to examine further the relation between DFT (target space theory) and the sigma model we proposed.

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A Proof of protobialgebroid structure on $(L_{B\beta}, L^{\star}_{B\beta})$

In this appendix we prove that $(L_{B\beta}, L^{\star}_{B\beta})$ with the ingredients (brackets, anchors and twists) given in section 2.2.1 is a protobialgebroid, i.e. it satisfies the properties 1 to 4 of Definition 2.1.

Proof of property 1. For $X = X^i\Theta_i$ and $Y = Y^i\Theta_i$ we compute:

$$\begin{aligned}
[X, fY]_{L_{B\beta}} &= e^B e^\beta \Big([e^{-\beta}e^{-B}X, e^{-\beta}e^{-B}fY]_{\text{Lie}} + \beta(e^{-\beta}e^{-B}(\phi(X, fY, \cdot)), \cdot) \Big) \\
&= e^B e^\beta \Big([e^{-\beta}e^{-B}X, fe^{-\beta}e^{-B}Y]_{\text{Lie}} + f\beta(e^{-\beta}e^{-B}(\phi(X, Y, \cdot)), \cdot) \Big) \\
&= e^B e^\beta \Big(f[e^{-\beta}e^{-B}X, e^{-\beta}e^{-B}Y]_{\text{Lie}} + (e^{-\beta}e^{-B}X(f))e^{-\beta}e^{-B}Y \\
&\quad + f\beta(e^{-\beta}e^{-B}(\phi(X, Y, \cdot)), \cdot) \Big) \\
&= f\Big(e^B e^\beta \big([e^{-\beta}e^{-B}X, e^{-\beta}e^{-B}Y]_{\text{Lie}} + \beta(e^{-\beta}e^{-B}(\phi(X, Y, \cdot)), \cdot) \big) \Big) \\
&\quad + e^B e^\beta(\rho(X)f)e^{-\beta}e^{-B}Y \\
&= f[X, Y]_{L_{B\beta}} + (\rho(X)f)Y \ .
\end{aligned} \tag{A.1}$$

Similarly the proof for $\eta = \eta_i E^i$ and $\xi = \xi_i E^i$.

Proof of property 2. Essentially this is already covered by construction in the main text. As a cross check we compute:

$$\begin{aligned}\rho([X,Y]_{L_{B\beta}}) &= \rho\Big(e^B e^\beta\big([e^{-\beta}e^{-B}X, e^{-\beta}e^{-B}Y]_{\text{Lie}} + \beta(e^{-\beta}e^{-B}(\phi(X,Y,\cdot)),\cdot)\big)\Big)\\ &\overset{(2.18),\,(2.19)}{=} \rho\Big(e^B e^\beta\big([\rho(X),\rho(Y)]_{\text{Lie}} + \rho_\star(\phi(X,Y,\cdot))\big)\Big)\\ &\overset{(2.18)}{=} [\rho(X),\rho(Y)]_{\text{Lie}} + \rho_\star(\phi(X,Y,\cdot))\end{aligned} \tag{A.2}$$

and the proof is complete. Similarly for the corresponding property on $L^\star_{B\beta}$.

Proof of property 3. We directly apply the general expressions for the derivations on $L_{B\beta}$ and $L^\star_{B\beta}$:

$$\begin{aligned}\mathrm{d}_{L_{B\beta}}\omega(X_1,\dots,X_{p+1}) &= \sum_{i=1}^{p+1}(-1)^{i+1}\rho(X_i)\omega(X_1,\dots,\hat{X}_i,\dots,X_{p+1}) +\\ &\quad + \sum_{i<j}(-1)^{i+j}\omega([X_i,X_j]_{L_{B\beta}},X_1,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_{p+1})\ .\\ \mathrm{d}_{L^\star_{B\beta}}\Omega(\eta_1,\dots,\eta_{p+1}) &= \sum_{i=1}^{p+1}(-1)^{i+1}\rho_\star(\eta_i)\Omega(\eta_1,\dots,\hat{\eta}_i,\dots,\eta_{p+1}) +\\ &\quad + \sum_{i<j}(-1)^{i+j}\Omega([\eta_i,\eta_j]_{L^\star_{B\beta}},\eta_1,\dots,\hat{\eta}_i,\dots,\hat{\eta}_j,\dots,\eta_{p+1})\,,\end{aligned}$$

for arbitrary $\omega\in\Gamma(\wedge^p L^\star_{B\beta})$ and $\Omega\in\Gamma(\wedge^p L_{B\beta})$, to compute the derivations of the basis elements $E^i\in\Gamma(\wedge^1 L^\star_{B\beta})$ and $\Theta_i\in\Gamma(\wedge^1 L_{B\beta})$

$$\mathrm{d}_{L_{B\beta}}E^i = -\frac{1}{2}(f^i_{jk}-\beta^{il}\phi_{ljk})E^j\wedge E^k\,, \tag{A.3}$$

$$\mathrm{d}_{L^\star_{B\beta}}\Theta_i = -\frac{1}{2}(\theta_i\beta^{jk}+2\beta^{jm}f^k_{im}+\beta^{jl}\beta^{km}\phi_{ilm})\Theta_j\wedge\Theta_k\ . \tag{A.4}$$

