

# Well-posedness, blow-up phenomena, and global solutions for the $b$ -equation

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**Abstract.** In the paper we first establish the local well-posedness for a family of nonlinear dispersive equations, the so called  $b$ -equation. Then we describe the precise blow-up scenario. Moreover, we prove that for the  $b$ -equation we do have the coexistence of global in time solutions and blow-up phenomena: Depending on the initial data solutions may exist for ever, while other data force the solution to produce a singularity in finite time. Finally, we prove the uniqueness and existence of global weak solution to the equation provided the initial data satisfy certain sign conditions.

## 1. Introduction

In the paper we study the following nonlinear dispersive equation:

$$(1.1) \quad \begin{cases} u_t - \alpha^2 u_{txx} + c_0 u_x + (b+1)uu_x + \Gamma u_{xxx} = \alpha^2 (bu_x u_{xx} + uu_{xxx}), & t > 0, x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

where  $c_0, b, \Gamma, \alpha$  are arbitrary real constants. Using the notation  $y := u - \alpha^2 u_{xx}$ , we can rewrite Eq. (1.1) as follows:

$$(1.2) \quad \begin{cases} y_t + c_0 u_x + u y_x + b u_x y + \Gamma u_{xxx} = 0, & t > 0, x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}. \end{cases}$$

The  $b$ -equation (1.2) can be derived as the family of asymptotically equivalent shallow water wave equations that emerges at quadratic order accuracy for any  $b \neq -1$  by an appropriate Kodama transformation, cf. [21], [22]. For the case  $b = -1$ , the corresponding Kodama transformation is singular and the asymptotic ordering is violated, cf. [21], [22]. The solutions of the  $b$ -equation (1.2) with  $c_0 = \Gamma = 0$  were studied numerically for various values of  $b$  in [27], [28], where  $b$  was taken as a bifurcation parameter. The symmetry conditions necessary for integrability of the  $b$ -equation (1.2) was investigated in [41]. The KdV equation, the Camassa-Holm equation and the Degasperis-Procesi equation are the only three integrable equations in the  $b$ -equation (1.2), which was shown in [18], [19] by using Painlevé analysis. The  $b$ -equation with  $c_0 = \Gamma = 0$  admits peakon solutions for any  $b \in \mathbb{R}$ , cf. [18], [27], [28].

If  $\alpha = 0$  and  $b = 2$ , then Eq. (1.2) becomes the well-known KdV equation which describes the unidirectional propagation of waves at the free surface of shallow water under the influence of gravity, cf. [20]. In this model  $u(t, x)$  represents the wave's height above a flat bottom,  $x$  is proportional to distance in the direction of propagation and  $t$  is proportional to the elapsed time. The KdV equation is completely integrable and its solitary waves are solitons [40]. The Cauchy problem of the KdV equation has been the subject of a number of studies, and a satisfactory local or global (in time) existence theory is now in hand (for example, see [32], [44]). It is shown that the KdV equation is globally well-posed for  $u_0 \in L^2(\mathbb{R})$ , cf. [44]. It is observed that the KdV equation does not accommodate wave breaking (by wave breaking we understand that the wave remains bounded while its slope becomes unbounded in finite time [46]).

For  $b = 2$  and  $\Gamma = 0$ , Eq. (1.2) becomes the Camassa-Holm equation, modelling the unidirectional propagation of shallow water waves over a flat bottom. Again  $u(t, x)$  stands for the fluid velocity at time  $t$  in the spatial  $x$  direction and  $c_0$  is a nonnegative parameter related to the critical shallow water speed ([4], [20], [29]). The Camassa-Holm equation is also a model for the propagation of axially symmetric waves in hyperelastic rods ([14], [16]). It has a bi-Hamiltonian structure ([25], [34]) and is completely integrable ([4], [8]). Its solitary waves are smooth if  $c_0 > 0$  and peaked in the limiting case  $c_0 = 0$ , cf. [5]. The orbital stability of the peaked solitons is proved in [13], and that of the smooth solitons in [15]. The explicit interaction of the peaked solitons is given in [2].

The Cauchy problem for the Camassa-Holm equation has been studied extensively. It has been shown that this equation is locally well-posed ([9], [35], [43]) for initial data  $u_0 \in H^s(\mathbb{R})$ ,  $s > 3/2$ . More interestingly, it has global strong solutions ([7], [9]) and also finite time blow-up solutions ([7], [9], [10], [35]). On the other hand, it has global weak solutions in  $H^1(\mathbb{R})$  ([3], [11], [12], [47]). The advantage of the Camassa-Holm equation in comparison with the KdV equation lies in the fact that the Camassa-Holm equation has peaked solitons and models wave breaking ([5], [10]).

If  $b = 3$  and  $c_0 = \Gamma = 0$  in Eq. (1.2), then we find the Degasperis-Procesi equation [19]. The formal integrability of the Degasperis-Procesi equation was obtained in [17] by constructing a Lax pair. It has a bi-Hamiltonian structure with an infinite sequence of conserved quantities and admits exact peakon solutions which are analogous to the Camassa-Holm peakons [17].

The Degasperis-Procesi equation can be regarded as a model for nonlinear shallow water dynamics and its asymptotic accuracy is the same as for the Camassa-Holm shallow water equation ([21], [22]). An inverse scattering approach for computing  $n$ -peakon solutions to the Degasperis-Procesi equation was presented in [38]. Its traveling wave solution was investigated in [33], [45].

The Cauchy problem for the Degasperis-Procesi equation has been studied recently. Local well-posedness of this equation is established in [50] for initial data  $u_0 \in H^s(\mathbb{R})$ ,  $s > 3/2$ . Similar to the Camassa-Holm equation, the Degasperis-Procesi equation has also global strong solutions ([23], [36], [51]) and also finite time blow-up solutions ([23], [36], [50], [51]). On the other hand, it has global weak solutions in  $H^1(\mathbb{R})$  ([23], [51]) and global entropy weak solutions belonging to the class  $L^1(\mathbb{R}) \cap BV(\mathbb{R})$  and to the class  $L^2(\mathbb{R}) \cap L^4(\mathbb{R})$ , cf. [6].

Although the Degasperis-Procesi equation is similar to the Camassa-Holm equation in several aspects, these two equations are truly different. One of the novel features of the Degasperis-Procesi different from the Camassa-Holm equation is that it has not only peakon solutions [17] and periodic peakon solutions [52], but also shock peakons [37] and the periodic shock waves [24].

Despite the abundant literature on the above three special cases of Eq. (1.2), in the case of  $\alpha > 0$  and  $b, c_0, \Gamma \in \mathbb{R}$ , the Cauchy problem for Eq. (1.2) seems not have been discussed so far. The aim of this paper is to establish the local well-posedness for the *b*-equation, to derive a precise blowup scenario, to prove that the equation has strong solutions which exist globally in time and blow up in finite time, and to show the uniqueness and existence of global weak solution to the equation provided the initial data satisfy certain sign conditions.

The local well-posedness for Eq. (1.1) is obtained by applying Kato's semigroup approach [30]. The local well-posedness results for the Camassa-Holm equation ([9], [43]) and for the Degasperis-Procesi equation (see [50]) are special cases of our result.

Using delicate energy estimates, we present a precise blow-up scenario for Eq. (1.2), which depends only on the value of *b*. It not only covers the corresponding results for the Camassa-Holm equation in [7], [48] and the Degasperis-Procesi equation in [50], but also presents another different possible blow-up mechanism, i.e., if  $b < 1/2$ , then the solution to the *b*-equation (1.2) blows up in finite time if and only if the slope of the solution becomes unbounded from above in finite time. This precise blow-up behaviors of the *b*-equations is much more precise than the blow-up scenario  $\limsup_{t \rightarrow T} \|u_x\|_{L^\infty} = +\infty$ , which is quite a common PDEs blow-up scenario for nonlinear hyperbolic (see [1], [46]).

The derivation of global strong solutions from local results is a matter of a priori estimates. By using a continuous family of diffeomorphisms of the line and three different conservation laws associated to the *b*-equation (1.2), we obtain several different global existence results for the different values of *b*. Two of these global existence results holding for all  $b \in \mathbb{R}$  are very useful for us to pursue the existence and uniqueness of global weak solutions to the *b*-equation.

By applying a novel a priori estimate for solutions, which was used first for the wave-breaking of the Degasperis-Procesi equation in [36], we obtain a quite nice blow-up result for strong solutions to the *b*-equation for  $b \geq 3$ , provided the initial data is odd and satisfies some sign conditions. Based on the steepening lemma developed in [4], [7], we present another common blow-up result for strong solutions to the *b*-equation for  $1 < b \leq 3$ , provided the slope of the odd initial data is nonpositive. These two blow-up results together with Theorem 4.3 below give a clear picture for global smooth solutions and blowing-up smooth solutions of the *b*-equation for all  $b \geq 0$ .

Referring to an approximation procedure used first for the solutions to the Camassa-Holm equation [12], a partial integration result in Bochner spaces and Helly's theorem together with the obtained global existence results and two useful a priori estimates for strong solutions to Eq. (1.2), we obtain the uniqueness and existence of global weak solution to the equation provided the initial data satisfy certain sign conditions. The

obtained theorems provide a suitable mathematical framework to study further peakon solutions, which appear in the *b*-equation (1.2) with  $c_0 = \Gamma = 0$  for any  $b \in \mathbb{R}$ , cf. [18], [27], [28].

Our paper is organized as follows. In Section 2, we establish the local well-posedness of the Cauchy problem associated with Eq. (1.2). In Section 3, we derive a precise blow-up scenario for the *b*-equation (1.2). In section 4, we investigate the global existence of strong solutions to Eq. (1.2). In Section 5, we study blow-up phenomena of strong solution to Eq. (1.2). The last section is devoted to prove the uniqueness and existence of global weak solution to Eq. (1.2), provided the initial data satisfy certain sign conditions.

## 2. Local well-posedness

In the section, we establish local well-posedness for Cauchy problem of Eq. (1.1) in  $H^r(\mathbb{R})$ ,  $r > 3/2$ .

We first introduce some notations. Let  $*$  denote the convolution and let  $[A, B] = AB - BA$  denote the commutator of the linear operators  $A$  and  $B$ . Let  $\|\cdot\|_Z$  denote the norm of the Banach space  $Z$ . For convenience, let  $\|\cdot\|_r$  and  $(\cdot, \cdot)_r$  denote the norm and the inner product of  $H^r(\mathbb{R})$ ,  $r \geq 0$ , respectively. Let  $X$  and  $Y$  be Hilbert spaces such that  $Y$  is continuously and densely embedded in  $X$  and let  $Q: Y \rightarrow X$  be a topological isomorphism.

