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RESEARCH ARTICLE

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Böttcher coordinates at wild superattracting fixed points

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Abstract

Let p be a prime number, let $g(x) = x^{p^2} + p^{r+2}x^{p^2+1}$ with $r \in \mathbb{Z}_{\geqslant 0}$, and let $\phi(x) = x + O(x^2)$ be the Böttcher coordinate satisfying $\phi(g(x)) = \phi(x)^{p^2}$. Salerno and Silverman conjectured that the radius of convergence of $\phi^{-1}(x)$ in $\mathbb{C}_p$ is $p^{-p^{-r}/(p-1)}$. In this article, we confirm that this conjecture is true by showing that it is a special case of our more general result.

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1 | INTRODUCTION

Let K be a field of characteristic 0 and let $g(x) = x^d + O(x^{d+1}) \in K[\![x]\!]$ with $d \geqslant 2$. Then there is a unique *Böttcher coordinate* $\phi(x) = x + O(x^2) \in K[\![x]\!]$ satisfying $\phi(g(x)) = \phi(x)^d$. It can be seen that

$$\phi(x) = \lim_{n\to\infty} g^n(x)^{1/d^n},$$

where the root is chosen such that $g^n(x)^{1/d^n} = x + O(x^2) \in K[\![x]\!]$.

Although the Böttcher coordinate over $K = \mathbb{C}$ has become a fundamental tool in the area of complex dynamics (see, e.g., [6, chapter 9] for more details), its analogue over $K = \mathbb{C}_p$ has only been studied from the last decade. Ingram [4] used p-adic Böttcher coordinates to study arboreal Galois representations. DeMarco et al. [1] used p-adic Böttcher coordinates to prove a theorem of unlikely intersections. Salerno and Silverman [7] studied the integrality properties of some p-adic Böttcher coordinates. In particular, they proposed the following conjecture [7, Conjecture 27].

Conjecture 1.1 (Salerno and Silverman). *Let p be a prime number, let*

$$g(x) = x^{p^2} + p^{r+2} x^{p^2+1}$$

with $r \in \mathbb{Z}_{\geqslant 0}$, and let $\phi(x) = x + O(x^2)$ be the Böttcher coordinate satisfying $\phi(g(x)) = \phi(x)^{p^2}$. Then the radius of convergence of $\phi^{-1}(x)$ in $\mathbb{C}_p$ is $p^{-p^{-r}/(p-1)}$.

In this article, we will prove a generalization of Conjecture 1.1. Before stating our main results, we first briefly explain how we approach the solution of this problem.

Let $f_c(z) = z^d - c$ for some $c \in \mathbb{C}_p$ and let

$$\varphi_c(z) = z\left(1 + \sum_{n=1}^{\infty} \frac{a_n}{z^{nd}}\right) \tag{1.1}$$

satisfy the functional equation

$$f_c(\varphi_c(z)) = \varphi_c(z^d). \tag{1.2}$$

Note that here $\varphi_c(z)$ is the inverse of the Böttcher coordinate, not the Böttcher coordinate itself. Let $x = z^{-d}$, then (1.2) can be simplified as

$$\left(1 + \sum_{n=1}^{\infty} a_n x^n\right)^d = 1 + cx + \sum_{n=1}^{\infty} a_n x^{nd}. \tag{1.3}$$

Let $g_c(x) = x^d + cx^{d+1}$ for some $c \in \mathbb{C}_p$ and let

$$\phi_c(x) = x\left(1 + \sum_{n=1}^{\infty} b_n x^n\right)$$

satisfy the Böttcher equation

$$\phi_c(g_c(x)) = \phi_c(x)^d.$$

Then it can be simplified as

$$\left(1 + \sum_{n=1}^{\infty} b_n x^n\right)^d = 1 + cx + \sum_{n=1}^{\infty} b_n x^{nd}(1 + cx)^{n+1}. \tag{1.4}$$

Instead of working on $g(x)$ and $\phi(x)$ directly, we will work on their generalizations $g_c(x)$ and $\phi_c(x)$. Therefore, we need to study (1.4) and, in particular, the properties of $v_p(b_n)$, where v_p is the p-adic valuation in $\mathbb{C}_p$. The key idea of our proofs is to consider (1.4) as a *perturbation* of (1.3). First we show that under some conditions on d and c, the values of $v_p(a_n)$ can be explicitly obtained. Then we show that under the same conditions, the perturbation is small enough so that $v_p(b_n) = v_p(a_n)$, which enables us to determine the radii of convergence of $\phi_c(x)$ and $\phi_c^{-1}(x)$.

The conditions mentioned above can be summarized as follows. The parameters p, N, d, and c will be used repeatedly throughout the whole article.

Condition A. Assume that p is a prime number, $N = 0$, d is a *multiple* of p, and

$$v_p(c) < v_p(d) + \frac{v_p((d-1)!)}{d-1}. \tag{1.5}$$

Condition B. Assume that p is a prime number, $N \geqslant 1$ is an integer, d is a *power* of p, and

$$N v_p(d) + \frac{v_p((d-1)!)}{d-1} < v_p(c) < (N+1)v_p(d) + \frac{v_p((d-1)!)}{d-1}. \tag{1.6}$$

Now we are ready to give the main theorems of this article.

Theorem 1.2. *Let p, N, d, and c satisfy Condition A or B. Then the maximal convergent open disks of $\varphi_c(z)$ and $\varphi_c^{-1}(z)$ are both $D(\infty, r_N^{1/d}) = \{z \in \mathbb{C}_p \, : \, |z|_p > r_N^{1/d}\}$, where*

$$r_N = \left(|c/d^{N+1}|_p p^{1/(p-1)} \right)^{1/d^N} > 1. \tag{1.7}$$

Moreover, $\varphi_c(z)$ gives a bijective isometry from $D(\infty, r_N^{1/d})$ onto itself.

