

\mathcal{PT} deformation of angular Calogero models

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ABSTRACT: The rational Calogero model based on an arbitrary rank- n Coxeter root system is spherically reduced to a superintegrable angular model of a particle moving on S^{n-1} subject to a very particular potential singular at the reflection hyperplanes. It is outlined how to find conserved charges and to construct intertwining operators. We deform these models in a \mathcal{PT} -symmetric manner by judicious complex coordinate transformations, which render the potential less singular. The \mathcal{PT} deformation does not change the energy eigenvalues but in some cases adds a previously unphysical tower of states. For integral couplings the new and old energy levels coincide, which roughly doubles the previous degeneracy and allows for a conserved nonlinear supersymmetry charge. We present the details for the generic rank-two (A_2 , G_2) and all rank-three Coxeter systems (AD_3 , BC_3 and H_3), including a reducible case ($A_1^{\otimes 3}$).

KEYWORDS: Field Theories in Lower Dimensions, Integrable Field Theories, Conformal and W Symmetry, Discrete Symmetries

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1 Introduction and summary

The rational Calogero model (for a review, see [1]) generalizes to any root system of a (finite-dimensional) Lie algebra or, better, to any Coxeter root system. Given such a system of rank n , it describes a conformal particle moving in \mathbb{R}^n under the influence of a very special potential. Since this potential has a universal inverse-square radial dependence and otherwise depends only on the angular coordinates (of S^{n-1}), a spherical reduction to its angular subsystem, the angular Calogero model, is natural. Like the full model on \mathbb{R}^n , the reduced dynamics on S^{n-1} is superintegrable, so that it enjoys $2n-3$ integrals of motion, which are however not in involution. Recently, the angular models have been analyzed in some detail, both classically and quantum mechanically [2–12].

It has been known for a long time that hermiticity is not an essential feature of a Hamiltonian for its spectrum to be real. For instance, it suffices that the Hamiltonian commutes with an antilinear involution (one example is provided by the \mathcal{PT} operator where \mathcal{P} correspond to the parity operator and \mathcal{T} the time reversal operator) which also leaves the eigenfunctions invariant (“unbroken \mathcal{PT} symmetry”) [13]. Such a non-hermitian Hamiltonian is related to a hermitian one by a (non-unitary) similarity transformation, which may be impossibly complicated. Often, however, there exists a family H_ϵ of non-hermitian \mathcal{PT} -invariant Hamiltonians representing a deformation of a hermitian H_0 . In this case we speak of a “ \mathcal{PT} deformation”, with the parameter ϵ measuring the deviation from hermiticity. For rational Calogero models, a particularly nice set of \mathcal{PT} deformations can be generated by a specific *complex* orthogonal deformation of the coordinates in the expression for the Hamiltonian. If such a \mathcal{PT} deformation is in accordance with the Coxeter reflection symmetry of the system, integrability will be preserved. This kind of \mathcal{PT} deformation has been applied to the full rational Calogero model about ten years ago by Fring and Znojil [14], and corresponding complex root systems were constructed by Fring and Smith thereafter [15–17]. For a review of \mathcal{PT} deformations of integrable models, see [18].

It is worth recalling the relevant part (for this paper) of the Calogero model’s long history:

- 1971 Calogero [19]:
Solution of the one-dimensional N-body problem ... inversely quadratic pair potentials
- 1981 Olshanetsky & Perelomov [20, 21]:
Classical integrable finite-dimensional systems related to Lie algebras (1983: quantum)
- 1983 Wojciechowski [22]:
Superintegrability of the Calogero-Moser system
- 1989 Dunkl [23]:
Differential-difference operators associated to reflection groups
- 1990 Chalykh & Veselov [24]:
Commutative rings of partial differential operators and Lie algebras, supercompleteness
- 1991 Heckman [25]:
Elementary construction for commuting charges and intertwiners (shift operators)
- 2003 M. Feigin [2]:
Intertwining relations for the spherical parts of generalized Calogero operators
- 2008 A. Fring, M. Znojil [14]:
 \mathcal{PT} -symmetric deformations of Calogero models
- 2008 Hakobyan, Nersessian, Yeghikyan [3]:
The cuboctahedric Higgs oscillator from the rational Calogero model (classical)

- 2010 A. Fring, M. Smith [15–17]:
Complex root systems in the Calogero model
- 2013 M. Feigin, Lechtenfeld, Polychronakos [9]:
The quantum angular Calogero-Moser model (spectra, eigenstates)
- 2013 Correa, Lechtenfeld, Plyushchay [10]:
Nonlinear supersymmetry in the quantum Calogero model
- 2014 M. Feigin, Hakobyan [11]:
On the algebra of Dunkl angular momentum operators
- 2015 Correa, Lechtenfeld [12]:
The tetrahedric angular Calogero model

The present paper describes the superintegrable spherical reduction of the rational quantum Calogero model for any Coxeter root system (section 2) and some of its complex \mathcal{PT} deformations (section 3). The emphasis is on the Weyl-singlet energy spectrum including degeneracy and eigenstates, and on the conserved charges and intertwiners, in particular for a coupling strength $g(g-1)$ with $g \in \mathbb{Z}$. We discuss all features in some detail for the rank-two cases of A_2 and G_2 (sections 4 and 5) and for all rank-three cases, i.e. AD_3 , BC_3 and H_3 (sections 6, 7 and 9), as well as for A_1^3 as a reducible example (section 8). Tables of low-lying states are collected in the appendix.

Our results generalize those of [12] to general Coxeter root systems, in particular to the non-simply-laced case, where two independent couplings wrongly suggest the existence of long-root and short-root intertwiners. Instead, we find that all intertwiners respecting the reflection symmetry either shift both couplings or only one of them, so not all states with integral couplings can be connected. We identify a geometric condition for complex orthogonal coordinate transformations to yield a \mathcal{PT} deformation (with \mathcal{P} given by a Coxeter element) and display the simplest solutions. It turns out that such deformations reduce the singularities of the angular Calogero potential from codimension one to codimension two. We also present a nonlinear \mathcal{PT} deformation which may completely remove those singularities (it does so for rank three). In such a situation, the non-normalizable eigenstates (formally given by sending $g \mapsto 1-g$ for $g \in \mathbb{N}$) become normalizable and have to be added to the spectrum.¹ Not only does this roughly double the state degeneracy, but it also gives rise to new ‘odd’ conserved charges, which connect the old and the new states. We display these effects for the generic rank-two and all rank-three Coxeter systems.

2 The angular rational Calogero model

The well known rational Calogero model describing n interacting identical particles moving on \mathbb{R} can be formulated for any finite reflection group W , with the multi-particle potential encoded in the associated Coxeter root system $\mathcal{R} \subset \mathbb{R}^n$. Since this interaction is not translation invariant² it is more natural to view such systems as a single particle moving

¹In \mathcal{PT} -deformed theories there exist different notions of state norms. They all agree on normalizability for our states. Regarding operators, we do not consider any conjugation properties besides \mathcal{P} and \mathcal{T} .

²The A_n model describes the *relative* coordinates of $n+1$ particles after decoupling the center of mass.

in \mathbb{R}^n under the influence of a rather particular external potential determined by \mathcal{R} . As the Hamiltonian is homogeneous under a common coordinate rescaling (the couplings are dimensionless) the model may be reduced over the $(n-1)$ -sphere. The result is what we have named the *angular* Calogero model, since it describes a particle moving on S^{n-1} , parametrized by angular coordinates $\vec{\theta}$ only. Because hyperspherical coordinates are rather unwieldy however, we prefer to employ the homogeneous \mathbb{R}^n coordinates $x = (x^i)$ with $i = 1, \dots, n$ and define

$$\sum_{i=1}^n (x^i)^2 =: r^2. \tag{2.1}$$

In terms of the latter, the angular Calogero Hamiltonian takes the form

$$H = \frac{1}{2}L^2 + U \quad \text{with} \quad L^2 = -\sum_{i<j} (x^i \partial_j - x^j \partial_i)^2 \quad \text{and} \quad U = r^2 \sum_{\alpha \in \mathcal{R}_+} \frac{g_\alpha (g_\alpha - 1) \alpha \cdot \alpha}{2(\alpha \cdot x)^2} \tag{2.2}$$

where \mathcal{R}_+ is the positive half of \mathcal{R} , $g_\alpha \in \mathbb{R}$ are the couplings, and \cdot is the standard scalar product in \mathbb{R}^n . Due to the invariance of the Hamiltonian under $g_\alpha + 1 \leftrightarrow -g_\alpha$, it suffices to consider $g_\alpha \geq \frac{1}{2}$, but we shall not impose this restriction because intermediate results do not reflect this symmetry. Each positive root α contributes a term of the form $\cos^{-2} \phi_\alpha$, where ϕ_α is the geodesic distance to $\alpha/\sqrt{\alpha \cdot \alpha}$. This so-called Higgs oscillator potential [26, 27] is singular on a great S^{n-2} , where the hyperplane orthogonal to α cuts our $(n-1)$ -sphere into two hemispheres. Taken together, these singular loci of codimension one tessellate the $(n-1)$ -sphere, and our particle is confined to a given Weyl chamber, with its wave function vanishing at the walls (except for $g=0$). The potential breaks the $SO(n)$ invariance of L^2 to its discrete subgroup W , so the energy eigenstates fall into W representations. Motivated by the physical interpretation, we admit only singlet states, i.e. wave functions are either totally symmetric or totally antisymmetric under Coxeter reflections.

The Weyl-invariant spectrum of H has been derived in [9] (see also the appendices of [19]),

$$H v_{\{\ell\}} = E_\ell v_{\{\ell\}} \quad \text{with} \quad \{\ell\} = (\ell_3, \ell_4, \dots, \ell_{n+1}) \quad \text{and} \quad \ell = d_3 \ell_3 + d_4 \ell_4 + \dots + d_{n+1} \ell_{n+1} \tag{2.3}$$

where $d_2=2, d_3, \dots, d_{n+1}$ are the degrees of the basic homogeneous W -invariant polynomials $\sigma_2 = \sum_i (x^i)^2, \sigma_3, \dots, \sigma_{n+1}$ and the quantum numbers $\ell_3, \ell_4, \dots, \ell_{n+1}$ are nonnegative integers.³ Note that σ_2 does not contribute because ℓ_2 labels the radial excitations. The energy depends only on the ‘deformed angular momentum’ q ,

$$E_\ell = \frac{1}{2} q (q + n - 2) \quad \text{with} \quad q = \ell + \sum_{\alpha \in \mathcal{R}_+} g_\alpha. \tag{2.4}$$

For vanishing couplings, $H = \frac{1}{2}L^2$, and $q = \ell$ is the familiar total angular momentum for a free particle on S^{n-1} . Nevertheless, the degeneracy of E_ℓ is greatly reduced by W -invariance to the number of partitions of ℓ into integers from the set $\{d_3, \dots, d_{n+1}\}$.

³The unconventional labelling is chosen to match with the standard choice for the $A_1 \oplus A_n$ model.

The angular wave function $v_{\{\ell\}}^{(g)}$ for couplings $g = \{g_\alpha\}$ can be constructed in the following way [9]. First, we split off a suitable power of r and a ‘Vandermonde factor’,

$$v_{\{\ell\}}^{(g)} = r^{-q} \Delta^g h_{\{\ell\}}^{(g)}(x) \quad \text{with} \quad \Delta^g = \prod_{\alpha \in \mathcal{R}_+} (\alpha \cdot x)^{g_\alpha} \quad (2.5)$$

and obtain a homogenous polynomial $h_{\{\ell\}}^{(g)}$ of degree ℓ in x . Second, the latter is a W -invariant Dunkl-deformed harmonic function given by

$$h_{\{\ell\}}^{(g)}(x) = r^{n-2+2q} \left(\prod_{\mu=3}^{n+1} \sigma_\mu(\{\tilde{\mathcal{D}}_i\})^{\ell_\mu} \right) r^{2-n-2(q-\ell)}, \quad (2.6)$$

where

$$\tilde{\mathcal{D}}_i = \partial_i + \sum_{\alpha \in \mathcal{R}_+} \frac{g_\alpha \alpha_i}{\alpha \cdot x} (1 - s_\alpha) = \Delta^{-g} \mathcal{D}_i \Delta^g \quad (2.7)$$

denotes the Dunkl differential-reflection operator [23, 28], which involves the Coxeter reflections s_α about the hyperplane $\alpha \cdot x = 0$. The tilde signifies the so-called potential-free frame, which is related to the ‘potential frame’ by a similarity transformation with Δ^g ,

$$\mathcal{D}_i = \Delta^g \tilde{\mathcal{D}}_i \Delta^{-g} = \partial_i - \sum_{\alpha \in \mathcal{R}_+} \frac{g_\alpha \alpha_i}{\alpha \cdot x} s_\alpha. \quad (2.8)$$

In particular, for the ground state one has

$$h_{\{0\}}^{(g)} = 1 \quad \implies \quad v_{\{0\}}^{(g)} = r^{-q} \Delta^g = \prod_{\alpha \in \mathcal{R}_+} \left(\alpha \cdot \frac{x}{r} \right)^{g_\alpha}, \quad (2.9)$$

and hence the full ground-state wave function is totally symmetric (antisymmetric) under Coxeter reflections for even (odd) integer values of g_α . Since all other ingredients besides Δ^g in (2.5) are completely symmetric, this symmetry property of the integer- g_α ground state extends to all excited states above it. The degeneracy of the energy levels decreases with growing values of g_α . Furthermore, the reflection symmetry $g_\alpha+1 \leftrightarrow -g_\alpha$ of the Hamiltonian (2.2) is broken since one tower of states is Weyl symmetric while the other one is antisymmetric. However, due to singularities at $\alpha \cdot x = 0$ coming from the Vandermonde factor in (2.5), for $g_\alpha < 0$ the formal eigenstates are not normalizable (i.e. not in $L_2(S^{n-1})$) and thus unphysical. In other words, the singularities in the potential U enforce boundary conditions, which admit only one of the two symmetry types. The free case is an exception, because then those boundary conditions are absent, and so both values $g_\alpha = 0$ and $g_\alpha = 1$ contribute to the same spectrum, leading to a rough doubling of the states.

Our Hamiltonian and other conserved quantities are conveniently constructed from the algebra of Dunkl-deformed angular momenta,

$$\mathcal{L}_{ij} = x^i \mathcal{D}_j - x^j \mathcal{D}_i, \quad (2.10)$$

which yields

$$-\frac{1}{2} \sum_{i < j} \mathcal{L}_{ij}^2 = \mathcal{H} - \frac{1}{2} S(S+n-2) \quad (2.11)$$

with

$$\mathcal{H} = \frac{1}{2}L^2 + r^2 \sum_{\alpha \in \mathcal{R}_+} \frac{\alpha \cdot \alpha / 2}{(\alpha \cdot x)^2} g_\alpha (g_\alpha - s_\alpha) \quad \text{and} \quad S = \sum_{\alpha} g_\alpha s_\alpha. \quad (2.12)$$

The restriction ‘res’ to W -symmetric functions provides the Hamiltonian,

$$-\frac{1}{2} \text{res} \left(\sum_{i < j} \mathcal{L}_{ij}^2 \right) = \text{res}(\mathcal{H}) - \frac{1}{2} \sum_{\alpha} g_\alpha \left(\sum_{\alpha} g_\alpha + n - 2 \right) = H - E_0. \quad (2.13)$$

As was shown in [2], the center of the algebra generated by $\{\mathcal{L}_{ij}\}$ is spanned by \mathcal{H} and the constants. Therefore, any polynomial \mathcal{C} built from the \mathcal{L}_{ij} will commute with \mathcal{H} . If such a polynomial is Weyl invariant, then its restriction yields a conserved quantity,

$$\mathcal{C} \text{ Weyl invariant} \quad \Longrightarrow \quad [C, H] = 0 \quad \text{for} \quad C = \text{res}(\mathcal{C}). \quad (2.14)$$

It is not clear whether some combinations of these are in involution or how to classify them.

It is actually more fruitful to investigate Weyl antiinvariant polynomials in \mathcal{L}_{ij} , since they give rise to intertwiners (shift operators) which connect Hamiltonians and eigenspaces differing by unit values in the couplings. To be more precise, let us split the set of positive roots into Weyl orbits,

$$\mathcal{R}_+ = \mathcal{R}' \cup \mathcal{R}'', \quad (2.15)$$

where one of the following four situations occurs:

case	A	B	C	D
\mathcal{R}'	all positive roots	long positive roots	short positive roots	empty
\mathcal{R}''	empty	short positive roots	long positive roots	all positive roots

Because all couplings g_α in a given Weyl orbit must coincide, we can have at most two different values, g' and g'' . The objects of interest are polynomials \mathcal{M} in \mathcal{L}_{ij} which are Weyl antiinvariant under \mathcal{R}' reflections but Weyl invariant under \mathcal{R}'' reflections. Because the structure of (2.12) implies that

$$\text{res}(\mathcal{M} \mathcal{H}^{(g', g'')}) = M H^{(g', g'')} \quad \text{but} \quad \text{res}(\mathcal{H}^{(g', g'')} \mathcal{M}) = H^{(g'+1, g'')} M \quad (2.16)$$

the commutation of \mathcal{M} and \mathcal{H} qualifies $M = \text{res}(\mathcal{M})$ as an intertwiner,

$$\mathcal{M} \left\{ \begin{array}{l} \mathcal{R}' \text{ antiinvariant} \\ \mathcal{R}'' \text{ invariant} \end{array} \right\} \quad \Longrightarrow \quad M H^{(g', g'')} = H^{(g'+1, g'')} M. \quad (2.17)$$

Note that \mathcal{M} and M depend on (g', g'') , which we have suppressed. This operator relation may be applied to W -noninvariant states. Hence, M maps $H^{(g', g'')}$ eigenstates of energy $E_{\ell}^{(g', g'')}$ to $H^{(g'+1, g'')}$ eigenstates of (the same) energy $E_{\ell'}^{(g'+1, g'')}$ with $\ell' = \ell - |\mathcal{R}'|$ (see (2.4)). In particular,

$$M v_{\{\ell\}}^{(g', g'')} = \sum_{\substack{\{\ell'\} \\ \ell' = \ell - |\mathcal{R}'|}} c_{\{\ell'\}}^{\{\ell\}} v_{\{\ell'\}}^{(g'+1, g'')} \quad (2.18)$$

with some coefficients $c_{\{\ell'\}}^{\{\ell\}} \in \mathbb{R}$. Generically, such a map M has a nonempty kernel. The

action on the deformed harmonic polynomials $h_{\{\ell\}}^{(g)}$ is obtained by passing to the potential-free frame,

$$\widetilde{M} h_{\{\ell\}}^{(g',g'')} = \sum_{\substack{\{\ell'\} \\ \ell'=\ell-|\mathcal{R}'|}} c_{\{\ell'\}}^{\{\ell\}} \Delta h_{\{\ell'\}}^{(g'+1,g'')} \quad \text{with} \quad \widetilde{M} = \Delta^{-g} M \Delta^g. \quad (2.19)$$

It is a nontrivial problem for a given Coxeter group W to identify a complete set of intertwiners, their algebra and its generators. We remark that case D does not shift any coupling and describes the constants of motion C mentioned above, while case A pertains to the simply-laced Coxeter groups. When both couplings g' and g'' are integer, repeated intertwining may relate all quantities with their analogs in the free theory, which allows one to generate analytic expressions for all wave functions.