Let  $\alpha > 0$  be given. With  $y = u - \alpha^2 u_{xx}$ , Eq. (1.2) takes the form of a quasi-linear evolution equation of hyperbolic type:

$$(2.1) \quad \begin{cases} y_t + \left(u - \frac{\Gamma}{\alpha^2}\right) y_x + b u_x y + \left(c_0 + \frac{\Gamma}{\alpha^2}\right) u_x = 0, & t > 0, x \in \mathbb{R}, \\ y(0, x) = u_0(x) - \alpha^2 \partial_x^2 u_0(x), & x \in \mathbb{R}. \end{cases}$$

Note that if  $p(x) := \frac{1}{2\alpha} e^{-|\frac{x}{\alpha}|}$ ,  $x \in \mathbb{R}$ , then  $(1 - \alpha^2 \partial_x^2)^{-1} f = p * f$  for all  $f \in L^2(\mathbb{R})$  and  $p * y = u$ . Using this relation, we can rewrite Eq. (2.1) as follows:

$$(2.2) \quad \begin{cases} u_t + \left(u - \frac{\Gamma}{\alpha^2}\right) u_x = -\partial_x p * \left(\frac{b}{2} u^2 + \frac{(3-b)\alpha^2}{2} u_x^2 + \left(c_0 + \frac{\Gamma}{\alpha^2}\right) u\right), & t > 0, x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

or in the equivalent form:

$$(2.3) \quad \begin{cases} u_t + \left(u - \frac{\Gamma}{\alpha^2}\right) u_x = -\partial_x (1 - \alpha^2 \partial_x^2)^{-1} \left(\frac{b}{2} u^2 + \frac{(3-b)\alpha^2}{2} u_x^2 + \left(c_0 + \frac{\Gamma}{\alpha^2}\right) u\right), & t > 0, x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}. \end{cases}$$

**Theorem 2.1.** *Given  $u_0 \in H^r(\mathbb{R})$ ,  $r > 3/2$ , there exists a  $T = T(\alpha, b, c_0, \Gamma, \|u_0\|_r) > 0$ , and a unique solution  $u$  to Eq. (1.1) (or Eq. (2.3)) such that*

$$u = u(\cdot, u_0) \in C([0, T]; H^r(\mathbb{R})) \cap C^1([0, T]; H^{r-1}(\mathbb{R})).$$

*The solution depends continuously on the initial data, i.e. the mapping*

$$u_0 \rightarrow u(\cdot, u_0) : H^r(\mathbb{R}) \rightarrow C([0, T]; H^r(\mathbb{R})) \cap C^1([0, T]; H^{r-1}(\mathbb{R}))$$

*is continuous. Moreover,  $T$  may be chosen independent of  $r$  in the following sense. If  $u = u(\cdot, u_0) \in C([0, T]; H^r(\mathbb{R})) \cap C^1([0, T]; H^{r-1}(\mathbb{R}))$  to Eq. (1.1) (or Eq. (2.3)), and if  $u_0 \in H^{r'}(\mathbb{R})$  for some  $r' \neq r$ ,  $r' > 3/2$ , then for the same  $T$ ,*

$$u \in C([0, T]; H^{r'}(\mathbb{R})) \cap C^1([0, T]; H^{r'-1}(\mathbb{R})).$$

*Furthermore, if  $u_0 \in H^\infty(\mathbb{R}) = \bigcap_{r \geq 0} H^r(\mathbb{R})$ , then  $u \in C([0, T]; H^\infty(\mathbb{R}))$ .*

*Proof.* Given  $u \in H^r(\mathbb{R})$ ,  $r > \frac{3}{2}$ , define the operator  $A(u) := \left(u - \frac{\Gamma}{\alpha^2}\right) \partial_x$ . By [48], Lemma 2.6, we know that  $A(u)$  is quasi-m-accretive, uniformly on bounded sets in  $H^{r-1}(\mathbb{R})$ . Moreover, we have<sup>1)</sup>

$$A(u) \in L(H^r(\mathbb{R}), H^{r-1}(\mathbb{R}))$$

and for all  $u, z, w \in H^r(\mathbb{R})$ ,

$$\|(A(u) - A(z))w\|_{r-1} \leq \mu_1 \|u - z\|_{r-1} \|w\|_r.$$

Define the operator  $B(u) := [(1 - \partial_x^2)^{\frac{1}{2}}, u \partial_x] (1 - \partial_x^2)^{-\frac{1}{2}}$ . By Lemma 2.7 in [48], we find

$$B(u) \in L(H^{r-1}(\mathbb{R}))$$

and for all  $u, z \in H^r(\mathbb{R})$ ,  $w \in H^{r-1}(\mathbb{R})$ ,

$$\|(B(u) - B(z))w\|_{r-1} \leq \mu_2 \|u - z\|_r \|w\|_{r-1}.$$

Define  $f(u) = -\partial_x (1 - \alpha^2 \partial_x^2)^{-1} \left( \frac{b}{2} u^2 + \frac{(3-b)\alpha^2}{2} u_x^2 + \left( c_0 + \frac{\Gamma}{\alpha^2} \right) u \right)$ . Following the lines of the proof of [49], Lemma 2.8, we can prove that  $f$  is bounded on bounded sets in  $H^r(\mathbb{R})$ , and satisfies

$$(a) \quad \|f(y) - f(z)\|_r \leq \mu_3 \|y - z\|_r, \quad \forall y, z \in H^r(\mathbb{R}),$$

$$(b) \quad \|f(y) - f(z)\|_{r-1} \leq \mu_4 \|y - z\|_{r-1}, \quad \forall y, z \in H^r(\mathbb{R}).$$

<sup>1)</sup> We write  $L(X, Y)$  for the space of all bounded linear operators mapping  $X$  into  $Y$ . In case  $X = Y$  we use the notation  $L(X)$ .

Set  $Y = H^r(\mathbb{R})$ ,  $X = H^{r-1}(\mathbb{R})$ , and  $Q = \Lambda = (1 - \partial_x^2)^{\frac{1}{2}}$ . Obviously,  $Q$  is an isomorphism of  $Y$  onto  $X$ . Following the proof of [48], Theorem 2.2, in view of Kato's theory for abstract quasilinear evolution equation of hyperbolic type, cf. [48], Theorem 2.1, one can obtain the local well-posedness of Eq. (1.1) (or (2.3)) in  $H^r(\mathbb{R})$ ,  $r > 3/2$ . Finally, as in the proof of [48], Theorem 2.3, one can show that  $T$  may be chosen independent of  $r$ . This completes the proof of Theorem 2.1.  $\square$

**Remark 2.1.** Theorem 2.1 covers the recent local well-posedness results for the Camassa-Holm equation [9], [43] and the Degasperis-Procesi equation [50].

### 3. Precise blow-up scenario

In this section, we present the precise blow-up scenario for solutions to the *b*-equation (2.1).

We first recall the following two lemmas.

**Lemma 3.1** ([31]). *If  $s > 0$ , then  $H^s(\mathbb{R}) \cap L^\infty(\mathbb{R})$  is an algebra. Moreover*

$$\|fg\|_s \leq c(\|f\|_{L^\infty(\mathbb{R})}\|g\|_s + \|f\|_s\|g\|_{L^\infty(\mathbb{R})}),$$

where  $c$  is a constant depending only on  $s$ .

**Lemma 3.2** ([31]). *If  $s > 0$ , then*

$$\|[\Lambda^s, f]g\|_{L^2(\mathbb{R})} \leq c(\|\partial_x f\|_{L^\infty(\mathbb{R})}\|\Lambda^{s-1}g\|_{L^2(\mathbb{R})} + \|\Lambda^s f\|_{L^2(\mathbb{R})}\|g\|_{L^\infty(\mathbb{R})}),$$

where  $c$  is a constant depending only on  $s$ .

Then we prove the following useful result.

**Theorem 3.1.** *Let  $\alpha > 0$  and  $u_0 \in H^r(\mathbb{R})$ ,  $r > 3/2$  be given and assume that  $T$  is the existence time of the corresponding solution with the initial data  $u_0$ . If there exists  $M > 0$  such that*

$$\|u_x(t, x)\|_{L^\infty(\mathbb{R})} \leq M, \quad t \in [0, T),$$

then the  $H^r(\mathbb{R})$ -norm of  $u(t, \cdot)$  does not blow up on  $[0, T)$ .

*Proof.* Let  $u$  be the solution to Eq. (1.1) with initial data  $u_0 \in H^r(\mathbb{R})$ ,  $r > 3/2$ , and let  $T$  be the maximal existence time of the solution  $u$ , which is guaranteed by Theorem 2.1. Throughout this proof,  $c > 0$  stands for a generic constant depending only on  $\alpha$ ,  $r$ ,  $b$  and  $\left(c_0 + \frac{\Gamma}{\alpha^2}\right)$ .

Applying the operator  $\Lambda^r$  to Eq. (2.3), multiplying by  $\Lambda^r u$ , and integrating over  $\mathbb{R}$ , we obtain

$$(3.1) \quad \frac{d}{dt} \|u\|_r^2 = -2(uu_x, u)_r + \frac{2\Gamma}{\alpha^2} (u_x, u)_r + 2(u, f_1(u))_r + 2(u, f_2(u))_r,$$

where  $f_1(u) = -\partial_x(1 - \alpha^2 \partial_x^2)^{-1} \left( \frac{b}{2} u^2 \right) = -b(1 - \alpha^2 \partial_x^2)^{-1} (uu_x)$  and

$$f_2(u) = -\partial_x(1 - \alpha^2 \partial_x^2)^{-1} \left( \frac{(3-b)\alpha^2}{2} u_x^2 + \left( c_0 + \frac{\Gamma}{\alpha^2} \right) u \right).$$

Let us estimate the first term of the right-hand side of Eq. (3.1).<sup>2)</sup>

$$(3.2) \quad \begin{aligned} |(uu_x, u)_r| &= |(\Lambda^r(u\partial_x u), \Lambda^r u)_0| \\ &= |([\Lambda^r, u]\partial_x u, \Lambda^r u)_0 + (u\Lambda^r \partial_x u, \Lambda^r u)_0| \\ &\leq \|[\Lambda^r, u]\partial_x u\|_0 \|\Lambda^r u\|_0 + \frac{1}{2} |(u_x \Lambda^r u, \Lambda^r u)_0| \\ &\leq \left( c \|u_x\|_{L^\infty(\mathbb{R})} + \frac{1}{2} \|u_x\|_{L^\infty(\mathbb{R})} \right) \|u\|_r^2 \\ &\leq c \|u_x\|_{L^\infty(\mathbb{R})} \|u\|_r^2. \end{aligned}$$

Here, we applied Lemma 3.2 with  $s = r$ . Secondly, we find for the second term of the right-hand side of Eq. (3.1) that

$$(3.3) \quad (u_x, u)_r = (\Lambda^r u_x, \Lambda^r u)_0 = (\partial_x \Lambda^r u, \Lambda^r u)_0 = 0.$$

Furthermore, we estimate the third term of the right-hand side of Eq. (3.1) in the following way:

$$(3.4) \quad \begin{aligned} |(f_1(u), u)_r| &= |b| |(\Lambda^r(1 - \alpha^2 \partial_x^2)^{-1}(u\partial_x u), \Lambda^r u)_0| \\ &\leq |b| \max\{1, \alpha^{-2}\} |(\Lambda^{r-1}(u\partial_x u), \Lambda^{r-1} u)_0| \\ &\leq c |([\Lambda^{r-1}, u]\partial_x u, \Lambda^{r-1} u)_0 + (u\Lambda^{r-1} \partial_x u, \Lambda^{r-1} u)_0| \\ &\leq c \left( \|[\Lambda^{r-1}, u]\partial_x u\|_0 \|\Lambda^{r-1} u\|_0 + \frac{1}{2} |(u_x \Lambda^{r-1} u, \Lambda^{r-1} u)_0| \right) \\ &\leq c \left( c \|u_x\|_{L^\infty(\mathbb{R})} + \frac{1}{2} \|u_x\|_{L^\infty(\mathbb{R})} \right) \|u\|_{r-1}^2 \\ &\leq c \|u_x\|_{L^\infty(\mathbb{R})} \|u\|_r^2. \end{aligned}$$

Here, we applied Lemma 3.2 with  $s = r - 1$ . Finally, let us estimate the fourth term of the right-hand side of Eq. (3.1).

<sup>2)</sup> A similar estimate for  $(uu_x, u)_r$  has been derived in [35].

$$\begin{aligned}
(3.5) \quad |(f_2(u), u)_r| &\leq \|f_2(u)\|_r \|u\|_r \\
&\leq c \left( \frac{|(3-b)\alpha^2|}{2} \|u_x^2\|_{r-1} + \left| c_0 + \frac{\Gamma}{\alpha^2} \right| \|u\|_{r-1} \right) \|u\|_r \\
&\leq c (\|u_x\|_{L^\infty(\mathbb{R})} \|u\|_{r-1} + \|u\|_{r-1}) \|u\|_r \\
&\leq c (\|u_x\|_{L^\infty(\mathbb{R})} + 1) \|u\|_r^2,
\end{aligned}$$

where we again apply Lemma 3.1 with  $s = r$ . Combining inequalities (3.2)–(3.5) with (3.1), we obtain

$$\frac{d}{dt} \|u\|_r^2 \leq c(M + 1) \|u\|_r^2.$$

An application of Gronwall's inequality yields

$$(3.6) \quad \|u(t)\|_r^2 \leq \exp(c(M + 1)t) \|u(0)\|_r^2.$$

This completes the proof of the theorem.  $\square$

Next, we present the precise blow-up scenario for the *b*-equation.

