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Matthias Paulsen

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ON THE RATIONALITY OF QUADRIC SURFACE BUNDLES

by Matthias PAULSEN

Abstract. — For any standard quadric surface bundle over $\mathbb{P}^2$, we show that the locus of rational fibres is dense in the moduli space.

Résumé. — Pour tout faisceau de surface quadrique standard sur $\mathbb{P}^2$, nous montrons que le lieu des fibres rationnelles est dense dans l'espace des modules.

1. Introduction

In [14], Hassett, Pirutka, and Tschinkel gave the first example of a family $\mathcal{X} \to B$ of smooth complex projective varieties such that for a very general $b \in B$, the fibre $\mathcal{X}_b$ is not stably rational, while the locus of $b \in B$ where $\mathcal{X}_b$ is rational is dense in B for the Euclidean topology. Specifically, they considered the family of smooth complex hypersurfaces in $\mathbb{P}^2 \times \mathbb{P}^3$ defined by a homogeneous polynomial of bidegree $(2,2)$. Their result is remarkable as it shows that rationality of the fibres is in general not a closed property on the base. In particular, rationality is not deformation invariant in smooth families.

In order to prove stable irrationality of a very general member, they used the specialization method of Voisin [26] and Colliot-Thélène–Pirutka [7], which allowed to disprove stable rationality in several other families as well, see e. g. [27] for an overview.

Subsequently, other smooth families containing both rational and stably irrational fibres were identified, for example in [1, 12, 13, 15, 18, 19]. Typically, it is easy to provide certain rational members in the studied families.

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However, this does not exclude that the locus of rational fibres is contained in a proper closed subset of the base. In only a few cases, it was shown that the locus of rational fibres is dense in the moduli space.

The fourfolds considered in [14] and [15] are (birational to) quadric surface bundles over $\mathbb{P}^2$ of types $(2,2,2,2)$ and $(0,2,2,4)$, respectively. Here, a quadric surface bundle of type (d_0,d_1,d_2,d_3) for integers $d_0,d_1,d_2,d_3 \geqslant 0$ of the same parity is given by an equation of the form

$$\sum_{0\leqslant i,j\leqslant 3} a_{ij} y_i y_j = 0 \tag{1.1}$$

where $a_{ij}=a_{ji}$ is a homogeneous polynomial of degree $\frac{1}{2}(d_i+d_j)$ in the three coordinates of $\mathbb{P}^2$ and y_0,y_1,y_2,y_3 denote local trivializations of a split vector bundle $\mathcal{E}$ on $\mathbb{P}^2$ of rank 4, see Section 3 for a more precise definition. The quadric surface bundle $X \subset \mathbb{P}(\mathcal{E})$ over $\mathbb{P}^2$ defined by equation (1.1) is also called a *standard quadric surface bundle*. Apart from the examples in [14] and [15], many other fourfolds are birational to standard quadric surface bundles. For instance, a hypersurface in $\mathbb{P}^5$ of degree $d+2$ with multiplicity d along a plane for some integer $d \geqslant 1$ is birational to a quadric surface bundle of type $(d,d,d,d+2)$, see e. g. [19, Lemma 23].

The smooth quadric surface bundles of fixed type (d_0,d_1,d_2,d_3) are parametrized by a non-empty Zariski open subset $B \subset \mathbb{P}(V)$ in the projectivization of the complex vector space

$$V = \bigoplus_{0\leqslant i\leqslant j\leqslant 3} H^0\left(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}\left(\tfrac{1}{2}(d_i+d_j)\right)\right). \tag{1.2}$$

We may then consider the universal family $\mathcal{X} \to B$ of smooth quadric surface bundles over $\mathbb{P}^2$ of type (d_0,d_1,d_2,d_3).

Using his improvement [19] of the specialization method, Schreieder proved in [18] that a very general quadric surface bundle of type (d_0,d_1,d_2,d_3) is not stably rational except for the two cases $(1,1,1,3)$ and $(0,2,2,2)$ (up to reordering) which remain open and for trivial cases where the quadric surface bundle always has a rational section and is hence rational. This vastly generalizes the irrationality results of [14] and [15] to a natural class of families of quadric surface bundles over $\mathbb{P}^2$.

The aim of this article is to prove the corresponding density assertion for any standard quadric surface bundle over $\mathbb{P}^2$, thus showing that in this large class of families the locus of rational fibres is never contained in a proper closed subset of the moduli space. Concretely, we will prove the following:

Theorem 1.1. — *Let $d_0, d_1, d_2, d_3 \geqslant 0$ be integers of the same parity and let $\mathcal{X} \to B \subset \mathbb{P}(V)$ be the family of smooth quadric surface bundles over $\mathbb{P}^2$ of type (d_0, d_1, d_2, d_3) as above. Then the set*

$$\{b \in B \mid \mathcal{X}_b \text{ is rational}\}$$

is dense in B for the Euclidean topology.

The first case where such a density result was proven was for type $(0, 2, 2, 4)$ and is due to Voisin [25, Section 2], see also [19, Proposition 25]. The case of type $(2, 2, 2, 2)$ was shown in [14]. In particular, Theorem 1.1 generalizes their density result to hypersurfaces in $\mathbb{P}^2 \times \mathbb{P}^3$ of bidegree $(d, 2)$ for arbitrary $d \geqslant 0$. Our result also gives an affirmative answer to the question raised in [19, Remark 49].

In order to prove Theorem 1.1, we follow Voisin's approach that has later been used in [14, Section 6] and [13, Section 2.3]. Using a theorem of Springer [20] and the fact that the integral Hodge conjecture is known in codimension two for quadric bundles over surfaces [8, Corollaire 8.2], we obtain a Hodge theoretic criterion guaranteeing the rationality of smooth quadric surface bundles over $\mathbb{P}^2$. This leads to the study of a Noether–Lefschetz locus in the variation of Hodge structure associated to the family $\mathcal{X} \to B$ in question. In [23, Proposition 5.20], Voisin stated an infinitesimal condition for the density of such loci, based on Green's proof in [5, Section 5] of an analogous density result in the context of the Noether–Lefschetz theorem. In our case, the criterion asks for a class $\overline{\lambda} \in H^{2,2}_{\mathrm{van}}(\mathcal{X}_b)$ at some base point $b \in B$ such that the infinitesimal period map evaluated at $\overline{\lambda}$

$$\overline{\nabla}_b(\overline{\lambda})\colon T_{B,b} \to H^{1,3}_{\mathrm{van}}(\mathcal{X}_b)$$

is surjective.

Since a standard quadric surface bundle over $\mathbb{P}^2$ is a toric variety, we can apply [2, Theorem 10.13] to describe $\overline{\nabla}_b(\overline{\lambda})$ as a multiplication map in a homogeneous quotient of a bigraded polynomial ring. Therefore, the desired density result reduces to an elementary statement about polynomials. This problem was solved in [14] and [13] with explicit computations. Of course, a different technique is required to handle a whole class of families rather than a specific one. The main contribution of this paper consists thus in solving this problem to which Theorem 1.1 reduces to via general arguments. An important ingredient of our proof is a result about the strong Lefschetz property of certain complete intersections which was proven in [11, Proposition 30].