3 \mathcal{PT} -symmetric complex coordinate deformations

We implement a complex deformation of the (angular) coordinates $\vec{\theta}$ through a family of complex linear maps

$$\Gamma(\epsilon) : \mathbb{R}^n \rightarrow \mathbb{C}^n \quad \text{with} \quad \Gamma(0) = \text{id} \quad (3.1)$$

which respect the standard scalar product of \mathbb{R}^n , so

$$\Gamma(\epsilon)^\top = \Gamma(\epsilon)^{-1}. \quad (3.2)$$

Hence, $\Gamma(\epsilon) \in \text{SO}(n, \mathbb{C})$, but because real coordinate rotations are inessential our family is parametrized by the coset $\text{SO}(n, \mathbb{C})/\text{SO}(n, \mathbb{R})$ of real dimension $\frac{1}{2}n(n-1)$,

$$\Gamma(\epsilon) = \exp\left\{\sum_{i<j} \epsilon_{ij} G_{ij}\right\} \quad \text{with} \quad G_{ij} : x^k \mapsto i(\delta^{kj} x^i - \delta^{ki} x^j), \quad (3.3)$$

and thus we also have

$$\Gamma(\epsilon)^* = \Gamma(\epsilon)^\top = \Gamma(-\epsilon). \quad (3.4)$$

A coordinate change effected by $\Gamma(\epsilon)$,

$$(x^1, x^2, \dots, x^n)^\top = x \mapsto \Gamma(\epsilon) x =: x(\epsilon), \quad (3.5)$$

leaves r^2 and the kinetic term $\frac{1}{2}L^2$ invariant but generates a complex deformation $U \mapsto U(\epsilon)$ of the angular potential (2.2), via

$$\alpha \cdot x \mapsto \alpha \cdot \Gamma(\epsilon) x = \Gamma(\epsilon)^\top \alpha \cdot x, \quad (3.6)$$

which may also be interpreted as a complex (dual) deformation of the roots α . Formally, the deformed Hamiltonian $H(\epsilon)$ is isospectral to $H = H(0)$, and its W -invariant eigenfunctions are simply given by

$$v_{\{\ell\}\epsilon}^{(g)} = r^{-q} \Delta_\epsilon^g h_{\{\ell\}\epsilon}^{(g)} \quad \text{with} \quad \Delta_\epsilon^g = \prod_{\alpha \in \mathcal{R}_+} (\alpha \cdot x(\epsilon))^{g_\alpha} \quad \text{and} \quad h_{\{\ell\}\epsilon}^{(g)}(x) = h_{\{\ell\}}^{(g)}(x(\epsilon)). \quad (3.7)$$

Our Hamiltonian is \mathcal{PT} symmetric if there exist two involutions, one linear (\mathcal{P}) and one antilinear (\mathcal{T}), under whose combined action it is invariant. For \mathcal{T} we take the conventional choice of complex conjugation. In the context of Calogero models, a natural \mathcal{P} transformation is provided by some element s of order 2 in the Coxeter group W . The kinetic term $\frac{1}{2}L^2$ is separately invariant under \mathcal{P} and \mathcal{T} but, in order for $U(\epsilon)$ to be \mathcal{PT} invariant, the action of the involutive Coxeter element s on the deformed coordinate $x(\epsilon)$ has to be undone by complex conjugation, implying

$$\mathcal{P}\Gamma(\epsilon) = \mathcal{T}\Gamma(\epsilon) \quad \implies \quad s\Gamma(\epsilon)s = \Gamma(\epsilon)^* = \Gamma(-\epsilon). \quad (3.8)$$

On the Lie-algebra level this condition reads

$$\{s, \epsilon:G\} = 0 \quad \Leftrightarrow \quad s(\epsilon:G)s = -\epsilon:G \quad \Leftrightarrow \quad P_{\pm}(\epsilon:G)P_{\pm} = 0 \quad (3.9)$$

with $\epsilon:G = \sum_{i<j} \epsilon_{ij}G_{ij}$ and projectors

$$P_- = \frac{1}{2}(1 - s) \quad \text{and} \quad P_+ = \frac{1}{2}(1 + s) \quad (3.10)$$

on the -1 and $+1$ eigenspaces of s , respectively. It means that $\epsilon:G$ intertwines between those two eigenspaces, and so

$$\text{rank}(\epsilon:G) = \min(2\text{rank}(P_-), 2\text{rank}(P_+)). \quad (3.11)$$

If s is just a Coxeter reflection s_{γ} pertaining to some (positive) root γ , then we can say a bit more. Since in this case P_- is of rank one, it follows that $\epsilon:G$ is of rank two only and parallel to γ ,

$$\epsilon:G = -i\epsilon \hat{\gamma} \wedge \hat{\eta} \in su(1,1) \quad \text{with} \quad (\epsilon:G)\gamma \sim \eta \perp \gamma \quad (3.12)$$

for some real vector η , carrying $n-1$ parameters. The hats denote unit vectors, and the overall scale has been absorbed into a single parameter ϵ . For this situation, the infinitesimal transformation can be integrated explicitly to

$$\Gamma(\epsilon) = \exp\{-i\epsilon \hat{\gamma} \wedge \hat{\eta}\} = P_{\gamma \wedge \eta}^{\perp} - P_{\gamma \wedge \eta}(\cosh(\epsilon) - i \sinh(\epsilon) \hat{\gamma} \wedge \hat{\eta}), \quad (3.13)$$

with the help of projectors $P_{\gamma \wedge \eta}$ and $P_{\gamma \wedge \eta}^{\perp}$ onto the plane spanned by γ and η and orthogonal to it, respectively. This is just a complex rotation (boost) in the plane determined by γ and η . A similar analysis applies in the co-rank-one case, i.e. when P_+ is of rank one. In adapted coordinates,

$$\Gamma(\epsilon) = e^{\epsilon:G} = \begin{pmatrix} \cosh(\epsilon) & -i \sinh(\epsilon) & 0 \cdots 0 \\ i \sinh(\epsilon) & \cosh(\epsilon) & 0 \cdots 0 \\ 0 & 0 & \mathbb{1}_{n-2} \\ \vdots & \vdots & & \\ 0 & 0 & & \end{pmatrix}. \quad (3.14)$$

The complex deformation greatly improves the singularities of U by generically increasing their codimension from one to two. The singularity relation $\alpha \cdot \Gamma(\epsilon) x = 0$ decomposes into a real and imaginary part giving two conditions,

$$\alpha \cdot x = 0 \quad \text{and} \quad \alpha \cdot (\epsilon:G) \cdot x = 0 \quad \text{mod } O(\epsilon^2), \quad (3.15)$$

leaving an S^{n-3} plus its antipode as the singular locus for each positive root α contributing to U . Specializing to \mathcal{PT} -symmetric deformations (3.9), the second condition may be empty if α lies in the kernel of $\epsilon:G$. However, such a situation can be avoided by a slight change in the parameters ϵ_{ij} . For the case of $s = s_\gamma$, the singular loci appear at

$$\alpha \cdot (P_{\gamma \wedge \eta}^\perp x + \cosh(\epsilon) P_{\gamma \wedge \eta}) x = 0 \quad \text{and} \quad \alpha \cdot (\gamma \wedge \eta) P_{\gamma \wedge \eta} x = 0. \quad (3.16)$$

The second condition gets lost if α lies in the kernel of $P_{\gamma \wedge \eta}$, i.e. if

$$\alpha \cdot (\gamma \wedge \eta) = 0, \quad (3.17)$$

However, by a suitable (generic) choice of η one can tilt the plane spanned by γ and η such as to avoid any roots and so evade this degenerate situation.

The deformation also ameliorates the singularities in the unphysical wave functions for negative values of the couplings. From the form of (3.7) it is clear that Δ_ϵ vanishes at antipodal pairs $(x_\alpha, -x_\alpha)$ obeying (3.16), for each $\alpha \in \mathcal{R}_+$. Hence, on a collection of $(n-3)$ -spheres in S^{n-1} our wave functions have nodes for positive values of g_α , but they still blow up for negative couplings when $n > 2$. Hence, for rank 3 and larger, the formal energy eigenstates at $g_\alpha < 0$ remain non-normalizable under the linear deformation (3.3). Passing to the deformed metric under which H becomes hermitian unfortunately does not change this, and so the \mathcal{PT} deformation in general does not enlarge the degeneracy of the energy spectrum. An exception occurs for $n=2$, which will be outlined below.

The conserved quantities and intertwiners naturally carry over to the deformed situation,

$$C_\epsilon = \text{res}(\mathcal{C}_\epsilon) \quad \text{and} \quad M_\epsilon = \text{res}(\mathcal{M}_\epsilon), \quad (3.18)$$

built from ‘doubly deformed’ angular momenta $\mathcal{L}_{ij}^\epsilon$ made from $x(\epsilon)$ and

$$\mathcal{D}_i^\epsilon = (\Gamma(\epsilon) \partial)_i - \sum_{\alpha \in \mathcal{R}_+} \frac{g_\alpha \alpha^i}{\alpha \cdot \Gamma(\epsilon) x} s_\alpha^\epsilon \quad \text{with} \quad s_\alpha^\epsilon = \Gamma(\epsilon) s_\alpha \Gamma(-\epsilon) \quad (3.19)$$

in the case of a linear deformation. Therefore, the superintegrability of the model is unchanged.

One may consider also nonlinear complex deformations of the coordinates. A particular one consists in a complex shift of each angle in a hyperspherical parametrization,

$$\begin{aligned} x^1(\epsilon) &= r \cos(\phi_1 + i\epsilon_1), \\ x^2(\epsilon) &= r \sin(\phi_1 + i\epsilon_1) \cos(\phi_2 + i\epsilon_2), \\ x^3(\epsilon) &= r \sin(\phi_1 + i\epsilon_1) \sin(\phi_2 + i\epsilon_2) \cos(\phi_3 + i\epsilon_3), \\ &\dots \\ x^{n-1}(\epsilon) &= r \sin(\phi_1 + i\epsilon_1) \sin(\phi_2 + i\epsilon_2) \cdots \sin(\phi_{n-2} + i\epsilon_{n-2}) \cos(\phi_{n-1} + i\epsilon_{n-1}), \\ x^n(\epsilon) &= r \sin(\phi_1 + i\epsilon_1) \sin(\phi_2 + i\epsilon_2) \cdots \sin(\phi_{n-2} + i\epsilon_{n-2}) \sin(\phi_{n-1} + i\epsilon_{n-1}). \end{aligned} \quad (3.20)$$

Such a deformation will (for $n > 2$) also modify the kinetic term $\frac{1}{2}L^2$. The obvious choice for \mathcal{P} is

$$\phi_i \mapsto -\phi_i \quad \Leftrightarrow \quad x^i \mapsto (-1)^{i+1}x^i. \quad (3.21)$$

The correspondingly deformed Hamiltonian is \mathcal{PT} invariant if this transformation is a symmetry of the root system, i.e. if it is contained in the Coxeter group extended by the outer automorphisms (symmetries of the Dynkin diagram).

The advantage of such a deformation is that the singular locus of the potential $U(\epsilon)$ and thus the zero set of the Vandermonde Δ_ϵ may be empty. This renders the formal energy eigenstates for $g_\alpha < 0$ normalizable and, hence, produces new towers of physical states for negative couplings. Due to $H(-g_\alpha) = H(g_\alpha + 1)$, these new states enlarge the state space for $g_\alpha > 1$. For integral g_α we can connect the two towers by a string of intertwiners.⁴ In the enlarged state space then acts an additional, ‘odd’ conserved charge,

$$\begin{aligned} Q_\epsilon^{(g',g'')} &= M_\epsilon^{(g'-1,g'')} M_\epsilon^{(g'-2,g'')} \dots M_\epsilon^{(1-g',g'')} \\ \Rightarrow Q_\epsilon^{(g',g'')} H_\epsilon^{(g',g'')} &= Q_\epsilon^{(g',g'')} H_\epsilon^{(1-g',g'')} = H_\epsilon^{(g',g'')} Q_\epsilon^{(g',g'')}, \end{aligned} \quad (3.22)$$

which intertwines between the $g' > 0$ and $g' \leq 0$ towers. In the potential-free frame,

$$\tilde{Q}_\epsilon^{(g',g'')} = \Delta_\epsilon^{-g',-g''} Q_\epsilon \Delta_\epsilon^{1-g',g''} : h_{\{\ell\}_\epsilon}^{(1-g',g'')} \mapsto h_{\{\ell'\}_\epsilon}^{(g',g'')} \quad (3.23)$$

relates the two Dunkl- and \mathcal{PT} -deformed harmonic polynomials to each other. Note that in contrast to $Q_\epsilon^{(g',g'')}$, the potential-free intertwiner $\tilde{Q}_\epsilon^{(g',g'')}$ is not conserved. The new odd charge squares to a polynomial in the conserved ‘even’ charges C and extends the algebra of conserved quantities to a nonlinear supersymmetric one. Due to the \mathcal{PT} regularization of the negative-coupling states, $Q_\epsilon^{(g',g'')}$ now has a regular action in the state space. In general there exist more than one intertwiner, giving rise to various such odd charges.

4 A_2 model

The simplest case to consider is the A_2 model, which is based on the roots

$$\mathcal{R}_+ = \left\{ e_1, \frac{1}{2}(e_1 + \sqrt{3}e_2), \frac{1}{2}(-e_1 + \sqrt{3}e_2) \right\}, \quad (4.1)$$

yielding the Coxeter reflections

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}. \quad (4.2)$$

Its spherical reduction yields the Pöschl-Teller model, which describes a particle on S^1 in the potential

$$U = \frac{9}{2} g(g-1) r^6 (x^1)^{-2} ((x^1)^2 - 3(x^2)^2)^{-2} = g(g-1) \frac{18(w\bar{w})^3}{(w^3 + \bar{w}^3)^2} = \frac{9}{2} g(g-1) \cos^{-2}(3\phi), \quad (4.3)$$

⁴This is also possible for odd half-integral couplings but does not yield an independent charge.

where we introduced a complex homogeneous \mathbb{R}^2 coordinate w and polar coordinates (r, ϕ) ,

$$w := x^1 + ix^2 = r e^{i\phi} \quad \Leftrightarrow \quad x^1 = r \cos \phi \quad \text{and} \quad x^2 = r \sin \phi. \quad (4.4)$$

Since A_2 is simply-laced, all couplings must coincide, $g_\alpha = g$. The two basic homogeneous polynomials invariant under $W = S_3$ are

$$\sigma_2 = (x^1)^2 + (x^2)^2 = w \bar{w} = r^2 \quad \text{and} \quad \sigma_3 = 3(x^1)^2 x^2 - (x^2)^3 \sim w^3 - \bar{w}^3 \sim r^3 \sin(3\phi). \quad (4.5)$$

Hence, $d_3 = 3$, $\{\ell\} = \ell_3$ and $\ell = 3\ell_3$, and we have the S_3 -invariant spectrum

$$E_\ell = \frac{1}{2} q^2 \quad \text{with} \quad q = \ell + 3g = 3(\ell_3 + g) \quad \text{and} \quad \text{deg}(E_\ell) = 1, \quad (4.6)$$

where ‘deg’ denotes the degeneracy. For $g > 0$ this implies $E_{\min} = \frac{9}{2}g^2$, but for $g < 0$ the spectrum goes down to zero energy. The Vandermonde factor takes the simple form

$$\Delta \sim (x^1)^3 - 3x^1(x^2)^2 \sim w^3 + \bar{w}^3 \sim r^3 \cos(3\phi), \quad (4.7)$$

and the Dunkl operator in the potential-free frame reads ($\rho = e^{2\pi i/3}$)

$$\tilde{\mathcal{D}}_w = \partial_w + \frac{3g w^2}{w^3 + \bar{w}^3} - g \left\{ \frac{1}{w + \bar{w}} s_0 + \frac{\rho}{\rho w + \bar{\rho} \bar{w}} s_+ + \frac{\bar{\rho}}{\bar{\rho} w + \rho \bar{w}} s_- \right\} \quad (4.8)$$

with the Coxeter reflections

$$s_0 : w \mapsto -\bar{w}, \quad s_+ : w \mapsto -\rho \bar{w}, \quad s_- : w \mapsto -\bar{\rho} \bar{w}. \quad (4.9)$$

Thus, the S_3 -invariant wave functions in the potential-free frame are (with $r^0 \rightarrow \ln r$)

$$\begin{aligned} h_\ell^{(g)} &= r^{2\ell+6g} (\tilde{\mathcal{D}}_w^3 - \tilde{\mathcal{D}}_{\bar{w}}^3)^{\ell_3} r^{-6g} \\ &\sim \sum_{k=0}^{\ell_3} \frac{\Gamma(1+\ell_3) \Gamma(g+k) \Gamma(g+\ell_3-k)}{\Gamma(2g+\ell_3) \Gamma(g) \Gamma(1+k) \Gamma(1+\ell_3-k)} w^{\ell-3k} (-\bar{w})^{3k} \\ &= \frac{\Gamma(g+\ell_3)}{\Gamma(2g+\ell_3)} {}_2F_1 \left(g, -\ell_3; 1-g-\ell_3; \left(-\frac{\bar{w}}{w} \right)^3 \right) w^\ell \\ &= \frac{\Gamma(g) \Gamma(\ell_3+1)}{\Gamma(2g+\ell_3)} P_{\ell_3}^{(-g-\ell_3, 2g-1)} \left(1+2 \left(\frac{\bar{w}}{w} \right)^3 \right) (-w)^\ell, \end{aligned} \quad (4.10)$$

expressed in terms of the hypergeometric function ${}_2F_1$ or the Jacobi polynomials $P_n^{(\alpha, \beta)}$. The gamma-function prefactors are irrelevant for $g > 0$ but are chosen such as to enable an analytic continuation to $g < 0$, which will become relevant in a while. A table of states for small values of ℓ can be found in appendix A.

The Dunkl-deformed angular momentum is given by

$$\mathcal{L} \equiv \mathcal{L}_{12} = x^1 \mathcal{D}_2 - x^2 \mathcal{D}_1 = i(w \mathcal{D}_w - \bar{w} \mathcal{D}_{\bar{w}}) \quad (4.11)$$

with $\mathcal{D}_w = \tilde{\mathcal{D}}_w - \frac{3g w^2}{w^3 + \bar{w}^3}$. From this we can build only one algebraically independent S_3 -symmetric polynomial (case D),

$$\mathcal{C}_2 = \mathcal{L}^2 = -2\mathcal{H} + g^2(s_0 + s_+ + s_-)^2, \quad (4.12)$$

whose restriction C_2 to S_3 -symmetric functions provides the Pöschl-Teller Hamiltonian minus its ground-state energy. The single basic S_3 -antiinvariant polynomial (case A) is \mathcal{L} itself, from which we get

$$\mathcal{M}_1 = \mathcal{L} \quad \Rightarrow \quad M_1 \equiv \text{res}(\mathcal{L}) = i(w\partial_w - \bar{w}\partial_{\bar{w}}) - 3ig \frac{w^3 - \bar{w}^3}{w^3 + \bar{w}^3} = \partial_\phi + 3g \tan(3\phi), \quad (4.13)$$

which obeys

$$M_1^{(g)} H^{(g)} = H^{(g+1)} M_1^{(g)} \quad \text{and} \quad M_1^{(1-g)} H^{(g)} = H^{(g-1)} M_1^{(1-g)}. \quad (4.14)$$

Because \mathcal{M}_1 is linear in \mathcal{L} , in this case it is also true that

$$\widetilde{M}_1 \equiv \Delta^{-g} M_1 \Delta^g = \text{res}(\widetilde{\mathcal{L}}) = i \text{res}(w\widetilde{\mathcal{D}}_w - \bar{w}\widetilde{\mathcal{D}}_{\bar{w}}) = L = i(w\partial_w - \bar{w}\partial_{\bar{w}}) = \partial_\phi, \quad (4.15)$$

which exceptionally does not depend on g . The ladder relation for the deformed harmonic polynomials (remember $\text{deg}(E_\ell) = 1$),

$$\widetilde{M}_1 h_\ell^{(g)} = \Delta h_{\ell-3}^{(g+1)} \quad \Leftrightarrow \quad \partial_\phi h_\ell^{(g)} = r^3 \cos(3\phi) h_{\ell-3}^{(g+1)}, \quad (4.16)$$

may for positive integer g be iterated to generate them from the free ($g=0$) ones,

$$\begin{aligned} h_\ell^{(g>0)} &= (\Delta^{-1} \widetilde{M}_1)^g h_{\ell+3g}^{(0)} = r^{-3g} (\cos^{-1}(3\phi) \partial_\phi)^g h_{\ell+3g}^{(0)} \\ &\sim ((w^3 + \bar{w}^3)^{-1} (w\partial_w - \bar{w}\partial_{\bar{w}}))^g (w^{\ell+3g} + (-\bar{w})^{\ell+3g}), \end{aligned} \quad (4.17)$$

which reproduces the analytic expression (4.10). Eventually, the iteration hits the kernel of \widetilde{M}_1 , i.e. $h_0^{(g)} = 1$ corresponding to the ground state, where it ceases.

The $g < 0$ states can as well be obtained directly from (4.14), which also implies that

$$\widetilde{M}_1 \Delta^{2g-1} h_\ell^{(g)} = \Delta^{2g-2} h_{\ell+3}^{(g-1)} \quad \Leftrightarrow \quad (\partial_\phi - 3(2g-1) \tan(3\phi)) h_\ell^{(g)} = r^{-3} \cos^{-1}(3\phi) h_{\ell+3}^{(g-1)}. \quad (4.18)$$

Its iteration for negative integer g produces

$$\begin{aligned} h_\ell^{(g<0)} &= \Delta^{-2g} (\widetilde{M}_1 \Delta^{-1})^{-g} h_{\ell+3g}^{(0)} = r^{-3g} \cos^{-2g}(3\phi) (\partial_\phi \cos^{-1}(3\phi))^{-g} h_{\ell+3g}^{(0)} \\ &\sim (w^3 + \bar{w}^3)^{-2g} ((w\partial_w - \bar{w}\partial_{\bar{w}}) (w^3 + \bar{w}^3)^{-1})^{-g} (w^{\ell+3g} + (-\bar{w})^{\ell+3g}), \end{aligned} \quad (4.19)$$

which may be checked to reproduce the analytic continuation of (4.10) to $g < 0$. However, without \mathcal{PT} deformation the full wave functions $v_\ell^{(g<0)}$ are not normalizable. For illustration, in appendix A we display the polynomials $h_\ell^{(g)}$ for $g = -2, -1, 0, 1, 2$ and $q \leq 12$.

Let us take a look at the possible \mathcal{PT} involutions and the compatible complex deformations for the Pöschl-Teller model. The only order-2 elements in S_3 are the Coxeter reflections about the lines perpendicular to the roots, so without loss of generality we may fix \mathcal{P} as the action of s_0 , which belongs to the root $\gamma = \sqrt{2}e_1$ and is a reflection on the x^2 -axis,

$$\hat{\gamma} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \Leftrightarrow \quad \mathcal{P} : s_0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (4.20)$$

Obviously, P_- and P_+ project onto the x^1 axis and the x^2 axis, respectively. As usual, \mathcal{T} is complex conjugation, but please be aware that this does *not* swap w with \bar{w} because the complex linear combination of the real coordinates x^1 and x^2 is unaffected by \mathcal{T} .

