Annali di Matematica Pura ed Applicata (1923 -)
https://doi.org/10.1007/s10231-024-01437-z

Descent of tautological sheaves from Hilbert schemes to Enriques manifolds

Fabian Reede[1]

Received: 21 December 2022 / Accepted: 16 February 2024

Abstract

Let X be a K3 surface which doubly covers an Enriques surface S. If $n \in \mathbb{N}$ is an odd number, then the Hilbert scheme of n-points $X^{[n]}$ admits a natural quotient $S_{[n]}$. This quotient is an Enriques manifold in the sense of Oguiso and Schröer. In this paper we construct slope stable sheaves on $S_{[n]}$ and study some of their properties.

Keywords Enriques manifolds · Stable sheaves · Moduli spaces

Mathematics Subject Classification Primary: 14F06 · Secondary: 14F08 · 14J28 · 14D20

In 1896 Federigo Enriques gave examples of smooth projective surfaces with irregularity $q = 0$ and geometric genus $p_g = 0$ which are not rational. Therefore these surfaces were counterexamples to a conjecture by Max Noether, which stated that surfaces with $q = p_g = 0$ are rational. Nowadays such a surface is called an Enriques surface.

The canonical bundle ω_S of an Enriques surface S has order two in the Picard group of S. The induced double cover turns out to be a K3 surface (a two dimensional hyperkähler manifold), hence it is the universal cover of S. On the other hand, every K3 surface X which admits a fixed point free involution doubly covers an Enriques surface S.

Mimicking this correspondence Oguiso and Schröer defined higher dimensional analogues of Enriques surfaces, the so called Enriques manifolds in [1]. To be precise a connected complex manifold that is not simply connected and whose universal cover is a hyperkäler manifold is called an Enriques manifold.

The following class of examples is of interest to us in this work: take an odd natural number $n \in \mathbb{N}$ and an Enriques surface S. We have the induced K3 surface X with a fixed point free involution ι such that $S = X/\iota$. Since n is odd we get an induced fixed point free involution $\iota^{[n]}$ on the Hilbert scheme of n-points $X^{[n]}$. The quotient of $X^{[n]}$ by the involution $\iota^{[n]}$ is an Enriques manifold $S_{[n]}$ of dimension $2n$. We have an étale Galois cover $\rho : X^{[n]} \to S_{[n]}$.

In this article we construct and study stable sheaves on Enriques manifolds of type $S_{[n]}$. The main idea is to start with slope stable sheaves on $X^{[n]}$ and check if they descend to

✉ Fabian Reede
reede@math.uni-hannover.de

1 Institut für Algebraische Geometrie, Leibniz Universität Hannover, Welfengarten 1, Hannover 30167, Deutschland

Published online: 15 March 2024

$S_{[n]}$. Known examples of stable sheaves on $X^{[n]}$ are given by the tautological bundles $E^{[n]}$ associated to slope stable locally free sheaves E on X.

For example, we prove that $E^{[n]}$ descends to $S_{[n]}$ if and only if E descends to S. If $E^{[n]}$ descends we have $E^{[n]} \cong \rho^* F_{[n]}$ for some locally free sheaf $F_{[n]}$ on $S_{[n]}$. We then show that it is possible to find an ample divisor $D \in \mathrm{Amp}(S_{[n]})$ such that $F_{[n]}$ is slope stable with respect to D. Finally using results from Kim [2] and Yoshioka [3], we are able to prove that, given certain conditions are satisfied, we have in fact a morphism

$$(-)_{[n]} : \mathrm{M}_{S,d}(v, L) \to \mathcal{M}_{S_{[n]},D}(v_{[n]}), \quad F \mapsto F_{[n]}$$

between a moduli spaces of stable sheaves on S and moduli space of stable sheaves on $S_{[n]}$. This morphism identifies the former moduli space as a smooth connected component in the latter.

This paper consists of four sections. In Sect. 1 we generalize some results concerning tautological bundles on Hilbert schemes of points. Section 2 contains results about the descent of tautological sheaves from $X^{[n]}$ to $S_{[n]}$. We compute certain Ext-spaces in Sect. 3. In the final Sect. 4 we study the stability of sheaves on Enriques manifolds of type $S_{[n]}$.

1 Stability of tautological sheaves on Hilbert schemes of points

Let X be a smooth projective surface. The Hilbert scheme $X^{[n]} := \mathrm{Hilb}^n(X)$ classifies length n subschemes in X, that is

$$X^{[n]} = \left\{[Z] \mid Z \subset X,\ \dim(Z) = 0 \text{ and } \dim(\mathrm{H}^0(Z, \mathcal{O}_Z)) = n\right\}.$$

In fact $X^{[n]}$ is smooth itself and has dimension $2n$, see [4, Theorem 2.4]. Moreover $X^{[n]}$ is a fine moduli space for the classification of length n subschemes and comes with the universal length n subscheme

$$\mathcal{Z} = \left\{(x, [Z]) \in X \times X^{[n]} \mid x \in \mathrm{supp}(Z)\right\} \subset X \times X^{[n]}.$$

The universal subscheme $\mathcal{Z}$ comes with two projections $p : \mathcal{Z} \to X^{[n]}$ and $q : \mathcal{Z} \to X$. Note that the morphism p is finite and flat of degree n.

To any locally free sheaf E of rank r on X one can associate the so called tautological vector bundle $E^{[n]}$ on $X^{[n]}$ via

$$E^{[n]} := p_* q^* E.$$

As p is finite and flat of degree n the sheaf $E^{[n]}$ is indeed locally free and has rank nr. The fiber at $[Z] \in X^{[n]}$ can be computed to be

$$E^{[n]} \otimes \mathcal{O}_{[Z]} \cong \mathrm{H}^0(Z, E_{|Z}).$$

Remark 1.1 Note that the definition of $E^{[n]}$ makes sense for E a coherent sheaf on X or even a complex $E \in \mathrm{D^b}(X)$ in the derived category of X, see [5, Definition 2.4].

In [6, Theorem 1.4, Theorem 4.9] Stapleton proves that if $h \in \mathrm{Amp}(X)$ is an ample divisor on X and $E \not\cong \mathcal{O}_X$ is a slope stable (with respect to h) locally free sheaf, then there is $H \in \mathrm{Amp}(X^{[n]})$ such that the associated tautological bundle $E^{[n]}$ is slope stable with respect to H on $X^{[n]}$.

In fact Stapleton's result remains true, if we drop the locally free condition and allow for torsion free sheaves, see for example [7, Proposition 2.4] for a first step toward the following observation:

Lemma 1.2 *Assume E is torsion free and slope stable with respect to $h \in \mathrm{Amp}(X)$ such that its double dual satisfies $E^{**} \neq \mathcal{O}_X$, then the associated tautological sheaf $E^{[n]}$ is slope stable with respect to some $H \in \mathrm{Amp}(X^{[n]})$.*

Proof Since X is a smooth projective surface and E is torsion free we can canonically embed E into its double dual. This gives an exact sequence

$$0 \longrightarrow E \longrightarrow E^{**} \longrightarrow Q \longrightarrow 0. \tag{1}$$

Here E^{**} is locally free and also slope stable with respect to h. Furthermore Q has support of codimension two.