**Theorem 3.2.** *Assume that  $\alpha > 0$  and  $u_0 \in H^r(\mathbb{R})$ ,  $r > 3/2$ . If  $b = 1/2$ , then every solution will exist globally in time. If  $b > 1/2$ , then the solution blows up in finite time if and only if the slope of the solution becomes unbounded from below in finite time. If  $b < 1/2$ , then the solution blows up in finite time if and only if the slope of the solution becomes unbounded from above in finite time.*

*Proof.* Applying Theorem 2.1 and a simple density argument, it suffices to consider the case  $r = 3$ . Let  $T > 0$  be the maximal time of existence of the solution  $u$  to Eq. (2.2) with initial data  $u_0 \in H^3(\mathbb{R})$ . From Theorem 2.1 we know that  $u \in C([0, T]; H^3(\mathbb{R})) \cap C^1([0, T]; H^2(\mathbb{R}))$ .

Multiplying Eq. (2.2) by  $y = u - \alpha^2 u_{xx}$ , and integrating by parts, we get

$$(3.7) \quad \frac{d}{dt} \int_{\mathbb{R}} y^2 dx = -2b \int_{\mathbb{R}} y^2 u_x dx - 2 \int_{\mathbb{R}} u y y_x dx = (1 - 2b) \int_{\mathbb{R}} u_x y^2 dx.$$

Here, we used the relations:  $\int_{\mathbb{R}} y u_x dx = 0$  and  $\int_{\mathbb{R}} y y_x dx = 0$ . Note that

$$(3.8) \quad \min\{1, \sqrt{2}\alpha, \alpha^2\} \|u(t, \cdot)\|_2 \leq \|y(t, \cdot)\|_{L^2} \leq \max\{1, \sqrt{2}\alpha, \alpha^2\} \|u(t, \cdot)\|_2.$$

From (3.7), we see that if  $b = 1/2$ , then we have

$$\begin{aligned}
\|u_x(t, \cdot)\|_{L^\infty} &\leq \|u(t, \cdot)\|_2 \leq (\min\{1, \sqrt{2}\alpha, \alpha^2\})^{-1} \|y(t, \cdot)\|_{L^2} \\
&= (\min\{1, \sqrt{2}\alpha, \alpha^2\})^{-1} \|y(0, \cdot)\|_{L^2} < \infty.
\end{aligned}$$

This implies, in view of Theorem 3.1, that every solution exists globally in time.



If  $b > 1/2$  and the slope of the solution is bounded from below or if  $b < 1/2$  and the slope of the solution is bounded from above on  $[0, T) \times \mathbb{R}$ , then there exists  $M > 0$  such that

$$\frac{d}{dt} \int_{\mathbb{R}} y^2 dx \leq M \int_{\mathbb{R}} y^2 dx.$$

By means of Gronwall's inequality, we have

$$\|y(t, \cdot)\|_{L^2(\mathbb{R})} \leq \|y(0, \cdot)\|_{L^2(\mathbb{R})} \exp\{Mt\}, \quad \forall t \in [0, T).$$

In view of (3.8) and Theorem 3.1, we see that the solution does not blow up in finite time.

On the other hand, by Theorem 2.1 and Sobolev's imbedding theorem, we see that if the slope of the solution becomes unbounded from below or from above in finite time, then the solution will blow up in finite time. This completes the proof of the theorem.  $\square$

**Remark 3.1.** Theorem 3.2 not only covers the corresponding results for the Camassa-Holm equation in [7], [48] and the Degasperis-Procesi equation in [50], but also presents another different possible blow-up mechanism, i.e., if  $b < 1/2$ , then the solution to the  $b$ -equation (1.2) blows up in finite time if and only if the slope of the solution becomes unbounded from above in finite time.

#### 4. Global strong solutions

In this section, we will show that there exist global strong solutions to Eq. (1.2) for any  $b \in \mathbb{R}$ , provided the initial data  $u_0$  and the parameters  $\alpha$ ,  $c_0$ , and  $\Gamma$  satisfy suitable conditions.

Let  $u_0 \in H^r(\mathbb{R})$ ,  $r > 3/2$ . Then there exists a unique solution

$$u \in C([0, T); H^r(\mathbb{R})) \cap C^1([0, T); H^{r-1}(\mathbb{R}))$$

to Eq. (1.1) with initial data  $u_0$ , defined for the existence time  $T > 0$ , cf. Theorem 2.1. Thus, we can consider the differential equation

$$(4.1) \quad \begin{cases} q_t = u(t, q) - \frac{\Gamma}{\alpha^2}, & t \in [0, T), \\ q(0, x) = x, & x \in \mathbb{R}. \end{cases}$$

Solutions to (4.1) may be viewed as Lagrange coordinates associated to the  $b$ -equation.

**Lemma 4.1.** *Let  $u_0 \in H^r(\mathbb{R})$ ,  $r > 3/2$ , and let  $T > 0$  be the existence time of the corresponding solution  $u$  to Eq. (1.1). Then the Eq. (4.1) has a unique solution  $q \in C^1([0, T) \times \mathbb{R}, \mathbb{R})$ . Moreover, the map  $q(t, \cdot)$  is an increasing diffeomorphism of  $\mathbb{R}$  with  $q_x(t, x) > 0$  for  $(t, x) \in [0, T) \times \mathbb{R}$ .*

*Proof.* Due to  $u(t, x) \in C^1([0, T); H^{r-1}(\mathbb{R}))$  and  $H^{r-1}(\mathbb{R}) \subset C(\mathbb{R})$ , we see that both functions  $u(t, x) - \frac{\Gamma}{\alpha^2}$  and  $u_x(t, x)$  are bounded, Lipschitz in the space variable  $x$ , and of

class  $C^1$  in time. Therefore, well-known classical results in the theory of ordinary differential equations yield that Eq. (4.1) has a unique solution  $q \in C^1([0, T] \times \mathbb{R}, \mathbb{R})$ . Differentiation of Eq. (4.1) with respect to  $t$  yields

$$(4.2) \quad \begin{cases} \frac{d}{dt} q_x = q_{xt} = u_x(t, q) q_x, & t \in [0, T], \\ q_x(0, x) = 1, & x \in \mathbb{R}. \end{cases}$$

The solution to Eq. (4.2) is given by

$$(4.3) \quad q_x(t, x) = \exp\left(\int_0^t u_x(s, q(s, x)) ds\right), \quad (t, x) \in [0, T] \times \mathbb{R}.$$

For arbitrarily fixed  $T' \in (0, T)$ , Sobolev's imbedding theorem implies that

$$\sup_{(s, x) \in [0, T'] \times \mathbb{R}} |u_x(s, x)| < \infty.$$

Thus, we infer from Eq. (4.3) that there exists a constant  $K > 0$  such that  $q_x(t, x) \geq e^{-tK} > 0$  for  $(t, x) \in [0, T'] \times \mathbb{R}$ . This completes the proof of the lemma.  $\square$

**Lemma 4.2.** *Assume that  $u_0 \in H^r(\mathbb{R})$ ,  $r > 3/2$ . Let  $T > 0$  be the existence time of the corresponding solution  $u$  to Eq. (1.1). If  $b \neq 0$ , then we have*

$$(4.4) \quad \left(y(t, q(t, x)) + b^{-1}\left(c_0 + \frac{\Gamma}{\alpha^2}\right)\right) [q_x(t, x)]^b = \left(y_0(x) + b^{-1}\left(c_0 + \frac{\Gamma}{\alpha^2}\right)\right),$$

where  $(t, x) \in [0, T] \times \mathbb{R}$  and  $y = u - \alpha^2 u_{xx}$ . If  $c_0 + \frac{\Gamma}{\alpha^2} = 0$ , then for all  $b \in \mathbb{R}$  we have

$$(4.5) \quad y(t, q(t, x)) [q_x(t, x)]^b = y_0(x).$$

*Proof.* As in the proof of Theorem 3.2 it suffices to prove the above lemma for  $r = 3$ . Let  $T > 0$  be the maximal existence time of the solution  $u$  with initial data  $u_0 \in H^3(\mathbb{R})$ .

Differentiating the left-hand side of Eq. (4.4) with respect to time variable  $t$ , in view of the relations (4.1), (4.2) and (2.1), we obtain

$$\begin{aligned} & \frac{d}{dt} \left\{ \left(y(t, q(t, x)) + b^{-1}\left(c_0 + \frac{\Gamma}{\alpha^2}\right)\right) [q_x(t, x)]^b \right\} \\ &= (y_t(t, q) + y_x q_t) [q_x]^b + \left(y(t, q) + b^{-1}\left(c_0 + \frac{\Gamma}{\alpha^2}\right)\right) b [q_x]^{b-1} q_{xt} \\ &= \left(y_t(t, q) + \left(u - \frac{\Gamma}{\alpha^2}\right) y_x + b y(t, q) u_x(t, q) + \left(c_0 + \frac{\Gamma}{\alpha^2}\right) u_x(t, q)\right) [q_x]^b \\ &= 0. \end{aligned}$$

This proves (4.4). Similarly, we can prove (4.5) and complete the proof of the lemma.  $\square$

**Lemma 4.3.** *Let  $u_0 \in H^r(\mathbb{R})$ ,  $r > 3/2$  be given. If  $y_0 := (u_0 - \alpha^2 u_{0,xx}) \in L^1(\mathbb{R})$ , then, as long as the solution  $u(t, \cdot)$  with initial data  $u_0$  given by Theorem 2.1 exists, we have*

$$\int_{\mathbb{R}} u(t, x) dx = \int_{\mathbb{R}} u_0 dx = \int_{\mathbb{R}} y_0 dx = \int_{\mathbb{R}} y(t, x) dx.$$

*Proof.* Again it suffices to consider the case  $r = 3$ . Let  $T$  be the maximal time of existence of the solution  $u$  to Eq. (2.2) with initial data  $u_0 \in H^3(\mathbb{R})$ .

Note that  $u_0 = p * y_0$  and  $y_0 = (u_0 - \alpha^2 u_{0,xx}) \in L^1(\mathbb{R})$ . By Young's inequality, we get

$$\|u_0\|_{L^1(\mathbb{R})} = \|p * y_0\|_{L^1(\mathbb{R})} \leq \|p\|_{L^1(\mathbb{R})} \|y_0\|_{L^1(\mathbb{R})} \leq \|y_0\|_{L^1(\mathbb{R})}.$$

Integrating Eq. (2.2) by parts, we get

$$\frac{d}{dt} \int_{\mathbb{R}} u dx = - \int_{\mathbb{R}} \left( u - \frac{\Gamma}{\alpha^2} \right) u_x dx - \int_{\mathbb{R}} \partial_x p * \left( \frac{b}{2} u^2 + \frac{3-b}{2} u_x^2 + \left( c_0 + \frac{\Gamma}{\alpha^2} \right) u \right) dx = 0.$$

It follows that

$$\int_{\mathbb{R}} u dx = \int_{\mathbb{R}} u_0 dx.$$

Due to  $y = u - \alpha^2 u_{xx}$ , we have

$$\begin{aligned} \int_{\mathbb{R}} y dx &= \int_{\mathbb{R}} u dx - \alpha^2 \int_{\mathbb{R}} u_{xx} dx = \int_{\mathbb{R}} u dx \\ &= \int_{\mathbb{R}} u_0 dx = \int_{\mathbb{R}} u_0 dx - \alpha^2 \int_{\mathbb{R}} u_{0,xx} dx = \int_{\mathbb{R}} y_0 dx. \end{aligned}$$

This completes the proof of the lemma.  $\square$

We now present our first global existence results.

**Theorem 4.1.** *Let  $u_0 \in H^r(\mathbb{R})$   $r > \frac{3}{2}$  be given and assume  $c_0 + \frac{\Gamma}{\alpha^2} = 0$ . If  $y_0 := u_0 - \alpha^2 \partial_x^2 u_0 \in L^1(\mathbb{R})$  is nonnegative, then the corresponding solution to Eq. (2.2) is defined globally in time. Moreover,  $I(u) = \int_{\mathbb{R}} u dx$  is a conservation law, and that for all  $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$ , we have:*

(i)  $y(t, x) \geq 0, u(t, x) \geq 0$  and

$$\|y_0\|_{L^1(\mathbb{R})} = \|y(t)\|_{L^1(\mathbb{R})} = \|u(t, \cdot)\|_{L^1(\mathbb{R})} = \|u_0\|_{L^1(\mathbb{R})}.$$

(ii)  $\|u_x(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\alpha} \|u_0\|_{L^1(\mathbb{R})}$  and

$$\|u(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \|u(t, \cdot)\|_1 \leq \max\{\alpha, \alpha^{-1}\} e^{\frac{|b-2|r}{2x}} \|u_0\|_{L^1(\mathbb{R})} \|u_0\|_1.$$

*Proof.* As we mentioned before that we only need to prove the above theorem for  $r = 3$ . Let  $T > 0$  be the maximal existence time of the solution  $u$  with initial data  $u_0 \in H^3(\mathbb{R})$ .