Theorem 1.3. *Let p, N, d, and c satisfy Condition A or B. Then the maximal convergent open disks of $\phi_c(x)$ and $\phi_c^{-1}(x)$ are both $D(0, r_N^{-1}) = \{x \in \mathbb{C}_p \, : \, |x|_p < r_N^{-1}\}$, where r_N is the same value given by (1.7). Moreover, $\phi_c(x)$ gives a bijective isometry from $D(0, r_N^{-1})$ onto itself.*

In Conjecture 1.1, we have $d = p^2$ and $c = p^{r+2}$ with $r \in \mathbb{Z}_{\geqslant 0}$, so we can take $N = \lfloor (r+1)/2 \rfloor$ to satisfy Condition A or B. Then by Theorem 1.3, the radius of convergence of $\phi^{-1}(x)$ is

$$r_N^{-1} = \left(|c/d^{N+1}|_p p^{1/(p-1)} \right)^{-1/d^N} = p^{-p^{-r}/(p-1)},$$

as conjectured by Salerno and Silverman.

Corollary 1.4. *Conjecture 1.1 is true.*

We remark that the technical Conditions A and B are crucial for Theorems 1.2 and 1.3.

Remark 1.5. If $p = d = c = 2$, then $f_2(z) = z^2 - 2$ is a Chebyshev map. Now $\varphi_2(z) = z + z^{-1}$ and

$$\varphi_2^{-1}(z) = z\left(1 - \sum_{n=1}^{\infty} \frac{C_n}{z^{2n}}\right), \text{ where } C_n = \frac{(2n-2)!}{(n-1)!n!}$$

are known as the Catalan numbers. Their maximal convergent open disks are $D(\infty, 0)$ and $D(\infty, 1)$, respectively. On the other hand, the maximal convergent open disks of $\varphi_c(z)$ and $\varphi_c^{-1}(z)$ in Theorem 1.2 are always identical.

Remark 1.6. If $d = p$ and c is a multiple of p, then by [7, Theorem 4], both $\phi_c(x)$ and $\phi_c^{-1}(x)$ have integral coefficients so that they are convergent on the open unit disk $D(0,1)$. On the other hand, $D(0, r_N^{-1})$ in Theorem 1.3 is always strictly smaller than $D(0,1)$.

Theorem 1.2 can also be interpreted in a different way. For any $c \in \mathbb{C}_p$, let

$$B(c) = \{z \in \mathbb{C}_p \,:\, f_c^{\circ n}(z) \to \infty \text{ as } n \to \infty\}$$

be the *basin of infinity* of $f_c(z)$. We say that $B(c_1)$ and $B(c_2)$ are *analytically conjugate* if there is a bijective analytic map $\Phi_{c_1,c_2} : B(c_1) \to B(c_2)$ whose inverse is also analytic such that

$$f_{c_2}(\Phi_{c_1,c_2}(z)) = \Phi_{c_1,c_2}(f_{c_1}(z)). \tag{1.8}$$

We know that $f_c(z)$ has good reduction if and only if

$$v_p(c) \geqslant 0 \Leftrightarrow B(c) = D(\infty, 1) \Leftrightarrow 0 \notin B(c),$$

so Theorem 1.2 tells us $\varphi_c(z)$ does not give an analytic conjugacy between $B(0)$ and $B(c)$. Indeed, we are able to prove the following more general result.

Theorem 1.7. *Let p, N, d, and $c = c_2$ satisfy Condition A or B. Let c_1 satisfy $v_p(c_1) \geqslant 0$ and*

$$v_p(c_1^{d-1} - c_2^{d-1}) = v_p(c_2^{d-1}). \tag{1.9}$$

Then $B(c_1)$ and $B(c_2)$ are not analytically conjugate.

We remark that Theorem 1.7 is inspired by the work of DeMarco and Pilgrim [2], although in this article we only consider the most basic cases. A discussion for the analytic conjugacy between the basins of infinity of two tame polynomials can be found in [5].

The structure of this article is as follows: In Section 2, we prove some preliminary lemmas that will be needed later. In Sections 3, 4, and 5, we prove Theorems 1.2, 1.3, and 1.7, respectively.

2 | SOME PRELIMINARY LEMMAS

In this section, we prove some preliminary lemmas that will be needed later.

Lemma 2.1. *We have $(d-1)!^{kn_k}(dk!)^{n_k} n_k!$ divides $(dkn_k)!$ for any $d, k \geqslant 1$ and $n_k \geqslant 0$.*

Proof. We have

$$(d-1)!^{kn_k}(dk!)^{n_k} n_k! = \prod_{i=1}^{n_k} (idk)(d-1)!^k(k-1)!$$

divides

$$\prod_{i=1}^{n_k}\left((idk)\prod_{j=(i-1)dk+1}^{idk-1} j\right) = (dkn_k)!.$$

□

Lemma 2.2 (Legendre). *Let $s_p(n)$ be the sum of the digits in the base-p expansion of n. Then*

$$v_p(n!) = \frac{n - s_p(n)}{p-1}.$$

Lemma 2.3. *Let p be a prime number and let d be a power of p. If $n_0 + n_1 d = m_0 + m_1 d$ for some $0 \leqslant n_0 < d$ and $n_1, m_0, m_1 \geqslant 0$, then*

$$v_p\left(\frac{m_0!m_1!}{n_0!n_1!}\right) \leqslant (n_1 - m_1)v_p(d!).$$

Proof. By Lemma 2.2, we have

$$\text{Left-hand side} = \frac{(m_0 - n_0 + m_1 - n_1) - (s_p(m_0) - s_p(n_0) + s_p(m_1) - s_p(n_1))}{p-1}$$

and

$$\text{Right-hand side} = \frac{(n_1 - m_1)(d-1)}{p-1} = \frac{m_0 - n_0 + m_1 - n_1}{p-1}.$$

As d is a power of p, the base-p and base-d expansions are compatible. Hence,

$$\begin{aligned}(p-1)(\text{Right-hand side} - \text{Left-hand side}) &= s_p(m_0) - s_p(n_0) + s_p(m_1) - s_p(n_1)\\ &= s_p(m_0 - n_0) + s_p(m_1) - s_p(n_1)\\ &= s_p((n_1 - m_1)d) + s_p(m_1) - s_p(n_1)\\ &= s_p(n_1 - m_1) + s_p(m_1) - s_p(n_1) \geqslant 0.\end{aligned}$$