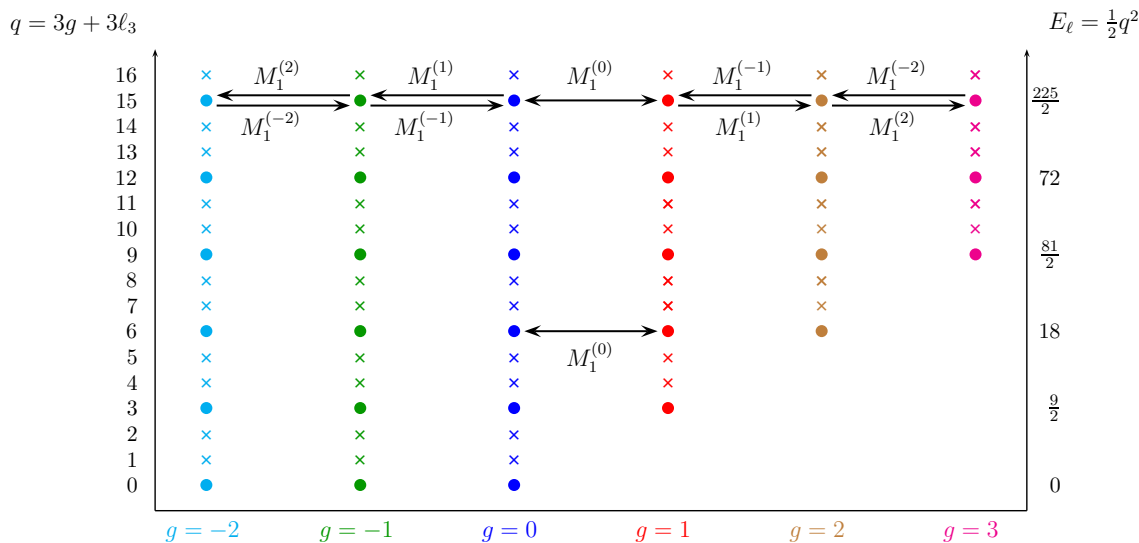


Figure 1. Action of the intertwiner $M_1^{(g)}$ on the A_2 spectrum for small values of g .

The coset $\text{SO}(2, \mathbb{C})/\text{SO}(2, \mathbb{R})$ is one-dimensional and parametrized as

$$\Gamma(\epsilon) = e^{\epsilon G} = \exp\left\{\epsilon \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right\} = \begin{pmatrix} \cosh(\epsilon) & -i \sinh(\epsilon) \\ i \sinh(\epsilon) & \cosh(\epsilon) \end{pmatrix} = \cosh(\epsilon) \mathbb{1} + \sinh(\epsilon) G. \quad (4.21)$$

Since there is just one plane, necessarily $\hat{\eta} = e_2$ and $P_{\gamma \wedge \eta} = \mathbb{1}$. Clearly, s_0 and G anticommute, and so all such complex deformations

$$(x^1, x^2) \mapsto (x^1(\epsilon), x^2(\epsilon)) = (\cosh(\epsilon)x^1 - i \sinh(\epsilon)x^2, \cosh(\epsilon)x^2 + i \sinh(\epsilon)x^1) \quad (4.22)$$

are \mathcal{PT} symmetric. In polar coordinates, this deformation takes a particularly simple form,

$$(r, \phi) \mapsto (r(\epsilon), \phi(\epsilon)) = (r, \phi + i\epsilon), \quad (4.23)$$

but for the complex combinations (w, \bar{w}) one has to keep in mind that \mathcal{T} does not conjugate

$$w(\epsilon) = e^{-\epsilon} w \quad \text{or} \quad \bar{w}(\epsilon) = e^\epsilon \bar{w} \quad (4.24)$$

but only flips the sign of ϵ . For any root α contributing to the potential, the singular locus of $U(\epsilon)$ for $\epsilon \neq 0$ lies at

$$\text{sing}(\alpha) = \{x \mid \alpha \cdot x = 0 \quad \& \quad \alpha \cdot Gx = 0\} = \emptyset \quad \forall \alpha, \quad (4.25)$$

since iG is a $\pi/2$ rotation in our plane. Hence, the deformed potential

$$U(\epsilon, \phi) = 9g(g-1) \frac{1 + \cosh(6\epsilon) \cos(6\phi) + i \sinh(6\epsilon) \sin(6\phi)}{(\cosh(6\epsilon) + \cos(6\phi))^2} \quad (4.26)$$

as well as the deformed wave functions (see (4.10))

$$v_{\ell \epsilon}^{(g)}(w, \bar{w}) = r^{-q} \Delta_\epsilon^g h_\ell^{(g)}(e^{-\epsilon} w, e^\epsilon \bar{w}) \quad (4.27)$$

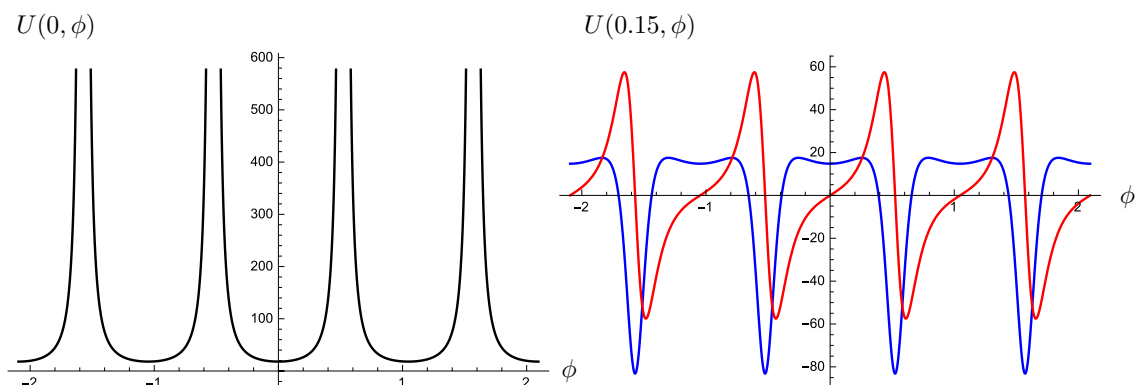


Figure 2. Singular ($\epsilon=0$) and regularized ($\epsilon=0.15$) A_2 potential $U(\epsilon, \phi)$ for $g=2$. The blue curve displays $\text{Re } U$, the red one shows $\text{Im } U$.

for $g < 0$ are free of singularities because

$$\Delta_\epsilon \sim e^{-3\epsilon} w^3 + e^{3\epsilon} \bar{w}^3 \sim r^3 (\cosh(3\epsilon) \cos(3\phi) - i \sinh(3\epsilon) \sin(3\phi)) \quad (4.28)$$

is regular everywhere. Because the complex deformation is merely a constant shift of the polar angle, the angular momentum and the potential-free intertwiner exceptionally remain undeformed,

$$\widetilde{M}_{1\epsilon} = L_\epsilon = i(w\partial_w - \bar{w}\partial_{\bar{w}}) = \partial_\phi. \quad (4.29)$$

Our intertwiner \widetilde{M}_1 has a simple kernel. Since

$$h_0^{(g>0)} = 1 \quad \text{and} \quad h_{6|g|}^{(g\leq 0)} = (w\bar{w})^{3|g|} = r^{6|g|}, \quad (4.30)$$

\widetilde{M}_1 at any fixed g annihilates this one state but no other one. Our \mathcal{PT} deformation leads to a rough doubling of the energy eigenstates, because the spectrum of $H^{(g)}$ now has to be joined with that of $H^{(1-g)}$. So, for a given $g > \frac{1}{2}$, we encounter two towers of states with $E = \frac{1}{2}q^2$, for

$$q = \ell + 3g \quad \text{and} \quad q = \ell + 3(1-g) \quad \text{with} \quad \ell = 3\ell_3 = 0, 3, 6, \dots, \quad (4.31)$$

where the second tower yields negative q for $\ell < 3(g-1)$. When g is integral or half-integral, the two towers meet, so the degeneracy doubles. However, it turns out that flipping the sign of q yields the same state again, and so for positive integral g the level degeneracy becomes

$$\text{deg}(E_q^{(g)}) = \begin{cases} 1 & \text{for } q = 0, 3, 6, \dots, 3(g-1) \\ 2 & \text{for } q = 3g, 3(g+1), 3(g+2), \dots \end{cases}. \quad (4.32)$$

The new states are again given by (4.10), where in the limit of negative integral g the zeros of the Jacobi polynomial are cancelled by poles of the prefactor, so a careful limit has to be taken. Such a state structure is common for systems possessing a hidden supersymmetric structure [29, 30], which is indeed the case here and revealed by the additional ‘odd’ conserved charge

$$Q_\epsilon^{(g)} = M_{1\epsilon}^{(g-1)} M_{1\epsilon}^{(g-2)} \dots M_{1\epsilon}^{(2-g)} M_{1\epsilon}^{(1-g)} = \Delta_\epsilon^g (\Delta_\epsilon^{-1} \partial_\phi)^{2g-1} \Delta_\epsilon^{g-1} = \Delta_\epsilon^g \widetilde{Q}_\epsilon^{(g)} \Delta_\epsilon^{g-1}. \quad (4.33)$$

In the potential-free frame, it simplifies to

$$\tilde{Q}_\epsilon^{(g)} = (\Delta_\epsilon^{-1} \partial_\phi)^{2g-1} = ((e^{-3\epsilon} w^3 + e^{3\epsilon} \bar{w}^3)^{-1} i(w \partial_w - \bar{w} \partial_{\bar{w}}))^{2g-1} : h_{\ell_\epsilon}^{(1-g)} \mapsto h_{\ell-3(2g-1)\epsilon}^{(g)} \quad (4.34)$$

and clearly obeys the intertwining relation

$$\tilde{Q}_\epsilon^{(g)} \tilde{H}_\epsilon^{(1-g)} = \tilde{H}_\epsilon^{(g)} \tilde{Q}_\epsilon^{(g)}, \quad (4.35)$$

relating the deformed harmonic polynomials at couplings $1-g$ and g . Since the transition from $h_\ell^{(g)}$ to $v_\ell^{(g)}$ involves the (g -dependent) factor of Δ^g and $\tilde{H}^{(1-g)} \neq \tilde{H}^{(g)}$, only in the potential frame this intertwining relation becomes a commutation relation,

$$[Q_\epsilon^{(g)}, H_\epsilon^{(g)}] = [Q_\epsilon^{(g)}, H_\epsilon^{(1-g)}] = 0. \quad (4.36)$$

The g singlet states (for $q < 3g$) are annihilated by $Q_\epsilon^{(g)}$,

$$Q_\epsilon^{(g)} v_{\ell_\epsilon}^{(1-g)} = 0 \quad \text{for } \ell_3 = g-1, g, g+1, \dots, 2g-2, \quad (4.37)$$

at energies

$$E_q = \frac{1}{2} q^2 = \frac{9}{2} (\ell_3 + 1 - g)^2 = \frac{9}{2} j^2 \quad \text{for } j = 0, 1, \dots, g-1. \quad (4.38)$$

For all other states, $Q_\epsilon^{(g)}$ maps the doublet partners to each other. The square of $Q_\epsilon^{(g)}$ is a polynomial in the Hamiltonian,

$$(Q_\epsilon^{(g)})^2 \propto \prod_{j=1-g}^{g-1} \left(H_\epsilon^{(g)} - \frac{9}{2} j^2 \right) = H_\epsilon^{(g)} \prod_{j=1}^{g-1} \left(H_\epsilon^{(g)} - \frac{9}{2} j^2 \right)^2, \quad (4.39)$$

which also reveals the properties of the combined spectrum.

5 G_2 model

The A_2 model is the first of an infinite list of dihedral $I_2(p)$ models, with

$$I_2(2) = A_1 \oplus A_1, \quad I_2(3) = A_2, \quad I_2(4) = BC_2, \quad I_2(6) = G_2, \quad (5.1)$$

and where for odd p all couplings must coincide while for even p the root system decomposes into two $I_2(\frac{p}{2})$ subsystems with two couplings g_S and g_L . Let us illustrate the latter situation on the G_2 example, since it can be obtained by a superposition of two A_2 systems (with a $\pi/2$ rotation),

$$\mathcal{R}_+ = \left\{ e_1, \frac{1}{2}(e_1 + \sqrt{3}e_2), \frac{1}{2}(-e_1 + \sqrt{3}e_2), \sqrt{3}e_2, \frac{1}{2}(3e_1 + \sqrt{3}e_2), \frac{1}{2}(-3e_1 + \sqrt{3}e_2) \right\}, \quad (5.2)$$

presented in the previous section. The corresponding Coxeter reflections read

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}. \quad (5.3)$$

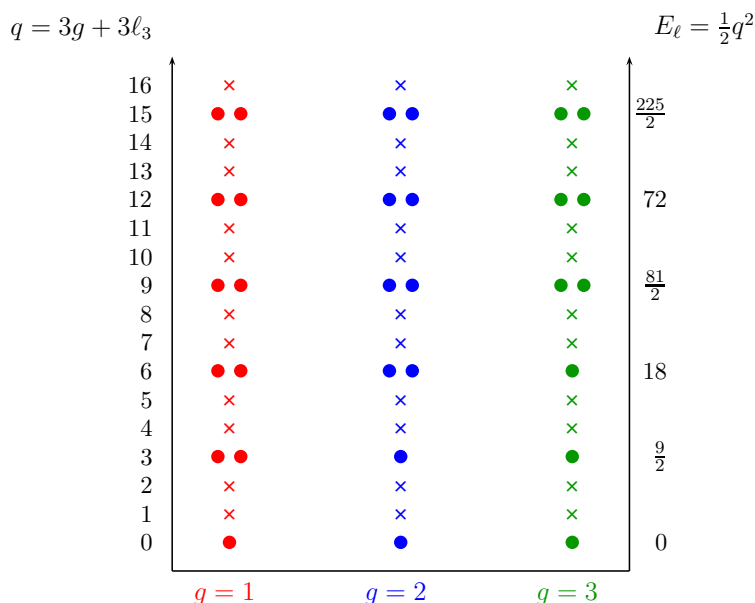


Figure 3. Joint spectrum of $H_\epsilon^{(g)}$ and $H_\epsilon^{(1-g)}$ after the \mathcal{PT} deformation for the A_2 model.

The potential is easily derived,

$$\begin{aligned}
 U &= \frac{9}{2} g_S(g_S-1) r^6 (x^1)^{-2} ((x^1)^2 - 3(x^2)^2)^{-2} + \frac{9}{2} g_L(g_L-1) r^6 (x^2)^{-2} ((x^2)^2 - 3(x^1)^2)^{-2} \\
 &= g_S(g_S-1) \frac{18 (w\bar{w})^3}{(w^3 + \bar{w}^3)^2} - g_L(g_L-1) \frac{18 (w\bar{w})^3}{(w^3 - \bar{w}^3)^2} \\
 &= \frac{9}{2} g_S(g_S-1) \cos^{-2}(3\phi) + \frac{9}{2} g_L(g_L-1) \sin^{-2}(3\phi),
 \end{aligned} \tag{5.4}$$

and exhibits the two subsystems. The Coxeter group is the dihedral group \mathcal{D}_6 with 12 elements, which maps short roots to short roots and long roots to long roots. The two basic \mathcal{D}_6 -invariant homogeneous polynomials are

$$\sigma_2 = w\bar{w} = r^2 \quad \text{and} \quad \sigma_6 \sim w^6 + \bar{w}^6 \sim r^6 \cos(6\phi). \tag{5.5}$$

Hence, $d_3 = 6$, $\{\ell\} = \ell_3$ and $\ell = 6\ell_3$, and we have the \mathcal{D}_6 -invariant spectrum

$$E_\ell = \frac{1}{2} q^2 \quad \text{with} \quad q = \ell + 3g_S + 3g_L = 3(2\ell_3 + g_S + g_L) \quad \text{and} \quad \deg(E_\ell) = 1. \tag{5.6}$$

For $g_L=0$ or $g_S=0$, we fall back to the Pöschl-Teller model, but only its ‘even’ states survive the more restrictive Weyl invariance requirement, as ℓ must be a multiple of 6 now. Compared to the Pöschl-Teller model, the density of energy eigenstates is cut in half. The Vandermonde factorizes,

$$\Delta = \Delta_S \Delta_L \quad \text{with} \quad \Delta_S \sim w^3 + \bar{w}^3 \sim r^3 \cos(3\phi) \quad \text{and} \quad \Delta_L \sim w^3 - \bar{w}^3 \sim r^3 \sin(3\phi), \tag{5.7}$$

and the (potential-free) Dunkl operator reads

$$\begin{aligned} \tilde{\mathcal{D}}_w = \partial_w + \frac{3g_S w^2}{w^3 + \bar{w}^3} - g_S \left\{ \frac{1}{w + \bar{w}} s_0 + \frac{\rho}{\rho w + \bar{\rho} \bar{w}} s_+ + \frac{\bar{\rho}}{\bar{\rho} w + \rho \bar{w}} s_- \right\} \\ + \frac{3g_L w^2}{w^3 - \bar{w}^3} - g_L \left\{ \frac{1}{w - \bar{w}} \bar{s}_0 + \frac{\rho}{\rho w - \bar{\rho} \bar{w}} \bar{s}_+ + \frac{\bar{\rho}}{\bar{\rho} w - \rho \bar{w}} \bar{s}_- \right\} \end{aligned} \quad (5.8)$$

with the additional Coxeter reflections

$$\bar{s}_0 : w \mapsto +\bar{w}, \quad \bar{s}_+ : w \mapsto +\rho\bar{w}, \quad \bar{s}_- : w \mapsto +\bar{\rho}\bar{w} . \quad (5.9)$$

With these ingredients, the wave functions in the potential-free frame can be constructed,

$$\begin{aligned} h_\ell^{(g_S, g_L)} = r^{2\ell + 6g_S + 6g_L} (\tilde{\mathcal{D}}_w^6 + \tilde{\mathcal{D}}_{\bar{w}}^6)^{\ell_3} r^{-6g_S - 6g_L} \\ \sim P_{\ell_3}^{(g_S - \frac{1}{2}, g_L - \frac{1}{2})} \left(\frac{1}{2} \left(\frac{w}{\bar{w}} \right)^3 + \frac{1}{2} \left(\frac{\bar{w}}{w} \right)^3 \right) (w \bar{w})^\ell \quad \text{with } \ell = 6\ell_3 . \end{aligned} \quad (5.10)$$

Since only even powers of w or \bar{w} occur, its form is a bit simpler than (4.10). Some low-lying wave functions are given explicitly in appendix B.