If  $y_0(x) \geq 0$ , then the identity (4.5) ensures that  $y(t, x) \geq 0$  for all  $t \in [0, T)$ . Noticing that  $u = p * y$  and the positivity of  $p$ , we infer that  $u(t, x) \geq 0$  for all  $t \in [0, T)$ . By Lemma 4.3, we obtain

$$(4.6) \quad -\alpha u_x(t, x) + \int_{-\infty}^x u(t, x) dx = \int_{-\infty}^x (u - \alpha^2 u_{xx}) dx = \int_{-\infty}^x y dx \\ \leq \int_{-\infty}^{\infty} y dx = \int_{\mathbb{R}} y_0 dx = \int_{\mathbb{R}} u_0 dx.$$

Therefore, from (4.6) we find that

$$(4.7) \quad u_x(t, x) \geq -\frac{1}{\alpha} \int_{\mathbb{R}} u_0 dx = -\frac{1}{\alpha} \|u_0\|_{L^1(\mathbb{R})}, \quad \forall (t, x) \in [0, T) \times \mathbb{R}.$$

On the other hand, by  $y(t, x) \geq 0$  for all  $t \in [0, T)$ , we obtain

$$\alpha u_x(t, x) - \int_{-\infty}^x u dx = - \int_{-\infty}^x (u - \alpha^2 u_{xx}) dx = - \int_{-\infty}^x y dx \leq 0.$$

By the above inequality and  $u(t, x) \geq 0$  for all  $t \in [0, T)$ , we get

$$(4.8) \quad u_x(t, x) \leq \frac{1}{\alpha} \int_{-\infty}^x u dx \leq \frac{1}{\alpha} \int_{\mathbb{R}} u dx = \frac{1}{\alpha} \int_{\mathbb{R}} u_0 dx = \frac{1}{\alpha} \|u_0\|_{L^1(\mathbb{R})}.$$

Thus, (4.7) and (4.8) imply that

$$(4.9) \quad |u_x(t, x)| \leq \|u_x(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\alpha} \|u_0\|_{L^1(\mathbb{R})}, \quad \forall (t, x) \in [0, T) \times \mathbb{R}.$$

By Theorem 3.2 and (4.9), we deduce that  $T = \infty$ . Recalling finally Lemma 4.3, we get assertion (i).

Multiplying (1.1) by  $u$  and integrating by parts, we obtain

$$(4.10) \quad \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} (u^2(t, x) + \alpha^2 u_x^2(t, x)) dx \\ = \alpha^2 \int_{\mathbb{R}} (u^2 u_{xxx} + buu_x u_{xx}) dx \\ = \left(1 - \frac{b}{2}\right) \alpha^2 \int_{\mathbb{R}} u_x^3 dx \leq \left|1 - \frac{b}{2}\right| \|u_x(t, \cdot)\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} \alpha^2 u_x^2 dx.$$

In view of (4.9)–(4.10), an application of Gronwall's inequality leads to

$$(4.11) \quad \int_{\mathbb{R}} (u^2(t, x) + \alpha^2 u_x^2(t, x)) dx \leq e^{\frac{|2-b|t}{\alpha} \|u_0\|_{L^1(\mathbb{R})}} \int_{\mathbb{R}} (u_0^2 + \alpha^2 u_{0,x}^2) dx.$$

Consequently,

$$(4.12) \quad \|u(t, \cdot)\|_1 \leq \max\{\alpha, \alpha^{-1}\} e^{\frac{|2-b|t}{2\alpha} \|u_0\|_{L^1(\mathbb{R})}} \|u_0\|_1.$$

On the other hand,

$$(4.13) \quad u^2(t, x) = \int_{-\infty}^x 2uu_x dx \leq \int_{\mathbb{R}} (u^2 + u_x^2) dx = \|u(t, \cdot)\|_1^2.$$

Combining (4.12) with (4.13), we obtain assertion (ii). This completes the proof of the theorem.  $\square$

In a similar way to the proof of Theorem 4.1, we can get the following global existence result.

**Theorem 4.2.** *Let  $u_0 \in H^r(\mathbb{R})$ ,  $r > \frac{3}{2}$  be given and assume that  $c_0 + \frac{\Gamma}{\alpha^2} = 0$ . If  $y_0 := u_0 - \alpha^2 \partial_x^2 u_0 \in L^1(\mathbb{R})$  is nonpositive, then the corresponding solution to Eq. (2.2) is defined globally in time. Moreover,  $I(u) = \int_{\mathbb{R}} u dx$  is a conservation law, and that for all  $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$ , we have:*

(i)  $y(t, x) \leq 0$ ,  $u(t, x) \leq 0$  and

$$\|y_0\|_{L^1(\mathbb{R})} = \|y(t)\|_{L^1(\mathbb{R})} = \|u(t, \cdot)\|_{L^1(\mathbb{R})} = \|u_0\|_{L^1(\mathbb{R})}.$$

(ii)  $\|u_x(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\alpha} \|u_0\|_{L^1(\mathbb{R})}$  and

$$\|u(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \|u(t, \cdot)\|_1 \leq \max\{\alpha, \alpha^{-1}\} e^{\frac{|b-2|t}{2\alpha} \|u_0\|_{L^1(\mathbb{R})}} \|u_0\|_1.$$

**Remark 4.1.** Theorems 4.1–4.2 cover the global existence results of strong solutions to the Camassa-Holm equation in [9] and the Degasperis-Procesi equation in [51].

We now present a further result on global existence for the *b*-equation.

**Theorem 4.3.** *Let  $0 \leq b \leq 1$  and  $c_0 + \frac{\Gamma}{\alpha^2} = 0$  be given and assume that  $u_0 \in H^r(\mathbb{R}) \cap W^{2, \frac{1}{b}}(\mathbb{R})$ ,  $r > \frac{3}{2}$ . Then the corresponding solution to Eq. (2.2) is defined globally in time.*

*Proof.* It suffices to prove the above theorem for  $r = 3$ . Let  $u_0 \in H^3(\mathbb{R})$ , and let  $T > 0$  be the maximal existence time of the solution  $u$  with initial data  $u_0$ . By (4.5) in Lemma 4.2, we get the following conservative quantities [27]:

$$(4.14) \quad \|y(t, x)\|_{L^{\frac{1}{b}}(\mathbb{R})} = \|y(0, x)\|_{L^{\frac{1}{b}}(\mathbb{R})} \quad \text{if } 0 < b \leq 1,$$

and

$$(4.15) \quad \|y(t, x)\|_{L^\infty(\mathbb{R})} = \|y(0, x)\|_{L^\infty(\mathbb{R})} \quad \text{if } b = 0.$$

Set  $p = 1/b$  if  $0 < b \leq 1$  and  $p = +\infty$  if  $b = 0$ . Note that  $u - \alpha^2 u_{xx} = y(t, x)$ . By the  $L^p$ -theory for linear elliptic equations (see e.g. [26]), we obtain  $u(t, \cdot) \in W^{2,p}(\mathbb{R})$ . By Sobolev imbedding theorem, we see that for  $p \geq 1$ ,  $W^{2,p}(\mathbb{R}) \subset C_B^1(\mathbb{R})$ . Thus, (4.14) and (4.15) imply that  $\|u_x(t, \cdot)\|_{L^\infty(\mathbb{R})}$  is uniformly bounded for all  $t \in [0, T]$ . By Theorem 3.1, we deduce that the corresponding solution to Eq. (2.2) is defined globally in time. This completes the proof of the theorem.  $\square$

We finally offer a fourth global existence result.

**Theorem 4.4.** *Let  $b = -1/2n$ , for some  $n = 1, 2, \dots$  and assume that  $c_0 = \Gamma = 0$ . If  $u_0 \in H^r(\mathbb{R}) \cap W^{3, -\frac{1}{b}}(\mathbb{R})$ ,  $r > 3/2$ , then the corresponding solution to Eq. (2.2) is defined globally in time.*

*Proof.* Again we consider only the case  $r = 3$ . Let  $u_0 \in H^3(\mathbb{R})$ , and let  $T > 0$  be the maximal existence time of the solution  $u$  with initial data  $u_0$ . Note that if  $c_0 = \Gamma = 0$  and  $b \neq 0$ , then we have the following conservation law (see [18]):

$$(4.16) \quad \int_{\mathbb{R}} y^{-\frac{1}{b}}(t, x) \left( \frac{y_x^2(t, x)}{b^2 y^2(t, x)} + 1 \right) dx = \int_{\mathbb{R}} y^{-\frac{1}{b}}(0, x) \left( \frac{y_x^2(0, x)}{b^2 y^2(0, x)} + 1 \right) dx.$$

Since  $b = -1/2n$ , for some  $n = 1, 2, \dots$ , it follows from (4.16) and Hölder's inequality that

$$(4.17) \quad \begin{aligned} & \int_{\mathbb{R}} y^{2n-2}(t, x) (y_x^2(t, x) + 4n^2 y^2(t, x)) dx \\ &= \int_{\mathbb{R}} y^{2n-2}(0, x) (y_x^2(0, x) + 4n^2 y^2(0, x)) dx \\ &\leq \|y(0, x)\|_{L^{2n}(\mathbb{R})}^{2n-2} \|y_x(0, x)\|_{L^{2n}(\mathbb{R})}^2 + 4n^2 \|y(0, x)\|_{L^{2n}(\mathbb{R})}^{2n}. \end{aligned}$$

By the  $L^p$  theory for linear elliptic equations and Sobolev imbedding theorem, in view of (4.17), we conclude that  $\|u_x(t, \cdot)\|_{L^\infty(\mathbb{R})}$  is uniformly bounded for all  $t \in [0, T]$ . By Theorem 3.1, we deduce that the corresponding solution to Eq. (1.1) is defined globally in time. This completes the proof of the theorem.  $\square$

## 5. Blow-up results

In this section we address the question of the formation of singularities for solutions to Eq. (1.1). We will present two blow-up results for the  $b$ -equation.

Let us first consider the following situation.

**Theorem 5.1.** *Let  $u_0 \in H^r(\mathbb{R})$ ,  $r > 3/2$  be given and assume that  $c_0 = \Gamma = 0$  and  $b \geq 3$ . If  $u_0 \not\equiv 0$  is odd such that*

$$\begin{cases} y_0(x) = u_0(x) - \alpha^2 u_{0,xx}(x) \geq 0 & \text{for } x \leq 0, \\ y_0(x) = u_0(x) - \alpha^2 u_{0,xx}(x) \leq 0 & \text{for } x \geq 0, \end{cases}$$

then the corresponding solution to Eq. (2.2) blows up in finite time. Moreover, the maximal existence time of the corresponding solution is strictly less than  $-1/u_x(0, 0)$ .

*Proof.* Again, we only need to show that the above theorem holds for  $r = 3$ . Let  $T > 0$  be the maximal time of existence of the solution  $u$  to Eq. (2.2) with the initial data  $u_0 \in H^3(\mathbb{R})$ .

