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Geometric post-Newtonian description of massive spin-half particles in curved spacetime

Ashkan Alibabaei[1,2], **Philip K Schwartz**[1,*] and **Domenico Giulini**[1,3]

[1] Institute for Theoretical Physics, Leibniz University Hannover, Appelstraße 2, 30167 Hannover, Germany
[2] Institute of Quantum Optics, Leibniz University Hannover, Welfengarten 1, 30167 Hannover, Germany
[3] Center of Applied Space Technology and Microgravity, University of Bremen, Am Fallturm 1, 28359 Bremen, Germany

E-mail: philip.schwartz@itp.uni-hannover.de

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Abstract
We consider the Dirac equation coupled to an external electromagnetic field in curved four-dimensional spacetime with a given timelike worldline γ representing a classical clock. We use generalised Fermi normal coordinates in a tubular neighbourhood of γ and expand the Dirac equation up to, and including, the second order in the dimensionless parameter given by the ratio of the geodesic distance to the radii defined by spacetime curvature, linear acceleration of γ, and angular velocity of rotation of the employed spatial reference frame along γ. With respect to the time measured by the clock γ, we compute the Dirac Hamiltonian to that order. On top of this 'weak-gravity' expansion we then perform a post-Newtonian expansion up to, and including, the second order of $1/c$, corresponding to a 'slow-velocity' expansion with respect to γ. As a result of these combined expansions we give the weak-gravity post-Newtonian expression for the Pauli Hamiltonian of a spin-half particle in an external electromagnetic field. This extends and partially corrects recent results from the literature, which we discuss and compare in some detail.

* Author to whom any correspondence should be addressed.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Modern experiments allow to probe the interface between quantum and gravitational physics at a rapidly growing degree of accuracy. A proper theoretical description of such experiments would ideally be based on a higher-level theory encompassing quantum mechanics as well as general relativity as appropriate limiting cases. However, as is well known, such a higher-level theory is still elusive. Hence, we cannot simply 'compute' the impact of a classical gravitational field, described by a (generally curved) spacetime metric, upon the dynamics of a quantum system. Rather, depending on the context, we must 'deduce' the influence of the gravitational field on the dynamics of the quantum system from general principles that we expect to be robust and eventually realised in the higher-level theory. This is, in a nutshell, the generally accepted strategy today for exploring the interface between quantum and gravitational physics, to which the present study also subscribes.

On the one hand, exploration of this interface is, of course, of fundamental theoretical interest, not least since by such exploration one hopes to gain insight into how to combine gravitational and quantum physics on a broader level. On the other hand, a systematic understanding of this interface is also necessary in view of ongoing progress in quantum-mechanical experiments, whose increasing accuracy makes the consideration of post-Newtonian gravitational effects inevitable. As recent examples of interest in 'novel' (i.e. previously not considered) gravitational effects in experiments, we mention the question of the gravitational contribution to high-precision measurements of the g-factor of an electron stored in a Penning trap (also referred to as a 'geonium atom') [1–4], and the recent results of qBOUNCE, which is a Ramsey-type gravitational resonance spectroscopy experiment using ultra-cold neutrons to test the neutron's coupling to the gravitational field of the earth in the micrometre range [5].

In the field of matter-wave interferometry, post-Newtonian gravitational effects have recently even become a direct object of investigation, expected to be observed in the foreseeable future with the current generation of matter-wave interferometers [6, 7]: for systems possessing internal degrees of freedom, post-Newtonian effects are expected to induce a coupling between these internal degrees of freedom and the system's external degrees of freedom [8–11]. In interferometry with such systems, dubbed 'quantum clock interferometry', these couplings may be observed and/or exploited for, e.g., tests of (certain aspects of) the equivalence principle [12–14].

In such examples, and in the more general context of gravitational effects in quantum systems, it is important to base one's estimates of possible gravity effects on a *well-defined* and *systematic* approximation scheme. Without such a controlled scheme, a deviation of experimental observations from expectations might be either (a) a result of the underlying theory being indeed 'wrong' (in the appropriate sense), or (b) simply an artefact of an unsystematic way of deriving the alleged 'theoretical predictions'. That is: only by employing a consistent and systematic scheme one can guarantee a complete and redundancy-free account of the (in our case relativistic) corrections that one derives, as a necessary condition for properly testing the underlying theory.

It is the aim of this paper to present such a scheme for a massive spin-half particle obeying the Dirac equation in curved spacetime. Our scheme is based on the assumed existence of a distinguished reference wordline γ, which, e.g., may be thought of as that of a clock in the laboratory or a distinguished particle. In a tubular neighbourhood of γ we use generalised Fermi normal coordinates [15–17] with reference to γ and an adapted (meaning the unit timelike vector is parallel to the tangent of γ) orthonormal frame along it. The coordinates are 'generalised' in the sense that we will allow the worldline γ to be accelerated, and the orthonormal frame to rotate, i.e. its Fermi–Walker derivative need not vanish. The approximation procedure then consists of two steps which are logically independent *a priori*.

In the first step we perform a 'weak-gravity expansion', which means that we expand the fields in the tubular neighbourhood of γ in terms of a dimensionless parameter given by the ratio of the spacelike geodesic distance to γ to the radii that are defined by spacetime curvature, acceleration of γ, and the angular velocity of rotation of the chosen frame along γ. We recall that the radius associated with γ's acceleration a is given by c^2/a and that the radius associated with the frame's angular velocity ω (against a Fermi–Walker transported one) is c/ω. The curvature radius is given by the inverse of the modulus of the typical Riemann-tensor components with respect to the orthonormal frame. As first derivatives of the Riemann-tensor will also appear, we also need to control these against *third* powers of the geodesic distance. Our expansion hypotheses are summarised in the expression (3.6). Consistently performing this expansion is the content of section 3, leading to the Dirac Hamiltonian (3.11), which is our first main result. Note that a 'Hamiltonian' refers to a 'time' with respect to which it generates the evolution of the dynamical quantities. In our case, that time is given by the proper time along γ, i.e. time read by the 'clock', extended to the tubular neighbourhood along spacelike geodesics.

In the second step we perform a 'slow-velocity' expansion by means of a formal power series expansion in terms of $1/c$, i.e. a *post-Newtonian expansion*. More specifically, we will expand positive-frequency solutions of the (classical) Dirac equation as formal power series in c^{-1}, similar to the corresponding expansion for the Klein–Gordon equation as discussed in, e.g., [11, 18, 19], and in a broader context in [20]. For the case of γ being a stationary worldline in a stationary spacetime, this expansion may be considered a post-Newtonian description of the one-particle sector of the massive Dirac quantum field theory. *A priori* this 'slow-velocity' approximation is an independent expansion on top of the former. But for the system moving under the influence of the gravitational field the latter approximation is only consistent with the former if the relative acceleration of the system against the reference set by γ stays bounded as $1/c \to 0$. This implies that the curvature tensor components with respect to the adapted orthonormal frame should be considered as being of order c^{-2}. The coupled expansions then lead us to the Pauli Hamiltonian (4.17), which is the second main result of our paper.

Clearly, our work should be considered in the context of previous work by others. In 1980, Parker [21, 22] presented explicit expressions for the energy shifts suffered by a one-electron atom in free fall within a general gravitational field, the only restriction imposed on the latter being that its time-rate of change be sufficiently small so as to allow stationary atomic states and hence well-defined energy levels. Parker also used Fermi normal coordinates, though standard ones, i.e. with respect to *non-rotating* frames along a *geodesic* curve γ. He then gave an explicit expression for the Dirac Hamiltonian to what he calls 'first order in the [dimensionful] curvature', which in our language means second order in the dimensionless ratio of geodesic distance to curvature radius. Regarding the 'slow-velocity' approximation, Parker considers only the leading-order terms, i.e. the Newtonian limit instead of a post-Newtonian expansion.

The restriction to non-rotating frames along γ and geodesic γ was lifted by Ito [4] in 2021, who aimed for estimating the inertial and gravitational effects upon g-factor measurements of a Dirac particle in a Penning trap. To that end he presented an expansion in generalised Fermi normal coordinates of the Dirac Hamiltonian also including terms to second order in the ratio (geodesic distance)/(curvature radius), but only to first order in the ratios (geodesic distance)/(acceleration radius), where 'acceleration radius' refers to both acceleration of γ and the rotation of the frames along γ as explained above. Ito also considers a 'non-relativistic limit' by performing a Fouldy–Wouthuysen transformation [23] with a transformation operator expanded as a formal power series in $1/m$ (the inverse mass of the fermionic particle). In dimensionless terms, the latter corresponds to a simultaneous expansion in v/c as well as the ratio (Compton wavelength)/(geodesic distance).

Finally we mention the work of Perche and Neuser [24] from 2021, who generalise Parker's work [22] in allowing the reference curve γ to be accelerating, though the frame along it is still assumed to be non-rotating (Fermi–Walker transported). For vanishing acceleration of γ, their result for the Dirac Hamiltonian coincides with that of Parker. Similar to Ito [4], they consider a 'non-relativistic limit' by means of an expansion in ratios of relevant energies to the rest energy, which effectively amounts to an expansion in $1/m$. Let it be mentioned already at this point that in section 4.1 we will show explicitly that the expansion as presented in [24] is *not* equivalent to the post-Newtonian expansion in $1/c$ that we employ. This we believe, however, to be rooted in the expansion in [24] being inconsistently applied; when taking proper care of all appearing terms, the expansion method of [24] is consistent with the corresponding truncation of our results.

Our paper is an extension of those approaches, in that it also includes inertial effects from acceleration and rotation to consistently the same order as gravitational effects resulting from curvature, namely to order $((\text{geodesic distance})/(\text{charecteristic radii}))^2$. We will find some inconsistencies in the approximations of the aforementioned paper that result in the omission of terms which we will restore. Our paper is partly based on the master's thesis [25]. Here we use the opportunity to correct some oversights in the calculation of order-x^2 terms in that thesis, that we will further comment on below (cf footnote 5).

To sum up, our paper is organised as follows: In section 2, we recall the Dirac equation in curved spacetime. In section 3, we implement the first step of our approximation procedure by expressing the Dirac equation in generalised Fermi normal coordinates corresponding to an accelerated reference worldline γ and orthonormal, possibly rotating frames along it. In section 4, we implement the second step, namely the 'slow-velocity' post-Newtonian expansion in $1/c$. This step should be contrasted with the mentioned $1/m$-expansions by others or expansions relying on Foldy–Wouthuysen transformations. In particular, this includes a comparison of our resulting Hamiltonian to that obtained in [24], which we discuss in some detail in section 4.1, where we argue for an inconsistency within the calculation in [24]. We conclude in section 5. Details of calculations and lengthy expressions are collected in appendices A–D.

2. The Dirac equation in curved spacetime

We consider a massive spin-half field ψ in a general curved background spacetime[4] (M, g), coupled to background electromagnetism, as described by the minimally coupled Dirac equation

$$\left(\mathrm{i}\gamma^I (\mathrm{e}_I)^\mu (\nabla_\mu - \mathrm{i}qA_\mu) - mc\right)\psi = 0. \tag{2.1}$$

Here A_μ are the components of the electromagnetic four-potential, ∇ is the Levi-Civita covariant derivative of the spacetime metric g, extended to Dirac spinor fields, m is the mass of the field, and q is its electric charge. Note that we set $\hbar = 1$, but keep explicit the velocity of light c. A detailed exposition of the Dirac equation in curved spacetime may be found in [27], to which we refer for further background information.

The Dirac equation takes the above local form with respect to a choice of *tetrad* $(\mathrm{e}_I) = (\mathrm{e}_0, \mathrm{e}_i)$, i.e. a local orthonormal frame of vector fields. Explicitly, this means that the vector fields satisfy

$$g(\mathrm{e}_I, \mathrm{e}_J) = \eta_{IJ} \tag{2.2}$$

where $(\eta_{IJ}) = \mathrm{diag}(-1,1,1,1)$ are the components of the Minkowski metric in Lorentzian coordinates. The gamma matrices γ^I appearing in the Dirac equation (2.1) are the standard Minkowski-spacetime gamma matrices $\gamma^I \in \mathrm{End}(\mathbb{C}^4)$, which satisfy the Clifford algebra relation

$$\left\{\gamma^I, \gamma^J\right\} = -2\eta^{IJ}\mathbb{1}_4 \tag{2.3}$$

with $\{\cdot,\cdot\}$ denoting the anti-commutator. The Dirac representation of the Lorentz algebra $\mathrm{Lie}(\mathrm{SO}(1,3))$ on $\mathbb{C}^4$ is given by

$$\mathrm{Lie}(\mathrm{SO}(1,3)) \ni \left(X^I{}_J\right) \mapsto -\frac{1}{2}X_{IJ}S^{IJ} \in \mathrm{End}\left(\mathbb{C}^4\right), \tag{2.4a}$$

with the generators $S^{IJ} \in \mathrm{End}(\mathbb{C}^4)$ given by

$$S^{IJ} = \frac{1}{4}\left[\gamma^I, \gamma^J\right]. \tag{2.4b}$$

Thus, the spinor covariant derivative is represented with respect to the chosen tetrad by

$$\nabla_\mu\psi = \partial_\mu\psi + \Gamma_\mu \cdot \psi, \tag{2.5a}$$

with the spinor representation of the local connection form explicitly given by

$$\Gamma_\mu = -\frac{1}{2}\omega_{\mu IJ}S^{IJ} \tag{2.5b}$$

[4] Of course, for the very notion of spinor fields to make sense, we need to assume the spacetime to be equipped with a spin structure, i.e. a double cover of its orthonormal frame bundle such that the covering homomorphism is in trivialisations given by the double covering of the (homogeneous) Lorentz group $\mathcal{L}_+^\uparrow = \mathrm{SO}_0(1,3)$ by the spin group $\mathrm{Spin}(1,3) = \mathrm{SL}(2,\mathbb{C})$. Dirac spinor fields are then sections of the Dirac spinor bundle, which is the vector bundle associated to the spin structure with respect to the $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ representation of the spin group. As is well-known, the existence of a spin structure is for four-dimensional non-compact spacetimes equivalent to the spacetime manifold being parallelisable [26].

in terms of the local connection form $\omega_\mu{}^I{}_J$ of the Levi-Civita connection with respect to the tetrad, defined by

$$\nabla \mathrm{e}_I = \omega^J{}_I \otimes \mathrm{e}_J\,, \tag{2.6a}$$

i.e. in components

$$\nabla_\mu (\mathrm{e}_I)^\nu = \omega_\mu{}^J{}_I (\mathrm{e}_J)^\nu\,. \tag{2.6b}$$

Due to the local connection form taking values in the Lorentz algebra, i.e. satisfying $\omega_{\mu IJ} = -\omega_{\mu JI}$, the spinor representation of the connection form may be explicitly expressed as

$$\Gamma_\mu = -\frac{1}{2}\omega_{\mu IJ} S^{IJ} = -\frac{1}{2}\omega_{\mu 0i}\gamma^0\gamma^i - \frac{1}{4}\omega_{\mu ij}\gamma^i\gamma^j. \tag{2.7}$$

As said in the introduction, in the following we will describe a systematic approximation scheme for the one-particle sector of the massive Dirac theory from the point of view of an observer moving along a fixed timelike reference worldline γ, which will proceed in two conceptually independent steps. The first step, which is described in section 3 and implements a 'weak-gravity' approximation by expanding the Dirac equation in (generalised) Fermi normal coordinates, is actually valid without restricting to the one-particle theory.

Only for the second step, the 'slow-velocity' post-Newtonian expansion in section 4, we will restrict to the one-particle theory. For this, we assume the spacetime and the reference worldline γ to be (approximately) stationary, such that there is a well-defined (approximate) notion of particles in quantum field theory and we may meaningfully restrict to the one-particle sector of the theory. This sector is then effectively described by positive-frequency *classical* solutions of the Dirac equation, which we will approximate by the post-Newtonian expansion.

3. 'Weak-gravity' expansion in generalised Fermi normal coordinates

As the first step of our scheme, we will implement a 'weak-gravity' approximation of the Dirac equation with respect to a timelike reference worldline γ and orthonormal spacelike vector fields $(\mathrm{e}_i(\tau))$ defined along γ which are orthogonal to the tangent $\mathrm{e}_0(\tau) := c^{-1}\dot\gamma(\tau)$. The approximation works by expressing the Dirac equation in *generalised Fermi normal coordinates* with respect to γ and (e_i). These coordinates are constructed as follows (compare figure 1): in a neighbourhood of γ, each point p is connected to γ by a unique spacelike geodesic. The temporal coordinate of p is the proper time parameter τ of the starting point of this geodesic, defined with respect to some fixed reference point on γ, and the spatial coordinates of p are the components x^i of the initial direction of the geodesic with respect to the basis $(\mathrm{e}_i(\tau))$. Phrased in terms of the exponential map, this means that the coordinate functions $(x^\mu) = (c\tau, x^i)$ are defined by the implicit equation

$$p = \exp\left(x^i(p)\,\mathrm{e}_i(\tau(p))\right). \tag{3.1}$$

These coordinates are adapted to an observer along γ who defines 'spatial directions' using the basis (e_i). Note that differently to classical Fermi normal coordinates [15] we allow for the worldline γ to be accelerated—i.e. γ need not be a geodesic—, as well as for the basis (e_i) to be rotating with respect to gyroscopes—i.e. the (e_i) need not be Fermi–Walker transported along γ. Generalised Fermi normal coordinates may be seen as the best analogue of inertial coordinates that exists for an arbitrarily moving observer carrying an arbitrarily rotating basis in a general curved spacetime.