□

Lemma 2.4. *Let p be a prime number, let d be a power of p, and let N be a nonnegative integer. If $n \geqslant 1$ can be decomposed as*

$$n = \sum_{k=0}^{N} n_k d^k \text{ with } 0 \leqslant n_k < d \text{ for any } 0 \leqslant k < N \text{ and } n_N \geqslant 0,$$

then

$$v_p(n!) = \sum_{k=0}^{N} v_p(d^k!^{n_k} n_k!).$$

Proof. As d is a power of p, the base-p and base-d expansions are compatible. Hence, by Lemma 2.2, we have

$$v_p(n!) = \frac{n - s_p(n)}{p-1} = \sum_{k=0}^{N} \frac{n_k d^k - s_p(n_k d^k)}{p-1}$$

and

$$\sum_{k=0}^{N} v_p(d^k!^{n_k} n_k!) = \sum_{k=0}^{N} \frac{n_k(d^k - 1) + n_k - s_p(n_k)}{p-1} = \sum_{k=0}^{N} \frac{n_k d^k - s_p(n_k d^k)}{p-1},$$

which are equal. □

Lemma 2.5. *Let $d \in \mathbb{Z} \backslash \{0\}$ and let*

$$F(z) = z\left(1 + \sum_{n=1}^{\infty} \frac{\alpha_n}{z^{nd}}\right)$$

be a formal power series. Then

$$F^{-1}(z) = z\left(1 + \sum_{n=1}^{\infty} \frac{\beta_n}{z^{nd}}\right),$$

where

$$\beta_n = -\frac{1}{nd-1} \sum_{\sum_{k=1}^{n} k m_k = n} \left(\binom{nd-1}{\sum_{k=1}^{n} m_k} \binom{\sum_{k=1}^{n} m_k}{m_1, \dots, m_n} \prod_{k=1}^{n} \alpha_k^{m_k} \right).$$

Proof. Let $[z^n]F^{-1}(z)$ be the coefficient of z^n in $F^{-1}(z)$. By the Lagrange–Bürmann formula,

$$\begin{aligned} \beta_n = [z^{-nd+1}]F^{-1}(z) &= \frac{1}{-nd+1}[z^{-nd}]\left(\frac{z}{F(z)}\right)^{-nd+1} \\ &= -\frac{1}{nd-1}[z^{-nd}]\left(1 + \sum_{k=1}^{\infty} \frac{\alpha_k}{z^{kd}}\right)^{nd-1}. \end{aligned}$$

Then we expand this power series to get the result. □

3 | PROOF OF THEOREM 1.2

In this section, we focus on the properties of a_n and give the proof of Theorem 1.2. First we show that we can compute a_n inductively from (1.3).

Proposition 3.1. *The sequence a_n satisfies the following inductive relations.*

(1) *For any $1 \leqslant n < d$, we have $a_n = \binom{1/d}{n} c^n$, where*

$$\binom{1/d}{n} = \frac{\prod_{j=0}^{n-1}(1/d - j)}{n!}.$$

(2) *For any $d^i \leqslant n < d^{i+1}$ with $i \geqslant 1$, we have*

$$a_n = \sum_{n_0 + d\sum_{k=1}^{d^i-1} kn_k = n} \alpha(n_0, n_1, \dots, n_{d^i-1}),$$

where the summation is taken over all nonnegative d^i-tuples $(n_0, n_1, \dots, n_{d^i-1})$ such that

$$n_0 + d\sum_{k=1}^{d^i-1} kn_k = n \tag{3.1}$$

and

$$\alpha(n_0, n_1, \dots, n_{d^i-1}) = \frac{c^{n_0}}{d^{n_0} n_0!} \prod_{k=1}^{d^i-1} \frac{a_k^{n_k}}{d^{n_k} n_k!} \prod_{j=0}^{\sum_{k=0}^{d^i-1} n_k - 1} (1 - jd).$$

Proof. Let

$$\left(1 + \sum_{n=1}^{\infty} a_n' x^n\right)^d = 1 + cx, \tag{3.2}$$

then

$$1 + \sum_{n=1}^{\infty} a_n' x^n = (1 + cx)^{1/d} = 1 + \sum_{n=1}^{\infty} \binom{1/d}{n} c^n x^n$$

and $a_n' = \binom{1/d}{n} c^n$ for any $n \geqslant 1$. Considering the difference of (1.3) and (3.2), we get

$$\left(\sum_{n=1}^{\infty}(a_n - a_n')x^n\right)\left(\sum_{i=0}^{d-1}\left(1 + \sum_{n=1}^{\infty} a_n x^n\right)^i \left(1 + \sum_{n=1}^{\infty} a_n' x^n\right)^{d-1-i}\right) = \sum_{n=1}^{\infty} a_n x^{nd}.$$

Comparing the degrees on both sides, we get $a_n = a'_n = \binom{1/d}{n}c^n$ for any $1 \leqslant n < d$. Moreover, let

$$\left(1 + \sum_{n=1}^{\infty} a''_n x^n\right)^d = 1 + cx + \sum_{n=1}^{d^i-1} a_n x^{nd},$$

then

$$1 + \sum_{n=1}^{\infty} a''_n x^n = 1 + \sum_{j=1}^{\infty} \binom{1/d}{j}\left(cx + \sum_{n=1}^{d^i-1} a_n x^{nd}\right)^j.$$

By the same reasoning as above, for any $d^i \leqslant n < d^{i+1}$, we have

$$\begin{aligned} a_n = a''_n &= \sum_{n_0 + d\sum_{k=1}^{d^i-1} kn_k = n} \binom{1/d}{\sum_{k=0}^{d^i-1} n_k}\binom{\sum_{k=0}^{d^i-1} n_k}{n_0, n_1, \dots, n_{d^i-1}} c^{n_0} \prod_{k=1}^{d^i-1} a_k^{n_k} \\ &= \sum_{n_0 + d\sum_{k=1}^{d^i-1} kn_k = n} \left(\frac{c^{n_0}}{d^{n_0} n_0!} \prod_{k=1}^{d^i-1} \frac{a_k^{n_k}}{d^{n_k} n_k!} \prod_{j=0}^{\sum_{k=0}^{d^i-1} n_k - 1} (1 - jd)\right). \end{aligned}$$

□

An immediate corollary of Proposition 3.1 is that a_n can be considered as a polynomial of degree n in c. This corollary, however, will not be used in the sequel. More results of this type can be found in [3, section 2.4.1].