By [8, Corollary 6] the functor $(-)^{[n]} : \mathrm{Coh}(X) \to \mathrm{Coh}(X^{[n]})$ is exact. So we get an exact sequence on $X^{[n]}$

$$0 \longrightarrow E^{[n]} \longrightarrow (E^{**})^{[n]} \longrightarrow Q^{[n]} \longrightarrow 0.$$

By our assumptions $(E^{**})^{[n]}$ is slope stable with respect to some $H \in \mathrm{Amp}(X^{[n]})$. But $Q^{[n]}$ has support of codimension two in $X^{[n]}$ so that $E^{[n]}$ is isomorphic to $(E^{**})^{[n]}$ in codimension one and thus must be also be slope stable with respect to H. $\square$

The previous lemma shows that for every slope stable E with $E^{**} \not\cong \mathcal{O}_X$ there is $H \in \mathrm{Amp}(X^{[n]})$ such that the tautological sheaf $E^{[n]}$ is slope stable with respect to H. Since E belongs to some moduli space $\mathrm{M}_{X,h}(r, \mathrm{c}_1, \mathrm{c}_2)$, one may ask how H varies if E varies in its moduli. We can answer this question in the case that all sheaves classified by $\mathrm{M}_{X,h}(r, \mathrm{c}_1, \mathrm{c}_2)$ are locally free.

Proposition 1.3 *If $(r, \mathrm{c}_1, \mathrm{c}_2) \neq (1, 0, 0)$ is chosen such that for every $[E] \in M_{X,h}(r, \mathrm{c}_1, \mathrm{c}_2)$ the sheaf E is slope stable and locally free, then there is $H \in \mathrm{Amp}(X^{[n]})$ such that $E^{[n]}$ is slope stable with respect to H for all $[E] \in \mathrm{M}_{X,h}(r, \mathrm{c}_1, \mathrm{c}_2)$.*

Proof By a result of Stapleton, see [6, Theorem 1.4], we know that for $[E] \in \mathrm{M}_{X,h}(r, \mathrm{c}_1, \mathrm{c}_2)$ the locally free sheaf $E^{[n]}$ is slope stable with respect to the induced nef divisor $h_n \in \mathrm{NS}(X^{[n]})$. It is also well known that the Hilbert - Chow morphism $\mathsf{HC} : X^{[n]} \to X^{(n)}$ is semi-small and that $q : \mathcal{Z} \to X$ is flat, see [9, Theorem 2.1].

The proof is now exactly the same as for tautological bundles on the generalized Kummer variety $\mathsf{Kum}_n(A)$ associated to an abelian surface A, see [10, Proposition 2.9]. $\square$

Remark 1.4 The condition that all sheaves in $\mathrm{M}_{X,h}(r, \mathrm{c}_1, \mathrm{c}_2)$ are slope stable can be achieved (for example) in the following two different ways: the first is by a special choice of the numerical invariants, see [11, Lemma 1.2.14]. The second way is by choosing a special ample class h, see [11, Theorem 4.C.3].

To find a moduli space such that all sheaves are locally free, one can do the following: if the tuple (r, c_1) is fixed, then by Bogomolov's inequality the second Chern class is bounded from below, see [11, Theorem 3.4.1]. Choose the minimal c_2, then every sheaf in $\mathrm{M}_{X,h}(r, \mathrm{c}_1, \mathrm{c}_2)$ is locally free. Indeed, if such an E is not locally free, then E^{**} is locally free, stable with respect to h and has the same tuple (r, c_1), but it has smaller c_2 by exact sequence (1) as $\mathrm{c}_2(Q) < 0$, contradicting minimality. See also [11, Remark 6.1.9] for a similar argument.

Now let X be a K3 surface. Denote the Mukai vectors of E and $E^{[n]}$ by v respectively $v^{[n]} \in \mathrm{H}^*(X^{[n]}, \mathbb{Q})$. If $E^{[n]}$ is slope stable, then it belongs to the moduli space $\mathcal{M}_{X^{[n]},H}(v^{[n]})$ of semistable sheaves on $X^{[n]}$ with Mukai vector $v^{[n]}$. In fact we can generalize [7, Corollary 4.6] to get the following

Theorem 1.5 *If $v \neq v(\mathcal{O}_X)$ is a Mukai vector such that for every $[E] \in \mathrm{M}_{X,h}(v)$ the sheaf E is slope stable, locally free and $h^i(X, E) = 0$ for $i = 1, 2$, then the functor $(-)^{[n]}$ induces a morphism*

$$(-)^{[n]} : \mathrm{M}_{X,h}(v) \to \mathcal{M}_{X^{[n]},H}(v^{[n]}), \ \ [E] \mapsto [E^{[n]}]$$

which identifies $\mathrm{M}_{X,h}(v)$ with a smooth connected component of $\mathcal{M}_{X^{[n]},H}(v^{[n]})$.

Proof First note that the map $[E] \mapsto [E^{[n]}]$ is indeed a regular morphism, see for example [12, Proposition 2.1]. Furthermore this morphism is injective on closed points, which follows immediately from [13, Theorem 1.1] (see also [9, Theorem 1.2] for a generalization).

By [5, Corollary 4.2 (11)] we find

$$\mathrm{Ext}^1_{X^{[n]}}(E^{[n]}, F^{[n]}) \cong \mathrm{Ext}^1_X(E, F)$$

since $h^0(X, E^\vee) = h^2(X, E) = 0$ as well as $h^1(X, E^\vee) = h^1(X, E) = 0$. For $E = F$ this isomorphisms translates to

$$\dim(T_{[E^{[n]}]}\mathcal{M}_{X^{[n]},H}(v^{[n]})) = \dim(T_{[E]}\mathrm{M}_{X,h}(v)).$$

These two facts imply that we can identify $\mathrm{M}_{X,h}(v)$ with a smooth connected component in $\mathcal{M}_{X^{[n]},H}(v^{[n]})$. $\square$

2 Descent of tautological sheaves to Enriques manifolds

Let G be a finite group. Consider an étale Galois cover $\varphi : Y \to Z$ with Galois group G, that is there is a free G-action on Y such that $Z = Y/G$ and φ is the quotient map. In this situation there is an equivalence between the categories $\mathrm{Coh}(Z)$ of coherent sheaves on Z and $\mathrm{Coh}^G(Y)$ of G-equivariant coherent sheaves on Y given by the functors

$$\varphi^* : \mathrm{Coh}(Z) \to \mathrm{Coh}^G(Y), \ \ E \mapsto \varphi^* E \text{ and}$$
$$\varphi^G_* : \mathrm{Coh}^G(Y) \to \mathrm{Coh}(Z), \ \ F \mapsto (\varphi_*(F))^G$$

We say that a coherent sheaf E on Y descends to Z if E is in the image of φ^*, that is there is a coherent sheaf F on Z together with an isomorphism $E \cong \varphi^*(F)$.

A coherent sheaf E on X is said to be G-invariant, if there are isomorphisms $E \cong g^*E$ for every $g \in G$. A G-equivariant coherent sheaf is G-invariant, but the converse is not true. For our purposes the following will suffice, see [14, Lemma 1]:

Proposition 2.1 *Assume that G is a cyclic group and E is a simple G-invariant coherent sheaf on Y, then E descends to Z.*

Remark 2.2 Recall that if (X, ι) is a pair consisting of a K3 surface and a fixed point free involution, then $G = \langle \iota \rangle \cong \mathbb{Z}/2\mathbb{Z}$ acts freely on X and the quotient S is an Enriques surface. The morphism $\pi : X \to S$ is an étale $\mathbb{Z}/2\mathbb{Z}$-Galois cover.

On the other hand if S is an Enriques surface, then its canonical bundle ω_S is 2-torsion. One can consider the induced canonical cover $\phi : \tilde{S} := \mathrm{Spec}(\mathcal{O}_S \oplus \omega_S) \to S$. The morphism ϕ is an étale $\mathbb{Z}/2\mathbb{Z}$-Galois cover and $\tilde{S}$ is a K3 surface with fixed point free involution, the covering involution of ϕ. Furthermore $\phi_* \mathcal{O}_{\tilde{S}} \cong \mathcal{O}_S \oplus \omega_S$.