Green's and Voisin's infinitesimal density criterion has been employed in many different situations since its first use in [5, Section 5]. For instance, Voisin used it in [24] when proving the integral Hodge conjecture for $(2,2)$-classes on uniruled or Calabi–Yau threefolds. More recently, a real analogue of the criterion was applied in [3] to prove that sums of three squares are dense among bivariate positive semidefinite real polynomials.

There exist different strategies for verifying the surjectivity of the infinitesimal period map. While [3] follows the approach of [6] by constructing components of the Noether–Lefschetz locus of maximal codimension, Kim gave in [16, Theorem 2] a new proof of the density theorem from [5, Section 5] by proving a statement about the Jacobian rings appearing in the description of $\overline{\nabla}_b(\overline{\lambda})$. The most general arguments are due to Voisin, for example in [22] and [24].

We use the method of computing the infinitesimal period map explicitly, as done in [16]. However, we solve the underlying algebraic problem in a different manner than in [16, Section 3]. Our approach involving the strong Lefschetz property, the use of which seems to be new in this area, further allows to give a short proof for the density of the original Noether–Lefschetz locus for surfaces in $\mathbb{P}^3$, thus simplifying the arguments of [16] considerably.

The article is structured as follows. In Section 2, we relate the rationality of smooth quadric surface bundles over $\mathbb{P}^2$ to the cohomology group $H^{2,2}$ and explain how Green's and Voisin's infinitesimal density criterion applies in our situation. In Section 3, we interpret standard quadric surface bundles as toric hypersurfaces in order to give an explicit representation of $\overline{\nabla}_b(\overline{\lambda})$. This cumulates in Proposition 3.1, where we formulate a non-trivial statement concerning a bigraded polynomial ring which is sufficient for showing Theorem 1.1. In Section 4, we provide some tools for studying the surjectivity of polynomial multiplication maps and demonstrate their power by giving a simple proof for the density of the classical Noether–Lefschetz locus. Finally, in Section 5 we use the previous preparations in order to prove Proposition 3.1, from which our main result follows.

Unless otherwise stated, we always work over the field of complex numbers. A variety is defined to be an integral separated scheme of finite type over a field. A quadric surface bundle over $\mathbb{P}^2$ is a complex projective variety X together with a flat morphism $\pi\colon X \to \mathbb{P}^2$ such that the generic fibre X_η is a smooth quadric surface over the function field $\mathbb{C}(\mathbb{P}^2)$. If X is a smooth complex projective variety and $Z \subset X$ is a subvariety of codimension k, we denote by $[Z] \in H^{k,k}(X,\mathbb{Z})$ the Poincaré dual of the homology class of Z.

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2. A Density Criterion

Let us consider a smooth quadric surface bundle $\pi\colon X \to \mathbb{P}^2$. Since $\mathbb{P}^2$ is rational, X is rational (over $\mathbb{C}$) as soon as the generic fibre X_η is rational over the function field $k = \mathbb{C}(\mathbb{P}^2)$. It is well known that this follows from the existence of a k-point on the smooth quadric surface X_η. Now we can use the following theorem of Springer [20]:

Proposition 2.1 (Springer). — *Let Q be a quadric hypersurface over a field k and let K/k be a finite field extension of odd degree. If Q has a K-point, then Q has a k-point.*

It therefore suffices to find a K-point on X_η for some field extension K/k of odd degree. This can be achieved through an odd degree multisection of π, i. e. a surface $Z \subset X$ such that $[Z] \cup \pi^*[p] \in H^{4,4}(X, \mathbb{Z}) \cong \mathbb{Z}$ is odd where $[p] \in H^{2,2}(\mathbb{P}^2, \mathbb{Z}) \cong \mathbb{Z}$ denotes the cohomology class of a closed point, since the function field $K = \mathbb{C}(Z)$ is such a field extension then.

The integral Hodge conjecture was proven for $(2, 2)$-classes on quadric bundles over surfaces by Colliot-Thélène and Voisin [8, Corollaire 8.2]. We use the following special case:

Proposition 2.2 (Colliot-Thélène–Voisin). — *Let $\pi\colon X \to \mathbb{P}^2$ be a smooth quadric surface bundle. Then the integral Hodge conjecture holds for $H^{2,2}(X, \mathbb{Z})$, i. e. any integral Hodge class $\alpha \in H^{2,2}(X, \mathbb{Z})$ is an integral linear combination $\alpha = \sum n_i[Z_i]$ for surfaces $Z_i \subset X$.*

This allows us to transform the assertion of π having an odd degree multisection into a Hodge theoretic condition (see also [14, Proposition 6]):

Corollary 2.3. — *Let $\pi\colon X \to \mathbb{P}^2$ be a smooth quadric surface bundle. Then X is rational if there exists an integral Hodge class $\alpha \in H^{2,2}(X, \mathbb{Z})$ such that $\alpha \cup \pi^*[p]$ is odd.*

Now let us consider the family $\mathcal{X} \to B$ of smooth quadric surface bundles over $\mathbb{P}^2$ of type (d_0, d_1, d_2, d_3) for fixed integers $d_j \geqslant 0$ of the same parity.

In order to prove Theorem 1.1, it is enough by Corollary 2.3 to show that the Noether–Lefschetz locus

$$\{b \in B \mid \exists\, \alpha \in H^{2,2}(\mathcal{X}_b, \mathbb{Z}) \colon \alpha \cup \pi_b^*[p] \equiv 1 \pmod 2\}$$

is dense in B for the Euclidean topology, where $\pi_b \colon \mathcal{X}_b \to \mathbb{P}^2$ denotes the quadric bundle structure on the fibre $\mathcal{X}_b$.

Since it is easier to compute, we consider instead the *vanishing cohomology*

$$H^4_{\mathrm{van}}(\mathcal{X}_b, \mathbb{C}) = \{\alpha \in H^4(\mathcal{X}_b, \mathbb{C}) \mid \alpha \cup \iota^*\beta = 0 \;\; \forall\, \beta \in H^4(\mathbb{P}(\mathcal{E}), \mathbb{C})\}$$

where the map $\iota^* \colon H^4(\mathbb{P}(\mathcal{E}), \mathbb{C}) \hookrightarrow H^4(\mathcal{X}_b, \mathbb{C})$ is induced by inclusion and is injective by the Lefschetz hyperplane theorem, provided that not all d_j are simultaneously zero(1). This construction is also applicable to the Hodge groups $H^{p,q}$ and gives a decomposition

$$H^4_{\mathrm{van}}(\mathcal{X}_b, \mathbb{C}) = \bigoplus_{p+q=4} H^{p,q}_{\mathrm{van}}(\mathcal{X}_b)\,.$$

We then want to show that the possibly smaller locus

$$\{b \in B \mid \exists\, \alpha \in H^{2,2}_{\mathrm{van}}(\mathcal{X}_b, \mathbb{Z}) \colon \alpha \cup \pi_b^*[p] \equiv 1 \pmod 2\} \tag{2.1}$$

is dense in B for the Euclidean topology. To achieve this, we utilise a variant of Voisin's description in [23, Proposition 5.20] of an infinitesimal density criterion due to Green [5, Section 5].

