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Quantum SU(2|1) supersymmetric Calogero-Moser spinning systems

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Abstract: SU(2|1) supersymmetric multi-particle quantum mechanics with additional semi-dynamical spin degrees of freedom is considered. In particular, we provide an $\mathcal{N}=4$ supersymmetrization of the quantum U(2) spin Calogero-Moser model, with an intrinsic mass parameter coming from the centrally-extended superalgebra $\widehat{su}(2|1)$. The full system admits an SU(2|1) covariant separation into the center-of-mass sector and the quotient. We derive explicit expressions for the classical and quantum SU(2|1) generators in both sectors as well as for the total system, and we determine the relevant energy spectra, degeneracies, and the sets of physical states.

Keywords: Extended Supersymmetry, Field Theories in Lower Dimensions, Matrix Models, Superspaces

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Contents

1 Introduction

The many-particle Calogero-Moser systems [1–6] and their generalizations occupy a distinguished place in the contemporary theoretical and mathematical physics. Apart from such notable mathematical properties, as the classical and quantum integrability, these systems possess a wide range of physical applications which are hard to enumerate. Among these applications, it is worth to mention, e.g., a close connection between the algebra of observables in the Calogero system and the higher-spin algebra, as was pointed out in [7, 8]. In the same papers, there was revealed an important role of Calogero-like models for describing particles with fractional statistics. Another widely known applications of the Calogero-Moser systems concern the black hole physics. It was suggested [9] that the

Calogero-Moser systems can provide a microscopic description of the extreme Reissner-Nordström black hole in the near-horizon limit. It was argued that, from the M-theory perspective, an important role in this correspondence should be played by various $\mathcal{N}=4$ supersymmetric extensions of the Calogero-Moser models. Supersymmetric Calogero-Moser systems have also further applications in string theory (see, for example, [10]) and $\mathcal{N}=4$ super Yang-Mills theory [11, 12].

Keeping in mind these physical and mathematical motivations, it seems of great interest to construct and study new versions of supersymmetric Calogero-type systems.

In a recent paper [13], there was proposed the superfield matrix model of SU(2|1) supersymmetric mechanics[1] as a new $\mathcal{N}=4$ extension of $d = 1$ Calogero-Moser multi-particle system. This matrix model is a massive generalization of the multiparticle $\mathcal{N}=4$ model constructed and studied in [24, 25]. It is naturally formulated in $d = 1$ harmonic superspace [26] and is described by the following set of $\mathcal{N}=4$ harmonic superfields:

- n^2 commuting general superfields $\mathcal{X}^a_b = (\widetilde{\mathcal{X}^b_a})$, $a, b = 1, \ldots, n$ combined into an hermitian $n{\times}n$-matrix superfield $\mathcal{X} = (\mathcal{X}^b_a)$ which transform in adjoint representation of U(n) and represent off-shell SU(2|1) multiplets $(\mathbf{1}, \mathbf{4}, \mathbf{3})$;

- n commuting analytic complex superfields $\mathcal{Z}^+_a$ forming U(n) spinor $\mathcal{Z}^+ = (\mathcal{Z}^+_a)$, $\widetilde{\mathcal{Z}}^+ = (\widetilde{\mathcal{Z}}^{+a})$ and representing off-shell SU(2|1) multiplets $(\mathbf{4}, \mathbf{4}, \mathbf{0})$;

- n^2 non-propagating analytic "topological" gauge superfields $V^{++} = (V^{++}{}^b_a)$, $(\widetilde{V^{++}{}^b_a}) = V^{++}{}^a_b$.

The matrix superfield $\mathcal{X} = \mathcal{X}(\zeta_H)$ is defined on the SU(2|1) harmonic superspace $\zeta_H \equiv \left(t_A, \theta^\pm, \bar\theta^\pm, w^\pm_i\right)$ $(i = 1, 2)$, while the analytic superfields $\mathcal{Z}^+$, $\widetilde{\mathcal{Z}}^+$ and V^{++} on the analytic harmonic subspace $\zeta_A = \left(t_A, \bar\theta^+, \theta^+, w^\pm_i\right) \subset \zeta_H$. The relevant superfield action is written as

$$S_{\rm matrix} = -\frac{1}{4}\int \mu_H \mathrm{Tr}\left(\mathcal{X}^2\right) + \frac{1}{2}\int \mu^{(-2)}_A \mathcal{V}_0 \widetilde{\mathcal{Z}}^{+a}\mathcal{Z}^+_a + \frac{i}{2}\, c\int \mu^{(-2)}_A\, \mathrm{Tr}\, V^{++}\,, \tag{1.1}$$

where the invariant integration measures are written as

$$\mu_H = dw\, dt_A\, d\bar\theta^- d\theta^- d\bar\theta^+ d\theta^+ \left(1 + m\,\theta^+\bar\theta^- - m\,\theta^-\bar\theta^+\right), \qquad \mu^{(-2)}_A = dw\, dt_A\, d\bar\theta^+ d\theta^+\,. \tag{1.2}$$

The mass-dimension parameter m is encoded in the centrally-extended superalgebra $\widehat{su}(2|1)$ as the contraction parameter to the flat $\mathcal{N}=4, d = 1$ superalgebra.[2] It does not explicitly appear in (1.1) but comes out in the component action from the measure μ_H and the θ-expansion of the superfields $\mathcal{X}$ as a result of solving the appropriate SU(2|1) covariant

[1] This kind of deformed $\mathcal{N}=4$ supersymmetric mechanics was introduced and studied in [14–21]. Some of the SU(2|1) models can be derived by a dimensional reduction from $\mathcal{N}=1$ Lagrangians on the curved $d = 4$ manifold $\mathbb{R}\times S^3$ [22, 23].

[2] It was shown in [20], that the centrally extended superalgebra $\widehat{su}(2|1)$ can be represented as a semi-direct sum of $su(2|1)$ and an extra R-symmetry generator: $\widehat{su}(2|1) \simeq su(2|1) \supsetplus u(1)$. The central charge is a combination of the R-symmetry generator and the internal U(1) generator of $su(2|1)$. In the models under consideration the central charge operator is identified with the canonical Hamiltonian.

constraints (for more details, see [13, 19]).The local U(n) transformations of the involved superfields are given by

$$\mathcal{X}' = e^{i\lambda}\mathcal{X}e^{-i\lambda}, \quad \mathcal{Z}^{+\prime} = e^{i\lambda}\mathcal{Z}^{+}, \quad V^{++\prime} = e^{i\lambda}\, V^{++}\, e^{-i\lambda} - i\, e^{i\lambda}(D^{++}e^{-i\lambda})\,. \tag{1.3}$$

The superfield $\mathcal{V}_0\,(\zeta_A)$ is a prepotential for the singlet part $\mathrm{Tr}\,(\mathcal{X})$ of the matrix superfield $\mathcal{X}$ (see details in [13]). The constant c in (1.1) is a parameter of the model. After quantization, it specifies external SU(2) spins of the physical states, $c\ \rightarrow\ 2s+1 \in \mathbb{Z}_{>0}\,,$ which implies that the set of these states splits into irreducible SU(2) multiplets.

The matrix $d=1$ superfield $\mathcal{X}$ has the physical component fields $X = (X_a{}^b) = X^{\dagger}$, $\Psi^k = (\Psi^k{}_a{}^b)$ and auxiliary bosonic component fields, the superfields $\mathcal{Z}^+_a, \widetilde{\mathcal{Z}}^{+a}$ have the bosonic components $Z'^k = (Z'^k_a)$, $\bar{Z}'_k = (\bar{Z}'^a_k) = (Z'^k)^{\dagger}$ and auxiliary fermionic fields. Choosing the WZ gauge $V^{++} = 2i\,\theta^+\bar{\theta}^+ A(t_A)$, eliminating auxiliary fields and redefining the spinor fields as $Z'^i_a \rightarrow Z^i_a/\left(\mathrm{Tr}(X)\right)^{1/2}$, we obtain from (1.1) the on-shell component action

$$S_{\rm matrix} = S_b + S_f\,, \tag{1.4}$$

$$\begin{aligned} S_b =& \frac{1}{2}\,\mathrm{Tr}\int dt\left(\nabla X\nabla X - m^2X^2\right) - c\int dt\,\mathrm{Tr}A \\ &+\frac{i}{2}\int dt\left(\nabla\bar{Z}_k\, Z^k - \bar{Z}_k\nabla Z^k\right) + \int dt\,\frac{S^{(ik)}S_{(ik)}}{4(X_0)^2}\,, \end{aligned} \tag{1.5}$$

$$S_f = \frac{1}{2}\,\mathrm{Tr}\int dt\Big[i\left(\bar{\Psi}_k\nabla\Psi^k - \nabla\bar{\Psi}_k\Psi^k\right) + 2m\bar{\Psi}_k\Psi^k\Big] - \int dt\,\frac{\Psi_0^{(i}\bar{\Psi}_0^{k)}S_{(ik)}}{(X_0)^2}\,. \tag{1.6}$$

Here,

$$\begin{aligned} &X_0 := \frac{1}{\sqrt{n}}\,\mathrm{Tr}(X), \qquad \Psi^i_0 := \frac{1}{\sqrt{n}}\,\mathrm{Tr}(\Psi^i), \qquad \bar{\Psi}^i_0 := \frac{1}{\sqrt{n}}\,\mathrm{Tr}(\bar{\Psi}^i)\,, \\ &S_{(ik)} := \bar{Z}_{(i}Z_{k)} := \bar{Z}^a_{(i}Z_{k)a}\,, \end{aligned} \tag{1.7}$$

and $(\nabla\bar{Z}_kZ^k){:=}\nabla\bar{Z}^a_kZ^k_a$. The U($n$) gauge-covariant derivatives in (1.5), (1.6) are defined by

$$\nabla X = \dot{X} + i[A,X]\,, \qquad \nabla\Psi^i = \dot{\Psi}^i + i[A,\Psi^i]\,, \qquad \nabla\bar{\Psi}_i = \dot{\bar{\Psi}}_i + i[A,\bar{\Psi}_i]\,. \tag{1.8}$$

$$\nabla Z^k = \dot{Z}^k + iA\,Z^k\,, \qquad \nabla\bar{Z}_k = \dot{\bar{Z}}_k - i\bar{Z}_k\,A\,. \tag{1.9}$$

The basic novel feature of the action (1.4) as compared to the more conventional actions of supersymmetric mechanics is the presence of the semi-dynamical spin variables Z^k_a [24],[3] which has a drastic impact on the structure of the relevant space of quantum states. These variables define extra SU(2) symmetries with the generators (1.7), with respect to which the physical states carry additional spin quantum numbers and so form the appropriate SU(2) multiplets. The diagonal $su(2)$ algebra is an essential part of the "internal" algebra $su(2) \subset \widehat{su}(2|1)$. Also, note the presence of the oscillator-type terms in (1.5) and (1.6), with the intrinsic parameter m as the relevant frequency.

[3]The kinetic term of the variables Z^k_a in the action (1.5) is of the first-order in the time derivatives, in contrast to the dynamical variable $X_a{}^b$ with the second-order kinetic term. Just for this reason we call Z^k_a, $\bar{Z}^a_k$ semi-dynamical variables. In the Hamiltonian (see below), they appear only in the interaction terms and enter through the SU(2) current $S_{(ik)}$.

The simplest one-particle (n=1) case of the system (1.4) was quantized in a recent paper [27]. Here we consider the quantum version of the system (1.4) for an arbitrary n.

As shown in [13], at the classical level the system (1.4) describes an SU(2|1) supersymmetric extension of the U(2)-spin Calogero-Moser model [28–33] generalizing the Calogero-Moser system of refs. [1–6] to the case with additional internal (spin) degrees of freedom. Therefore, the basic purpose of the present paper can be formulated as a construction of new quantum multi-particle spinning Calogero-Moser type system with deformed $\mathcal{N}$=4, $d = 1$ supersymmetry.

The quantization of the Calogero-type multi-particle systems can be accomplished by the two methods, basically leading to the same result. One method [7, 8, 30, 31, 33, 34] is based on the construction of the Dunkl operators for a given system. Using such operators makes it possible to represent a multiparticle system as an oscillator-like system for which the Dunkl operators play the role of generalized momentum operators. Another way of quantizing multi-particle systems is based on considering matrix systems with additional gauge symmetries [30–33, 37–39]. The elimination of some degrees of freedom in such matrix systems results in the standard multi-particle Calogero-type systems. Due to the oscillator nature of matrix operators, the quantization of matrix systems is simpler and the main task of this approach consists in finding solutions of the constraints generating gauge symmetries. In this paper, we will mainly stick to the second method. We will present the explicit expressions of the multi-particle operators of deformed $\mathcal{N}$=4 supersymmetry, in the matrix case and for the reduced system.

The plan of the paper is as follows. In section 2 we construct the Hamiltonian formalism for the matrix system (1.4) and show that the model indeed describes SU(2|1) supersymmetrization of the U(2)-spin Calogero-Moser model [28–33]. In section 3 we find, by Noether procedure, the supercharges of the underlying $\widehat{su}(2|1)$ superalgebra, in matrix case and for a system with the reduced phase variables space. In the latter case $\widehat{su}(2|1)$ is closed up to the constraints generating some residual gauge invariances. In section 4 we construct a quantum realization of the deformed $\mathcal{N}$=4, $d = 1$ superalgebra $\widehat{su}(2|1)$ for the multi-particle Calogero-Moser system. In the case of the reduced system with n bosonic position coordinates such a superalgebra is closed up to the generators of the $[\mathrm{U}(1)]^n$ gauge symmetry, like in the classical case. This $\widehat{su}(2|1)$ superalgebra is represented as a sum of two $\widehat{su}(2|1)$ superalgebras. One $\widehat{su}(2|1)$ acts in the center-of-mass sector, whereas the other operates only on the super-variables parametrizing the quotient over this sector. The spin operators are common for both these superalgebras. In sections 5–7 we analyze the energy spectrum in all cases: for the center-of-mass subsystem, for the system with relative supercoordinates and in the general case, when all position operators are included. The last section 8 contains a Summary and outlook.

2 Hamiltonian analysis and gauge fixing

The action (1.4) yields the canonical Hamiltonian

$$H_{\text{total}} = H_{\text{matrix}} - \mathrm{Tr}(A\,G)\,, \tag{2.1}$$

where

$$H_{\text{matrix}} = \frac{1}{2}\,\mathrm{Tr}\Big(P^2 + m^2X^2 - 2m\bar\Psi_k\Psi^k\Big) - \frac{S^{(ik)}S_{(ik)}}{4(X_0)^2} + \frac{\Psi_0^{(i}\bar\Psi_0^{k)}S_{(ik)}}{(X_0)^2}\,. \tag{2.2}$$

The Hamiltonian (2.1) involves the matrix momentum $P_a{}^b \equiv (\nabla X)_a{}^b$ and another matrix quantity

$$G_a{}^b \equiv i\,[X,P]_a{}^b + \Big\{\bar\Psi_k,\Psi^k\Big\}_a{}^b + Z_a^k\bar Z_k^b - c\,\delta_a{}^b\,. \tag{2.3}$$

The action (1.4) also produces the primary constraints

$$P_{Z_k^a} + \frac{i}{2}\,\bar Z_k^a \approx 0\,, \qquad P_{\bar Z_a^k} - \frac{i}{2}\,Z_a^k \approx 0\,, \tag{2.4}$$

$$P_{\Psi k\,a}{}^b - \frac{i}{2}\,\bar\Psi_{k\,a}{}^b \approx 0\,, \qquad P_{\bar\Psi}{}^k{}_a{}^b - \frac{i}{2}\,\Psi^k{}_a{}^b \approx 0\,, \tag{2.5}$$

$$P_{A\,a}{}^b \approx 0\,. \tag{2.6}$$

The constraints (2.4), (2.5) are second class and so we introduce Dirac brackets for them. As the result, we eliminate the momenta $P_{Z_k^a}$, $P_{\Psi k\,a}{}^b$ and their c.c. The residual variables obey the Dirac brackets

$$\Big\{X_a{}^b, P_c{}^d\Big\}^* = \delta_a^d\delta_c^b\,, \qquad \Big\{Z_a^k, \bar Z_l^b\Big\}^* = i\delta_l^k\delta_a^b\,, \qquad \Big\{\Psi^k{}_a{}^b, \bar\Psi_{l\,c}{}^d\Big\}^* = -i\delta_l^k\delta_a^d\delta_c^b\,. \tag{2.7}$$

Requiring the constraints (2.6) to be preserved by the Hamiltonian (2.1) generates secondary constraints

$$G_a{}^b \approx 0\,. \tag{2.8}$$

Despite the presence of the constant c in (2.3) these constraints are first class: with respect to the Dirac brackets (2.7) they form $u(n)$ algebra,

$$\Big\{G_a{}^b, G_c{}^d\Big\}^* = i\left(\delta_a^d G_c^b - \delta_c^b G_a^d\right), \tag{2.9}$$

and so produce the U(n) invariance of the action (1.4)

$$X' = e^{i\alpha}\,X\,e^{-i\alpha}\,, \quad \Psi'^k = e^{i\alpha}\,\Psi^k\,e^{-i\alpha}\,, \quad Z'^k = e^{i\alpha}Z^k\,, \quad A' = e^{i\alpha}\,A\,e^{-i\alpha} - i\,e^{i\alpha}(\partial_t e^{-i\alpha}), \tag{2.10}$$

where $\alpha_a{}^b(t) \in u(n)$ are $d{=}1$ gauge parameters.

In the first-order formulation, the system (1.4) is represented by the action

$$S_{\text{matrix}} = \int dt\,L_{\text{matrix}}\,, \tag{2.11}$$

$$L_{\text{matrix}} = \mathrm{Tr}\Big(P\dot X\Big) + \frac{i}{2}\,\mathrm{Tr}\Big(\bar\Psi_k\dot\Psi^k - \dot{\bar\Psi}_k\Psi^k\Big) + \frac{i}{2}\Big(\dot{\bar Z}_k^a Z_a^k - \bar Z_k^a\dot Z_a^k\Big) - H_{\text{matrix}} + \mathrm{Tr}(AG)\,, \tag{2.12}$$

where H_{matrix} was defined in (2.2).