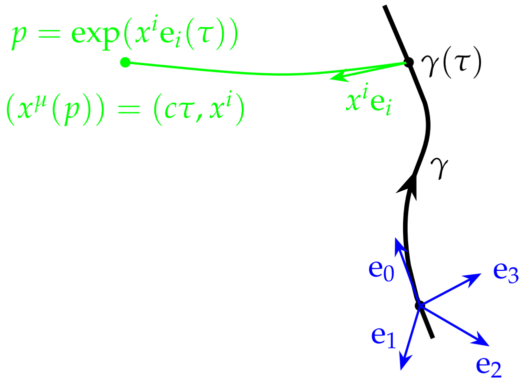

Figure 1. The construction of generalised Fermi normal coordinates.

The acceleration $a(\tau)$ of γ is the covariant derivative of $\dot\gamma(\tau)$ along γ, i.e. the vector field

$$a(\tau) = \nabla_{\dot\gamma(\tau)}\dot\gamma(\tau) \tag{3.2a}$$

along γ, which is everywhere orthogonal to $\dot\gamma = c\mathrm{e}_0$. Note that we take the covariant derivative with respect to the worldline's four-velocity $\dot\gamma(\tau)$, such that the physical dimension of the components a^μ will really be that of an acceleration (given that the coordinate functions have the dimension of length). The angular velocity of the observer's spatial basis vector fields (e_i) (with respect to non-rotating directions, i.e. Fermi–Walker transported ones) is another vector field along γ that is everywhere orthogonal to $\dot\gamma = c\mathrm{e}_0$; we denote it by $\omega(\tau)$. It is defined by

$$(\nabla_{\dot\gamma}\mathrm{e}_I)^\mu = -\left(c^{-2}a^\mu\dot\gamma_\nu - c^{-2}\dot\gamma^\mu a_\nu + c^{-1}\varepsilon_{\rho\sigma}{}^\mu{}_\nu\dot\gamma^\rho\omega^\sigma\right)\mathrm{e}_I^\nu\,, \tag{3.2b}$$

where both sides of the equation are evaluated along γ, and ε denotes the volume form of the spacetime metric g. The covariant derivatives of a and ω along γ will be denoted by

$$b(\tau) := \nabla_{\dot\gamma(\tau)}a(\tau)\,, \quad \eta(\tau) := \nabla_{\dot\gamma(\tau)}\omega(\tau)\,. \tag{3.3}$$

When working in generalised Fermi normal coordinates we will denote the timelike coordinate which has the dimension of length by $s = c\tau$, since it is an extension of the proper length function along γ. In index notation, we will use s as the timelike coordinate index and reserve 0 for use as the timelike index for orthonormal frame components.

The components of the spacetime metric g in generalised Fermi normal coordinates may be expressed as formal power series in the geodesic distance to γ according to [16]

$$g_{ss} = -1 - 2c^{-2}\boldsymbol{a}\cdot\boldsymbol{x} - c^{-4}(\boldsymbol{a}\cdot\boldsymbol{x})^2 - R_{0l0m}x^lx^m + c^{-2}(\boldsymbol{\omega}\times\boldsymbol{x})^2 + \mathrm{O}\left(\|\boldsymbol{x}\|^3\right), \tag{3.4a}$$

$$g_{si} = c^{-1}(\boldsymbol{\omega}\times\boldsymbol{x})_i - \frac{2}{3}R_{0lim}x^lx^m + \mathrm{O}\left(\|\boldsymbol{x}\|^3\right), \tag{3.4b}$$

$$g_{ij} = \delta_{ij} - \frac{1}{3}R_{iljm}x^lx^m + \mathrm{O}\left(\|\boldsymbol{x}\|^3\right). \tag{3.4c}$$

Here, in addition to the acceleration $a^i(\tau)$ of γ and the angular velocity $\omega^i(\tau)$ of the spatial basis (e_i), the curvature tensor $R_{IJKL}(\tau)$ evaluated along γ appears as well; the components are taken with respect to the orthonormal basis $(\mathrm{e}_0, \mathrm{e}_i)$ along γ. We also have used standard 'three-vector' notation for geometric operations taking place in the three-dimensional vector space

$\Sigma_\tau = (\mathrm{e}_0(\tau))^\perp = \mathrm{span}\{\mathrm{e}_i(\tau)\} \subset T_{\gamma(\tau)}M$ of the observer's local 'spatial directions', endowed with the Euclidean metric $\delta_\tau := g|_{\Sigma_\tau}$ induced by g: we write

$$\boldsymbol{v}\cdot\boldsymbol{w} := \delta_{ij}v^i w^j, \quad \|\boldsymbol{v}\| := \sqrt{\delta_{ij}v^i v^j}, \quad (\boldsymbol{v}\times\boldsymbol{w})_i := \varepsilon_{ijk}v^j w^k \tag{3.5}$$

for the scalar product, the norm, and the vector product with respect to this metric. Note that with respect to the orthonormal basis (e_i), the components δ_{ij} of the induced metric and ε_{ijk} of its volume form are just given by the Kronecker delta and the totally antisymmetric three-dimensional Levi-Civita symbol, respectively.

The expansion in powers of the geodesic distance to γ implements the desired approximation in terms of 'weak gravity' and 'weak inertial effects': we expand according to

$$R_{IJKL}\cdot\|\boldsymbol{x}\|^2 \ll 1, \qquad \frac{\boldsymbol{a}}{c^2}\cdot\boldsymbol{x} \ll 1, \qquad \frac{\boldsymbol{\omega}}{c}\cdot\boldsymbol{x} \ll 1, \qquad \frac{R_{IJKL;M}}{R_{NOPQ}}\cdot\|\boldsymbol{x}\| \ll 1, \tag{3.6}$$

i.e. for the expansion to be valid at a point, the geodesic distance to γ has to be small compared to the curvature radius of spacetime, the 'acceleration radius' of γ, the 'angular velocity radius' of the spatial reference vector fields, and the characteristic length scale on which the curvature changes. This also gives a precise analytical meaning to the formal expansion in the dimensionful parameter $\|\boldsymbol{x}\|$: the actual dimensionless quantity in which we expand is the ratio of $\|\boldsymbol{x}\|$ to the minimum of the characteristic geometric lengths defined by the spacetime curvature, acceleration $\boldsymbol{a}$, angular velocity $\boldsymbol{\omega}$, and rate of change of the curvature, as given in (3.6). For the sake of brevity, in the following we will speak of terms of n^{th} order in the geodesic distance to γ simply as being of 'order x^n', and correspondingly use the shorthand notation $\mathrm{O}(x^n) := \mathrm{O}(\|\boldsymbol{x}\|^n)$.

Our goal is to expand the Dirac equation (2.1) systematically to order x^2. To make precise what we mean by this, first recall that for the local formulation (2.1) of the Dirac equation to be possible, we have to choose a tetrad (e_I) not only along the reference worldline γ, but also away from it. This choice of tetrad is an additional input into the approximation procedure, on top of the choice of local coordinate system. However, in our situation there is a natural choice for the tetrad: on γ, we choose it to be given by the basis $(c^{-1}\dot\gamma, \mathrm{e}_i)$ with respect to which the generalised Fermi normal coordinates are defined; away from γ, we extend the vector fields by parallel transport along spacelike geodesics. The explicit form of the tetrad components in coordinates will be computed at a later stage. With a choice of tetrad, we may rewrite the Dirac equation (2.1) in the Schrödinger-like form

$$\mathrm{i}\partial_\tau\psi = H_{\text{Dirac}}\psi \tag{3.7a}$$

with the Dirac Hamiltonian

$$H_{\text{Dirac}} = (g^{ss})^{-1}\gamma^J(\mathrm{e}_J)^s\left(\mathrm{i}\gamma^I(\mathrm{e}_I)^i\,c\,(D_i+\Gamma_i) - mc^2\right) - \mathrm{i}c\,(\Gamma_s - \mathrm{i}qA_s), \tag{3.7b}$$

where we used that $\partial_s = c^{-1}\partial_\tau$ and that $(g^{ss})^{-1}\gamma^J(\mathrm{e}_J)^s = \left(-\gamma^J(\mathrm{e}_J)^s\right)^{-1}$, and where $D_i = \partial_i - \mathrm{i}qA_i$ denotes the spatial electromagnetic covariant derivative. It is *this* Dirac Hamiltonian that we will expand to order x^2 in the following. Note that the partial derivative $\partial_i = \frac{\partial}{\partial x^i}$ in the operator D_i effectively is of order x^{-1} when acting on functions, such that in the following calculation, it is important to keep track of terms of the form $x^l x^m x^n D_i$, which despite their superficial appearance are in fact of order x^2.

We are now going to compute all objects appearing in the Dirac Hamiltonian (3.7*b*) to those orders in x which are necessary to obtain the total Hamiltonian to order x^2.

In order to be able to expand covariant derivatives and the local connection form to order x^2, we need to know the Christoffel symbols in our coordinate system to order x^2. Note that these cannot be obtained from the metric components as given in (3.4): there the metric is given to order x^2, such that its derivatives can only be known to order x^1. However, extending the work in [16], the Christoffel symbols to order x^2 (and the metric to order x^3) in generalised Fermi normal coordinates were calculated in [17]. The Christoffel symbols are given in the appendix in (A.1) (note that some calculational errors were made in [17], which we corrected in (A.1)).

We may now compute the coordinate components of our tetrad (e_I). Recall that we define the tetrad by extending the vector fields $(c^{-1}\dot{\gamma}, \mathrm{e}_i)$ along γ into a neighbourhood of γ by parallel transport along spacelike geodesics. Since spacelike geodesics take a simple form in generalised Fermi normal coordinates, the parallel transport equation may explicitly be solved perturbatively using the Christoffel symbols (A.1). This calculation is straightforward, but quite lengthy; it yields the tetrad components

$$
\begin{aligned}
(\mathrm{e}_0)^s &= 1 - c^{-2}\boldsymbol{a}\cdot\boldsymbol{x} + c^{-4}(\boldsymbol{a}\cdot\boldsymbol{x})^2 - \frac{1}{2}R_{0l0m}x^l x^m - \frac{1}{6}R_{0l0m;n}x^l x^m x^n \\
&\quad + \frac{5}{6}c^{-2}(\boldsymbol{a}\cdot\boldsymbol{x})R_{0l0m}x^l x^m - c^{-6}(\boldsymbol{a}\cdot\boldsymbol{x})^3 + \mathrm{O}(x^4), && (3.8a)\\
(\mathrm{e}_0)^i &= -c^{-1}(\boldsymbol{\omega}\times\boldsymbol{x})^i + c^{-3}(\boldsymbol{a}\cdot\boldsymbol{x})(\boldsymbol{\omega}\times\boldsymbol{x})^i + \frac{1}{2}R_{0l}{}^{i}{}_{m}x^l x^m + \frac{1}{6}R_{0l}{}^{i}{}_{m;n}x^l x^m x^n \\
&\quad + \frac{1}{2}c^{-1}(\boldsymbol{\omega}\times\boldsymbol{x})^i R_{0l0m}x^l x^m - c^{-5}(\boldsymbol{\omega}\times\boldsymbol{x})^i(\boldsymbol{a}\cdot\boldsymbol{x})^2 - \frac{1}{3}c^{-2}(\boldsymbol{a}\cdot\boldsymbol{x})R_{0l}{}^{i}{}_{m}x^l x^m \\
&\quad + \mathrm{O}(x^4), && (3.8b)\\
(\mathrm{e}_i)^s &= -\frac{1}{6}R_{0lim}x^l x^m - \frac{1}{12}R_{0lim;n}x^l x^m x^n + \frac{1}{6}c^{-2}(\boldsymbol{a}\cdot\boldsymbol{x})R_{0lim}x^l x^m + \mathrm{O}(x^4), && (3.8c)\\
(\mathrm{e}_i)^j &= \delta_i^j + \frac{1}{6}R^j{}_{lim}x^l x^m + \frac{1}{12}R^j{}_{lim;n}x^l x^m x^n + \frac{1}{6}c^{-1}(\boldsymbol{\omega}\times\boldsymbol{x})^j R_{0lim}x^l x^m + \mathrm{O}(x^4). && (3.8d)
\end{aligned}
$$

From this, we may compute the components of the dual frame as

$$
\begin{aligned}
(\mathrm{e}^0)_s &= 1 + c^{-2}\boldsymbol{a}\cdot\boldsymbol{x} + \frac{1}{2}R_{0l0m}x^l x^m + \mathrm{O}(x^3), && (3.9a)\\
(\mathrm{e}^0)_i &= \frac{1}{6}R_{0lim}x^l x^m + \mathrm{O}(x^3), && (3.9b)\\
(\mathrm{e}^i)_s &= c^{-1}(\boldsymbol{\omega}\times\boldsymbol{x})^i - \frac{1}{2}R^i{}_{l0m}x^l x^m + \mathrm{O}(x^3), && (3.9c)\\
(\mathrm{e}^i)_j &= \delta_j^i - \frac{1}{6}R^i{}_{ljm}x^l x^m + \mathrm{O}(x^3). && (3.9d)
\end{aligned}
$$

Note that we have computed the dual frame components only to order x^2 (instead of going to order x^3 as would have been possible from (3.8)), since this suffices for our goal, namely the expansion of the Dirac Hamiltonian (3.7b) to order x^2.

Now we have the required information in order to calculate the local connection form $\omega_\mu{}^I{}_J$ according to (2.6) to order x^2, which is given in the appendix in (A.2). From this, we can directly obtain its spinor representation Γ_μ according to (2.7).

We will also need the component g^{ss} of the inverse metric to order x^3. Using the frame (3.8), we may easily compute this according to $g^{ss} = -((\mathrm{e}_0)^s)^2 + \delta^{ij}(\mathrm{e}_i)^s(\mathrm{e}_j)^s$, yielding

$$
g^{ss} = -1 + 2c^{-2}\boldsymbol{a}\cdot\boldsymbol{x} - 3c^{-4}(\boldsymbol{a}\cdot\boldsymbol{x})^2 + 4c^{-6}(\boldsymbol{a}\cdot\boldsymbol{x})^3 + R_{0l0m}x^l x^m
$$

$$+\frac{1}{3}R_{0l0m;n}x^lx^mx^n-\frac{8}{3}c^{-2}\left(\boldsymbol{a}\cdot\boldsymbol{x}\right)R_{0l0m}x^lx^m+\mathrm{O}\left(x^4\right). \tag{3.10}$$

We thus have obtained all ingredients to express the Dirac equations (2.1) and (3.7) in generalised Fermi normal coordinates and our chosen tetrad to order x^2. Inserting g^{ss}, the tetrad components, and the spinor representation of the local connection form as computed above into the Dirac Hamiltonian (3.7*b*), by a tedious but straightforward calculation, employing standard identities for products of three gamma matrices, we obtain the explicit form of the Dirac Hamiltonian as

$$\begin{aligned}
H_{\mathrm{Dirac}} &= \gamma^0\left\{mc^2+m\boldsymbol{a}\cdot\boldsymbol{x}+\frac{mc^2}{2}R_{0l0m}x^lx^m\right\}-\gamma^i\left\{\frac{mc^2}{6}R_{0lim}x^lx^m\right\}\\
&+\mathbb{1}\left\{-qA_\tau+\mathrm{i}\left(\boldsymbol{\omega}\times\boldsymbol{x}\right)^iD_i-\frac{\mathrm{i}c}{2}R_{0l}{}^{i}{}_{m}x^lx^mD_i+\frac{\mathrm{i}c^{-1}}{4}\left(\boldsymbol{a}\cdot\boldsymbol{x}\right)R_{0l}x^l+\frac{\mathrm{i}c}{12}R_{0l;m}x^lx^m\right.\\
&\left.-\frac{\mathrm{i}c}{6}R_{0l}{}^{i}{}_{m;n}x^lx^mx^nD_i-\frac{\mathrm{i}c^{-1}}{6}\left(\boldsymbol{a}\cdot\boldsymbol{x}\right)R_{0l}{}^{i}{}_{m}x^lx^mD_i\right\}\\
&-\gamma^0\gamma^j\left\{\mathrm{i}cD_j+\frac{\mathrm{i}c^{-1}}{2}a_j+\mathrm{i}c^{-1}\left(\boldsymbol{a}\cdot\boldsymbol{x}\right)D_j+\frac{\mathrm{i}c}{4}\left(R_{0j0l}-R_{jl}\right)x^l+\frac{\mathrm{i}c}{2}R_{0l0m}x^lx^mD_j\right.\\
&+\frac{\mathrm{i}c}{6}R^i{}_{ljm}x^lx^mD_i+\frac{\mathrm{i}c}{12}\left(R_{0j0l;m}-2R_{jl;m}\right)x^lx^m-\frac{\mathrm{i}c^{-1}}{4}\left(\boldsymbol{a}\cdot\boldsymbol{x}\right)R_{jl}x^l\\
&+\frac{\mathrm{i}c}{6}R_{0l0m;n}x^lx^mx^nD_j+\frac{\mathrm{i}c}{12}R^i{}_{ljm;n}x^lx^mx^nD_i+\frac{\mathrm{i}c^{-1}}{6}\left(\boldsymbol{a}\cdot\boldsymbol{x}\right)R_{0l0m}x^lx^mD_j\\
&\left.+\frac{\mathrm{i}c^{-1}}{6}\left(\boldsymbol{a}\cdot\boldsymbol{x}\right)R^i{}_{ljm}x^lx^mD_i\right\}+\gamma^i\gamma^j\left\{-\frac{\mathrm{i}}{4}\varepsilon_{ijk}\omega^k+\frac{\mathrm{i}c}{4}R_{0ijl}x^l+\frac{\mathrm{i}c}{6}R_{0lim}x^lx^mD_j\right.\\
&\left.+\frac{\mathrm{i}c}{12}R_{0ijl;m}x^lx^m+\frac{\mathrm{i}c}{12}R_{0lim;n}x^lx^mx^nD_j+\frac{\mathrm{i}c^{-1}}{6}\left(\boldsymbol{a}\cdot\boldsymbol{x}\right)R_{0lim}x^lx^mD_j\right\}+\mathrm{O}\left(x^3\right).
\end{aligned} \tag{3.11}$$

As already stated in the introduction, this is our first main result. Here $A_\tau=cA_s$ is the electric scalar potential with respect to our coordinates. Recall that the partial derivative operator ∂_i appearing in D_i is effectively of order x^{-1} when acting on functions, such we need to keep terms of the form $x^lx^mx^nD_i$ (since they are of order x^2). The terms in the Hamiltonian are ordered, in each pair of curly brackets, by order in spatial geodesic distance x to the worldline, with those terms of a given order that include a D_i appearing after those without[5].