Differentiating Eq. (2.2) with respect to  $x$ , in view of  $-\alpha^2 \partial_x^2 p * f = p * f - f$  and  $c_0 = \Gamma = 0$ , we have

$$(5.1) \quad u_{tx} = -uu_{xx} + \frac{b}{2\alpha^2}u^2 + \frac{1-b}{2}u_x^2 - \frac{1}{\alpha^2}p * \left( \frac{b}{2}u^2 + \frac{(3-b)\alpha^2}{2}u_x^2 \right).$$

Note that if  $c_0 = \Gamma = 0$ , then Eq. (2.2) possesses the symmetry  $(u, x) \rightarrow (-u, -x)$ . Since  $u_0(x)$  is odd, the uniqueness of solutions implies that  $u(t, x)$  is odd too. By  $y = u - \alpha^2 u_x$ , we know that  $y(t, x)$  is also odd. Furthermore, we have

$$(5.2) \quad u_{tx}(t, 0) = \frac{1-b}{2}u_x^2(t, 0) - \frac{1}{\alpha^2}p * \left( \frac{b}{2}u^2 + \frac{(3-b)\alpha^2}{2}u_x^2 \right)(t, 0).$$

Since the function  $q(t, x)$  is an increasing diffeomorphism of  $\mathbb{R}$  with  $q_x(t, x) > 0$  with respect to  $t$ , we infer from the assumption of the theorem and (4.5) in Lemma 4.2 that for  $t \in [0, T)$ , we obtain

$$(5.3) \quad \begin{cases} y(t, x) \geq 0 & \text{if } x \leq q(t, 0) = 0, \\ y(t, x) \leq 0 & \text{if } x \geq q(t, 0) = 0, \end{cases}$$

and  $y(t, q(t, 0)) = 0, t \in [0, T)$ . By  $u = p * y$ , we have

$$(5.4) \quad u(t, x) = \frac{1}{2\alpha} e^{-\frac{x}{\alpha}} \int_{-\infty}^x e^{\frac{\xi}{\alpha}} y(t, \xi) d\xi + \frac{1}{2\alpha} e^{\frac{x}{\alpha}} \int_x^{\infty} e^{-\frac{\xi}{\alpha}} y(t, \xi) d\xi.$$

Differentiating Eq. (5.4) with respect to  $x$  yields for  $(t, x) \in [0, T) \times \mathbb{R}$ ,

$$(5.5) \quad \alpha u_x(t, x) = -\frac{1}{2\alpha} e^{-\frac{x}{\alpha}} \int_{-\infty}^x e^{\frac{\xi}{\alpha}} y(t, \xi) d\xi + \frac{1}{2\alpha} e^{\frac{x}{\alpha}} \int_x^{\infty} e^{-\frac{\xi}{\alpha}} y(t, \xi) d\xi.$$

From (5.4) and (5.5), we conclude that

$$(5.6) \quad u^2(t, x) - \alpha^2 u_x^2(t, x) = \frac{1}{\alpha^2} \int_{-\infty}^x e^{\frac{\xi}{\alpha}} y(t, \xi) d\xi \int_x^{\infty} e^{-\frac{\xi}{\alpha}} y(t, \xi) d\xi.$$

By the definition of  $p(x)$ , we have

$$\begin{aligned}
(5.7) \quad p * (u^2 - \alpha^2 u_x^2)(t, 0) &= \frac{1}{2\alpha} \int_{-\infty}^{\infty} e^{-\frac{|\eta|}{\alpha}} (u^2(t, \eta) - \alpha^2 u_\eta^2(t, \eta)) d\eta \\
&= \frac{1}{2\alpha} \int_{-\infty}^0 e^{\frac{\eta}{\alpha}} (u^2(t, \eta) - \alpha^2 u_\eta^2(t, \eta)) d\eta \\
&\quad + \frac{1}{2\alpha} \int_0^{\infty} e^{-\frac{\eta}{\alpha}} (u^2(t, \eta) - \alpha^2 u_\eta^2(t, \eta)) d\eta.
\end{aligned}$$

Further, we deduce from (5.3) and (5.6) that

$$\begin{aligned}
(5.8) \quad &\int_{-\infty}^0 e^{\frac{\eta}{\alpha}} (u^2(t, \eta) - \alpha^2 u_\eta^2(t, \eta)) d\eta \\
&= \frac{1}{\alpha^2} \int_{-\infty}^0 e^{\frac{\eta}{\alpha}} \left( \int_{\eta}^{\infty} e^{-\frac{\xi}{\alpha}} y(t, \xi) d\xi \right) \left( \int_{-\infty}^{\eta} e^{\frac{\xi}{\alpha}} y(t, \xi) d\xi \right) d\eta \\
&= \frac{1}{\alpha^2} \int_{-\infty}^0 e^{\frac{\eta}{\alpha}} \left( \int_0^{\infty} e^{-\frac{\xi}{\alpha}} y(t, \xi) d\xi \right) \left( \int_{-\infty}^{\eta} e^{\frac{\xi}{\alpha}} y(t, \xi) d\xi \right) d\eta \\
&\quad + \frac{1}{\alpha^2} \int_{-\infty}^0 e^{\frac{\eta}{\alpha}} \left( \int_{\eta}^0 e^{-\frac{\xi}{\alpha}} y(t, \xi) d\xi \right) \left( \int_{-\infty}^{\eta} e^{\frac{\xi}{\alpha}} y(t, \xi) d\xi \right) d\eta \\
&\geq \frac{1}{\alpha^2} \int_{-\infty}^0 e^{\frac{\eta}{\alpha}} \left( \int_0^{\infty} e^{-\frac{\xi}{\alpha}} y(t, \xi) d\xi \right) \left( \int_{-\infty}^{\eta} e^{\frac{\xi}{\alpha}} y(t, \xi) d\xi \right) d\eta \\
&\geq \frac{1}{\alpha} \int_0^{\infty} e^{-\frac{\xi}{\alpha}} y(t, \xi) d\xi \int_{-\infty}^0 e^{\frac{\xi}{\alpha}} y(t, \xi) d\xi \\
&= \alpha (u^2(t, 0) - \alpha^2 u_x^2(t, 0)).
\end{aligned}$$

Similarly, one can prove that

$$(5.9) \quad \int_0^{\infty} e^{-\frac{\eta}{\alpha}} (u^2(t, \eta) - \alpha^2 u_\eta^2(t, \eta)) d\eta \geq \alpha (u^2(t, 0) - \alpha^2 u_x^2(t, 0)).$$

Combining (5.7)–(5.9), in view of the oddness of  $u(t, x)$ , we obtain

$$(5.10) \quad p * (u^2 - \alpha^2 u_x^2)(t, 0) \geq u^2(t, 0) - \alpha^2 u_x^2(t, 0) = -\alpha^2 u_x^2(t, 0).$$

By (5.2) and (5.10), we have

$$\begin{aligned}
(5.11) \quad u_{tx}(t, 0) &= \frac{1-b}{2} u_x^2(t, 0) - \frac{1}{\alpha^2} p * \left( \frac{b}{2} u^2 + \frac{(3-b)\alpha^2}{2} u_x^2 \right) (t, 0) \\
&= \frac{1-b}{2} u_x^2(t, 0) - \frac{b-3}{2\alpha^2} p * (u^2 - \alpha^2 u_x^2)(t, 0) - \frac{3}{2\alpha^2} p * (u^2)(t, 0) \\
&\leq -u_x^2(t, 0).
\end{aligned}$$

The assumptions of the theorem and (5.4) imply now



$$u_x(0, 0) = -\frac{1}{2\alpha^2} \int_{-\infty}^0 e^{\frac{\eta}{\alpha}} y_0(\eta) d\eta + \frac{1}{2\alpha^2} \int_0^{\infty} e^{-\frac{\eta}{\alpha}} y_0(\eta) d\eta < 0.$$

It then follows from (5.11) that

$$u_x(t, 0) \leq u_x(0, 0) < 0, \quad \forall t \in [0, T].$$

Thus, solving the differential inequality (5.11), we get

$$\frac{1}{u_x(0, 0)} - \frac{1}{u_x(t, 0)} + t \leq 0, \quad \forall t \in [0, T].$$

Note that  $-\frac{1}{u_x(t, 0)} > 0$ . Thus we have

$$T < -\frac{1}{u_x(0, 0)}.$$

This completes the proof of the theorem.  $\square$

**Theorem 5.2.** *Let  $u_0 \in H^r(\mathbb{R})$ ,  $r > 3/2$  be given and assume that  $c_0 = \Gamma = 0$  and  $1 < b \leq 3$ . If  $u(0, x) \not\equiv 0$  is odd and  $u_x(0, 0) \leq 0$ , then the corresponding solution to Eq. (2.2) blows up in finite time. Moreover, if  $u_x(0, 0) < 0$ , then the maximal existence time of the corresponding solution is strictly less than  $-\frac{2}{(b-1)u_x(0, 0)}$ .*

*Proof.* As before, we only consider the case  $r = 3$ . Let  $T > 0$  be the maximal time of existence of the solution  $u$  to Eq. (2.2) with the initial data  $u_0 \in H^3(\mathbb{R})$ . Since  $u_0(x)$  is odd, it follows that  $u(t, x)$  is odd as well. By  $y = u - u_{xx}$ , we know that  $y(t, x)$  is odd. Following the same arguments as in the proof of Theorem 5.1, we see that Eq. (5.2) is also valid in the present situation. Since  $1 < b \leq 3$ , it follows from (5.2) that

$$(5.12) \quad u_{tx}(t, 0) \leq -\frac{b-1}{2} u_x^2(t, 0)$$

and

$$(5.13) \quad u_{tx}(t, 0) \leq -\frac{b}{2\alpha^2} p * (u^2)(t, 0).$$

By the uniqueness of  $u(t, x)$  and the assumption  $u(0, x) \not\equiv 0$ , we have

$$(5.14) \quad -\frac{b}{2\alpha^2} p * (u^2)(t, 0) < 0, \quad \forall t \in [0, T].$$

If  $u_x(0, 0) \leq 0$ , then the continuity of  $u_x(t, 0)$  together with (5.13) and (5.14) ensure that there exists  $t_1 \in [0, T)$  such that  $u_x(t_1, 0) < 0$ . Thus, following the same arguments at the end of Theorem 5.1, we get the desired conclusions.  $\square$

**Remark 5.1.** Theorems 5.1–5.2 show that there exist smooth solutions to the *b*-equation for any  $b > 1$  that blow up in finite time, while Theorem 4.3 shows that in the case  $0 \leq b \leq 1$  every smooth solution to the *b*-equation exists globally in time. This gives a clear picture for global smooth solutions and blowing-up smooth solutions of the *b*-equation for all  $b \geq 0$ .

**Remark 5.2.** By Theorems 4.1–4.2 and Theorems 5.1–5.2, we see that the lifespan of strong solutions of the *b*-equation for  $b > 1$  is not affected by the smoothness of the initial data, but by the shape of the initial data.

**Remark 5.3.** Up to now, we were not able to present a clear picture for global smooth solutions and blowing-up smooth solutions of the *b*-equation for negative values of *b*. This will be the topic of further research.

## 6. Global weak solutions

In this section, we first recall a result on partial integration in Bochner spaces, see e.g. [39] and useful approximation lemmas presented in [12]. We then show that there exists a unique global weak solution to Eq. (1.1) provided the initial data  $u_0$  satisfies certain sign conditions.

Let us introduce some notations. The duality bracket between  $H^1(\mathbb{R})$  and  $H^{-1}(\mathbb{R})$  is always denoted by  $\langle \cdot, \cdot \rangle$ . We write  $M(\mathbb{R})$  for the space of Radon measures on  $\mathbb{R}$  with bounded total variation. The cone of positive measures is denoted by  $M^+(\mathbb{R})$ . Let  $BV(\mathbb{R})$  stand for the space of functions with bounded variation and write  $\mathbb{V}(f)$  for the total variation of  $f \in BV(\mathbb{R})$ . Finally, let  $\{\rho_n\}_{n \geq 1}$  denote the mollifiers

$$\rho_n(x) := \left( \int_{\mathbb{R}} \rho(\xi) d\xi \right)^{-1} n\rho(nx), \quad x \in \mathbb{R}, n \geq 1,$$

where  $\rho \in C_c^\infty(\mathbb{R})$  is defined by

$$\rho(x) := \begin{cases} e^{\frac{1}{x^2-1}}, & \text{for } |x| < 1, \\ 0, & \text{for } |x| \geq 1. \end{cases}$$

Note that Eq. (1.2) has peakon solutions with corners at their peaks, cf. [18], [27], [28]. Obviously, such solutions are not strong solutions to Eq. (2.2). In order to provide a mathematical framework for the study of peakon solutions, we shall give the notion of weak solutions to Eq. (2.2).