Let us fix a partial gauge for the transformations (2.10). To this end, we introduce the following notation for the matrix entries of X and P:

$$\begin{array}{lll} x_a := X_a{}^a\,, & p_a := P_a{}^a & (\text{no summation over } a)\,, \\ x_a{}^b := X_a{}^b\,, & p_a{}^b := P_a{}^b & \text{for} \quad a \neq b\,, \\ x_a{}^a := 0\,, & p_a{}^a := 0 & (\text{no summation over } a)\,, \end{array} \tag{2.13}$$

i.e., ${X_a}^b = x_a\delta_a^b + {x_a}^b$, ${P_a}^b = p_a\delta_a^b + {p_a}^b$ and $X_0 = \frac{1}{\sqrt{n}}\sum_{a=1}^n x_a$. Note that

$$\mathrm{Tr}P^2 = \sum_a p_a p_a + \sum_{a\neq b} {p_a}^b {p_b}^a\,, \quad \mathrm{Tr}(XP) = \sum_a x_a p_a + \sum_{a\neq b} {x_a}^b {p_b}^a\,.$$

In the notation (2.13) the constraints (2.8) take the form

$$G_a{}^b = i(x_a - x_b){p_a}^b - i(p_a - p_b){x_a}^b + i({x_a}^c{p_c}^b - {p_a}^c{x_c}^b) + {T_a}^b \approx 0 \tag{2.14}$$

for $a \neq b$ and

$$G_a{}^a = i({x_a}^c{p_c}^a - {p_a}^c{x_c}^a) + {T_a}^a - c \approx 0 \qquad \text{(no summation over } a) \tag{2.15}$$

for the diagonal elements of G, with

$${T_a}^b := Z_a^k \bar{Z}_k^b + \left\{\bar{\Psi}_k, \Psi^k\right\}_a{}^b. \tag{2.16}$$

Provided that the Calogero-like conditions $x_a \neq x_b$ are fulfilled, we can impose the gauge

$${x_a}^b \approx 0\,, \qquad a \neq b\,, \tag{2.17}$$

for the constraints (2.14). Then we introduce Dirac brackets for the constraints (2.14), (2.17) and eliminate ${x_a}^b$ by (2.17) and ${p_a}^b$ by (2.14):

$${p_a}^b = \frac{i\,{T_a}^b}{x_a - x_b}\,, \qquad a \neq b\,. \tag{2.18}$$

Due to the resolved form of gauge-fixing conditions, new Dirac brackets for the remaining variables coincide with (2.7):

$$\{x_a, p_b\}^{**} = \delta_{ab}\,, \qquad \left\{Z_a^k, \bar{Z}_l^b\right\}^{**} = i\delta_l^k\delta_a^b\,, \qquad \left\{{\Psi^k}_a{}^b, {\bar{\Psi}_l}{}_c{}^d\right\}^{**} = -i\delta_l^k\delta_a^d\delta_c^b\,. \tag{2.19}$$

In the gauge (2.17), the constraints (2.15) become

$$T_a - c := {T_a}^a - c = Z_a^k \bar{Z}_k^a + \left\{\bar{\Psi}_k, \Psi^k\right\}_a{}^a - c \approx 0 \qquad \text{(no summation over } a) \tag{2.20}$$

and they generate local $[\mathrm{U}(1)]^n$ transformations of Z_a^k and ${\Psi^k}_a{}^b$ with $a \neq b$. Preservation of the conditions (2.17), $\dot{x}_a{}^b = \{{x_a}^b, H_{\text{total}}\} = 0$, allows one to express

$${A_a}^b = \frac{i\,{p_a}^b}{x_a - x_b} = -\frac{{T_a}^b}{(x_a - x_b)^2}\,, \qquad a \neq b\,. \tag{2.21}$$

Inserting (2.17), (2.14) and (2.21) into (2.2), we arrive at the reduced total Hamiltonian

$$H^{(red)} = H_{\mathrm{C-M}} - \sum_a A_a\Big(T_a - c\Big)\,, \tag{2.22}$$

where $A_a = A_a^a$ (no summation over a) and the generalized Calogero-Moser Hamiltonian is defined as

$$\begin{aligned} H_{\mathrm{C-M}} = {} & \frac{1}{2}\sum_a \left(p_a p_a + m^2 x_a x_a\right) + \frac{1}{2}\sum_{a\neq b}\frac{{T_a}^b{T_b}^a}{(x_a - x_b)^2} - m\mathrm{Tr}\left(\bar{\Psi}_k\Psi^k\right) \\ & - \frac{S^{(ik)}S_{(ik)}}{4(X_0)^2} + \frac{\Psi_0^{(i}\bar{\Psi}_0^{k)}S_{(ik)}}{(X_0)^2}\,. \end{aligned} \tag{2.23}$$

The same final result can be attained in a different way. Eliminating ${A_a}^b$, $a{\neq}b$, by the equations of motion ${A_a}^b = -{T_a}^b/(x_a - x_b)^2$ we obtain that the action (1.4) in the gauge (2.17) takes the form

$$
\begin{aligned}
S_{\mathrm{C-M}} = \int dt \Big\{ & \frac{1}{2}\sum_a \Big(\dot{x}_a \dot{x}_a - m^2 x_a x_a\Big) - \frac{i}{2}\sum_a \Big(\bar{Z}^a_k \dot{Z}^k_a - \dot{\bar{Z}}{}^a_k Z^k_a\Big) + \sum_a A_a\Big(T_a - c\Big) \\
& + \mathrm{Tr}\Big[\frac{i}{2}\Big(\bar{\Psi}_k \dot{\Psi}^k - \dot{\bar{\Psi}}_k \Psi^k\Big) + m \bar{\Psi}_k \Psi^k\Big] \\
& - \frac{1}{2}\sum_{a\neq b} \frac{{T_a}^b {T_b}^a}{(x_a - x_b)^2} + \frac{S^{(ik)} S_{(ik)}}{4(X_0)^2} - \frac{\Psi_0^{(i} \bar{\Psi}_0^{k)} S_{(ik)}}{(X_0)^2} \Big\}.
\end{aligned} \tag{2.24}
$$

The action (2.24) produces the Hamiltonian (2.23), the constraints (2.20) and the brackets (2.19).

The important ingredients of the action (2.24) are bilinear combinations of Z^k_a and $\bar{Z}^a_k$ with the external SU(2) indices

$$
{S_{ak}}^j := \bar{Z}^a_k Z^j_a \quad (\text{no summation over } a)\,, \qquad {S_k}^j := \sum_a {S_{ak}}^j\,. \tag{2.25}
$$

With respect to the Dirac brackets (2.7) the objects ${S_{ak}}^j$ for each index a form $u(2)$ algebras

$$
\Big\{ {S_{ai}}^j, {S_{bk}}^l \Big\}^* = i\delta_{ab} \Big[\delta^j_k\, {S_{ai}}^l - \delta^l_i\, {S_{ak}}^j \Big]. \tag{2.26}
$$

The object ${S_k}^j$ forms the "diagonal" $u(2)$ algebra in the product of above ones

$$
\Big\{ {S_i}^j, {S_k}^l \Big\}^* = i \Big[\delta^j_k\, {S_i}^l - \delta^l_i\, {S_k}^j \Big]. \tag{2.27}
$$

The triplets of the quantities (2.25) (see also (1.7))

$$
S^{(kj)}_a := \bar{Z}^{a(k} Z^{j)}_a, \qquad S^{(kj)} := \sum_a S^{(kj)}_a \tag{2.28}
$$

generate $su(2)$ algebras

$$
\Big\{ S^{(ij)}_a, S^{(kl)}_b \Big\}^* = -i\delta_{ab} \Big[\varepsilon^{ik}\, S^{(jl)}_a + \varepsilon^{jl}\, S^{(ik)}_a \Big], \tag{2.29}
$$

$$
\Big\{ S^{(ij)}, S^{(kl)} \Big\}^* = -i \Big[\varepsilon^{ik}\, S^{(jl)} + \varepsilon^{jl}\, S^{(ik)} \Big]. \tag{2.30}
$$

Below we will also use the brackets

$$
\Big\{ S_{(ij)}, Z^k_a \Big\}^* = -i\delta^k_{(i} Z_{aj)}\,, \qquad \Big\{ S^{(ij)}, \bar{Z}^a_k \Big\}^* = i\delta^{(i}_k \bar{Z}^{aj)}\,. \tag{2.31}
$$

One more matrix present in the action (2.24) is ${T_a}^b$ defined in (2.16). These quantities form $u(n)$ algebra (2.9) with respect to the Dirac brackets:

$$
\Big\{ {T_a}^b, {T_c}^d \Big\}^* = i \Big(\delta^d_a {T_c}^b - \delta^b_c {T_a}^d \Big). \tag{2.32}
$$

JHEP04(2018)043

The odd matrix variables are transformed by adjoint $u(n)$ representation:

$$\left\{T_a{}^b, \Psi^k{}_c{}^d\right\}^* = i\left(\delta_a^d \Psi^k{}_c{}^b - \delta_c^b \Psi^k{}_a{}^d\right), \quad \left\{T_a{}^b, \Psi_0^k\right\}^* = 0\,, \tag{2.33}$$

$$\left\{T_a{}^b, \bar\Psi^k{}_c{}^d\right\}^* = i\left(\delta_a^d \bar\Psi^k{}_c{}^b - \delta_c^b \bar\Psi^k{}_a{}^d\right), \quad \left\{T_a{}^b, \bar\Psi_0^k\right\}^* = 0\,. \tag{2.34}$$

These $u(n)$ transformations commute with $u(2)$ transformations generated by $S_{ak}{}^j$:

$$\left\{T_a{}^b, S_{ai}{}^j\right\}^* = 0\,. \tag{2.35}$$

Let us consider the bosonic core of the system (2.24) and demonstrate that it corresponds just to the spin Calogero-Moser model. Omitting terms with fermionic variables, we find

$$\begin{aligned} S^{(\mathrm{bose})}_{\mathrm{C-M}} = \int dt \Bigg\{ &\frac{1}{2}\sum_a \left(\dot x_a \dot x_a - m^2 x_a x_a\right) - \frac{i}{2}\sum_a \left(\bar Z_k^a \dot Z_a^k - \dot{\bar Z}{}_k^a Z_a^k\right) \\ &+ \sum_a A_a\left(Z_k^a Z_a^k - c\right) - \frac{1}{2}\sum_{a\neq b} \frac{\mathrm{Tr}\,(S_a S_b)}{(x_a - x_b)^2} + \frac{S^{(ik)}S_{(ik)}}{4(X_0)^2} \Bigg\}, \end{aligned} \tag{2.36}$$

where

$$\mathrm{Tr}\,(S_a S_b) := S_{ak}{}^j S_{bj}{}^k \tag{2.37}$$

and $S_{ak}{}^j$ are defined in (2.25). The analogous reduction of the Hamiltonian (2.23) yields

$$H_{\mathrm{C-M}} = \frac{1}{2}\sum_a \left(p_a p_a + m^2 x_a x_a\right) + \frac{1}{2}\sum_{a\neq b} \frac{\mathrm{Tr}\,(S_a S_b)}{(x_a - x_b)^2} - \frac{S^{(ik)}S_{(ik)}}{4(X_0)^2}\,. \tag{2.38}$$

The Hamiltonian (2.38) contains a potential in the center-of-mass sector with the coordinate X_0 (the last term in (2.38)). Modulo this extra potential, the bosonic limit of the system constructed is none other than the U(2)-spin Calogero-Moser model which is a massive generalization of the U(2)-spin Calogero model [28, 29, 31–33]. Thus the system (2.24) with the Hamiltonian (2.23) describes SU(2|1) supersymmetric extension of the U(2)-spin Calogero-Moser model.

3 Supercharges

In this section we will find the classical expressions for the generators of the deformed $\mathcal{N}=4$ supersymmetry (SU(2|1) supersymmetry) for the n-particle systems, both in the matrix formulation and in the case of the reduced system with n position coordinates.

3.1 Matrix system

The odd SU(2|1) transformations of the component matrix fields entering (1.4) are as follows[4]

$$\begin{aligned} &\delta X = -\,\epsilon_k \Psi^k + \bar\epsilon^k \bar\Psi_k\,, \\ &\delta\Psi^k = i\,\bar\epsilon^k\left(\nabla X + im\,X\right) + \frac{\bar\epsilon_j S^{(jk)}}{X_0}\,\mathbb{1}\,, \quad \delta\bar\Psi_k = -i\,\epsilon_k\left(\nabla X - im\,X\right) + \frac{\epsilon^j S_{(jk)}}{X_0}\,\mathbb{1}\,, \end{aligned} \tag{3.1}$$

[4]These transformations are a sum of the initial linear supertranslations plus extra compensating gauge transformations (1.3) with $\lambda = 2i\left(\bar\theta^+\epsilon^- - \theta^+\bar\epsilon^-\right) A$ which are required for preserving WZ gauge for V^{++}.

whereas the supertranslations of the spin fields are represented by the SU(2) rotations

$$\delta Z^k_a = \omega^{(kj)} Z_{aj}\,, \qquad \delta \bar Z^a_k = -\omega_{(kj)} \bar Z^{aj} \tag{3.2}$$

with the composite parameters

$$\omega^{(kj)} = \frac{\epsilon^{(k}\Psi_0^{j)} + \bar\epsilon^{(k}\bar\Psi_0^{j)}}{X_0}\,.$$

Under the transformations (3.1), (3.2) and $\delta A = 0$ the action (1.4) transforms as

$$\delta S_{\rm matrix} = \int dt \dot\Lambda_1\,,$$
$$\Lambda_1 = -\frac{\epsilon_k}{2}\,{\rm Tr}\Big[(\nabla X + im X)\,\Psi^k\Big] + \frac{\bar\epsilon^k}{2}\,{\rm Tr}\Big[(\nabla X - im X)\,\bar\Psi_k\Big] - \frac{i}{2}\omega^{(kj)} S_{(kj)}\,. \tag{3.3}$$

Using (3.1), (3.2) and (3.3) we obtain the following expressions for Noether supercharges:

$$\begin{aligned} \mathcal{Q}^k &= {\rm Tr}\Big[\,(P - im\,X)\;\Psi^k\Big] + \frac{i\,S^{(kj)}\,\Psi_{0\,j}}{X_0}\,, \\ \bar{\mathcal{Q}}_k &= {\rm Tr}\Big[\,(P + im\,X)\,\bar\Psi_k\Big] - \frac{i\,S_{(kj)}\,\bar\Psi_0^j}{X_0}\,, \end{aligned} \tag{3.4}$$

where $P = \nabla X$. The generators (3.4) constitute an $\widehat{su}(2|1)$ superalgebra with respect to the Dirac brackets (2.7)

$$\left\{\mathcal{Q}^i, \bar{\mathcal{Q}}_k\right\}^* = -2i\,\delta^i_k\,H - 2im\left(I^i_k - \delta^i_k\,F\right)\,, \quad \left\{\mathcal{Q}^i, \mathcal{Q}^k\right\}^* = 0\,, \quad \left\{\bar{\mathcal{Q}}_i, \bar{\mathcal{Q}}_k\right\}^* = 0\,. \tag{3.5}$$

Here, $H = H_{\rm matrix}$, where $H_{\rm matrix}$ was defined in (2.2), and also the $su(2)$ and $u(1)$ generators are present:

$$I^i_k = \varepsilon_{kj}\left[S^{(ij)} + {\rm Tr}\Big(\Psi^{(i}\bar\Psi^{j)}\Big)\right]\,, \tag{3.6}$$

$$F = \frac{1}{2}\,{\rm Tr}\Big(\Psi^k\bar\Psi_k\Big). \tag{3.7}$$

The Hamiltonian H commutes with all other generators and so can be identified with the central charge operator of $\widehat{su}(2|1)$. The rest of Dirac brackets among the generators (2.2), (3.4), (3.6), (3.7) is given by the relations

$$\left\{H, \mathcal{Q}^k\right\}^* = \left\{H, \bar{\mathcal{Q}}_k\right\}^* = \left\{H, I^i_k\right\}^* = \left\{H, F\right\}^* = 0\,, \tag{3.8}$$

$$\left\{F, \mathcal{Q}^k\right\}^* = -\frac{i}{2}\,\mathcal{Q}^k\,, \qquad \left\{F, \bar{\mathcal{Q}}_k\right\}^* = \frac{i}{2}\,\bar{\mathcal{Q}}_k\,, \qquad \left\{F, I^i_k\right\}^* = 0\,, \tag{3.9}$$

$$\left\{I^i_k, \mathcal{Q}^j\right\}^* = -\frac{i}{2}\left(\delta^j_k\mathcal{Q}^i + \varepsilon^{ij}\mathcal{Q}_k\right)\,, \qquad \left\{I^i_k, \bar{\mathcal{Q}}_j\right\}^* = \frac{i}{2}\left(\delta^i_j\bar{\mathcal{Q}}_k + \varepsilon_{kj}\bar{\mathcal{Q}}^i\right)\,, \tag{3.10}$$

$$\left\{I^i_k, I^j_l\right\}^* = i\left(\delta^i_l I^j_k - \delta^j_k I^i_l\right)\,. \tag{3.11}$$

Note that the first-order action (2.11) is invariant, up to the surface term $\delta S_{\rm matrix} = \int dt\dot\Lambda_1$ (with the substitution $\nabla X = P$ in Λ_1), under the transformations (3.1), (3.2), $\delta A = 0$ and

$$\delta P = -im\left(\epsilon_k\Psi^k + \bar\epsilon^k\bar\Psi_k\right) - i\,\frac{\epsilon_k S^{(kj)}\Psi_{0\,j} + \bar\epsilon^k S_{(kj)}\bar\Psi_0^j}{X_0}\,\mathbb{1}\,. \tag{3.12}$$

It is worth pointing out that $\delta H = 0$ and $\delta G_a{}^b = 0$ under these transformations.

3.2 Reduced system in the standard Calogero-Moser representation

Let us compute the $\widehat{su}(2|1)$ charges for the reduced system (2.24) which follows from the matrix formulation after imposing the gauge (2.17).

On the pattern of (2.13), we introduce the following notation for the entries of Ψ^k and $\bar{\Psi}_k$:

$$\begin{aligned} \psi_a^k &:= {\Psi^k}_a{}^a\,, \qquad & \bar{\psi}_{ak} &:= {\bar{\Psi}_{k a}}{}^a \qquad & &(\text{no summation over } a)\,, \\ {\psi^k}_a{}^b &:= {\Psi^k}_a{}^b\,, \qquad & {\bar{\psi}_{ka}}{}^b &:= {\bar{\Psi}_{ka}}{}^b \qquad & &\text{for} \quad a \neq b\,. \end{aligned} \tag{3.13}$$

Note that $\mathrm{Tr}(P\Psi^k) = \sum\limits_a p_a \psi_a^k + \sum\limits_{a\neq b} {p_a}^b {\psi^k}_b{}^a$ and $\Psi_0^k = \dfrac{1}{\sqrt{n}} \sum\limits_{a=1}^n \psi_a^k$, $\bar{\Psi}_0^k = \dfrac{1}{\sqrt{n}} \sum\limits_{a=1}^n \bar{\psi}_a^k$.

In the gauge (2.17), supertranslations are a sum of the transformations (3.1), (3.2) and the additional compensating gauge transformations (2.10) with the composite parameters

$$\alpha_a{}^b = i\,\frac{\epsilon_k \psi_a^{kb} - \bar{\epsilon}^k \bar{\psi}_{ka}{}^b}{x_a - x_b} \quad \text{for} \quad a \neq b\,, \qquad \alpha_a{}^b = 0 \quad \text{for} \quad a = b\,. \tag{3.14}$$

These transformations preserve the conditions (2.17) and have the following explicit form

$$\delta x_a = -\,\epsilon_k \psi_a^k + \bar{\epsilon}^k \bar{\psi}_{ak}\,, \tag{3.15}$$

$$\begin{aligned} \delta \psi_a^k &= i\,\bar{\epsilon}^k \left(\dot{x}_a + im\,x_a\right) + \frac{\bar{\epsilon}_j S^{(jk)}}{X_0} + i \sum_b \left(\alpha_a{}^b {\psi^k}_b{}^a - {\psi^k}_a{}^b \alpha_b{}^a \right), \\ \delta \bar{\psi}_{ak} &= -i\,\epsilon_k \left(\dot{x}_a - im\,x_a\right) + \frac{\epsilon^j S_{(jk)}}{X_0} + i \sum_b \left(\alpha_a{}^b \bar{\psi}_{kb}{}^a - \bar{\psi}_{ka}{}^b \alpha_b{}^a \right), \end{aligned} \tag{3.16}$$

$$\begin{aligned} \delta {\psi^k}_a{}^b &= -\frac{\bar{\epsilon}^k T_a{}^b}{x_a - x_b} - i\alpha_a{}^b \left(\psi_a^k - \psi_b^k \right) + i \sum_c \left(\alpha_a{}^c {\psi^k}_c{}^b - {\psi^k}_a{}^c \alpha_c{}^b \right), \\ \delta \bar{\psi}_{ka}{}^b &= \frac{\epsilon_k T_a{}^b}{x_a - x_b} - i\alpha_a{}^b \left(\bar{\psi}_{ak} - \bar{\psi}_{bk} \right) + i \sum_c \left(\alpha_a{}^c \bar{\psi}_{kc}{}^b - \bar{\psi}_{ka}{}^c \alpha_c{}^b \right), \end{aligned} \tag{3.17}$$

$$\delta Z_a^k = \omega^{(kj)} Z_{aj} + i \sum_b \alpha_a{}^b Z_b^k\,, \qquad \delta \bar{Z}_k^a = -\omega_{(kj)} \bar{Z}^{aj} - i \sum_b \bar{Z}_k^b \alpha_b{}^a\,, \tag{3.18}$$

$$\delta A_a = i \sum_b \frac{\alpha_a{}^b T_b{}^a + T_a{}^b \alpha_b{}^a}{x_a - x_b}\,. \tag{3.19}$$

An important property is that the constraints (2.20) are invariant with respect to these supersymmetry transformations, $\delta T_a = 0$. Also, $\delta \Big[\sum_a A_a \left(T_a - c \right) \Big] = 0$.