Note that setting $\omega=0$ and ignoring quadratic terms in a^i and R_{IJKL} as well as terms involving covariant derivatives of the curvature tensor, our Dirac Hamiltonian (3.11) reproduces the Dirac Hamiltonian from [24].

4. Post-Newtonian expansion

As the second step of our approximation scheme, we will now perform a post-Newtonian 'slow-velocity' expansion of the Dirac equation with respect to our reference worldline γ.

[5] Here we correct some omissions that occurred in the master's thesis [25] concerning terms of order x^2. Consequently our Dirac Hamiltonian (3.11) differs from that in [25].

In order to perform the post-Newtonian expansion systematically, we are going to implement it as a formal power series expansion[6] in the parameter c^{-1}, where c is the velocity of light[7]. Such formal expansions are a well-established device to implement Newtonian limits and post-Newtonian expansions of (locally) Poincaré-relativistic physics in a mathematically controlled manner: they appear, of course, in the İnönü–Wigner contraction from the Poincaré to the Galilei group [28], and have been applied, e.g., to systematically develop the post-Newtonian expansion of the Klein–Gordon equation [11, 18–20], or to discuss the rigorous post-Newtonian expansion of General Relativity and its modifications in the context of Newton–Cartan gravity (geometrised Newtonian gravity) [29–34]. In order to obtain a consistent post-Newtonian expansion[8], we need to treat the orthonormal-basis components of the curvature tensor and its covariant derivative as being of order c^{-2}, i.e.

$$R_{IJKL} = \mathrm{O}\left(c^{-2}\right), R_{IJKL;M} = \mathrm{O}\left(c^{-2}\right). \tag{4.1}$$

Since we have already introduced a formal power series expansion in x (i.e. in spacelike geodesic distance to our reference worldline γ), in the following we will encounter expressions that are 'doubly expanded' as power series in powers of both c^{-1} and x[9]. When writing down such expansions, we will order their terms as follows: first, we group and sort the terms by order of c^{-1}, and second, the terms comprising such a coefficient of a power c^{-n} will be sorted by order of x. We will also use the notation $\mathrm{O}(c^{-n}x^m)$ for terms that are of order at least n in the c^{-1}-expansion and order at least m in the x-expansion—e.g., we have $c^{-2}x^4 + c^{-3}x^3 = \mathrm{O}(c^{-2}x^3)$. For example, the expansion of some quantity X might look like

$$X = A + B_i x^i + C_{ij} x^i x^j + c^{-1}\left(E + F_i x^i\right) + \mathrm{O}\left(c^{-1}x^2\right) \tag{4.2}$$

(which would in particular imply that X has *vanishing* coefficients for all powers $c^{-n}x$ with $n \geqslant 2$).

Considering the Dirac Hamiltonian H_{Dirac} that appears in the Dirac equation $\mathrm{i}\partial_\tau\psi = H_{\mathrm{Dirac}}\psi$ in generalised Fermi normal coordinates, as computed in (3.11), we may of course read off its expansion as a power series in c^{-1} directly from (3.11)—we just need to keep in mind that we treat the curvature tensor as being of order c^{-2} according to (4.1). However, this expansion of the Dirac Hamiltonian in powers of c^{-1} is of no direct physical relevance for perturbation theory in the parameter c^{-1}: from (3.11), we directly obtain $H_{\mathrm{Dirac}} = \gamma^0 mc^2 + \mathrm{O}(c^1)$, such that when expanding the Dirac spinor field as a formal power series $\psi = \sum_{k=0}^{\infty} c^{-k}\psi^{(k)}$, the Dirac equation tells us at the lowest occurring order in c^{-1}, namely c^2, that $0 = \gamma^0 m\psi^{(0)}$, i.e. $\psi^{(0)} = 0$. At the next order c^1, it then implies $\psi^{(1)} = 0$, etc—meaning that the Dirac equation

[6] More precisely, since for some objects terms of negative order in c^{-1} will appear, it is an expansion as formal *Laurent* series. We will however continue to use the term 'power series', since most of our series will only have terms of non-negative order in c^{-1}.

[7] Of course, analytically speaking, a 'Taylor expansion' in a dimensionful parameter like c^{-1} does not make sense (even more so since c is a constant of nature); only for *dimensionless* parameters can a meaningful 'small-parameter approximation' be made. In physical realisations of the limit from (locally) Poincaré- to Galilei-symmetric theories, this means that the corresponding small parameter has to be chosen as, e.g., the ratio of some typical velocity of the system under consideration to the speed of light. In the following, however, we will ignore such issues and simply expand in c^{-1} as a formal 'deformation' parameter.

[8] From a purely formal perspective, not assigning those c^{-1}-orders to the curvature components would lead to the expanded positive-frequency Dirac equation that we consider later not having perturbative solutions. However, as already stated in the introduction, this assumption may also be viewed from a physical angle: in order for the acceleration of a system relative to γ, as given by the geodesic deviation equation, to stay bounded in the formal limit $c \to \infty$, we need to assume that $R_{0i0j} = \mathrm{O}(c^{-2})$.

[9] Formally, they will be valued in the formal Laurent/power series ring $\mathbb{R}((c^{-1}, x]]$.

has no non-trivial perturbative solutions of this form. Hence, in order to obtain a meaningful 'slow-velocity' approximation to the Dirac theory, we need to make a different perturbative ansatz for the spinor field. This will be a WKB-like 'positive frequency' ansatz.

Conceptionally, we now restrict from the full Dirac quantum field theory to its (effective) one-particle sector, which is a well-defined notion if we assume the spacetime to be stationary. The one-particle sector is effectively described by *classical* positive-frequency solutions of the Dirac equation, where 'positive frequency' is defined with respect to the stationarity Killing field [35][10]. It is those positive-frequency solutions whose field equation of motion we will expand in the following in powers of c^{-1}. A similar post-Newtonian expansion scheme for the Klein–Gordon equation may be found in [11, 18, 19]; a more general discussion of such schemes is given in [20].

Note that in any realistic situation, in which the theory contains interactions, this description can only be an approximation: the energy of all processes taking place has to be small enough such as to stay below the threshold of pair production, such that the system does not leave the one-particle sector. Therefore, such a post-Newtonian expansion always has to be considered a low-energy approximation.

In the following, we will define positive frequencies with respect to the coordinate time τ of the generalised Fermi normal coordinates introduced in section 3; therefore, for the relationship between positive-frequency classical solutions and the one-particle sector of the quantum theory to (approximately) hold, we need the timelike vector field $\partial/\partial\tau$ to be (approximately) Killing. The geometric meaning of this is briefly discussed in appendix B. Note, however, that the definition of positive-frequency solutions with respect to some 'time translation' vector field and the post-Newtonian expansion of such solutions of course also works for time translation vector fields which are *not* Killing, i.e. in a non-stationary situation, in which it still allows to view the full 'relativistic' positive-frequency Dirac equation as a formal deformation of its (locally) Galilei-symmetric Newtonian limit. In particular, as long as we are in an *approximately* stationary situation and the vector field is *approximately* Killing, the expansion will still give an approximate description of the one-particle sector of quantum field theory.

The WKB-like positive frequency ansatz that we will make for the Dirac field will lead, due to the lowest c^{-1} orders of the Dirac equation, to a split of the Dirac spinor into two two-component spinor fields with coupled equations of motion. One of these components can then, order by order in c^{-1}, be eliminated in terms of the other, which will in the end lead to a Pauli equation for the remaining two-spinor field, with gravitational and inertial 'corrections'. We are going to carry out this expansion to order c^{-2}, and in doing so, we want to keep the expansion in spacelike geodesic distance to the reference worldline γ such that the resulting Pauli Hamiltonian contains terms to order x^2, as it was the case for the Dirac Hamiltonian in (3.11). However, in the decoupling/elimination process described above, the to-be-eliminated component of the Dirac spinor field will be spatially differentiated once. Therefore, to achieve our goal of a consistent expansion of the final Hamiltonian to order x^2, we actually need to know those terms in the Dirac Hamiltonian which are of order up to c^{-1} in the c^{-1}-expansion *not only to order x^2, but to order x^3*. Employing the methods from [17], one can calculate the order-x^3 terms in the Christoffel symbols in generalised Fermi normal coordinates of c^{-1}-expansion order up to c^{-2} with a comparably small amount of work; and while doing so, one

[10] Often, this is called consideration of the 'first-quantised theory'—a historically grown name that sometimes unfortunately tends to create conceptual confusion. For details and caveats of how and why the one-particle sector of the quantum field theory is described by the positive-frequency classical theory, we refer to the extensive discussion in the monograph by Wald [35].

can actually convince oneself that all x-dependent terms in the Christoffel symbols are actually of order at least c^{-2}. The resulting Christoffel symbols, to order x^3 in the c^{-2} terms and to order x^2 in the higher-c^{-1}-order ones, are given in the appendix in (C.1). Using these further expanded Christoffel symbols, we can go through the further steps of the calculation of the Dirac Hamiltonian from section 3, thus computing the Dirac Hamiltonian to order x^3 in the c^{-1} terms and to order x^2 in the higher-c^{-1}-order ones. The expressions for the frame, the connection form and the inverse metric component g^{ss} arising as intermediate results in this process are given in appendix C; the resulting Dirac Hamiltonian is given in (C.6). *This* Dirac Hamiltonian will give rise, when carrying out our systematic expansion of the positive-frequency Dirac equation in powers of c^{-1}, to a consistently derived Pauli Hamiltonian to order x^2 and c^{-2}.

As the first step for implementing the expansion, we make for the Dirac field the WKB-like ansatz[11]

$$\psi = \mathrm{e}^{\mathrm{i}c^2 S}\tilde{\psi} \text{ with } S = \mathrm{O}\left(c^0\right), \tilde{\psi} = \sum_{k=0}^{\infty} c^{-k}\tilde{\psi}^{(k)}. \tag{4.3}$$

This ansatz we then insert into the Dirac equation $\mathrm{i}\partial_\tau\psi = H_{\mathrm{Dirac}}\psi$, with the Dirac Hamiltonian given by (C.6). The resulting equation we multiply with $\mathrm{e}^{-\mathrm{i}c^2 S}$ and compare coefficients of different powers of c^{-1}. At the lowest occurring order c^3, we obtain the equation $0 = \gamma^0\gamma^i(\partial_i S)\tilde{\psi}^{(0)}$, which in order to allow for non-trivial solutions $\tilde{\psi}$ enforces $\partial_i S = 0$, i.e. the function S depends only on time. At the next order c^2, we then obtain the equation

$$-(\partial_\tau S)\tilde{\psi}^{(0)} = \gamma^0 m\tilde{\psi}^{(0)}. \tag{4.4}$$

Since γ^0 has eigenvalues ± 1, for non-trivial solutions $\tilde{\psi}$ of the Dirac equation to exist we need $\partial_\tau S = \pm m$. Since we are interested in positive-frequency solutions of the Dirac equation, we choose $S = -m\tau$, discarding the constant of integration (which would lead to an irrelevant global phase). The preceding equation then tells us that the component of $\tilde{\psi}^{(0)}$ which lies in the -1 eigenspace of γ^0 has to vanish.

In the following, we will work in the Dirac representation for the gamma matrices, in which they are given by

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{4.5}$$

in terms of the Pauli matrices σ^i, such that Dirac spinors may be decomposed as

$$\psi = \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} \tag{4.6}$$

in terms of their components ψ_A, ψ_B lying in the $+1$ and -1 eigenspace of γ^0, respectively. (Note that $\psi_{A,B}$ are represented by functions taking values in $\mathbb{C}^2$.)

Summing up the above, our ansatz for the Dirac field now takes the form

$$\psi = \mathrm{e}^{-\mathrm{i}mc^2\tau}\begin{pmatrix} \tilde{\psi}_A \\ \tilde{\psi}_B \end{pmatrix}, \quad \tilde{\psi}_{A,B} = \sum_{k=0}^{\infty} c^{-k}\tilde{\psi}_{A,B}^{(k)}, \tag{4.7}$$

and we know that $\tilde{\psi}_B^{(0)} = 0$. Inserting this into the Dirac equation and multiplying with $\mathrm{e}^{\mathrm{i}mc^2\tau}$, we obtain two coupled equations for $\tilde{\psi}_{A,B}$, which are given in the appendix in (D.1). Now

[11] Note that in the master's thesis [25] on which the present article is based, a different notational convention was used in which $\tilde{\psi}^{(k)}$ includes the factor of c^{-k}.

comparing in these equations the coefficients of different orders of c^{-1}, we may order by order read off equations for the $\tilde{\psi}^{(k)}_{A,B}$. These allow to eliminate $\tilde{\psi}_B$ in favour of $\tilde{\psi}_A$, for which we will obtain a post-Newtonian Pauli equation.

More explicitly, this proceeds as follows. (D.1*a*) at order c^1 yields

$$0 = -\mathrm{i}\sigma^j D_j \tilde{\psi}^{(0)}_B \,, \tag{4.8}$$

which is trivially satisfied since $\tilde{\psi}^{(0)}_B = 0$. (D.1*b*) at order c^1 gives

$$2m\tilde{\psi}^{(1)}_B = -\mathrm{i}\sigma^j D_j \tilde{\psi}^{(0)}_A \tag{4.9}$$

and thus allows us to express $\tilde{\psi}^{(1)}_B$ in terms of $\tilde{\psi}^{(0)}_A$. We can carry on to the next order: (D.1*a*) at order c^0 yields

$$\left\{\mathrm{i}\partial_\tau + qA_\tau - m\boldsymbol{a}\cdot\boldsymbol{x} - \frac{mc^2}{2}R_{0l0m}x^l x^m - \mathrm{i}\left(\boldsymbol{\omega}\times\boldsymbol{x}\right)^i D_i + \frac{1}{2}\boldsymbol{\sigma}\cdot\boldsymbol{\omega} + \mathrm{O}\left(x^3\right)\right\}\tilde{\psi}^{(0)}_A = -\mathrm{i}\sigma^j D_j \tilde{\psi}^{(1)}_B. \tag{4.10}$$

Using (4.9), this may be rewritten as a Pauli equation

$$\mathrm{i}\partial_t \tilde{\psi}^{(0)}_A = H^{(0)}\tilde{\psi}^{(0)}_A \tag{4.11}$$

for $\tilde{\psi}^{(0)}_A$, with lowest-order Hamiltonian

$$H^{(0)} = -\frac{1}{2m}\left(\boldsymbol{\sigma}\cdot\boldsymbol{D}\right)^2 + m\boldsymbol{a}\cdot\boldsymbol{x} + \frac{mc^2}{2}R_{0l0m}x^l x^m + \mathrm{i}\left(\boldsymbol{\omega}\times\boldsymbol{x}\right)^i D_i - \frac{1}{2}\boldsymbol{\sigma}\cdot\boldsymbol{\omega} - qA_\tau + \mathrm{O}\left(x^3\right). \tag{4.12}$$

Next, (D.1*b*) at order c^0 allows us to express $\tilde{\psi}^{(2)}_B$ in terms of $\tilde{\psi}^{(1)}_A$ and $\tilde{\psi}^{(0)}_A$:

$$2m\tilde{\psi}^{(2)}_B = -\mathrm{i}\sigma^j D_j \tilde{\psi}^{(1)}_A + \left(\frac{mc^2}{6}\sigma^i R_{0lim}x^l x^m + \frac{mc^2}{12}\sigma^i R_{0lim;n}x^l x^m x^n + \mathrm{O}\left(x^4\right)\right)\tilde{\psi}^{(0)}_A. \tag{4.13}$$

Note that since $\tilde{\psi}^{(2)}_B$ will be differentiated once in the following calculation, here we need to include the term of order x^3 for later consistency, i.e. in order to be able to obtain the final Hamiltonian to order x^2. This is why we needed to know the low-c^{-1}-order terms of the Dirac Hamiltonian to order x^3, and not just order x^2. The same will happen at several later stages of the computation.