The Dunkl-deformed angular momentum is given by

$$\mathcal{L} = i(w\mathcal{D}_w - \bar{w}\mathcal{D}_{\bar{w}}) \quad \text{with} \quad \mathcal{D}_w = \tilde{\mathcal{D}}_w - \frac{3g_S w^2}{w^3 + \bar{w}^3} - \frac{3g_L w^2}{w^3 - \bar{w}^3} \quad (5.11)$$

and essentially squares to the Hamiltonian,

$$\mathcal{C}_2 = \mathcal{L}^2 = -2\mathcal{H} + [g_S(s_0 + s_+ + s_-) + g_L(\bar{s}_0 + \bar{s}_+ + \bar{s}_-)]^2, \quad (5.12)$$

via $H = \text{res}(\mathcal{H})$. Again, for generic g this is the only conserved charge (case D). Like before, \mathcal{L} is Weyl antiinvariant (case A), thus providing the basic intertwiner

$$\begin{aligned} M_1 \equiv \text{res}(\mathcal{L}) = i(w\partial_w - \bar{w}\partial_{\bar{w}}) - 3ig_S \frac{w^3 - \bar{w}^3}{w^3 + \bar{w}^3} - 3ig_L \frac{w^3 + \bar{w}^3}{w^3 - \bar{w}^3} \\ = \partial_\phi + 3g_S \tan(3\phi) - 3g_L \cot(3\phi) . \end{aligned} \quad (5.13)$$

The intertwining relations read

$$\begin{aligned} M_1^{(g_S, g_L)} H^{(g_S, g_L)} &= H^{(g_S+1, g_L+1)} M_1^{(g_S, g_L)}, \\ M_1^{(1-g_S, 1-g_L)} H^{(g_S, g_L)} &= H^{(g_S-1, g_L-1)} M_1^{(1-g_S, 1-g_L)}, \\ M_1^{(1-g_S, g_L)} H^{(g_S, g_L)} &= H^{(g_S-1, g_L+1)} M_1^{(1-g_S, g_L)}, \\ M_1^{(g_S, 1-g_L)} H^{(g_S, g_L)} &= H^{(g_S+1, g_L-1)} M_1^{(g_S, 1-g_L)}. \end{aligned} \quad (5.14)$$

and again the potential-free intertwiner trivializes,

$$\widetilde{M}_1 \equiv \Delta_L^{-g_L} \Delta_S^{-g_S} M_1 \Delta_S^{g_S} \Delta_L^{g_L} = L = i(w\partial_w - \bar{w}\partial_{\bar{w}}) = \partial_\phi. \quad (5.15)$$

The corresponding ladder relations for the wave functions are

$$\begin{aligned}
 \partial_\phi h_\ell^{(g_S, g_L)} &= \Delta_S \Delta_L h_{\ell-6}^{(g_S+1, g_L+1)}, \\
 \partial_\phi \Delta_S^{2g_S-1} h_\ell^{(g_S, g_L)} &= \Delta_S^{2g_S-2} \Delta_L h_\ell^{(g_S-1, g_L+1)}, \\
 \partial_\phi \Delta_L^{2g_L-1} h_\ell^{(g_S, g_L)} &= \Delta_S \Delta_L^{2g_L-2} h_\ell^{(g_S+1, g_L-1)}, \\
 \partial_\phi \Delta_S^{2g_S-1} \Delta_L^{2g_L-1} h_\ell^{(g_S, g_L)} &= \Delta_S^{2g_S-2} \Delta_L^{2g_L-2} h_{\ell+6}^{(g_S-1, g_L-1)},
 \end{aligned} \tag{5.16}$$

with special relations for the vanishing of one of the couplings,

$$\begin{aligned}
 \partial_\phi h_\ell^{(g_S, 0)} &= \Delta_S h_{\ell-3}^{(g_S+1, 0)} & \text{and} & & \partial_\phi \Delta_S^{2g_S-1} h_\ell^{(g_S, 0)} &= \Delta_S^{2g_S-2} h_{\ell+3}^{(g_S-1, 0)}, \\
 \partial_\phi h_\ell^{(0, g_L)} &= \Delta_L h_{\ell-3}^{(0, g_L+1)} & \text{and} & & \partial_\phi \Delta_L^{2g_L-1} h_\ell^{(0, g_L)} &= \Delta_L^{2g_L-2} h_{\ell+3}^{(0, g_L-1)},
 \end{aligned} \tag{5.17}$$

where we intermediately allow Weyl ‘half-invariant’ states at $\ell = 3, 9, 12, \dots$. For integral couplings the above relations may be iterated for the alternative wave function reconstruction

$$\begin{aligned}
 h_\ell^{(g_S, g_L)} &= \begin{cases} (\Delta_S^{-1} \Delta_L^{-1} \partial_\phi)^{g_L} (\Delta_S^{-1} \partial_\phi)^{g_S-g_L} h_{\ell+3g_S+3g_L}^{(0,0)} & \text{for } g_S \geq g_L \geq 0 \\ (\Delta_S^{-1} \Delta_L^{-1} \partial_\phi)^{g_S} (\Delta_L^{-1} \partial_\phi)^{g_L-g_S} h_{\ell+3g_S+3g_L}^{(0,0)} & \text{for } g_L \geq g_S \geq 0 \end{cases}, \\
 h_\ell^{(g_S, g_L)} &= \begin{cases} \Delta_S^{-2g_S} (\Delta_L^{-1} \partial_\phi \Delta_S^{-1})^{g_L} (\partial_\phi \Delta_S^{-1})^{-g_S-g_L} h_{\ell+3g_S+3g_L}^{(0,0)} & \text{for } -g_S \geq g_L \geq 0 \\ \Delta_S^{-2g_S} (\Delta_L^{-1} \partial_\phi \Delta_S^{-1})^{-g_S} (\Delta_L^{-1} \partial_\phi)^{g_S+g_L} h_{\ell+3g_S+3g_L}^{(0,0)} & \text{for } g_L \geq -g_S \geq 0 \end{cases}
 \end{aligned} \tag{5.18}$$

and similarly for the four other domains of (g_S, g_L) , starting from

$$h_{\ell+3g_S+3g_L}^{(0,0)} \sim w^{\ell+3g_S+3g_L} + (-\bar{w})^{\ell+3g_S+3g_L} \quad \text{with } \ell = 0, 6, 12, \dots \tag{5.20}$$

When g_S and g_L are non-negative, the wave functions are normalizable. For integral couplings, the \mathcal{D}_6 -invariant energy spectrum $E_\ell = \frac{1}{2}q^2$ is non-empty only for

$$q = \begin{cases} 0 \pmod 6 & \text{if } g_S+g_L \text{ is even} \\ 3 \pmod 6 & \text{if } g_S+g_L \text{ is odd} \end{cases} \quad \text{and} \quad q \geq 3(g_S + g_L). \tag{5.21}$$

When a coupling turns negative, the zeros of the corresponding Vandermonde factor render the full wave function $v_\ell^{(g_S, g_L)}$ non-normalizable. In order to make these states physical, we turn to the \mathcal{PT} deformation.

The order-2 elements in \mathcal{D}_6 are precisely the 6 root reflections, so there are only two inequivalent cases, corresponding to $\hat{\gamma} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and to $\hat{\gamma} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$,

$$\mathcal{P} : s_0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \bar{\mathcal{P}} : \bar{s}_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.22}$$

However, since all linear complex coordinate deformations are admissible in two dimensions, the discussion is identical to the A_2 case, and the potential and wave functions lose all their singularities,

$$\begin{aligned}
 U(\epsilon) &= 9 g_S (g_S - 1) \frac{1 + \cosh(6\epsilon) \cos(6\phi) + i \sinh(6\epsilon) \sin(6\phi)}{(\cosh(6\epsilon) + \cos(6\phi))^2} \\
 &+ 9 g_L (g_L + 1) \frac{1 - \cosh(6\epsilon) \cos(6\phi) - i \sinh(6\epsilon) \sin(6\phi)}{(\cosh(6\epsilon) - \cos(6\phi))^2},
 \end{aligned} \tag{5.23}$$

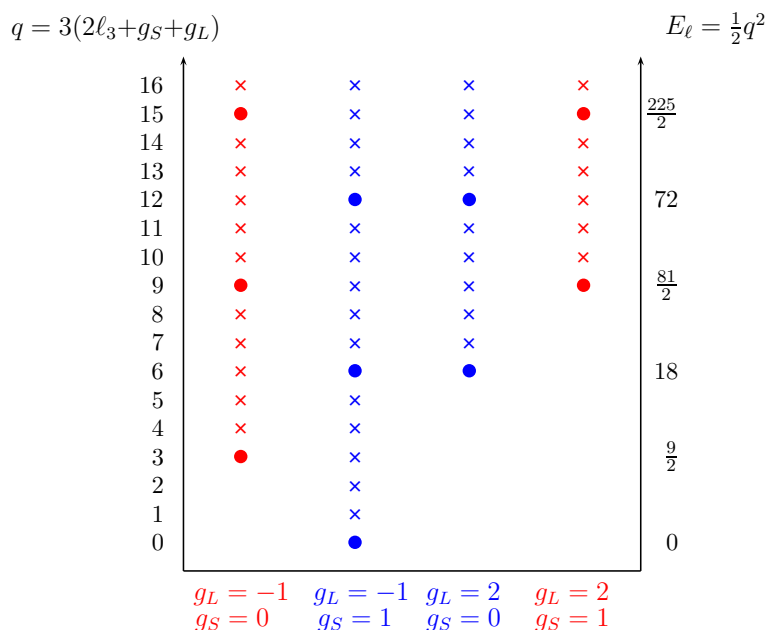


Figure 4. The spectra of the G_2 Hamiltonians $H_\epsilon^{(g_S, g_L)}$ for the four towers should be joined for the full \mathcal{PT} -symmetric extension. The blue and red towers are distinguished by q taking odd and even integer values, respectively.

$$\begin{aligned} \Delta_{S\epsilon} &\sim e^{-3\epsilon} w^3 + e^{3\epsilon} \bar{w}^3 \sim r^3 (\cosh(3\epsilon) \cos(3\phi) - i \sinh(3\epsilon) \sin(3\phi)), \\ \Delta_{L\epsilon} &\sim e^{-3\epsilon} w^3 - e^{3\epsilon} \bar{w}^3 \sim r^3 (\cosh(3\epsilon) \sin(3\phi) + i \sinh(3\epsilon) \cos(3\phi)). \end{aligned} \quad (5.24)$$

The \mathcal{PT} deformation now leads to an approximate quadrupling of the eigenstates because

$$H_\epsilon^{(g_S, g_L)} = H_\epsilon^{(1-g_S, g_L)} = H_\epsilon^{(g_S, 1-g_L)} = H_\epsilon^{(1-g_S, 1-g_L)} \quad (5.25)$$

tells us to join four towers of states. Let us look at positive integral couplings (g_S, g_L) . Then, (5.6) implies that the first and fourth tower from (5.25) coincide, and likewise do the second and third tower. Depending on whether $g_S + g_L$ is even or odd, one pair of towers sits at $q = 0, 6, 12, \dots$ and the other one at $q = 3, 9, 15, \dots$. Therefore, the density of energy eigenstates is about the same as in the A_2 model. Like in the latter though, some states are missing for small values of q , since the towers do not reach all the way down to zero (see (5.21)).

When g_S and g_L are positive integers, we can write down an additional ‘odd’ conserved charge

$$Q_\epsilon^{(g_S, g_L)} : v_\ell^{(1-g_S, 1-g_L)} \mapsto v_{\ell-6(g_S+g_L-1)}^{(g_S, g_L)} \epsilon, \quad (5.26)$$

whose explicit form reads

$$\begin{aligned}
 Q_\epsilon^{(g_S, g_L)} &= \tag{5.27} \\
 &= \begin{cases} \left(\prod_{j=g_S+g_L-1}^{2g_S-2} M_{1\epsilon}^{(1-g_S-j, 2-g_L-2g_S+j)} \right) \left(\prod_{i=0}^{g_S+g_L-2} M_{1\epsilon}^{(1-g_S-i, 1-g_L-i)} \right) & \text{for } g_S \geq g_L \\ \left(\prod_{j=g_S+g_L-1}^{2g_L-2} M_{1\epsilon}^{(2-g_S-2g_L+j, 1-g_L-j)} \right) \left(\prod_{i=0}^{g_S+g_L-2} M_{1\epsilon}^{(1-g_S-i, 1-g_L-i)} \right) & \text{for } g_L \geq g_S \end{cases}
 \end{aligned}$$

where the product order must be assumed from right to left due to noncommuting action of the intertwining operators. The potential-free form reads

$$\tilde{Q}_\epsilon^{(g_S, g_L)} = \begin{cases} (\Delta_{S\epsilon}^{-1} \Delta_{L\epsilon}^{-1} \partial_\phi)^{g_L} (\Delta_{S\epsilon}^{-1} \partial_\phi)^{2(g_S-g_L)} (\Delta_{S\epsilon}^{-1} \Delta_{L\epsilon}^{-1} \partial_\phi)^{g_L-1} & \text{for } g_S \geq g_L \\ (\Delta_{S\epsilon}^{-1} \Delta_{L\epsilon}^{-1} \partial_\phi)^{g_S} (\Delta_{L\epsilon}^{-1} \partial_\phi)^{2(g_L-g_S)} (\Delta_{S\epsilon}^{-1} \Delta_{L\epsilon}^{-1} \partial_\phi)^{g_S-1} & \text{for } g_L \geq g_S \end{cases}. \tag{5.28}$$

The form (5.27) or (5.28) represents an action $(g_S, g_L) \mapsto (1-g_S, 1-g_L)$ on the couplings. Analogously to (4.35), $\tilde{Q}_\epsilon^{(g_S, g_L)}$ obeys an intertwining relation, while $Q_\epsilon^{(g_S, g_L)}$ commutes with the potential-frame Hamiltonian as in (4.36). There exist other admissible actions like $(1-g_S, g_L) \mapsto (g_S, 1-g_L)$ which only produce different factorizations of the same operator (5.27) but no new conserved charges, see figure 5. For $g_S \geq g_L$, $Q_\epsilon^{(g_S, g_L)}$ annihilates the singlet states with energies

$$E(g_S, g_L; j) = \begin{cases} \frac{9}{2}j^2 & \text{for } j - g_S + g_L < 0 \\ \frac{9}{2}(g_L - g_S + j)^2 & \text{for } j - g_S + g_L \geq 0 \end{cases}, \quad j = 1, \dots, g_S - 1. \tag{5.29}$$

For $g_L \geq g_S$, the roles of g_L and g_S are reversed. In analogy with the A_2 case, (5.27) squares to a polynomial in the Hamiltonian [31],

$$(Q_\epsilon^{(g_S, g_L)})^2 \propto \begin{cases} H_\epsilon^{(g_S, g_L)} \prod_{j=1}^{g_S-1} (H_\epsilon^{(g_S, g_L)} - E(g_S, g_L; j))^2 & \text{for } g_S \geq g_L \geq 0 \\ H_\epsilon^{(g_S, g_L)} \prod_{j=1}^{g_L-1} (H_\epsilon^{(g_S, g_L)} - E(g_L, g_S; j))^2 & \text{for } g_L \geq g_S \geq 0 \end{cases}. \tag{5.30}$$

The structure presented in the last two sections are easily generalized to all dihedral $I_2(p)$ models. Essentially, w^3 is replaced by w^p or $w^{p/2}$, $\ell = p \ell_3$, ρ becomes a p th root of unity, and the intertwiner shifts $(g_S, g_L, \ell) \rightarrow (g_S+1, g_L+1, \ell-p)$. The wave-function formulæ (4.10) and (5.10) generalize without any change after the first line.

6 AD_3 model

A much richer and less trivial situation appears one dimension higher, i.e. at rank 3. To reduce index cluttering, we redenote the coordinates as

$$x = (x^1, x^2, x^3) =: (x, y, z). \tag{6.1}$$

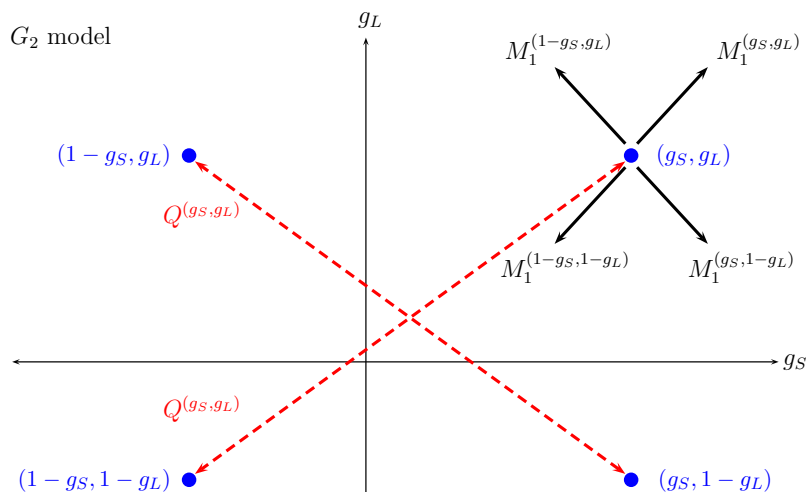


Figure 5. Action of the intertwining operators and ‘odd’ conserved charges in the G_2 model.

The set of positive roots can be chosen as

$$\mathcal{R}_+ = \{e_x + e_y, e_x - e_y, e_x + e_z, e_x - e_z, e_y + e_z, e_y - e_z\}. \quad (6.2)$$

We consider the A_3 (or D_3) Calogero model spherically reduced to what we have named the tetrahexahedric model. Here, a particle moves on the 2-sphere with the potential

$$U = 2g(g-1)(x^2 + y^2 + z^2) \left(\frac{x^2 + y^2}{(x^2 - y^2)^2} + \frac{y^2 + z^2}{(y^2 - z^2)^2} + \frac{z^2 + x^2}{(z^2 - x^2)^2} \right). \quad (6.3)$$

Since $W = S_4$ has just one orbit on the root system, again we have a single coupling, $g_\alpha = g$. It can be generated by

$$s_{x-y} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad s_{y-z} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad s_{y+z} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}. \quad (6.4)$$

The three basic S_4 -invariant homogeneous polynomials read

$$\sigma_2 = x^2 + y^2 + z^2 =: r^2, \quad \sigma_3 = xyz, \quad \sigma_4 = x^4 + y^4 + z^4, \quad (6.5)$$

and therefore $\{\ell\} = (\ell_3, \ell_4)$ and $\ell = 3\ell_3 + 4\ell_4$, and the energy levels take the form

$$E_\ell = \frac{1}{2}q(q+1) \quad \text{with} \quad q = \ell + 6g = 3\ell_3 + 4\ell_4 + 6g \quad (6.6)$$

for S_4 -invariant states, with a degeneracy given by

$$\deg(E_\ell) = \left\lfloor \frac{\ell}{12} \right\rfloor + \begin{cases} 0 & \text{for } \ell = 1, 2, 5 \pmod{12} \\ 1 & \text{for } \ell = \text{else} \pmod{12} \end{cases}. \quad (6.7)$$

The Vandermonde reads

$$\Delta = (x^2 - y^2)(y^2 - z^2)(z^2 - x^2), \quad (6.8)$$

and the Dunkl operators in the potential-free frame are given by

$$\begin{aligned}
 \tilde{\mathcal{D}}_x &= \partial_x + \frac{g}{x+y}(1-s_{x+y}) + \frac{g}{x-y}(1-s_{x-y}) + \frac{g}{x+z}(1-s_{x+z}) + \frac{g}{x-z}(1-s_{x-z}), \\
 \tilde{\mathcal{D}}_y &= \partial_y + \frac{g}{y+z}(1-s_{y+z}) + \frac{g}{y-z}(1-s_{y-z}) + \frac{g}{y+x}(1-s_{x+y}) + \frac{g}{y-x}(1-s_{x-y}), \\
 \tilde{\mathcal{D}}_z &= \partial_z + \frac{g}{z+x}(1-s_{x+z}) + \frac{g}{z-x}(1-s_{x-z}) + \frac{g}{z+y}(1-s_{y+z}) + \frac{g}{z-y}(1-s_{y-z}),
 \end{aligned} \tag{6.9}$$

with the Coxeter reflections

$$\begin{aligned}
 s_{x+y} &: (x, y, z) \mapsto (-y, -x, +z), & s_{x-y} &: (x, y, z) \mapsto (+y, +x, +z), \\
 s_{x+z} &: (x, y, z) \mapsto (-z, +y, -x), & s_{x-z} &: (x, y, z) \mapsto (+z, +y, +x), \\
 s_{y+z} &: (x, y, z) \mapsto (+x, -z, +y), & s_{y-z} &: (x, y, z) \mapsto (+x, +z, +y).
 \end{aligned} \tag{6.10}$$

These ingredients enter the S_4 -invariant energy eigenfunctions $v_{\{\ell\}}^{(g)} = r^{-q} \Delta^g h_{\{\ell\}}^{(g)}(x)$ in

$$h_{\{\ell\}}^{(g)}(x, y, z) = r^{2\ell+12g+1} (\tilde{\mathcal{D}}_x \tilde{\mathcal{D}}_y \tilde{\mathcal{D}}_z)^{\ell_3} (\tilde{\mathcal{D}}_x^4 + \tilde{\mathcal{D}}_y^4 + \tilde{\mathcal{D}}_z^4)^{\ell_4} r^{-1-12g}, \tag{6.11}$$

for which we cannot offer a more explicit expression. The lowest-energy wave functions are given in the table of appendix C. Their degeneracies and corresponding quantum numbers (ℓ_3, ℓ_4) are listed below, where the notation $(\ell_3, \ell_4)^*$ identifies the $q < 0$ states.

$g=-2$	deg	(ℓ_3, ℓ_4)	$g=-1$	deg	(ℓ_3, ℓ_4)	$g \geq 0$	deg	(ℓ_3, ℓ_4)
$E = 0$	3	$(4, 0), (0, 3), (1, 2)^*$	$E = 0$	1	$(2, 0)$	$E = \frac{1}{2}6g(6g+1)$	1	$(0, 0)$
$E = 1$	2	$(3, 1), (2, 1)^*$	$E = 1$	2	$(1, 1), (0, 1)^*$	$E = \frac{1}{2}(6g+3)(6g+4)$	1	$(1, 0)$
$E = 3$	2	$(2, 2), (3, 0)^*$	$E = 3$	2	$(0, 2), (1, 0)^*$	$E = \frac{1}{2}(6g+4)(6g+5)$	1	$(0, 1)$
$E = 6$	3	$(5, 0), (1, 3), (0, 2)^*$	$E = 6$	1	$(3, 0)$	$E = \frac{1}{2}(6g+6)(6g+7)$	1	$(2, 0)$
$E=10$	3	$(4, 1), (0, 4), (1, 1)^*$	$E=10$	1	$(2, 1)$	$E = \frac{1}{2}(6g+7)(6g+8)$	1	$(1, 1)$

The Dunkl-deformed angular momenta,

$$\mathcal{L}_x \equiv \mathcal{L}_{yz} = y\mathcal{D}_z - z\mathcal{D}_y, \quad \mathcal{L}_y \equiv \mathcal{L}_{zx} = z\mathcal{D}_x - x\mathcal{D}_z, \quad \mathcal{L}_z \equiv \mathcal{L}_{xy} = x\mathcal{D}_y - y\mathcal{D}_x \tag{6.12}$$

with $\mathcal{D}_i = \tilde{\mathcal{D}}_i - g \partial_i \ln \Delta$ (amounting to dropping the ‘1’s in (6.9)), get permuted under the action of S_4 , with an odd number of sign flips thrown in. The ring of Weyl invariant polynomials in $\{\mathcal{L}_x, \mathcal{L}_y, \mathcal{L}_z\}$ (case D) is generated by

$$\mathcal{C}_k = \mathcal{L}_x^k + \mathcal{L}_y^k + \mathcal{L}_z^k \quad \text{for } k = 0, 2, 4, 6, \tag{6.13}$$

where

$$\mathcal{C}_2 = -2\mathcal{H} + S(S+1) \quad \text{with} \quad S = g \sum_{\alpha} s_{\alpha}, \tag{6.14}$$

giving rise to three algebraically independent conserved quantities, $C_k = \text{res}(\mathcal{C}_k)$ for $k = 2, 4, 6$, see also [12]. Their algebra seems to be freely generated, modulo the center spanned by C_2 .