On B we consider the holomorphic vector bundle $\mathcal{H}$ with fibre $\mathcal{H}_b = H^4_{\mathrm{van}}(\mathcal{X}_b, \mathbb{C})$ at $b \in B$. By Ehresmann's lemma, $\mathcal{H}$ is trivial over any contractible open subset of B. The vector bundle $\mathcal{H}$ is flat with respect to the Gauß–Manin connection $\nabla \colon \mathcal{H} \to \mathcal{H} \otimes \Omega_B$. Since $H^{4,0}_{\mathrm{van}}(\mathcal{X}_b) = H^{0,4}_{\mathrm{van}}(\mathcal{X}_b) = 0$ for all $b \in B$, each fibre of $\mathcal{H}$ has a Hodge filtration of weight 2. It is well known that the Hodge filtration on the fibres of $\mathcal{H}$ induces a filtration

$$F^2\mathcal{H} \subset F^1\mathcal{H} \subset F^0\mathcal{H} = \mathcal{H}$$

by holomorphic subbundles. These satisfy Griffiths' transversality condition

$$\nabla\left(F^p\mathcal{H}^k\right) \subset F^{p-1}\mathcal{H}^k \otimes \Omega_B$$

for all p and hence ∇ gives rise to an $\mathcal{O}_B$-linear map

$$\overline{\nabla} \colon \mathcal{H}^{1,1} \to \mathcal{H}^{0,2} \otimes \Omega_B$$

(1) If $d_j = 0$ for all j, Theorem 1.1 is trivial because a quadric surface bundle of type $(0,0,0,0)$ is the product of $\mathbb{P}^2$ with a quadric surface in $\mathbb{P}^3$ and hence rational.

on the quotients $\mathcal{H}^{p,2-p} = F^p\mathcal{H}/F^{p+1}\mathcal{H}$. Fibrewise, we obtain by adjunction the infinitesimal period map

$$\overline{\nabla}_b \colon T_{B,b} \to \operatorname{Hom}\big(\mathcal{H}_b^{1,1}, \mathcal{H}_b^{0,2}\big)$$

for all $b \in B$. Note that we may identify $\mathcal{H}_b^{p,q}$ with $H_{\mathrm{van}}^{p+1,q+1}(\mathcal{X}_b)$ for $p+q=2$.

Let $\mathcal{H}_{\mathbb{R}}$ be the real vector bundle on B with fibre $\mathcal{H}_{\mathbb{R},b} = H^4_{\mathrm{van}}(\mathcal{X}_b, \mathbb{R})$ at $b \in B$. Then we have $\mathcal{H}_b = \mathcal{H}_{\mathbb{R},b} \otimes_{\mathbb{R}} \mathbb{C}$ for all $b \in B$. Similarly, for the real vector subbundle

$$\mathcal{H}_{\mathbb{R}}^{1,1} = \mathcal{H}_{\mathbb{R}} \cap F^1\mathcal{H} \subset \mathcal{H}_{\mathbb{R}}$$

with fibre $\mathcal{H}_{\mathbb{R},b}^{1,1} = H_{\mathrm{van}}^{2,2}(\mathcal{X}_b, \mathbb{R})$ at $b \in B$ we have $\mathcal{H}_b^{1,1} \cong \mathcal{H}_{\mathbb{R},b}^{1,1} \otimes_{\mathbb{R}} \mathbb{C}$ for all $b \in B$. The last identification is given by the restricted projection

$$p \colon \mathcal{H}_{\mathbb{R}}^{1,1} \subset F^1\mathcal{H} \to F^1\mathcal{H}/F^2\mathcal{H} = \mathcal{H}^{1,1}\,.$$

For all $b \in B$, let us consider the discrete subset

$$D\mathcal{H}_b = \{\alpha \in H^4_{\mathrm{van}}(\mathcal{X}_b, \mathbb{Z}) \mid \alpha \cup \pi_b^*[p] \equiv 1 \pmod 2\} \subset \mathcal{H}_{\mathbb{R},b}\,.$$

Since $D\mathcal{H}_b$ is defined by a topological property of $\mathcal{X}_b$ which is compatible with the local trivializations of $\mathcal{X}$ from Ehresmann's lemma (it does in particular not depend on the Hodge filtration on $\mathcal{H}_b$), we obtain a fibre subbundle $D\mathcal{H} \subset \mathcal{H}_{\mathbb{R}}$ which is trivial over any contractible open subset of B. Note that the locus (2.1) is precisely the image of the projection map $D\mathcal{H} \cap \mathcal{H}_{\mathbb{R}}^{1,1} \to B$. Our variant of [23, Proposition 5.20] can now be stated as follows:

Proposition 2.4 (Green–Voisin). — *Suppose there exists $b \in B$ and $\overline{\lambda} \in \mathcal{H}_b^{1,1}$ such that the infinitesimal period map evaluated at $\overline{\lambda}$*

$$\overline{\nabla}_b(\overline{\lambda}) \colon T_{B,b} \to \mathcal{H}_b^{0,2}$$

is surjective. Then the projection of $D\mathcal{H} \cap \mathcal{H}_{\mathbb{R}}^{1,1}$ is dense in B for the Euclidean topology.

Proof. — We first observe that the surjectivity condition is a Zariski open property on $\overline{\lambda} \in \mathcal{H}^{1,1} = \mathcal{H}_{\mathbb{R}}^{1,1} \otimes_{\mathbb{R}} \mathbb{C}$. Hence, the condition is fulfilled on a dense open subset of the real classes $p(\mathcal{H}_{\mathbb{R}}^{1,1}) \subset \mathcal{H}^{1,1}$. Therefore, it suffices to show the statement locally around $b \in B$ where $\overline{\lambda} = p(\lambda)$ satisfies the hypothesis for some $\lambda \in \mathcal{H}_{\mathbb{R},b}^{1,1}$. By shrinking B, we may assume that the vector bundle $\mathcal{H}_{\mathbb{R}}$ is trivial over B, i. e. $\mathcal{H}_{\mathbb{R}} \cong B \times \mathcal{H}_{\mathbb{R},b}$. By [23, Lemma 5.21], the composed map

$$\phi \colon \mathcal{H}_{\mathbb{R}}^{1,1} \hookrightarrow \mathcal{H}_{\mathbb{R}} \cong B \times \mathcal{H}_{\mathbb{R},b} \to \mathcal{H}_{\mathbb{R},b}$$

obtained via inclusion, isomorphism and projection is a submersion at $\lambda \in \mathcal{H}^{1,1}_{\mathbb{R}}$. As shown in [19, Lemma 20], there are smooth quadric surface bundles $\mathcal{X}_u$ of type (d_0, d_1, d_2, d_3) which admit a rational section and hence $D\mathcal{H}_u \neq \emptyset$. Since B is connected, it follows that $D\mathcal{H}_b \neq \emptyset$. By definition, $D\mathcal{H}_b$ is a coset of a subgroup of $H^4_{\mathrm{van}}(\mathcal{X}_b, \mathbb{Z})$ of index 2. Therefore, $\mathbb{R}^* D\mathcal{H}_b$ is dense in $\mathcal{H}_{\mathbb{R},b} = H^4_{\mathrm{van}}(\mathcal{X}_b, \mathbb{Z}) \otimes \mathbb{R}$. Since ϕ is a submersion, the preimage $\phi^{-1}(\mathbb{R}^* D\mathcal{H}_b)$ is dense around $\lambda \in \mathcal{H}^{1,1}_{\mathbb{R}}$. But this precisely means $(\mathbb{R}^* D\mathcal{H}) \cap \mathcal{H}^{1,1}_{\mathbb{R}}$ is dense in $\mathcal{H}^{1,1}_{\mathbb{R}}$ around λ. Hence, its projection is dense around $b \in B$. But the projections of $D\mathcal{H} \cap \mathcal{H}^{1,1}_{\mathbb{R}}$ and $(\mathbb{R}^* D\mathcal{H}) \cap \mathcal{H}^{1,1}_{\mathbb{R}}$ agree because $\mathcal{H}^{1,1}_{\mathbb{R}}$ is a real vector bundle. □