(D.1*a*) at order c^{-1} will then give an equation for $\tilde{\psi}^{(1)}_A$, which may be rewritten in the Pauli-like form

$$\mathrm{i}\partial_t \tilde{\psi}^{(1)}_A = H^{(0)}\tilde{\psi}^{(1)}_A + H^{(1)}\tilde{\psi}^{(0)}_A \,. \tag{4.14}$$

Due to the nature of the expansion, the lowest-order operator $H^{(0)}$ read off here will be the same as the one from the previous order. Detailed expressions may be found in appendix D.

Continuing, (D.1*b*) at order c^{-1} allows to express $\tilde{\psi}_B^{(3)}$ in terms of $\tilde{\psi}_A^{(2)}$, $\tilde{\psi}_A^{(1)}$, $\tilde{\psi}_A^{(0)}$, and $\tilde{\psi}_B^{(1)}$, which in turn may be expressed in terms of $\tilde{\psi}_A^{(0)}$ by (4.9). (D.1*a*) at order c^{-2} can then be rewritten as the Pauli-like equation

$$\mathrm{i}\partial_t\tilde{\psi}_A^{(2)} = H^{(0)}\tilde{\psi}_A^{(2)} + H^{(1)}\tilde{\psi}_A^{(1)} + H^{(2)}\tilde{\psi}_A^{(0)}\,. \tag{4.15}$$

Again, we know that $H^{(0)}$ and $H^{(1)}$ are the same as determined before; the operator $H^{(2)}$ will contain new information. Detailed expressions may again be found in appendix D. Note that in the process of expressing $\tilde{\psi}_B^{(3)}$ in terms of the $\tilde{\psi}_A$, one term arises for which we need to re-use the Pauli equation (4.11) for $\tilde{\psi}_A^{(0)}$ in order to fully eliminate the time derivative in the resulting expression.

The three Pauli-like equations (4.11), (4.14) and (4.15) now may be combined into a Pauli equation

$$\mathrm{i}\partial_t\tilde{\psi}_A = H_{\mathrm{Pauli}}\tilde{\psi}_A \tag{4.16a}$$

with Hamiltonian

$$H_{\mathrm{Pauli}} = H^{(0)} + c^{-1}H^{(1)} + c^{-2}H^{(2)} + \mathrm{O}\left(c^{-3}\right). \tag{4.16b}$$

Explicitly, the post-Newtonian Pauli Hamiltonian reads

$$\begin{aligned}
H_{\mathrm{Pauli}} = &\left\{ -\frac{1}{2m} - \frac{1}{2mc^2}\boldsymbol{a}\cdot\boldsymbol{x} - \frac{1}{4m}R_{0l0m}x^l x^m - \frac{1}{8m}R_{0l0m;n}x^l x^m x^n \right.\\
&\left. - \frac{1}{24m}R_{0k0l;mn}x^k x^l x^m x^n \right\}(\boldsymbol{\sigma}\cdot\boldsymbol{D})^2 - \frac{1}{8m^3c^2}(\boldsymbol{\sigma}\cdot\boldsymbol{D})^4\\
&+ \left\{ -\frac{1}{6m}R^i{}_l{}^j{}_m x^l x^m - \frac{1}{12m}R^i{}_l{}^j{}_{m;n}x^l x^m x^n - \frac{1}{40m}R^i{}_k{}^j{}_{l;mn}x^k x^l x^m x^n \right\} D_i D_j\\
&+ \left\{ \mathrm{i}\,(\boldsymbol{\omega}\times\boldsymbol{x})^j - \frac{2\mathrm{i}c}{3}R_{0l}{}^j{}_m x^l x^m - \frac{\mathrm{i}c}{4}R_{0l}{}^j{}_{m;n}x^l x^m x^n - \frac{1}{4mc^2}a^j - \frac{\mathrm{i}}{4mc^2}(\boldsymbol{\sigma}\times\boldsymbol{a})^j \right.\\
&+ \frac{1}{12m}\left(4R^j{}_l + R_0{}^j{}_{0l}\right)x^l + \frac{\mathrm{i}}{8m}\sigma^k\left(-2\varepsilon^{ij}{}_k R_{0l0i} + \varepsilon^{im}{}_k R^j{}_{lim}\right)x^l\\
&+ \frac{1}{24m}\Big(5R^j{}_{l;m} - 3R_0{}^j{}_{0l;m} - R_{0l0m}{}^{;j} - R^j{}_l{}^i{}_{m;i} - \mathrm{i}\varepsilon^{ij}{}_k\sigma^k\left(2R_{0i0l;m} + R_{0l0m;i}\right)\\
&+ 2\mathrm{i}\varepsilon^{in}{}_k\sigma^k R^j{}_{lin;m}\Big)x^l x^m + \frac{1}{120m}\left(9R^j{}_{l;mn} - 6R_0{}^j{}_{0l;mn} - 5R_{0l0m}{}^{;j}{}_n - 3R^j{}_l{}^i{}_{m;in}\right)x^l x^m x^n\\
&\left. + \frac{\mathrm{i}}{96m}\sigma^k\left(-4\varepsilon^{ij}{}_k\left(R_{0i0l;mn} + R_{0l0m;ni}\right) + 3\varepsilon^{ir}{}_k R^j{}_{lir;mn}\right)x^l x^m x^n \right\} D_j\\
&- qA_\tau + m\boldsymbol{a}\cdot\boldsymbol{x} + \frac{mc^2}{2}R_{0l0m}x^l x^m - \frac{1}{2}\boldsymbol{\sigma}\cdot\boldsymbol{\omega} + \frac{\mathrm{i}c}{3}R_{0l}x^l - \frac{c}{4}\varepsilon^{ij}{}_k\sigma^k R_{0lij}x^l\\
&+ \frac{\mathrm{i}c}{24}\left(5R_{0l;m} - R_{0l}{}^i{}_{m;i}\right)x^l x^m - \frac{c}{8}\varepsilon^{ij}{}_k\sigma^k R_{0lij;m}x^l x^m + \frac{1}{8m}R + \frac{1}{4m}R_{00}\\
&+ \frac{1}{16m}\left(R_{;l} + 2R^i{}_{l;i}\right)x^l + \frac{\mathrm{i}}{24m}\varepsilon^{ij}{}_k\sigma^k\left(R_{0i0l;j} - 2R_{il;j}\right)x^l
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{48m}\left(R_{;lm}+4R^{i}{}_{l;im}+\mathrm{i}\varepsilon^{ij}{}_{k}\sigma^{k}\left(R_{0i0l;jm}-3R_{il;jm}\right)\right)x^{l}x^{m}\\
&-\frac{q}{4m^{2}c^{2}}\sigma^{i}\sigma^{j}D_{i}E_{j}-\frac{q}{12m}\left(R_{lm}+R_{0l0m}\right)x^{l}x^{m}\boldsymbol{\sigma}\cdot\boldsymbol{B}+\frac{q}{12m}\sigma^{j}R_{iljm}x^{l}x^{m}B^{i}\\
&+\frac{\mathrm{i}q}{4m^{2}c^{2}}\boldsymbol{\sigma}\cdot(\boldsymbol{\omega}\times\boldsymbol{B})+\frac{q}{2m^{2}c^{2}}\boldsymbol{\omega}\cdot\boldsymbol{B}+\frac{q}{4m^{2}c^{2}}\left(\omega_{j}x^{i}-\omega^{i}x_{j}\right)D_{i}B^{j}\\
&+\frac{\mathrm{i}q}{4m^{2}c^{2}}\left(\boldsymbol{\sigma}\cdot(\boldsymbol{\omega}\times\boldsymbol{x})\right)\boldsymbol{B}\cdot\boldsymbol{D}-\frac{\mathrm{i}q}{4m^{2}c^{2}}\sigma^{j}\left(\boldsymbol{\omega}\times\boldsymbol{x}\right)\cdot\boldsymbol{D}B_{j}+\mathrm{O}\left(c^{-3}\right)+\mathrm{O}\left(x^{3}\right),
\end{aligned}
\tag{4.17}
$$

where $E_i = \partial_i A_\tau - \partial_\tau A_i$ is the electric field and $B^i = \varepsilon^{ijk}\partial_j A_k$ is the magnetic field (note that up to higher-order corrections, these are indeed the electromagnetic field components in an orthonormal basis). Note that in the expressions $D_i E_j$, $D_i B^j$, and $\boldsymbol{D}B_j$, the D_i acts on the product of the electric/magnetic field and the $\tilde{\psi}_A$ on which the Hamiltonian acts. The post-Newtonian Pauli Hamiltonian (4.17) is the second main result of this paper.

The terms in the Hamiltonian are ordered as follows: the terms involving electromagnetic fields come in the end, the terms without in the beginning. The latter are grouped by the form of the spatial derivative operators (built from D_i) appearing in them. In each of these groups, the terms are ordered as explained before (4.2): first, they are sorted by order of c^{-1}, and for each c^{-1}-order, the terms are sorted by order of x.

The lowest-order terms in the Hamiltonian, marked in green in (4.17), have clear interpretations: we have the usual 'Newtonian' kinetic-energy term $-\frac{1}{2m}(\boldsymbol{\sigma}\cdot\boldsymbol{D})^2$ for a Pauli particle minimally coupled to electromagnetism, the coupling $-qA_\tau$ to the electric scalar potential, the 'Newtonian' gravitational coupling $m(\boldsymbol{a}\cdot\boldsymbol{x}+\frac{c^2}{2}R_{0l0m}x^l x^m)$ to a potential including an acceleration and a tidal force term, and the spin–rotation coupling $-\frac{1}{2}\boldsymbol{\sigma}\cdot\boldsymbol{\omega}$. Note also that the Hamiltonian contains the special-relativistic correction to kinetic energy, $-\frac{1}{8m^3c^2}(\boldsymbol{\sigma}\cdot\boldsymbol{D})^4$. The other terms are higher-order inertial and gravitational corrections.

Note that the scalar product of our quantum theory, with respect to which the Hamiltonian (4.17) needs to be interpreted, is *not* simply the standard L^2 scalar product of $\mathbb{C}^2$-valued Pauli wavefunctions

$$
\langle\tilde{\phi}_A,\tilde{\psi}_A\rangle_{\mathrm{L}^2} := \int \mathrm{d}^3x\,\overline{\tilde{\phi}_A(\boldsymbol{x})}^{T}\tilde{\psi}_A(\boldsymbol{x})\,. \tag{4.18}
$$

Rather, the correct scalar product is that coming from the original Dirac theory: we start with the original Dirac scalar product

$$
\langle\phi,\psi\rangle_{\mathrm{Dirac}} := \int_\Sigma \mathrm{dvol}_\Sigma\, n_\mu\,\overline{\phi}^{T}\gamma^{I}(\mathrm{e}_I)^{0}\gamma^{J}(\mathrm{e}_J)^{\mu}\,\psi \tag{4.19}
$$

and compute its expansion in x and c^{-1} that arises from inserting our post-Newtonian ansatz (4.7) for the Dirac field and expressing $\tilde{\phi}_B$ and $\tilde{\psi}_B$ in terms of $\tilde{\phi}_A$ and $\tilde{\psi}_A$. With respect to *this* scalar product, the Hamiltonian is automatically Hermitian, since the Dirac scalar product in the full theory is conserved under time evolution.

Our post-Newtonian quantum theory also comes with a natural position operator, which in this representation of the Hilbert space is given by multiplication of 'wave functions' by coordinate position x^a. This operator arises as the post-Newtonian equivalent of that operator in the one-particle sector of the full Dirac theory which multiplies the Dirac fields by coordinate position x^a. For the case of the reference worldline γ being an inertial worldline in Minkowski spacetime and a non-rotating frame, that operator is, in fact, the Newton–Wigner position operator [36, 37].

4.1. Comparison to previous results by others

We now want to compare our post-Newtonian Hamiltonian (4.17) to that obtained in [24]. In order to do so we proceed as follows: we first recall the hypotheses on which the expansion in [24] is based. These we then use to further approximate our result in accordance with these hypotheses. Then, finally, we compare the result so obtained with that of [24]. We shall find a difference which we interpret as an inconsistency in [24].

Now, the approximation hypotheses in [24] that go beyond those imposed by us fall into three classes: First, concerning 'weak gravity', they assume $\omega = 0$ (no frame rotation), they neglect quadratic terms in a^i and R_{IJKL}, and, finally, they also do not consider terms involving covariant derivatives of the curvature tensor. Second, as regards their 'non-relativistic approximation', they neglect terms of quadratic or higher order in m^{-1}. Third, they trace over the spin degrees of freedom, i.e. compute $\frac{1}{2}\mathrm{tr}(H_{\mathrm{Pauli}})$, in order to obtain what in [24, p 16] was called 'the Hamiltonian [...] compatible with the description of a Schrödinger wavefunction'[12]. In units with $c = 1$, as used in [24], the result of applying this procedure to our Hamiltonian reads

$$\begin{aligned} \frac{1}{2}\mathrm{tr}\left(H_{\mathrm{Pauli}}\right) = & \left\{-\frac{1}{2m} - \frac{1}{2m}\boldsymbol{a}\cdot\boldsymbol{x} - \frac{1}{4m}R_{0l0m}x^l x^m\right\}\boldsymbol{D}^2 \\ & + \left\{-\frac{2\mathrm{i}}{3}R^j{}_{l0m}x^l x^m - \frac{1}{4m}a^j + \frac{1}{3m}R^j{}_l x^l + \frac{1}{12m}R_{0l0}{}^j x^l\right\}D_j \\ & - qA_\tau + m\boldsymbol{a}\cdot\boldsymbol{x} + \frac{m}{2}R_{0l0m}x^l x^m + \frac{\mathrm{i}}{3}R_{0l}x^l + \frac{1}{8m}R \\ & + \frac{1}{4m}R_{00} - \frac{1}{6m}R^i{}_l{}^j{}_m x^l x^m D_j D_i \ . \end{aligned} \tag{4.20}$$

This is *different* from the resulting Hamiltonian $\mathcal{H}$ from [24], with the difference reading

$$\frac{1}{2}\mathrm{tr}\left(H_{\mathrm{Pauli}}\right) - \mathcal{H} = \left\{\frac{1}{4m}\boldsymbol{a}\cdot\boldsymbol{x} + \frac{1}{8m}R_{0l0m}x^l x^m\right\}\boldsymbol{D}^2 + \left\{\frac{1}{2m}a^j + \frac{1}{2m}R_{0l0}{}^j x^l\right\}D_j + \frac{1}{4m}R_{00} \ . \tag{4.21}$$

This difference arises precisely from that term in the computation of our second-order Hamiltonian $H^{(2)}$ for which we had to re-use the lowest-order Pauli equation for $\tilde{\psi}_A^{(0)}$: in the final Pauli Hamiltonian, this term amounts to a contribution of

$$-\frac{1}{4m^2c^2}\left(-\mathrm{i}\boldsymbol{\sigma}\cdot\boldsymbol{D}\right)\left\{\mathrm{i}q\boldsymbol{\sigma}\cdot\boldsymbol{E} + \left(-\mathrm{i}\boldsymbol{\sigma}\cdot\boldsymbol{D}\right)\left(H^{(0)} + qA_\tau\right)\right\}; \tag{4.22}$$

due to $H^{(0)}$ containing terms proportional to m, this expression contains terms proportional to m^{-1}, which yield exactly the difference term (4.21).

[12] This method of tracing over the spin degrees of freedom in order to obtain a Hamiltonian acting on single-component (complex-number-valued) wavefunctions is used in [24] without further justification beyond the goal of acting on $\mathbb{C}$-valued functions. We do not believe this method to be of general physical validity for the following reason: the unitary time evolution described by the full post-Newtonian Pauli Hamiltonian contains interactions between the position and spin degrees of freedom. Therefore, the effective time evolution which we would obtain by ignoring the spin, i.e. by taking the partial trace of the total density matrix over the spin degrees of freedom, would no longer be unitary. Consequently, it cannot be described by a Schrödinger equation with respect to some Hamiltonian. Of course, this general argument does not exclude that, depending on the context, an approximately unitary time evolution for some specific initial states does indeed exist, but such an argument is not given in [24]. Nevertheless, for the sake of comparison to [24], we still apply the tracing procedure which we consider physically unwarranted.

Closely examining the calculation of [24], one can exactly pinpoint the step of this calculation at which the above term has been neglected: in appendix C of [24], going from equation (C3) to (C4), an inverse operator of the form $\frac{1}{2m}\left(1+\frac{\mathrm{i}\partial_T+qA_0}{2m}+(\text{terms linear in } a^i \text{ and } R)\right)^{-1}$ is evaluated via a perturbative expansion (in the notation of [24], $\mathrm{i}\partial_T$ is the 'non-relativistic energy' operator, i.e. the total energy of the Dirac solution minus the rest energy). The authors of [24] argue that when expanding (with respect to small quotients of involved energies), 'the rest mass of the system tends to be much larger than any of the terms that show up in the expansion', such that 'in a power expansion of the inverse operator in equation (C3), it makes sense to neglect terms that will contribute with order m^{-2}'. Following this argument, the term involving $\frac{\mathrm{i}\partial_T}{2m}$ is neglected. However, by following the ensuing calculation one can check that if it were *not* neglected at this point, this term would in the end lead to a contribution to the final Pauli Hamiltonian of the form

$$-\frac{1}{4m^2}\left(-\mathrm{i}\boldsymbol{\sigma}\cdot\boldsymbol{D}\right)^2 H^{(0)}+\mathrm{O}\left(m^{-2}\right) \tag{4.23}$$

in our notation, which to the order of approximation used in [24] is precisely the term noted above in (4.22).