In [1] Oguiso and Schröer generalized the notion of an Enriques surface to that of an Enriques manifold by mimicking the above constructions:

Definition 2.3 A manifold Y is called an Enriques manifold if it is a connected complex manifold that is not simply connected and whose universal cover is a hyperkähler manifold.

Remark 2.4 In [15] the authors also gave a definition of higher dimensional Enriques varieties, which slightly differs from the one of Enriques manifolds in [1].

Remark 2.5 An Enriques manifold is of even dimension, say $\dim(Y) = 2n$. The fundamental group $\pi_1(Y)$ is finite of order d with $d \mid n+1$. This number d is called the index of Y. In addition Y is projective and the canonical bundle ω_Y has finite order d and generates the torsion group of $\mathrm{Pic}(Y)$, see [1, Sect. 2].

We will work with the following class of Enriques manifolds, see [1, Proposition 4.1]:

Example 2.6 Let (X, ι) be a pair consisting of a K3 surface together with a fixed point free involution ι on X. Then X covers the Enriques surface $S = X/\iota$. If $n \in \mathbb{N}$ is odd, then (X, ι) induces the pair $(X^{[n]}, \iota^{[n]})$ of the Hilbert scheme of n-points on X and the induced fixed point free involution $\iota^{[n]}$ on $X^{[n]}$. Thus $G = \left\langle \iota^{[n]} \right\rangle \cong \mathbb{Z}/2\mathbb{Z}$ acts freely on $X^{[n]}$ and the quotient $S_{[n]}$ is an Enriques manifold with index $d = 2$ coming with an étale $\mathbb{Z}/2\mathbb{Z}$-cover $\rho : X^{[n]} \to S_{[n]}$.

We want to study the descent of sheaves from X to S respectively from $X^{[n]}$ to $S_{[n]}$. To do this we need the following lemma:

Lemma 2.7 *There is an isomorphism of functors from* $\mathrm{Coh}(X)$ *to* $\mathrm{Coh}(X^{[n]})$*:*

$$(\iota^{[n]})^* \left((-)^{[n]} \right) \cong (\iota^*(-))^{[n]}.$$

Proof Recall that $(-)^{[n]} = \mathrm{FM}_{\mathcal{O}_{\mathcal{Z}}}(-)$ can be written as the Fourier—Mukai transform with kernel the structure sheaf of universal family $\mathcal{Z}$ in $X \times X^{[n]}$, see for example [9, Sect. 2.3]. Define a group isomorphism

$$\mu : \langle \iota \rangle \to \left\langle \iota^{[n]} \right\rangle, \quad \iota \mapsto \iota^{[n]}$$

and note that this is a so-called c-isomorphism, see [16, Definitions 3.1 and 3.3]. By the definition of the universal family we see that there is an isomorphism

$$(\iota \times \mu(\iota))^* \mathcal{O}_{\mathcal{Z}} = (\iota \times \iota^{[n]})^* \mathcal{O}_{\mathcal{Z}} \cong \mathcal{O}_{\mathcal{Z}}.$$

Thus $\mathcal{O}_{\mathcal{Z}}$ is μ-invariant, see [16, Definition 3.4], which implies

$$(\iota^{[n]})^* \left(\mathrm{FM}_{\mathcal{O}_{\mathcal{Z}}}(-) \right) \cong \mathrm{FM}_{\mathcal{O}_{\mathcal{Z}}}(\iota^*(-))$$

by [16, Lemma 3.6 (iii)]. □

We can now prove the main result of this section:

Theorem 2.8 *Assume (X, ι) is a K3 surface together with a fixed point free involution and let $n \in \mathbb{N}$ be an odd number. If a torsion free sheaf E on X is simple, then the associated tautological sheaf $E^{[n]}$ on $X^{[n]}$ descends to $S_{[n]}$ if and only if E descends to S.*

Proof First we note that if E is simple then $E^{[n]}$ is also simple. Indeed by [5, Corollary 4.2 (11)] there is an isomorphism

$$\mathrm{End}_{X^{[n]}}(E^{[n]}) \cong \mathrm{End}_X(E) \oplus \mathrm{H}^0(X, E^*) \otimes \mathrm{H}^0(X, E).$$

Since E is simple the second summand must vanish, since otherwise E would have an endomorphism, which has image of rank one and thus is no homothety.

Proposition 2.1 shows

$$E^{[n]} \text{ decends to } S_{[n]} \Leftrightarrow (\iota^{[n]})^* E^{[n]} \cong E^{[n]}.$$

By Lemma 2.7 we get

$$(\iota^{[n]})^* E^{[n]} \cong E^{[n]} \Leftrightarrow (\iota^* E)^{[n]} \cong E^{[n]}.$$

But [9, Theorem 1.2] shows

$$(\iota^* E)^{[n]} \cong E^{[n]} \Leftrightarrow \iota^* E \cong E.$$

Thus $E^{[n]}$ descends to $S_{[n]}$ if and only if E descends to S. $\square$

The theorem shows that given a simple ι-invariant torsion free sheaf E on X then there is $F \in \mathrm{Coh}(S)$ and $G \in \mathrm{Coh}(S_{[n]})$ such that

$$E \cong \pi^* F \ \text{ as well as } \ E^{[n]} \cong \rho^* G.$$

In fact, there is a close relationship between the sheaves F and G: as $\mathcal{O}_{\mathcal{Z}}$ is μ-invariant on $X \times X^{[n]}$, the structure sheaf $\mathcal{O}_{\mathcal{Z}}$ is naturally μ-linearizable on $\mathcal{Z}$, hence so is $\mathcal{O}_{\mathcal{Z}}$ as a sheaf on $X \times X^{[n]}$.

Therefore by [16, Proposition 4.2] the functor $(-)^{[n]}$ descends to a functor

$$(-)_{[n]} : \mathrm{D^b}(S) \to \mathrm{D^b}(S_{[n]})$$

together with a commutative diagram

$$\begin{tikzcd}
\mathrm{D^b}(S) \arrow[r, "\pi^*", "\cong"'] \arrow[d, "(-)_{[n]}"] \arrow[rr, bend left, "\pi^*"] & \mathrm{D}^b_{\iota}(X) \arrow[r, "\mathrm{For}"] \arrow[d, "(-)^{[n]}_{\mathbb{Z}/2\mathbb{Z}}"] & \mathrm{D^b}(X) \arrow[d, "(-)^{[n]}"] \\
\mathrm{D^b}(S_{[n]}) \arrow[r, "\rho^*", "\cong"'] \arrow[rr, bend right, "\rho^*"'] & \mathrm{D}^b_{\iota^{[n]}}(X^{[n]}) \arrow[r, "\mathrm{For}"] & \mathrm{D^b}(X^{[n]})
\end{tikzcd} \tag{2}$$

Here For is the functor forgetting the linearizations.

That is if we start with a simple sheaf E on X, which descends to S i.e. $E \cong \pi^* F$, then $E^{[n]}$ descends to $S_{[n]}$ with $E^{[n]} \cong \rho^* F_{[n]}$.

Remark 2.9 As $\mathcal{O}_{\mathcal{Z}}$ has two choices of a μ-linearization (differing by the non-trivial character), there are actually two choices of the descent $(-)_{[n]} : \mathrm{D^b}(S) \to \mathrm{D^b}(S_{[n]})$ (differing by tensor product by $\omega_{S_{[n]}}$).

We end this section by giving a more explicit description of $(-)_{[n]}$ similar to $(-)^{[n]}$. For this recall that by [12, 2.4] we have

$$(-)^{[n]} = \mathrm{FM}_{\mathcal{O}_{\mathcal{Z}}}(-) = p_{X^{[n]}*}(p_X^*(-)),$$

where $p_X : \mathcal{Z} \to X$ and $p_{X^{[n]}} : \mathcal{Z} \to X^{[n]}$ are the projections.