Actually, the above proof works for any fibre bundle $D\mathcal{H} \subset \mathcal{H}_{\mathbb{R}}$, trivial over contractible open subsets of B, such that $\mathbb{R}^* D\mathcal{H}_b$ is dense in $\mathcal{H}_{\mathbb{R},b}$ for some $b \in B$. This leads to a more general version of Proposition 2.4, which can be found in [17, Section 3.3].

3. Computation of the Cohomology

We first give a more precise definition of standard quadric surface bundles over $\mathbb{P}^2$, following [19, Section 3.5]. Let

$$\mathcal{E} = \bigoplus_{j=0}^{3} \mathcal{O}_{\mathbb{P}^2}(-r_j)$$

be a split vector bundle on $\mathbb{P}^2$ for integers $r_j \geqslant 0$ and let $q\colon \mathcal{E} \to \mathcal{O}_{\mathbb{P}^2}(d)$ be a quadratic form for some integer $d \geqslant 0$, i. e. a global section of $\mathrm{Sym}^2 \mathcal{E}^\vee \otimes \mathcal{O}_{\mathbb{P}^2}(d)$. Let us assume that the quadratic form q_η at the generic point $\eta \in \mathbb{P}^2$ is non-degenerate and that $q_s \neq 0$ for all $s \in \mathbb{P}^2$. Then the zero set $X \subset \mathbb{P}(\mathcal{E})$ of q is a quadric surface bundle over $\mathbb{P}^2$. Since the vector bundle $\mathrm{Sym}^2 \mathcal{E}^\vee \otimes \mathcal{O}_{\mathbb{P}^2}(d)$ only depends on the integers $d_j = 2r_j + d$ for $j \in \{0, 1, 2, 3\}$, we call X a standard quadric surface bundle of type (d_0, d_1, d_2, d_3). Conversely, quadric surface bundles of type (d_0, d_1, d_2, d_3) for given integers $d_j \geqslant 0$ exist whenever d_0, d_1, d_2, d_3 are of the same parity(2). Since

$$H^0\big(\mathbb{P}^2, \mathrm{Sym}^2 \mathcal{E}^\vee \otimes \mathcal{O}_{\mathbb{P}^2}(d)\big) \cong \bigoplus_{0 \leqslant i \leqslant j \leqslant 3} H^0\big(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(r_i + r_j + d)\big) = V$$

where V was defined in (1.2), X can be described by an equation of the form (1.1) where y_j is a local trivialization of $\mathcal{O}_{\mathbb{P}^2}(-r_j)$.

(2) One can always ensure $d \in \{0, 1\}$, but this is not needed in our arguments.

We now aim to interpret (1.1) differently as a global equation inside the polynomial ring

$$S = \mathbb{C}[x_0, x_1, x_2; y_0, y_1, y_2, y_3]$$

endowed with a non-standard bigrading. By [9, Example 7.3.5], the total space $\mathbb{P}(\mathcal{E})$ is a toric variety associated to a fan Σ in $\mathbb{R}^2 \times \mathbb{R}^3$ and has coordinate ring S. If u_1, u_2 and v_1, v_2, v_3 denote the standard basis vectors of $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively, then the seven 1-dimensional cones of Σ are generated by $u_0, u_1, u_2, v_0, v_1, v_2, v_3$ where

$$u_0 = -\sum_{k=1}^{2} u_k + \sum_{j=1}^{3} (r_j - r_0) v_j \quad \text{and} \quad v_0 = -\sum_{j=1}^{3} v_j \,.$$

Further, the maximal cones of Σ are given by

$$\langle u_0, \dots, \widehat{u}_k, \dots, u_2, v_0, \dots, \widehat{v}_j, \dots, v_3 \rangle \,, \quad k \in \{0,1,2\} \,, \quad j \in \{0,1,2,3\} \,.$$

By [2, Definition 1.7], we have $\mathrm{Cl}(\Sigma) \cong \mathbb{Z}^7 / \operatorname{Im} C$ where

$$C = \begin{pmatrix} -1 & -1 & r_1 - r_0 & r_2 - r_0 & r_3 - r_0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{Hom}(\mathbb{Z}^5, \mathbb{Z}^7) \,.$$

It is easy to see that the surjection

$$\begin{aligned} \mathbb{Z}^7 &\to \mathbb{Z}^2 \\ (m_0, m_1, m_2, n_0, n_1, n_2, n_3) &\mapsto \left(\sum_{k=0}^{2} m_k - \sum_{j=0}^{3} n_j r_j, \sum_{j=0}^{3} n_j \right) \end{aligned}$$

has kernel $\operatorname{Im} C$. Hence, this map descends to an isomorphism $\mathrm{Cl}(\Sigma) \cong \mathbb{Z}^2$ and endowes the coordinate ring S with the non-standard bigrading

$$\deg x_k = (1, 0) \,, \quad \deg y_j = (-r_j, 1)$$

for $k \in \{0,1,2\}$ and $j \in \{0,1,2,3\}$. For $m, n \in \mathbb{Z}$, we denote by $S(m,n)$ the subspace of homogeneous polynomials of bidegree (m,n) in S. This gives a decomposition

$$S = \bigoplus_{m,n \in \mathbb{Z}} S(m,n)$$

into finite dimensional $\mathbb{C}$-vector spaces.

A quadratic form $q\colon \mathcal{E} \to \mathcal{O}_{\mathbb{P}^2}(d)$ corresponds to an element in $S(d,2)$. In this way, the local description (1.1) of the zero set of q can be seen globally as a defining equation for a toric hypersurface $X \subset \mathbb{P}(\mathcal{E})$.

This allows us to compute the middle cohomology groups of a smooth quadric surface bundle $\pi\colon X \to \mathbb{P}^2$ of type (d_0,d_1,d_2,d_3) defined by a polynomial $f \in S(d,2)$ via the method of [2, Theorem 10.13], which generalizes the work of Griffiths [10] to toric hypersurfaces. We have canonical isomorphisms

$$H^{1,3}_{\mathrm{van}}(X) \cong R(t,4) \quad \text{and} \quad H^{2,2}_{\mathrm{van}}(X) \cong R(t-d,2)$$

where

$$t = 4d - 3 + r_0 + r_1 + r_2 + r_3$$

and where R denotes the Jacobian ring of f, i. e. the quotient of S by all partial derivatives of f.