We thus see that the $\frac{\mathrm{i}\partial_T}{2m}$ term ought *not* to be neglected in going from (C3) to (C4) in [24], since in the end it leads to terms that are *of the same order* as the other correction terms. A more direct formulation of the argument against neglecting this term is to note that $\mathrm{i}\partial_T$ acting on ψ_A (in the notation of [24]) induces terms proportional to m, such that $-\mathrm{i}\partial_T/(4m^2)$ is not actually of order m^{-2}, but of order m^{-1}.

One may also formulate our argument against the neglection without referring to expanding in m^{-1} at all, speaking only about quotients of energies instead, in the spirit of [24]: if one were to neglect the term $-\mathrm{i}\partial_T/(4m^2)=-\frac{1}{2m}\cdot\frac{\text{'non-relativistic energy'}}{2(\text{rest energy})}$ in going from (C3) to (C4) in [24], then one would as well have to neglect the terms $-\frac{1}{2m}\left(\frac{1}{2}a_jx^j+\frac{1}{4}R_{k0m0}x^kx^m\right)=-\frac{1}{2m}\cdot\frac{m(a_jx^j+R_{k0m0}x^kx^m/2)}{2m}=-\frac{1}{2m}\cdot\frac{\text{corrections in 'non-relativistic energy'}}{2(\text{rest energy})}$. These last terms, however, clearly have to be kept in the calculation since they contribute at a relevant order, and indeed are kept in [24].

Thus, we come to the conclusion that the difference between the result of [24] and our result when truncated to linear approximation order is due to an undue neglection in [24], which without further justification seems to render the approximation used in [24] inconsistent. In our opinion, this exemplifies that a mathematically clear systematic approximation scheme with spelled-out assumptions—such as ours, based on (formal) power series expansions in deformation parameters—reduces possibilities for conceptual errors in approximative calculations.

5. Conclusion

Deducing the impact of classical gravitational fields (in the sense of general relativity) onto the dynamical evolution of quantum systems is a non-trivial task of rapidly increasing theoretical interest given the acceleration that we currently witness in experimental areas, like g-factor measurements [1–4], atom interferometry [6–9, 12–14], and metrology.

The relatively simple case of a single spin-half particle in an external gravitational field that we dealt with here provides a good example of the nature and degree on non-triviality immediately encountered. Given the many much further reaching claims that emerge from various 'approaches' to a theory of quantum gravity proper this may be read as a call for some restraint. On the other hand, just listing longer and longer strings of corrections to Hamiltonians will in the end also lead us nowhere without a consistent interpretational scheme that eventually allows us to communicate the physical significance of each term to our experimental colleagues. In this respect we tried hard to consistently stay within a well-defined scheme, so as to produce each term of a given, well-defined order once and only once. In that respect we also wish to refer to our discussion in [20].

Closest to our approach are the papers that we already discussed in the introduction. We claim to have improved on them concerning not only the order of approximation but also concerning the systematics. We showed that even within the larger (and hence more restricting) set of approximation-hypotheses assumed in the most recent of these papers [24], their list of terms for the final Hamiltonian is not complete. Ours, we believe, is.

Finally we wish to mention a characteristic difficulty concerning the interpretation of interaction terms in Hamiltonians in the context of general relativity. It has to do with the changing interpretation of coordinates once the Hamiltonian refers to different metrics. More precisely, consider two Hamiltonian functions being given, one of which takes into account the interaction with the gravitational field to a higher degree than the other; then, strictly speaking, it is not permissible to address the additional terms as the sole expression of the higher order interaction, the reason being that together with the higher degree of approximation to the metric, the metric meaning of the coordinates, too, has also changed at the same time. Again we refer to [20] for a more extensive discussion, also providing a typical example.

Data availability statement

No new data were created or analysed in this study.

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Appendix A. The connection in generalised Fermi normal coordinates

The Christoffel symbols in generalised Fermi normal coordinates were calculated to second order in the geodesic distance to the reference worldline in [17]. Note that in this reference, some calculational errors were made, which we have corrected in the following and marked in red. The Christoffel symbols are given by

$$\begin{aligned}\Gamma^s_{ss} &= c^{-3}\left(\boldsymbol{b}\cdot\boldsymbol{x}+2\boldsymbol{a}\cdot(\boldsymbol{\omega}\times\boldsymbol{x})\right)+\frac{1}{2}R_{0l0m;0}x^l x^m+\frac{1}{3}c^{-2}a^i R_{0lim}x^l x^m\\ &\quad -c^{-5}\left(\boldsymbol{b}\cdot\boldsymbol{x}+2\boldsymbol{a}\cdot(\boldsymbol{\omega}\times\boldsymbol{x})\right)(\boldsymbol{a}\cdot\boldsymbol{x})+2c^{-1}R_{0i0j}\left(\boldsymbol{\omega}\times\boldsymbol{x}\right)^i x^j+\mathrm{O}\left(x^3\right),\end{aligned} \tag{A.1a}$$

$$\begin{aligned}\Gamma^s_{si} &= c^{-2}a_i-c^{-4}a_i\left(\boldsymbol{a}\cdot\boldsymbol{x}\right)+R_{0i0j}x^j+\frac{1}{6}\left(R_{0l0m;i}+2R_{0i0l;m}\right)x^l x^m-\frac{2}{3}c^{-2}\left(\boldsymbol{a}\cdot\boldsymbol{x}\right)R_{0i0j}x^j\\ &\quad -\frac{1}{3}c^{-2}a_i R_{0l0m}x^l x^m+c^{-6}a_i\left(\boldsymbol{a}\cdot\boldsymbol{x}\right)^2-\frac{1}{3}c^{-1}\left(\boldsymbol{\omega}\times\boldsymbol{x}\right)^k\left(R_{0ilk}+R_{0kli}\right)x^l+\mathrm{O}\left(x^3\right),\end{aligned} \tag{A.1b}$$

$$\Gamma^s_{ij} = \frac{1}{3}\left\{2R_{0(ij)k} + \frac{1}{4}\left(5R_{0(ij)k;l} - R_{0kl(i;j)}\right)x^l - 2c^{-2}(\boldsymbol{a}\cdot\boldsymbol{x})R_{0(ij)k}\right\}x^k + \mathrm{O}\left(x^3\right), \tag{A.1c}$$

$$\begin{aligned}\Gamma^i_{ss} &= c^{-2}a^i + R_0{}^i{}_{0j}x^j + c^{-2}(\boldsymbol{\eta}\times\boldsymbol{x})^i + c^{-4}(\boldsymbol{a}\cdot\boldsymbol{x})a^i + c^{-2}(\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{x}))^i - \frac{1}{2}R_{0l0m}{}^{;i}x^lx^m \\ &\quad + R_0{}^i{}_{0l;m}x^lx^m + 2c^{-2}(\boldsymbol{a}\cdot\boldsymbol{x})R_0{}^i{}_{0j}x^j - \frac{1}{3}c^{-2}a^jR^i{}_{ljm}x^lx^m \\ &\quad - c^{-4}(\boldsymbol{\omega}\times\boldsymbol{x})^i(\boldsymbol{b}\cdot\boldsymbol{x} + 2\boldsymbol{a}\cdot(\boldsymbol{\omega}\times\boldsymbol{x})) - 2c^{-1}(\boldsymbol{\omega}\times\boldsymbol{x})^kR_{0j}{}^i{}_kx^j + \mathrm{O}\left(x^3\right),\end{aligned} \tag{A.1d}$$

$$\begin{aligned}\Gamma^i_{sj} &= -c^{-1}\varepsilon^i{}_{jk}\omega^k - R_{0kj}{}^ix^k - c^{-3}(\boldsymbol{\omega}\times\boldsymbol{x})^ia_j + \left\{+\frac{1}{6}R_{0j}{}^i{}_{l;m} - \frac{1}{2}R_{0l}{}^i{}_{j;m} - \frac{1}{6}R_{0l}{}^i{}_{m;j}\right\}x^lx^m \\ &\quad - \frac{1}{3}c^{-2}(\boldsymbol{a}\cdot\boldsymbol{x})\left(R_{0k}{}^i{}_j + R_0{}^i{}_{kj}\right)x^k + \frac{1}{3}c^{-2}a_jR_{0l}{}^i{}_mx^lx^m - c^{-1}(\boldsymbol{\omega}\times\boldsymbol{x})^iR_{0j0k}x^k \\ &\quad - \frac{1}{3}c^{-1}(\boldsymbol{\omega}\times\boldsymbol{x})^l\left(R_{lk}{}^i{}_j + R_l{}^i{}_{kj}\right)x^k + c^{-5}a_j(\boldsymbol{a}\cdot\boldsymbol{x})(\boldsymbol{\omega}\times\boldsymbol{x})^i + \mathrm{O}\left(x^3\right),\end{aligned} \tag{A.1e}$$

$$\Gamma^i_{jk} = -\frac{1}{3}\left\{2R^i{}_{(jk)l} + \frac{1}{4}\left(5R^i{}_{(jk)l;m} - R^i{}_{lm(j;k)}\right)x^m + 2c^{-1}(\boldsymbol{\omega}\times\boldsymbol{x})^iR_{0(jk)l}\right\}x^l + \mathrm{O}\left(x^3\right). \tag{A.1f}$$

The local connection form with respect to the frame (3.8) is given by

$$\omega_\mu{}^0{}_0 = 0, \tag{A.2a}$$

$$\begin{aligned}\omega_s{}^0{}_i &= c^{-2}a_i + R_{0i0l}x^l + \frac{1}{2}c^{-2}(\boldsymbol{a}\cdot\boldsymbol{x})R_{0i0l}x^l + \frac{1}{2}c^{-1}(\boldsymbol{\omega}\times\boldsymbol{x})^kR_{0ikl}x^l \\ &\quad + \frac{1}{2}R_{0i0l;m}x^lx^m + \mathrm{O}\left(x^3\right),\end{aligned} \tag{A.2b}$$

$$\omega_i{}^0{}_j = \frac{1}{2}R_{0jil}x^l + \frac{1}{3}R_{0jil;m}x^lx^m + \mathrm{O}\left(x^3\right), \tag{A.2c}$$

$$\omega_\mu{}^i{}_0 = \delta^{ij}\omega_\mu{}^0{}_j\,, \tag{A.2d}$$

$$\begin{aligned}\omega_s{}^i{}_j &= -c^{-1}\varepsilon^i{}_{jk}\omega^k - R^i{}_{j0l}x^l - \frac{1}{2}c^{-2}(\boldsymbol{a}\cdot\boldsymbol{x})R^i{}_{j0l}x^l - \frac{1}{2}c^{-1}(\boldsymbol{\omega}\times\boldsymbol{x})^kR^i{}_{jkl}x^l \\ &\quad - \frac{1}{2}R^i{}_{j0l;m}x^lx^m + \mathrm{O}\left(x^3\right),\end{aligned} \tag{A.2e}$$

$$\omega_k{}^i{}_j = -\frac{1}{2}R^i{}_{jkl}x^l - \frac{1}{3}R^i{}_{jkl;m}x^lx^m + \mathrm{O}\left(x^3\right). \tag{A.2f}$$

Appendix B. Stationarity with respect to the generalised Fermi normal coordinate time translation field

In the following, we are going to briefly discuss the geometric interpretation of the possible condition that the metric be stationary with respect to the time coordinate τ of the generalised Fermi normal coordinates introduced in section 3, i.e. that the timelike vector field $\partial/\partial\tau$ be Killing. Note that, as explained in the main text, the post-Newtonian expansion in c^{-1} of section 4 is *still* a meaningful approximation procedure if stationarity does *not* hold, formulating the Dirac theory as a deformation of its Newtonian limit.

As a first step, stationarity with respect to $\partial/\partial\tau$ of course means that the reference worldline γ has to be stationary.

Away from the worldline, $\partial/\partial\tau$ being Killing means that the metric components (3.4) need be independent of coordinate time τ; i.e. we need the components a^i, ω^i, R_{IJKL} of the acceleration of γ, the angular velocity of the spatial basis (e_i) and the curvature to be constant along the reference worldline γ:

$$\dot{a}^i(\tau) = 0, \quad \dot{\omega}^i(\tau) = 0, \quad \dot{R}_{IJKL}(\tau) = 0. \tag{B.1}$$

Note, however, that the components are taken *with respect to the generalised Fermi normal coordinates*; therefore, to see the true geometric meaning of these conditions, we need to rewrite them covariantly.

By direct computation, for the covariant derivatives of acceleration and angular velocity we have

$$b^i(\tau) = \left(\nabla_{\dot{\gamma}(\tau)} a(\tau)\right)^i = \dot{a}^i(\tau) + c\Gamma^i_{sj}(\gamma(\tau))\, a^j(\tau) = \dot{a}^i(\tau) + (\boldsymbol{\omega}(\tau) \times \boldsymbol{a}(\tau))^i\,, \tag{B.2}$$

$$\eta^i(\tau) = \left(\nabla_{\dot{\gamma}(\tau)} \omega(\tau)\right)^i = \dot{\omega}^i(\tau) + c\Gamma^i_{sj}(\gamma(\tau))\, \omega^j(\tau) = \dot{\omega}^i(\tau)\,. \tag{B.3}$$

Thus, we see that stationarity of the metric with respect to the time translation vector field given by generalised Fermi normal coordinates implies that the angular velocity ω of the spatial reference vectors be covariantly constant along the reference worldline γ. However, in the case of ω being non-zero, in the generic case the worldline's acceleration a need *not* be covariantly constant—it has to itself rotate with angular velocity ω, such that its components with respect to the rotating basis are constant. This may sound somewhat artificial, but note that for example one could satisfy this condition with a covariantly constant acceleration a and spatial basis vectors (e_i) that rotate around the axis given by a.

Of course, the condition of constancy of the curvature components along γ can be rewritten in terms of covariant derivatives of the curvature tensor as well; however, this does not lead to any great insight, so we will refrain from doing so here.

Appendix C. The Dirac Hamiltonian up to $\mathrm{O}(c^{-1}x^4) + \mathrm{O}(c^{-2}x^3)$

In the main text, for a consistent post-Newtonian expansion of the Dirac Hamiltonian leading to a resulting Pauli Hamiltonian known to order c^{-2} and x^2, we need to know the Dirac Hamiltonian to order x^3 in those terms of order up to c^{-1} in the c^{-1}-expansion. Going through the derivation of [17], one can convince oneself that all x-dependent terms in the Christoffel symbols in generalised Fermi normal coordinates are of order at least c^{-2} when expanding also in c^{-1}; and employing the methods from [17], one can go to higher order and calculate the order-x^3 terms to order c^{-2}. The resulting Christoffel symbols read as follows, with the newly calculated terms marked in blue (note that we use the ordering of terms and the $\mathrm{O}(c^{-n}x^m)$ notation as explained in the main text before (4.2)):

$$
\begin{aligned}
\Gamma^s_{ss} &= c^{-2}\left(\frac{c^2}{2}R_{0l0m;0}x^l x^m + \frac{c^2}{6}R_{0l0m;n0}x^l x^m x^n\right) \\
&\quad + c^{-3}\left(\boldsymbol{b}\cdot\boldsymbol{x} + 2\boldsymbol{a}\cdot(\boldsymbol{\omega}\times\boldsymbol{x}) + 2c^2 R_{0i0j}(\boldsymbol{\omega}\times\boldsymbol{x})^i x^j\right) + c^{-4}\left(\frac{c^2}{3}a^i R_{0lim}x^l x^m\right) \\
&\quad - c^{-5}\left((\boldsymbol{b}\cdot\boldsymbol{x} + 2\boldsymbol{a}\cdot(\boldsymbol{\omega}\times\boldsymbol{x}))(\boldsymbol{a}\cdot\boldsymbol{x})\right) + \mathrm{O}\left(c^{-2}x^4\right) + \mathrm{O}\left(c^{-3}x^3\right),
\end{aligned} \tag{C.1a}
$$

$$
\begin{aligned}
\Gamma^s_{si} &= c^{-2}\left(a_i + c^2 R_{0i0j}x^j + \frac{c^2}{6}\left(R_{0l0m;i} + 2R_{0i0l;m}\right)x^l x^m + \frac{c^2}{12}\left(R_{0i0l;mn} + R_{0l0m;ni}\right)x^l x^m x^n\right) \\
&\quad + c^{-3}\left(-\frac{c^2}{3}(\boldsymbol{\omega}\times\boldsymbol{x})^k\left(R_{0ilk} + R_{0kli}\right)x^l\right) + c^{-4}\left(-a_i(\boldsymbol{a}\cdot\boldsymbol{x}) - \frac{2c^2}{3}(\boldsymbol{a}\cdot\boldsymbol{x})R_{0i0j}x^j\right. \\
&\quad \left. - \frac{c^2}{3}a_i R_{0l0m}x^l x^m\right) + c^{-6}a_i(\boldsymbol{a}\cdot\boldsymbol{x})^2 + \mathrm{O}\left(c^{-2}x^4\right) + \mathrm{O}\left(c^{-3}x^3\right),
\end{aligned} \tag{C.1b}
$$