The group $G = \mathbb{Z}/2\mathbb{Z}$ acts freely on X via ι with quotient S, freely on $X^{[n]}$ via $\iota^{[n]}$ with quotient $S_{[n]}$ and thus also freely on $X \times X^{[n]}$ via $\iota \times \iota^{[n]}$. As the universal family $\mathcal{Z} \hookrightarrow$

$X \times X^{[n]}$ is G-invariant, we get a closed subvariety $\mathcal{Z}/G \hookrightarrow (X \times X^{[n]})/G$. Furthermore the projections p_X and $p_{X^{[n]}}$ are G-equivariant. By [17, Lemma 2.3.3] we get cartesian squares

$$\begin{tikzcd} X \arrow[d, "\pi"'] & \mathcal{Z} \arrow[l, "p_X"'] \arrow[r, "p_{X^{[n]}}"] \arrow[d, "\alpha"] & X^{[n]} \arrow[d, "\rho"] \\ S & \mathcal{Z}/G \arrow[l, "p_S"'] \arrow[r, "p_{S_{[n]}}"] & S_{[n]} \end{tikzcd} \tag{3}$$

Theorem 2.10 *The functor* $(-)_{[n]} : \mathrm{D^b}(S) \to \mathrm{D^b}(S_{[n]})$ *has the following description:*

$$(-)_{[n]} = p_{S_{[n]}*}(p_S^*(-)).$$

Proof From diagram (2) we see that $\rho^*((-)_{[n]}) = (\pi^*(-))^{[n]}$. Since $\rho_*(\rho^*(-))^G = \mathrm{id}$ we find

$$\begin{aligned}(-)_{[n]} &= \rho_*((\pi^*(-))^{[n]})^G = \rho_*(p_{X^{[n]}*}(p_X^*(\pi^*(-))))^G \\ &= p_{S_{[n]}*}(\alpha_*(\alpha^*(p_S^*(-))))^G = p_{S_{[n]}*}(\alpha_*(\alpha^*(p_S^*(-)))^G) \\ &= p_{S_{[n]}*}(p_S^*(-)).\end{aligned}$$

Here we used the commutativity of diagram (3), the fact that G acts trivially on $\mathcal{Z}/G$ and $S_{[n]}$ hence by [5, Equation (5)] we have $(-)^G p_{S_{[n]}*} = p_{S_{[n]}*}(-)^G$ and the G-equivariant projection formula. □

3 Computation of certain extension spaces

In [18, Theorem 3.17] Krug gave explicit formulas for homological invariants of tautological objects in $\mathrm{D^b}(X^{[n]})$ in terms of those in $\mathrm{D^b}(X)$, for example for $E, F \in \mathrm{D^b}(X)$ there is an isomorphism of graded vector spaces:

$$\begin{aligned}\mathrm{Ext}^*_{X^{[n]}}(E^{[n]}, F^{[n]}) \cong &\mathrm{Ext}^*_X(E, F) \otimes S^{n-1}\mathrm{H}^*(X, \mathcal{O}_X) \\ &\oplus \mathrm{Ext}^*_X(E, \mathcal{O}_X) \otimes \mathrm{Ext}^*_X(\mathcal{O}_X, F) \otimes S^{n-2}\mathrm{H}^*(X, \mathcal{O}_X).\end{aligned}$$

See also [5, Sect. 4] for a considerably simplified proof of this formula.

In this section we want to find homological invariants of sheaves of the form $G_{[n]}$ on $S_{[n]}$ in terms of the sheaf G on S. It is certainly possible to find a general formula similar to Krug's result, but to keep formulas and proofs short and readable and since it is enough for our purposes, we will restrict our attention to Hom- and Ext^1- spaces as well as sheaves without higher cohomology. We will use the notations and results from [5].

We start by studying how Krug's result behaves with respect to the group actions by $\mathbb{Z}/2\mathbb{Z}$ on $X^{[n]}$ via $\iota^{[n]}$ and on X via ι. We will denote the various versions of the group $G = \mathbb{Z}/2\mathbb{Z}$ in the following by their nontrivial element, that is by ι or $\iota^{[n]}$ etc.

Lemma 3.1 *Assume* (X, ι) *is a K3 surface together with a fixed point free involution. For* ι*-equivariant coherent sheaves* $E, F \in \mathrm{Coh}_\iota(X)$ *there is an isomorphism of graded vector spaces:*

$$\begin{aligned}\left(\mathrm{Ext}^*_{X^{[n]}}(E^{[n]}, F^{[n]})\right)^{\iota^{[n]}} \cong &\left(\mathrm{Ext}^*_X(E, F) \otimes S^{n-1}\mathrm{H}^*(X, \mathcal{O}_X)\right)^{\iota} \\ &\oplus \left(\mathrm{Ext}^*_X(E, \mathcal{O}_X) \otimes \mathrm{Ext}^*_X(\mathcal{O}_X, F) \otimes S^{n-2}\mathrm{H}^*(X, \mathcal{O}_X)\right)^{\iota}.\end{aligned}$$

Proof Note that on the right hand side of the formula we take invariants with respect to the actions induced by the linearizations of E, F and $\mathcal{O}_X$. On the left hand side we take invariants with respect to the *induced* linearizations on $E^{[n]}$ and $F^{[n]}$. The existence of the induced linearizations follows from the right-hand side of diagram (2).

By [5, Theorem 3.6] there is an isomorphism of functors

$$(-)^{[n]} \cong \Psi \circ \mathsf{C}, \tag{1}$$

where $\mathsf{C} : \mathrm{Coh}(X) \to \mathrm{Coh}_{\mathfrak{S}_n}(X^n)$ is the exact functor with

$$\mathsf{C}(E) := \mathsf{Ind}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} \mathrm{pr}_1^* E \cong \bigoplus_{i=1}^{n} \mathrm{pr}_i^* E.$$

Furthermore $\Psi : \mathrm{D}^{\mathrm{b}}_{\mathfrak{S}_n}(X^n) \to \mathrm{D}^{\mathrm{b}}(X^{[n]})$ is the Fourier - Mukai transform with kernel the structure sheaf of the isospectral Hilbert scheme $I^n X$. Here the isosprectral Hilbert scheme is the reduced fiber product $I^n X := (X^{[n]} \times_{S^n X} X^n)_{\mathrm{red}}$ of the quotient map $\nu : X^n \to S^n X$ to the symmetric power and the Hilbert - Chow morphism $\mu : X^{[n]} \to S^n X$. This Fourier - Mukai transform is an equivalence, see [5, Proposition 2.8] and satisfies

$$\left(\iota^{[n]}\right)^* \circ \Psi = \Psi \circ \left(\iota^{\times n}\right)^* \tag{2}$$

see for example [16, Sect. 5.6]. Here $\iota^{\times n}$ is the induced involution on X^n.

We have the following chain of isomorphisms:

$$\begin{aligned}
\left(\mathrm{Ext}^*_{X^{[n]}}(E^{[n]}, F^{[n]})\right)^{\iota^{[n]}} &\cong \left(\mathrm{Ext}^*_{X^{[n]}}(\Psi(\mathsf{C}(E)), \Psi(\mathsf{C}(F)))\right)^{\iota^{[n]}} \\
&\cong \left(\mathrm{Ext}^*_{X^n,\mathfrak{S}_n}(\mathsf{C}(E), \mathsf{C}(F))\right)^{\iota^{\times n}} \\
&\cong \left(\mathrm{Ext}^*_{X^n,\mathfrak{S}_{n-1}}(\mathrm{pr}_1^* E, \mathrm{pr}_1^* F)\right)^{\iota^{\times n}} \oplus \left(\mathrm{Ext}^*_{X^n,\mathfrak{S}_{n-2}}(\mathrm{pr}_1^* E, \mathrm{pr}_2^* F)\right)^{\iota^{\times n}}
\end{aligned}$$

Here the first isomorphism is (1). The second isomorphism uses that Ψ is an equivalence and (2). The last isomorphism can be extracted from [5, Proposition 4.1].