Now we return to the family $\mathcal{X} \to B$ of smooth quadric surface bundles of type (d_0,d_1,d_2,d_3). If we identify $T_{B,b} \cong (S/fS)(d,2)$ where $f \in S(d,2)$ is the defining equation of $\mathcal{X}_b$ for some $b \in B$, then the infinitesimal period map

$$\overline{\nabla}_b\colon T_{B,b} \otimes H^{2,2}_{\mathrm{van}}(\mathcal{X}_b) \to H^{1,3}_{\mathrm{van}}(\mathcal{X}_b)$$

is given, up to a sign, as the multiplication map

$$(S/fS)(d,2) \otimes R(t-d,2) \to R(t,4)\,.$$

This was first shown for hypersurfaces in projective space by Carlson and Griffiths [4], see also [23, Theorem 6.13]. In order to show that the assumption of Proposition 2.4 holds and thus to prove Theorem 1.1, it therefore suffices to provide polynomials $f \in S(d,2)$ and $g \in S(t-d,2)$ such that the quadric surface bundle $\{f=0\} \subset \mathbb{P}(\mathcal{E})$ is smooth and the composed map $S(d,2) \to R(t,4)$ given by multiplication with g followed by projection is surjective. By Bertini's theorem, the hypersurface $\{f=0\} \subset \mathbb{P}(\mathcal{E})$ is smooth for a general polynomial $f \in S(d,2)$. The surjectivity part is equivalent to claiming that the ideal generated by g and all partial derivatives of f contains all polynomials in $S(t,4)$. Consequently, we reduced Theorem 1.1 to the following statement:

Proposition 3.1. — *For general polynomials $f \in S(d,2)$ and $g \in S(t-d,2)$, the ideal in S generated by the polynomials*

$$\frac{\partial f}{\partial x_0}, \quad \frac{\partial f}{\partial x_1}, \quad \frac{\partial f}{\partial x_2}, \quad \frac{\partial f}{\partial y_0}, \quad \frac{\partial f}{\partial y_1}, \quad \frac{\partial f}{\partial y_2}, \quad \frac{\partial f}{\partial y_3}, \quad g$$

contains all polynomials in $S(t,4)$.

The remaining part of the paper is devoted to the proof of this proposition.

4. Preparations

The property that a homogeneous ideal in a bigraded polynomial ring (or more generally, an arbitrarily graded $\mathbb{C}$-algebra) contains all polynomials of a certain bidegree is, as we now show, a Zariski open condition on its generators if their bidegrees are fixed.

Lemma 4.1. — *Let G be an Abelian group and let A be a G-graded $\mathbb{C}$-algebra whose homogeneous components $A(m)$ are finite dimensional $\mathbb{C}$-vector spaces for all $m \in G$. Let $m_0, \ldots, m_k \in G$. Then the set*

$$\{(f_1, \ldots, f_k) \in A(m_1) \oplus \cdots \oplus A(m_k) \mid A(m_0) \subset f_1 A + \cdots + f_k A\}$$

is Zariski open.

Proof. — The condition on $(f_1, \ldots, f_k)$ is equivalent to saying that the $\mathbb{C}$-linear map

$$\begin{aligned} A(m_0 - m_1) \oplus \cdots \oplus A(m_0 - m_k) &\to A(m_0) \\ (g_1, \ldots, g_k) &\mapsto f_1 g_1 + \cdots + f_k g_k \end{aligned}$$

is surjective. This map is represented by a matrix B with $r = \dim_{\mathbb{C}} A(m_0)$ rows, whose entries are linear polynomials in the coefficients of $f_1, \ldots, f_k$. The locus in $A(m_1) \oplus \cdots \oplus A(m_k)$ where this linear map is not surjective is precisely where the determinants of all $(r \times r)$-submatrices of B vanish (in particular, it is the whole affine space if B has less than r columns) and thus Zariski closed. Therefore, the set in question is open for the Zariski topology. □

Since taking partial derivatives is a linear and hence Zariski continuous map between the respective $\mathbb{Z}^2$-graded pieces of S, Lemma 4.1 shows that the desired condition in Proposition 3.1 is Zariski open on f and g.

Apart from S, we will often apply Lemma 4.1 to the polynomial ring $\mathbb{C}[x_0, x_1, x_2]$ together with its usual grading. In this situation, we can give sufficient criteria whether three or four polynomials satisfy the Zariski open condition in the lemma. More generally, for $n \geqslant 0$ we can give such criteria for $n + 1$ and $n + 2$ polynomials in the graded polynomial ring

$$P_n = \mathbb{C}[x_0, \ldots, x_n] = \bigoplus_{m \geqslant 0} P_n(m)\,.$$

Lemma 4.2. — *If $f_0, \dots, f_n \in P_n$ form a complete intersection, i. e. they have no common zero in $\mathbb{P}^n$, then*

$$P_n(m) \subset f_0 P_n + \cdots + f_n P_n$$

for all $m \geqslant m_0 + \cdots + m_n - n$ where $f_j \in P_n(m_j)$ for $j \in \{0, \dots, n\}$.

Proof. — This immediately follows from Macaulay's Theorem (see for example [23, Section 6.2.2]) which tells us that the quotient of P_n by the ideal generated by $f_0, \dots, f_n$ is a graded Gorenstein ring with socle degree $\sum(m_j - 1)$, and hence its m-th graded piece is zero-dimensional for all $m \geqslant \sum m_j - n$. □

To state a sufficient criterion whether $n+2$ polynomials in P_n belong to the Zariski open set in Lemma 4.1, we use the so called *strong Lefschetz property*, see e. g. [21]. A quotient Q of P_n by homogeneous polynomials $f_0, \dots, f_n \in P_n$ is said to have the strong Lefschetz property if there exists a linear homogeneous polynomial $\ell \in P_n(1)$ such that the map $Q(m) \to Q(m+i)$ given by multiplication with ℓ^i has maximal rank for all $m, i \geqslant 0$. The polynomial ℓ is then called a *strong Lefschetz element* for the system $f_0, \dots, f_n$.

Lemma 4.3. — *If $f_0, \dots, f_n \in P_n$ form a complete intersection having the strong Lefschetz property and $f_{n+1} \in P_n$ is a power of a strong Lefschetz element for $f_0, \dots, f_n$, then*

$$P_n(m) \subset f_0 P_n + \cdots + f_{n+1} P_n$$

for all $m \geqslant \frac{1}{2}(m_0 + \cdots + m_{n+1} - n - 1)$ where $f_j \in P_n(m_j)$ for $j \in \{0, \dots, n+1\}$.