$$
\begin{aligned}
\Gamma^s_{ij} &= c^{-2}\left(\frac{c^2}{3}\left\{2R_{0(ij)k} + \frac{1}{4}\left(5R_{0(ij)k;l} - R_{0kl(i;j)}\right)x^l\right\}x^k + \frac{c^2}{20}\left(3R_{0(ij)l;mn} - R_{0lm(i;j)n}\right)x^l x^m x^n\right) \\
&\quad + c^{-4}\left(-\frac{2c^2}{3}(\boldsymbol{a}\cdot\boldsymbol{x})R_{0(ij)l}x^l\right) + \mathrm{O}\left(c^{-2}x^4\right) + \mathrm{O}\left(c^{-3}x^3\right),
\end{aligned} \tag{C.1c}
$$

$$
\begin{aligned}
\Gamma^i_{ss} &= c^{-2}\left(a^i + c^2 R_0{}^i{}_{0j}x^j + (\boldsymbol{\eta}\times\boldsymbol{x})^i + (\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{x}))^i + \frac{c^2}{2}\left(2R_0{}^i{}_{0l;m} - R_{0l0m}{}^{;i}\right)x^l x^m\right. \\
&\quad \left. + \frac{c^2}{6}\left(2R_0{}^i{}_{0l;mn} - R_{0l0m;n}{}^i\right)x^l x^m x^n\right) + c^3\left(-2c(\boldsymbol{\omega}\times\boldsymbol{x})^k R_{0j}{}^i{}_k x^j\right) \\
&\quad + c^{-4}\left((\boldsymbol{a}\cdot\boldsymbol{x})a^i + 2c^2(\boldsymbol{a}\cdot\boldsymbol{x})R_0{}^i{}_{0j}x^j - \frac{c^2}{3}a^j R^i{}_{ljm}x^l x^m\right. \\
&\quad \left. - (\boldsymbol{\omega}\times\boldsymbol{x})^i\left(\boldsymbol{b}\cdot\boldsymbol{x} + 2\boldsymbol{a}\cdot(\boldsymbol{\omega}\times\boldsymbol{x})\right)\right) + \mathrm{O}\left(c^{-2}x^4\right) + \mathrm{O}\left(c^{-3}x^3\right),
\end{aligned} \tag{C.1d}
$$

$$
\begin{aligned}
\Gamma^i_{sj} &= -c^{-1}\varepsilon^i{}_{jk}\omega^k + c^{-2}\left(-c^2 R_{0k}{}^i{}_j x^k + c^2\left\{\frac{1}{6}R_{0j}{}^i{}_{l;m} - \frac{1}{2}R_{0l}{}^i{}_{j;m} - \frac{1}{6}R_{0l}{}^i{}_{m;j}\right\}x^l x^m\right. \\
&\quad \left. + \frac{c^2}{12}\left(R_{0j}{}^i{}_{l;mn} - 2R_{0l}{}^i{}_{j;mn} - R_{0l}{}^i{}_{m;nj}\right)x^l x^m x^n\right) \\
&\quad + c^{-3}\left(-(\boldsymbol{\omega}\times\boldsymbol{x})^i a_j - \frac{c^2}{3}(\boldsymbol{\omega}\times\boldsymbol{x})^l\left(R_{lk}{}^i{}_j + R_l{}^i{}_{kj}\right)x^k - c^2(\boldsymbol{\omega}\times\boldsymbol{x})^i R_{0j0k}x^k\right) \\
&\quad + c^{-4}\left(-\frac{c^2}{3}(\boldsymbol{a}\cdot\boldsymbol{x})\left(R_{0k}{}^i{}_j + R_0{}^i{}_{kj}\right)x^k + \frac{c^2}{3}a_j R_{0l}{}^i{}_m x^l x^m\right) + c^{-5}a_j(\boldsymbol{a}\cdot\boldsymbol{x})(\boldsymbol{\omega}\times\boldsymbol{x})^i \\
&\quad + \mathrm{O}\left(c^{-2}x^4\right) + \mathrm{O}\left(c^{-3}x^3\right),
\end{aligned} \tag{C.1e}
$$

$$\Gamma^i_{jk} = c^{-2}\Bigg(-\frac{c^2}{3}\left\{2R^i{}_{(jk)l} + \frac{1}{4}\left(5R^i{}_{(jk)l;m} - R^i{}_{lm(j;k)}\right)x^m\right\}x^l - \frac{c^2}{20}\left(3R^i{}_{(jk)l;mn} - R^i{}_{lm(j;k)n}\right)x^l x^m x^n\Bigg) + c^{-3}\left(2c^2\left(\boldsymbol{\omega}\times\boldsymbol{x}\right)^i R_{0(jk)l}x^l\right) + \mathrm{O}\left(c^{-2}x^4\right) + \mathrm{O}\left(c^{-3}x^3\right). \tag{C.1f}$$

Note that, according to (4.1), we have treated the curvature tensor as being of order c^{-2}.

Using the above Christoffel symbols, one can compute the parallely transported frame (3.8) to higher order of expansion, which reads

$$(\mathrm{e}_0)^s = 1 + c^{-2}\left(-\boldsymbol{a}\cdot\boldsymbol{x} - \frac{c^2}{2}R_{0l0m}x^l x^m - \frac{c^2}{6}R_{0l0m;n}x^l x^m x^n - \frac{c^2}{24}R_{0k0l;mn}x^k x^l x^m x^n\right) + c^{-4}\left(\left(\boldsymbol{a}\cdot\boldsymbol{x}\right)^2 + \frac{5c^2}{6}\left(\boldsymbol{a}\cdot\boldsymbol{x}\right)R_{0l0m}x^l x^m\right) - c^{-6}\left(\boldsymbol{a}\cdot\boldsymbol{x}\right)^3 + \mathrm{O}\left(c^{-2}x^5\right) + \mathrm{O}\left(c^{-3}x^4\right), \tag{C.2a}$$

$$(\mathrm{e}_0)^i = -c^{-1}\left(\boldsymbol{\omega}\times\boldsymbol{x}\right)^i + c^{-2}\left(\frac{c^2}{2}R_{0l}{}^i{}_m x^l x^m + \frac{c^2}{6}R_{0l}{}^i{}_{m;n}x^l x^m x^n + \frac{c^2}{24}R_{0k}{}^i{}_{l;mn}x^k x^l x^m x^n\right) + c^{-3}\left(\left(\boldsymbol{a}\cdot\boldsymbol{x}\right)\left(\boldsymbol{\omega}\times\boldsymbol{x}\right)^i + \frac{c^2}{2}\left(\boldsymbol{\omega}\times\boldsymbol{x}\right)^i R_{0l0m}x^l x^m\right) + c^{-4}\left(-\frac{c^2}{3}\left(\boldsymbol{a}\cdot\boldsymbol{x}\right)R_{0l}{}^i{}_m x^l x^m\right) - c^{-5}\left(\boldsymbol{\omega}\times\boldsymbol{x}\right)^i\left(\boldsymbol{a}\cdot\boldsymbol{x}\right)^2 + \mathrm{O}\left(c^{-2}x^5\right) + \mathrm{O}\left(c^{-3}x^4\right), \tag{C.2b}$$

$$(\mathrm{e}_i)^s = c^{-2}\left(-\frac{c^2}{6}R_{0lim}x^l x^m - \frac{c^2}{12}R_{0lim;n}x^l x^m x^n - \frac{c^2}{40}R_{0kil;mn}x^k x^l x^m x^n\right) + c^{-4}\left(\frac{c^2}{6}\left(\boldsymbol{a}\cdot\boldsymbol{x}\right)R_{0lim}x^l x^m\right) + \mathrm{O}\left(c^{-2}x^5\right) + \mathrm{O}\left(c^{-3}x^4\right), \tag{C.2c}$$

$$(\mathrm{e}_i)^j = \delta^j_i + c^{-2}\left(\frac{c^2}{6}R^j{}_{lim}x^l x^m + \frac{c^2}{12}R^j{}_{lim;n}x^l x^m x^n + \frac{c^2}{40}R^j{}_{kil;mn}x^k x^l x^m x^n\right) + c^{-3}\left(\frac{c^2}{6}\left(\boldsymbol{\omega}\times\boldsymbol{x}\right)^j R_{0lim}x^l x^m\right) + \mathrm{O}\left(c^{-2}x^5\right) + \mathrm{O}\left(c^{-3}x^4\right). \tag{C.2d}$$

For the dual frame, we also obtain that the x dependence starts at order c^{-2}:

$$\left(\mathrm{e}^0\right)_s = 1 + c^{-2}\left(\boldsymbol{a}\cdot\boldsymbol{x} + \frac{c^2}{2}R_{0l0m}x^l x^m\right) + \mathrm{O}\left(c^{-2}x^3\right) \tag{C.3a}$$

$$\left(\mathrm{e}^0\right)_i = c^{-2}\left(\frac{c^2}{6}R_{0lim}x^l x^m\right) + \mathrm{O}\left(c^{-2}x^3\right) \tag{C.3b}$$

$$\left(\mathrm{e}^i\right)_s = c^{-1}\left(\boldsymbol{\omega}\times\boldsymbol{x}\right)^i + c^{-2}\left(-\frac{c^2}{2}R^i{}_{l0m}x^l x^m\right) + \mathrm{O}\left(c^{-2}x^3\right) \tag{C.3c}$$

$$\left(\mathrm{e}^i\right)_j = \delta^i_j + c^{-2}\left(-\frac{c^2}{6}R^i{}_{ljm}x^l x^m\right) + \mathrm{O}\left(c^{-2}x^3\right). \tag{C.3d}$$

From this, we can compute the higher-order corrections to the connection form, the non-trivial components of which read

$$\omega_s{}^0{}_i = c^{-2}\left(a_i + c^2 R_{0i0l}x^l + \frac{c^2}{2}R_{0i0l;m}x^l x^m + \frac{c^2}{6}R_{0i0l;mn}x^l x^m x^n\right) + c^{-3}\left(\frac{c^2}{2}(\boldsymbol{\omega}\times\boldsymbol{x})^k R_{0ikl}x^l\right) + c^{-4}\left(\frac{c^2}{2}(\boldsymbol{a}\cdot\boldsymbol{x})R_{0i0l}x^l\right) + \mathrm{O}\left(c^{-2}x^4\right) + \mathrm{O}\left(c^{-3}x^3\right), \tag{C.4a}$$

$$\omega_i{}^0{}_j = c^{-2}\left(\frac{c^2}{2}R_{0jil}x^l + \frac{c^2}{3}R_{0jil;m}x^l x^m + \frac{c^2}{8}R_{0jil;mn}x^l x^m x^n\right) + \mathrm{O}\left(c^{-2}x^4\right) + \mathrm{O}\left(c^{-3}x^3\right), \tag{C.4b}$$

$$\omega_s{}^i{}_j = -c^{-1}\varepsilon^i{}_{jk}\omega^k + c^{-2}\left(-c^2 R^i{}_{j0l}x^l - \frac{c^2}{2}R^i{}_{j0l;m}x^l x^m - \frac{c^2}{6}R^i{}_{j0l;mn}x^l x^m x^n\right) + c^{-3}\left(-\frac{c^2}{2}(\boldsymbol{\omega}\times\boldsymbol{x})^k R^i{}_{jkl}x^l\right) + c^{-4}\left(-\frac{c^2}{2}(\boldsymbol{a}\cdot\boldsymbol{x})R^i{}_{j0l}x^l\right) + \mathrm{O}\left(c^{-2}x^4\right) + \mathrm{O}\left(c^{-3}x^3\right), \tag{C.4c}$$

$$\omega_k{}^i{}_j = c^{-2}\left(-\frac{c^2}{2}R^i{}_{jkl}x^l - \frac{c^2}{3}R^i{}_{jkl;m}x^l x^m - \frac{c^2}{8}R^i{}_{jkl;mn}x^l x^m x^n\right) + \mathrm{O}\left(c^{-2}x^4\right) + \mathrm{O}\left(c^{-3}x^3\right). \tag{C.4d}$$

The component of the inverse metric that is needed for the computation of the Dirac Hamiltonian takes the following form including the newly computed higher-order corrections:

$$g^{ss} = -1 + c^{-2}\left(2\boldsymbol{a}\cdot\boldsymbol{x} + c^2 R_{0l0m}x^l x^m + \frac{c^2}{3}R_{0l0m;n}x^l x^m x^n + \frac{c^2}{12}R_{0k0l;mn}x^k x^l x^m x^n\right) + c^{-4}\left(-3(\boldsymbol{a}\cdot\boldsymbol{x})^2 - \frac{8c^2}{3}(\boldsymbol{a}\cdot\boldsymbol{x})R_{0l0m}x^l x^m\right) + c^{-6}4(\boldsymbol{a}\cdot\boldsymbol{x})^3 + \mathrm{O}\left(c^{-2}x^5\right) + \mathrm{O}\left(c^{-3}x^4\right). \tag{C.5}$$

Using all these ingredients, we can finally compute the Dirac Hamiltonian in our coordinates and frame to the necessary order (as for the original Dirac Hamiltonian (3.11), the computation is rather tedious, but straightforward):

$$\begin{aligned} H_{\mathrm{Dirac}} = \gamma^0&\left\{mc^2 + c^0\left(m\boldsymbol{a}\cdot\boldsymbol{x} + \frac{mc^2}{2}R_{0l0m}x^l x^m + \frac{mc^2}{6}R_{0l0m;n}x^l x^m x^n\right) + \mathrm{O}\left(c^0x^4\right)\right\} \\ &- \gamma^i\left\{c^0\left(\frac{mc^2}{6}R_{0lim}x^l x^m + \frac{mc^2}{12}R_{0lim;n}x^l x^m x^n\right) + \mathrm{O}\left(c^0x^4\right)\right\} \\ &+ \mathbb{1}\left\{-qA_\tau + \mathrm{i}(\boldsymbol{\omega}\times\boldsymbol{x})^i D_i + c^{-1}\left(-\frac{\mathrm{i}c^2}{2}R_{0l}{}^i{}_m x^l x^m D_i + \frac{\mathrm{i}c^2}{12}R_{0l;m}x^l x^m\right.\right. \\ &\left. - \frac{\mathrm{i}c^2}{6}R_{0l}{}^i{}_{m;n}x^l x^m x^n D_i + \frac{\mathrm{i}c^2}{24}R_{0l;mn}x^l x^m x^n - \frac{\mathrm{i}c^2}{24}R_{0k}{}^i{}_{l;mn}x^k x^l x^m x^n D_i\right) \\ &\left. + c^{-3}\left(\frac{\mathrm{i}c^2}{4}(\boldsymbol{a}\cdot\boldsymbol{x})R_{0l}x^l - \frac{\mathrm{i}c^2}{6}(\boldsymbol{a}\cdot\boldsymbol{x})R_{0l}{}^i{}_m x^l x^m D_i\right)\right\} \end{aligned}$$

$$
\begin{aligned}
&-\gamma^0\gamma^j\Big\{ciD_j+c^{-1}\Big(\frac{i}{2}a_j+i(\boldsymbol{a}\cdot\boldsymbol{x})D_j+\frac{ic^2}{4}(R_{0j0l}-R_{jl})x^l+\frac{ic^2}{2}R_{0l0m}x^lx^mD_j\\
&+\frac{ic^2}{6}R^i{}_{ljm}x^lx^mD_i+\frac{ic^2}{12}(R_{0j0l;m}-2R_{jl;m})x^lx^m+\frac{ic^2}{6}R_{0l0m;n}x^lx^mx^nD_j\\
&+\frac{ic^2}{12}R^i{}_{ljm;n}x^lx^mx^nD_i+\frac{ic^2}{48}(R_{0j0l;mn}-3R_{jl;mn})x^lx^mx^n\\
&+\frac{ic^2}{24}R_{0k0l;mn}x^kx^lx^mx^nD_j+\frac{ic^2}{40}R^i{}_{kjl;mn}x^kx^lx^mx^nD_i\Big)\\
&+c^{-3}\Big(-\frac{ic^2}{4}(\boldsymbol{a}\cdot\boldsymbol{x})R_{jl}x^l+\frac{ic^2}{6}(\boldsymbol{a}\cdot\boldsymbol{x})R_{0l0m}x^lx^mD_j+\frac{ic^2}{6}(\boldsymbol{a}\cdot\boldsymbol{x})R^i{}_{ljm}x^lx^mD_i\Big)\Big\}\\
&+\gamma^i\gamma^j\Big\{-\frac{i}{4}\varepsilon_{ijk}\omega^k+c^{-1}\Big(\frac{ic^2}{4}R_{0ijl}x^l+\frac{ic^2}{6}R_{0lim}x^lx^mD_j+\frac{ic^2}{12}R_{0ijl;m}x^lx^m\\
&+\frac{ic^2}{12}R_{0lim;n}x^lx^mx^nD_j+\frac{ic^2}{48}R_{0ijl;mn}x^lx^mx^n+\frac{ic^2}{40}R_{0kil;mn}x^kx^lx^mx^nD_j\Big)\\
&+c^{-3}\frac{ic^2}{6}(\boldsymbol{a}\cdot\boldsymbol{x})R_{0lim}x^lx^mD_j\Big\}+\mathrm{O}(c^{-1}x^4)+\mathrm{O}(c^{-2}x^3).
\end{aligned}
\tag{C.6}
$$