The variation of the action (2.24) under the supersymmetry transformations (3.15)–(3.18) reads

$$\delta S_{C-M} = \int dt \dot{\Lambda}_2\,,$$

where

$$\Lambda_2 = -\,\frac{\epsilon_k}{2} \sum_a \left(\dot{x}_a + im\,x_a\right) \psi_a^k + \frac{\bar{\epsilon}^k}{2} \sum_a \left(\dot{x}_a - im\,x_a\right) \bar{\psi}_{ak} + \frac{1}{2} \sum_{a\neq b} \alpha_a{}^b T_b{}^a - \frac{i}{2}\,\omega^{(kj)} S_{(kj)}\,. \tag{3.20}$$

The corresponding Noether supercharges are found to be

$$
\begin{aligned}
Q^k &= \sum_a (p_a - imx_a)\,\psi^k_a + i\sum_{a\neq b}\frac{T_a{}^b\,\psi^k{}_b{}^a}{x_a - x_b} + \frac{i\,S^{(kj)}\,\Psi_{0\,j}}{X_0}\,,\\
\bar{Q}_k &= \sum_a (p_a + imx_a)\,\bar\psi_{ak} + i\sum_{a\neq b}\frac{T_a{}^b\,\bar\psi_{kb}{}^a}{x_a - x_b} - \frac{i\,S_{(kj)}\,\bar\Psi^j_0}{X_0}\,,
\end{aligned}
\tag{3.21}
$$

where $p_a = \dot{x}_a$. These expressions can be also obtained by inserting (2.17), (2.18) into (3.4) and turning to the notations (2.13), (3.13).

With respect to the Dirac brackets (2.19), the generators (3.21) form, up to the residual $[\mathrm{U}(1)]^n$ gauge transformations generated by (2.20), the following $\widehat{su}(2|1)$ superalgebra

$$
\begin{aligned}
\left\{Q^i, \bar{Q}_k\right\}^{**} &= -2i\,\delta^i_k\,H - 2im\left(I^i_k - \delta^i_k\,F\right) + 2i\sum_{a\neq b}\frac{\psi^i{}_a{}^b\bar\psi_{kb}{}^a}{(x_a - x_b)^2}\,(T_a - T_b)\,,\\
\left\{Q^i, Q^k\right\}^{**} &= 2i\sum_{a\neq b}\frac{\psi^i{}_a{}^b\psi^k{}_b{}^a}{(x_a - x_b)^2}\,(T_a - T_b)\,,\\
\left\{\bar{Q}_i, \bar{Q}_k\right\}^{**} &= 2i\sum_{a\neq b}\frac{\bar\psi_{ia}{}^b\bar\psi_{kb}{}^a}{(x_a - x_b)^2}\,(T_a - T_b)\,.
\end{aligned}
\tag{3.22}
$$

Here we used that the last relation in (2.19), being cast in the notation (2.13), (3.13), amounts to the relations $\left\{\psi^i_a, \bar\psi_{b\,k}\right\}^{**} = -i\delta^i_k\delta_{ab}$, $\left\{\psi^i{}_a{}^b, \bar\psi_{k\,c}{}^d\right\}^{**} = -i\delta^i_k\delta^d_a\delta^b_c$, In (3.22), $H = H_{\mathrm{C-M}}$, with $H_{\mathrm{C-M}}$ defined by (2.23), and the generators I^i_k, F were defined in (3.6), (3.7).

The Hamiltonian (2.23) commutes with the supercharges (3.21) modulo the first-class constraints (2.20):

$$
\left\{Q^k, H\right\}^{**} = 2\sum_{a\neq b}\frac{T_a{}^b\psi^k{}_b{}^a}{(x_a - x_b)^3}\,(T_a - T_b)\,,\qquad \left\{\bar{Q}_k, H\right\}^{**} = 2\sum_{a\neq b}\frac{T_a{}^b\bar\psi_{kb}{}^a}{(x_a - x_b)^3}\,(T_a - T_b)\,.
\tag{3.23}
$$

The generators I^i_k, F satisfy the same Dirac brackets as in (3.8), (3.9), (3.10), and (3.11).

4 Quantum multi-particle $\widehat{su}(2|1)$ superalgebra

Quantum $su(2|1)$ superalgebra obtained by quantizing the Dirac brackets (3.5), (3.8), (3.9), (3.10), (3.11), is formed by the following non-vanishing (anti)commutators:

$$
\begin{aligned}
&\left\{Q^i, \bar{\mathbf{Q}}_k\right\} = 2\,\delta^i_k\,\mathbf{H} + 2m\left(\mathbf{I}^i_k - \delta^i_k\,\mathbf{F}\right),\\
&\left[\mathbf{F}, \mathbf{Q}^k\right] = \frac{1}{2}\,\mathbf{Q}^k\,, \qquad\qquad \left[\mathbf{F}, \bar{\mathbf{Q}}_k\right] = -\frac{1}{2}\,\bar{\mathbf{Q}}_k\,,\\
&\left[\mathbf{I}^i_k, \mathbf{Q}^j\right] = \frac{1}{2}\left(\delta^j_k\mathbf{Q}^i + \varepsilon^{ij}\mathbf{Q}_k\right), \qquad \left[\mathbf{I}^i_k, \bar{\mathbf{Q}}_j\right] = -\frac{1}{2}\left(\delta^i_j\bar{\mathbf{Q}}_k + \varepsilon_{kj}\bar{\mathbf{Q}}^i\right),\\
&\left[\mathbf{I}^i_k, \mathbf{I}^j_l\right] = \delta^j_k\mathbf{I}^i_l - \delta^i_l\mathbf{I}^j_k\,.
\end{aligned}
\tag{4.1}
$$

The second- and third-order Casimir operators of $su(2|1)$ are defined by the expressions [21]

$$\mathbf{C}_2 = \left(\frac{1}{m}\,\mathbf{H} - \mathbf{F}\right)^2 - \frac{1}{2}\,\mathbf{I}^i_k\mathbf{I}^k_i + \frac{1}{4m}\,\left[\mathbf{Q}^i, \bar{\mathbf{Q}}_i\right], \tag{4.2}$$

$$\mathbf{C}_3 = \left(\mathbf{C}_2 + \frac{1}{2}\right)\left(\frac{1}{m}\,\mathbf{H} - \mathbf{F}\right) + \frac{1}{8m}\left\{\delta^j_i\left(\frac{1}{m}\,\mathbf{H} - \mathbf{F}\right) - \mathbf{I}^j_i\right\}\left[\mathbf{Q}^i, \bar{\mathbf{Q}}_j\right]. \tag{4.3}$$

In this section we will present the explicit form of this deformed $\mathcal{N}$=4 supersymmetry algebra for multiparticle system constructed in the previous sections. We will do it for the matrix formulation of this system and for the reduced system with n position coordinates.

4.1 Matrix formulation

4.1.1 Supercharges of the $\widehat{su}(2|1)$ superalgebra

In the matrix formulation, the n-particle system is described by quantum operators $\mathbf{X}_a{}^b$, $\mathbf{P}_a{}^b$; $\mathbf{\Psi}^i{}_a{}^b$, $\bar{\mathbf{\Psi}}_{i\,a}{}^b$; $\mathbf{Z}^i_a$, $\bar{\mathbf{Z}}^b_i$ which satisfy the quantum counterpart of the Dirac brackets algebra (2.7):

$$\left[\mathbf{X}_a{}^b, \mathbf{P}_c{}^d\right] = i\,\delta^d_a\delta^b_c\,, \qquad \left[\mathbf{Z}^k_a, \bar{\mathbf{Z}}^b_j\right] = -\,\delta^k_j\delta^b_a\,, \qquad \left\{\mathbf{\Psi}^k{}_a{}^b, \bar{\mathbf{\Psi}}_{j\,c}{}^d\right\} = \delta^k_j\delta^d_a\delta^b_c\,. \tag{4.4}$$

The quantum supercharges are uniquely restored by the classical expressions (3.4):

$$\begin{aligned}
\mathbf{Q}^k &= \mathrm{Tr}\Big[\left(\mathbf{P} - im\,\mathbf{X}\right)\mathbf{\Psi}^k\Big] + \frac{i\,\mathbf{S}^{(kj)}\,\mathbf{\Psi}_{0\,j}}{\mathbf{X}_0}\,, \\
\bar{\mathbf{Q}}_k &= \mathrm{Tr}\Big[\left(\mathbf{P} + im\,\mathbf{X}\right)\bar{\mathbf{\Psi}}_k\Big] - \frac{i\,\mathbf{S}_{(kj)}\,\bar{\mathbf{\Psi}}^j_0}{\mathbf{X}_0}\,,
\end{aligned} \tag{4.5}$$

where the $su(2)$ generators are

$$\mathbf{S}^{(ik)} = \sum_a \mathbf{Z}^{(i}_a\bar{\mathbf{Z}}^{k)a}\,. \tag{4.6}$$

These generators form the quantum algebra of the corresponding diagonal external algebra (2.30). The closure of the generators (4.5) is the full $\widehat{su}(2|1)$ superalgebra (4.1) with the following even generators

$$\mathbf{H} = \mathbf{H}^{\rm bose} + \mathbf{H}^{\rm fermi}\,, \tag{4.7}$$

$$\mathbf{H}^{\rm bose} = \frac{1}{2}\,\mathrm{Tr}\left(\mathbf{P}^2 + m^2\mathbf{X}^2\right) - \frac{n\,\mathbf{S}^{(ik)}\mathbf{S}_{(ik)}}{4(\mathbf{X}_0)^2}\,, \tag{4.8}$$

$$\mathbf{H}^{\rm fermi} = \frac{m}{2}\,\mathrm{Tr}\left[\mathbf{\Psi}^k, \bar{\mathbf{\Psi}}_k\right] + \frac{\mathbf{\Psi}^i_0\bar{\mathbf{\Psi}}^k_0\mathbf{S}_{(ik)}}{(\mathbf{X}_0)^2}\,, \tag{4.9}$$

$$\mathbf{I}^i_k = \varepsilon_{kj}\left[\mathbf{S}^{(ij)} + \mathrm{Tr}\left(\mathbf{\Psi}^{(i}\bar{\mathbf{\Psi}}^{j)}\right)\right], \tag{4.10}$$

$$\mathbf{F} = \frac{1}{4}\,\mathrm{Tr}\left[\mathbf{\Psi}^k, \bar{\mathbf{\Psi}}_k\right]. \tag{4.11}$$

The set of physical states of the matrix system is singled out by the n^2 constraints

$$\mathbf{G}_a{}^b = \left(i\,[\mathbf{X},\mathbf{P}]_a{}^b + \left\{\bar{\mathbf{\Psi}}_k, \mathbf{\Psi}^k\right\}_a{}^b + \mathbf{Z}^k_a\bar{\mathbf{Z}}^b_k\right)_{\rm W} - (2q+1)\;\delta_a{}^b \simeq 0\,, \tag{4.12}$$

which are quantum counterparts of the classical constraints (2.8) (the subscript "W" denotes Weyl-ordering) and should be imposed on the wave functions. The constant $(2q+1)$ present in (4.12) differs from the classical constant c due to ordering ambiguities. The operators (4.12) form $u(n)$ algebra

$$\left[\mathbf{G}_a{}^b, \mathbf{G}_c{}^d\right] = \delta_c{}^b \mathbf{G}_a{}^d - \delta_a{}^d \mathbf{G}_c{}^b\,. \tag{4.13}$$

It is important that all constants appearing in the diagonal part of $\mathbf{G}_a{}^b$, i.e. at $a{=}b$, are equal to $(2q+1)$. A corollary of (4.12) is that $u(1)$ generator

$$\sum_a \mathbf{G}_a{}^a = \sum_a \mathbf{Z}_a^k \bar{\mathbf{Z}}_k^a - 2nq \simeq 0 \tag{4.14}$$

includes spin $\mathbf{Z}$-operators only.

As we will see below, the $u(n)$ constraints (4.12) have a transparent meaning: the physical states are $su(n)$ singlets. The constraint (4.14) fixes the homogeneity degree of the physical states with respect to spin variables, whence $2q \in \mathbb{Z}_{>0}$.

4.1.2 Separation of the center-of-mass sector

Let us split the matrix quantities as

$$\begin{aligned}
\mathbf{X}_a{}^b &= \frac{1}{\sqrt{n}}\,\delta_a^b \mathbf{X}_0 + \hat{\mathbf{X}}_a{}^b\,, & \mathbf{P}_a{}^b &= \frac{1}{\sqrt{n}}\,\delta_a^b \mathbf{P}_0 + \hat{\mathbf{P}}_a{}^b\,, \\
\mathbf{\Psi}^k{}_a{}^b &= \frac{1}{\sqrt{n}}\,\delta_a^b \mathbf{\Psi}_0^k + \hat{\mathbf{\Psi}}^k{}_a{}^b\,, & \bar{\mathbf{\Psi}}_k{}_a{}^b &= \frac{1}{\sqrt{n}}\,\delta_a^b \bar{\mathbf{\Psi}}_{0\,k} + \hat{\bar{\mathbf{\Psi}}}_k{}_a{}^b\,,
\end{aligned} \tag{4.15}$$

with

$$\mathbf{X}_0 = \frac{1}{\sqrt{n}} \sum_a \mathbf{X}_a{}^a\,, \quad \mathbf{P}_0 = \frac{1}{\sqrt{n}} \sum_a \mathbf{P}_a{}^a\,, \quad \mathbf{\Psi}_0^k = \frac{1}{\sqrt{n}} \sum_a \mathbf{\Psi}^k{}_a{}^a\,, \quad \bar{\mathbf{\Psi}}_{0\,k} = \frac{1}{\sqrt{n}} \sum_a \bar{\mathbf{\Psi}}_k{}_a{}^a \tag{4.16}$$

being the center-of-mass operators and

$$\begin{aligned}
\hat{\mathbf{X}}_a{}^b &= \mathbf{X}_a{}^b - \frac{1}{\sqrt{n}}\,\delta_a^b \mathbf{X}_0\,, & \hat{\mathbf{P}}_a{}^b &= \mathbf{P}_a{}^b - \frac{1}{\sqrt{n}}\,\delta_a^b \mathbf{P}_0\,, \\
\hat{\mathbf{\Psi}}^k{}_a{}^b &= \mathbf{\Psi}^k{}_a{}^b - \frac{1}{\sqrt{n}}\,\delta_a^b \mathbf{\Psi}_0^k\,, & \hat{\bar{\mathbf{\Psi}}}_k{}_a{}^b &= \bar{\mathbf{\Psi}}_k{}_a{}^b - \frac{1}{\sqrt{n}}\,\delta_a^b \bar{\mathbf{\Psi}}_{0\,k}
\end{aligned} \tag{4.17}$$

the traceless parts of matrix operators.

In terms of the variables (4.16), (4.17) the supercharges (4.5) are represented as

$$\mathbf{Q}^k = \mathbf{Q}_0^k + \hat{\mathbf{Q}}^k\,, \qquad \bar{\mathbf{Q}}_k = \bar{\mathbf{Q}}_{0k} + \hat{\bar{\mathbf{Q}}}_k\,, \tag{4.18}$$

where

$$\mathbf{Q}_0^k = \left(\mathbf{P}_0 - im\mathbf{X}_0\right) \mathbf{\Psi}_0^k + \frac{i\,\mathbf{S}^{(kj)}\,\mathbf{\Psi}_{0\,j}}{\mathbf{X}_0}\,, \quad \bar{\mathbf{Q}}_{0k} = \left(\mathbf{P}_0 + im\mathbf{X}_0\right) \bar{\mathbf{\Psi}}_{0\,k} - \frac{i\,\mathbf{S}_{(kj)}\,\bar{\mathbf{\Psi}}_0^j}{\mathbf{X}_0} \tag{4.19}$$

involve only the center-of-mass operators (4.16) and spin variables, whereas

$$\hat{\mathbf{Q}}^k = \mathrm{Tr}\Big[\left(\hat{\mathbf{P}} - im\,\hat{\mathbf{X}}\right)\hat{\mathbf{\Psi}}^k\Big]\,, \qquad \hat{\bar{\mathbf{Q}}}_k = \mathrm{Tr}\Big[\left(\hat{\mathbf{P}} + im\,\hat{\mathbf{X}}\right)\hat{\bar{\mathbf{\Psi}}}_k\Big] \tag{4.20}$$

depend on the traceless parts (4.17).

The even operators (4.7), (4.8), (4.9), (4.10), (4.11) admit a similar splitting

$$\mathbf{H} = \mathbf{H}_0 + \hat{\mathbf{H}}\,, \qquad \mathbf{I}^i_k = \mathbf{I}_0{}^i_k + \hat{\mathbf{I}}^i_k\,, \qquad \mathbf{F} = \mathbf{F}_0 + \hat{\mathbf{F}}\,. \tag{4.21}$$

Here,

$$\mathbf{H}_0 = \frac{1}{2}\left((\mathbf{P}_0)^2 + m^2\,(\mathbf{X}_0)^2\right) + \frac{m}{2}\left[\mathbf{\Psi}^k_0, \bar{\mathbf{\Psi}}_{0\,k}\right] - \frac{\mathbf{S}^{(ik)}\mathbf{S}_{(ik)}}{4\,(\mathbf{X}_0)^2} + \frac{\mathbf{S}_{(ik)}\mathbf{\Psi}^i_0\bar{\mathbf{\Psi}}^k_0}{(\mathbf{X}_0)^2}\,, \tag{4.22}$$

$$\mathbf{I}_0{}^i_k = \varepsilon_{kj}\left[\mathbf{S}^{(ij)} + \mathbf{\Psi}^{(i}_0\bar{\mathbf{\Psi}}^{j)}_0\right], \tag{4.23}$$

$$\mathbf{F}_0 = \frac{1}{4}\left[\mathbf{\Psi}^k_0, \bar{\mathbf{\Psi}}_{k\,0}\right], \tag{4.24}$$

and

$$\hat{\mathbf{H}} = \frac{1}{2}\,\mathrm{Tr}\left(\hat{\mathbf{P}}^2 + m^2\hat{\mathbf{X}}^2\right) + \frac{m}{2}\,\mathrm{Tr}\left[\hat{\mathbf{\Psi}}^k, \hat{\bar{\mathbf{\Psi}}}_k\right], \tag{4.25}$$

$$\hat{\mathbf{I}}^i_k = \varepsilon_{kj}\mathrm{Tr}\left(\hat{\mathbf{\Psi}}^{(i}\hat{\bar{\mathbf{\Psi}}}^{j)}\right), \tag{4.26}$$

$$\hat{\mathbf{F}} = \frac{1}{4}\,\mathrm{Tr}\left[\hat{\mathbf{\Psi}}^k, \hat{\bar{\mathbf{\Psi}}}_k\right]. \tag{4.27}$$

The sets ($\mathbf{Q}^k_0$, $\bar{\mathbf{Q}}_{0k}$, $\mathbf{H}_0$, $\mathbf{I}_0{}^i_k$, $\mathbf{F}_0$) and ($\hat{\mathbf{Q}}^k$, $\hat{\bar{\mathbf{Q}}}_k$, $\hat{\mathbf{H}}$, $\hat{\mathbf{I}}^i_k$, $\hat{\mathbf{F}}$) form $\widehat{su}(2|1)$ superalgebras (4.1) on their own, with the vanishing mutual (anti)commutators: $\left\{\mathbf{Q}^i_0, \hat{\mathbf{Q}}^k\right\} = \left\{\mathbf{Q}^i_0, \hat{\bar{\mathbf{Q}}}_k\right\} = 0$, etc. Thus, we have singled out the center-of-mass sector from the total system. Note that the $\widehat{su}(2|1)$ generators $\hat{\mathbf{Q}}^k$, $\hat{\bar{\mathbf{Q}}}_k$, $\hat{\mathbf{H}}$, $\hat{\mathbf{I}}^i_k$, $\hat{\mathbf{F}}$ have no action on the spin operators $\mathbf{Z}$ which in fact remain in the center-of-mass sector.