Proof. — As in Lemma 4.2, the quotient Q of P_n by $f_0, \dots, f_n$ is a graded Gorenstein ring with socle degree $s = \sum(m_j - 1)$. Macaulay's Theorem also shows that $\dim_{\mathbb{C}} Q(i) = \dim_{\mathbb{C}} Q(s-i)$ for all $i \in \mathbb{Z}$. Because of the strong Lefschetz property, $\dim_{\mathbb{C}} Q(i)$ needs to be increasing for $i \leqslant \frac{s}{2}$ and decreasing for $i \geqslant \frac{s}{2}$. The claimed statement is equivalent to saying that the map $Q(m - m_{n+1}) \to Q(m)$ given by multiplication with f_{n+1} is surjective. Since f_{n+1} is a power of a strong Lefschetz element, it suffices to show $\dim_{\mathbb{C}} Q(m - m_{n+1}) \geqslant \dim_{\mathbb{C}} Q(m)$. This is clear if $m - m_{n+1} \geqslant \frac{s}{2}$. For $m - m_{n+1} \leqslant \frac{s}{2}$, we have $\dim_{\mathbb{C}} Q(m) = \dim_{\mathbb{C}} Q(s-m) \leqslant \dim_{\mathbb{C}} Q(m - m_{n+1})$ because $s - m \leqslant m - m_{n+1}$ holds due to the given bound on m. □

To make use of Lemma 4.3, it is convenient to have a rich source of complete intersections enjoying the strong Lefschetz property. The following important result, proved in 1980 by Stanley [21] and independently in

1987 by Watanabe [28], was the starting point for the theory of Lefschetz properties:

Proposition 4.4 (Stanley–Watanabe). — *A monomial complete intersection $x_0^{m_0}, \dots, x_n^{m_n}$ in P_n with $m_0, \dots, m_n \geqslant 0$ has the strong Lefschetz property for all $n \geqslant 0$.*

Stanley's proof goes as follows: If we interpret the graded quotient $Q = \bigoplus_{m \geqslant 0} Q(m)$ of P_n by the monomials $x_0^{m_0}, \dots, x_n^{m_n}$ as the cohomology ring in even degree $H^{2\bullet}(X, \mathbb{C})$ of the Kähler manifold $X = \mathbb{P}^{m_0-1} \times \dots \times \mathbb{P}^{m_n-1}$, the linear polynomial $\ell = x_1 + \dots + x_n$ corresponds to the cohomology class of a Kähler form on X and the strong Lefschetz property for ℓ precisely translates into the hard Lefschetz theorem for X, hence also the name of this condition.

It is known for $n \leqslant 1$ and conjectured for $n \geqslant 2$ that actually all complete intersections in P_n have the strong Lefschetz property. For $n = 2$, the following partial result proven in [11, Proposition 30] satisfies our needs for the proof of Proposition 3.1:

Proposition 4.5 (Harima–Watanabe). — *If f_0, f_1, $f_2 \in \mathbb{C}[x_0, x_1, x_2]$ form a complete intersection such that f_0 is a power of a linear polynomial, then f_0, f_1, f_2 has the strong Lefschetz property.*

As a motivating example, we show how Lemmas 4.2 and 4.3 can be used to give a short proof for the density of the classical Noether–Lefschetz locus for surfaces in $\mathbb{P}^3$. For this, we do not need Proposition 4.5, but only the earlier result stated in Proposition 4.4. Since the setup here is a lot easier than in the case of standard quadric surface bundles, this will also be a good preparation for the more involved arguments in Section 5.

Theorem 4.6 (Ciliberto–Harris–Miranda–Green). — *For $d \geqslant 4$, let $\mathcal{X} \to B \subset \mathbb{P}(P_3(d))$ be the universal family of smooth surfaces in $\mathbb{P}^3$ of degree d. Then the Noether–Lefschetz locus*

$$\{b \in B \,|\, \mathrm{Pic}(\mathcal{X}_b) \supsetneq \mathbb{Z} \cdot \mathcal{O}_{\mathbb{P}^3}(1)|_{\mathcal{X}_b}\} = \left\{b \in B \,\middle|\, H^{1,1}_{\mathrm{van}}(\mathcal{X}_b, \mathbb{Z}) \neq 0\right\} ,$$

i. e. those surfaces containing curves which are no complete intersections, is dense in B for the Euclidean topology.

Proof. — By Green's and Voisin's infinitesimal density criterion, it suffices to show that there exists a point $b \in B$ and a class $\overline{\lambda} \in H^{1,1}_{\mathrm{van}}(\mathcal{X}_b)$ such that

$$\overline{\nabla}_b(\overline{\lambda}) \colon T_{B,b} \to H^{0,2}_{\mathrm{van}}(\mathcal{X}_b)$$

is surjective. For a surface $X \subset \mathbb{P}^3$ defined by a polynomial $f \in P_3(d)$, Griffiths [10] has shown that

$$H^{0,2}_{\mathrm{van}}(X) \cong R(3d-4) \quad \text{and} \quad H^{1,1}_{\mathrm{van}}(X) \cong R(2d-4)$$

where R denotes the Jacobian ring of f, i.e. the quotient of P_3 by the partial derivatives of f. If we identify $T_{B,b} \cong (P_3/fP_3)(d)$ where $f \in P_3(d)$ is the defining equation of $\mathcal{X}_b$ for some $b \in B$, Carlson and Griffiths [4] proved that the infinitesimal period map

$$\overline{\nabla}_b \colon T_{B,b} \otimes H^{1,1}_{\mathrm{van}}(\mathcal{X}_b) \to H^{0,2}_{\mathrm{van}}(\mathcal{X}_b)$$

is given, up to a sign, as the multiplication map

$$(P_3/fP_3)(d) \otimes R(2d-4) \to R(3d-4)\,.$$

Therefore, it suffices to find polynomials $f \in P_3(d)$ and $g \in P_3(2d-4)$ such that the surface $\{f = 0\} \subset \mathbb{P}^3$ is smooth and the ideal generated by g and the partial derivatives of f contains the whole of $P_3(3d-4)$.

One can achieve this with the smooth Fermat surface defined by

$$f = x_0^d + x_1^d + x_2^d + x_3^d\,,$$

which was also used in [16, Section 3]. Since the complete intersection consisting of the partial derivatives of f has the strong Lefschetz property by Proposition 4.4, we can take g to be a power of a corresponding strong Lefschetz element and obtain via Lemma 4.3

$$P_3(m) \subset x_0^{d-1}P_3 + x_1^{d-1}P_3 + x_2^{d-1}P_3 + x_3^{d-1}P_3 + gP_3$$

for all $m \geqslant \frac{1}{2}(4(d-1) + 2d - 4 - 4) = 3d - 6$. Since $3d - 4 \geqslant 3d - 6$, this finishes the proof. $\square$