Appendix D. Details of the post-Newtonian expansion

The equations that arise from the Dirac equation when inserting the post-Newtonian ansatz (4.7) are

$$
\begin{aligned}
&\Big\{iD_\tau-m\boldsymbol{a}\cdot\boldsymbol{x}-\frac{mc^2}{2}R_{0l0m}x^lx^m-\frac{mc^2}{6}R_{0l0m;n}x^lx^mx^n-i(\boldsymbol{\omega}\times\boldsymbol{x})^iD_i+\frac{1}{2}\boldsymbol{\sigma}\cdot\boldsymbol{\omega}\\
&+c^{-1}\Big(\frac{ic^2}{2}R_{0l}{}^i{}_mx^lx^mD_i-\frac{ic^2}{6}R_{0l;m}x^lx^m-\frac{c^2}{12}\varepsilon^{ij}{}_k\sigma^kR_{0ijl;m}x^lx^m+\frac{ic^2}{6}R_{0l}{}^i{}_{m;n}x^lx^mx^nD_i\\
&-\frac{ic^2}{16}R_{0l;mn}x^lx^mx^n-\frac{c^2}{48}\varepsilon^{ij}{}_k\sigma^kR_{0ijl;mn}x^lx^mx^n+\frac{ic^2}{24}R_{0k}{}^i{}_{l;mn}x^kx^lx^mx^nD_i\\
&+\sigma^i\sigma^j\Big[\frac{ic^2}{4}R_{0ijl}x^l+\frac{ic^2}{6}R_{0lim}x^lx^mD_j+\frac{ic^2}{12}R_{0lim;n}x^lx^mx^nD_j+\frac{ic^2}{40}R_{0kil;mn}x^kx^lx^mx^nD_j\Big]\Big)\\
&+c^{-3}\Big(-\frac{ic^2}{4}(\boldsymbol{a}\cdot\boldsymbol{x})R_{0l}x^l+\frac{ic^2}{6}(\boldsymbol{a}\cdot\boldsymbol{x})R_{0l}{}^i{}_mx^lx^mD_i+\frac{ic^2}{6}(\boldsymbol{a}\cdot\boldsymbol{x})\sigma^i\sigma^jR_{0lim}x^lx^mD_j\Big)\\
&+\mathrm{O}(c^0x^4)+\mathrm{O}(c^{-2}x^3)\Big\}\tilde\psi_A\\
&=-\sigma^j\Big\{icD_j+\frac{mc^2}{6}R_{0ljm}x^lx^m+\frac{mc^2}{12}R_{0ljm;n}x^lx^mx^n+c^{-1}\Big(\frac{i}{2}a_j+i(\boldsymbol{a}\cdot\boldsymbol{x})D_j+\frac{ic^2}{4}(R_{0j0l}-R_{jl})x^l\\
&+\frac{ic^2}{2}R_{0l0m}x^lx^mD_j+\frac{ic^2}{6}R^i{}_{ljm}x^lx^mD_i+\frac{ic^2}{12}(R_{0j0l;m}-2R_{jl;m})x^lx^m+\frac{ic^2}{6}R_{0l0m;n}x^lx^mx^nD_j\\
&+\frac{ic^2}{12}R^i{}_{ljm;n}x^lx^mx^nD_i+\frac{ic^2}{48}(R_{0j0l;mn}-3R_{jl;mn})x^lx^mx^n+\frac{ic^2}{24}R_{0k0l;mn}x^kx^lx^mx^nD_j\\
&+\frac{ic^2}{40}R^i{}_{kjl;mn}x^kx^lx^mx^nD_i\Big)+c^{-3}\Big(-\frac{ic^2}{4}(\boldsymbol{a}\cdot\boldsymbol{x})R_{jl}x^l+\frac{ic^2}{6}(\boldsymbol{a}\cdot\boldsymbol{x})R_{0l0m}x^lx^mD_j\\
&+\frac{ic^2}{6}(\boldsymbol{a}\cdot\boldsymbol{x})R^i{}_{ljm}x^lx^mD_i\Big)+\mathrm{O}(c^0x^4)+\mathrm{O}(c^{-2}x^3)\Big\}\tilde\psi_B,
\end{aligned}
\tag{D.1a}
$$

$$
\begin{aligned}
&\Big\{2mc^2 + \mathrm{i}D_\tau + m\boldsymbol{a}\cdot\boldsymbol{x} + \frac{mc^2}{2}R_{0l0m}x^lx^m + \frac{mc^2}{6}R_{0l0m;n}x^lx^mx^n - \mathrm{i}(\boldsymbol{\omega}\times\boldsymbol{x})^iD_i + \frac{1}{2}\boldsymbol{\sigma}\cdot\boldsymbol{\omega}\\
&\quad + c^{-1}\Big(\frac{\mathrm{i}c}{2}R_{0l}{}^i{}_mx^lx^mD_i - \frac{\mathrm{i}c^2}{6}R_{0l;m}x^lx^m - \frac{c^2}{12}\varepsilon^{ij}{}_k\sigma^kR_{0ijl;m}x^lx^m + \frac{\mathrm{i}c^2}{6}R_{0l}{}^i{}_{m;n}x^lx^mx^nD_i\\
&\quad - \frac{\mathrm{i}c^2}{16}R_{0l;mn}x^lx^mx^n - \frac{c^2}{48}\varepsilon^{ij}{}_k\sigma^kR_{0ijl;mn}x^lx^mx^n + \frac{\mathrm{i}c^2}{24}R_{0k}{}^i{}_{l;mn}x^kx^lx^mx^nD_i\\
&\quad + \sigma^i\sigma^j\Big[\frac{\mathrm{i}c^2}{4}R_{0ijl}x^l + \frac{\mathrm{i}c^2}{6}R_{0lim}x^lx^mD_j + \frac{\mathrm{i}c^2}{12}R_{0lim;n}x^lx^mx^nD_j + \frac{\mathrm{i}c^2}{40}R_{0kil;mn}x^kx^lx^mx^nD_j\Big]\Big)\\
&\quad + c^{-3}\Big(-\frac{\mathrm{i}c^2}{4}(\boldsymbol{a}\cdot\boldsymbol{x})R_{0l}x^l + \frac{\mathrm{i}c^2}{6}(\boldsymbol{a}\cdot\boldsymbol{x})R_{0l}{}^i{}_mx^lx^mD_i + \frac{\mathrm{i}c^2}{6}(\boldsymbol{a}\cdot\boldsymbol{x})\sigma^i\sigma^jR_{0lim}x^lx^mD_j\Big)\\
&\quad + \mathrm{O}(c^0x^4) + \mathrm{O}(c^{-2}x^3)\Big\}\tilde\psi_B\\
&= -\sigma^j\Big\{\mathrm{i}cD_j - \frac{mc^2}{6}R_{0ljm}x^lx^m - \frac{mc^2}{12}R_{0ljm;n}x^lx^mx^n + c^{-1}\Big(\frac{\mathrm{i}}{2}a_j + \mathrm{i}(\boldsymbol{a}\cdot\boldsymbol{x})D_j + \frac{\mathrm{i}c^2}{4}(R_{0j0l} - R_{jl})x^l\\
&\quad + \frac{\mathrm{i}c^2}{2}R_{0l0m}x^lx^mD_j + \frac{\mathrm{i}c^2}{6}R^i{}_{ljm}x^lx^mD_i + \frac{\mathrm{i}c^2}{12}(R_{0j0l;m} - 2R_{jl;m})x^lx^m + \frac{\mathrm{i}c^2}{6}R_{0l0m;n}x^lx^mx^nD_j\\
&\quad + \frac{\mathrm{i}c^2}{12}R^i{}_{ljm;n}x^lx^mx^nD_i + \frac{\mathrm{i}c^2}{48}(R_{0j0l;mn} - 3R_{jl;mn})x^lx^mx^n + \frac{\mathrm{i}c^2}{24}R_{0k0l;mn}x^kx^lx^mx^nD_j\\
&\quad + \frac{\mathrm{i}c^2}{40}R^i{}_{kjl;mn}x^kx^lx^mx^nD_i\Big) + c^{-3}\Big(-\frac{\mathrm{i}c^2}{4}(\boldsymbol{a}\cdot\boldsymbol{x})R_{jl}x^l + \frac{\mathrm{i}c^2}{6}(\boldsymbol{a}\cdot\boldsymbol{x})R_{0l0m}x^lx^mD_j\\
&\quad + \frac{\mathrm{i}c^2}{6}(\boldsymbol{a}\cdot\boldsymbol{x})R^i{}_{ljm}x^lx^mD_i\Big) + \mathrm{O}(c^0x^4) + \mathrm{O}(c^{-2}x^3)\Big\}\tilde\psi_A\,, \qquad \text{(D.1}b\text{)}
\end{aligned}
$$

where $D_\tau = \partial_\tau - \mathrm{i}qA_\tau, D_i = \partial_i - \mathrm{i}qA_i$ denotes the electromagnetic covariant derivative. Note that we used the Pauli matrix identity $\sigma^i\sigma^j = \delta^{ij}\mathbb{1} + \mathrm{i}\varepsilon^{ij}{}_k\sigma^k$ for the simplifications $\sigma^i\sigma^j\varepsilon_{ijk}\omega^k = 2\mathrm{i}\boldsymbol{\sigma}\cdot\boldsymbol{\omega}$ and $\sigma^i\sigma^jR_{0ijl;m} = -R_{0l;m} + \mathrm{i}\varepsilon^{ij}{}_k\sigma^kR_{0ijl;m}$.

At order c^{-1}, (D.1a) yields

$$
\begin{aligned}
&\Big\{\mathrm{i}D_\tau - m\boldsymbol{a}\cdot\boldsymbol{x} - \frac{mc^2}{2}R_{0l0m}x^lx^m - \mathrm{i}\left(\boldsymbol{\omega}\times\boldsymbol{x}\right)^iD_i + \frac{1}{2}\boldsymbol{\sigma}\cdot\boldsymbol{\omega} + \mathrm{O}\left(x^3\right)\Big\}\tilde\psi_A^{(1)}\\
&+ \Big\{\frac{\mathrm{i}c^2}{2}R_{0l}{}^i{}_mx^lx^mD_i - \frac{\mathrm{i}c^2}{6}R_{0l;m}x^lx^m - \frac{c^2}{12}\varepsilon^{ij}{}_k\sigma^kR_{0ijl;m}x^lx^m + \frac{\mathrm{i}c^2}{6}R_{0l}{}^i{}_{m;n}x^lx^mx^nD_i\\
&+ \frac{\mathrm{i}c^2}{4}\sigma^i\sigma^jR_{0ijl}x^l + \frac{\mathrm{i}c^2}{6}\sigma^i\sigma^jR_{0lim}x^lx^mD_j + \frac{\mathrm{i}c^2}{12}\sigma^i\sigma^jR_{0lim;n}x^lx^mx^nD_j + \mathrm{O}\left(x^3\right)\Big\}\tilde\psi_A^{(0)}\\
&\quad = -\mathrm{i}\sigma^jD_j\tilde\psi_B^{(2)} + \Big\{-\frac{mc^2}{6}\sigma^iR_{0lim}x^lx^m - \frac{mc^2}{12}\sigma^iR_{0lim;n}x^lx^mx^n + \mathrm{O}\left(x^4\right)\Big\}\tilde\psi_B^{(1)}. \qquad \text{(D.2)}
\end{aligned}
$$

Using (4.9) and (4.13) to eliminate the $\tilde{\psi}_B$, this may be rewritten as

$$
\begin{aligned}
&\left\{\mathrm{i}D_\tau - m\boldsymbol{a}\cdot\boldsymbol{x} - \frac{mc^2}{2}R_{0l0m}x^lx^m - \mathrm{i}\left(\boldsymbol{\omega}\times\boldsymbol{x}\right)^i D_i + \frac{1}{2}\boldsymbol{\sigma}\cdot\boldsymbol{\omega} + \mathrm{O}\left(x^3\right)\right\}\tilde{\psi}_A^{(1)} \\
&\quad + \left\{\frac{\mathrm{i}c^2}{2}R_{0l}{}^i{}_m x^lx^mD_i - \frac{\mathrm{i}c^2}{6}R_{0l;m}x^lx^m - \frac{c^2}{12}\varepsilon^{ij}{}_k\sigma^kR_{0ijl;m}x^lx^m + \frac{\mathrm{i}c^2}{6}R_{0l}{}^i{}_{m;n}x^lx^mx^nD_i\right. \\
&\quad \left. + \frac{\mathrm{i}c^2}{4}\sigma^i\sigma^jR_{0ijl}x^l + \frac{\mathrm{i}c^2}{6}\sigma^i\sigma^jR_{0lim}x^lx^mD_j + \frac{\mathrm{i}c^2}{12}\sigma^i\sigma^jR_{0lim;n}x^lx^mx^nD_j + \mathrm{O}\left(x^3\right)\right\}\tilde{\psi}_A^{(0)} \\
&= -\frac{1}{2m}\left(\boldsymbol{\sigma}\cdot\boldsymbol{D}\right)^2\tilde{\psi}_A^{(1)} + \left\{-\frac{\mathrm{i}c^2}{12}\sigma^j\sigma^iR_{0lim}D_j\left(x^lx^m\cdot\right) - \frac{\mathrm{i}c^2}{24}\sigma^j\sigma^iR_{0lim;n}D_j\left(x^lx^mx^n\cdot\right)\right. \\
&\quad \left. + \frac{\mathrm{i}c^2}{12}\sigma^i\sigma^jR_{0lim}x^lx^mD_j + \frac{\mathrm{i}c^2}{24}\sigma^i\sigma^jR_{0lim;n}x^lx^mx^nD_j + \mathrm{O}\left(x^3\right)\right\}\tilde{\psi}_A^{(0)}.
\end{aligned} \tag{D.3}
$$

From this, we can read off the next-to-leading-order Hamiltonian $H^{(1)}$ according to (4.14), giving

$$
\begin{aligned}
H^{(1)} &= -\frac{\mathrm{i}c^2}{2}R_{0l}{}^i{}_mx^lx^mD_i + \frac{\mathrm{i}c^2}{6}R_{0l;m}x^lx^m + \frac{c^2}{12}\varepsilon^{ij}{}_k\sigma^kR_{0ijl;m}x^lx^m - \frac{\mathrm{i}c^2}{6}R_{0l}{}^i{}_{m;n}x^lx^mx^nD_i \\
&\quad - \frac{\mathrm{i}c^2}{4}\sigma^i\sigma^jR_{0ijl}x^l - \frac{\mathrm{i}c^2}{12}R_{0lim}\left(\sigma^i\sigma^jx^lx^mD_j + \sigma^j\sigma^iD_j\left(x^lx^m\cdot\right)\right) \\
&\quad - \frac{\mathrm{i}c^2}{24}R_{0lim;n}\left(\sigma^i\sigma^jx^lx^mx^nD_j + \sigma^j\sigma^iD_j\left(x^lx^mx^n\cdot\right)\right) + \mathrm{O}\left(x^3\right) \\
&= \frac{\mathrm{i}c^2}{3}R_{0l}x^l - \frac{c^2}{4}\varepsilon^{ij}{}_k\sigma^kR_{0lij}x^l - \frac{2\mathrm{i}c^2}{3}R^j_{0lm}x^lx^mD_j + \frac{\mathrm{i}c^2}{24}\left(5R_{0l;m} - R_{0l}{}^i{}_{m;i}\right)x^lx^m \\
&\quad - \frac{c^2}{8}\varepsilon^{ij}{}_k\sigma^kR_{0lij;m}x^lx^m - \frac{\mathrm{i}c^2}{4}R_{0l}{}^j{}_{m;n}x^lx^mx^nD_j + \mathrm{O}\left(x^3\right),
\end{aligned} \tag{D.4}
$$

where we again used $\sigma^i\sigma^j = \delta^{ij}\mathbb{1} + \mathrm{i}\varepsilon^{ij}{}_k\sigma^k$ for simplifications, as well as the Bianchi identities. The difference of this result to the corresponding one in the master's thesis [25] on which the present article is based, arising from oversights in [25] regarding the consistent calculation of the order x^2 terms, consists solely in the appearance of the terms containing covariant derivatives of the curvature tensor. Note that $H^{(0)}$ read off from (D.3) is the same as the one calculated above in (4.12).