Corollary 3.2. *For any $n \geqslant 1$, we have $a_n \in \frac{1}{n!}\mathbb{Z}[c/d]$ with the leading term $\binom{1/d}{n}c^n$.*

Proof. By Proposition 3.1, the assertion is true for any $1 \leqslant n < d$. Now we assume that it is true for any $1 \leqslant n < d^i$ and use induction to show that it is also true for any $d^i \leqslant n < d^{i+1}$. For each $(n_0, n_1, \dots, n_{d^i-1})$ such that (3.1) holds and $n_0 \neq n$, we have

$$\deg_c \alpha(n_0, n_1, \dots, n_{d^i-1}) = n_0 + \sum_{k=1}^{d^i-1} kn_k < n.$$

Hence, the leading term of a_n is given by $\alpha(n, 0, \dots, 0) = \binom{1/d}{n}c^n$. Also, by the induction hypothesis, we know that

$$\begin{aligned} \alpha(n_0, n_1, \dots, n_{d^i-1}) &\in \frac{1}{n_0!}\prod_{k=1}^{d^i-1} \frac{1}{(dk!)^{n_k} n_k!}\mathbb{Z}[c/d] \\ &\subseteq \frac{1}{n_0!}\prod_{k=1}^{d^i-1} \frac{1}{(dkn_k)!}\mathbb{Z}[c/d] \text{ by Lemma 2.1,} \\ &\subseteq \frac{1}{n!}\mathbb{Z}[c/d] \text{ by (3.1).} \end{aligned}$$

This completes the proof. □

The following proposition is the most important step of this article. It shows that under Condition A or B, we are able to obtain all values of $v_p(a_n)$ simultaneously rather than successively.

Proposition 3.3. *Let p, N, d, and c satisfy Condition A or B. Then*

(1) *for any* $0 \leqslant k \leqslant N$, *we have*

$$v_p(a_{d^k}) = v_p\left(\frac{c}{d^{k+1}}\right)$$

(2) *if* $n \geqslant 1$ *can be decomposed as*

$$n = \sum_{k=0}^{N} n_k d^k \text{ with } 0 \leqslant n_k < d \text{ for any } 0 \leqslant k < N \text{ and } n_N \geqslant 0, \tag{3.3}$$

then

$$v_p(a_n) = \sum_{k=0}^{N} v_p\left(\frac{a_{d^k}^{n_k}}{n_k!}\right)$$

(3) *consequently, for any* $n \geqslant 1$, *we have*

$$v_p(a_n) = v_p\left(\frac{c^n}{d^n n!}\right) - \sum_{k=1}^{N}\left((d-1)v_p\left(\frac{c}{d^k}\right) - v_p((d-1)!)\right)\left\lfloor\frac{n}{d^k}\right\rfloor.$$

Proof. By Proposition 3.1, the assertions are true for any $1 \leqslant n < d$. Now we assume that they are true for any $1 \leqslant n < d^i$ and use induction to show that they are also true for any $d^i \leqslant n < d^{i+1}$.

We know that each partition σ of n with a particular form gives a summand $\alpha(\sigma)$ of a_n. We call (3.3) the canonical partition σ_{can} of n. We claim that $v_p(\alpha(\sigma)) > v_p(\alpha(\sigma_{\text{can}}))$ unless $\sigma = \sigma_{\text{can}}$.

Let σ be an arbitrary partition $n = m_0 + d\sum_{j=1}^{d^i-1} jm_j$ and, for each j, let $j = \sum_{k=0}^{N} m_{j,k}d^k$ be the canonical partition of j. Then we can produce another partition σ_0 that is given by

$$\begin{aligned} n = m_0 + d\sum_{j=1}^{d^i-1} jm_j &= m_0 + d\sum_{j=1}^{d^i-1}\left(\sum_{k=0}^{N} m_{j,k}d^k\right)m_j \\ &= m_0 + d\sum_{k=0}^{N}\left(d^k\sum_{j=1}^{d^i-1} m_j m_{j,k}\right) = m_0 + d\sum_{k=0}^{N} d^k M_{d^k}, \end{aligned}$$

where

$$M_{d^k} = \sum_{j=1}^{d^i-1} m_j m_{j,k}. \tag{3.4}$$

Now

$$
\begin{aligned}
v_p(\alpha(\sigma)) &= v_p\left(\frac{c^{m_0}}{d^{m_0}m_0!}\prod_{j=1}^{d^i-1}\frac{a_j^{m_j}}{d^{m_j}m_j!}\right) \text{as } p \mid d,\\
&= v_p\left(\frac{c^{m_0}}{d^{m_0}m_0!}\right) + \sum_{j=1}^{d^i-1}\left(m_j\sum_{k=0}^{N} v_p\left(\frac{a_{d^k}^{m_{j,k}}}{m_{j,k}!}\right) - v_p(d^{m_j}m_j!)\right) \text{by induction,}\\
&= v_p\left(\frac{c^{m_0}}{d^{m_0}m_0!}\right) + \sum_{k=0}^{N} v_p(a_{d^k}^{M_{d^k}}) - \sum_{k=0}^{N}\sum_{j=1}^{d^i-1} v_p(m_{j,k}!^{m_j}) - \sum_{j=1}^{d^i-1} v_p(d^{m_j}m_j!)
\end{aligned}
$$

and

$$
v_p(\alpha(\sigma_0)) = v_p\left(\frac{c^{m_0}}{d^{m_0}m_0!}\right) + \sum_{k=0}^{N} v_p(a_{d^k}^{M_{d^k}}) - \sum_{k=0}^{N} v_p(d^{M_{d^k}}M_{d^k}!).
$$