It is of importance that the constraints (4.12) involve in fact only the traceless parts (4.17) of the matrix operators (apart from the spin variable operators). Indeed, they can be rewritten in the form

$$\mathbf{G}_a{}^b = i\left[\hat{\mathbf{X}}, \hat{\mathbf{P}}\right]_a{}^b + \left\{\hat{\bar{\mathbf{\Psi}}}_k, \hat{\mathbf{\Psi}}^k\right\}_a{}^b + \mathbf{Z}^k_a\bar{\mathbf{Z}}^b_k - \left(2q + n - \frac{1}{n}\right)\delta_a{}^b \simeq 0\,. \tag{4.28}$$

However, due to the presence of the same spin variables in the center-of-mass sector, these constraints are applicable also to the corresponding quantum states and so accomplish a link between the two sectors.

4.2 Quantum algebra for SU(2|1) spinning Calogero-Moser system

4.2.1 $\widehat{su}(2|1)$ superalgebra with n dynamical bosons

The quantum counterpart of the multiparticle system from section 3.2 is described by the quantum operators $\mathbf{x}_a$, $\mathbf{p}_a$; $\psi^i{}_a$, $\bar{\psi}_{i\,a}$; $\psi^i{}_a{}^b$, $\bar{\psi}_{i\,a}{}^b$, $a \neq b$; $\mathbf{Z}^i_a$, $\bar{\mathbf{Z}}^a_i$ which satisfy the algebra

$$\begin{aligned} &[\mathbf{x}_a, \mathbf{p}_b] = i\,\delta_{ab}\,, && \left[\mathbf{Z}^k_a, \bar{\mathbf{Z}}^b_j\right] = -\,\delta^k_j\delta^b_a\,,\\ &\left\{\psi^k{}_a, \bar{\psi}_{j\,b}\right\} = \delta^k_j\delta_{ab}\,, && \left\{\psi^k{}_a{}^b, \bar{\psi}_{j\,c}{}^d\right\} = \delta^k_j\delta^d_a\delta^b_c\ \ (a \neq b, c \neq d)\,. \end{aligned} \tag{4.29}$$

Performing the Weyl-ordering in the quantum counterpart of (3.21), we obtain the quantum supercharges:

$$\begin{aligned}
\mathbf{Q}^k &= \sum_a (\mathbf{p}_a - im\mathbf{x}_a)\,\psi^k{}_a + \frac{i\,\mathbf{S}^{(kj)}\,\mathbf{\Psi}_{0\,j}}{\mathbf{X}_0} - \frac{i}{2}\sum_{a\neq b}\frac{\psi^k{}_a - \psi^k{}_b}{\mathbf{x}_a - \mathbf{x}_b} + i\sum_{a\neq b}\frac{\mathbf{T}_a{}^b\,\psi^k{}_b{}^a}{\mathbf{x}_a - \mathbf{x}_b}\,,\\
\bar{\mathbf{Q}}_k &= \sum_a (\mathbf{p}_a + im\mathbf{x}_a)\,\bar\psi_{k\,a} - \frac{i\,\mathbf{S}_{(kj)}\,\bar{\mathbf{\Psi}}^j_0}{\mathbf{X}_0} - \frac{i}{2}\sum_{a\neq b}\frac{\bar\psi_{k\,a} - \bar\psi_{k\,b}}{\mathbf{x}_a - \mathbf{x}_b} + i\sum_{a\neq b}\frac{\mathbf{T}_a{}^b\,\bar\psi_{k\,b}{}^a}{\mathbf{x}_a - \mathbf{x}_b}\,,
\end{aligned} \tag{4.30}$$

where

$$\mathbf{T}_a{}^b = \mathbf{Z}^k_a\bar{\mathbf{Z}}^b_k + \left(\psi^k_a - \psi^k_b\right)\bar\psi_{ka}{}^b + \left(\bar\psi_{ak} - \bar\psi_{bk}\right)\psi^k{}_a{}^b + \sum_{c\neq a,\,c\neq b}\left(\psi^k{}_a{}^c\bar\psi_{kc}{}^b + \bar\psi_{ka}{}^c\psi^k{}_c{}^b\right) \tag{4.31}$$

are quantum counterparts of (2.16) at $a \neq b$ and

$$\mathbf{X}_0 = \frac{1}{\sqrt{n}}\sum_a \mathbf{x}_a\,,\qquad \mathbf{\Psi}^i_0 = \frac{1}{\sqrt{n}}\sum_a \psi^i{}_a\,,\qquad \bar{\mathbf{\Psi}}_{0\,i} = \frac{1}{\sqrt{n}}\sum_a \bar\psi_{i\,a}\,. \tag{4.32}$$

Computing the anticommutators of the supercharges (4.30),

$$\left\{\mathbf{Q}^i, \bar{\mathbf{Q}}_k\right\} = 2\,\delta^i_k\,\mathbf{H}_n + 2m\left(\mathbf{I}^i_k - \delta^i_k\,\mathbf{F}\right) - 2\sum_{a\neq b}\frac{\psi^i{}_a{}^b\bar\psi_{kb}{}^a}{(\mathbf{x}_a - \mathbf{x}_b)^2}\left(\mathbf{T}_a - \mathbf{T}_b\right), \tag{4.33}$$

$$\left\{\mathbf{Q}^i, \mathbf{Q}^k\right\} = -2\sum_{a\neq b}\frac{\psi^i{}_a{}^b\psi^k{}_b{}^a}{(\mathbf{x}_a - \mathbf{x}_b)^2}\left(\mathbf{T}_a - \mathbf{T}_b\right), \tag{4.34}$$

$$\left\{\bar{\mathbf{Q}}_i, \bar{\mathbf{Q}}_k\right\} = -2\sum_{a\neq b}\frac{\bar\psi_{ia}{}^b\bar\psi_{kb}{}^a}{(\mathbf{x}_a - \mathbf{x}_b)^2}\left(\mathbf{T}_a - \mathbf{T}_b\right) \tag{4.35}$$

we find the explicit form of the quantum even generators

$$\begin{aligned}
\mathbf{H} &= \frac{1}{2}\sum_a\left(\mathbf{p}_a\mathbf{p}_a + m^2\mathbf{x}_a\mathbf{x}_a\right) + \frac{m}{2}\sum_a\left[\psi^k{}_a, \bar\psi_{k\,a}\right] + \frac{m}{2}\sum_{a\neq b}\left[\psi^k{}_a{}^b, \bar\psi_{k\,b}{}^a\right]\\
&\quad - \frac{\mathbf{S}^{(ik)}\mathbf{S}_{(ik)}}{4\,(\mathbf{X}_0)^2} + \frac{\mathbf{S}_{(ik)}\,\mathbf{\Psi}^i_0\bar{\mathbf{\Psi}}^k_0}{(\mathbf{X}_0)^2} + \frac{1}{2}\sum_{a\neq b}\frac{\mathbf{T}_a{}^b\mathbf{T}_b{}^a}{(\mathbf{x}_a - \mathbf{x}_b)^2}\,,
\end{aligned} \tag{4.36}$$

$$\mathbf{I}^i_k = \varepsilon_{kj}\left[\mathbf{S}^{(ij)} + \sum_a \psi^{(i}{}_a\bar\psi^{j)}{}_a + \sum_{a\neq b}\psi^{(i}{}_a{}^b\bar\psi^{j)}{}_b{}^a\right], \tag{4.37}$$

$$\mathbf{F} = \frac{1}{4}\sum_a\left[\psi^k{}_a, \bar\psi_{k\,a}\right] + \frac{1}{4}\sum_{a\neq b}\left[\psi^k{}_a{}^b, \bar\psi_{k\,b}{}^a\right]. \tag{4.38}$$

The commutators of the generator $\mathbf{H}$ with odd generators $\mathbf{Q}^i$, $\bar{\mathbf{Q}}_i$ are a quantum generalization of (3.23). The remaining generators $\mathbf{I}^i_k$, $\mathbf{F}$ obey the same commutation relations as in (4.1).

From the (anti)commutators obtained we observe that the generators (4.30), (4.36), (4.37), (4.38) form the $\widehat{su}(2|1)$ superalgebra (4.1) up to the differences $(\mathbf{T}_a - \mathbf{T}_b)$. However,

JHEP04(2018)043

recalling the constraints (2.20), this reduced system is specified also by the conditions

$$\mathbf{T}_a - 2q - 2\,(n-1) = \mathbf{Z}^k_a \bar{\mathbf{Z}}^a_k + \sum_{c \neq a} \left(\psi^k{}_a{}^c \bar{\psi}_{kc}{}^a - \psi^k{}_c{}^a \bar{\psi}_{ka}{}^c \right) - 2q \simeq 0\,, \tag{4.39}$$

which must be superimposed on the physical states. Therefore, the differences $(\mathbf{T}_a - \mathbf{T}_b)$ are vanishing on the physical states, and the physical sector of the relevant Hilbert space is closed under $\widehat{\mathrm{SU}}(2|1)$ symmetry.

It is important that the quantum constraints (4.39) commute with the $\widehat{su}(2|1)$ generators:

$$\left[\mathbf{T}_a, \mathbf{Q}^i\right] = \left[\mathbf{T}_a, \bar{\mathbf{Q}}_i\right] = 0\,. \tag{4.40}$$

In addition, the quantities (4.31) satisfy the algebra

$$\left[\mathbf{T}_a{}^b, \mathbf{T}_c{}^d\right] = \delta^b_c\, \mathbf{T}_a{}^d - \delta^d_a\, \mathbf{T}_c{}^b\,, \tag{4.41}$$

where $\mathbf{T}_a{}^a = \mathbf{T}_a$ at fixed a.

It is instructive to be convinced that the numerator in the last term in (4.36) is indeed reduced to that for U(2) spin Calogero-Moser system [33], when applied to the bosonic wave functions Φ_{bos} defined by the conditions

$$\bar{\psi}_{i\,a}\Phi_{bos} = \bar{\psi}_{i\,a}{}^b\Phi_{bos} = 0\,.$$

It is easy to check that in this case

$$\mathbf{T}_a{}^b \;\Rightarrow\; \mathbf{Z}^k_a \bar{\mathbf{Z}}^b_k + 2(n-1)\delta^b_a$$

and the constraint (4.39) is reduced to

$$\mathbf{Z}^k_a \bar{\mathbf{Z}}^a_k - 2q \simeq 0\,.$$

Now, taking into account that in the numerator in (4.36) $a \neq b$, it is easy to check that

$$\frac{1}{2}\,\mathbf{T}_a{}^b \mathbf{T}_b{}^a \;\Rightarrow\; -\frac{1}{2}\,\mathbf{S}^{(ij)}_a \mathbf{S}_{b\,(ij)} + q\,(q+1)\,, \qquad a \neq b\,, \tag{4.42}$$

where $\mathbf{S}^{(ij)}_a = \mathbf{Z}^{(i}_a \bar{\mathbf{Z}}^{j)\,a}$ (no summation over a). The operators $\mathbf{S}^{(ij)}_a$ are just the quantum version of $S^{(ij)}_a$ defined in (2.28). Foe each value of the index a they generate $su(2)$ algebras and commute with each other for $a \neq b$. The expression (4.42) coincides with that appearing in the rational U(2) spin Calogero-Moser model, with q being the pairwise spin coupling constant.[5]

[5] Following [33], this model can be referred to as the reduced matrix U(2) spin Calogero-Moser model, with $2q \in \mathbb{Z}_{>0}$. There exists another type of U(s) spin models, the so-called "exchange-operator models" [33–36], for which the spin coupling constant is an arbitrary number. Our SU(2|1) supersymmetric multi-particle system yields just the first type of U(2) spin models in the bosonic sector.

JHEP04(2018)043

4.2.2 Division into subsystems

Using the simple identity

$$\sum_a \mathbf{K}_a \mathbf{M}_a = \frac{1}{n} \sum_a \mathbf{K}_a \sum_b \mathbf{M}_b + \frac{1}{2n} \sum_{a\neq b} (\mathbf{K}_a - \mathbf{K}_b)(\mathbf{M}_a - \mathbf{M}_b)$$

which is valid for arbitrary n-vector operators $\mathbf{K}_a$, $\mathbf{M}_a$, $a = 1, \ldots, n$, and introducing the center-of-mass quantities (4.32) and

$$\mathbf{P}_0 = \frac{1}{\sqrt{n}} \sum_a \mathbf{p}_a \,, \tag{4.43}$$

we can represent the charges (4.30) as the sums

$$\mathbf{Q}^k = \mathbf{Q}^k_0 + \mathbb{Q}^k \,, \qquad \bar{\mathbf{Q}}_k = \bar{\mathbf{Q}}_{0k} + \bar{\mathbb{Q}}_k \,. \tag{4.44}$$

The first items $\mathbf{Q}^k_0$, $\bar{\mathbf{Q}}_{0k}$ in these sums were defined in (4.19), and they involve only the central-of-mass supercoordinates, whereas the second items $\mathbb{Q}^k$, $\bar{\mathbb{Q}}_k$ depend only on the differences of the supercoordinates:

$$\begin{aligned}
\mathbb{Q}^k &= \frac{1}{2n} \sum_{a\neq b} \Big[(\mathbf{p}_a - \mathbf{p}_b) - im\,(\mathbf{x}_a - \mathbf{x}_b) \Big] \Big(\boldsymbol{\psi}^k{}_a - \boldsymbol{\psi}^k{}_b \Big) \\
&\quad - \frac{i}{2} \sum_{a\neq b} \frac{\boldsymbol{\psi}^k{}_a - \boldsymbol{\psi}^k{}_b}{\mathbf{x}_a - \mathbf{x}_b} + i \sum_{a\neq b} \frac{\mathbf{T}_a{}^b \, \boldsymbol{\psi}^k{}_b{}^a}{\mathbf{x}_a - \mathbf{x}_b} \,, \\
\bar{\mathbb{Q}}_k &= \frac{1}{2n} \sum_{a\neq b} \Big[(\mathbf{p}_a - \mathbf{p}_b) + im\,(\mathbf{x}_a - \mathbf{x}_b) \Big] \big(\bar{\boldsymbol{\psi}}_{k\,a} - \bar{\boldsymbol{\psi}}_{k\,b} \big) \\
&\quad - \frac{i}{2} \sum_{a\neq b} \frac{\bar{\boldsymbol{\psi}}_{k\,a} - \bar{\boldsymbol{\psi}}_{k\,b}}{\mathbf{x}_a - \mathbf{x}_b} + i \sum_{a\neq b} \frac{\mathbf{T}_a{}^b \, \bar{\boldsymbol{\psi}}_{k\,b}{}^a}{\mathbf{x}_a - \mathbf{x}_b} \,.
\end{aligned} \tag{4.45}$$

Since $[\mathbf{S}^{(ij)}, \mathbf{T}_a{}^b] = 0$, $\mathbf{Q}^k_0$, $\bar{\mathbf{Q}}_{0k}$ anticommute with second $\mathbb{Q}^k$, $\bar{\mathbb{Q}}_k$:

$$\Big\{ \mathbf{Q}^k_0, \mathbb{Q}^j \Big\} = \Big\{ \mathbf{Q}^k_0, \bar{\mathbb{Q}}_j \Big\} = \big\{ \bar{\mathbf{Q}}_{0k}, \mathbb{Q}^j \big\} = \big\{ \bar{\mathbf{Q}}_{0k}, \bar{\mathbb{Q}}_j \big\} = 0 \,. \tag{4.46}$$

This implies that the bosonic generators (4.36), (4.37), (4.38) can also be represented as similar sums,

$$\mathbf{H} = \mathbf{H}_0 + \mathbb{H} \,, \qquad \mathbf{I}^i_k = \mathbf{I}_0{}^i_k + \mathbb{I}^i_k \,, \qquad \mathbf{F} = \mathbf{F}_0 + \mathbb{F} \,, \tag{4.47}$$

where $\mathbf{H}_0, \mathbf{I}_0{}^i_k$ and $\mathbf{F}_0$ are given by eqs. (4.22), (4.23), (4.24) and so involve only the center-of-mass coordinates, while the rest of operators is defined by the expressions

$$\begin{aligned}
\mathbb{H} &= \frac{1}{4n} \sum_{a\neq b} \Big((\mathbf{p}_a - \mathbf{p}_b)^2 + m^2 (\mathbf{x}_a - \mathbf{x}_b)^2 \Big) + \frac{1}{2} \sum_{a\neq b} \frac{\mathbf{T}_a{}^b \mathbf{T}_b{}^a}{(\mathbf{x}_a - \mathbf{x}_b)^2} \\
&\quad + \frac{m}{4n} \sum_{a\neq b} \Big[\Big(\boldsymbol{\psi}^k{}_a - \boldsymbol{\psi}^k{}_b \Big), \big(\bar{\boldsymbol{\psi}}_{k\,a} - \bar{\boldsymbol{\psi}}_{k\,b} \big) \Big] + \frac{m}{2} \sum_{a\neq b} \Big[\boldsymbol{\psi}^k{}_a{}^b, \bar{\boldsymbol{\psi}}_{k\,b}{}^a \Big] \,,
\end{aligned} \tag{4.48}$$

$$\mathbb{I}^i_k = \varepsilon_{kj} \left[\frac{1}{2n} \sum_{a\neq b} \Big(\boldsymbol{\psi}^{(i}{}_a - \boldsymbol{\psi}^{(i}{}_b \Big) \Big(\bar{\boldsymbol{\psi}}^{j)}{}_a - \bar{\boldsymbol{\psi}}^{j)}{}_b \Big) + \sum_{a\neq b} \boldsymbol{\psi}^{(i}{}_a{}^b \bar{\boldsymbol{\psi}}^{j)}{}_b{}^a \right] , \tag{4.49}$$

$$\mathbb{F} = \frac{1}{8n} \sum_{a\neq b} \Big[\Big(\boldsymbol{\psi}^k{}_a - \boldsymbol{\psi}^k{}_b \Big), \big(\bar{\boldsymbol{\psi}}_{k\,a} - \bar{\boldsymbol{\psi}}_{k\,b} \big) \Big] + \frac{1}{4} \sum_{a\neq b} \Big[\boldsymbol{\psi}^k{}_a{}^b, \bar{\boldsymbol{\psi}}_{k\,b}{}^a \Big] \,. \tag{4.50}$$

The sets of the generators ($\mathbf{Q}_0^k$, $\bar{\mathbf{Q}}_{0k}$, $\mathbf{H}_0$, $\mathbf{I}_0{}^i_k$, $\mathbf{F}_0$) and ($\mathbb{Q}^k$, $\bar{\mathbb{Q}}_k$, $\mathbb{H}$, $\mathbb{I}^i_k$, $\mathbb{F}$) form two separate mutually (anti)commuting $\widehat{su}(2|1)$ superalgebras. Note that second set generates an $\widehat{su}(2|1)$ superalgebra up to the constraints, as in (4.33), (4.34), (4.35). Also, note that the "internal" SU(2) generators (4.49) (appearing in the anticommutator of supercharges) act on the indices i, j of the fermionic operators $\boldsymbol{\psi}^i{}_a - \boldsymbol{\psi}^i{}_b$, $\bar{\boldsymbol{\psi}}_{i\,a} - \bar{\boldsymbol{\psi}}_{i\,b}$, $\boldsymbol{\psi}^i{}_a{}^b$, $\bar{\boldsymbol{\psi}}_{i\,a}{}^b$, $a \neq b$, while the indices i, j of the spin operators $\mathbf{Z}^i_a$, $\bar{\mathbf{Z}}^a_i$ are subject to the action of the external SU(2) generators (4.6).

4.3 Subsystems of $\mathcal{N}$=4 supersymmetric Calogero-Moser model

We have found that the $\mathcal{N}$=4 supersymmetric n-particle Calogero-Moser system is a direct sum of two subsystems with different realizations of the $\widehat{su}(2|1)$ generators.