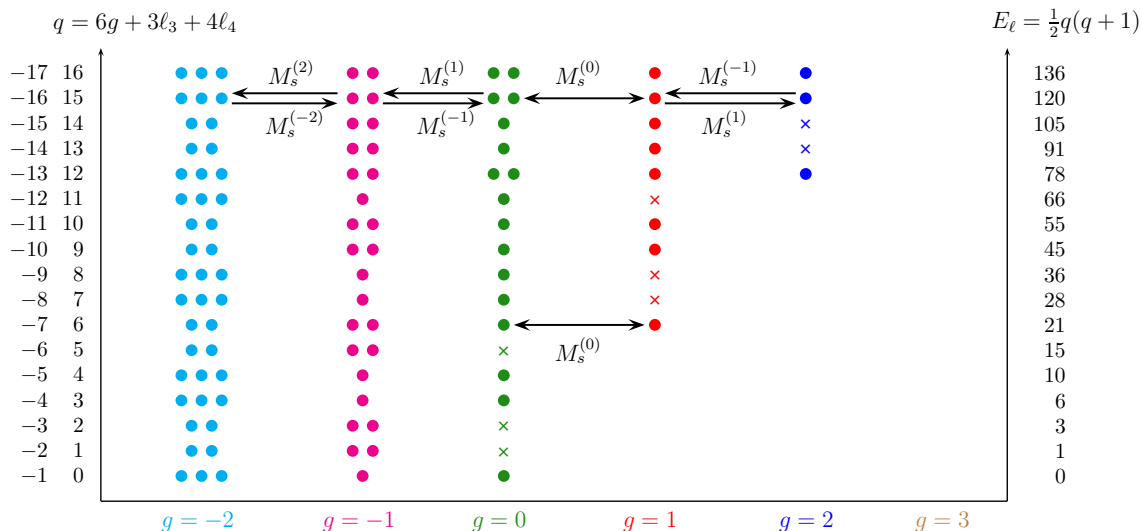


Figure 6. Spectrum of $H_\epsilon^{(g)}$ and action of the interwiners for the AD_3 model.

The basic Weyl antiinvariants built from $\{\mathcal{L}_x, \mathcal{L}_y, \mathcal{L}_z\}$ (case A) are

$$\begin{aligned} \mathcal{M}_3 &= \mathcal{L}_x \mathcal{L}_y \mathcal{L}_z + \mathcal{L}_x \mathcal{L}_z \mathcal{L}_y + \mathcal{L}_y \mathcal{L}_z \mathcal{L}_x + \mathcal{L}_y \mathcal{L}_x \mathcal{L}_z + \mathcal{L}_z \mathcal{L}_x \mathcal{L}_y + \mathcal{L}_z \mathcal{L}_y \mathcal{L}_x, \\ \mathcal{M}_6 &= \{\mathcal{L}_x^4, \mathcal{L}_y^2\} - \{\mathcal{L}_y^4, \mathcal{L}_x^2\} + \{\mathcal{L}_y^4, \mathcal{L}_z^2\} - \{\mathcal{L}_z^4, \mathcal{L}_y^2\} + \{\mathcal{L}_z^4, \mathcal{L}_x^2\} - \{\mathcal{L}_x^4, \mathcal{L}_z^2\}, \end{aligned} \tag{6.15}$$

and all higher ones are words in these and the \mathcal{C}_k . Their restriction to S_4 -symmetric functions produces two independent intertwiners, M_3 and M_6 , which obey the same relations (4.14). Their potential-free version⁵

$$\widetilde{M}_s = \Delta^{-g} M_s \Delta^g \quad \text{for } s = 3, 6 \tag{6.16}$$

can be employed to step up the energy eigenfunctions in the coupling,

$$\widetilde{M}_s h_{\{\ell\}}^{(g)} = \sum_{\substack{\{\ell'\} \\ \ell' = \ell - 6}} c_{\{\ell'\}}^s \Delta h_{\{\ell'\}}^{(g+1)}. \tag{6.17}$$

In this way, eventually all states with positive integer coupling can be reached. This may not be true for the (more numerous) negative integer coupling states, some of which can be found by applying the adjoint intertwiner. In contrast to the previous section, \widetilde{M}_s now depend on the value of g , which prevents a nice closed formula like (4.10) for the polynomials $h_{\{\ell\}}^{(g)}$.

What are the possibilities for a linear realization of \mathcal{PT} transformations? The Coxeter group $W = S_4$ contains one rank-zero involution (the identity), 6 rank-one involutions (the Coxeter reflections), and 3 rank-two involutions (π rotations on one of the three basic planes). The unique rank-three involution (the negative identity) is the outer automorphism of A_3 , hence it is not in S_4 but generates its double cover. Vanishing rank or

⁵Note that \widetilde{M}_s are *not* the restrictions of the corresponding polynomials $\widetilde{\mathcal{M}}_s$ in the potential-free frame.

co-rank of P_- does not admit a compatible complex deformation. The three-dimensional coset $\text{SO}(3, \mathbb{C})/\text{SO}(3, \mathbb{R})$ is parametrized as

$$\Gamma(\epsilon, e) = \exp\left\{-i\epsilon \begin{pmatrix} 0 & w & -v \\ -w & 0 & u \\ v & -u & 0 \end{pmatrix}\right\} = \begin{pmatrix} c+(1-c)u^2 & (1-c)uw-isw & (1-c)uw+isv \\ (1-c)vu+isw & c+(1-c)v^2 & (1-c)vw-isu \\ (1-c)wu-isv & (1-c)wv+isu & c+(1-c)w^2 \end{pmatrix} \quad (6.18)$$

where

$$e = \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \quad u^2 + v^2 + w^2 = 1 \quad \text{and} \quad c \equiv \cosh \epsilon, \quad s \equiv \sinh \epsilon. \quad (6.19)$$

Clearly, any nonvanishing G is of rank two. Degeneracy in the singular locus $\alpha \cdot x(\epsilon) = 0$ occurs only when e is parallel to some root α .

For $\text{rank}(P_-) = 1$, without loss of generality we choose \mathcal{P} to permute x and y , i.e.

$$\mathcal{P} = s_{x-y} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \eta = \begin{pmatrix} w/2 \\ w/2 \\ -v \end{pmatrix}, \quad (6.20)$$

with free real parameters u and v . Compatibility of (6.18) with the rank-one involution (6.20) requires merely $u = v$. The simplest option is $(u, v, w) = (0, 0, 1)$, which copies the $n=2$ case into the xy plane,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}(\epsilon) = \begin{pmatrix} cx - isy \\ cy + isx \\ z \end{pmatrix}. \quad (6.21)$$

Since no root is orthogonal to this plane, our option is generic, and each singular locus has a nontrivial imaginary part. This is not the case for another option, $(u, v, w) = \pm(1, 1, 0)/\sqrt{2}$, since this unit vector is parallel to a root.

For $\text{rank}(P_-) = 2$, we may take \mathcal{P} to rotate by π in the yz plane, so effectively

$$\mathcal{P} = s_{y+z}s_{y-z} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \eta = \begin{pmatrix} 0 \\ w \\ -v \end{pmatrix}, \quad (6.22)$$

The deformation (6.18) is consistent with (6.22) precisely if $u = 0$. Specializing once more to $(u, v, w) = (0, 0, 1)$, without loss of generality, we again arrive at the boost (6.21). Also in this case, there are some degenerate options, namely $(u, v, w) = \pm(0, 1, 1)/\sqrt{2}$ and $(u, v, w) = \pm(0, 1, -1)/\sqrt{2}$.

The singular set of the deformed potential

$$\frac{U(\epsilon)}{2g(g-1)} = \frac{1}{\sin^2\theta \cos^2 2(\phi+i\epsilon)} + \frac{\cos^2\theta + \sin^2\theta \cos^2(\phi+i\epsilon)}{(\cos^2\theta - \sin^2\theta \cos^2(\phi+i\epsilon))^2} + \frac{\cos^2\theta + \sin^2\theta \sin^2(\phi+i\epsilon)}{(\cos^2\theta - \sin^2\theta \sin^2(\phi+i\epsilon))^2} \quad (6.23)$$

consists of 6 antipodal pairs $(x_\alpha, -x_\alpha)$ of points,

$$\text{sing} = \pm \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ \nu \\ c\nu \end{pmatrix}, \begin{pmatrix} 0 \\ -\nu \\ c\nu \end{pmatrix}, \begin{pmatrix} \nu \\ 0 \\ c\nu \end{pmatrix}, \begin{pmatrix} -\nu \\ 0 \\ c\nu \end{pmatrix} \right\} \quad \text{with} \quad \nu = 1/\sqrt{1+c^2}, \quad (6.24)$$

where the first two pairs coincide. With increasing deformation parameter ϵ , the other four pairs move from the location of the roots (outside the xy plane) to the north and south poles. Clearly, the singular Vandermonde factor Δ_ϵ^g keeps the energy eigenstates unphysical for negative values of g . Hence, only the free state spaces at $g=0$ and $g=1$ should be combined, so that its degeneracy becomes

$$\text{deg}(E_\ell^{(1)}) = \left\lfloor \frac{\ell}{6} \right\rfloor + \begin{cases} 0 & \text{for } \ell = 1, 2, 5 \pmod{6} \\ 1 & \text{for } \ell = 0, 3, 4 \pmod{6} \end{cases}. \quad (6.25)$$

When the potential is turned on, the linear \mathcal{PT} deformation hence does not alter the degeneracy of the energy spectrum but smoothly modifies the states.

With a nonlinear \mathcal{PT} deformation of the type (3.20) we may, however, completely remove the wave-function and potential singularities. For the case at hand, it reads

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}(\epsilon_1, \epsilon_2) = r \begin{pmatrix} \sin(\theta+i\epsilon_1) \cos(\phi+i\epsilon_2) \\ \sin(\theta+i\epsilon_1) \sin(\phi+i\epsilon_2) \\ \cos(\theta+i\epsilon_1) \end{pmatrix} = r \begin{pmatrix} c_1 c_2 x - i c_1 s_2 y + s_1 s_2 \frac{z y}{\rho} + i s_1 c_2 \frac{z x}{\rho} \\ c_1 c_2 y + i c_1 s_2 x - s_1 s_2 \frac{z x}{\rho} + i s_1 c_2 \frac{z y}{\rho} \\ c_1 z - i s_1 \rho \end{pmatrix} \quad (6.26)$$

$$\text{with } c_i = \cosh(\epsilon_i), \quad s_i = \sinh(\epsilon_i) \quad \text{and} \quad \rho = \sqrt{x^2 + y^2}. \quad (6.27)$$

For $\epsilon_1 = 0$ we come back to the linear complex boost in the xy plane. The \mathcal{P} involution is chosen as the outer automorphism

$$\mathcal{P} : (\theta, \phi) \mapsto (-\theta, -\phi) \quad \Leftrightarrow \quad (x, y, z) \mapsto (x, -y, z), \quad (6.28)$$

and it is easy to see that the deformed Hamiltonian $H(\epsilon)$ is \mathcal{PT} symmetric. We should note, however, that the above deformation modifies the kinetic term,

$$\begin{aligned} L_\epsilon^2 &= -\partial_\theta^2 - \frac{c_1 \cos(\theta) - i s_1 \sin(\theta)}{c_1 \sin(\theta) + i s_1 \cos(\theta)} \partial_\theta - \frac{1}{(c_1 \sin(\theta) + i s_1 \cos(\theta))^2} \partial_\phi^2 \\ &= -\partial_\theta^2 + \frac{\sin(2\theta) - i \sinh(2\epsilon_1)}{\cos(2\theta) - \cosh(2\epsilon_1)} \partial_\theta - 2 \frac{1 - \cosh(2\epsilon_1) \cos(2\theta) - i \sinh(2\epsilon_1) \sin(2\theta)}{(\cos(2\theta) - \cosh(2\epsilon_1))^2} \partial_\phi^2. \end{aligned} \quad (6.29)$$

Because with this deformation the Vandermonde is nowhere vanishing,

$$\begin{aligned} \Delta_\epsilon &\sim r^6 \sin^2(\theta+i\epsilon_1) \cos^4(\theta+i\epsilon_1) \cos^2(2\phi+2i\epsilon_2) \\ &\quad \times (\tan^2(\theta+i\epsilon_1) \cos^2(\phi+i\epsilon_2) - 1) (\tan^2(\theta+i\epsilon_1) \sin^2(\phi+i\epsilon_2) - 1), \end{aligned} \quad (6.30)$$

all state spaces at $g < 0$ become physical, and so we should combine the tower for any $g > \frac{1}{2}$ with the one at $1-g$. In contrast to the A_2 model, both branches for $1-g < 0$ contribute, and for positive integral g one finds (demanding S_4 invariance) that

$$\text{deg}(E_\ell) = \begin{cases} g-1 + \begin{cases} 0 & \text{for } q+6g = 0, 3, 4, 7, 8, 11 \pmod{12} \\ 1 & \text{for } q+6g = 1, 2, 5, 6, 9, 10 \pmod{12} \end{cases} & \text{if } q < 6g-6 \\ \left\lfloor \frac{q}{6} \right\rfloor + \begin{cases} 0 & \text{for } q = 1, 2, 5 \pmod{6} \\ 1 & \text{for } q = 0, 3, 4 \pmod{6} \end{cases} & \text{if } q \geq 6g-6 \end{cases} \quad (6.31)$$

which demonstrates the doubling (for large enough energy) compared to (6.7).

In this situation we encounter additional ‘odd’ conserved charges (suppressing ϵ)

$$Q_{\{s\}}^{(g)} = M_{s_{g-1}}^{(g-1)} M_{s_{g-2}}^{(g-2)} \dots M_{s_{2-g}}^{(2-g)} M_{s_{1-g}}^{(1-g)} \quad \text{with } \{s\} = \{s_i\} \quad \text{and} \quad s_i \in \{3, 6\}. \quad (6.32)$$

They square to polynomials in the even charges C_2, C_4 and C_6 , e.g.

$$\begin{aligned} (Q_{333}^{(2)})^2 &\propto 8C_6^3 - 36C_2 C_4 C_6^2 + 12C_2^3 C_6^2 + 54C_2^2 C_4^2 C_6 - 36C_2^4 C_4 C_6 + 6C_2^6 C_6 \\ &\quad - 27C_2^3 C_4^3 + 27C_2^5 C_4^2 - 9C_2^7 C_4 + C_2^9 + \text{lower-order terms}. \end{aligned} \quad (6.33)$$

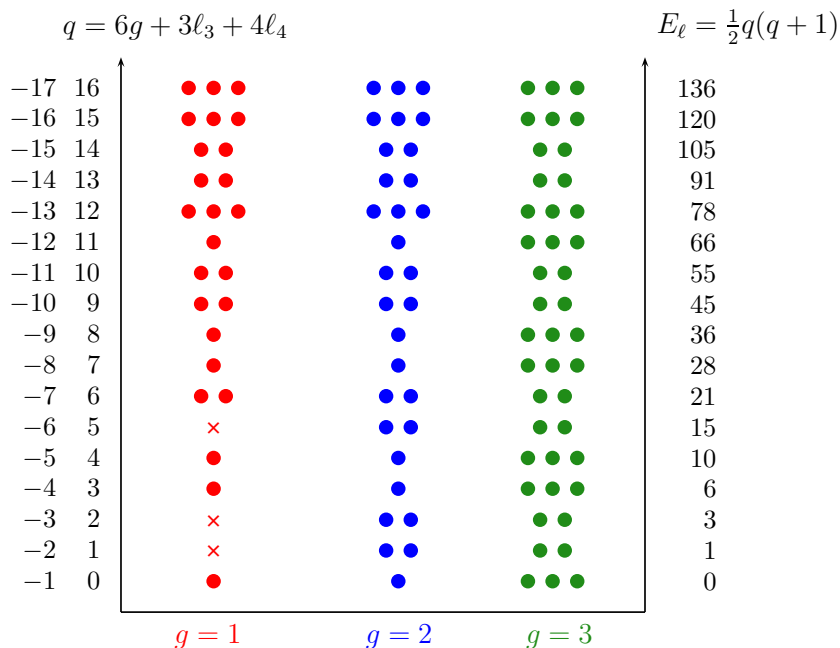


Figure 7. Joint spectrum of $H_\epsilon^{(g)}$ and $H_\epsilon^{(1-g)}$ after the \mathcal{PT} deformation in the AD_3 model.

7 BC_3 model

To understand the non-simply-laced situation at rank-three, we study the model based on the BC_3 Coxeter system. It is obtained by extending the AD_3 root system to

$$\mathcal{R}_+ = \{e_x+e_y, e_x-e_y, e_x+e_z, e_x-e_z, e_y+e_z, e_y-e_z, e_x, e_y, e_z\}, \quad (7.1)$$

yielding the potential

$$U = 2g_L(g_L-1)r^2 \left(\frac{x^2+y^2}{(x^2-y^2)^2} + \frac{y^2+z^2}{(y^2-z^2)^2} + \frac{z^2+x^2}{(z^2-x^2)^2} \right) + \frac{1}{2}g_S(g_S-1)r^2 \left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} \right). \quad (7.2)$$

The Coxeter group $W = S_4 \times \mathbb{Z}_2$ enlarges the previous S_4 by reflections on the basic coordinate planes, and it may be generated by

$$s_{x-y} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad s_{y-z} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad s_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (7.3)$$

The basic invariant polynomials are⁶

$$\sigma_2 = x^2 + y^2 + z^2 =: r^2, \quad \sigma_3 = x^2 y^2 z^2, \quad \sigma_4 = x^4 + y^4 + z^4, \quad (7.4)$$

which leads to $\ell = 6\ell_3 + 4\ell_4$ and W -invariant energy levels

$$E_\ell = \frac{1}{2}q(q+1) \quad \text{with} \quad q = \ell + 6g_L + 3g_S = 6\ell_3 + 4\ell_4 + 6g_L + 3g_S \quad (7.5)$$

⁶The choice of σ_3 is not unique. Other options are $x^6+y^6+z^6$ or $\sigma_2\sigma_4 - (x^6+y^6+z^6)$.

and a degeneracy $\deg(E_\ell) = 0$ when ℓ is odd and

$$\deg(E_\ell) = \left\lfloor \frac{\ell}{12} \right\rfloor + \begin{cases} 0 & \text{for } \ell = 2 \pmod{12} \\ 1 & \text{for } \ell = \text{else} \pmod{12} \end{cases} \quad (7.6)$$

when ℓ is even. Putting $g_S = 0$, we are back to the AD_3 case, but its states with odd ℓ_3 and thus odd ℓ are absent here. The Vandermonde splits,

$$\Delta = \Delta_L \Delta_S \quad \text{with} \quad \Delta_L = (x^2 - y^2)(y^2 - z^2)(z^2 - x^2) \quad \text{and} \quad \Delta_S = x y z. \quad (7.7)$$

The Dunkl operators $\tilde{\mathcal{D}}_i$ can be obtained from (6.9) by specifying $g \rightarrow g_L$ and adding a term $\frac{g_S}{x^i}(1-s_i)$ with the additional Coxeter reflections

$$s_x : (x, y, z) \mapsto (-x, y, z), \quad s_y : (x, y, z) \mapsto (x, -y, z), \quad s_z : (x, y, z) \mapsto (x, y, -z) \quad (7.8)$$

complementing (6.10). For the W -invariant energy eigenfunctions $v_{\{\ell\}}^{(g)} = r^{-q} \Delta^g h_{\{\ell\}}^{(g)}(x)$ we must construct the degree- ℓ homogeneous polynomials

$$h_{\{\ell\}}^{(g)}(x, y, z) = r^{2\ell+12g_L+6g_S+1} (\tilde{\mathcal{D}}_x \tilde{\mathcal{D}}_y \tilde{\mathcal{D}}_z)^{2\ell_3} (\tilde{\mathcal{D}}_x^4 + \tilde{\mathcal{D}}_y^4 + \tilde{\mathcal{D}}_z^4)^{\ell_4} r^{-1-12g_L-6g_S}. \quad (7.9)$$

Comparing with the AD_3 case, apart from the extended Dunkl operators this formula is very similar to (6.11), but all odd- ℓ states have disappeared. The following tables show the states and degeneracy at small values of the energy for a few values of g_S and g_L , where again a * denotes the $q < 0$ states. We see that the latter appear even when only one of the couplings is negative. Some of the wave functions can be calculated explicitly from the table in appendix D.