We look at the first summand, the second working similarly. First note that

$$\mathrm{pr}_1^* E = E \boxtimes \mathcal{O}_X \boxtimes \cdots \boxtimes \mathcal{O}_X.$$

Applying the Künneth formula shows

$$\begin{aligned}
\mathrm{Ext}^*_{X^n,\mathfrak{S}_{n-1}}(\mathrm{pr}_1^* E, \mathrm{pr}_1^* F) &= \mathrm{Ext}^*_{X^n,\mathfrak{S}_{n-1}}(E \boxtimes \mathcal{O}_X \boxtimes \cdots \boxtimes \mathcal{O}_X, F \boxtimes \mathcal{O}_X \boxtimes \cdots \boxtimes \mathcal{O}_X) \\
&\cong \left(\mathrm{Ext}^*_X(E, F) \otimes \mathrm{H}^*(X, \mathcal{O}_X)^{\otimes n-1}\right)^{\mathfrak{S}_{n-1}}
\end{aligned}$$

But the group $\mathbb{Z}/2\mathbb{Z}$ acts on sheaves of the form $\mathrm{pr}_1^* E$ by definition of $\iota^{\times n}$ as

$$\left(\iota^{\times n}\right)^* \mathrm{pr}_1^* E = \iota^* E \boxtimes \iota^* \mathcal{O}_X \boxtimes \cdots \boxtimes \iota^* \mathcal{O}_X$$

that is simply by the pullback via ι on each factor in the box product. Since the action of $\mathbb{Z}/2\mathbb{Z}$ via $\iota^{\times n}$ and the $\mathfrak{S}_n$ action commute we finally see that:

$$\left(\mathrm{Ext}^*_{X^n,\mathfrak{S}_{n-1}}(\mathrm{pr}_1^* E, \mathrm{pr}_1^* F)\right)^{\iota^{\times n}} \cong \left(\mathrm{Ext}^*_X(E, F) \otimes S^{n-1}\mathrm{H}^*(X, \mathcal{O}_X)\right)^{\iota}. \qquad \square$$

□

Theorem 3.2 *Let (X,ι) be a K3 surface together with a fixed point free involution and let $n\in\mathbb{N}$ be an odd number. If $G,H\in\mathrm{Coh}(S)$ are such that π^*G and π^*H have no higher cohomology (here $S=X/\iota$ is the associated Enriques surface), then*

$$\mathrm{Hom}_{S_{[n]}}(G_{[n]},H_{[n]})\cong\mathrm{Hom}_S(G,H)\ \ \textit{and}\ \ \mathrm{Ext}^1_{S_{[n]}}(G_{[n]},H_{[n]})\cong\mathrm{Ext}^1_S(G,H).$$

Proof Define $E:=\pi^*G$ and $F:=\pi^*H$. It follows from diagram (2) that $E^{[n]}\cong\rho^*G_{[n]}$ and $F^{[n]}\cong\rho^*H_{[n]}$. We therefore have an isomorphism

$$\mathrm{Ext}^*_{S_{[n]}}(G_{[n]},H_{[n]})\cong\left(\mathrm{Ext}^*_{X^{[n]}}(\rho^*G_{[n]},\rho^*H_{[n]})\right)^{\iota^{[n]}}\cong\left(\mathrm{Ext}^*_{X^{[n]}}(E^{[n]},F^{[n]})\right)^{\iota^{[n]}}.$$

By Lemma 3.1 the last space is isomorphic to

$$\left(\mathrm{Ext}^*_X(E,F)\otimes S^{n-1}\mathrm{H}^*(X,\mathcal{O}_X)\right)^{\iota}\oplus\left(\mathrm{Ext}^*_X(E,\mathcal{O}_X)\otimes\mathrm{H}^*(X,F)\otimes S^{n-2}\mathrm{H}^*(X,\mathcal{O}_X)\right)^{\iota}\quad(3)$$

We begin investigating the first summand. The natural $\mathbb{Z}/2$-linearization of $\mathcal{O}_X$ induces an $\mathbb{Z}/2$-linearization on $\pi_*\mathcal{O}_X\cong\mathcal{O}_S\oplus\omega_S$ given by the generator of $\mathbb{Z}/2$ acting by $+1$ on $\mathcal{O}_S$ and by -1 on ω_S, see for example [19, Remarks on p.72]. Hence ι acts as $+1$ on $\mathrm{H}^0(X,\mathcal{O}_X)\cong\mathrm{H}^0(S,\mathcal{O}_S)$ and by -1 on $\mathrm{H}^2(X,\mathcal{O}_X)\cong\mathrm{H}^2(S,\omega_S)$. Furthermore, by the adjunction between π^* and π_* together with the projection formula, we get a splitting

$$\mathrm{Ext}^*_X(E,F)\cong\mathrm{Ext}^*_S(G,H)\oplus\mathrm{Ext}^*_S(G,H\otimes\omega_S).$$

where ι acts as $+1$ on the first summand and by -1 on the second summand.

Thus writing $\mathrm{H}^*(X,\mathcal{O}_X)=\mathbb{C}[t]/(t^2)$ with $\deg(t)=2$ we get

$$S^{n-1}\mathrm{H}^*(X,\mathcal{O}_X)=\mathbb{C}[t]/(t^n),\ \ \deg(t)=2$$

and ι acts as $+1$ on the constants and as -1 on t.

We can now compute the invariants and find

$$\left(\mathrm{Ext}^*_S(G,H)\otimes\mathbb{C}[t]/(t^n)\right)^{\iota}=\mathrm{Ext}^*_S(G,H)\otimes\mathbb{C}[t^2]/(t^n),\ \ \deg(t)=2$$

as well as

$$\left(\mathrm{Ext}^*_S(G,H\otimes\omega_S)\otimes\mathbb{C}[t]/(t^n)\right)^{\iota}=\mathrm{Ext}^*_S(G,H\otimes\omega_S)\otimes t\mathbb{C}[t^2]/(t^n),\ \ \deg(t)=2.$$

Looking at the components in degree zero and one sees

$$\left(\left(\mathrm{Ext}^*_X(E,F)\otimes S^{n-1}\mathrm{H}^*(X,\mathcal{O}_X)\right)^{\iota}\right)_0\cong\mathrm{Hom}_S(G,H)\ \text{ as well as}$$
$$\left(\left(\mathrm{Ext}^*_X(E,F)\otimes S^{n-1}\mathrm{H}^*(X,\mathcal{O}_X)\right)^{\iota}\right)_1\cong\mathrm{Ext}^1_S(G,H).$$

Next we study the second summand in (3): since E and F have no higher cohomology we have

$$\mathrm{Ext}^*_X(E,\mathcal{O}_X)\otimes\mathrm{H}^*(X,F)\cong\mathrm{Ext}^2_X(E,\mathcal{O}_X)\otimes\mathrm{H}^0(X,F)$$

which already lives in degree two. As we also have

$$S^{n-2}\mathrm{H}^*(X,\mathcal{O}_X)=\mathbb{C}[t]/(t^{n-1}),\ \ \deg(t)=2,$$

we see that the second summand in (3) can possibly have nontrivial components starting in degrees at least two. Especially for $k\in\{0,1\}$ we find

$$\left(\left(\mathrm{Ext}^*_X(E,\mathcal{O}_X)\otimes\mathrm{H}^*(X,E)\otimes S^{n-2}\mathrm{H}^*(X,\mathcal{O}_X)\right)^{\iota}\right)_k=0.$$