5. Proof of Proposition 3.1

Without loss of generality, let $r_0 \leqslant r_1 \leqslant r_2 \leqslant r_3$. Let us recall from Section 3 that $d_j = 2r_j + d$ for $j \in \{0,1,2,3\}$ and $t = 4d - 3 + \sum r_j$. By Lemma 4.1, the property stated in Proposition 3.1 is Zariski open on f and g. Hence, it suffices to show the existence of polynomials $f \in S(d,2)$ and $g \in S(t-d,2)$ such that the homogeneous ideal $I \subset S$ generated by

$$\frac{\partial f}{\partial x_0}, \quad \frac{\partial f}{\partial x_1}, \quad \frac{\partial f}{\partial x_2}, \quad \frac{\partial f}{\partial y_0}, \quad \frac{\partial f}{\partial y_1}, \quad \frac{\partial f}{\partial y_2}, \quad \frac{\partial f}{\partial y_3}, \quad g$$

contains all polynomials in $S(t,4)$. Let

$$f = f_0 y_0^2 + f_1 y_1^2 + f_2 y_2^2 + f_3 y_3^2 \in S(d,2)$$

where $f_j \in S(d_j, 0)$ are general for $j \in \{0,1,2,3\}$. Further let

$$g = g_{11}y_1^2 + g_{33}y_3^2 + \sum_{0\leqslant i<j\leqslant 3} g_{ij}y_iy_j \in S(t-d,2)$$

where $g_{ij} \in S(t-d+r_i+r_j, 0)$ are general for $i,j \in \{0,1,2,3\}$. Instead of proving directly that $S(t,4) \subset I$, we will consider the homogeneous ideal

$$J = \bigoplus_{m,n\in\mathbb{Z}} \{r \in S(m,n) \mid rS \cap S(t,4) \subset I\},$$

and aim to show $J = S$. One can think of J as all relations which hold if a polynomial of bidegree $(t,4)$ is considered modulo I. Since $I \subset J$, the following congruences hold:

$$f_jy_j \equiv 0 \pmod{J}, \quad j \in \{0,1,2,3\}, \tag{5.1}$$

$$\frac{\partial f_0}{\partial x_k}y_0^2 + \frac{\partial f_1}{\partial x_k}y_1^2 + \frac{\partial f_2}{\partial x_k}y_2^2 + \frac{\partial f_3}{\partial x_k}y_3^2 \equiv 0 \pmod{J}, \quad k \in \{0,1,2\}, \tag{5.2}$$

$$g_{11}y_1^2 + g_{33}y_3^2 + \sum_{0\leqslant i<j\leqslant 3} g_{ij}y_iy_j \equiv 0 \pmod{J}. \tag{5.3}$$

It suffices to show $S(t,4) \subset J$. For this it is enough to prove the following four claims for all permutations σ of $\{0,1,2,3\}$:

$$y_{\sigma(0)}y_{\sigma(1)}y_{\sigma(2)} \in J, \quad y_{\sigma(0)}^3y_{\sigma(1)} \in J, \quad y_{\sigma(0)}^2y_{\sigma(1)}^2 \in J, \quad y_{\sigma(0)}^4 \in J.$$

The proof of each of these claims will constitute one of the four steps 5.1, 5.2, 5.3, and 5.4 below. In each step, it suffices to show that any monomial of bidegree $(t,4)$ containing the specified variables y_j can be reduced to 0 modulo J using the congruences (5.1), (5.2), (5.3), and the previous steps. In fact, the assertion $r_0 \leqslant r_1 \leqslant r_2 \leqslant r_3$ and the congruence (5.3) will not be used in the first two steps, so we are allowed to restrict ourselves to the case $\sigma = \mathrm{id}$ in these two steps.

5.1. First step

We have $y_{\sigma(0)}y_{\sigma(1)}y_{\sigma(2)} \in J$ for all permutations σ of $\{0,1,2,3\}$.

Proof. — Without loss of generality, let $\sigma = \mathrm{id}$. We first note that

$$S(d_0+d_1+d_2-2, 0) \subset f_0S + f_1S + f_2S. \tag{5.4}$$

This follows from Lemmas 4.1 and 4.2 because there are complete intersections f_0, f_1, f_2 in

$$P_2 = \mathbb{C}[x_0,x_1,x_2] = \bigoplus_{m\geqslant 0} S(m,0).$$

Now let us take a monomial $hy_0y_1y_2y_j \in S(t,4)$ where $j \in \{0,1,2,3\}$ and $h \in S(t+r_0+r_1+r_2+r_j,0)$. We may assume that $r_j > 0$ or $d > 0$, since for $d_j = 2r_j+d = 0$ we have $y_j \equiv 0 \pmod J$ by (5.1) and hence $hy_0y_1y_2y_j \equiv 0 \pmod J$. In view of (5.1) and (5.4), it suffices to show that

$$t + r_0 + r_1 + r_2 + r_j \geqslant d_0 + d_1 + d_2 - 2\,.$$

This is equivalent to

$$2r_0 + 2r_1 + 2r_2 + r_3 + r_j + 4d - 3 \geqslant 2r_0 + 2r_1 + 2r_2 + 3d - 2$$

or just $r_3 + r_j + d \geqslant 1$, which is true because $r_j > 0$ or $d > 0$. $\square$

5.2. Second step

We have $y^3_{\sigma(0)}y_{\sigma(1)} \in J$ for all permutations σ of $\{0,1,2,3\}$.

Proof. — Without loss of generality, let $\sigma = \mathrm{id}$. Multiplying (5.2) with y_0y_1 and using step 5.1 yields

$$\left(\frac{\partial f_0}{\partial x_k}y_0^2 + \frac{\partial f_1}{\partial x_k}y_1^2\right) y_0y_1 \equiv 0 \pmod J\,, \quad k \in \{0,1,2\}\,. \tag{5.5}$$

We introduce the new polynomial ring $T = \mathbb{C}[x_0,x_1,x_2;z_0,z_1]$ with the bigrading

$$\deg x_k = (1,0)\,, \quad \deg z_j = (-d_j,1)$$

for $k \in \{0,1,2\}$ and $j \in \{0,1\}$.

Claim. — We have

$$\begin{aligned} T(d_0+d_1-3,1) \\ \subset f_0T + f_1T + \left(\frac{\partial f_0}{\partial x_0}z_0 + \frac{\partial f_1}{\partial x_0}z_1\right)T + \left(\frac{\partial f_0}{\partial x_1}z_0 + \frac{\partial f_1}{\partial x_1}z_1\right)T\,. \end{aligned} \tag{5.6}$$

Proof of the claim. — The claim is true if $d_0 = 0$ or $d_1 = 0$ because f_0 or f_1 is a unit then. If $d_0, d_1 > 0$, setting $f_0 = (x_0+x_1)^{d_0} + x_2^{d_0}$ and $f_1 = (x_0-x_1)^{d_1} + x_2^{d_1}$ yields

$$\left(\frac{\partial f_0}{\partial x_0}z_0 + \frac{\partial f_1}{\partial x_0}z_1\right) + \left(\frac{\partial f_0}{\partial x_1}z_0 + \frac{\partial f_1}{\partial x_1}z_1\right) = 2d_0(x_0+x_1)^{d_0-1}z_0\,.$$