If $\sigma \neq \sigma_0$, then there is some $j \notin \{d^k : 0 \leqslant k \leqslant N\}$ such that $m_j \neq 0$. Therefore,

$$
\begin{aligned}
v_p(\alpha(\sigma)) - v_p(\alpha(\sigma_0)) &= \sum_{k=0}^{N} v_p(d^{M_{d^k}}M_{d^k}!) - \sum_{k=0}^{N}\sum_{j=1}^{d^i-1} v_p(m_{j,k}!^{m_j}) - \sum_{j=1}^{d^i-1} v_p(d^{m_j}m_j!)\\
&\geqslant \sum_{k=0}^{N} v_p(d^{M_{d^k}}M_{d^k}!) - \sum_{k=0}^{N}\sum_{j=1}^{d^i-1} v_p(m_{j,k}!^{m_j}) - \sum_{j=1}^{d^i-1} v_p(d^{m_j}) - \sum_{k=0}^{N}\sum_{\substack{j=1\\ m_{j,k}\neq 0}}^{d^i-1} v_p(m_j!)\\
&= \sum_{j=1}^{d^i-1}\left(\sum_{k=0}^{N} m_{j,k} - 1\right) m_j v_p(d) + \sum_{k=0}^{N}\left(v_p(M_{d^k}!) - \sum_{\substack{j=1\\ m_{j,k}\neq 0}}^{d^i-1} v_p(m_{j,k}!^{m_j}m_j!)\right)\\
&\geqslant \sum_{j=1}^{d^i-1}\left(\sum_{k=0}^{N} m_{j,k} - 1\right) m_j v_p(d) \text{ by Lemma 2.1 and (3.4),}\\
&> 0 \text{ as } \sigma \neq \sigma_0.
\end{aligned}
$$

Next, for each $1 \leqslant j \leqslant N$, we let σ_j be the partition

$$
n = \sum_{k=0}^{j-1} n_k d^k + N_j d^j + \sum_{k=j}^{N} M_{d^k} d^{k+1}.
$$

We also let $N_0 = m_0$ and $a_{d^{-1}} = c$. For any $1 \leqslant j \leqslant N$, if $\sigma_{j-1} \neq \sigma_j$, then we have

$$
N_{j-1} + M_{d^{j-1}} d = n_{j-1} + N_j d \tag{3.5}
$$

and

$$\begin{aligned}
v_p(\alpha(\sigma_{j-1})) - v_p(\alpha(\sigma_j)) &= v_p\left(\frac{a_{d^{j-2}}^{N_{j-1}}}{d^{N_{j-1}}N_{j-1}!}\frac{a_{d^{j-1}}^{M_{d^{j-1}}}}{d^{M_{d^{j-1}}}M_{d^{j-1}}!}\right) - v_p\left(\frac{a_{d^{j-2}}^{n_{j-1}}}{d^{n_{j-1}}n_{j-1}!}\frac{a_{d^{j-1}}^{N_j}}{d^{N_j}N_j!}\right) \\
&= v_p\left(\frac{(c/d^j)^{N_{j-1}}}{N_{j-1}!}\frac{(c/d^j)^{M_{d^{j-1}}}}{d^{M_{d^{j-1}}}M_{d^{j-1}}!}\right) - v_p\left(\frac{(c/d^j)^{n_{j-1}}}{n_{j-1}!}\frac{(c/d^j)^{N_j}}{d^{N_j}N_j!}\right) \text{ by induction,} \\
&= (N_j - M_{d^{j-1}})\left((d-1)v_p\left(\frac{c}{d^j}\right) + v_p(d)\right) - v_p\left(\frac{N_{j-1}!M_{d^{j-1}}!}{n_{j-1}!N_j!}\right) \text{ by (3.5),} \\
&> (N_j - M_{d^{j-1}})v_p(d!) - v_p\left(\frac{N_{j-1}!M_{d^{j-1}}!}{n_{j-1}!N_j!}\right) \text{ by the left-hand side of (1.6) and } \sigma_{j-1} \neq \sigma_j, \\
&\geqslant 0 \text{ by Lemma 2.3. (Here we need the condition } d \text{ is a power of } p.)
\end{aligned}$$

Next, if $\sigma_N \neq \sigma_{\text{can}}$, then by the same reasoning as above, we have

$$\begin{aligned}
v_p(\alpha(\sigma_N)) - v_p(\alpha(\sigma_{\text{can}})) &= -M_{d^N}\left((d-1)v_p\left(\frac{c}{d^{N+1}}\right) + v_p(d)\right) - v_p\left(\frac{N_N!M_{d^N}!}{n_N!}\right) \\
&> v_p\left(\frac{n_N!}{N_N!M_{d^N}!d!^{M_{d^N}}}\right) \text{ by (1.5), the right-hand side of (1.6), and } \sigma_N \neq \sigma_{\text{can}}, \\
&\geqslant 0 \text{ by Lemma 2.1.}
\end{aligned}$$

We have shown that $v_p(\alpha(\sigma)) > v_p(\alpha(\sigma_0)) > \ldots > v_p(\alpha(\sigma_N) > v_p(\alpha(\sigma_{\text{can}}))$, so

$$v_p(a_n) = v_p(\alpha(\sigma_{\text{can}})) = v_p\left(\frac{c^{n_0}}{d^{n_0}n_0!}\right) + \sum_{k=1}^{N} v_p\left(\frac{a_{d^{k-1}}^{n_k}}{d^{n_k}n_k!}\right),$$

which implies parts (1) and (2) immediately. Part (3) is a corollary of parts (1) and (2) because

$$\begin{aligned}
v_p(a_n) &= \sum_{k=0}^{N} v_p\left(\frac{a_{d^k}^{n_k}}{n_k!}\right) = \sum_{k=0}^{N} n_k v_p\left(\frac{c}{d^{k+1}}\right) - \sum_{k=0}^{N} v_p(n_k!) \\
&= \sum_{k=0}^{N} n_k v_p\left(\frac{c}{d^{k+1}}\right) + \sum_{k=0}^{N} n_k v_p(d^k!) - v_p(n!) \text{ by Lemma 2.4,} \\
&= \sum_{k=0}^{N-1}\left(\left\lfloor\frac{n}{d^k}\right\rfloor - \left\lfloor\frac{n}{d^{k+1}}\right\rfloor d\right)v_p\left(\frac{cd^k!}{d^{k+1}}\right) + \left\lfloor\frac{n}{d^N}\right\rfloor v_p\left(\frac{cd^N!}{d^{N+1}}\right) - v_p(n!) \\
&= v_p\left(\frac{c^n}{d^n n!}\right) + \sum_{k=1}^{N}\left\lfloor\frac{n}{d^k}\right\rfloor v_p\left(\frac{cd^k!}{d^{k+1}}\right) - \sum_{k=1}^{N}\left\lfloor\frac{n}{d^k}\right\rfloor d v_p\left(\frac{cd^{k-1}!}{d^k}\right)
\end{aligned}$$

$$= v_p\left(\frac{c^n}{d^n n!}\right) - \sum_{k=1}^{N} \left\lfloor \frac{n}{d^k} \right\rfloor \left((d-1)v_p\left(\frac{c}{d^k}\right) - v_p\left(\frac{d^k!}{d(d^{k-1}!)^d}\right)\right)$$

$$= v_p\left(\frac{c^n}{d^n n!}\right) - \sum_{k=1}^{N} \left\lfloor \frac{n}{d^k} \right\rfloor \left((d-1)v_p\left(\frac{c}{d^k}\right) - v_p((d-1)!)\right).$$

This completes the proof. □

From Proposition 3.3, we can deduce that the sequence $v_p(a_n)/n$ has a negative limit.