$g_S=-1$	deg	(ℓ_3, ℓ_4)	$g_S=-1$	deg	(ℓ_3, ℓ_4)	$g_S=-1$	deg	(ℓ_3, ℓ_4)
$g_L=-2$			$g_L=-1$			$g_L=0$		
$E = 0$	1	(1, 2)*	$E = 0$	1	(0, 2)*	$E = 0$	0	
$E = 1$	2	(2, 1), (0, 4)	$E = 1$	1	(1, 1)	$E = 1$	1	(0, 1)
$E = 3$	2	(2, 0)*, (0, 3)*	$E = 3$	1	(1, 0)*	$E = 3$	1	(0, 0)*
$E = 6$	2	(3, 0), (1, 3)	$E = 6$	2	(2, 0), (0, 3)	$E = 6$	1	(1, 0)
$E=10$	1	(1, 1)*	$E=10$	1	(0, 1)*	$E = 10$	0	

$g_L=-1$	deg	(ℓ_3, ℓ_4)	$g_L=-1$	deg	(ℓ_3, ℓ_4)	$g_L=-1$	deg	(ℓ_3, ℓ_4)
$g_S=-2$			$g_S=-1$			$g_S=0$		
$E = 0$	2	(2, 0), (0, 3)	$E = 0$	1	(0, 2)*	$E = 0$	1	(1, 0)
$E = 1$	1	(1, 1)*	$E = 1$	1	(1, 1)	$E = 1$	1	(0, 1)*
$E = 3$	1	(1, 2)*	$E = 3$	1	(1, 0)*	$E = 3$	1	(0, 2)
$E = 6$	1	(0, 2)*	$E = 6$	2	(2, 0), (0, 3)	$E = 6$	0	
$E=10$	2	(2, 1), (0, 4)	$E=10$	1	(0, 1)*	$E = 10$	1	(1, 1)

The Dunkl-deformed angular momenta

$$\mathcal{L}_i = \epsilon_{ijk} x^j \mathcal{D}_k \quad \text{with} \quad \mathcal{D}_i = \widetilde{\mathcal{D}}_i - g_L \partial_i \ln \Delta_L - g_S \partial_i \ln \Delta_S \quad (7.10)$$

do not differ much from those of the AD_3 model. The Coxeter reflections permute them and can flip the sign of any number of them. Therefore, the Weyl invariant polynomials in $\{\mathcal{L}_x, \mathcal{L}_y, \mathcal{L}_z\}$ are the same as in the AD_3 case, generated by $\{C_0, C_2, C_4, C_6\}$, and the conserved charges agree with the previous ones, except that the constituting Dunkl operators have been extended by the short-root terms. What about Weyl antiinvariants, corresponding to cases A, B or C in section 2? Unfortunately, because

$$s_x s_y s_z : (\mathcal{L}_x, \mathcal{L}_y, \mathcal{L}_z) \mapsto (\mathcal{L}_x, \mathcal{L}_y, \mathcal{L}_z), \quad (7.11)$$

there do not exist \mathcal{L}_i polynomials which are antiinvariant under the short-root reflections. Besides, an intertwiner shifting g_S by unity would connect states with an even value of q to states with an odd one, which is incompatible with (7.5). Therefore, besides case D (the invariants) we can only realize case B, which copies the AD_3 intertwining situation. As a result, the two basic AD_3 intertwiners M_3 and M_6 , based on (6.15) with the \mathcal{L}_i pertaining to the BC_3 system, will obey the relations

$$\begin{aligned} M_s^{(g_L, g_S)} H^{(g_L, g_S)} &= H^{(g_L+1, g_S)} M_s^{(g_L, g_S)}, \\ M_s^{(1-g_L, g_S)} H^{(g_L, g_S)} &= H^{(g_L-1, g_S)} M_s^{(1-g_L, g_S)} \end{aligned} \quad (7.12)$$

but do not shift the g_S value. Therefore, iterating the \widetilde{M}_s action, we can produce the polynomials $h_{\{\ell\}}^{(g_L, g_S)}$ only from $h_{\{\ell'\}}^{(0, g_S)}$. The latter states are those of the $A_1^{\oplus 3}$ model, to be discussed next. Of course, intertwining operators may be constructed state-by-state by brute force, but those will not respect the Weyl symmetry.⁷

The discussion of \mathcal{PT} deformations may be completely borrowed from the previous section. The additional $\text{rank}(P_-)=1$ option of $\mathcal{P} = s_x$ does not produce anything new. Under the nonlinear deformation (6.26), again the Vandermonde loses its zeros, and the negative- g state spaces become physical. So for positive integral values of g_L and g_S , we must combine two state towers at

$$(g_L, g_S) \ \& \ (1-g_L, g_S) \quad \text{as well as} \quad (g_L, 1-g_S) \ \& \ (1-g_L, 1-g_S), \quad (7.13)$$

where one pair has states only at even q and the other pair only at odd q . For $q \geq 6(g_L-1) + 3(g_S-1)$, the irregularities due to missing low-energy states disappear, and the degeneracy grows approximately like $\frac{\ell}{6}$ both for even and odd q values. For $g_L \in \mathbb{Z}$ there appear ‘odd’ conserved charges $Q_{\{s\}}^{(g_L, g_S)}$ mapping $(1-g_L, g_S) \mapsto (g_L, g_S)$. They are formally identical to those of the AD_3 model. Analogous odd operators connecting the states at $1-g_S$ and g_S do not exist since the two pairs of towers have disjoint spectra.

⁷An example can be found in [2]. Note that this is in contrast with rank-one systems, where all states with integral couplings can be related, like in [32, 33].

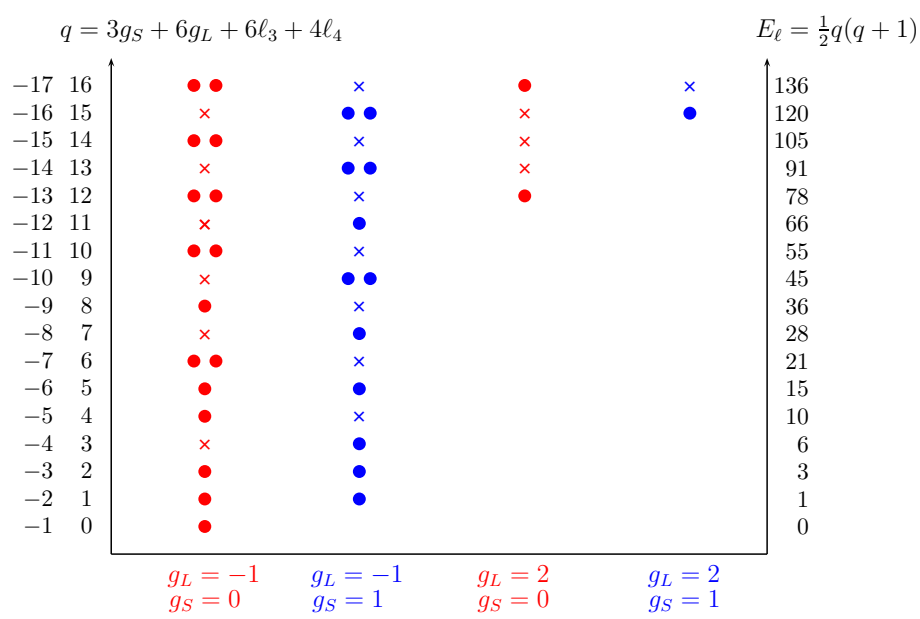


Figure 8. Spectrum for $(g_L, g_S) = (2, 1)$ comprising four towers for the \mathcal{PT} -extended BC_3 model. The blue and red towers carry odd and even integer values of q , respectively.

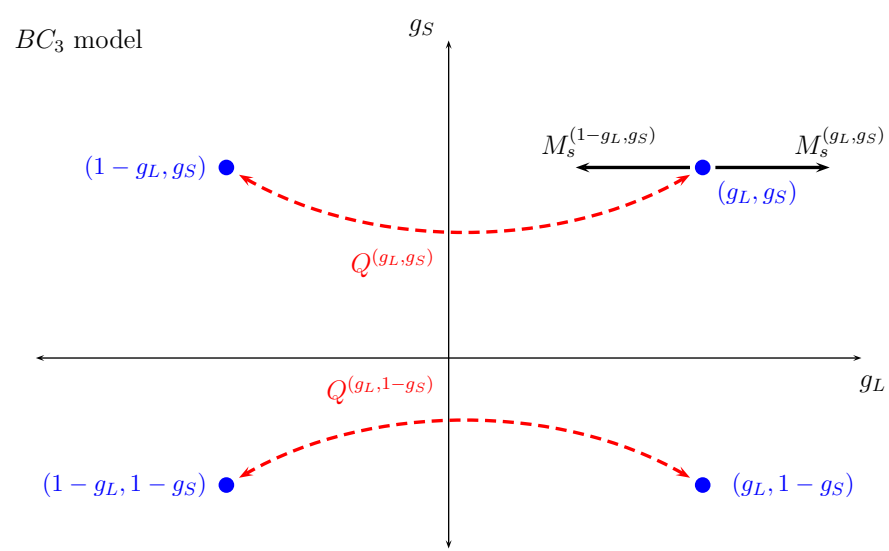


Figure 9. Action of the intertwining operators and ‘odd’ conserved charges in the BC_3 model.

8 $A_1^{\oplus 3}$ model

The previous section reduced the AD_3 system to the $A_1^{\oplus 3}$ system of short roots,

$$\mathcal{R}_+ = \{e_x, e_y, e_z\}. \quad (8.1)$$

When the radial excitations are included, this model is reducible and decomposes into three copies of the rank-one system with inverse-square potential and coinciding couplings $g_s = g$. However, the spherical reduction couples the three subsystems to a potential

$$U = \frac{1}{2}g(g-1)r^2\left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right) = \frac{1}{2}g(g-1)\left(3 + \frac{x^2+y^2}{z^2} + \frac{y^2+z^2}{x^2} + \frac{z^2+x^2}{y^2}\right). \quad (8.2)$$

The Coxeter group $W = \mathbb{Z}_2^3$ consists merely of the 3 reflections about the elementary coordinate planes,

$$s_x : (x, y, z) \mapsto (-x, y, z), \quad s_y : (x, y, z) \mapsto (x, -y, z), \quad s_z : (x, y, z) \mapsto (x, y, -z), \quad (8.3)$$

and the basic invariant polynomials can be taken as⁸

$$\sigma_2 = x^2 + y^2 + z^2 =: r^2, \quad \sigma_3 = x^2, \quad \sigma_4 = y^2, \quad (8.4)$$

and thus

$$E_\ell = \frac{1}{2}q(q+1) \quad \text{with} \quad q = \ell + 3g = 2(\ell_3 + \ell_4) + 3g \quad (8.5)$$

for the W -invariant states, with a degeneracy

$$\text{deg}(E_\ell) = \frac{\ell}{2} + 1. \quad (8.6)$$

This is consistent with the fact that only the spherical-harmonic combinations

$$Y_{\ell,0} \quad \text{and} \quad Y_{\ell,m} + Y_{\ell,-m} \quad \text{for} \quad \ell, m = 0, 2, 4, \dots \quad (8.7)$$

are Weyl invariant. The Vandermonde is simply $\Delta = xyz$, and the potential-free wave functions arise from

$$h_{\{\ell\}}^{(g)}(x, y, z) = r^{2\ell+6g+1} \tilde{\mathcal{D}}_x^{2\ell_3} \tilde{\mathcal{D}}_y^{2\ell_4} r^{-1-6g} \quad (8.8)$$

with

$$\tilde{\mathcal{D}}_x = \partial_x + \frac{g}{x}(1-s_x), \quad \tilde{\mathcal{D}}_y = \partial_y + \frac{g}{y}(1-s_y), \quad \tilde{\mathcal{D}}_z = \partial_z + \frac{g}{z}(1-s_z). \quad (8.9)$$

With the above choice of symmetric polynomials we could find the following formulæ for the states with $\ell_4 = 0$,

$$\begin{aligned} h_{(\ell_3,0)}^{(g)}(x, y, z) &= \sum_{i=0}^{\ell_3} \frac{2^{\ell_3} (-1)^{-i+\ell_3} \Gamma(\ell_3+1) \Gamma(g+\ell_3+\frac{1}{2}) \Gamma(2g+\ell_3+1)}{\Gamma(i+1) \Gamma(2g+i+1) \Gamma(-i+\ell_3+1) \Gamma(g-i+\ell_3+\frac{1}{2})} x^{2(\ell_3-i)} (y^2+z^2)^i \\ &= x^{2\ell_3} {}_2F_1\left(-\ell_3, -g-\ell_3+\frac{1}{2}; 2g+1; -\frac{y^2+z^2}{x^2}\right), \end{aligned} \quad (8.10)$$

⁸The choice of σ_3 and σ_4 is ambiguous; other possibilities are $\sigma_3 = x^2 - y^2$ and $\sigma_4 = x^2 + y^2 - 2z^2$ or $\sigma_3 = x^2$ and $\sigma_4 = x^2 + y^2$.

and due to the symmetry $\ell_3 \leftrightarrow \ell_4$ plus $x \leftrightarrow y$ we can obtain the $\ell_3 = 0$ states,

$$h_{(0,\ell_4)}^{(g)}(x, y, z) = h_{(\ell_4,0)}^{(g)}(y, x, z). \quad (8.11)$$

Below we present the low-lying degeneracies and quantum numbers at $g \geq -2$. Their explicit form can be found in appendix E, where without loss of generality we restrict to $\ell_3 \geq \ell_4$.

$g=-2$	deg	(ℓ_3, ℓ_4)	$g=-1$	deg	(ℓ_3, ℓ_4)	$g \geq 0$	deg	(ℓ_3, ℓ_4)
$E=0$	4	(3, 0), (2, 1), (1, 2), (0, 3)	$E=0$	2	(1, 0)*, (0, 1)*	$E = \frac{1}{2}3g(3g+1)$	1	(0, 0)
$E=1$	3	(2, 0)*, (1, 1)*, (0, 2)*	$E=1$	3	(2, 0), (1, 1), (0, 2)	$E = \frac{1}{2}(3g+2)(3g+3)$	2	(1, 0), (0, 1)
$E=3$	5	(4, 0), (3, 1), ..., (1, 3), (0, 4)	$E=3$	1	(0, 0)*	$E = \frac{1}{2}(3g+4)(3g+5)$	3	(2, 0), (1, 1), (0, 2)
$E=6$	2	(1, 0)*, (0, 1)*	$E=6$	4	(3, 0), (2, 1), (1, 2), (0, 3)	$E = \frac{1}{2}(3g+6)(3g+7)$	4	(3, 0), (2, 1), (1, 2), (0, 3)
$E=10$	6	(5, 0), (4, 1), ..., (1, 4), (0, 5)	$E=10$	0		$E = \frac{1}{2}(3g+8)(3g+9)$	5	(4, 0), (3, 1), ..., (1, 3), (0, 4)
$E=15$	1	(0, 0)*	$E=15$	5	(4, 0), (3, 1), ..., (1, 3), (0, 4)	$E = \frac{1}{2}(3g+10)(3g+11)$	6	(5, 0), (4, 1), ..., (1, 4), (0, 5)

The Dunkl-deformed angular momenta have the simple form

$$\mathcal{L}_x = y\partial_z - z\partial_y - g\left(\frac{y}{z}s_z - \frac{z}{y}s_y\right), \quad \mathcal{L}_y = z\partial_x - x\partial_z - g\left(\frac{z}{x}s_x - \frac{x}{z}s_z\right), \quad \mathcal{L}_z = x\partial_y - y\partial_x - g\left(\frac{x}{y}s_y - \frac{y}{x}s_x\right), \quad (8.12)$$

and any word in \mathcal{L}_i^2 and $\mathcal{L}_x\mathcal{L}_y\mathcal{L}_z$ (and permutations) will restrict to a conserved quantity. As was argued in the previous section, there exist neither Weyl antiinvariant polynomials in \mathcal{L}_i nor intertwiners shifting g by unity.⁹ As a consequence, an ‘odd’ conserved charge for integral g cannot be constructed in this way.

A linear \mathcal{PT} deformation of the type (6.21) (but with a non-coordinate plane) still leaves three pairs of singular points in the potential U_ϵ , while the nonlinear deformation (6.26) yields the fully regularized potential

$$U_{\epsilon_1, \epsilon_2} = \frac{1}{2}g(g-1)\left(\frac{4}{\sin^2(\theta+i\epsilon_2)\sin^2(2\phi+2i\epsilon_2)} + \frac{1}{\cos^2(\theta+i\epsilon_1)}\right). \quad (8.13)$$

This revives the negative- g state spaces and lets us combine the towers at $1-g$ and g . The result is a linearly (with q) growing W -invariant spectrum both for even and odd values of q ,

$$\deg(E_\ell) = \left\{ \begin{array}{ll} \frac{1}{2}(q - 3g + 2) & \text{for } q+g \text{ even} \\ \frac{1}{2}(q + 3g - 1) & \text{for } q+g \text{ odd} \end{array} \right\} \quad \text{when } q \geq 3(g-1). \quad (8.14)$$

The $A_1^{\oplus 3}$ model is the simplest of an infinite reducible series, based on $A_1 \oplus I_2(p)$. We leave it to the reader to work out the details for $p > 2$.

9 H_3 model

Finally, to fully cover the rank-3 landscape, let us turn to the non-crystallographic case of H_3 . Abbreviating the golden ratio and its algebraic conjugate,

$$\tau = \frac{1}{2}(1+\sqrt{5}) \quad \text{and} \quad \bar{\tau} = \frac{1}{2}(1-\sqrt{5}), \quad (9.1)$$

⁹The intertwiners proposed in [2] are not W invariant.

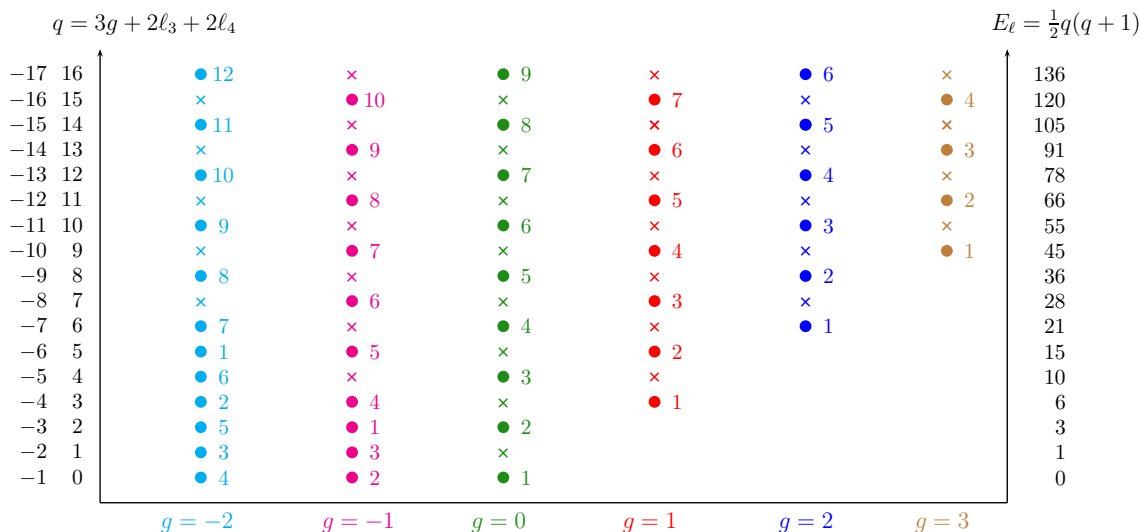


Figure 10. Low-lying energy spectrum for the $A_1^{\oplus 3}$ model. The levels are labeled with their degeneracy. States at $g < 0$ become physical only under a \mathcal{PT} deformation, which adds them to the tower at $1-g$.

the set of 15 positive roots,¹⁰

$$\mathcal{R}_+ = \{e_x \pm \tau e_y \pm \bar{\tau} e_z, e_y \pm \tau e_z \pm \bar{\tau} e_x, e_z \pm \tau e_x \pm \bar{\tau} e_y, e_x, e_y, e_z\}. \quad (9.2)$$

Accordingly, the potential takes the form

$$U = \frac{1}{2}g(g-1)r^2 \left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} \right) + 2g(g-1)r^2 \left(\frac{1}{(x+\tau y+\bar{\tau}z)^2} + \frac{1}{(x+\tau y-\bar{\tau}z)^2} + \frac{1}{(x-\tau y+\bar{\tau}z)^2} + \frac{1}{(x-\tau y-\bar{\tau}z)^2} + \text{cyclic} \right) \quad (9.3)$$

with 15 double poles. The Coxeter group is the icosahedral group I of 120 elements, and it may be generated by the elements

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1 & -\tau & -\bar{\tau} \\ \tau & -\bar{\tau} & -1 \\ -\bar{\tau} & 1 & \tau \end{pmatrix}. \quad (9.4)$$

We can choose three basic invariant polynomials of degrees 2, 6 and 10, for instance

$$\begin{aligned} \sigma_2(x, y, z) &= x^2 + y^2 + z^2 =: r^2, \\ \sigma_3(x, y, z) &= (\tau - \bar{\tau})(x^6 + y^6 + z^6) - 15\bar{\tau}(x^2y^4 + y^2z^4 + z^2x^4) + 30x^2y^2z^2, \\ \sigma_4(x, y, z) &= x^{10} + y^{10} + z^{10} + 30x^2y^2z^2(x^2y^2 + y^2z^2 + z^2x^2) + 15(\tau + 1)(x^8y^2 + y^8z^2 + z^8x^2) \\ &\quad + 30(\tau + 1)(x^6y^4 + y^6z^4 + z^6x^4) - 60\tau x^2y^2z^2(x^4 + y^4 + z^4). \end{aligned} \quad (9.5)$$