Therefore we must have the desired isomorphisms

$$\mathrm{Hom}_{S_{[n]}}(G_{[n]}, H_{[n]}) \cong \mathrm{Hom}_S(G, H) \text{ and } \mathrm{Ext}^1_{S_{[n]}}(G_{[n]}, H_{[n]}) \cong \mathrm{Ext}^1_S(G, H). \quad \square$$

□

4 Stable sheaves on Enriques manifolds

In this section we want to study the slope stability of sheaves of the form $F_{[n]}$ on $S_{[n]}$. For this we first recall the following fact: let $\varphi : Y \to Z$ be an étale Galois cover with finite Galois group G then there is the following relationship between slopes with respect to $h \in \mathrm{Amp}(Z)$:

$$\mu_{\varphi^* h}(\varphi^* F) = |G|\, \mu_h(F). \tag{4}$$

Using this fact we can prove the following lemma:

Lemma 4.1 *Let E be a torsion free coherent sheaf on Y, slope stable with respect to $\varphi^* h$ for some $h \in \mathrm{Amp}(Z)$. If E descends to Z, that is $E \cong \varphi^* F$, then F is slope stable with respect to h.*

Proof Let $H \subset F$ be a subsheaf of F. Then $\varphi^* H$ is a subsheaf of $\varphi^* F \cong E$. Since E is slope stable with respect to $\varphi^* h$ we have

$$\mu_{\varphi^* h}(\varphi^* H) < \mu_{\varphi^* h}(E) = \mu_{\varphi^* h}(\varphi^* F)$$

which by (4) implies

$$\mu_h(H) < \mu_h(F).$$

Hence F is slope stable with respect to h. □

For the rest of this section we let (X, ι) be a K3 surface together with a fixed point free involution ι. We denote the associated Enriques surface by S.

To prove the main theorem in this section we need the following isomorphism:

$$\mathrm{NS}(X^{[n]}) \cong \mathrm{NS}(X)_n \oplus \mathbb{Z}\delta.$$

Remark 4.2 The summand $\mathrm{NS}(X)_n$ is constructed as follows: take $d \in \mathrm{NS}(X)$ and consider the element

$$D^n := \sum_{i=1}^{n} \mathrm{pr}_i^* d \in \mathrm{NS}(X^n).$$

This element is $\mathfrak{S}_n$-invariant and thus descends to the symmetric product $S^n X$ by [20, Lemma 6.1]. More exactly, there is an element $D_n \in \mathrm{NS}(S^n X)$ such that $\nu^* D_n = D^n$ for the quotient map $\nu : X^n \to S^n X$. Then we define $d_n := \mu^* D_n$, where $\mu : X^{[n]} \to S^n X$ is the Hilbert - Chow morphism.

By [21, Sect. 3] the involution $\iota^{[n]}$ acts on $\mathrm{NS}(X)_n$ via:

$$\left(\iota^{[n]}\right)^* (d_n) = \left(\iota^* d\right)_n. \tag{5}$$

We are now ready to prove the main result of this section:

Theorem 4.3 *Assume $E \in \mathrm{Coh}(X)$ satisfies $E^{**} \ncong \mathcal{O}_X$, is torsion free and slope stable with respect to $h = \pi^* d$ for some $d \in \mathrm{Amp}(S)$. If E descends to S, that is $E \cong \pi^* F$ for some $F \in \mathrm{Coh}(S)$, then the induced torsion free sheaf $F_{[n]}$ is slope stable with respect to some ample divisor D on $S_{[n]}$.*

Proof By the results of Stapleton in [6] and in Sect. 1 we know that for a given slope stable torsion free sheaf E on X with $E^{**} \neq \mathcal{O}_X$, the associated tautological sheaf $E^{[n]}$ is slope stable on $X^{[n]}$.

By Theorem 2.8 the sheaf $E^{[n]}$ descends to $S_{[n]}$ if and only if E descends to S. In this case $E^{[n]} \cong \rho^* F_{[n]}$. Now by Theorem 4.1 the sheaf $F_{[n]}$ is slope stable with respect to some $D \in \mathrm{Amp}(S_{[n]})$ if $E^{[n]}$ is slope stable with respect to $H \in \mathrm{Amp}(X^{[n]})$ of the form $H = \rho^* D$ for some $D \in \mathrm{Amp}(S_{[n]})$.

To see that we find such a $D \in \mathrm{Amp}(S_{[n]})$, we note that the divisor H is described quite explicitly in [6, Proposition 4.8]: it is of the form

$$H = h_n + \epsilon A$$

for an *arbitrary* ample divisor A on $X^{[n]}$ and ϵ sufficiently small. We choose A of the form $A = \rho^* C$ for some $C \in \mathrm{Amp}(S_{[n]})$. By (5) we also have

$$\left(\iota^{[n]}\right)^* (h_n) = (\iota^* h)_n = \left(\iota^* \pi^* d\right)_n = \left(\pi^* d\right)_n = h_n$$

which implies that we must have that $h_n = \rho^* B$ for some divisor B on $S_{[n]}$. Putting both facts together shows

$$H = \rho^* D \;\text{ for }\; D = B + \epsilon C.$$

It remains to see that D is ample. But since ρ is finite and surjective D is ample if and only if $\rho^* D = H$ is ample, see [22, Proposition I.4.4]. □

In the rest of this section we want to study the moduli spaces containing the slope stable sheaves F on S and $F_{[n]}$ on $S_{[n]}$. For this we let $v \in \mathrm{H}^*_{\mathrm{alg}}(S, \mathbb{Z})$ be a Mukai vector on S, that is $v = \mathrm{ch}(F)\sqrt{\mathrm{td}(S)}$ for some $F \in \mathrm{Coh}(S)$. Here

$$\mathrm{H}^*_{\mathrm{alg}}(S, \mathbb{Z}) = \mathrm{H}^0(S, \mathbb{Z}) \oplus \mathrm{Num}(S) \oplus \frac{1}{2}\mathbb{Z}\xi_S.$$

where ξ_S denotes the fundamental class of S.

We begin with the following result:

Theorem 4.4 *Let F be a torsion free coherent sheaf with $F \ncong F \otimes \omega_S$. If F is slope stable with respect to $d \in \mathrm{Amp}(S)$, $F^{**} \ncong \mathcal{O}_S$ and $F^{**} \ncong \omega_S$, then $F_{[n]}$ is a slope stable torsion free coherent sheaf on $S_{[n]}$.*

Proof The assumptions imply that F is simple and that $\mathrm{Hom}_S(F, F \otimes \omega_S) = 0$. Hence $E := \pi^* F$ is is simple due to the formula

$$\mathrm{Hom}_X(E, E) \cong \mathrm{Hom}_S(F, F) \oplus \mathrm{Hom}_S(F, F \otimes \omega_S).$$

By [11, Lemma 3.2.3], the sheaf E is polystable with respect to $h = \pi^* d$. Being simple and polystable, E is stable.

Since $E^{**} \ncong \mathcal{O}_X$ the sheaf $E^{[n]}$ is slope stable with respect to some $H \in \mathrm{Amp}(X^{[n]})$ and descends to $S_{[n]}$ via $E^{[n]} \cong \rho^* F_{[n]}$. Now Theorem 4.3 implies that $F_{[n]}$ is slope stable with respect to some $D \in \mathrm{Amp}(S_{[n]})$ satisfying $\rho^* D = H$. □

Remark 4.5 Every torsion free coherent sheaf F of odd rank satisfies the condition $F \ncong F \otimes \omega_S$.