Proposition 3.4. *Let p, N, d, and c satisfy Condition A or B. Then the sequence $v_p(a_n)$ is subadditive and*

$$\lim_{n\to\infty} \frac{v_p(a_n)}{n} = \inf_n \frac{v_p(a_n)}{n} = \frac{v_p(c/d^{N+1})}{d^N} - \frac{1}{(p-1)d^N} < 0.$$

Proof. The subadditivity of $v_p(a_n)$ can be easily seen from Proposition 3.3. Therefore, by Fekete's lemma, the limit of $v_p(a_n)/n$ exists and is equal to the infimum of $v_p(a_n)/n$. By Proposition 3.3 and Lemma 2.2,

$$\inf_n \frac{v_p(a_n)}{n} = \inf_n \left(v_p\left(\frac{c}{d}\right) - \frac{n - s_p(n)}{(p-1)n} - \frac{1}{n}\sum_{k=1}^{N} \left((d-1)v_p\left(\frac{c}{d^k}\right) - v_p((d-1)!)\right) \left\lfloor \frac{n}{d^k} \right\rfloor \right)$$

$$= v_p\left(\frac{c}{d}\right) - \frac{1}{p-1} - \sum_{k=1}^{N} \left((d-1)v_p\left(\frac{c}{d^k}\right) - v_p((d-1)!)\right) \frac{1}{d^k}$$

$$= \frac{v_p(a_{d^N})}{d^N} - \frac{1}{(p-1)d^N} = \frac{v_p(c/d^{N+1})}{d^N} - \frac{1}{(p-1)d^N}.$$

Moreover, the limit is negative because

$$\frac{v_p(c/d^{N+1})}{d^N} < \frac{v_p((d-1)!)}{(d-1)d^N} \text{ by (1.5) and the right-hand side of (1.6),}$$

$$= \frac{(d-1) - s_p(d-1)}{(p-1)(d-1)d^N} < \frac{1}{(p-1)d^N} \text{ by Lemma 2.2.}$$

This completes the proof. □

The last ingredient needed for the proof of Theorem 1.2 is the following inequality.

Proposition 3.5. *Let p, N, d, and c satisfy Condition A or B. If $n = \sum_{k=1}^{n} k m_k$, where $m_k \geqslant 0$ for any $1 \leqslant k \leqslant n$, then*

$$v_p(a_n) \leqslant \sum_{k=1}^{n} v_p\left(\frac{a_k^{m_k}}{m_k!}\right).$$

Proof. Let

$$e(n) = \sum_{k=1}^{N} \left((d-1) v_p\left(\frac{c}{d^k}\right) - v_p((d-1)!) \right) \left\lfloor \frac{n}{d^k} \right\rfloor. \tag{3.6}$$

It is clear that the sequence $e(n)$ is superadditive. Then

$$\begin{aligned}\sum_{k=1}^{n} v_p\left(\frac{a_k^{m_k}}{m_k!}\right) &= \sum_{k=1}^{n} v_p\left(\frac{c^{km_k}}{d^{km_k}}\right) - \sum_{k=1}^{n} v_p(k!^{m_k} m_k!) - \sum_{k=1}^{n} m_k e(k) \text{ by Proposition 3.3,} \\ &\geqslant v_p\left(\frac{c^n}{d^n}\right) - v_p(n!) - e(n) \text{ by Lemma 2.1,} \\ &= v_p(a_n) \text{ by Proposition 3.3.}\end{aligned}$$

□

Now we are ready to give the proof of Theorem 1.2.

Proof of Theorem 1.2. By (1.1) and Proposition 3.4, $\varphi_c(z)$ is convergent when

$$|z|_p^d > \lim_{n\to\infty} |a_n|_p^{1/n} = \lim_{n\to\infty} p^{-v_p(a_n)/n} = r_N.$$

By Lemma 2.5,

$$\varphi_c^{-1}(z) = z\left(1 + \sum_{n=1}^{\infty} \frac{a'_n}{z^{nd}}\right),$$

where

$$a'_n = -\sum_{\sum_{k=1}^n km_k = n} \left(\prod_{j=2}^{\sum_{k=1}^n m_k} (nd-j) \prod_{k=1}^{n} \frac{a_k^{m_k}}{m_k!} \right).$$

By Proposition 3.5, $v_p(a'_n) \geqslant v_p(a_n)$ for any $n \geqslant 1$. Now we want to show that $v_p(a'_n) = v_p(a_n)$ for infinitely many n, which will then imply

$$\liminf_{n\to\infty} \frac{v_p(a'_n)}{n} = \liminf_{n\to\infty} \frac{v_p(a_n)}{n}$$

and the maximal convergent open disks of $\varphi_c(z)$ and $\varphi_c^{-1}(z)$ are the same. We claim that if n is a power of p and $m_n = 0$, then

$$v_p\left(\prod_{j=2}^{\sum_{k=1}^n m_k} (nd-j) \prod_{k=1}^{n} \frac{a_k^{m_k}}{m_k!} \right) > v_p(a_n).$$