Hence, $\ell = 6\ell_3 + 10\ell_4$, and the I -invariant energy levels are given by

$$E_\ell = \frac{1}{2}q(q+1) \quad \text{with} \quad q = \ell + 15g = 6\ell_3 + 10\ell_4 + 15g \quad (9.6)$$

¹⁰All four sign combinations appear. These roots do not lie in a half-space, but this is irrelevant here.

and a degeneracy $\deg(E_\ell) = 0$ when ℓ is odd and

$$\deg(E_\ell) = \left\lfloor \frac{\ell}{30} \right\rfloor + \begin{cases} 0 & \text{for } \ell = 2, 4, 8, 14 \pmod{30} \\ 1 & \text{for } \ell = \text{else} \pmod{30} \end{cases} \quad (9.7)$$

when ℓ is even. The Vandermonde factor $\Delta = \Delta_1 \Delta_2$ is split in terms of $\Delta_1 = x y z$ and

$$\begin{aligned} \Delta_2 &= \prod_{\epsilon_{1,2}=0,1} (x+(-1)^{\epsilon_1} \tau y + (-1)^{\epsilon_2} \bar{\tau} z) (\tau x + (-1)^{\epsilon_1} \bar{\tau} y + (-1)^{\epsilon_2} z) (\bar{\tau} x + (-1)^{\epsilon_1} y + (-1)^{\epsilon_2} \tau z) \\ &= x^{12} - (13 - \sqrt{5}) x^{10} y^2 - (13 + \sqrt{5}) x^{10} z^2 + \frac{1}{2} (113 - 11\sqrt{5}) x^8 y^4 + \frac{1}{2} (113 + 11\sqrt{5}) x^8 z^4 \\ &\quad + 50 x^8 y^2 z^2 - 84 x^6 y^6 - (90 - 66\sqrt{5}) x^6 y^4 z^2 - (90 + 66\sqrt{5}) x^6 y^2 z^4 + 126 x^4 y^4 z^4 \\ &\quad + \text{cyclic permutations.} \end{aligned} \quad (9.8)$$

The analytical computation of the energy eigenfunctions $v_{\{\ell\}}^{(g)} = r^{-q} \Delta^g h_{\{\ell\}}^{(g)}(x)$ in

$$h_{\{\ell\}}^{(g)}(x, y, z) = r^{2\ell+15g+1} \sigma_3(\tilde{\mathcal{D}}_x, \tilde{\mathcal{D}}_y, \tilde{\mathcal{D}}_z)^{\ell_3} \sigma_4(\tilde{\mathcal{D}}_x, \tilde{\mathcal{D}}_y, \tilde{\mathcal{D}}_z)^{\ell_4} r^{-1-15g} \quad (9.9)$$

becomes quite more complicated in contrast with the previous cases. Because of the sum over the 15 positive roots, the Dunkl operators yield quite tedious expressions considering also that the invariant polynomials are given in terms of powers of differential operators of order 6 and 10. We present here the simplest wave functions of order 6, with $(\ell_3, \ell_4) = (1, 0)$,

$$\begin{aligned} h_{\{\ell=6\}}^{(g)} &= (1 + 2g)[600] - 3\tau(7\tau + 8\bar{\tau} + (30\tau + 32\bar{\tau})g)[420] \\ &\quad - 3\bar{\tau}(7\bar{\tau} + 8\tau + (30\bar{\tau} + 32\tau)g)[240] + 2(15 + 62g)[222], \end{aligned} \quad (9.10)$$

and of order 10, with $(\ell_3, \ell_4) = (0, 1)$,

$$\begin{aligned} h_{\{\ell=10\}}^{(g)} &= (1 + 2g)^2[1000] + 8(63 + 285g + 310g^2)[622] + 10(63 + 278g + 288g^2)[442] \\ &\quad + \kappa(\tau, \bar{\tau})[640] + \kappa(\bar{\tau}, \tau)[460] + \lambda(\tau, \bar{\tau})[820] + \lambda(\bar{\tau}, \tau)[280], \end{aligned} \quad (9.11)$$

where we defined $[rst] := x^r y^s z^t + x^t y^r z^s + x^s y^t z^r$ and abbreviated

$$\kappa(\tau, \bar{\tau}) = -\frac{126}{5}\tau + \frac{336}{5}\bar{\tau} + (-144\tau + 284\bar{\tau})g + (-200\tau + 280\bar{\tau})g^2, \quad (9.12)$$

$$\lambda(\tau, \bar{\tau}) = -\left(\frac{53}{5}\tau + \frac{162}{5}\bar{\tau} + (72\tau + 148\bar{\tau})g + (100\tau + 160\bar{\tau})g^2 \right). \quad (9.13)$$

It is possible to check that they are symmetric under the simultaneous transposition of variables plus $\tau \rightarrow \bar{\tau}$, concretely

$$(x, y, z, \tau) \rightarrow (y, x, z, \bar{\tau}), \quad (x, y, z, \tau) \rightarrow (z, y, x, \bar{\tau}), \quad (x, y, z, \tau) \rightarrow (x, z, y, \bar{\tau}). \quad (9.14)$$

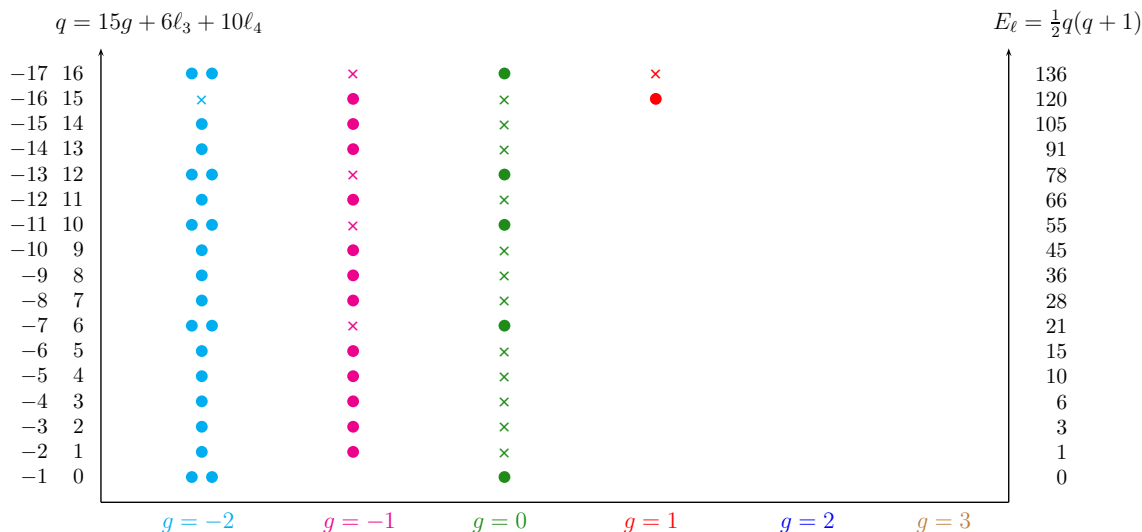


Figure 11. Low-lying energy spectrum for the H_3 model. The towers at $g=2$ and $g=3$ are invisible because their spectrum begins at $E_\ell = 465$ and $E_\ell = 1035$ respectively. States at $g < 0$ become physical only under a \mathcal{PT} deformation, which adds them to the tower at $1-g$.

The low-lying degeneracies and quantum numbers at $g \geq -2$ are presented in the following table and also illustrated in figure 11.

$g=-2$	deg	(ℓ_3, ℓ_4)	$g=-1$	deg	(ℓ_3, ℓ_4)	$g \geq 0$	deg	(ℓ_3, ℓ_4)
$E = 0$	2	$(5, 0), (0, 3)$	$E = 0$	0		$E = \frac{1}{2}15g(15g+1)$	1	$(0, 0)$
$E = 1$	1	$(3, 1)^*$	$E = 1$	1	$(1, 1)$	$E = \frac{1}{2}(15g+6)(15g+7)$	1	$(1, 0)$
$E = 3$	1	$(2, 2)$	$E = 3$	1	$(2, 0)^*$	$E = \frac{1}{2}(15g+10)(15g+11)$	1	$(0, 1)$
$E = 6$	1	$(1, 2)^*$	$E = 6$	1	$(3, 0)$	$E = \frac{1}{2}(15g+12)(15g+13)$	1	$(2, 0)$
$E=10$	1	$(4, 1)$	$E=10$	1	$(0, 1)^*$	$E = \frac{1}{2}(15g+16)(15g+17)$	1	$(1, 1)$
$E=15$	1	$(4, 0)^*$	$E=15$	1	$(0, 2)$	$E = \frac{1}{2}(15g+18)(15g+19)$	1	$(3, 0)$
$E=21$	2	$(6, 0), (1, 3)$	$E=21$	0		$E = \frac{1}{2}(15g+20)(15g+21)$	1	$(0, 2)$

10 Outlook

We have investigated the \mathcal{PT} deformation of the angular Calogero model firstly in general and secondly in detail for rank-two and rank-three systems. Among the different ways to introduce an antilinear symmetry like \mathcal{PT} , nonlinear complex deformations of the coordinates seem to be more effective for removing the singularities of the potential than linear ones. As a result of such a ‘ \mathcal{PT} regularization’, the energy spectrum gets enlarged due to the $g \mapsto 1-g$ invariance of the (potential-frame) Hamiltonian: the previously non-normalizable eigenstates at $g < 0$ become physical and have to be included. In non-simply-laced cases this holds separately for the short- and long-root couplings. For integer (or half-integer)

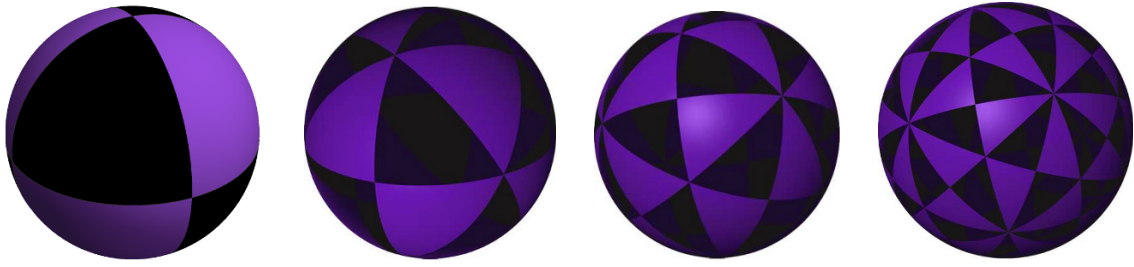


Figure 12. We close with the a visualisation of the Coxeter groups W for the $A_1^{\oplus 3}$, AD_3 , BC_3 and H_3 models, given by the Coxeter complexes for three orthogonal lines, the tetrahedron, the hexahedron/octahedron and the dodecahedron/icosahedron, respectively. This illustrates the close relation of irreducible rank-three Coxeter systems and platonic solids.

values of g , the energy levels at $1-g$ coincide with those at g , increasing the degeneracy of the latter. In this situation, a suitable product of intertwiners produces conserved charges, which act in a regular way thanks to the \mathcal{PT} regularization. When g is an integer, these charges represent ‘square roots’ of conserved charges defined for any g -value, which extends their algebra to a nonlinear \mathbb{Z}_2 -graded one. In the light of our results it is interesting to investigate how the energy spectra get modified for \mathcal{PT} -deformed trigonometric, hyperbolic or elliptic Calogero models. We plan to address these problems in the future.

Acknowledgments

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A A_2 states

ℓ	ℓ_3	$h_\ell^{(g)}$
0	0	1
1	3	$w^3 - \bar{w}^3$
2	6	$(g+1)(w^6 + \bar{w}^6) - 2gw^3\bar{w}^3$
3	9	$(w^3 - \bar{w}^3)[(2+g)(w^6 + \bar{w}^6) - 2(g-1)w^3\bar{w}^3]$
4	12	$(g+2)(g+3)(w^{12} + \bar{w}^{12}) - 4g(g+2)(w^9\bar{w}^3 + w^3\bar{w}^9) + 6g(1+g)w^6\bar{w}^6$
5	15	$(w^3 - \bar{w}^3)[(g+3)(g+4)(w^{12} + \bar{w}^{12}) - 4(g-1)(g+3)(w^9\bar{w}^3 + w^3\bar{w}^9) + 2(3g^2 + g + 6)w^6\bar{w}^6]$
6	18	$(g+3)(g+4)(g+5)(w^{18} + \bar{w}^{18}) - 6g(g+3)(g+4)(w^{15}\bar{w}^3 + w^3\bar{w}^{15}) + 15g(g+1)(g+3)(w^{12}\bar{w}^6 + w^6\bar{w}^{12}) - 20g(g+1)(g+2)w^9\bar{w}^9$
7	21	$(w^3 - \bar{w}^3)[(g+4)(g+5)(g+6)(w^{18} + \bar{w}^{18}) - 6(g-1)(g+4)(g+5)(w^{15}\bar{w}^3 + w^3\bar{w}^{15}) + 3(g+4)(5g^2 - g + 10)(w^{12}\bar{w}^6 + w^6\bar{w}^{12}) - 4(g-1)(5g^2 + 17g + 30)w^9\bar{w}^9]$

q	$h_\ell^{(-2)}$	$h_\ell^{(-1)}$	$h_\ell^{(0)}$	$h_\ell^{(1)}$	$h_\ell^{(2)}$
0	$w^6 - 4w^3\bar{w}^3 + \bar{w}^6$	$w^3 - \bar{w}^3$	1		
3	$w^3 - \bar{w}^3$	$w^3\bar{w}^3$	$w^3 - \bar{w}^3$	1	
6	$w^6\bar{w}^6$	$w^9 + 3w^6\bar{w} - 3w\bar{w}^6 - \bar{w}^9$	$w^6 + \bar{w}^6$	$w^3 - \bar{w}^3$	1
9	$w^{15} + 5w^{12}\bar{w}^3 + 10w^9\bar{w}^6 - 10w^6\bar{w}^9 - 5w^3\bar{w}^{12} - \bar{w}^{15}$	$w^{12} + 2w^9\bar{w}^3 + 2w^3\bar{w}^9 + \bar{w}^{12}$	$w^9 - \bar{w}^9$	$w^6 - w^3\bar{w}^3 + \bar{w}^6$	$w^3 - \bar{w}^3$
12	$w^{18} + 4w^{15}\bar{w}^3 + 5w^{12}\bar{w}^6 + 5w^6\bar{w}^{12} + 4w^3\bar{w}^{15} + \bar{w}^{18}$	$3w^{15} + 5w^{12}\bar{w}^3 - 5w^3\bar{w}^{12} - 3\bar{w}^{15}$	$w^{12} + \bar{w}^{12}$	$w^9 - w^6\bar{w}^3 + w^3\bar{w}^6 - w^9$	$3w^6 - 4w^3\bar{w}^3 + 3\bar{w}^6$
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots

Table 1. Low-lying wave functions $v_\ell^{(g)} = r^{-\ell-3g}\Delta^g h_\ell^{(g)}$ of the Pöschl-Teller model with $E_\ell = \frac{1}{2}q^2$ and $q = \ell+3g$.

B G_2 states

ℓ	ℓ_3	$h_\ell^{(g_S, g_L)}$
0	0	1
6	1	$(1 + g_S + g_L)(w^6 + \bar{w}^6) + 2(g_L - g_S)(w\bar{w})^3$
12	2	$(2 + g_S + g_L)(3 + g_S + g_L)(w^{12} + \bar{w}^{12}) + 4(g_L - g_S)(2 + g_S + g_L)(w^9\bar{w}^3 + w^3\bar{w}^9) + 2(3g_L^2 + 3g_L - 6g_Lg_S + 3g_S + 3g_S^2)(w\bar{w})^6$
18	3	$(3 + g_S + g_L)(4 + g_S + g_L)(5 + g_S + g_L)(w^{18} + \bar{w}^{18}) + 6(g_L - g_S)(3 + g_S + g_L)(4 + g_S + g_L)(w^{15}\bar{w}^3 + w^3\bar{w}^{15})$ $+ 3(3 + g_S + g_L)(5g_L^2 + 5g_L - 6g_Lg_S + 5g_S + 5g_S^2)(w^{12}\bar{w}^6 + w^6\bar{w}^{12}) + 4(g_L - g_S)(5g_L^2 + 15g_L + 2g_Lg_S + 10 + 15g_S + 5g_S^2)(w\bar{w})^9$
24	4	$(4 + g_S + g_L)(5 + g_S + g_L)(6 + g_S + g_L)(7 + g_S + g_L)(w^{24} + \bar{w}^{24}) + (g_L - g_S)(4 + g_S + g_L)(5 + g_S + g_L)(6 + g_S + g_L)(w^{21}\bar{w}^3 + w^3\bar{w}^{21})$ $+ 4(4 + g_S + g_L)(5 + g_S + g_L)(7g_L^2 + 7g_L - 10g_Lg_S + 7g_S + 7g_S^2)(w^{18}\bar{w}^6 + w^6\bar{w}^{18})$ $+ 8(g_L - g_S)(4 + g_S + g_L)(7g_L^2 + 21g_L - 2g_Lg_S + 14 + 21g_S + 7g_S^2)(w^{15}\bar{w}^9 + w^9\bar{w}^{15})$ $+ (70g_S^4 + 420g_S^3 + 770g_S^2 + 420g_S - 40g_S^3g_L + 36g_S^2g_L^2 - 84g_S^2g_L - 40g_Sg_L^3 - 84g_Sg_L^2 - 124g_Sg_L + 420g_L + 770g_L^2 + 420g_L^3 + 70g_L^4)(w\bar{w})^{12}$

Table 2. Deformed harmonic polynomials for low-lying wave functions of the G_2 model at general couplings g_L and g_S .

C AD_3 states

ℓ	ℓ_3	ℓ_4	$h_\ell^{(g)}$
0	0	0	{000}
3	1	0	{111}
4	0	1	$(1+2g)\{400\} - (3+8g)\{220\}$
6	2	0	$(1+2g)\{600\} - 3(5+8g)\{420\} + 2(3+4g)(5+9g)\{222\}$
7	1	1	$(3+2g)\{511\} - (5+8g)\{331\}$
8	0	2	$(1+2g)(3+2g)\{800\} - 4(7+8g)(3+2g)\{620\} + 3(35+56g+24g^2)\{440\} + 12g(7+8g)\{422\}$
9	3	0	$3(3+2g)\{711\} - 9(7+8g)\{531\} + 2(35+69g+36g^2)\{333\}$
10	2	1	$(1+2g)(3+2g)\{1000\} - 5(9+8g)(3+2g)\{820\} + 2(63+149g+76g^2)\{640\} + 4(3+g)(126+239g+108g^2)\{622\} - 6(315+914g+892g^2+288g^3)\{442\}$
11	1	2	$(3+2g)(5+2g)\{911\} - 4(9+8g)(5+2g)\{731\} + 9(21+24g+8g^2)\{551\} + 12g(9+8g)\{533\}$
12	4	0	$4(1+2g)(3+2g)^2\{1200\} - 24(11+8g)(3+2g)^2\{1020\} + 3(815+2244g+1904g^2+512g^3)\{840\} + (4893+11868g+9296g^2+2368g^3)\{660\} +$ $3(15375+40696g+38928g^2+15872g^3+2304g^4)\{822\} - 6(35875+114060g+135440g^2+70976g^3+13824g^4)\{642\} +$ $(179375+658280g+972744g^2+725280g^3+273024g^4+41472g^5)\{444\}$
12	0	3	$-32(1+2g)(3+2g)(707+1774g+1296g^2+288g^3)\{1200\} + 192(3+2g)(11+8g)(707+1774g+1296g^2+288g^3)\{1020\} +$ $384(82425+272562g+359094g^2+235132g^3+76320g^4+9792g^5)\{840\} + 32(894789+2962624g+3859432g^2+2483776g^3+793728g^4+101376g^5)\{660\} +$ $-96(60795+331284g+626140g^2+537952g^3+214848g^4+32256g^5)\{822\} + 192(141855+471268g+675384g^2+543616g^3+243072g^4+46080g^5)\{642\} +$ $96(-236425-635390g-560488g^2-113008g^3+77184g^4+29952g^5)\{444\}$
ℓ	ℓ_3	ℓ_4	$h_\ell^{(-2)}$
13	3	1	$334\{553\} + 176\{733\} + 106\{751\} + 25\{931\} - \{1111\}$
14	2	2	$1780\{644\} + 880\{662\} + 1010\{842\} + 95\{860\} + 64\{1022\} + 17\{1040\} - 7\{1220\} + \{1400\}$
15	5	0	$5\{555\} + 18\{753\} + 3\{771\} + 3\{933\} + 3\{951\}$
15	1	3	$229\{555\} + 826\{753\} + 101\{771\} + 151\{933\} + 116\{951\} + 18\{1131\} - \{1311\}$
16	4	1	$234\{664\} + 153\{844\} + 157\{862\} + 6\{880\} + 77\{1042\} + 7\{1060\} + 5\{1222\} + \{1240\}$

Table 3. Low-lying polynomials for the AD_3 model. The notation is $\{rst\} := x^r y^s z^t + x^r y^t z^s + x^s y^t z^r + x^s y^r z^t + x^t y^r z^s + x^t y^s z^r$.