The generators ($\mathbf{Q}_0^k$, $\bar{\mathbf{Q}}_{0k}$, $\mathbf{H}_0$, $\mathbf{I}_0{}^i_k$, $\mathbf{F}_0$) act in the sector of the center-of-mass operators ($\mathbf{X}_0$, $\mathbf{P}_0$, $\boldsymbol{\Psi}_0^i$, $\bar{\boldsymbol{\Psi}}_{0\,i}$) and the spin operators ($\mathbf{Z}^i_a$, $\bar{\mathbf{Z}}^a_i$). The second set of the $\widehat{su}(2|1)$ generators ($\hat{\mathbf{Q}}^k$, $\hat{\bar{\mathbf{Q}}}_k$, $\hat{\mathbf{H}}$, $\hat{\mathbf{I}}^i_k$, $\hat{\mathbf{F}}$) act, in the matrix formulation, within the sector of the traceless operators ($\hat{\mathbf{X}}$, $\hat{\mathbf{P}}$, $\hat{\boldsymbol{\Psi}}^i$, $\hat{\bar{\boldsymbol{\Psi}}}_i$). Physical states in this subsystem are specified also by the spin operators ($\mathbf{Z}^i_a$, $\bar{\mathbf{Z}}^a_i$) which are present in the $u(n)$ constraints (4.28). These constraints also specify physical states in the center-of-mass sector involving the same spin operators. In the reduced formulation, the generators ($\mathbb{Q}^k$, $\bar{\mathbb{Q}}_k$, $\mathbb{H}$, $\mathbb{I}^i_k$, $\mathbb{F}$) are spanned by the set of operators ($\mathbf{x}_a - \mathbf{x}_b$, $\mathbf{p}_a - \mathbf{p}_b$, $\boldsymbol{\psi}^i{}_a - \boldsymbol{\psi}^i{}_b$, $\bar{\boldsymbol{\psi}}_{i\,a} - \bar{\boldsymbol{\psi}}_{i\,b}$, $\boldsymbol{\psi}^i{}_a{}^b$, $\bar{\boldsymbol{\psi}}_{i\,a}{}^b$, $a \neq b$; $\mathbf{Z}^i_a$, $\bar{\mathbf{Z}}^a_i$). It should be pointed out that the spin operators $\mathbf{Z}^i_a$, $\bar{\mathbf{Z}}^a_i$ have a non-zero action on the physical states with $q \neq 0$ for all subsystems defined above and listed below.

The just described structure of the considered system suggests that we can consider three subsystems:

I) The center-of-mass sector spanned by the quantum operators ($\mathbf{X}_0$, $\mathbf{P}_0$, $\boldsymbol{\Psi}_0^i$, $\bar{\boldsymbol{\Psi}}_{0\,i}$, $\mathbf{Z}^i_a$, $\bar{\mathbf{Z}}^a_i$) and the symmetry operators ($\mathbf{Q}_0^k$, $\bar{\mathbf{Q}}_{0k}$, $\mathbf{H}_0$, $\mathbf{I}_0{}^i_k$, $\mathbf{F}_0$);

II) The pure Calogero-Moser multi-particle sector with the center-of-mass sector separated. It is spanned by the quantum operators ($\hat{\mathbf{X}}$, $\hat{\mathbf{P}}$, $\hat{\boldsymbol{\Psi}}^i$, $\hat{\bar{\boldsymbol{\Psi}}}_i$) in the matrix formulation or by ($\mathbf{x}_a - \mathbf{x}_b$, $\mathbf{p}_a - \mathbf{p}_b$, $\boldsymbol{\psi}^i{}_a - \boldsymbol{\psi}^i{}_b$, $\bar{\boldsymbol{\psi}}_{i\,a} - \bar{\boldsymbol{\psi}}_{i\,b}$, $\boldsymbol{\psi}^i{}_a{}^b$, $\bar{\boldsymbol{\psi}}_{i\,a}{}^b$, $a \neq b$) in the reduced formulation. In both formulations, this subsystem also involves the spin operators $\mathbf{Z}^i_a$, $\bar{\mathbf{Z}}^a_i$. The SU(2|1) symmetry generators are ($\hat{\mathbf{Q}}^k$, $\hat{\bar{\mathbf{Q}}}_k$, $\hat{\mathbf{H}}$, $\hat{\mathbf{I}}^i_k$, $\hat{\mathbf{F}}$) or ($\mathbb{Q}^k$, $\bar{\mathbb{Q}}_k$, $\mathbb{H}$, $\mathbb{I}^i_k$, $\mathbb{F}$);

III) The full Calogero-Moser multi-particle system which contains the center-of-mass sector and so is spanned by the set of all quantum operators. The $\widehat{\mathrm{SU}}(2|1)$ symmetry generators are sums of the $\widehat{\mathrm{SU}}(2|1)$ generators acting in the two previously defined sectors.

Now we are prepared to determine the energy spectrum of all these systems.

5 Center-of-mass subsystem with n sets of spin variables

In this section we consider the subsystem **I)** which describes the center-of-mass sector with the Hamiltonian $\mathbf{H}_0$ (4.22).

The center-of-mass supercoordinates (4.32), (4.43) satisfy the following (anti)commutation relations

$$[\mathbf{X}_0, \mathbf{P}_0] = i\,, \qquad \{\mathbf{\Psi}_0^i, \bar{\mathbf{\Psi}}_{0\,k}\} = \delta_k^i\,, \tag{5.1}$$

while those for the spin variables read

$$\left[\mathbf{Z}_a^i, \bar{\mathbf{Z}}_k^b\right] = -\,\delta_k^i \delta_a^b\,. \tag{5.2}$$

We will use the following realization of the operator relations (5.1), (5.2)

$$\mathbf{X}_0 = x_0\,, \quad \mathbf{P}_0 = -i\frac{\partial}{\partial x_0}\,, \quad \mathbf{\Psi}_0^i = \psi_0^i\,, \quad \bar{\mathbf{\Psi}}_{0\,i} = \frac{\partial}{\partial \psi_0^i}\,, \tag{5.3}$$

$$\mathbf{Z}_a^i = z_a^i\,, \quad \bar{\mathbf{Z}}_i^a = \frac{\partial}{\partial z_a^i}\,, \tag{5.4}$$

where x_0 is a real commuting variable, z_a^i are complex commuting variables and ψ_0^i are complex Grassmann variables. In this realization the Hamiltonian (4.22) takes the form

$$\mathbf{H}_0 = \frac{1}{2}\left(-\frac{\partial^2}{\partial {x_0}^2} + m^2 {x_0}^2\right) + m\left(\psi_0^i \frac{\partial}{\partial \psi_0^i} - 1\right) + \frac{1}{{x_0}^2}\left(-\frac{1}{4}\mathbf{S}^{(ik)}\mathbf{S}_{ik} + \mathbf{S}^{(ik)}\psi_{0\,i}\frac{\partial}{\partial \psi_0^k}\right), \tag{5.5}$$

where $\mathbf{S}_{(ij)} = \sum_a z_{a\,(i}\frac{\partial}{\partial z_a^{j)}}$. Wave function $\Phi^{(2q)}(x_0, z_a^i, \psi_0^i)$ is subject to the n constraints originating from (4.28):

$$\mathbf{G}_a{}^a\, \Phi^{(2q)} = \left(z_a^k \frac{\partial}{\partial z_a^k} - 2q\right)\Phi^{(2q)} = 0\,, \qquad a = 1, \ldots, n\,. \tag{5.6}$$

The solution of eqs. (5.6) is a function which is homogeneous of degree $2q$ with respect to each set of spin variables z_a^k. So the number q taking positive integer and half-integer values can be treated as a spin associated with every SU(2) group generated by the quantum generators

$$\mathbf{S}_a^{(ij)} = z_{a\,(i}\frac{\partial}{\partial z_a^{j)}}\,, \qquad \text{(no summation over } a\text{)}. \tag{5.7}$$

The number s will be associated with the diagonal SU(2) group generated by

$$\mathbf{S}^{(ij)} = \sum_{a=1}^{n} \mathbf{S}_a^{(ij)}, \tag{5.8}$$

and, in what follows, will be referred to as "SU(2) spin s". Since $\Phi^{(2q)}$ is transformed in the direct product of n spin q SU(2) representations, the maximal external SU(2) spin is just $s = nq$. It will be convenient to expand $\Phi^{(2q)}$ into irreducible multiplets of the diagonal SU(2), with spins running in the intervals $0, 1 \ldots nq$ (for $2nq$ even) or $1/2, 3/2 \ldots nq$ (for $2nq$ odd).

JHEP04(2018)043

As an illustration, we dwell on two lower-n cases.

$\boldsymbol{n=2.}$ In this case the wave function $\Phi_0(x_0, z_1^i, z_2^i, \psi_0^i)$ is subject to two constraints

$$(\mathbf{T}_1 - 2q)\,\Phi_0^{(2q)} = \left(z_1^k\frac{\partial}{\partial z_1^k} - 2q\right)\Phi_0^{(2q)} = 0\,,\;(\mathbf{T}_2 - 2q)\,\Phi_0^{(2q)} = \left(z_2^k\frac{\partial}{\partial z_2^k} - 2q\right)\Phi_0^{(2q)} = 0\,. \tag{5.9}$$

Their general solution is

$$\Phi_0^{(2q)} = (z_1 z_2)^{2q}\,\phi + \sum_{s=1}^{2q} (z_1 z_2)^{2q-s}\, z_1^{i_1}\ldots z_1^{i_s}\, z_2^{i_{s+1}}\ldots z_2^{i_{2s}}\,\Phi_{(i_1\ldots i_{2s})}\,, \tag{5.10}$$

where $(z_1 z_2) := z_1^i z_{2i}$. The component wave functions ϕ, $\Phi_{(i_1\ldots i_{2s})}$ in the expansion (5.10)are functions of x_0, ψ_0^i. They form irreducible SU(2) multiplets with spins $s = 0, 1, \ldots 2q$. Their expansions with respect to ψ_0^i are

$$\phi = a_+ + \psi_0^i b_i + (\psi_0)^2\, a_-\,, \tag{5.11}$$

$$\Phi_{(i_1\ldots i_{2s})} = A_{+(i_1\ldots i_{2s})} + \psi_{0\,(i_1}B_{i_2\ldots i_{2s})} + \psi_0^j C_{j(i_1\ldots i_{2s})} + (\psi_0)^2\, A_{-(i_1\ldots i_{2s})}\,. \tag{5.12}$$

All components in these expansions are functions of x_0 only. In the bosonic wave function $\Phi_0^{(2q)}$, the fields $a_\pm$, $A_{\pm(i_1\ldots i_{2s})}$ are bosonic, whereas b_i, $B_{(i_2\ldots i_{2s})}$, $C_{(ji_1\ldots i_{2s})}$ are fermionic. The SU(2) spins of the component wave functions are counted with respect to the "internal" SU(2) with the generators (4.23) which contain, besides the part acting on the bosonic spin variables, also the one acting on the fermionic variables.

Let us determine the eigenvalues of the Hamiltonian (5.5) on the wave function (5.10), i.e. solve the stationary Schrödinger equation

$$\mathbf{H}_0\,\Phi_0^{(2q,\ell)} = E_{s,\ell}\,\Phi_0^{(2q,\ell)}\,. \tag{5.13}$$

As a prerequisite, we adduce the following eigenvalue relations

$$-\frac{1}{2}\mathbf{S}^{(ij)}\mathbf{S}_{(ij)} z_1^{k_1}\ldots z_1^{k_p} z_2^{k_{p+1}}\ldots z_2^{k_{2s}} A_{(k_1\ldots k_{2s})} = s\,(s+1)\, z_1^{k_1}\ldots z_1^{k_p} z_2^{k_{p+1}}\ldots z_2^{k_{2s}} A_{(k_1\ldots k_{2s})}\,, \tag{5.14}$$

$$\mathbf{S}_{(ij)}\,(z_1 z_2) = 0\,, \qquad \mathbf{S}^{(ij)}\mathbf{S}_{(ij)}\,\psi_0^k z_{a\,k} = -\mathbf{S}^{(ij)}\psi_{0\,i}\,\frac{\partial}{\partial\psi_0^j}\,\psi_0^k z_{a\,k} = -\frac{3}{2}\,\psi_0^k z_{a\,k}\,, \tag{5.15}$$

$$\begin{aligned}\mathbf{S}^{(ij)}\psi_{0\,i}\,\frac{\partial}{\partial\psi_0^j}\,\psi_0^n z_1^{k_1}\ldots z_1^{k_p} z_2^{k_{p+1}}\ldots z_2^{k_{2s}} A_{(nk_1\ldots k_{2s})} &= \\ = -s\,\psi_0^n z_1^{k_1}\ldots z_1^{k_p} z_2^{k_{p+1}}\ldots z_2^{k_{2s}} A_{(nk_1\ldots k_{2s})}\,.&\end{aligned} \tag{5.16}$$

$$\begin{aligned}\mathbf{S}^{(ij)}\psi_{0\,i}\,\frac{\partial}{\partial\psi_0^j}\, z_1^{k_1}\ldots z_1^{k_p} z_2^{k_{p+1}}\ldots z_2^{k_{2s}}\psi_{0\,(k_1}A_{k_2\ldots k_{2s})} &= \\ = (s+1)\, z_1^{k_1}\ldots z_1^{k_p} z_2^{k_{p+1}}\ldots z_2^{k_{2s}}\psi_{0\,(k_1}A_{k_2\ldots k_{2s})}.&\end{aligned} \tag{5.17}$$

They can be easily checked and shown to be valid for an arbitrary $p \leq 2s$. Due to these relations, all the component fields in the expansions (5.11), (5.12) of the wave function (5.10) are eigenstates of the center-of-mass Hamiltonian (5.5) with the spin s of the diagonal SU(2) group given by $\mathbf{S}^{(ij)}$. The equation (5.13) amounts to the following equations for the component wave functions

$$
\begin{aligned}
\frac{1}{2}\left[-\frac{\partial^2}{\partial x_0^2}+m^2{x_0}^2\right] a_{\pm}^{(\ell)} &= (E_{0,\ell}\pm m)\, a_{\pm}^{(\ell)}\,,\\
\frac{1}{2}\left[-\frac{\partial^2}{\partial x_0^2}+m^2{x_0}^2\right] b_i^{(\ell)} &= E_{0,\ell}\, b_i^{(\ell)}\,,
\end{aligned}
\tag{5.18}
$$

$$
\begin{aligned}
\frac{1}{2}\left[-\frac{\partial^2}{\partial x_0^2}+m^2{x_0}^2+\frac{s(s+1)}{{x_0}^2}\right] A_{\pm(i_1\ldots i_{2s})}^{(\ell)} &= (E_{s,\ell}\pm m)\, A_{\pm(i_1\ldots i_{2s})}^{(\ell)}\,,\\
\frac{1}{2}\left[-\frac{\partial^2}{\partial x_0^2}+m^2{x_0}^2+\frac{(s+1)(s+2)}{{x_0}^2}\right] B_{(i_1\ldots i_{2s-1})}^{(\ell)} &= E_{s,\ell}\, B_{(i_1\ldots i_{2s-1})}^{(\ell)}\,,\\
\frac{1}{2}\left[-\frac{\partial^2}{\partial x_0^2}+m^2{x_0}^2+\frac{s(s-1)}{{x_0}^2}\right] C_{(i_1\ldots i_{2s+1})}^{(\ell)} &= E_{s,\ell}\, C_{(i_1\ldots i_{2s+1})}^{(\ell)}\,,
\end{aligned}
\tag{5.19}
$$

where (5.18) corresponds to $s=0$, while in (5.19) s runs over $2q\geq 1$ values, $s=1,\ldots,2q$.

The equations (5.18) for the fields $a_{\pm}^{(\ell)}$ and $b_i^{(\ell)}$ have the form

$$
\frac{1}{2}\left[-\frac{\partial^2}{\partial x_0^2}+m^2{x_0}^2\right] f^{(\ell)}(x_0) = \mathcal{E}_\ell\, f^{(\ell)}(x_0)
\tag{5.20}
$$

and describe the excitations of oscillators. The standard solutions of the equation (5.20) are given via Hermite polynomials H_ℓ as

$$
f^{(\ell)}(x_0) = \mathrm{H}_\ell\,(x_0)\exp\left(-m{x_0}^2/2\right), \qquad \ell=0,1,2,\ldots\,,
\tag{5.21}
$$

and have the energies

$$
\mathcal{E}_\ell = m\,(\ell+1/2)\,.
\tag{5.22}
$$

Thus, the energy spectrum reads

$$
E_{0,\ell} = m\left(\ell-\frac{1}{2}\right).
\tag{5.23}
$$

It is worth pointing out that the solution for $\mathcal{N}=4$ supersymmetric harmonic oscillator (5.18) was originally given in [14].

The equations (5.19) for the fields $A_{\pm(i_1\ldots i_{2s})}$, $B_{(i_1\ldots i_{2s-1})}$ and $C_{(i_1\ldots i_{2s+1})}$ have the generic form

$$
\frac{1}{2}\left[-\frac{\partial^2}{\partial x_0^2}+m^2{x_0}^2+\frac{\gamma\,(\gamma-1)}{{x_0}^2}\right] f^{(\ell)}(x_0) = \mathcal{E}'_\ell\, f^{(\ell)}(x_0)\,,
\tag{5.24}
$$

where γ is a constant. It is the well-known equation describing quantum states of non-relativistic particle moving in a sum of the one-dimensional oscillator and conformal inverse-square potentials, and it has the following general solution (see, e.g., [1, 3, 40])

$$
f^{(\ell)}(x_0) = \sqrt{\frac{2\ell!}{\Gamma(\ell+\gamma+1/2)}}\,{x_0}^{\gamma}\,L_\ell^{(\gamma-1/2)}(m{x_0}^2)\exp(-m{x_0}^2/2)\,, \qquad \ell=0,1,2,\ldots\,,
\tag{5.25}
$$

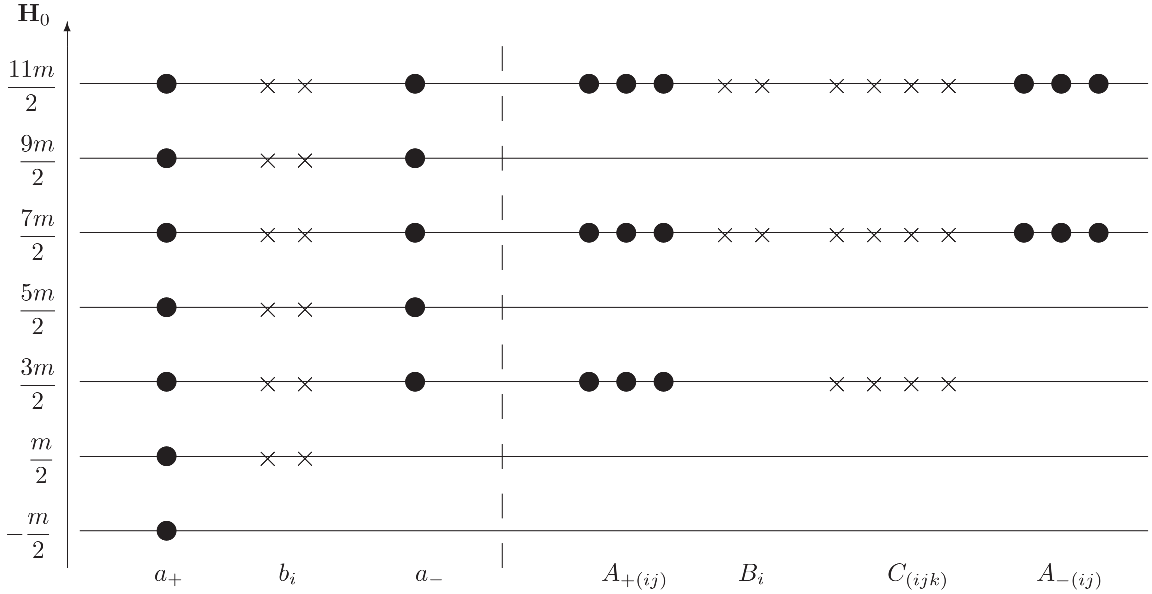

Figure 1. The degeneracy of energy levels of $\mathbf{H}_0$ for $n = 2$ and $q = 1/2$. Circles and crosses represent bosonic and fermionic states, respectively. On the left from the dotted vertical line the degeneracy corresponding to harmonic oscillator [14, 19] is shown. On the right side there is shown a sum of SU(2|1) representations specified by their spin values s and coinciding with those found in [27] for the relevant spin. For the considered simplest case of $q = 1/2$, spin s takes only one value, $s = 1$.

where $L_\ell^{(\gamma-1/2)}$ is a generalized Laguerre polynomial. The corresponding energy levels are

$$\mathcal{E}'_{\gamma,\ell} = m\left(2\ell + \gamma + \frac{1}{2}\right). \tag{5.26}$$

The general solution (5.25) was used in ref. [27] to reveal the energy spectrum of the one-particle system with one set of the spin variables. Each equation in the set (5.19) has the form of (5.24), the parameter γ being $s+1$, $s+2$ and s, respectively. Thus, the energy of the states of spins s and $s+1/2$ described by the wave functions $A_{+(i_1\ldots i_{2s})}$ and $C_{(i_1\ldots i_{2s+1})}$, is equal to

$$E_{s,\ell} = m\,(2\ell + s + 1/2)\,, \quad \ell = 0,1,2,\ldots\,, \qquad s = 1,\ldots,2q\,.$$

The energy of the states $A_{-(i_1\ldots i_{2s})}$ of spin s and the states $B_{(i_1\ldots i_{2s-1})}$ of spin $s+1/2$ is given only for excited states by the same expression

$$E_{s,\ell} = m\,(2\ell + s + 1/2)\,, \quad \ell = 1,2,3,\ldots\,, \qquad s = 1,\ldots,2q\,.$$

The lowest energy for these states corresponds to $s = 1$, $\ell = 0$ and equals

$$E_{\min} = \frac{3m}{2}\,. \tag{5.27}$$

At $q = 1/2$ we have the picture drawn in the figure 1.