Let us return to Eq. (2.2). If we set

$$F(u) := \left( \frac{u^2}{2} - \frac{\Gamma}{\alpha^2} u \right) + p * \left( \frac{b}{2} u^2 + \frac{3-b}{2} u_x^2 + \left( c_0 + \frac{\Gamma}{\alpha^2} \right) u \right),$$

then Eq. (2.2) can be rewritten as the conservation law

$$u_t + F(u)_x = 0, \quad u(0, x) = u_0, \quad t > 0, x \in \mathbb{R}.$$

In order to introduce the notion of weak solutions to the *b*-equation, let  $\psi \in C_0^\infty([0, T] \times \mathbb{R})$  denote the set of all the restrictions to  $[0, T] \times \mathbb{R}$  of smooth functions on  $\mathbb{R}^2$  with compact support contained in  $(-T, T) \times \mathbb{R}$ .

**Definition 6.1.** Let  $u_0 \in H^1(\mathbb{R})$ . If  $u$  belongs to  $L_{\text{loc}}^\infty([0, T]; H^1(\mathbb{R}))$  and satisfies the identity

$$\int_0^T \int_{\mathbb{R}} (u\psi_t + F(u)\psi_x) dx dt + \int_{\mathbb{R}} u_0(x)\psi(0, x) dx = 0$$

for all  $\psi \in C_0^\infty([0, T] \times \mathbb{R})$ , then  $u$  is called a weak solution to Eq. (2.2). If  $u$  is a weak solution on  $[0, T)$  for every  $T > 0$ , then it is called global weak solution to Eq. (2.2) (or Eq. (1.2)).

The following proposition is standard.

**Proposition 6.1.** (i) *Every strong solution is a weak solution.*

(ii) *If  $u$  is a weak solution and  $u \in C([0, T]; H^r(\mathbb{R})) \cap C^1([0, T]; H^{r-1}(\mathbb{R}))$ ,  $r > 3/2$ , then it is a strong solution.*

Let us now prepare the construction of global weak solutions.

**Lemma 6.1** ([39]). *Let  $T > 0$ . If*

$$f, g \in L^2((0, T); H^1(\mathbb{R})) \quad \text{and} \quad \frac{df}{dt}, \frac{dg}{dt} \in L^2((0, T); H^{-1}(\mathbb{R})),$$

then  $f, g$  are a.e. equal to a function continuous from  $[0, T]$  into  $L^2(\mathbb{R})$  and

$$\langle f(t), g(t) \rangle - \langle f(s), g(s) \rangle = \int_s^t \left\langle \frac{df(\tau)}{d\tau}, g(\tau) \right\rangle d\tau + \int_s^t \left\langle \frac{dg(\tau)}{d\tau}, f(\tau) \right\rangle d\tau$$

for all  $s, t \in [0, T]$ .

**Lemma 6.2** ([12]). *Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be uniformly continuous and bounded. If  $\mu \in M(\mathbb{R})$ , then*

$$\|\rho_n * \mu\|_{L^1(\mathbb{R})} \leq \|\rho_n\|_{L^1(\mathbb{R})} \|\mu\|_{M(\mathbb{R})} \leq \|\mu\|_{M(\mathbb{R})}$$

and

$$[\rho_n * (f\mu) - (\rho_n * f)(\rho_n * \mu)] \rightarrow 0, \quad \text{as } n \rightarrow \infty \text{ in } L^1(\mathbb{R}).$$

**Lemma 6.3** ([12]). *Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be uniformly continuous and bounded. If  $g \in L^\infty(\mathbb{R})$ , then*

$$[\rho_n * (fg) - (\rho_n * f)(\rho_n * g)] \rightarrow 0, \quad \text{as } n \rightarrow \infty \text{ in } L^\infty(\mathbb{R}).$$

**Lemma 6.4** ([12]). *Assume that  $u(t, \cdot)$  is uniformly bounded in  $W^{1,1}(\mathbb{R})$  for all  $t \in \mathbb{R}_+$ . Then for a.e.  $t \in \mathbb{R}_+$*

$$\frac{d}{dt} \int_{\mathbb{R}} |\rho_n * u| dx = \int_{\mathbb{R}} (\rho_n * u_t) \operatorname{sgn}(\rho_n * u) dx$$

and

$$\frac{d}{dt} \int_{\mathbb{R}} |\rho_n * u_x| dx = \int_{\mathbb{R}} (\rho_n * u_{xt}) \operatorname{sgn}(\rho_n * u_x) dx.$$

For the proof of Lemma 6.4 we refer to the arguments in [12], pages 55–56.

Let us first present an existence result for global weak solutions.

**Theorem 6.1.** *Let  $u_0 \in H^1(\mathbb{R})$  be given and assume further that  $c_0 + \frac{\Gamma}{\alpha^2} = 0$  and*

$$(u_0 - \alpha^2 u_{0,xx}) \in M^+(\mathbb{R}).$$

Then Eq. (2.2) has a weak solution

$$u \in W^{1,\infty}(\mathbb{R}_+ \times \mathbb{R}) \cap L_{\text{loc}}^\infty(\mathbb{R}_+; H^1(\mathbb{R}))$$

with initial data  $u(0) = u_0$  and

$$(u(t, \cdot) - \alpha^2 u_{xx}(t, \cdot)) \in M^+(\mathbb{R})$$

is uniformly bounded for all  $t \in \mathbb{R}_+$ . Moreover,  $I(u)$  is a conservation law.

*Proof.* Let  $u_0 \in H^1(\mathbb{R})$  and assume that  $y_0 := u_0 - \alpha^2 u_{0,xx} \in M^+(\mathbb{R})$ . Note that  $u_0 = p * y_0$ . Thus, given  $f \in L^\infty(\mathbb{R})$ , we have

$$\begin{aligned} (6.1) \quad \|u_0\|_{L^1(\mathbb{R})} &= \|p * y_0\|_{L^1(\mathbb{R})} = \sup_{\|f\|_{L^\infty(\mathbb{R})} \leq 1} \int_{\mathbb{R}} f(x) \int_{\mathbb{R}} p(x - \xi) dy_0(\xi) dx \\ &= \sup_{\|f\|_{L^\infty(\mathbb{R})} \leq 1} \int_{\mathbb{R}} (p * f)(\xi) dy_0(\xi) \\ &\leq \sup_{\|f\|_{L^\infty(\mathbb{R})} \leq 1} \|p * f\|_{L^\infty(\mathbb{R})} \|y_0\|_{M(\mathbb{R})} \\ &\leq \sup_{\|f\|_{L^\infty(\mathbb{R})} \leq 1} \|p\|_{L^1(\mathbb{R})} \|f\|_{L^\infty(\mathbb{R})} \|y_0\|_{M(\mathbb{R})} = \|y_0\|_{M(\mathbb{R})}. \end{aligned}$$

We first prove that there exists a solution  $u$  with initial data  $u_0$ , which belongs to  $W_{\text{loc}}^{1,\infty}(\mathbb{R}_+ \times \mathbb{R}) \cap L_{\text{loc}}^\infty(\mathbb{R}_+; H^1(\mathbb{R}))$ , satisfying Eq. (2.1) in the sense of distributions.

Let us define  $u_0^n := \rho_n * u_0 \in H^\infty(\mathbb{R})$  for  $n \geq 1$ . Obviously, we have

$$(6.2) \quad u_0^n \rightarrow u_0 \quad \text{in } H^1(\mathbb{R}) \quad \text{for } n \rightarrow \infty$$

and

$$(6.3) \quad \begin{aligned} \|u_0^n\|_1 &= \|\rho_n * u_0\|_1 \leq \|u_0\|_1, \quad \forall n \geq 1, \\ \|u_0^n\|_{L^1(\mathbb{R})} &= \|\rho_n * u_0\|_{L^1(\mathbb{R})} \leq \|u_0\|_{L^1(\mathbb{R})}, \quad \forall n \geq 1, \end{aligned}$$

in view of Young's inequality. Note that for all  $n \geq 1$ ,

$$y_0^n := u_0^n - \alpha^2 u_{0,xx}^n = \rho_n * (y_0) \geq 0.$$

Referring to the proof of (6.1), we have

$$(6.4) \quad \|y_0^n\|_{L^1(\mathbb{R})} = \|\rho_n * y_0\|_{L^1(\mathbb{R})} \leq \|y_0\|_{M(\mathbb{R})}, \quad \forall n \geq 1.$$

By Theorem 2.1 and Theorem 4.1, in view of the assumption  $c_0 + \frac{\Gamma}{\alpha^2} = 0$ , we obtain that there exists a unique strong solution to Eq. (2.2),

$$u^n = u^n(\cdot, u_0^n) \in C([0, \infty); H^r(\mathbb{R})) \cap C^1([0, \infty); H^{r-1}(\mathbb{R})), \quad \forall r \geq 3.$$

Using Theorem 4.1 (i)–(ii), Young's inequality and (6.3), we obtain

$$(6.5) \quad \begin{aligned} & \left\| \left( u^n(t) - \frac{\Gamma}{\alpha^2} \right) u_x^n(t) \right\|_{L^2(\mathbb{R})} \\ & \leq \left\| u^n(t) - \frac{\Gamma}{\alpha^2} \right\|_{L^\infty(\mathbb{R})} \|u_x^n(t)\|_{L^2(\mathbb{R})} \\ & \leq \|u^n(t)\|_1^2 + \left| \frac{\Gamma}{\alpha^2} \right| \|u^n(t)\|_1 \\ & \leq \max\{\alpha^2, \alpha^{-2}\} e^{\frac{|b-2t|}{\alpha} \|u_0\|_{L^1(\mathbb{R})}} \|u_0\|_1^2 + \left| \frac{\Gamma}{\alpha^2} \right| \max\{\alpha, \alpha^{-1}\} e^{\frac{|b-2t|}{2\alpha} \|u_0\|_{L^1(\mathbb{R})}} \|u_0\|_1, \end{aligned}$$

for all  $t \geq 0$  and  $n \geq 1$ . By Young's inequality and Theorem 4.1 (i)–(ii), we get

$$(6.6) \quad \begin{aligned} & \left\| \partial_x p * \left( \frac{b}{2} [u^n(t)]^2 + \frac{(3-b)\alpha^2}{2} [u_x^n(t)]^2 \right) \right\|_{L^2(\mathbb{R})} \\ & \leq \|p_x\|_{L^2(\mathbb{R})} \max\left\{ \frac{|b|}{2}, \frac{|3-b|\alpha^2}{2} \right\} \|u^n(t)\|_1^2 \\ & \leq \|p_x\|_{L^2(\mathbb{R})} \max\left\{ \frac{|b|}{2}, \frac{|3-b|\alpha^2}{2} \right\} \max\{\alpha^2, \alpha^{-2}\} e^{\frac{|b-2t|}{\alpha} \|u_0\|_{L^1(\mathbb{R})}} \|u_0\|_1^2, \end{aligned}$$

for all  $t \geq 0$  and  $n \geq 1$ . Based on (6.5) and (6.6) and Eq. (2.2), we find

$$\begin{aligned}
(6.7) \quad & \left\| \frac{d}{dt} u^n(t) \right\|_{L^2(\mathbb{R})} \\
& \leq \left( \|p_x\|_{L^2(\mathbb{R})} \max \left\{ \frac{|b|}{2}, \frac{|3-b|\alpha^2}{2} \right\} + 1 \right) \max \{ \alpha^2, \alpha^{-2} \} e^{\frac{|b-2|t}{\alpha} \|u_0\|_{L^1(\mathbb{R})}} \|u_0\|_1^2 \\
& \quad + \left| \frac{\Gamma}{\alpha^2} \right| \max \{ \alpha, \alpha^{-1} \} e^{\frac{|b-2|t}{2\alpha} \|u_0\|_{L^1(\mathbb{R})}} \|u_0\|_1,
\end{aligned}$$

for all  $t \geq 0$  and  $n \geq 1$ . In view of (6.6) and (6.7), we get

$$(6.8) \quad \int_0^T \int_{\mathbb{R}} ([u^n(t, x)]^2 + [u_x^n(t, x)]^2 + [u_t^n(t, x)]^2) dx dt \leq M,$$