D BC_3 states

ℓ	ℓ_3	ℓ_4	$h_\ell^{(g_L, g_S)}$
0	0	0	{000}
4	0	1	$(1+2g_L+2g_S)\{400\} - (3+8g_L+2g_S)\{220\}$
6	1	0	$(1+2g_S)(1+2g_L+2g_S)\{600\} - 3(1+2g_S)(5+8g_L+2g_S)\{420\} + 2(15+47g_L+36g_L^2+16g_S+22g_Lg_S+4g_S^2)\{222\}$
8	0	2	$(1+2g_L+2g_S)(3+2g_L+2g_S)\{800\} - 4(3+2g_L+2g_S)(7+8g_L+2g_S)\{620\} + 3(35+56g_L+24g_L^2+24g_S+16g_Lg_S+4g_S^2)\{440\} + 12g_L(7+8g_L+2g_S)\{422\}$
10	1	1	$(1+2g_S)(1+2g_L+2g_S)(3+2g_L+2g_S)\{1000\} - 5(1+2g_S)(3+2g_L+2g_S)(9+8g_L+2g_S)\{820\} + 2(1+2g_S)(63+149g_L+76g_L^2+32g_S+26g_Lg_S+4g_S^2)\{640\} +$ $4(378+843g_L+563g_L^2+108g_L^3+444g_S+688g_Lg_S+262g_L^2g_S+152g_S^2+116g_Lg_S^2+16g_S^3)\{622\} +$ $-6(315+914g_L+892g_L^2+288g_L^3+286g_S+484g_Lg_S+200g_L^2g_S+84g_S^2+64g_Lg_S^2+8g_S^3)\{442\}$
12	2	0	$4(3+2g_L+g_S)(1+2g_S)(3+2g_S)(1+2g_L+2g_S)(3+2g_L+2g_S)\{1200\} - 24(3+2g_L+g_S)(1+2g_S)(3+2g_S)(3+2g_L+2g_S)(11+8g_L+2g_S)\{1020\} +$ $3(1+2g_S)(3+2g_S)(815+2244g_L+1904g_L^2+512g_L^3+354g_S+720g_Lg_S+320g_L^2g_S+4g_S^2+16g_Lg_S^2 - 8g_S^3)\{840\} +$ $3(3+2g_S)(15375+40696g_L+38928g_L^2+15872g_L^3+2304g_L^4+22328g_S+45240g_Lg_S+29824g_L^2g_S+6400g_L^3g_S+11160g_S^2+15136g_Lg_S^2+5056g_L^2g_S^2+2336g_S^3+1568g_Lg_S^3+176g_S^4)\{822\}$ $+ (1+2g_S)(3+2g_S)(4893+11868g_L+9296g_L^2+2368g_L^3+4086g_S+6432g_Lg_S+2464g_L^2g_S+1132g_S^2+848g_Lg_S^2+104g_S^3)\{660\}$ $-6(3+2g_S)(35875+114060g_L+135440g_L^2+70976g_L^3+13824g_L^4+38432g_S+88920g_Lg_S+68544g_L^2g_S+17536g_L^3g_S+15304g_S^2+22864g_Lg_S^2+8512g_L^2g_S^2+2688g_S^3+1952g_Lg_S^3+176g_S^4)\{642\}$ $+ \beta_1\{444\}$
12	0	3	$(1+2g_L+2g_S)(3+2g_L+2g_S)(707+1774g_L+1296g_L^2+288g_L^3+990g_S+1584g_Lg_S+576g_L^2g_S+468g_S^2+360g_Lg_S^2+72g_S^3)\{1200\} +$ $-6(3+2g_L+2g_S)(11+8g_L+2g_S)(707+1774g_L+1296g_L^2+288g_L^3+990g_S+1584g_Lg_S+576g_L^2g_S+468g_S^2+360g_Lg_S^2+72g_S^3)\{1020\} +$ $+ \alpha_1\{840\} + \alpha_2\{820\} + \alpha_3\{660\} + \alpha_4\{642\} + \alpha_5\{444\}$

Table 4. Low-lying polynomials for the BC_3 model. The constants β_1 and α_m for $m = 1, \dots, 5$ are given below.

$$\begin{aligned}
\beta_1 &= 3(179375 + 658280g_L + 972744g_L^2 + 725280g_L^3 + 273024g_L^4 + 41472g_L^5 + 263910g_S + 755080g_Lg_S + 813232g_L^2g_S + 391936g_L^3g_S + 71424g_L^4g_S \\
&\quad + 153384g_S^2 + 322608g_Lg_S^2 + 226400g_L^2g_S^2 + 53120g_L^3g_S^2 + 44048g_S^3 + 60896g_Lg_S^3 + 21056g_L^2g_S^3 + 6256g_S^4 + 4288g_Lg_S^4 + 352g_S^5) \\
\alpha_1 &= 12(82425 + 272562g_L + 359094g_L^2 + 235132g_L^3 + 76320g_L^4 + 9792g_L^5 + 140802g_S + 367196g_Lg_S + 358868g_L^2g_S + 155808g_L^3g_S + 25344g_L^4g_S + \\
&\quad + 93944g_S^2 + 180904g_Lg_S^2 + 116136g_L^2g_S^2 + 24912g_L^3g_S^2 + 30384g_S^3 + 38160g_Lg_S^3 + 11952g_L^2g_S^3 + 4752g_S^4 + 2880g_Lg_S^4 + 288g_S^5) \\
\alpha_2 &= 3(60795 + 331284g_L + 626140g_L^2 + 537952g_L^3 + 214848g_L^4 + 32256g_L^5 + 123666g_S + 501240g_Lg_S + 685160g_L^2g_S + 386496g_L^3g_S + 77184g_L^4g_S + \\
&\quad + 93224g_S^2 + 270256g_Lg_S^2 + 239760g_L^2g_S^2 + 66816g_L^3g_S^2 + 32112g_S^3 + 60192g_Lg_S^3 + 26208g_L^2g_S^3 + 5040g_S^4 + 4608g_Lg_S^4 + 288g_S^5) \\
\alpha_3 &= +(-894789 - 2962624g_L - 3859432g_L^2 - 2483776g_L^3 - 793728g_L^4 - 101376g_L^5 - 1379182g_S - 3583344g_Lg_S - 3421424g_L^2g_S - 1425024g_L^3g_S - 218880g_L^4g_S + \\
&\quad - 828536g_S^2 - 1581472g_Lg_S^2 - 982368g_L^2g_S^2 - 198144g_L^3g_S^2 - 243792g_S^3 - 304704g_Lg_S^3 - 92736g_L^2g_S^3 - 35280g_S^4 - 21888g_Lg_S^4 - 2016g_S^5) \\
\alpha_4 &= -6(141855 + 471268g_L + 675384g_L^2 + 543616g_L^3 + 243072g_L^4 + 46080g_L^5 + 234514g_S + 610784g_Lg_S + 631568g_L^2g_S + 319104g_L^3g_S + 66816g_L^4g_S + \\
&\quad + 143624g_S^2 + 266656g_Lg_S^2 + 170784g_L^2g_S^2 + 39168g_L^3g_S^2 + 41328g_S^3 + 47232g_Lg_S^3 + 13248g_L^2g_S^3 + 5616g_S^4 + 2880g_Lg_S^4 + 288g_S^5) \\
\alpha_5 &= -3(-236425 - 635390g_L - 560488g_L^2 - 113008g_L^3 + 77184g_L^4 + 29952g_L^5 - 327810g_S - 644352g_Lg_S - 347696g_L^2g_S + 12672g_L^3g_S + 36864g_L^4g_S + \\
&\quad - 177176g_S^2 - 236560g_Lg_S^2 - 63072g_L^2g_S^2 + 10944g_L^3g_S^2 - 46512g_S^3 - 36864g_Lg_S^3 - 2880g_L^2g_S^3 - 5904g_S^4 - 2016g_Lg_S^4 - 288g_S^5)
\end{aligned}$$

E $A_1^{\oplus 3}$ states

ℓ	ℓ_3	ℓ_4	$h_\ell^{(g)}$
0	0	0	1
2	1	0	$-2x^2 + y^2 + z^2$
4	1	1	$4(g+1)(2g+1)x^4 - (2g+3)x^2((10g+9)y^2 - (2g+1)z^2) + (2g+1)(y^2+z^2)(4(g+1)y^2 - (2g+1)z^2)$
4	2	0	$8(g+1)(2g+1)x^4 - 8(g+1)(2g+3)x^2(y^2+z^2) + (2g+1)(2g+3)(y^2+z^2)^2$
6	2	1	$8(g+1)(2g+1)x^6 - 4x^4((4g(3g+10) + 29)y^2 + (2g+1)z^2) + x^2(y^2+z^2)((4g(9g+31) + 101)y^2 - (2g+1)(6g+11)z^2) - (2g+1)(y^2+z^2)^2((4g+6)y^2 - (2g+1)z^2)$
6	3	0	$-16(g+1)(2g+1)x^6 + 24(g+1)(2g+5)x^4(y^2+z^2) - 6(2g+3)(2g+5)x^2(y^2+z^2)^2 + (2g+1)(2g+5)(y^2+z^2)^3$
8	3	1	$32(g+1)(g+2)(2g+1)x^8 - 8(g+2)x^6((4g(7g+30) + 101)y^2 + (2g+1)(2g+11)z^2) + 12(g+2)(2g+5)x^4(y^2+z^2)((10g+29)y^2 - (2g+1)z^2) - (2g+5)x^2(y^2+z^2)^2((4g(13g+58) + 247)y^2 - (2g+1)(10g+23)z^2) + (2g+1)(2g+5)(y^2+z^2)^3(4(g+2)y^2 - (2g+1)z^2)$
8	2	2	$8(g+2)(2g+1)(2g+3)x^8 - 8(g+2)(2g+3)x^6((10g+29)y^2 - (2g+1)z^2) + 3(2g+5)x^4((4g(11g+52) + 237)y^4 - 2(2g+3)(2g+5)y^2z^2 - (4g(g+2) + 3)z^4) - 2(2g+3)x^2(y^2+z^2)(4(g+2)(10g+29)y^4 - (4g(7g+34) + 157)y^2z^2 + (2g+1)(2g+7)z^4) + (2g+1)(2g+3)(y^2+z^2)^2(8(g+2)y^4 - 8(g+2)y^2z^2 + (2g+1)z^4)$
8	4	0	$64(g+1)(g+2)(2g+1)x^8 - 128(g+1)(g+2)(2g+7)x^6(y^2+z^2) + 48(g+2)(2g+5)(2g+7)x^4(y^2+z^2)^2 - 16(g+2)(2g+5)(2g+7)x^2(y^2+z^2)^3 + (2g+1)(2g+5)(2g+7)(y^2+z^2)^4$
10	4	1	$-64(g+2)(2g+3)x^8((8g^2+42g+39)y^2 + (g+6)(2g+1)z^2) + 64(g+1)(g+2)(2g+1)(2g+3)x^{10} + 16(g+2)(2g+7)x^6(y^2+z^2)((4g(11g+59) + 267)y^2 + (3-4(g-1)g)z^2) - 8(2g+5)(2g+7)x^4(y^2+z^2)^2((2g+7)(14g+27)y^2 - (2g+1)(4g+9)z^2) + (2g+3)(2g+7)x^2(y^2+z^2)^3((4g(17g+93) + 489)y^2 - (2g+1)(14g+39)z^2) \& - (2g+1)(2g+3)(2g+7)(y^2+z^2)^4(2(2g+5)y^2 - (2g+1)z^2)$
10	3	2	$-(2g+7)x^4(y^2+z^2)((2g(292g^2+2138g+5103) + 7947)y^4 - 2(2g+3)(4g(17g+82) + 387)y^2z^2 + (2g+1)(2g+3)(2g+9)z^4) + 16(g+2)(2g+1)(2g+3)^2x^{10} - 8(g+2)(2g+3)x^8((4g(11g+59) + 267)y^2 + (3-4(g-1)g)z^2) + 2(2g+7)x^6((2g(2g(86g+615) + 2813) + 4161)y^4 + 2(2g+3)(4g(g+16) + 111)y^2z^2 - (2g+1)(2g+3)(10g+21)z^4) + 2(2g+3)(2g+7)x^2(y^2+z^2)^2(2(2g+5)(14g+27)y^4 - (2g+3)(22g+51)y^2z^2 + 2(4g(g+2) + 3)z^4) - (4g(g+2) + 3)(y^2+z^2)^3(8(g+2)(2g+5)y^4 - 4(2g+3)(2g+5)y^2z^2 + (4g(g+2) + 3)z^4)$
10	5	0	$-128(g+1)(g+2)(2g+1)(2g+3)x^{10} + 320(g+1)(g+2)(2g+3)(2g+9)x^8(y^2+z^2) - 160(g+2)(2g+3)(2g+7)(2g+9)x^6(y^2+z^2)^2 + 80(g+2)(2g+5)(2g+7)(2g+9)x^4(y^2+z^2)^3 - 10(2g+3)(2g+5)(2g+7)(2g+9)x^2(y^2+z^2)^4 + (2g+1)(2g+3)(2g+7)(2g+9)(y^2+z^2)^5$

Table 5. Low-lying polynomials for the $A_1^{\oplus 3}$ model at coinciding couplings for $\ell_3 \geq \ell_4$.

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References

- [1] A.P. Polychronakos, *Physics and mathematics of Calogero particles*, *J. Phys. A* **39** (2006) 12793 [[hep-th/0607033](https://arxiv.org/abs/hep-th/0607033)] [[INSPIRE](#)].
- [2] M.V. Feigin, *Intertwining relations for the spherical parts of generalized Calogero operators*, *Theor. Math. Phys.* **135** (2003) 497.
- [3] T. Hakobyan, A. Nersessian and V. Yeghikyan, *Cuboctahedric Higgs oscillator from the Calogero model*, *J. Phys. A* **42** (2009) 205206 [[arXiv:0808.0430](https://arxiv.org/abs/0808.0430)] [[INSPIRE](#)].
- [4] T. Hakobyan, S. Krivonos, O. Lechtenfeld and A. Nersessian, *Hidden symmetries of integrable conformal mechanical systems*, *Phys. Lett. A* **374** (2010) 801 [[arXiv:0908.3290](https://arxiv.org/abs/0908.3290)] [[INSPIRE](#)].
- [5] O. Lechtenfeld, A. Nersessian and V. Yeghikyan, *Action-angle variables for dihedral systems on the circle*, *Phys. Lett. A* **374** (2010) 4647 [[arXiv:1005.0464](https://arxiv.org/abs/1005.0464)] [[INSPIRE](#)].
- [6] T. Hakobyan, O. Lechtenfeld, A. Nersessian and A. Saghatelian, *Invariants of the spherical sector in conformal mechanics*, *J. Phys. A* **44** (2011) 055205 [[arXiv:1008.2912](https://arxiv.org/abs/1008.2912)] [[INSPIRE](#)].
- [7] T. Hakobyan, O. Lechtenfeld and A. Nersessian, *The spherical sector of the Calogero model as a reduced matrix model*, *Nucl. Phys. B* **858** (2012) 250 [[arXiv:1110.5352](https://arxiv.org/abs/1110.5352)] [[INSPIRE](#)].
- [8] T. Hakobyan, O. Lechtenfeld, A. Nersessian, A. Saghatelian and V. Yeghikyan, *Action-angle variables and novel superintegrable systems*, *Phys. Part. Nucl.* **43** (2012) 577 [[INSPIRE](#)].
- [9] M. Feigin, O. Lechtenfeld and A.P. Polychronakos, *The quantum angular Calogero-Moser model*, *JHEP* **07** (2013) 162 [[arXiv:1305.5841](https://arxiv.org/abs/1305.5841)] [[INSPIRE](#)].
- [10] F. Correa, O. Lechtenfeld and M. Plyushchay, *Nonlinear supersymmetry in the quantum Calogero model*, *JHEP* **04** (2014) 151 [[arXiv:1312.5749](https://arxiv.org/abs/1312.5749)] [[INSPIRE](#)].
- [11] M. Feigin and T. Hakobyan, *On Dunkl angular momenta algebra*, *JHEP* **11** (2015) 107 [[arXiv:1409.2480](https://arxiv.org/abs/1409.2480)] [[INSPIRE](#)].
- [12] F. Correa and O. Lechtenfeld, *The tetrahedric angular Calogero model*, *JHEP* **10** (2015) 191 [[arXiv:1508.04925](https://arxiv.org/abs/1508.04925)] [[INSPIRE](#)].
- [13] C.M. Bender and S. Boettcher, *Real spectra in non-Hermitian Hamiltonians having PT symmetry*, *Phys. Rev. Lett.* **80** (1998) 5243 [[physics/9712001](https://arxiv.org/abs/physics/9712001)] [[INSPIRE](#)].
- [14] A. Fring and M. Znojil, *PT-symmetric deformations of Calogero models*, *J. Phys. A* **41** (2008) 194010 [[arXiv:0802.0624](https://arxiv.org/abs/0802.0624)] [[INSPIRE](#)].
- [15] A. Fring and M. Smith, *Antilinear deformations of Coxeter groups, an application to Calogero models*, *J. Phys. A* **43** (2010) 325201 [[arXiv:1004.0916](https://arxiv.org/abs/1004.0916)] [[INSPIRE](#)].
- [16] A. Fring and M. Smith, *PT invariant complex E_8 root spaces*, *Int. J. Theor. Phys.* **50** (2011) 974 [[arXiv:1010.2218](https://arxiv.org/abs/1010.2218)] [[INSPIRE](#)].
- [17] A. Fring and M. Smith, *Non-Hermitian multi-particle systems from complex root spaces*, *J. Phys. A* **45** (2012) 085203 [[arXiv:1108.1719](https://arxiv.org/abs/1108.1719)] [[INSPIRE](#)].
- [18] A. Fring, *PT-symmetric deformations of integrable models*, *Phil. Trans. Roy. Soc. Lond. A* **371** (2013) 20120046 [[arXiv:1204.2291](https://arxiv.org/abs/1204.2291)] [[INSPIRE](#)].

- [19] F. Calogero, *Solution of the one-dimensional N body problems with quadratic and/or inversely quadratic pair potentials*, *J. Math. Phys.* **12** (1971) 419 [Erratum *ibid.* **37** (1996) 3646] [INSPIRE].
- [20] M.A. Olshanetsky and A.M. Perelomov, *Classical integrable finite dimensional systems related to Lie algebras*, *Phys. Rept.* **71** (1981) 313 [INSPIRE].
- [21] M.A. Olshanetsky and A.M. Perelomov, *Quantum integrable systems related to Lie algebras*, *Phys. Rept.* **94** (1983) 313 [INSPIRE].
- [22] S. Wojciechowski, *Superintegrability of the Calogero-Moser system*, *Phys. Lett. A* **95** (1983) 279.
- [23] C.F. Dunkl, *Differential-difference operators associated to reflection groups*, *Trans. Amer. Math. Soc.* **311** (1989) 167.
- [24] O.A. Chalykh and A.P. Veselov, *Commutative rings of partial differential operators and Lie algebras*, *Commun. Math. Phys.* **126** (1990) 597.
- [25] G.J. Heckman, *A remark on the Dunkl differential-difference operators*, in *Harmonic analysis on reductive groups*, W. Barker and P. Sally eds., *Progr. Math.* **101**, Birkhäuser, Boston U.S.A., (1991), pg. 181.
- [26] P.W. Higgs, *Dynamical symmetries in a spherical geometry. 1*, *J. Phys. A* **12** (1979) 309 [INSPIRE].
- [27] H.I. Leemon, *Dynamical symmetries in a spherical geometry. 2*, *J. Phys. A* **12** (1979) 489 [INSPIRE].
- [28] C.F. Dunkl and Y. Xu, *Orthogonal polynomials of several variables*, Cambridge University Press, Cambridge U.K., (2001).
- [29] M.S. Plyushchay, *Deformed Heisenberg algebra, fractional spin fields and supersymmetry without fermions*, *Annals Phys.* **245** (1996) 339 [hep-th/9601116] [INSPIRE].
- [30] M. Plyushchay, *Hidden nonlinear supersymmetries in pure parabosonic systems*, *Int. J. Mod. Phys. A* **15** (2000) 3679 [hep-th/9903130] [INSPIRE].
- [31] F. Correa and M.S. Plyushchay, *Spectral singularities in PT -symmetric periodic finite-gap systems*, *Phys. Rev. D* **86** (2012) 085028 [arXiv:1208.4448] [INSPIRE].
- [32] J.F. Cariñena and M.S. Plyushchay, *ABC of ladder operators for rationally extended quantum harmonic oscillator systems*, *J. Phys. A* **50** (2017) 275202 [arXiv:1701.08657] [INSPIRE].
- [33] J.F. Cariñena, L. Inzunza and M.S. Plyushchay, *Duality-induced rational extensions of the isotonic oscillator*, arXiv:1707.07357 [INSPIRE].