The $q = 1$ case encompasses the same states as for $q = 1/2$ ($s = 1$) depicted in figure 1, but also additional states with higher spins and higher energies. The similar pictures persist at lager q.

Summarizing the above discussion, we observe the basic distinction between the one-particle system of ref. [27] and the center-of-mass sector of n-particle system considered here. In the former case, the energy spectrum arises as a solution of the eigenvalue problems of the type (5.24), with the Hamiltonians involving a sum of the oscillator and the inverse square potentials. In the latter case, the energy spectrum contains as well pure oscillator excitations due to the presence of the eigenvalue problems of the type (5.20).

$\boldsymbol{n=3.}$ In this case there are three constraints of the type (5.6) and for integer q they lead to the following dependence of the wave function on the spin variables

$$\Phi_0^{(2q)} = [(z_a z_b)\,(z_a z_c)\,(z_b z_c)]^q\,\phi + \sum_{a\neq b, a\neq c, b\neq c} (z_a z_b)^{q-1}\,[(z_a z_c)\,(z_b z_c)]^q\,z_a^i z_b^k\,\Phi_{ab(ik)} + \ldots$$
$$+\; z_1^{i_1}\ldots z_1^{i_{2q}} z_2^{j_1}\ldots z_2^{j_{2q}} z_3^{k_1}\ldots z_3^{k_{2q}}\,\Phi_{(i_1\ldots i_{2q} j_1\ldots j_{2q} k_1 \ldots k_{2q})}\,. \tag{5.28}$$

The component wave functions in the expansion (5.28) are functions of x_0 and ψ_0^i, and they display the dependence on ψ_0^i similar to that in (5.12).

In the three-spinor case, the relations analogous to (5.14), (5.15), (5.16), (5.17) are also valid, the difference is that now an additional spin variable z_3^i appears in the products. As a result, in the energy spectrum we find the same states as in the $n = 2$ case, though with a bigger multiplicity (due to extra indices ab in (5.28)), as well as the states of higher spins due to the presence of the additional spin variable z_3^i.

In the case of half-integer q, the $n = 3$ wave function has an expansion in which the component wave functions carry odd numbers of spinor indices, as opposed to the expansion (5.28). For example, for $q = 1/2$ wave function is

$$\Phi_0^{(1)} = (z_2 z_3)\,z_1^i\,\Phi_{1\,i} + (z_1 z_3)\,z_2^i\,\Phi_{2\,i} + (z_1 z_2)\,z_3^i\,\Phi_{3\,i}\,. \tag{5.29}$$

The superwave functions $\Phi_{a\,i}(x_0, \psi_0)$ display the energy spectrum of the one-particle system of ref. [27], this time with the three-fold degeneracy.

The pictures for higher n are similar to those for $n = 2$ and $n = 3$, such that the number of states and the values of admissible spins are increasing at increasing n.

6 Calogero-Moser system without center-of-mass sector

As was mentioned in Introduction, there are two methods of finding the quantum energy spectrum of multiparticle Calogero-type systems: either by considering matrix models which produce physically equivalent Calogero-type systems after gauge-fixing and the corresponding reduction of phase space, or through introducing Dunkl operators and passing to a generalized oscillator system. In this section we apply the first method to quantize the $\mathcal{N} = 4$ spin Calogero-Moser model under consideration in the matrix formulation, with the center-of-mass sector detached (section 6.1). The case of the reduced-phase space is briefly addressed in section 6.2.

6.1 Quantization in matrix formulation

We consider the quantization of the matrix subsystem in which $\widehat{su}(2|1)$ superalgebra is formed by the generators ($\hat{\mathbf{Q}}^k$, $\hat{\bar{\mathbf{Q}}}_k$, $\hat{\mathbf{H}}$, $\hat{\mathbf{I}}^i_k$, $\hat{\mathbf{F}}$) defined in (4.20), (4.25), (4.26) and (4.27). The basic operators of this system are spin operators $\mathbf{Z}^i_a$, $\bar{\mathbf{Z}}^a_i$ and traceless matrix operators $\hat{\mathbf{X}}_a{}^b$, $\hat{\mathbf{P}}_a{}^b$, $\hat{\mathbf{\Psi}}_a{}^b$, $\hat{\bar{\mathbf{\Psi}}}_a{}^b$ subject to the constraints (4.28) (as usual, applied to the physical states).

Introducing creation and annihilation even operators

$$\begin{aligned} &\mathbf{A}_a{}^b = \frac{1}{\sqrt{2m}}\left(\hat{\mathbf{P}}_a{}^b - im\hat{\mathbf{X}}_a{}^b\right), \qquad \mathbf{A}^{+b}_a = \frac{1}{\sqrt{2m}}\left(\hat{\mathbf{P}}_a{}^b + im\hat{\mathbf{X}}_a{}^b\right), \\ &\mathrm{Tr}\left(\mathbf{A}\right) = \mathrm{Tr}\left(\mathbf{A}^+\right) = 0\,, \end{aligned} \tag{6.1}$$

we rewrite the Hamiltonian (4.25) in the form

$$\hat{\mathbf{H}} = \frac{m}{2}\,\mathrm{Tr}\left\{\mathbf{A}^+, \mathbf{A}\right\} + \frac{m}{2}\,\mathrm{Tr}\left[\hat{\mathbf{\Psi}}^k, \hat{\bar{\mathbf{\Psi}}}_k\right]. \tag{6.2}$$

In this notation, the supercharges (4.20) are rewritten as

$$\hat{\mathbf{Q}}^k = \sqrt{2m}\,\mathrm{Tr}\left(\mathbf{A}\,\hat{\mathbf{\Psi}}^k\right), \qquad \hat{\bar{\mathbf{Q}}}_k = \sqrt{2m}\,\mathrm{Tr}\left(\mathbf{A}^+\hat{\bar{\mathbf{\Psi}}}_k\right), \tag{6.3}$$

where quantum (anti)commutators of the involved operators are

$$\left[\mathbf{A}_a{}^b, \mathbf{A}^{+d}_c\right] = \delta_a{}^d\delta_c{}^b - \frac{1}{n}\,\delta_a{}^b\delta_c{}^d, \qquad \left\{\hat{\mathbf{\Psi}}^i{}_a{}^b, \hat{\bar{\mathbf{\Psi}}}_{jc}{}^d\right\} = \left(\delta_a{}^d\delta_c{}^b - \frac{1}{n}\,\delta_a{}^b\delta_c{}^d\right)\delta^i_j\,. \tag{6.4}$$

The constraints (4.28) take the form

$$\mathbf{G}_a{}^b = \left[\mathbf{A}^+, \mathbf{A}\right]_a{}^b + \left\{\hat{\bar{\mathbf{\Psi}}}_k, \hat{\mathbf{\Psi}}^k\right\}_a{}^b + \mathbf{Z}^k_a\bar{\mathbf{Z}}^b_k - \left(2q + n - \frac{1}{n}\right)\delta_a{}^b \simeq 0\,. \tag{6.5}$$

They involve the spin operators, with the non-vanishing commutator

$$\left[\mathbf{Z}^i_a, \bar{\mathbf{Z}}^b_k\right] = -\,\delta^i_k\delta^b_a\,. \tag{6.6}$$

The operators $\mathbf{Z}^i_a$, $\mathbf{A}^{+b}_a$, $\hat{\mathbf{\Psi}}^i{}_a{}^b$ form a full set of creation operators. Therefore, the general structure of the physical states is as follows

$$\mathbf{Z}^{i_1}_{a_1}\ldots\mathbf{Z}^{i_{k_1}}_{a_{k_1}}\mathbf{A}^{+\,c_1}_{b_1}\ldots\mathbf{A}^{+\;\,c_{k_2}}_{b_{k_2}}\hat{\mathbf{\Psi}}^{j_1}{}_{d_1}{}^{e_1}\ldots\hat{\mathbf{\Psi}}^{j_{k_3}}{}_{d_{k_3}}{}^{e_{k_3}}\,|0\rangle\,. \tag{6.7}$$

In the holomorphic realization,

$$\begin{aligned} &\mathbf{Z}^i_a = z^i_a\,, && \bar{\mathbf{Z}}^a_i = \frac{\partial}{\partial z^i_a}\,, \\ &\mathbf{A}^{+c}_b = \hat{a}^{+c}_b = a^{+c}_b - \frac{1}{n}\,\delta^c_b\,a^{+d}_d\,, && \mathbf{A}_b{}^c = \frac{\partial}{\partial\hat{a}^{+b}_c} = \frac{\partial}{\partial a^{+b}_c} - \frac{1}{n}\,\delta^c_b\,\frac{\partial}{\partial a^{+d}_d}\,, \\ &\hat{\mathbf{\Psi}}^i{}_b{}^c = \hat{\Psi}^i{}_b{}^c = \Psi^i{}_b{}^c - \frac{1}{n}\,\delta^c_b\,\Psi^i{}_d{}^d\,, && \hat{\bar{\mathbf{\Psi}}}_{ib}{}^c = \frac{\partial}{\partial\hat{\Psi}^i{}_c{}^b} = \frac{\partial}{\partial\Psi^i{}_c{}^b} - \frac{1}{n}\,\delta_b{}^c\frac{\partial}{\partial\Psi^i{}_d{}^d}\,, \end{aligned} \tag{6.8}$$

we deal with the traceless objects $\hat{a}^{+}$ and $\hat{\Psi}^{i}$. Then the physical states (6.7) are rewritten as

$$z^{i_1}_{a_1} \dots z^{i_{k_1}}_{a_{k_1}} \hat{a}^{+}_{b_1}{}^{c_1} \dots \hat{a}^{+}_{b_{k_2}}{}^{c_{k_2}} \hat{\Psi}^{j_1}{}_{d_1}{}^{e_1} \dots \hat{\Psi}^{j_{k_3}}{}_{d_{k_3}}{}^{e_{k_3}} \, |0\rangle \,. \tag{6.9}$$

The constraint (6.5) indicates that all physical states are singlets of SU(n) (see [33, 37–39]). This is also a direct consequence of vanishing of all Casimir operators on the states (6.7):

$$\mathbf{G}_a{}^b \mathbf{G}_b{}^c \dots \mathbf{G}_e{}^a \simeq 0 \,. \tag{6.10}$$

On the other hand, the states (6.7) belong to irreducible representations of the group SU(2) with the generators $\mathbf{S}_{(ij)}$ defined in (4.6) and the group SU(2) $\times$ U(1) with the generators (4.26), (4.27) acting only on fermionic fields.

Eigenvalues of the Hamiltonian (6.2) on the states (6.7) are specified by the numbers $N_{\mathbf{A}}$ and $N_{\mathbf{\Psi}}$ of the operators $\mathbf{A}^{+c}_{b}$ and $\hat{\mathbf{\Psi}}^{i}{}_{b}{}^{c}$:

$$E = m \left(N_{\mathbf{A}} + N_{\mathbf{\Psi}} - \frac{n^2 - 1}{2} \right) . \tag{6.11}$$

Here we will basically limit our consideration to the pure bosonic case, without odd operators $\hat{\mathbf{\Psi}}^{i}{}_{b}{}^{c}$, $\hat{\bar{\mathbf{\Psi}}}_{j b}{}^{c}$. The set of fermionic states can be generated by action of the supercharges on the subset of bosonic states. Examples of fermionic states will be constructed below for few simple particular cases.

The trace part of the constraints (6.5) leads to homogeneity of the physical states of degree $2qn$ with respect to the spin operators $\mathbf{Z}$. In addition, the property that physical states are the SU(n) singlets implies the following structure for them [33, 37–39]

$$\begin{aligned} \Phi^{(2q,s,\ell)} \simeq & \left[\mathrm{Tr} \left(\mathbf{A}^{+} \right)^2 \right]^{p_2} \left[\mathrm{Tr} \left(\mathbf{A}^{+} \right)^3 \right]^{p_3} \dots \left[\mathrm{Tr} \left(\mathbf{A}^{+} \right)^n \right]^{p_n} \\ & \times \prod_{r=0}^{2q-1} \left\{ \varepsilon^{a_1 a_2 \dots a_n} \left[\left(\mathbf{A}^{+} \right)^{l_{nr+1}} \mathbf{Z}^{i_{nr+1}} \right]_{a_1} \dots \left[\left(\mathbf{A}^{+} \right)^{l_{nr+n}} \mathbf{Z}^{i_{nr+n}} \right]_{a_n} \right\} |0\rangle \,, \end{aligned} \tag{6.12}$$

where $p_2, p_3 \dots, p_n$ are arbitrary integers and $0 \leq l_{nr+1} \leq l_{nr+2} \dots l_{nr+n} < n$. The wave function $\Phi^{(2q,s,\ell)}$ in (6.12) is given up to the coefficients $C_{(i_1 i_2 \dots i_{2s})}$, where the number $s \leq nq$ can be interpreted as SU(2) spin, with $2s$ being the number of symmetrized SU(2) indices of spin variables. It is worth pointing out that for $l_r < n/2$ the coincident degrees, $l_r = l_p$, are permitted. Besides, such a degree cannot appear more than once in the products of monomials

$$\left\{ \varepsilon^{a_1 a_2 \dots a_n} \left[\left(\mathbf{A}^{+} \right)^{l_{nr+1}} \mathbf{Z}^{i_{nr+1}} \right]_{a_1} \dots \left[\left(\mathbf{A}^{+} \right)^{l_{nr+n}} \mathbf{Z}^{i_{nr+n}} \right]_{a_n} \right\} , \qquad r = 0, 1 \dots 2q-1 \,. \tag{6.13}$$

For the highest spin $s = nq$, the degrees are given by

$$l_{nr+1} = 0 \,, \quad l_{nr+2} = 1 \,, \quad \dots \quad l_{nr+n} = n-1 \,, \qquad r = 0, 1 \dots 2q-1 \,. \tag{6.14}$$

Degeneracy analysis of bosonic wave functions for the quantum spin Calogero model was considered in [36], where it was noticed that some of possible spin states may vanish. Here we consider the matrix construction for the system with the center-of-mass sector

JHEP04(2018)043

detached,[6] where some of these spin states may also vanish. Listing all admissible degree numbers $l_{nr+1}, l_{nr+2}\ldots, l_{nr+n}$ is a rather complicated task.

On the states (6.12) the energy (6.11) take the values

$$E = m\left(\sum_{k=2}^{n} kp_k + \sum_{k=1}^{2qn} l_k - \frac{n^2-1}{2}\right). \tag{6.15}$$

The energy is maximal for the choice (6.14):

$$E_{(s=nq)} = m\left(\sum_{k=2}^{n} kp_k + (n-1)\,nq - \frac{n^2-1}{2}\right). \tag{6.16}$$

The minimal energy corresponds to the choice $p_2 = p_3\ldots p_n = 0$ and $l_{nr+1} = l_{nr+2} = 0$, $l_{nr+3} = l_{nr+4} = 1$, $l_{nr+5} = l_{nr+6} = 2$, etc:

$$\begin{aligned} E_{\min} &= m\left[n\left(\frac{n}{2}-1\right)q - \frac{n^2-1}{2}\right], \quad \text{for even } n\,, \\ E_{\min} &= m\left[\frac{(n-1)^2 q}{2} - \frac{n^2-1}{2}\right], \quad \text{for odd } n>1\,. \end{aligned} \tag{6.17}$$

The fermionic states are constructed with the help of the operators $\hat{\mathbf{\Psi}}^{i}{}_{b}{}^{c}$, on the pattern of (6.12). Such physical states have additional contributions $mN_{\mathbf{\Psi}}$ to the energy value (6.15). As was already mentioned, full wave functions can be generated from the bosonic states (6.7) by acting on them by the supercharges (6.3). Casimir operators (4.2), (4.2) take the following values on the states (6.7) and those produced from (6.7) by SU(2|1) supersymmetry transformation:

$$\begin{aligned} m^2\mathbf{C}_2 &= \left(E + \frac{(n^2-1)\,m}{2}\right)\left(E + \frac{(n^2-3)\,m}{2}\right), \\ m^3\mathbf{C}_3 &= \left(E + \frac{(n^2-2)\,m}{2}\right)\mathbf{C}_2\,. \end{aligned} \tag{6.18}$$

Casimirs can take zero eigenvalues only for $n=2$ at arbitrary q and for $n=3$ at $q=1/2$ (we consider $q>0$ in this paper). The corresponding sets of the quantum states belong to atypical representations of SU(2|1). For illustration, we will consider here these two cases in some detail.