where  $M$  is a positive constant depending only on  $b$ ,  $\alpha$ ,  $T$ ,  $\|p_x\|_{L^2(\mathbb{R})}$ , and  $\|u_0\|_1$ . It then follows from (6.8) that the sequence  $\{u^n\}_{n \geq 1}$  is uniformly bounded in the space  $H^1((0, T) \times \mathbb{R})$ . Thus, we can extract a subsequence such that

$$(6.9) \quad u^{n_k} \rightharpoonup u \quad \text{weakly in } H^1((0, T) \times \mathbb{R}) \quad \text{for } n_k \rightarrow \infty$$

and

$$(6.10) \quad u^{n_k} \rightarrow u \quad \text{a.e. on } (0, T) \times \mathbb{R} \quad \text{for } n_k \rightarrow \infty,$$

for some  $u \in H^1((0, T) \times \mathbb{R})$ . By Theorem 4.1 (i)–(ii) and (6.3), we have that for fixed  $t \in (0, T)$ , the sequence  $u_x^{n_k}(t, \cdot) \in BV(\mathbb{R})$  satisfies

$$\begin{aligned}
\mathbb{V}[u_x^{n_k}(t, \cdot)] &= \|u_{xx}^{n_k}\|_{L^1(\mathbb{R})} \\
&\leq \alpha^{-2} \|u^{n_k}(t, \cdot)\|_{L^1(\mathbb{R})} + \alpha^{-2} \|y^{n_k}(t, \cdot)\|_{L^1(\mathbb{R})} \leq 2\alpha^{-2} \|y_0\|_{M(\mathbb{R})}
\end{aligned}$$

and

$$(6.11) \quad \|u_x^{n_k}(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\alpha} \|u_0^{n_k}\|_{L^1(\mathbb{R})} \leq \frac{1}{\alpha} \|u_0\|_{L^1(\mathbb{R})} \leq \alpha^{-1} \|y_0\|_{M(\mathbb{R})}.$$

Applying Helly's theorem, cf. [42], we conclude that there exists a subsequence, denoted again  $\{u_x^{n_k}(t, \cdot)\}$ , which converges at every point to some function  $v(t, \cdot)$  of finite variation with

$$\mathbb{V}(v(t, \cdot)) \leq 2\alpha^{-2} \|y_0\|_{M(\mathbb{R})}.$$

Since for almost all  $t \in (0, T)$ ,  $u_x^{n_k}(t, \cdot) \rightarrow u_x(t, \cdot)$  in  $D'(\mathbb{R})$  in view of (6.10), it follows that  $v(t, \cdot) = u_x(t, \cdot)$  for a.e.  $t \in (0, T)$ . Therefore, we have

$$(6.12) \quad u_x^{n_k}(t, \cdot) \rightarrow u_x(t, \cdot) \quad \text{a.e. on } (0, T) \times \mathbb{R} \quad \text{for } n_k \rightarrow \infty,$$

and for a.e.  $t \in (0, T)$ ,

$$(6.13) \quad \mathbb{V}[u_x(t, \cdot)] = \|u_{xx}\|_{M(\mathbb{R})} \leq 2\alpha^{-2} \|y_0\|_{M(\mathbb{R})}.$$

By Theorem 4.1 (i)–(ii), we have

$$\begin{aligned} & \left\| \frac{b}{2} [u^n(t)]^2 + \frac{(3-b)\alpha^2}{2} [u_x^n(t)]^2 \right\|_{L^2(\mathbb{R})} \\ & \leq \frac{|b|}{2} \|u^n(t)\|_{L^2(\mathbb{R})} \|u^n(t)\|_{L^\infty(\mathbb{R})} + \frac{|3-b|\alpha^2}{2} \|u_x^n(t)\|_{L^2(\mathbb{R})} \|u_x^n(t)\|_{L^\infty(\mathbb{R})} \\ & \leq \left( \frac{|b|}{2} \|u^n(t)\|_{L^\infty(\mathbb{R})} + \frac{|3-b|\alpha^2}{2} \|u_x^n(t)\|_{L^\infty(\mathbb{R})} \right) \|u^n(t)\|_1 \\ & \leq \frac{|b|}{2} \max\{\alpha^2, \alpha^{-2}\} e^{\frac{|b-2t|}{\alpha} \|u_0\|_{L^1(\mathbb{R})}} \|u_0\|_1^2 \\ & \quad + \frac{|3-b|\alpha}{2} \|u_0\|_{L^1(\mathbb{R})} \max\{\alpha, \alpha^{-1}\} e^{\frac{|b-2t|}{2\alpha} \|u_0\|_{L^1(\mathbb{R})}} \|u_0\|_1. \end{aligned}$$

From the above inequality, we see that for fixed  $t \in (0, T)$  the sequence

$$\left\{ \frac{b}{2} [u^n(t)]^2 + \frac{(3-b)\alpha^2}{2} [u_x^n(t)]^2 \right\}_{n \geq 1}$$

is uniformly bounded in  $L^2(\mathbb{R})$ . Therefore, it has a subsequence

$$\left\{ \frac{b}{2} [u^{n_k}(t)]^2 + \frac{(3-b)\alpha^2}{2} [u_x^{n_k}(t)]^2 \right\}_{n_k \geq 1}$$

which converges weakly in  $L^2(\mathbb{R})$ . By (6.10) and (6.12), we deduce that the weak  $L^2(\mathbb{R})$ -limit is

$$\frac{b}{2} [u(t, \cdot)]^2 + \frac{(3-b)\alpha^2}{2} [u_x(t, \cdot)]^2.$$

Since  $p_x \in L^2(\mathbb{R})$ , it follows that

$$\begin{aligned} & \partial_x p * \left( \frac{b}{2} [u^{n_k}(t)]^2 + \frac{(3-b)\alpha^2}{2} [u_x^{n_k}(t)]^2 \right) \\ & \rightarrow \partial_x p * \left( \frac{b}{2} u^2 + \frac{(3-b)\alpha^2}{2} u_x^2 \right) \quad \text{as } n_k \rightarrow \infty. \end{aligned}$$

Thus, in view of (6.10), (6.12) and the above inequality, we obtain that  $u$  satisfies Eq. (2.2) with  $\left(c_0 + \frac{\Gamma}{\alpha^2}\right) = 0$  in  $D'((0, T) \times \mathbb{R})$ .

Since  $u_t^{n_k}(t, \cdot)$  is uniformly bounded in  $L^2(\mathbb{R})$  and  $\|u^{n_k}(t, \cdot)\|_1$  has a uniform bound for all  $t \in [0, T)$  and all  $n \geq 1$ , we have that the family  $t \mapsto u^{n_k}(t, \cdot) \in H^1(\mathbb{R})$  is weakly equicontinuous on  $[0, T]$ . An application of the Arzela-Ascoli theorem yields that  $\{u^{n_k}\}$  has a

subsequence, denoted again  $\{u^{n_k}\}$ , which converges weakly in  $H^1(\mathbb{R})$ , uniformly in  $t \in [0, T)$ . The limit function is  $u$ . Since  $T$  is arbitrary, it follows that  $u$  is locally and weakly continuous from  $\mathbb{R}_+$  into  $H^1(\mathbb{R})$  i.e.  $u \in C_{w, \text{loc}}(\mathbb{R}_+; H^1(\mathbb{R}))$ .

Note that for a.e.  $t \in \mathbb{R}_+$ ,  $u^{n_k}(t, \cdot) \rightharpoonup u(t, \cdot)$  weakly in  $H^1(\mathbb{R})$ . By Theorem 4.1 (i)–(ii), we have

$$(6.14) \quad \begin{aligned} \|u(t, \cdot)\|_{L^\infty(\mathbb{R})} &\leq \|u(t, \cdot)\|_1 \leq \liminf_{n_k \rightarrow \infty} \|u^{n_k}(t, \cdot)\|_1 \\ &\leq \max\{\alpha, \alpha^{-1}\} e^{\frac{|b-2|t}{2\alpha} \|u_0\|_{L^1(\mathbb{R})}} \|u_0\|_1, \end{aligned}$$

for a.e.  $t \in \mathbb{R}_+$ . The above inequality implies that

$$u \in L_{\text{loc}}^\infty(\mathbb{R}_+ \times \mathbb{R}) \cap L_{\text{loc}}^\infty(\mathbb{R}_+; H^1(\mathbb{R})).$$

Combining (6.11) with (6.12), we get

$$(6.15) \quad u_x \in L^\infty(\mathbb{R}_+ \times \mathbb{R}).$$

Next, we prove that  $I(u)$  is a conservation law, that

$$(u(t, \cdot) - \alpha^2 u_{xx}(t, \cdot)) \in M^+(\mathbb{R})$$

is uniformly bounded on  $\mathbb{R}$ , and that  $u(t, x) \in W^{1, \infty}(\mathbb{R}_+ \times \mathbb{R})$ .

Since  $u$  solves Eq. (2.2) in distributional sense, we have

$$\rho_n * u_t + \rho_n * (uu_x) + \rho_n * \partial_x p * \left( \frac{b}{2} u^2 + \frac{(3-b)\alpha^2}{2} u_x^2 \right) = 0,$$

for a.e.  $t \in \mathbb{R}_+$ . Integrating the above equation with respect to  $x$  on  $\mathbb{R}$ , we obtain

$$\frac{d}{dt} \int_{\mathbb{R}} \rho_n * u \, dx + \int_{\mathbb{R}} \rho_n * (uu_x) \, dx + \int_{\mathbb{R}} \rho_n * \partial_x p * \left( \frac{b}{2} u^2 + \frac{(3-b)\alpha^2}{2} u_x^2 \right) \, dx = 0.$$

Integration by parts yields

$$\frac{d}{dt} \int_{\mathbb{R}} \rho_n * u \, dx = 0, \quad t \in \mathbb{R}_+, n \geq 1.$$

Applying now Lemma 6.1, we get

$$\int_{\mathbb{R}} \rho_n * u(t, \cdot) \, dx = \int_{\mathbb{R}} \rho_n * u_0(\cdot) \, dx.$$

Note that

$$\lim_{n \rightarrow \infty} \|\rho_n * u(t, \cdot) - u(t, \cdot)\|_{L^1(\mathbb{R})} = \lim_{n \rightarrow \infty} \|\rho_n * u_0 - u_0\|_{L^1(\mathbb{R})} = 0.$$



It follows that for a.e.  $t \in \mathbb{R}_+$ ,

$$\int_{\mathbb{R}} u(t, \cdot) dx = \lim_{n \rightarrow \infty} \int_{\mathbb{R}} \rho_n * u(t, \cdot) dx = \lim_{n \rightarrow \infty} \int_{\mathbb{R}} \rho_n * u_0 dx = \int_{\mathbb{R}} u_0 dx$$

showing that  $I(u(t)) = \int_{\mathbb{R}} u dx$  is a conservation law.

Note that  $L^1(\mathbb{R}) \subset (L^\infty(\mathbb{R}))^* \subset (C_0(\mathbb{R}))^* = M(\mathbb{R})$ . By (6.13) and the conservation law  $I(u)$ , we get that

$$\begin{aligned} \|u(t, \cdot) - \alpha^2 u_{xx}(t, \cdot)\|_{M(\mathbb{R})} &\leq \|u(t, \cdot)\|_{L^1(\mathbb{R})} + \alpha^2 \|u_{xx}(t, \cdot)\|_{M(\mathbb{R})} \\ &\leq \|u_0\|_{L^1(\mathbb{R})} + 2\|y_0\|_{M(\mathbb{R})} \leq 3\|y_0\|_{M(\mathbb{R})}, \end{aligned}$$

for a.e.  $t \in \mathbb{R}_+$ . The above inequality shows that

$$(u(t, \cdot) - \alpha^2 u_{xx}(t, \cdot)) \in M(\mathbb{R})$$

is uniformly bounded on  $\mathbb{R}_+$ . For fixed  $T > 0$ , in view of (6.10) and (6.12), we have

$$[u^{n_k}(t, \cdot) - \alpha^2 u_{xx}^{n_k}(t, \cdot)] \rightarrow [u(t, \cdot) - \alpha^2 u_{xx}(t, \cdot)] \quad \text{in } D'(\mathbb{R}) \quad \text{for } n \rightarrow \infty,$$

for a.e.  $t \in [0, T]$ . Since  $u^{n_k}(t, x) - \alpha^2 u_{xx}^{n_k}(t, x) \geq 0$  for all  $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$ , we obtain that  $(u(t, \cdot) - \alpha^2 u_{xx}(t, \cdot)) \in M^+(\mathbb{R})$  for a.e.  $t \in \mathbb{R}_+$ .