Assume from now on, that S is an unnodal Enriques surface, that is S contains no smooth rational curves (that is no (-2)-curves). Note that in the moduli space of Enriques surfaces, a very general element will be unnodal by [23, Corollary 5.7].

Denote the moduli space of slope semistable sheaves (with respect to $d \in \mathrm{Amp}(S)$) with Mukai vector v on S by $\mathrm{M}_{S,d}(v)$. Assume that v is primitive and chosen such that every slope semistable sheaf is slope stable and the rank of v is odd. Then for a generic choice of $d \in \mathrm{Amp}(S)$ the moduli space $\mathrm{M}_{S,d}(v)$ is smooth of dimension $v^2 + 1$ and $\mathrm{M}_{S,d}(v) \neq \emptyset$ if and only if $v^2 \geq -1$, see [3, Proposition 4.2, Theorem 4.6 (i)].

Furthermore in this situation there is a decomposition

$$\mathrm{M}_{S,d}(v) = \mathrm{M}_{S,d}(v, L_1) \coprod \mathrm{M}_{S,d}(v, L_2) \tag{6}$$

where $\mathrm{M}_{S,d}(v, L_i)$ contains those $[E] \in \mathrm{M}_{S,d}(v)$ with $\det(E) = L_i$ where $L_2 = L_1 \otimes \omega_S$, that is $\mathrm{c}_1 = \mathrm{c}_1(L_1) = \mathrm{c}_1(L_2) \in \mathrm{Num}(S)$. By [3, Theorem 4.6.(ii)] for a general choice of $d \in \mathrm{Amp}(S)$ the moduli space $\mathrm{M}_{S,d}(v, L)$ is irreducible, that is a smooth projective variety.

We also assume that the Mukai vector is chosen such that for all $[F] \in \mathrm{M}_{S,d}(v, L)$ the sheaf F is locally free on S and does not have higher cohomology. Denote the Mukai vector of the associated sheaf $F_{[n]}$ on $S_{[n]}$ by $v_{[n]}$. If $F_{[n]}$ is slope stable with respect to some $D \in \mathrm{Amp}(S_{[n]})$, denote its moduli space by $\mathcal{M}_{S_{[n]},D}(v_{[n]})$.

Proposition 4.6 *If $v \neq v(\mathcal{O}_S) = v(\omega_S)$ then there is a class $D \in \mathrm{Amp}(S_{[n]})$ such that $F_{[n]}$ is slope stable with respect to D for all $[F] \in \mathrm{M}_{S,d}(v, L)$.*

Proof Since all sheaves classified by $\mathrm{M}_{S,d}(v, L)$ are locally free on S, so are all the $E = \pi^* F$ on X. Proposition 1.3 shows that there is one $H \in \mathrm{Amp}(X^{[n]})$ such that all $E^{[n]}$ are slope stable with respect to H since $E \ncong \mathcal{O}_X$. But then by the construction of $D \in \mathrm{Amp}(S_{[n]})$ with $H = \rho^* D$ in Theorem 4.3, it follows that there is one such desired D. □

We have the following corollary:

Corollary 4.7 *If $v \neq v(\mathcal{O}_S) = v(\omega_S)$, then functor $(-)_{[n]}$ induces a morphism*

$$(-)_{[n]} : \mathrm{M}_{S,d}(v, L) \to \mathcal{M}_{S_{[n]},D}(v_{[n]}),\ [F] \mapsto [F_{[n]}]$$

which identifies $\mathrm{M}_{S,d}(v, L)$ with a smooth connected component of $\mathcal{M}_{S_{[n]},D}(v_{[n]})$.

Proof We use the explicit description $(-)_{[n]} = p_{S_{[n]}*}(p_S^*(-))$ given by Theorem 2.10. Since p_X and $p_{X^{[n]}}$ are flat we know by faithfully flat descent for π resp. ρ that the induced projections p_S and $p_{S_{[n]}}$ are flat. Similarly since $p_{X^{[n]}}$ is a finite morphism so is $p_{S_{[n]}}$.

Using these facts together with Theorem 4.4 and Proposition 4.6 shows that Krug's argument in the proof of [12, Proposition 2.1] also works in this case. Hence $[F] \mapsto [F_{[n]}]$ is a regular morphism.

Similar to Theorem 1.5 it follows from Theorem 3.2 that $(-)_{[n]}$ is injective on closed points as $\mathrm{Hom}_{S_{[n]}}(F_{[n]}, G_{[n]}) \cong \mathrm{Hom}_S(F, G)$. By Theorem 3.2 we also have

$$\dim(\mathrm{Ext}^1_{S_{[n]}}(F_{[n]}, F_{[n]})) = \dim(\mathrm{Ext}^1_S(F, F)).$$

Both facts together imply that $(-)_{[n]}$ identifies $\mathrm{M}_{S,d}(v, L)$ with a smooth connected component of $\mathcal{M}_{S_{[n]},D}(v_{[n]})$. □

Remark 4.8 There is a decomposition

$$\mathcal{M}_{S_{[n]},D}(v_{[n]}) = \mathcal{M}_{S_{[n]},D}(v_{[n]}, \mathcal{L}_1) \coprod \mathcal{M}_{S_{[n]},D}(v_{[n]}, \mathcal{L}_2)$$

analogous to (6) and, depending on the choice of $(-)_{[n]}$ (see Remark 2.9), $\mathrm{M}_{S,d}(v, L)$ is mapped to a component of $\mathcal{M}_{S_{[n]},D}(v_{[n]}, \mathcal{L}_1)$ or a component of $\mathcal{M}_{S_{[n]},D}(v_{[n]}, \mathcal{L}_2)$.

Denote the Mukai vector of $E = \pi^* F$ on X by w, that is $w = \pi^* v$. In the rest of this section we want to study the fixed loci of ι^* in $\mathrm{M}_{X,h}(w)$ and $\left(\iota^{[n]}\right)^*$ in $\mathcal{M}_{X^{[n]},H}(w^{[n]})$. In our situation we have a well defined morphism

$$\pi^* : \mathrm{M}_{S,d}(v) \to \mathrm{M}_{X,h}(w), \quad F \mapsto \pi^* F$$

which has image in $\mathrm{Fix}(\iota^*)$. More exactly the image of π^* is the fixed locus of ι^* and the morphism restricts to an étale 2:1-morphism

$$\pi^* : \mathrm{M}_{S,d}(v) \to \mathrm{Fix}(\iota^*).$$

Furthermore $\mathrm{Fix}(\iota^*)$ is a Lagrangian subscheme in $\mathrm{M}_{X,h}(w)$, see for example [2, Theorem (1)] or [24, Theorem 2.3 (c)].

As the morphism $\pi^* : \mathrm{M}_{S,d}(v) \to \mathrm{Fix}(\iota^*)$ is an étale 2:1-morphism, the decomposition (6) shows that π^* induces an isomorphism $\mathrm{M}_{S,d}(v, L) \cong \mathrm{Fix}(\iota^*)$. As $\mathrm{M}_{S,d}(v, L)$ is irreducible, so is $\mathrm{Fix}(\iota^*)$.

Theorem 4.9 *The fixed locus* $\mathrm{Fix}(\iota^*)$ *is a smooth projective variety. The morphism* $(-)^{[n]}$ *in Theorem 1.5 restricts to a morphism*

$$(-)^{[n]} : \mathrm{Fix}(\iota^*) \to \mathrm{Fix}(\left(\iota^{[n]}\right)^*)$$

which identifies $\mathrm{Fix}(\iota^*)$ *with a smooth connected component of* $\mathrm{Fix}(\left(\iota^{[n]}\right)^*)$.