$n = 2$. In this case the Hamiltonian is written as

$$\hat{\mathbf{H}} = m\,\mathrm{Tr}\left(\mathbf{A}^{+}\mathbf{A}\right) + m\,\mathrm{Tr}\left(\hat{\mathbf{\Psi}}^{k}\hat{\bar{\mathbf{\Psi}}}_{k}\right) - \frac{3m}{2}\,. \tag{6.19}$$

Bosonic wave functions, from which the full set of the wave functions can be produced by the supercharges (6.4), are given by

$$\Phi^{(2q,s,\ell)} = \left[\mathrm{Tr}\left(\mathbf{A}^{+}\right)^2\right]^{\ell}\left(\varepsilon_{ij}\varepsilon^{ab}\mathbf{Z}^{i}_{a}\mathbf{Z}^{j}_{b}\right)^{2q-s}\mathbf{Z}^{i_1}_{a_1}\mathbf{Z}^{i_2}_{a_2}\ldots\mathbf{Z}^{i_{2s}}_{a_{2s}}\prod_{k=1}^{s}\varepsilon^{a_k b_k}\mathbf{A}^{+}{}_{b_k}{}^{a_{s+k}}C_{(i_1 i_2\ldots i_{2s})}\,|0\rangle\,, \tag{6.20}$$

[6]The construction of [36] implies reduction to the angular spin Calogero model by separating the radial coordinate, which corresponds in the quantum case to setting $p_2 = 0$ in (6.12).

where $C_{(i_1 i_2 \dots i_{2s})}$ are coefficients with $2s$ symmetric indices. The wave functions $\Phi^{(2q,s,\ell)}$ are eigenfunctions of the Hamiltonian (6.19),

$$\hat{\mathbf{H}}\Phi^{(2q,s,\ell)} = E_{(s,\ell)}\Phi^{(2q,s,\ell)}, \qquad s = 0, 1 \dots 2q\,, \tag{6.21}$$

with the energy eigenvalues

$$E_{(s,\ell)} = m\left(2\ell + s - \frac{3}{2}\right). \tag{6.22}$$

The wave functions on which Casimirs take zero values, i.e., those belonging to atypical representations of SU(2|1), correspond to the choice $s = 0$, $\ell = 0$:

$$\Phi^{(2q,0,0)} = \left(\varepsilon_{ij}\varepsilon^{ab}\mathbf{Z}^i_a\mathbf{Z}^j_b\right)^{2q} |0\rangle\,, \tag{6.23}$$

This ground state wave function is SU(2|1) singlet, since it is annihilated by both supercharges. There is still another atypical non-singlet bosonic state corresponding to $s = 1$, $\ell = 0$:

$$\Phi^{(2q,1,0)} = \left(\varepsilon_{k_1k_2}\varepsilon^{c_1c_2}\mathbf{Z}^{k_1}_{c_1}\mathbf{Z}^{k_2}_{c_2}\right)^{2q-1} \varepsilon^{ab}\mathbf{Z}^{(i_1}_a\mathbf{Z}^{i_2)}_d\mathbf{A}^{+d}_b\,|0\rangle\,, \tag{6.24}$$

which gives rise to the fundamental SU(2|1) representation. The other two components of this representation are generated from (6.24) by SU(2|1) supercharges:

$$\hat{\mathbf{Q}}^j\Phi^{(2q,1,0)} = \sqrt{2m}\left(\varepsilon_{k_1k_2}\varepsilon^{c_1c_2}\mathbf{Z}^{k_1}_{c_1}\mathbf{Z}^{k_2}_{c_2}\right)^{2q-1} \varepsilon^{ab}\mathbf{Z}^{(i_1}_a\mathbf{Z}^{i_2)}_d\hat{\mathbf{\Psi}}^j{}_b{}^d\,|0\rangle\,,\ \text{etc.} \tag{6.25}$$

$n = 3$, $q = 1/2$. The $n = 3$ Hamiltonian reads

$$\hat{\mathbf{H}} = \frac{m}{2}\,\mathrm{Tr}\left\{\mathbf{A}^+, \mathbf{A}\right\} + \frac{m}{2}\,\mathrm{Tr}\left[\hat{\mathbf{\Psi}}^k, \hat{\bar{\mathbf{\Psi}}}_k\right] = m\,\mathrm{Tr}\left(\mathbf{A}^+\mathbf{A}\right) + m\,\mathrm{Tr}\left(\hat{\mathbf{\Psi}}^k\hat{\bar{\mathbf{\Psi}}}_k\right) - 4m\,. \tag{6.26}$$

For $q = 1/2$, the bosonic wave functions $\Phi^{(2q,s,\ell)}$ as eigenfunctions of this Hamiltonian are constructed as

$$\begin{aligned}
\Phi^{(1,1/2,\ell)} &= \left[\mathrm{Tr}\left(\mathbf{A}^+\right)^2\right]^{p_2}\left[\mathrm{Tr}\left(\mathbf{A}^+\right)^3\right]^{p_3}\varepsilon^{a_1a_2a_3}\varepsilon_{ij}\mathbf{Z}^i_{a_1}\mathbf{Z}^j_{a_2}\left(\mathbf{A}^+\mathbf{Z}^k\right)_{a_3}C_k\,|0\rangle\,, \\
\Phi'^{(1,1/2,\ell)} &= \left[\mathrm{Tr}\left(\mathbf{A}^+\right)^2\right]^{p_2}\left[\mathrm{Tr}\left(\mathbf{A}^+\right)^3\right]^{p_3}\varepsilon^{a_1a_2a_3}\varepsilon_{ij}\left(\mathbf{A}^+\mathbf{Z}^i\right)_{a_1}\left(\mathbf{A}^+\mathbf{Z}^j\right)_{a_2}\mathbf{Z}^k_{a_3}C'_k\,|0\rangle\,, \\
\Phi^{(1,3/2,\ell)} &= \left[\mathrm{Tr}\left(\mathbf{A}^+\right)^2\right]^{p_2}\left[\mathrm{Tr}\left(\mathbf{A}^+\right)^3\right]^{p_3}\varepsilon^{a_1a_2a_3}\mathbf{Z}^i_{a_1}\left(\mathbf{A}^+\mathbf{Z}^j\right)_{a_2}\left(\left[\mathbf{A}^+\right]^2\mathbf{Z}^k\right)_{a_3}C_{(ijk)}\,|0\rangle\,,
\end{aligned} \tag{6.27}$$

with the coefficients C_k, C'_k, $C_{(ijk)}$. They have the following energy values

$$E_{(1/2,\ell)} = m\,(\ell - 3)\,, \qquad E'_{(1/2,\ell)} = m\,(\ell - 2)\,, \qquad E_{(3/2,\ell)} = m\,(\ell - 1)\,, \tag{6.28}$$

where $\ell = 2p_2 + 3p_3$. The minimal energy is achieved on the state

$$\Phi^{(1,1/2,0)} = \varepsilon^{a_1a_2a_3}\varepsilon_{ij}\mathbf{Z}^{i_1}_{a_1}\mathbf{Z}^{i_2}_{a_2}\left(\mathbf{A}^+\mathbf{Z}^k\right)_{a_3}|0\rangle \tag{6.29}$$

and it is equal to

$$E_{(1/2,0)} = -3m\,. \tag{6.30}$$

Casimir operators take zero values on this state. The action of the SU(2|1) supercharge,

$$\hat{\mathbf{Q}}^j \Phi^{(1,1/2,0)} = \varepsilon^{a_1a_2a_3}\varepsilon_{i_1i_2}\mathbf{Z}^{i_1}_{a_1}\mathbf{Z}^{i_2}_{a_2}\left(\hat{\mathbf{\Psi}}^j\mathbf{Z}^k\right)_{a_3}|0\rangle\,, \tag{6.31}$$

produces an additional fermionic state which, together with (6.29) and one more bosonic state generated by further action of supercharges on (6.31), constitute an atypical fundamental SU(2|1) supermultiplet.

6.2 Quantization of the reduced spinning Calogero-Moser system

We briefly discuss quantization of the reduced spinning Calogero-Moser multi-particle system without center-of-mass defined in section 4.2.1. More explicitly, we consider the two-particle case $n = 2$.

Introducing the holomorphic realization

$$\begin{aligned}
&\mathbf{Z}^i_a := z^i_a\,, \qquad \bar{\mathbf{Z}}^a_i := \frac{\partial}{\partial z^i_a}\,, \qquad \mathbf{p}_a := -i\partial_a = -i\,\frac{\partial}{\partial x_a}\,,\\
&\boldsymbol{\psi}^i{}_a = \psi^i{}_a, \qquad \bar{\boldsymbol{\psi}}_{i\,a} = \frac{\partial}{\partial \psi^i{}_a}, \qquad \boldsymbol{\psi}^i{}_b{}^c = \psi^i{}_b{}^c, \qquad \bar{\boldsymbol{\psi}}_{ib}{}^c = \frac{\partial}{\partial \psi^i{}_c{}^b}\,,
\end{aligned} \tag{6.32}$$

the differential realization of Hamiltonian (4.48) on physical states is given by

$$\sum_{a<b}\left[\frac{1}{2n}\left[-\left(\partial_a-\partial_b\right)^2+m^2\left(x_a-x_b\right)^2\right]+\frac{g_{ab}}{\left(x_a-x_b\right)^2}\right]+\text{const}\,. \tag{6.33}$$

Here g_{ab} are eigenvalues of the quantum operators $\frac{1}{2}\left\{\mathbf{T}_a{}^b,\mathbf{T}_b{}^a\right\}$ $(a<b)$, and they correspond to spin couplings of two interacting particles x_a and x_b. As was shown in section 4.2, one can represent g_{ab} on bosonic states via (5.7) as

$$g_{ab} = -\,\mathbf{S}^{(ij)}_a\mathbf{S}_{b\,(ij)} + 2q\,(q+1)\,, \qquad a<b\,. \tag{6.34}$$

This model is restricted to positive integer values of $2q$ and is referred to as "matrix model" in [33]. Arbitrary values of $2q$ can be achieved by applying the exchange operator formalism involving Dunkl operators (see, e.g., [35]). As was already mentioned, our SU(2|1) supersymmetric system yields just the U(2) spin matrix model as its bosonic core.

In the simpler case $g_{ab} = \vartheta\,(\vartheta \mp 1)$, quantization was given in [7, 8, 34] via Dunkl operators defined as

$$\mathcal{D}_a = \partial_a + \sum_{a(\neq b)}\frac{\vartheta}{x_a-x_b}\left(1-K_{ab}\right), \tag{6.35}$$

where K_{ab} is a permutation operator, $K_{ab}\,x_b = x_a\,K_{ab}$. Below we consider the simplest case $n = 2$, where $g_{12} = s\,(s+1)$ and the operator K_{12} becomes Klein-type operator acting on the relative coordinate $x_1 - x_2$ as

$$K_{12}\left(x_1-x_2\right) = \left(x_2-x_1\right)K_{12} = -\left(x_1-x_2\right)K_{12}\,, \qquad \left(K_{12}\right)^2 = 1\,. \tag{6.36}$$

Let us consider in details the two-particle system ($n = 2$). It is described by the algebra of quantum operators

$$\begin{gathered} [\mathbf{x}, \mathbf{p}] = i\,, \qquad\qquad \left[\mathbf{Z}^k_1, \bar{\mathbf{Z}}^1_j\right] = \left[\mathbf{Z}^k_2, \bar{\mathbf{Z}}^2_j\right] = -\,\delta^k_j\,, \\ \left\{\boldsymbol{\psi}^k, \bar{\boldsymbol{\psi}}_j\right\} = \delta^k_j\,, \qquad \left\{\boldsymbol{\psi}^k{}_1{}^2, \bar{\boldsymbol{\psi}}_{j\,2}{}^1\right\} = \left\{\boldsymbol{\psi}^k{}_2{}^1, \bar{\boldsymbol{\psi}}_{j\,1}{}^2\right\} = \delta^k_j\,, \end{gathered} \tag{6.37}$$

where

$$\mathbf{x} = \frac{1}{\sqrt{2}}\left(\mathbf{x}_1 - \mathbf{x}_2\right), \qquad \mathbf{p} = \frac{1}{\sqrt{2}}\left(\mathbf{p}_1 - \mathbf{p}_2\right), \tag{6.38}$$

$$\boldsymbol{\psi}^i = \frac{1}{\sqrt{2}}\left(\boldsymbol{\psi}^i{}_1 - \boldsymbol{\psi}^i{}_2\right), \quad \bar{\boldsymbol{\psi}}_i = \frac{1}{\sqrt{2}}\left(\bar{\boldsymbol{\psi}}_{i\,1} - \bar{\boldsymbol{\psi}}_{i\,2}\right). \tag{6.39}$$

Below we use the following realization for them

$$\begin{array}{llll} \mathbf{x} = x\,, & \mathbf{p} = -i\dfrac{\partial}{\partial x}\,, & \boldsymbol{\psi}^i = \psi^i\,, & \bar{\boldsymbol{\psi}}_i = \dfrac{\partial}{\partial \psi^i}\,, \\ \mathbf{Z}^i_1 = z^i_1\,, & \mathbf{Z}^i_2 = z^i_2\,, & \bar{\mathbf{Z}}^1_j = \dfrac{\partial}{\partial z^j_1}\,, & \bar{\mathbf{Z}}^2_j = \dfrac{\partial}{\partial z^j_2}\,, \\ \boldsymbol{\psi}^i{}_1{}^2 = \psi^i{}_1{}^2\,, & \boldsymbol{\psi}^i{}_2{}^1 = \psi^i{}_2{}^1\,, & \bar{\boldsymbol{\psi}}_{i\,1}{}^2 = \dfrac{\partial}{\partial \psi^i{}_2{}^1}\,, & \bar{\boldsymbol{\psi}}_{i\,2}{}^1 = \dfrac{\partial}{\partial \psi^i{}_1{}^2}\,. \end{array} \tag{6.40}$$

Here x, z^i_a and ψ^i, $\psi^i{}_a{}^b$, $a = 1, 2$ are complex commuting and fermionic anticommuting variables, respectively.

The Hamiltonian (4.48) without center of mass takes the form

$$\mathbb{H} = \frac{1}{2}\left(-\frac{\partial^2}{\partial x^2} + m^2 x^2\right) + m\left(\psi^k \frac{\partial}{\partial \psi^k} + \psi^k{}_1{}^2 \frac{\partial}{\partial \psi^k{}_1{}^2} + \psi^k{}_2{}^1 \frac{\partial}{\partial \psi^k{}_2{}^1} - 3\right) + \frac{\left\{\mathbf{T}_1{}^2, \mathbf{T}_2{}^1\right\}}{4x^2}\,. \tag{6.41}$$

where

$$\begin{aligned} \mathbf{T}_1{}^2 &= z^k_1 \frac{\partial}{\partial z^k_2} + \sqrt{2}\,\psi^k \frac{\partial}{\partial \psi^k{}_2{}^1} - \sqrt{2}\,\psi^k{}_1{}^2 \frac{\partial}{\partial \psi^k}\,, \\ \mathbf{T}_2{}^1 &= z^k_2 \frac{\partial}{\partial z^k_1} - \sqrt{2}\,\psi^k \frac{\partial}{\partial \psi^k{}_1{}^2} + \sqrt{2}\,\psi^k{}_2{}^1 \frac{\partial}{\partial \psi^k}\,. \end{aligned} \tag{6.42}$$

The operators (6.42) act in the following way on the variables entering the wave function

$$\begin{array}{llll} \mathbf{T}_1{}^2: & z^k_2 \to z^k_1\,, & \psi^k \to -\sqrt{2}\,\psi^k{}_1{}^2\,, & \psi^k{}_2{}^1 \to \sqrt{2}\,\psi^k\,; \\ \mathbf{T}_2{}^1: & z^k_1 \to z^k_2\,, & \psi^k \to \sqrt{2}\,\psi^k{}_2{}^1\,, & \psi^k{}_1{}^2 \to -\sqrt{2}\,\psi^k \end{array} \tag{6.43}$$

and give zero, while acting on other variables. Thus, the operators $\mathbf{T}_1{}^2\mathbf{T}_2{}^1$ and $\mathbf{T}_2{}^1\mathbf{T}_1{}^2$ transform all components in the expansion of the wave function into themselves, with some coefficients including the vanishing ones. Therefore, on all components the Hamiltonian (6.41) has the standard form with the oscillator and conformal potentials. As the result, we can find its energy spectrum.

This system is similar to the one we have considered in section 5, but it has a wider set of fermionic fields. The spin s is associated with the diagonal external SU(2) group (4.6).

In contrast to (4.23), the SU(2) subgroup (4.49) of SU(2|1) acts only on fermionic fields, which gives a different degeneracy picture for the supergroup SU(2|1).

Let us consider pure bosonic wave functions. They are given by

$$\Omega^{(2q)} = \sum_{\ell}^{\infty}\sum_{s=0}^{2q}\Omega^{(2q,s,\ell)},$$
$$\Omega^{(2q,s,\ell)}\left(x, z_1^i, z_2^j\right) = \left(z_1^k z_{k2}\right)^{2q-s} z_1^{i_1} z_1^{i_2}\ldots z_1^{i_s} z_2^{i_{s+1}}\ldots z_2^{i_{2s}} A^{(s,\ell)}_{(i_1 i_2\ldots i_{2s})}(x)\,, \tag{6.44}$$

and are subject to the constraints

$$\mathbf{T}_1\Omega^{(2q)} = \mathbf{T}_2\Omega^{(2q)} = 2q\,\Omega^{(q)}, \qquad 2q = 1,2,3\ldots\,,$$
$$\mathbf{T}_1 = z_1^k\frac{\partial}{\partial z_1^k} + {\psi^k}_1{}^2\frac{\partial}{{\psi^k}_1{}^2} - {\psi^k}_2{}^1\frac{\partial}{{\psi^k}_2{}^1}\,,$$
$$\mathbf{T}_2 = z_2^k\frac{\partial}{\partial z_2^k} - {\psi^k}_1{}^2\frac{\partial}{{\psi^k}_1{}^2} + {\psi^k}_2{}^1\frac{\partial}{{\psi^k}_2{}^1}\,. \tag{6.45}$$

The eigenvalue problem for the Hamiltonian (6.41) amounts to the equation

$$\frac{1}{2}\left[-\frac{\partial^2}{\partial x^2} + m^2x^2 + \frac{s\,(s+1)}{x^2} - 6m\right] A^{(s,\ell)}(x)_{(i_1 i_2\ldots i_{2s})} = E_{(s,\ell)}\,A^{(s,\ell)}_{(i_1 i_2\ldots i_{2s})}, \tag{6.46}$$

which is solved as[7]

$$A^{(s,\ell)}(x)_{(i_1 i_2\ldots i_{2s})} = C_{(i_1 i_2\ldots i_{2s})}\,x^{s+1}\,L_\ell^{(s+1/2)}\left(mx^2\right)\exp(-mx^2/2)\,, \qquad \ell = 0,1,2,\ldots\,,$$
$$E_{(s,\ell)} = m\left(2\ell + s - \frac{3}{2}\right). \tag{6.47}$$

The energy spectrum is consistent with the energy spectrum (6.22) calculated in the matrix formulation.

An alternative construction of wave functions can be given via creation and annihilation operators [7, 8, 34]. To solve the equation (6.46), we take the ansatz:

$$A^{(s,\ell)}_{(i_1 i_2\ldots i_{2s})} = C_{(i_1 i_2\ldots i_{2s})}\,x^{s+1}\,\Phi_+^{(s,\ell)}, \tag{6.48}$$

where $\Phi_+^{(s,\ell)}$ is an even function of x. Introducing the Klein operator $K := K_{12}$ (6.36) satisfying

$$K^2 = 1\,, \qquad Kx = -xK, \qquad K\partial_x = -\partial_x K, \qquad K\Phi_+^{(s,\ell)} = \Phi_+^{(s,\ell)}, \tag{6.49}$$

we define the creation and annihilation operators $\mathbf{a}^\pm$ through the Dunkl operator $\mathcal{D}$:

$$\mathbf{a}^\pm = \mp\mathcal{D} + mx\,, \qquad \mathcal{D} = \left[\partial_x + \frac{s+1}{x}(1-K)\right], \qquad K\mathbf{a}^\pm = -\,\mathbf{a}^\pm K\,. \tag{6.50}$$

[7] As was discussed in [27], the equation (6.46) has an additional solution which was thrown away for $s > 0$ due to the presence of singularities at $x = 0$. Here this additional solution must be thrown away even for the case $s = 0$, applying the same reasoning to the fermionic expansion of the full wave function.

Then eq. (6.46) is rewritten as

$$\frac{1}{2}\left[-\frac{\partial^2}{\partial x^2}+m^2x^2+\frac{s\,(s+1)}{x^2}-6m\right]A^{(s,\ell)}=x^{s+1}\mathbb{H}_+\Phi_+^{(s,\ell)},\tag{6.51}$$

where

$$\mathbb{H}_+=\frac{1}{2}\,\mathbf{a}^+\mathbf{a}^-+\frac{m}{2}\,(2sK+2K-5)\,,\qquad \left[\mathbb{H}_+,\mathbf{a}^\pm\right]=\pm m\mathbf{a}^\pm.\tag{6.52}$$

Taking into account that $K\Phi_+^{(s,\ell)}=\Phi_+^{(s,\ell)}$, the function $\Phi_+^{(s,\ell)}$ is expressed as

$$\Phi_+^{(s,\ell)}=\left(\mathbf{a}^+\right)^{2\ell}\Phi_+^{(s,0)},\qquad \mathbf{a}^-\,\Phi_+^{(s,0)}=0\,,\tag{6.53}$$

and the associate energy spectrum is

$$\mathbb{H}_+\Phi_+^{(s,\ell)}=E_{(s,\ell)}\Phi_+^{(s,\ell)},\qquad E_{(s,\ell)}=m\left(2\ell+s-\frac{3}{2}\right),\qquad \ell=0,1,2,\ldots.\tag{6.54}$$

One can also choose an alternative ansatz

$$A^{(s,\ell)}=x^s\,\Phi_-^{(s,\ell)},\tag{6.55}$$

satisfying

$$K\Phi_-^{(s,\ell)}=-\,\Phi_-^{(s,\ell)}.\tag{6.56}$$

The construction of wave functions via the creation and annihilation operators will give the same solution for the energy spectrum as (6.54). We skip details of this construction which is similar to the previous one.