Suppose not, then by Propositions 3.3 and 3.5,

$$0 = v_p\left(\prod_{j=2}^{\sum_{k=1}^{n} m_k}(nd-j)\prod_{k=1}^{n}\frac{a_k^{m_k}}{m_k!}\right) - v_p(a_n)$$

$$= \sum_{j=2}^{\sum_{k=1}^{n} m_k} v_p(nd-j) + \left(v_p(n!) - \sum_{k=1}^{n} v_p(k!^{m_k} m_k!)\right) + \left(e(n) - \sum_{k=1}^{n} m_k e(k)\right),$$

where $e(n)$ is given by (3.6). Therefore, we have

$$0 = (p-1)\left(v_p(n!) - \sum_{k=1}^{n} v_p(k!^{m_k} m_k!)\right)$$

$$= n - s_p(n) - \sum_{k=1}^{n}\left(m_k(k - s_p(k)) + m_k - s_p(m_k)\right) \text{ by Lemma 2.2,}$$

$$= \sum_{k=1}^{n}\left(m_k(s_p(k)-1) + s_p(m_k)\right) - 1 \text{ as } n \text{ is a power of } p.$$

It follows that there is exactly one $m_{k_0} \neq 0$ and $n = k_0 m_{k_0}$. If $m_n = 0$, then $m_{k_0} \geqslant p$ and

$$\sum_{j=2}^{\sum_{k=1}^{n} m_k} v_p(nd-j) \geqslant v_p(nd - m_{k_0}) > 0.$$

This is a contradiction, from which we conclude that $v_p(a_n') = v_p(a_n)$ if n is a power of p. Thus, the first assertion is proved. For the second assertion, we note that

$$\frac{\varphi_c(z) - \varphi_c(w)}{z - w} = 1 - \sum_{n=1}^{\infty}\sum_{i=1}^{nd-1}\frac{a_n}{z^i w^{nd-i}}.$$

If $z, w \in D(\infty, r_N^{1/d})$, then by Proposition 3.4, we have

$$\left|\frac{a_n}{z^i w^{nd-i}}\right|_p < \frac{|a_n|_p}{r_N^n} = \left(\frac{p^{-v_p(a_n)/n}}{\lim_{n\to\infty} p^{-v_p(a_n)/n}}\right)^n < 1.$$

Therefore, $|\varphi_c(z) - \varphi_c(w)|_p = |z - w|_p$ on $D(\infty, r_N^{1/d})$. □

4 | PROOF OF THEOREM 1.3

In this section, we focus on the properties of b_n and give the proof of Theorem 1.3. In addition to Proposition 3.5, we need two more inequalities.

Proposition 4.1. *Let p, N, d, and c satisfy Condition A or B. Then*

(1) *if* $d \mid n$*, then* $v_p(da_n) \leqslant v_p(a_{n/d})$*;*
(2) *if* $1 \leqslant i < n/d$*, then* $v_p(da_n) < v_p(a_i c^{n-id})$*.*

Proof. If $d \mid n$, we let

$$n/d = \sum_{k=0}^{N} m_k d^k \qquad \text{and} \qquad n = \sum_{k=0}^{N-2} m_k d^{k+1} + (m_{N-1} + m_N d)d^N$$

be the canonical partitions (3.3) of n/d and n. Then we have

$$\begin{aligned}
v_p(da_n) &= v_p(d) + \sum_{k=0}^{N-2} v_p\left(\frac{a_{d^{k+1}}^{m_k}}{m_k!}\right) + v_p\left(\frac{a_{d^N}^{m_{N-1}+m_N d}}{(m_{N-1}+m_N d)!}\right) \text{ by Proposition 3.3,}\\
&= v_p(d) + \sum_{k=0}^{N-2} v_p\left(\frac{a_{d^k}^{m_k}}{d^{m_k} m_k!}\right) + v_p\left(\frac{a_{d^{N-1}}^{m_{N-1}} a_{d^N}^{m_N d}}{d^{m_{N-1}}(m_{N-1}+m_N d)!}\right) \text{ by Proposition 3.3,}\\
&= v_p(d) + \sum_{k=0}^{N} v_p\left(\frac{a_{d^k}^{m_k}}{d^{m_k} m_k!}\right) + m_N v_p(a_{d^N}^{d-1} d) - v_p\left(\frac{(m_{N-1}+m_N d)!}{m_{N-1}! m_N!}\right)\\
&\leqslant v_p(d) + \sum_{k=0}^{N} v_p\left(\frac{a_{d^k}^{m_k}}{d^{m_k} m_k!}\right) + m_N v_p(a_{d^N}^{d-1} d) - m_N v_p(d!) \text{ by Lemmas 2.1 and 2.4,}\\
&= v_p(a_{n/d}) + \left(1 - \sum_{k=0}^{N} m_k\right) v_p(d) + m_N\left((d-1)v_p\left(\frac{c}{d^{N+1}}\right) - v_p((d-1)!)\right)\\
&\leqslant v_p(a_{n/d}) \text{ by (1.5) and the right-hand side of (1.6).}
\end{aligned}$$

If $1 \leqslant i < n/d$, then

$$\begin{aligned}
v_p(a_i c^{n-id}) &\geqslant v_p(da_{id} c^{n-id}) \text{ by part (1),}\\
&= v_p(d) + v_p\left(\frac{c^n}{d^{id}(id)!}\right) - e(id) \text{ by Proposition 3.3 and (3.6),}\\
&> v_p(d) + v_p\left(\frac{c^n}{d^n n!}\right) - e(n) \text{ as } id < n,\\
&= v_p(da_n) \text{ by Proposition 3.3.}
\end{aligned}$$

□

As mentioned in the introduction, we can consider (1.4) as a perturbation of the simpler Equation (1.3). Now we show that the perturbation is insignificant in the following sense.