Proof The fixed locus $\mathrm{Fix}(\iota^*)$ is smooth and projective since $\mathrm{M}_{X,h}(w)$ is smooth and projective. Since it is also irreducible, it is a smooth projective variety.

By Lemma 2.7 the morphism $(-)^{[n]}$ restricts to a morphism between the fixed loci. Since $(-)^{[n]}$ is injective on closed points, so is its restriction to $\mathrm{Fix}(\iota^*)$.

To identify $\mathrm{Fix}(\iota^*)$ as a smooth connected component it is therefore enough to prove

$$\dim\left(T_{[E]}\mathrm{Fix}(\iota^*)\right) = \dim\left(T_{[E^{[n]}]}\mathrm{Fix}(\left(\iota^{[n]}\right)^*)\right)$$

But a general fact says that the tangent space of the fixed locus satisfies

$$T_y\left(Y^G\right) \cong (T_y Y)^G,$$

see for example [25, Proposition 3.2]. As we have $E \cong \pi^* F$ for some sheaf F on S, this shows

$$T_{[E]}\mathrm{Fix}(\iota^*) \cong \left(T_{[E]}\mathrm{M}_{X,h}(w)\right)^{\iota} \cong \left(\mathrm{Ext}^1_X(E, E)\right)^{\iota} \cong \mathrm{Ext}^1_S(F, F).$$

A similar computation shows

$$T_{[E^{[n]}]}\mathrm{Fix}(\left(\iota^{[n]}\right)^*) \cong \left(\mathrm{Ext}^1_{X^{[n]}}(E^{[n]}, E^{[n]})\right)^{\iota^{[n]}} \cong \mathrm{Ext}^1_{S_{[n]}}(F_{[n]}, F_{[n]}) \cong \mathrm{Ext}^1_S(F, F)$$

by Theorem 3.2 since $E^{[n]} \cong \rho^* F_{[n]}$. ◻

Corollary 4.10 *The diagram (2) induces the commutative diagram:*

$$\begin{array}{ccc} \mathrm{Fix}(\iota^*) & \xrightarrow{(-)^{[n]}} & \mathrm{Fix}\big(\big(\iota^{[n]}\big)^*\big) \\ {\scriptstyle\pi^*}\uparrow & & \uparrow{\scriptstyle\rho^*} \\ \mathrm{M}_{S,d}(v,L) & \xrightarrow[(-)_{[n]}]{} & \mathcal{M}_{S_{[n]},D}(v_{[n]}) \end{array} .$$

Acknowledgements I thank Andreas Krug for answering my (many) questions regarding [5]. I also thank the referee for a very detailed report which improved the paper greatly.

Funding Open Access funding enabled and organized by Projekt DEAL.

References

1. Oguiso, K., Schröer, S.: Enriques manifolds. J. Reine Angew. Math. **661**, 215–235 (2011). https://doi.org/10.1515/CRELLE.2011.077
2. Kim, H.: Moduli spaces of stable vector bundles on Enriques surfaces. Nagoya Math. J. **150**, 85–94 (1998). https://doi.org/10.1017/S002776300002506X
3. Yoshioka, K.: Twisted stability and Fourier-Mukai transform. I. Compositio Math. **138**(3), 261–288 (2003). https://doi.org/10.1023/A:1027304215606
4. Fogarty, J.: Algebraic families on an algebraic surface. Am. J. Math. **90**, 511–521 (1968). https://doi.org/10.2307/2373541
5. Krug, A.: Remarks on the derived McKay correspondence for Hilbert schemes of points and tautological bundles. Math. Ann. **371**(1–2), 461–486 (2018). https://doi.org/10.1007/s00208-018-1660-5
6. Stapleton, D.: Geometry and stability of tautological bundles on Hilbert schemes of points. Algebra Number Theory **10**(6), 1173–1190 (2016). https://doi.org/10.2140/ant.2016.10.1173
7. Wandel, M.: Tautological sheaves: stability, moduli spaces and restrictions to generalised Kummer varieties. Osaka J. Math. **53**(4), 889–910 (2016)
8. Scala, L.: Some remarks on tautological sheaves on Hilbert schemes of points on a surface. Geom. Dedicata **139**, 313–329 (2009). https://doi.org/10.1007/s10711-008-9338-x
9. Krug, A., Rennemo, J. r.V.: Some ways to reconstruct a sheaf from its tautological image on a Hilbert scheme of points. Math. Nachr. **295**(1), 158–174 (2022). https://doi.org/10.1002/mana.201900351
10. Reede, F., Zhang, Z.: Stable vector bundles on generalized Kummer varieties. Forum Math. **34**(4), 1015–1031 (2022). https://doi.org/10.1515/forum-2021-0249
11. Huybrechts, D., Lehn, M.: The Geometry of Moduli Spaces of Sheaves, Cambridge Mathematical Library, 2nd edn., p. 325. Cambridge University Press, Cambridge (2010). https://doi.org/10.1017/CBO9780511711985
12. Krug, A.: Extension groups of tautological bundles on symmetric products of curves. Beitr. Algebra Geom. **64**(2), 493–530 (2023). https://doi.org/10.1007/s13366-022-00644-0
13. Biswas, I., Nagaraj, D.S.: Fourier-Mukai transform of vector bundles on surfaces to Hilbert scheme. J. Ramanujan Math. Soc. **32**(1), 43–50 (2017)
14. Ploog, D.: Equivariant autoequivalences for finite group actions. Adv. Math. **216**(1), 62–74 (2007). https://doi.org/10.1016/j.aim.2007.05.002
15. Boissière, S., Nieper-Wißkirchen, M., Sarti, A.: Higher dimensional Enriques varieties and automorphisms of generalized Kummer varieties. J. Math. Pures Appl. **95**(5), 553–563 (2011). https://doi.org/10.1016/j.matpur.2010.12.003
16. Krug, A., Sosna, P.: Equivalences of equivariant derived categories. J. Lond. Math. Soc. **92**(1), 19–40 (2015). https://doi.org/10.1112/jlms/jdv014

17. Haution, O.: Fixed point theorems involving numerical invariants. Compos. Math. **155**(2), 260–288 (2019). https://doi.org/10.1112/s0010437x18007911
18. Krug, A.: Extension groups of tautological sheaves on Hilbert schemes. J. Algebraic Geom. **23**(3), 571–598 (2014). https://doi.org/10.1090/S1056-3911-2014-00655-X
19. Mumford, D.: Abelian Varieties. Tata Institute of Fundamental Research Studies in Mathematics, vol. 5, p. 242. Published for the Tata Institute of Fundamental Research, Bombay by Oxford University Press, London, (1970)
20. Fogarty, J.: Algebraic families on an algebraic surface. II. The Picard scheme of the punctual Hilbert scheme. Am. J. Math. **95**, 660–687 (1973). https://doi.org/10.2307/2373734
21. Boissière, S., Sarti, A.: A note on automorphisms and birational transformations of holomorphic symplectic manifolds. Proc. Am. Math. Soc. **140**(12), 4053–4062 (2012). https://doi.org/10.1090/S0002-9939-2012-11277-8
22. Hartshorne, R.: Ample Subvarieties of Algebraic Varieties. Lecture Notes in Mathematics, Vol. 156, (1970), p. 256. Springer. Notes written in collaboration with C. Musili
23. Namikawa, Y.: Periods of Enriques surfaces. Math. Ann. **270**(2), 201–222 (1985). https://doi.org/10.1007/BF01456182
24. Nuer, H.: A note on the existence of stable vector bundles on Enriques surfaces. Selecta Math. (N.S.) **22**(3), 1117–1156 (2016). https://doi.org/10.1007/s00029-015-0218-6
25. Edixhoven, B.: Néron models and tame ramification. Compositio Math. **81**(3), 291–306 (1992)