Then we compute

$$[[\Theta_i,\Theta_j]_{L_{B\beta}},\Theta_k]_{L_{B\beta}} = \big(\theta_k(\beta^{mn}\phi_{mij}) - f^l_{ij}\beta^{nm}\phi_{mlk} - f^n_{lk}\beta^{lm}\phi_{mij} + \beta^{lm}\beta^{np}\phi_{mij}\phi_{pkl}\big)\Theta_n\,, \tag{A.5}$$

and

$$\phi(\mathrm{d}_{L^\star_{B\beta}}\Theta_i,\Theta_j,\Theta_k) = -\phi_{jkl}\big(\theta_i\beta^{ln} - 2f^{[l}_{im}\beta^{n]m} + \beta^{lp}\beta^{nm}\phi_{ipm}\big)\Theta_n\ . \tag{A.6}$$

Moreover,

$$\mathrm{d}_{L^\star_{B\beta}}\phi(\Theta_i,\Theta_j,\Theta_k) = \beta^{lm}\theta_m\phi_{ijk}\Theta_n\ . \tag{A.7}$$

These expressions deliver the result

$$\begin{aligned}&[[\Theta_i,\Theta_j]_{L_{B\beta}},\Theta_k]_{L_{B\beta}} + \text{c.p.} - \mathrm{d}_{L^\star_{B\beta}}\phi(\Theta_i,\Theta_j,\Theta_k) - \phi(\mathrm{d}_{L^\star_{B\beta}}\Theta_i,\Theta_j,\Theta_k) - \phi(\Theta_i,\mathrm{d}_{L^\star_{B\beta}}\Theta_j,\Theta_k)\\ &-\phi(\Theta_i,\Theta_j,\mathrm{d}_{L^\star_{B\beta}}\Theta_k) = \beta^{ml}\left(\theta_{[i}\phi_{jkl]} - \frac{3}{2}f^n_{[ij}\phi_{kl]n} + \frac{3}{2}\beta^{np}\phi_{n[ij}\phi_{kl]p}\right)\Theta_m\,,\end{aligned} \tag{A.8}$$

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which means that the property holds when the condition

$$\theta_{[i}\phi_{jkl]} - \frac{3}{2}\phi_{m[ij}(f^m_{kl]} - \beta^{nm}\phi_{\underline{n}kl]}) = 0 \tag{A.9}$$

is satisfied. A similar computation for the dual property yields the condition

$$\beta^{[l\underline{m}}\theta_m\psi^{ijk]} - \frac{3}{2}\psi^{m[jk}(\theta_m\beta^{li]} + \beta^{l\underline{n}}f^{i]}_{mn} + \beta^{l\underline{s}}\beta^{i]t}\phi_{mst}) = 0\ . \tag{A.10}$$

These two conditions are essentially Bianchi identities as will be clear from property 4.

Proof of property 4. Using the expansions

$$\begin{aligned}\phi &= \frac{1}{6}\phi_{ijk}E^i\wedge E^j\wedge E^k\,,\\ \psi &= \frac{1}{6}\psi^{ijk}\Theta_i\wedge\Theta_j\wedge\Theta_k\,,\end{aligned} \tag{A.11}$$

and the result (A.3) we compute

$$\begin{aligned}\mathrm{d}_{L_{B\beta}}\phi &= \frac{1}{6}(\mathrm{d}_{L_{B\beta}}\phi_{ijk})E^i\wedge E^j\wedge E^k + \frac{1}{2}\phi_{ijk}(\mathrm{d}_{L_{B\beta}}E^i)\wedge E^j\wedge E^k\\ &= \frac{1}{6}\left(\theta_l\phi_{ijk} - \frac{3}{2}\phi_{mjk}(f^m_{li} - \beta^{mn}\phi_{nli})\right)E^l\wedge E^i\wedge E^j\wedge E^k\,,\end{aligned} \tag{A.12}$$

which vanishes when the condition (A.9) is satisfied. This is essentially a Bianchi identity (and fully agrees with previous results, e.g. [5, 53]). It is simple to check that this Bianchi identity is satisfied in the example of section 2.2.2. On the other hand, using (A.4) we compute

$$\begin{aligned}\mathrm{d}_{L^\star_{B\beta}}\psi=&\frac{1}{6}(\mathrm{d}_{L^\star_{B\beta}}\psi^{ijk})\Theta_i\wedge\Theta_j\wedge\Theta_k + \frac{1}{2}\psi^{ijk}(\mathrm{d}_{L^\star_{B\beta}}\Theta_i)\wedge\Theta_j\wedge\Theta_k\\ =&\frac{1}{6}(\beta^{lm}\theta_m\psi^{ijk} - \frac{3}{2}\psi^{mjk}(\theta_m\beta^{li} + \beta^{ln}f^i_{mn} + \beta^{ls}\beta^{it}\phi_{mst})\Theta_l\wedge\Theta_i\wedge\Theta_j\wedge\Theta_k\,,\end{aligned} \tag{A.13}$$

which vanishes when the Bianchi identity (A.10) is satisfied. This is also true in the example of section 2.2.2.

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