(D.1*b*) at order c^{-1} gives the following:

$$
\begin{aligned}
&2m\tilde{\psi}_B^{(3)} + \left\{\mathrm{i}D_\tau + m\boldsymbol{a}\cdot\boldsymbol{x} + \frac{mc^2}{2}R_{0l0m}x^lx^m + \frac{mc^2}{6}R_{0l0m;n}x^lx^mx^n - \mathrm{i}\left(\boldsymbol{\omega}\times\boldsymbol{x}\right)^iD_i + \frac{1}{2}\boldsymbol{\sigma}\cdot\boldsymbol{\omega} + \mathrm{O}\left(x^4\right)\right\}\tilde{\psi}_B^{(1)} \\
&= -\mathrm{i}\sigma^jD_j\tilde{\psi}_A^{(2)} + \left\{\frac{mc^2}{6}\sigma^iR_{0lim}x^lx^m + \frac{mc^2}{12}\sigma^iR_{0lim;n}x^lx^mx^n + \mathrm{O}\left(x^4\right)\right\}\tilde{\psi}_A^{(1)} \\
&\quad - \sigma^j\left\{\frac{\mathrm{i}}{2}a_j + \mathrm{i}\left(\boldsymbol{a}\cdot\boldsymbol{x}\right)D_j + \frac{\mathrm{i}c^2}{4}\left(R_{0j0l} - R_{jl}\right)x^l + \frac{\mathrm{i}c^2}{2}R_{0l0m}x^lx^mD_j + \frac{\mathrm{i}c^2}{6}R^i{}_{ljm}x^lx^mD_i\right. \\
&\quad + \frac{\mathrm{i}c^2}{12}\left(R_{0j0l;m} - 2R_{jl;m}\right)x^lx^m + \frac{\mathrm{i}c^2}{6}R_{0l0m;n}x^lx^mx^nD_j + \frac{\mathrm{i}c^2}{12}R^i{}_{ljm;n}x^lx^mx^nD_i \\
&\quad \left. + \frac{\mathrm{i}c^2}{48}\left(R_{0j0l;mn} - 3R_{jl;mn}\right)x^lx^mx^n + \frac{\mathrm{i}c^2}{24}R_{0k0l;mn}x^kx^lx^mx^nD_j + \frac{\mathrm{i}c^2}{40}R^i{}_{kjl;mn}x^kx^lx^mx^nD_i + \mathrm{O}\left(x^4\right)\right\}\tilde{\psi}_A^{(0)}.
\end{aligned} \tag{D.5}
$$

With (4.9) to eliminate $\tilde{\psi}_B^{(1)}$, this can be used to express $\tilde{\psi}_B^{(3)}$ in terms of the $\tilde{\psi}_A$. Note however that this will involve the term $-\frac{\mathrm{i}D_\tau}{2m}\tilde{\psi}_B^{(1)} = -\frac{\mathrm{i}D_\tau}{4m^2}(-\mathrm{i}\boldsymbol{\sigma}\cdot\boldsymbol{D})\tilde{\psi}_A^{(0)}$, such that we need to re-use the Pauli equation (4.11) for $\tilde{\psi}_A^{(0)}$ to fully eliminate the time derivative in the resulting expression. Explicitly, the term in question evaluates to

$$\begin{aligned}\frac{-\mathrm{i}D_\tau}{4m^2}(-\mathrm{i}\boldsymbol{\sigma}\cdot\boldsymbol{D})\tilde{\psi}_A^{(0)} &= -\frac{1}{4m^2}\left\{[D_\tau,\boldsymbol{\sigma}\cdot\boldsymbol{D}] + (-\mathrm{i}\boldsymbol{\sigma}\cdot\boldsymbol{D})\,\mathrm{i}D_\tau\right\}\tilde{\psi}_A^{(0)} \\ &= -\frac{1}{4m^2}\left\{\mathrm{i}q\boldsymbol{\sigma}\cdot\boldsymbol{E} + (-\mathrm{i}\boldsymbol{\sigma}\cdot\boldsymbol{D})\left(H^{(0)} + qA_\tau\right)\right\}\tilde{\psi}_A^{(0)}\,, \end{aligned} \tag{D.6}$$

where $E_i = \partial_i A_\tau - \partial_\tau A_i$ is the electric field (note that up to higher-order corrections, these are indeed the electric field components in an orthonormal basis).

We finally need the next order of expansion in c^{-1} in order to compute the Hamiltonian at order c^{-2}. (D.1a) at order c^{-2} is

$$\begin{aligned}&\left\{\mathrm{i}D_\tau - m\boldsymbol{a}\cdot\boldsymbol{x} - \frac{mc^2}{2}R_{0l0m}x^l x^m - \mathrm{i}(\boldsymbol{\omega}\times\boldsymbol{x})^i D_i + \frac{1}{2}\boldsymbol{\sigma}\cdot\boldsymbol{\omega} + \mathrm{O}\left(x^3\right)\right\}\tilde{\psi}_A^{(2)} \\
&+\left\{\frac{\mathrm{i}c^2}{2}R^i_{0lm}x^l x^m D_i - \frac{\mathrm{i}c^2}{6}R_{0l;m}x^l x^m - \frac{c^2}{12}\varepsilon^{ij}{}_k\sigma^k R_{0ijl;m}x^l x^m + \frac{\mathrm{i}c^2}{6}R_{0l}{}^i{}_{m;n}x^l x^m x^n D_i\right. \\
&\left.+\frac{\mathrm{i}c^2}{4}\sigma^i\sigma^j R_{0ijl}x^l + \frac{\mathrm{i}c^2}{6}\sigma^i\sigma^j R_{0lim}x^l x^m D_j + \frac{\mathrm{i}c^2}{12}\sigma^i\sigma^j R_{0lim;n}x^l x^m x^n D_j + \mathrm{O}\left(x^3\right)\right\}\tilde{\psi}_A^{(1)} + \mathrm{O}\left(x^3\right)\tilde{\psi}_A^{(0)} \\
&= -\mathrm{i}\sigma^j D_j\tilde{\psi}_B^{(3)} + \left\{-\frac{mc^2}{6}\sigma^i R_{0lim}x^l x^m - \frac{mc^2}{12}\sigma^i R_{0lim;n}x^l x^m x^n + \mathrm{O}\left(x^4\right)\right\}\tilde{\psi}_B^{(2)} \\
&-\sigma^j\left\{\frac{\mathrm{i}}{2}a_j + \mathrm{i}(\boldsymbol{a}\cdot\boldsymbol{x})D_j + \frac{\mathrm{i}c^2}{4}(R_{0j0l} - R_{jl})x^l + \frac{\mathrm{i}c^2}{2}R_{0l0m}x^l x^m D_j + \frac{\mathrm{i}c^2}{6}R^i{}_{ljm}x^l x^m D_i\right. \\
&+\frac{\mathrm{i}c^2}{12}(R_{0j0l;m} - 2R_{jl;m})x^l x^m + \frac{\mathrm{i}c^2}{6}R_{0l0m;n}x^l x^m x^n D_j + \frac{\mathrm{i}c^2}{12}R^i{}_{ljm;n}x^l x^m x^n D_i \\
&+\frac{\mathrm{i}c^2}{48}(R_{0j0l;mn} - 3R_{jl;mn})x^l x^m x^n + \frac{\mathrm{i}c^2}{24}R_{0k0l;mn}x^k x^l x^m x^n D_j + \frac{\mathrm{i}c^2}{40}R^i{}_{kjl;mn}x^k x^l x^m x^n D_i \\
&\left.+\mathrm{O}\left(x^4\right)\right\}\tilde{\psi}_B^{(1)}.\end{aligned} \tag{D.7}$$

Now we use (4.9), (4.13), (D.5) and (D.6) to rewrite (D.7) just in terms of $\tilde{\psi}_A$ and read off the next-order Hamiltonian $H^{(2)}$ according to (4.15):

$$\begin{aligned}H^{(2)} = &-\frac{(-\mathrm{i}\boldsymbol{\sigma}\cdot\boldsymbol{D})}{4m^2}\left\{\mathrm{i}q\boldsymbol{\sigma}\cdot\boldsymbol{E} + (-\mathrm{i}\boldsymbol{\sigma}\cdot\boldsymbol{D})\left(H^{(0)} + qA_\tau\right)\right\} \\
&-\frac{(-\mathrm{i}\boldsymbol{\sigma}\cdot\boldsymbol{D})}{2m}\left\{m\boldsymbol{a}\cdot\boldsymbol{x} + \frac{mc^2}{2}R_{0l0m}x^l x^m + \frac{mc^2}{6}R_{0l0m;n}x^l x^m x^n - \mathrm{i}(\boldsymbol{\omega}\times\boldsymbol{x})^i D_i + \frac{1}{2}\boldsymbol{\sigma}\cdot\boldsymbol{\omega}\right. \\
&\left.+\mathrm{O}\left(x^4\right)\right\}\frac{(-\mathrm{i}\boldsymbol{\sigma}\cdot\boldsymbol{D})}{2m} - \frac{(-\mathrm{i}\boldsymbol{\sigma}\cdot\boldsymbol{D})}{2m}\sigma^j\left\{\frac{\mathrm{i}}{2}a_j + \mathrm{i}(\boldsymbol{a}\cdot\boldsymbol{x})D_j + \frac{\mathrm{i}c^2}{4}(R_{0j0l} - R_{jl})x^l\right. \\
&+\frac{\mathrm{i}c^2}{2}R_{0l0m}x^l x^m D_j + \frac{\mathrm{i}c^2}{6}R^i{}_{ljm}x^l x^m D_i + \frac{\mathrm{i}c^2}{12}(R_{0j0l;m} - 2R_{jl;m})x^l x^m + \frac{\mathrm{i}c^2}{6}R_{0l0m;n}x^l x^m x^n D_j \\
&+\frac{\mathrm{i}c^2}{12}R^i{}_{ljm;n}x^l x^m x^n D_i + \frac{\mathrm{i}c^2}{48}(R_{0j0l;mn} - 3R_{jl;mn})x^l x^m x^n + \frac{\mathrm{i}c^2}{24}R_{0k0l;mn}x^k x^l x^m x^n D_j \\
&\left.+\frac{\mathrm{i}c^2}{40}R^i{}_{kjl;mn}x^k x^l x^m x^n D_i + \mathrm{O}\left(x^4\right)\right\}\end{aligned}$$

$$
\begin{aligned}
&-\sigma^j\left\{\frac{\mathrm{i}}{2}a_j+\mathrm{i}\left(\boldsymbol{a}\cdot\boldsymbol{x}\right)D_j+\frac{\mathrm{i}c^2}{4}\left(R_{0j0l}-R_{jl}\right)x^l+\frac{\mathrm{i}c^2}{2}R_{0l0m}x^lx^mD_j+\frac{\mathrm{i}c^2}{6}R^i{}_{ljm}x^lx^mD_i\right.\\
&+\frac{\mathrm{i}c^2}{12}\left(R_{0j0l;m}-2R_{jl;m}\right)x^lx^m+\frac{\mathrm{i}c^2}{6}R_{0l0m;n}x^lx^mx^nD_j+\frac{\mathrm{i}c^2}{12}R^i{}_{ljm;n}x^lx^mx^nD_i\\
&+\frac{\mathrm{i}c^2}{48}\left(R_{0j0l;mn}-3R_{jl;mn}\right)x^lx^mx^n+\frac{\mathrm{i}c^2}{24}R_{0k0l;mn}x^kx^lx^mx^nD_j+\frac{\mathrm{i}c^2}{40}R^i{}_{kjl;mn}x^kx^lx^mx^nD_i\\
&\left.+\mathrm{O}\left(x^4\right)\right\}\frac{\left(-\mathrm{i}\boldsymbol{\sigma}\cdot\boldsymbol{D}\right)}{2m}. \qquad \text{(D.8)}
\end{aligned}
$$

Note that in the expression $-\frac{mc^2}{6}\sigma^i R_{0lim}x^lx^m\tilde{\psi}_B^{(2)}=-\frac{c^2}{12}\sigma^i R_{0lim}x^lx^m\{-\mathrm{i}\boldsymbol{\sigma}\cdot\boldsymbol{D}\tilde{\psi}_A^{(1)}+\mathrm{O}(x^2)\tilde{\psi}_A^{(0)}\}$ appearing in (D.7), the second term is off our order of approximation, so we neglected it when reading off $H^{(2)}$. Explicitly evaluating the above expression, we obtain the following order c^{-2} Hamiltonian:

$$
\begin{aligned}
H^{(2)}=&-\frac{1}{4m}\boldsymbol{a}\cdot\boldsymbol{D}-\frac{\mathrm{i}}{4m}\left(\boldsymbol{\sigma}\times\boldsymbol{a}\right)\cdot\boldsymbol{D}-\frac{1}{2m}\left(\boldsymbol{a}\cdot\boldsymbol{x}\right)\left(\boldsymbol{\sigma}\cdot\boldsymbol{D}\right)^2-\frac{c^2}{4m}R_{0l0m}x^lx^m\left(\boldsymbol{\sigma}\cdot\boldsymbol{D}\right)^2\\
&-\frac{c^2}{8m}R_{0l0m;n}x^lx^mx^n\left(\boldsymbol{\sigma}\cdot\boldsymbol{D}\right)^2-\frac{c^2}{24m}R_{0k0l;mn}x^kx^lx^mx^n\left(\boldsymbol{\sigma}\cdot\boldsymbol{D}\right)^2-\frac{1}{8m^3}\left(\boldsymbol{\sigma}\cdot\boldsymbol{D}\right)^4\\
&+\frac{c^2}{8m}R+\frac{c^2}{4m}R_{00}+\frac{c^2}{12m}\left(4R^j{}_l+R_0{}^j{}_{0l}\right)x^lD_j+\frac{\mathrm{i}c^2}{8m}\sigma^k\left(-2\varepsilon^{ij}{}_kR_{0l0i}+\varepsilon^{im}{}_kR^j{}_{lim}\right)x^lD_j\\
&-\frac{c^2}{6m}R^i{}_l{}^j{}_mx^lx^mD_iD_j+\frac{c^2}{16m}\left(R_{;l}+2R^i{}_{l;i}\right)x^l+\frac{\mathrm{i}c^2}{24m}\varepsilon^{ij}{}_k\sigma^k\left(R_{0i0l;j}-2R_{il;j}\right)x^l\\
&+\frac{c^2}{24m}\Big(5R^j{}_{l;m}-3R_0{}^j{}_{0l;m}-R_{0l0m}{}^{;j}-R^j{}_l{}^i{}_{m;i}-\mathrm{i}\varepsilon^{ij}{}_k\sigma^k\left(2R_{0i0l;m}+R_{0l0m;i}\right)\\
&+2\mathrm{i}\varepsilon^{in}{}_k\sigma^kR^j{}_{lin;m}\Big)x^lx^mD_j-\frac{c^2}{12m}R^i{}_l{}^j{}_{m;n}x^lx^mx^nD_iD_j\\
&+\frac{c^2}{48m}\left(R_{;lm}+4R^i{}_{l;im}+\mathrm{i}\varepsilon^{ij}{}_k\sigma^k\left(R_{0i0l;jm}-3R_{il;jm}\right)\right)x^lx^m\\
&+\frac{c^2}{120m}\left(9R^j{}_{l;mn}-6R_0{}^j{}_{0l;mn}-5R_{0l0m}{}^{;j}{}_n-3R^j{}_l{}^i{}_{m;in}\right)x^lx^mx^nD_j\\
&+\frac{\mathrm{i}c^2}{96m}\sigma^k\left(-4\varepsilon^{ij}{}_k\left(R_{0i0l;mn}+R_{0l0m;ni}\right)+3\varepsilon^{ir}{}_kR^j{}_{lir;mn}\right)x^lx^mx^nD_j\\
&-\frac{c^2}{40m}R^i{}_k{}^j{}_{l;mn}x^kx^lx^mx^nD_iD_j-\frac{q}{4m^2}\sigma^i\sigma^jD_iE_j-\frac{q}{12m}c^2\left(R_{lm}+R_{0l0m}\right)x^lx^m\boldsymbol{\sigma}\cdot\boldsymbol{B}\\
&+\frac{q}{12m}\sigma^jc^2R_{iljm}x^lx^mB^i+\frac{\mathrm{i}q}{4m^2}\boldsymbol{\sigma}\cdot\left(\boldsymbol{\omega}\times\boldsymbol{B}\right)+\frac{q}{2m^2}\boldsymbol{\omega}\cdot\boldsymbol{B}+\frac{q}{4m^2}\left(\omega_jx^i-\omega^ix_j\right)D_iB^j\\
&+\frac{\mathrm{i}q}{4m^2}\left(\boldsymbol{\sigma}\cdot\left(\boldsymbol{\omega}\times\boldsymbol{x}\right)\right)\boldsymbol{B}\cdot\boldsymbol{D}-\frac{\mathrm{i}q}{4m^2}\sigma^j(\boldsymbol{\omega}\times\boldsymbol{x})\cdot\boldsymbol{D}B_j+\mathrm{O}(x^3). \qquad \text{(D.9)}
\end{aligned}
$$

This is the final information that we need in order to calculate the Pauli Hamiltonian up to and including the order of c^{-2} (4.17). Note that we have used the identity $\sigma^i\sigma^j=\delta^{ij}\mathbb{1}+\mathrm{i}\varepsilon^{ij}{}_k\sigma^k$ multiple times for simplifications, as well as the Bianchi identities and $[D_i,D_j]=-\mathrm{i}q(\partial_iA_j-\partial_jA_i)=-\mathrm{i}q\varepsilon_{ijk}B^k$, where $B^i=\varepsilon^{ijk}\partial_jA_k$ is the magnetic field. We also used that covariant derivatives commute up to curvature terms, which are of higher order in c^{-1}.

Note that due to a calculational oversight, the terms explicitly containing the magnetic field were missing in the master's thesis [25] on which the present article is based. In [25], some oversights were also made regarding the consistency of the calculation of the terms of order x^2.

However, the only differences of (D.9) to the corresponding result in [25] that arise from these miscalculations of order x^2 terms are the appearance of all terms which contain covariant derivatives of the curvature tensor and the absence of the term $-\frac{\mathrm{i}c^2}{4}(\boldsymbol{\omega}\times\boldsymbol{x})^k(R_{kl}+R_{0k0l})x^l$ from [25].

ORCID iDs

Ashkan Alibabaei https://orcid.org/0000-0002-7437-1563
Philip K Schwartz https://orcid.org/0000-0001-7222-9099
Domenico Giulini https://orcid.org/0000-0003-3123-7257

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