Note that  $u(t, x) = p * (u(t, x) - \alpha^2 u_{xx}(t, x))$ . Thus we get

$$\begin{aligned} |u(t, x)| &= |p * (u(t, x) - \alpha^2 u_{xx}(t, x))| \\ &\leq \|p\|_{L^\infty(\mathbb{R})} \|u(t, \cdot) - \alpha^2 u_{xx}(t, \cdot)\|_{M(\mathbb{R})} \leq \frac{3}{2\alpha} \|y_0\|_{M(\mathbb{R})}. \end{aligned}$$

This shows that  $u(t, x) \in W^{1, \infty}(\mathbb{R}_+ \times \mathbb{R})$  in view of (6.15), and completes the proof of the theorem.  $\square$

We now present a uniqueness result for global weak solutions.

**Theorem 6.2.** *Let  $u_0 \in H^1(\mathbb{R})$  be given. Assume that  $c_0 + \frac{\Gamma}{\alpha^2} = 0$  and let*

$$u, v \in W^{1, \infty}(\mathbb{R}_+ \times \mathbb{R}) \cap L_{\text{loc}}^\infty(\mathbb{R}_+; H^1(\mathbb{R}))$$

*be two global weak solutions of (2.2) with initial data  $u_0$ . Assume further that  $(u(t, \cdot) - \alpha^2 u_{xx}(t, \cdot)) \in M^+(\mathbb{R})$  and  $(v(t, \cdot) - \alpha^2 v_{xx}(t, \cdot)) \in M^+(\mathbb{R})$  are uniformly bounded on  $\mathbb{R}_+$ . Then  $u = v$  for a.e.  $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$ .*

*Proof.* Set

$$N := \sup_{t \in \mathbb{R}_+} \{ \|u(t, \cdot) - \alpha^2 u_{xx}(t, \cdot)\|_{M(\mathbb{R})} + \|v(t, \cdot) - \alpha^2 v_{xx}(t, \cdot)\|_{M(\mathbb{R})} \}.$$

By the assumption of the theorem, we have that  $N < \infty$ . Thus, for all  $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$ , we obtain

$$(6.16) \quad \begin{aligned} |u(t, x)| &= |p * (u(t, \cdot) - \alpha^2 u_{xx}(t, \cdot))| \\ &\leq \|p\|_{L^\infty(\mathbb{R})} \|u(t, \cdot) - \alpha^2 u_{xx}(t, \cdot)\|_{M(\mathbb{R})} \leq \frac{N}{2\alpha} \end{aligned}$$

and

$$(6.17) \quad \begin{aligned} |u_x(t, x)| &= |p_x * (u(t, \cdot) - \alpha^2 u_{xx}(t, \cdot))| \\ &\leq \|p_x\|_{L^\infty(\mathbb{R})} \|u(t, \cdot) - \alpha^2 u_{xx}(t, \cdot)\|_{M(\mathbb{R})} \leq \frac{N}{2\alpha^2}. \end{aligned}$$

Similarly, we can obtain

$$(6.18) \quad |v(t, \cdot)| \leq \frac{N}{2\alpha}, \quad |v_x(t, \cdot)| \leq \frac{N}{2\alpha^2}, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}.$$

In view of (6.1), we have

$$(6.19) \quad \begin{aligned} \|u(t, \cdot)\|_{L^1(\mathbb{R})} &= \|p * [u(t, \cdot) - \alpha^2 u_{xx}(t, \cdot)]\|_{L^1(\mathbb{R})} \leq \|p\|_{L^1(\mathbb{R})} N = N, \\ \|u_x(t, \cdot)\|_{L^1(\mathbb{R})} &= \|p_x * [u(t, \cdot) - \alpha^2 u_{xx}(t, \cdot)]\|_{L^1(\mathbb{R})} \leq \|p_x\|_{L^1(\mathbb{R})} N = \frac{N}{\alpha}, \\ \|v(t, \cdot)\|_{L^1(\mathbb{R})} &\leq N, \quad \text{and} \quad \|v_x(t, \cdot)\|_{L^1(\mathbb{R})} \leq \frac{N}{\alpha}, \end{aligned}$$

for all  $t \geq 0$ . Let us now set

$$w(t, \cdot) = u(t, \cdot) - v(t, \cdot), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R},$$

and fix  $T > 0$ . Convoluting Eq. (2.2) for  $u$  and  $v$  with  $\rho_n$  and using Lemma 6.4, we obtain that

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} |\rho_n * w| dx &= \int_{\mathbb{R}} (\rho_n * w_t) \operatorname{sgn}(\rho_n * w) dx \\ &= - \int_{\mathbb{R}} [\rho_n * (wu_x)] \operatorname{sgn}(\rho_n * w) dx - \int_{\mathbb{R}} [\rho_n * (vw_x)] \operatorname{sgn}(\rho_n * w) dx \\ &\quad - \frac{\Gamma}{\alpha^2} \int_{\mathbb{R}} (\rho_n * w_x) \operatorname{sgn}(\rho_n * w) dx \\ &\quad - \frac{b}{2} \int_{\mathbb{R}} (\rho_n * p_x * [w(u+v)]) \operatorname{sgn}(\rho_n * w) dx \\ &\quad - \frac{(3-b)\alpha^2}{2} \int_{\mathbb{R}} (\rho_n * p_x * [w_x(u_x+v_x)]) \operatorname{sgn}(\rho_n * w) dx, \end{aligned}$$

for a.e.  $t \in [0, T]$  and all  $n \geq 1$ . Using (6.14), (6.16)–(6.19), Young’s inequality and Lemmas 6.2–6.3 and following the procedure described in [12], page 56–57, we deduce that

$$(6.20) \quad \frac{d}{dt} \int_{\mathbb{R}} |\rho_n * w| dx = C \int_{\mathbb{R}} |\rho_n * w| dx + C \int_{\mathbb{R}} |\rho_n * w_x| dx + R_n(t),$$

for a.e.  $t \in [0, T]$  and all  $n \geq 1$ , where  $C$  is a generic constant depending on  $\Gamma$ ,  $b$ ,  $\alpha$ , and  $N$  and where  $R_n(t)$  satisfies

$$(6.21) \quad \begin{cases} \lim_{n \rightarrow \infty} R_n(t) = 0, \\ |R_n(t)| \leq K(T), \quad n \geq 1, t \in [0, T]. \end{cases}$$

Here  $K(T)$  is a positive constant depending on  $\Gamma$ ,  $\alpha$ ,  $b$ ,  $T$ ,  $N$  and the  $H^1(\mathbb{R})$ -norms of  $u(0)$  and  $v(0)$ .

Similarly, convoluting Eq. (2.2) for  $u$  and  $v$  with  $\rho_{n,x}$  and using Lemma 6.4, we obtain that

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} |\rho_n * w_x| dx &= \int_{\mathbb{R}} (\rho_n * w_{xt}) \operatorname{sgn}(\rho_{n,x} * w) dx \\ &= - \int_{\mathbb{R}} [\rho_n * (w_x(u_x + v_x))] \operatorname{sgn}(\rho_{n,x} * w) dx - \int_{\mathbb{R}} [\rho_n * (wv_{xx})] \operatorname{sgn}(\rho_{n,x} * w) dx \\ &\quad - \int_{\mathbb{R}} [\rho_n * (uw_{xx})] \operatorname{sgn}(\rho_{n,x} * w) dx - \frac{\Gamma}{\alpha^2} \int_{\mathbb{R}} (\rho_n * w_{xx}) \operatorname{sgn}(\rho_{n,x} * w) dx \\ &\quad - \int_{\mathbb{R}} \rho_n * p_{xx} * \left( \frac{b}{2}(u^2 - v^2) + \frac{(3-b)\alpha^2}{2}(u_x^2 - v_x^2) \right) \operatorname{sgn}(\rho_{n,x} * w) dx, \end{aligned}$$

for a.e.  $t \in [0, T]$  and all  $n \geq 1$ . Using (6.14), (6.16)–(6.19), Young’s inequality, Lemmas 6.2–6.3 and the identity  $\alpha^2 p_{xx} * f = f - p * f$  and following the arguments given in [12], page 57–59, we deduce that

$$(6.22) \quad \frac{d}{dt} \int_{\mathbb{R}} |\rho_n * w_x| dx = C \int_{\mathbb{R}} |\rho_n * w| dx + C \int_{\mathbb{R}} |\rho_n * w_x| dx + R_n(t),$$

for a.e.  $t \in [0, T]$  and all  $n \geq 1$ , where  $C$  is a generic constant and  $R_n(t)$  satisfies (6.21).

Summing (6.20) and (6.22) and then using Gronwall’s inequality, we infer that

$$\begin{aligned} \int_{\mathbb{R}} (|\rho_n * w| + |\rho_n * w_x|)(t, x) dx &\leq \int_0^t e^{2C(t-s)} R_n(s) ds \\ &\quad + e^{2Ct} \int_{\mathbb{R}} (|\rho_n * w| + |\rho_n * w_x|)(0, x) dx, \end{aligned}$$

for all  $t \in [0, T]$  and  $n \geq 1$ . Note that  $w = u - v \in W^{1,1}(\mathbb{R})$ . In view of (6.21), an application of Lebesgue’s dominated convergence theorem yields

$$\int_{\mathbb{R}} (|w| + |w_x|)(t, x) dx \leq e^{2Ct} \int_{\mathbb{R}} (|w| + |w_x|)(0, x) dx,$$

for all  $t \in [0, T]$ . Since  $w(0) = w_x(0) = 0$ , it follows from the above inequality that  $u(t, x) = v(t, x)$  for all  $(t, x) \in [0, T] \times \mathbb{R}$ . This completes the proof of the theorem.  $\square$

Applying Theorem 4.2 and following the proof of Theorems 6.1–6.2, we get the following theorem.

**Theorem 6.3.** *Let  $u_0 \in H^1(\mathbb{R})$  be given. Assume that  $c_0 + \frac{\Gamma}{\alpha^2} = 0$  and*

$$(\alpha^2 u_{0,xx} - u_0) \in M^+(\mathbb{R}).$$

*Then Eq. (2.2) with  $c_0 + \frac{\Gamma}{\alpha^2} = 0$  has a unique weak solution*

$$u \in W^{1,\infty}(\mathbb{R}_+ \times \mathbb{R}) \cap L_{\text{loc}}^\infty(\mathbb{R}_+; H^1(\mathbb{R}))$$

*with initial data  $u(0) = u_0$  and*

$$(\alpha^2 u_{xx}(t, \cdot) - u(t, \cdot)) \in M^+(\mathbb{R})$$

*is uniformly bounded for all  $t \in \mathbb{R}_+$ . Moreover,  $I(u)$  is a conservation law.*

**Remark 6.1.** Theorems 6.1–6.3 correspond to the recent results for global weak solutions of the Camassa-Holm equation in [12] and cover the recent results for global weak solutions of the Degasperis-Procesi equation in [51].

**Example 6.1** (Peakon solutions). Consider Eq. (2.2) with  $c_0 = \Gamma = 0$ . Given the initial datum  $u_0(x) = ce^{-\frac{|x|}{\alpha}}$ ,  $c \in \mathbb{R}$ , a straightforward computation shows that

$$u_0 - \alpha^2 u_{0,xx} = 2c\alpha \delta(x) \in M_+(\mathbb{R}) \quad \text{if } c \geq 0$$

and

$$\alpha^2 u_{0,xx} - u_0 = -2c\alpha \delta(x) \in M_+(\mathbb{R}) \quad \text{if } c < 0.$$

One can also check that

$$u(t, x) = ce^{-\frac{|x-ct|}{\alpha}}$$

satisfies Eq. (2.2) in distributional sense. Theorems 6.1–6.3 show that  $u(t, x)$  is the unique global weak solution with the initial date  $u_0(x)$ . This weak solution is a peaked solitary wave.

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