7 Quantization of the full system (center-of-mass plus relative-coordinate sectors)

The energy spectrum of the unified system, which is a sum of the center-of-mass sector of section 5 and the relative coordinate system of section 6, can be found as a tensorial product of the spectra of these two subsystems.

In the ungauged matrix formulation, the bosonic wave functions are a generalization of (6.12) in the holomorphic realization (6.8):

$$\begin{aligned}\Phi^{(2q,s,l)}\sim{}& f_{(j_1\ldots j_{2s})}(x_0)\left[\mathrm{Tr}\,(\hat{a}^+)^2\right]^{p_2}\left[\mathrm{Tr}\,(\hat{a}^+)^3\right]^{p_3}\ldots\left[\mathrm{Tr}\,(\hat{a}^+)^n\right]^{p_n}\\ &\times\prod_{r=0}^{2q-1}\left\{\varepsilon^{a_1a_2\ldots a_n}\left[(\hat{a}^+)^{l_{nr+1}}\,z^{i_{nr+1}}\right]_{a_1}\ldots\left[(\hat{a}^+)^{l_{nr+n}}\,z^{i_{nr+n}}\right]_{a_n}\right\}|0\rangle\,.\end{aligned}\tag{7.1}$$

Their general structure is quite specified by the three requirements:

- The wave functions should be $\mathrm{U}(n)$ invariant as a consequence of the constraint (4.12) (or its equivalent form (4.28)). This means that all $\mathrm{U}(n)$ indices a should be contracted with the appropriate invariant tensors;

- They should be of degree $2qn$ with respect to the whole set of spin variables in virtue of the constraint (4.14);

- All free SU(2) indices of the spin variables should be symmetrized and contracted with the indices of $f_{(i_1\ldots i_{2s})}(x_0)$. The energy spectrum of admissible spins of these functions extends from $s=0$ to nq (for $2nq$ even) and from $s=1/2$ to nq (for $2nq$ odd).

All the fermionic wave functions can be obtained by action of the total supercharges on (7.1). The basic distinctions of the total system from the multi-particle system of section 6 concern the realizations of the SU(2) symmetry appearing in the anti-commutators of the supercharges as an internal subgroup of SU(2|1). In the system with the center-of-mass sector detached considered in section 6, this SU(2) symmetry is given by (4.49), acts only on the fermionic operators and gives rise just to degeneracy of the energy spectrum. In the total system, the internal SU(2) symmetry acts on the indices $i, j, \ldots$ of all components of the wave functions (7.1) and their fermionic completion.

Taking into account the analysis of the previous section, we see that the problem of description of all states in the unified case (the option **III)** in section 4.3) for an arbitrary n is rather complicated. At the same time, we can directly determine, for all possible cases, the full energy spectrum simply by applying the methods of the previous sections. Let us briefly describe the energy spectrum for the choice of $n=2$.

In this simplest case the matrix system is described by the Hamiltonian

$$\begin{aligned}\mathbf{H} = {}& \frac{1}{2}\left(-\frac{\partial^2}{\partial {x_0}^2} + m^2 {x_0}^2 - 1\right) + m\,\psi_0^i\,\frac{\partial}{\partial\psi_0^i} + \frac{1}{{x_0}^2}\left(-\frac{1}{4}\,\mathbf{S}^{(ik)}\mathbf{S}_{ik} + \mathbf{S}^{(ik)}\psi_{0\,i}\,\frac{\partial}{\partial\psi_0^k}\right) \\ & + m\,\mathrm{Tr}\left(\mathbf{A}^+\mathbf{A}\right) + m\,\mathrm{Tr}\left(\mathbf{\Psi}^k\bar{\mathbf{\Psi}}_k\right) - 2m\,. \end{aligned} \tag{7.2}$$

The traceless part of the general constraints (6.5),

$$\mathbf{G}_a{}^b = \frac{i}{n}\left[\mathbf{X}_0, \mathbf{P}_0\right]\delta_a{}^b + \left[\mathbf{A}^+, \mathbf{A}\right]_a{}^b + \left\{\bar{\mathbf{\Psi}}_k, \mathbf{\Psi}^k\right\}_a{}^b + \mathbf{Z}_a^k\bar{\mathbf{Z}}_k^b - (2q+n)\,\delta_a{}^b \simeq 0\,, \tag{7.3}$$

requires wave functions to be SU(n) scalars, while its trace part fixes the degree of homogeneity with respect to spin variables:

$$\sum_a \mathbf{G}_a{}^a = 0 \qquad \Rightarrow \qquad \sum_a \mathbf{Z}_a^k\bar{\mathbf{Z}}_k^a - 4q = 0\,. \tag{7.4}$$

The bosonic wave functions are constructed as

$$\Phi^{(2q,0,\ell,\ell_0)} = \mathrm{H}_{\ell_0}\left(x_0\right) e^{-\frac{m{x_0}^2}{2}} \left[\mathrm{Tr}\left(\mathbf{A}^+\right)^2\right]^\ell \left(\varepsilon_{ij}\varepsilon^{ab}\mathbf{Z}_a^i\mathbf{Z}_b^j\right)^{2q} |0\rangle\,, \tag{7.5}$$

$$\begin{aligned}\Phi^{(2q,s,\ell,\ell_0)} = {}& A^{(\ell_0)}_{(i_1 i_2\ldots i_{2s})}\left(x_0\right)\left[\mathrm{Tr}\left(\mathbf{A}^+\right)^2\right]^\ell \left(\varepsilon_{ij}\varepsilon^{ab}\mathbf{Z}_a^i\mathbf{Z}_b^j\right)^{2q-s} \mathbf{Z}_{a_1}^{(i_1}\mathbf{Z}_{a_2}^{i_2}\ldots\mathbf{Z}_{a_{2s}}^{i_{2s})} \\ & \times \prod_{k=1}^{s}\varepsilon^{a_k b_k}\mathbf{A}^{+}_{b_k}{}^{a_{s+k}}\,|0\rangle\,, \qquad s = 1, 2\ldots 2q\,, \end{aligned} \tag{7.6}$$

and possess the energies

$$
\begin{aligned}
E_{(0,\ell,\ell_0)} &= m\,(2\ell + \ell_0 - 2)\,, \\
E_{(s,\ell,\ell_0)} &= 2m\left(\ell + \ell_0 + s - \frac{1}{2}\right), \qquad s = 1, 2 \ldots 2q\,.
\end{aligned} \tag{7.7}
$$

The complete set of the quantum states is recovered through the action of SU(2|1) supercharges on the complete set of these bosonic wave functions.

The generic case in the reduced phase space formulation will be considered elsewhere.

8 Concluding remarks and outlook

In this paper, we presented the full quantum description of the SU(2|1) supersymmetric multi-particle Calogero-Moser system with spin variables. It was constructed by making use of the matrix formulation of this system. Due to the presence of spin variables, the system under consideration involves internal spin degrees of freedom and so provides $\mathcal{N}$=4 supersymmetrization of U(2) spin Calogero-Moser system, as opposed to the systems considered in refs. [41–43].

We obtained the explicit expressions for the classical and quantum charges of the mass-deformed $\mathcal{N}$=4 supersymmetry inherent to the multiparticle system considered. The crucial role in quantization of this system is played by the property that it became possible to single out the center-of-mass subsector in the full system. This allowed us to separately explore the case of the center of mass and the case without the center-of-mass variables. Knowing the energy spectrum in these two cases immediately allows one to derive the energy spectrum of the total system.

We computed the energy spectrum, exploiting the matrix formulation of the $\mathcal{N}$=4 supersymmetric U(2) spin Calogero-Moser system. An alternative way of quantizing such systems is to deal with the reduced system, involving the dynamical position coordinates only. Such a method [7, 8, 30, 31, 33, 34] (the "operator method" in the terminology by A. Polychronakos) widely uses the Dunkl operators for building the oscillator-like phase space of the multi-particle Calogero-type systems. Some simple examples of applying this equivalent method within the model considered here were already discussed in section 6.2. In the next publication we are planning to develop, in full generality, the applications of the operator method to the systems with spin variables. On this way we expect, in particular, to find out some new generalizations of the Dunkl operators and obtain a complete set of independent conserved quantities (integrals) for a rigorous proof of possible integrability of our system. One more direction for the future study is to construct and quantize multi-particle Calogero-type models with higher-rank deformed supersymmetries of the kind SU$(m|n)$ and to reveal their relationships with the integrable structures in $\mathcal{N}$=4 super Yang-Mills theory, e.g., along the lines of refs. [11, 12].

One more interesting problem is to elucidate a possible hidden superconformal symmetry of the multi-particle system considered. In the one-particle case, the corresponding quantum-mechanical (massive) system [13] was found to possess such a hidden $\mathcal{N}$=4 superconformal symmetry associated with the supergroup OSp(4|2) [27]. In the quantum

JHEP04(2018)043

domain, the corresponding superalgebra $osp(4|2)$ acts as a spectrum-generating algebra. The existence of an analogous extension of SU(2|1) symmetry in the multi-particle case is an open question. In general, one could expect as well a hidden $D(2,1;\alpha)$ supersymmetry for which OSp(4|2) is a particular case corresponding to the choice $\alpha = -1/2$. However, this possibility would require, from the very beginning, some nonlinear sigma model action for the superfields $\mathfrak{X}^a_b$ in (1.1) and, respectively, for the bosonic fields X^a_b in (1.5). The choice of OSp(4|2) is the unique one consistent with free kinetic terms for the bosonic fields, as long as one insists on the supercharges (4.30) being linear in fermionic variables [44]. Allowing for supercharge terms cubic in the fermionic operators will constrain their coefficient functions by the so-called WDVV equations [41–46]. It will be interesting to develop a superspace variant of this more general situation.

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References

[1] F. Calogero, *Solution of a three-body problem in one-dimension*, *J. Math. Phys.* **10** (1969) 2191 [INSPIRE].

[2] F. Calogero, *Ground state of one-dimensional N body system*, *J. Math. Phys.* **10** (1969) 2197 [INSPIRE].

[3] F. Calogero, *Solution of the one-dimensional N body problems with quadratic and/or inversely quadratic pair potentials*, *J. Math. Phys.* **12** (1971) 419 [INSPIRE].

[4] J. Moser, *Three integrable Hamiltonian systems connnected with isospectral deformations*, *Adv. Math.* **16** (1975) 197 [INSPIRE].

[5] M.A. Olshanetsky and A.M. Perelomov, *Classical integrable finite dimensional systems related to Lie algebras*, *Phys. Rept.* **71** (1981) 313 [INSPIRE].

[6] M.A. Olshanetsky and A.M. Perelomov, *Quantum Integrable Systems Related to Lie Algebras*, *Phys. Rept.* **94** (1983) 313 [INSPIRE].

[7] L. Brink, T.H. Hansson and M.A. Vasiliev, *Explicit solution to the N body Calogero problem*, *Phys. Lett.* **B 286** (1992) 109 [hep-th/9206049] [INSPIRE].

[8] L. Brink, T.H. Hansson, S. Konstein and M.A. Vasiliev, *The Calogero model: Anyonic representation, fermionic extension and supersymmetry*, *Nucl. Phys.* **B 401** (1993) 591 [hep-th/9302023] [INSPIRE].

[9] G.W. Gibbons and P.K. Townsend, *Black holes and Calogero models*, *Phys. Lett.* **B 454** (1999) 187 [hep-th/9812034] [INSPIRE].

[10] J. McGreevy, S. Murthy and H.L. Verlinde, *Two-dimensional superstrings and the supersymmetric matrix model*, *JHEP* **04** (2004) 015 [hep-th/0308105] [INSPIRE].

[11] A. Dabholkar, *Fermions and nonperturbative supersymmetry breaking in the one-dimensional superstring*, *Nucl. Phys.* **B 368** (1992) 283.

[12] A. Agarwal and A.P. Polychronakos, *BPS operators in $\mathcal{N} = 4$ SYM: Calogero models and 2D fermions*, *JHEP* **08** (2006) 034 [hep-th/0602049] [INSPIRE].

[13] S. Fedoruk and E. Ivanov, *Gauged spinning models with deformed supersymmetry*, *JHEP* **11** (2016) 103 [arXiv:1610.04202] [INSPIRE].

[14] A.V. Smilga, *Weak supersymmetry*, *Phys. Lett.* **B 585** (2004) 173 [hep-th/0311023] [INSPIRE].

[15] S. Bellucci and A. Nersessian, *(Super)oscillator on CP^N and constant magnetic field*, *Phys. Rev.* **D 67** (2003) 065013 [*Erratum ibid.* **D 71** (2005) 089901] [hep-th/0211070] [INSPIRE].

[16] S. Bellucci and A. Nersessian, *Supersymmetric Kähler oscillator in a constant magnetic field*, in *Proceedings, 5th International Workshop on Supersymmetries and Quantum Symmetries (SQS'03)*, Dubna, Russia, July 24–29, 2003, pp. 379–384 (2004) [hep-th/0401232] [INSPIRE].

[17] C. Römelsberger, *Counting chiral primaries in $N = 1$, $d = 4$ superconformal field theories*, *Nucl. Phys.* **B 747** (2006) 329 [hep-th/0510060] [INSPIRE].

[18] C. Romelsberger, *Calculating the Superconformal Index and Seiberg Duality*, arXiv:0707.3702 [INSPIRE].

[19] E. Ivanov and S. Sidorov, *Deformed Supersymmetric Mechanics*, *Class. Quant. Grav.* **31** (2014) 075013 [arXiv:1307.7690] [INSPIRE].

[20] E. Ivanov and S. Sidorov, *Super Kähler oscillator from* $\mathrm{SU}(2|1)$ *superspace*, *J. Phys.* **A 47** (2014) 292002 [arXiv:1312.6821] [INSPIRE].

[21] E. Ivanov and S. Sidorov, $\mathrm{SU}(2|1)$ *mechanics and harmonic superspace*, *Class. Quant. Grav.* **33** (2016) 055001 [arXiv:1507.00987] [INSPIRE].

[22] B. Assel, D. Cassani, L. Di Pietro, Z. Komargodski, J. Lorenzen and D. Martelli, *The Casimir Energy in Curved Space and its Supersymmetric Counterpart*, *JHEP* **07** (2015) 043 [arXiv:1503.05537] [INSPIRE].

[23] C.T. Asplund, F. Denef and E. Dzienkowski, *Massive quiver matrix models for massive charged particles in AdS*, *JHEP* **01** (2016) 055 [arXiv:1510.04398] [INSPIRE].

[24] S. Fedoruk, E. Ivanov and O. Lechtenfeld, *Supersymmetric Calogero models by gauging*, *Phys. Rev.* **D 79** (2009) 105015 [arXiv:0812.4276] [INSPIRE].

[25] S. Fedoruk, E. Ivanov and O. Lechtenfeld, *Superconformal Mechanics*, *J. Phys.* **A 45** (2012) 173001 [arXiv:1112.1947] [INSPIRE].

[26] E. Ivanov and O. Lechtenfeld, *$N = 4$ supersymmetric mechanics in harmonic superspace*, *JHEP* **09** (2003) 073 [hep-th/0307111] [INSPIRE].

[27] S. Fedoruk, E. Ivanov and S. Sidorov, *Deformed supersymmetric quantum mechanics with spin variables*, *JHEP* **01** (2018) 132 [arXiv:1710.02130] [INSPIRE].

[28] J. Gibbons and T. Hermsen, *A generalization of the Calogero-Mozer system*, *Physica* **D 11** (1984) 337.

[29] S. Wojciechowski, *An integrable marriage of the Euler equations with the Calogero-Mozer system*, *Phys. Lett.* **A 111** (1985) 101.

[30] A.P. Polychronakos, *Generalized statistics in one-dimension*, in *Topological Aspects of Low-dimensional Systems: Proceedings, Les Houches Summer School of Theoretical Physics, Session 69*, Les Houches, France, July 7–31 1998 (1999) [hep-th/9902157] [INSPIRE].

[31] A.P. Polychronakos, *Generalized Calogero models through reductions by discrete symmetries*, *Nucl. Phys.* **B 543** (1999) 485 [hep-th/9810211] [INSPIRE].

[32] A.P. Polychronakos, *Calogero-Moser models with noncommutative spin interactions*, *Phys. Rev. Lett.* **89** (2002) 126403 [hep-th/0112141] [INSPIRE].

[33] A.P. Polychronakos, *Physics and Mathematics of Calogero particles*, *J. Phys.* **A 39** (2006) 12793 [hep-th/0607033] [INSPIRE].

[34] A.P. Polychronakos, *Exchange operator formalism for integrable systems of particles*, *Phys. Rev. Lett.* **69** (1992) 703 [hep-th/9202057] [INSPIRE].

[35] J.A. Minahan and A.P. Polychronakos, *Integrable systems for particles with internal degrees of freedom*, *Phys. Lett.* **B 302** (1993) 265 [hep-th/9206046] [INSPIRE].

[36] M. Feigin, O. Lechtenfeld and A.P. Polychronakos, *The quantum angular Calogero-Moser model*, *JHEP* **07** (2013) 162 [arXiv:1305.5841] [INSPIRE].

[37] A.P. Polychronakos, *Integrable systems from gauged matrix models*, *Phys. Lett.* **B 266** (1991) 29 [INSPIRE].

[38] S. Hellerman and M. Van Raamsdonk, *Quantum Hall physics equals noncommutative field theory*, *JHEP* **10** (2001) 039 [hep-th/0103179] [INSPIRE].

[39] A.P. Polychronakos, *Quantum Hall states as matrix Chern-Simons theory*, *JHEP* **04** (2001) 011 [hep-th/0103013] [INSPIRE].

[40] A.M. Perelomov, *Algebraical approach to the solution of one-dimensional model of n interacting particles*, *Teor. Mat. Fiz.* **6** (1971) 364 [INSPIRE].

[41] S. Bellucci, A.V. Galajinsky and E. Latini, *New insight into WDVV equation*, *Phys. Rev.* **D 71** (2005) 044023 [hep-th/0411232] [INSPIRE].

[42] A. Galajinsky, O. Lechtenfeld and K. Polovnikov, *$N = 4$ superconformal Calogero models*, *JHEP* **11** (2007) 008 [arXiv:0708.1075] [INSPIRE].

[43] A. Galajinsky, O. Lechtenfeld and K. Polovnikov, *$N = 4$ mechanics, WDVV equations and roots*, *JHEP* **03** (2009) 113 [arXiv:0802.4386] [INSPIRE].

[44] S. Krivonos and O. Lechtenfeld, *Many-particle mechanics with $D(2, 1 : \alpha)$ superconformal symmetry*, *JHEP* **02** (2011) 042 [arXiv:1012.4639] [INSPIRE].

[45] S. Krivonos, O. Lechtenfeld and K. Polovnikov, *$N = 4$ superconformal n-particle mechanics via superspace*, *Nucl. Phys.* **B 817** (2009) 265 [arXiv:0812.5062] [INSPIRE].

[46] O. Lechtenfeld, K. Schwerdtfeger and J. Thürigen, *$N = 4$ Multi-Particle Mechanics, WDVV Equation and Roots*, *SIGMA* **7** (2011) 023 [arXiv:1011.2207] [INSPIRE].

JHEP04(2018)043