Proposition 4.2. *Let p, N, d, and c satisfy Condition A or B. Then $v_p(b_n) = v_p(a_n)$ for any $n \geqslant 1$.*

Proof. We use induction to show that $v_p(a_n - b_n) > v_p(a_n)$, which will then imply $v_p(b_n) = v_p(a_n)$. Considering the degree n terms of (1.3) and (1.4), we have

$$da_n + \sum_{\substack{\sum_{k=0}^{n-1} m_k = d \\ \sum_{k=0}^{n-1} km_k = n}} \binom{d}{m_0, m_1, \dots, m_{n-1}} \prod_{k=1}^{n-1} a_k^{m_k} = \begin{cases} a_{n/d}, & \text{if } d \mid n, \\ 0, & \text{if } d \nmid n, \end{cases}$$

and

$$db_n + \sum_{\substack{\sum_{k=0}^{n-1} m_k = d \\ \sum_{k=0}^{n-1} km_k = n}} \binom{d}{m_0, m_1, \dots, m_{n-1}} \prod_{k=1}^{n-1} b_k^{m_k} = \sum_{id \leqslant n} q(n,i) b_i c^{n-id}, \tag{4.1}$$

where $q(n,i) \in \mathbb{Z}$ and $q(n, n/d) = 1$ if $d \mid n$. Therefore,

$$d(a_n - b_n) + \sum_{\substack{\sum_{k=0}^{n-1} m_k = d \\ \sum_{k=0}^{n-1} km_k = n}} \binom{d}{m_0, m_1, \dots, m_{n-1}} \left(\prod_{k=1}^{n-1} a_k^{m_k} - \prod_{k=1}^{n-1} b_k^{m_k} \right)$$
$$= \begin{cases} a_{n/d} - b_{n/d} - \sum_{id<n} q(n,i) b_i c^{n-id}, & \text{if } d \mid n, \\ -\sum_{id<n} q(n,i) b_i c^{n-id}, & \text{if } d \nmid n. \end{cases}$$

By the induction hypothesis and Proposition 4.1, we have

$$v_p(a_{n/d} - b_{n/d}) > v_p(a_{n/d}) \geqslant v_p(da_n)$$

and

$$v_p(q(n,i) b_i c^{n-id}) \geqslant v_p(a_i c^{n-id}) > v_p(da_n). \tag{4.2}$$

By the induction hypothesis and Proposition 3.5, we have

$$v_p\left(\binom{d}{m_0, m_1, \dots, m_{n-1}} \left(\prod_{k=1}^{n-1} a_k^{m_k} - \prod_{k=1}^{n-1} b_k^{m_k} \right) \right)$$
$$= v_p\left(\binom{d}{m_0, m_1, \dots, m_{n-1}} \left(\prod_{k=1}^{n-1} a_k^{m_k} - \prod_{k=1}^{n-1} (a_k - (a_k - b_k))^{m_k} \right) \right)$$
$$> v_p\left(\binom{d}{m_0, m_1, \dots, m_{n-1}} \prod_{k=1}^{n-1} a_k^{m_k} \right) = v_p\left(\frac{d!}{m_0!} \prod_{k=1}^{n-1} \frac{a_k^{m_k}}{m_k!} \right) \geqslant v_p(da_n).$$

Combining these inequalities together, we get $v_p(a_n - b_n) > v_p(a_n)$ and $v_p(b_n) = v_p(a_n)$. □

A consequence of Proposition 4.2 is that Propositions 3.3, 3.4, and 3.5 remain true if we replace a_n by b_n. Therefore, the proof of Theorem 1.3 is essentially the same as the proof of Theorem 1.2.

5 | PROOF OF THEOREM 1.7

In this section, we give the proof of Theorem 1.7.

If $v_p(c_1) \geqslant 0$ and $\Phi_{c_1,c_2} : B(c_1) = D(\infty, 1) \to B(c_2)$ exists, then Φ_{c_1,c_2} must be of the form

$$\Phi_{c_1,c_2,\omega}(z) = \omega z \left(1 + \sum_{n=1}^{\infty} \frac{t_n}{z^{nd}} \right)$$

for some ω with $\omega^{d-1} = 1$. Let $x = z^{-d}$, then (1.8) can be simplified as

$$\begin{aligned}\left(1 + \sum_{n=1}^{\infty} t_n x^n \right)^d &= 1 + (\omega^{-1}c_2 - c_1)x + \sum_{n=1}^{\infty} \frac{t_n x^{nd}}{(1 - c_1 x)^{nd-1}} \\ &= 1 + (\omega^{-1}c_2 - c_1)x + \sum_{n=d}^{\infty} \sum_{id \leqslant n} q'(n,i) t_i c_1^{n-id} x^n,\end{aligned}$$

where $q'(n,i) \in \mathbb{Z}$ and $q'(n, n/d) = 1$ if $d \mid n$. We can imitate the proof of Proposition 4.2 to prove the following proposition.

Proposition 5.1. *Let p, N, d, and $c = \omega^{-1}c_2 - c_1$ satisfy Condition A or B. Let c_1 satisfy $v_p(c_1) \geqslant 0$ and $v_p(c_1) \geqslant v_p(c)$, then*

(1) *we have $v_p(t_n) = v_p(a_n)$ for any $n \geqslant 1$;*
(2) *the maximal convergent open disks of $\Phi_{c_1,c_2,\omega}(z)$ and $\Phi^{-1}_{c_1,c_2,\omega}(z)$ are both $D(\infty, r_N^{1/d})$, moreover, $\Phi_{c_1,c_2,\omega}(z)$ gives a bijective isometry from $D(\infty, r_N^{1/d})$ onto itself;*
(3) *$\Phi_{c_1,c_2,\omega}(z)$ does not give an analytic conjugacy between $B(c_1)$ and $B(c_2)$.*

Proof. The proof of part (1) is essentially the same as the proof of Proposition 4.2, except that we need to replace b_n by t_n, replace (4.1) by

$$dt_n + \sum_{\substack{\sum_{k=0}^{n-1} m_k = d \\ \sum_{k=0}^{n-1} k m_k = n}} \binom{d}{m_0, m_1, \dots, m_{n-1}} \prod_{k=1}^{n-1} t_k^{m_k} = \sum_{id \leqslant n} q'(n,i) t_i c_1^{n-id},$$

and replace (4.2) by

$$v_p(q'(n,i) t_i c_1^{n-id}) \geqslant v_p(a_i c^{n-id}) > v_p(d a_n).$$

The proof of part (2) is essentially the same as the proof of Theorem 1.2. Part (3) follows because $D(\infty, r_N^{1/d})$ is strictly smaller than $B(c_1) = D(\infty, 1)$. □

Now we are ready to give the proof of Theorem 1.7.

Proof of Theorem 1.7. By (1.9), we have $v_p(c_1) \geqslant v_p(c_2) = v_p(\omega^{-1}c_2 - c_1)$ for any ω with $\omega^{d-1} = 1$. By Proposition 5.1, none of $\Phi_{c_1,c_2,\omega}(z)$ gives an analytic conjugacy between $B(c_1)$ and $B(c_2)$. □

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