Since $(x_0+x_1)^{d_0-1}, f_0, f_1$ form a complete intersection in

$$P_2 = \mathbb{C}[x_0,x_1,x_2] = \bigoplus_{m\geqslant 0} T(m,0)\,,$$

Lemma 4.2 implies that the right-hand side of (5.6) contains all polynomials in $T(d_0+d_1-3,1)$ of type hz_0 where $h \in T(2d_0+d_1-3,0)$. Similarly,

$$\left(\frac{\partial f_0}{\partial x_0}z_0+\frac{\partial f_1}{\partial x_0}z_1\right)-\left(\frac{\partial f_0}{\partial x_1}z_0+\frac{\partial f_1}{\partial x_1}z_1\right)=2d_1(x_0-x_1)^{d_1-1}z_1$$

and $(x_0-x_1)^{d_1-1}, f_0, f_1$ are again a complete intersection, so all polynomials in $T(d_0+d_1-3,1)$ divisible by z_1 are contained in the right-hand side of (5.6) as well. Hence, the claim follows from Lemma 4.1 applied the polynomial ring T, since the coefficients of the four polynomials which are supposed to generate $T(d_0+d_1-3,1)$ depend linearly and thus Zariski continuously on those of the general polynomials f_0 and f_1. □

Now let us take a monomial $hy_0^3y_1 \in S(t,4)$ where $h \in S(t+3r_0+r_1,0)$. We have

$$t+3r_0+r_1 = 4r_0+2r_1+r_2+r_3+4d-3 \geqslant 4r_0+2r_1+3d-3 = 2d_0+d_1-3\,.$$

Therefore, as a consequence of (5.6) we obtain

$$hz_0 = h_0f_0+h_1f_1+h_2\left(\frac{\partial f_0}{\partial x_0}z_0+\frac{\partial f_1}{\partial x_0}z_1\right)+h_3\left(\frac{\partial f_0}{\partial x_1}z_0+\frac{\partial f_1}{\partial x_1}z_1\right)$$

for certain homogeneous polynomials $h_0,h_1,h_2,h_3 \in T$. Substituting z_j by y_j^2 for $j \in \{0,1\}$ and multiplying with y_0y_1, we get by (5.1) and (5.5)

$$\begin{aligned}
hy_0^3y_1 &= \widetilde{h}_0f_0y_0y_1+\widetilde{h}_1f_1y_0y_1\\
&\qquad + h_2\left(\frac{\partial f_0}{\partial x_0}y_0^2+\frac{\partial f_1}{\partial x_0}y_1^2\right)y_0y_1+h_3\left(\frac{\partial f_0}{\partial x_0}y_0^2+\frac{\partial f_1}{\partial x_0}y_1^2\right)y_0y_1\\
&\equiv \widetilde{h}_0y_1\cdot 0+\widetilde{h}_1y_0\cdot 0+h_2\cdot 0+h_3\cdot 0\equiv 0 \pmod J
\end{aligned}$$

where $\widetilde{h}_0$ and $\widetilde{h}_1$ denote the results of the substitution inside h_0 and h_1. □

5.3. Third step

We have $y_{\sigma(0)}^2y_{\sigma(1)}^2 \in J$ for all permutations σ of $\{0,1,2,3\}$.

Proof. — Multiplying (5.3) with y_iy_j for $0 \leqslant i < j \leqslant 3$ and using the previous steps, we obtain

$$g_{ij}y_i^2y_j^2 \equiv 0 \pmod J\,. \tag{5.7}$$

For the following definition, we assume $d_0>0$ at first. For $j \in \{0,1,2,3\}$, let $\widehat{A}_j$ be the (3×3)-matrix where we leave out the j-th column (counted from 0) of the matrix

$$\left(\frac{\partial f_j}{\partial x_k}\right)_{\substack{k\in\{0,1,2\}\\ j\in\{0,1,2,3\}}}.$$

A straightforward calculation shows that (5.2) implies

$$\big(\det \widehat{A}_j\big) y_i^2 \equiv \varepsilon_{ij}\big(\det \widehat{A}_i\big) y_j^2 \pmod J\,, \quad i,j \in \{0,1,2,3\} \tag{5.8}$$

where $\det \widehat{A}_j \in S(d_0+d_1+d_2+d_3-d_j-3,0)$ for $j \in \{0,1,2,3\}$ and $\varepsilon_{ij} \in \{\pm 1\}$ is a sign depending on $i,j \in \{0,1,2,3\}$.

For $d_0 = 0$, both sides of (5.8) would be zero since $\frac{\partial f_0}{\partial x_k} = 0$ for $k \in \{0,1,2\}$. Therefore, in the case $d_0 = 0$ we define the matrix $\widehat{A}_j$ for $j \in \{1,2,3\}$ to be the (2×2)-matrix where one leaves out the j-th column (counted from 1) of the matrix

$$\left(\frac{\partial f_j}{\partial x_k}\right)_{\substack{k \in \{0,1\} \\ j \in \{1,2,3\}}}.$$

Because (5.1) implies $y_0 \equiv 0 \pmod J$ in this case, one can still conclude from (5.2) that

$$\big(\det \widehat{A}_j\big) y_i^2 \equiv \varepsilon_{ij}\big(\det \widehat{A}_i\big) y_j^2 \pmod J\,, \quad i,j \in \{1,2,3\} \tag{5.9}$$

where $\det \widehat{A}_j \in S(d_1+d_2+d_3-d_j-2,0)$ for $j \in \{1,2,3\}$ and $\varepsilon_{ij} \in \{\pm 1\}$ may be different for $i,j \in \{1,2,3\}$.

Let us first suppose that $\{\sigma(0),\sigma(1)\} = \{1,2\}$. Multiplying (5.3) with y_2^2 and using steps 5.1 and 5.2 yields

$$g_{11} y_1^2 y_2^2 + g_{33} y_2^2 y_3^2 \equiv 0 \pmod J\,. \tag{5.10}$$

Let us consider the polynomial ring $U = \mathbb{C}[x_0,x_1,x_2;z_1,z_3]$ with the bigrading

$$\deg x_k = (1,0)\,, \quad \deg z_j = (-d_j,1)$$

for $k \in \{0,1,2\}$ and $j \in \{1,3\}$. We claim that

$$U(t-d+2r_2,1) \subset K\,, \tag{5.11}$$

where K denotes the ideal in U generated by

$$f_1 z_1\,, \quad f_2\,, \quad f_3 z_3\,, \quad g_{12} z_1\,, \quad g_{23} z_3\,, \quad g_{11} z_1 + g_{33} z_3\,,$$
$$\big(\det \widehat{A}_3\big) z_1 - \varepsilon_{13}\big(\det \widehat{A}_1\big) z_3\,.$$

Since the coefficients of these seven polynomials in U depend algebraically on those of $f_0, f_1, f_2, f_3, g_{11}, g_{12}, g_{23}, g_{33}$, Lemma 4.1 with $A = U$ shows that it is enough to provide a special choice for the general polynomials $f_j, g_{ij} \in \mathbb{C}[x_0,x_1,x_2]$ making (5.11) true.

Claim. — This can be achieved in the following way, where $\mu, \nu \in U(1,0)$ denote suitable strong Lefschetz elements of complete intersections that will be specified later:

$$\begin{aligned} f_0 &= x_0^{d_0} & g_{11} &= x_2^{t-d+2r_1} \\ f_1 &= x_0^{d_1} & g_{12} &= \nu^{t-d+r_1+r_2} \\ f_2 &= x_0^{d_2} + x_1^{d_2} & g_{23} &= \mu^{t-d+r_2+r_3} \\ f_3 &= x_0^{d_3} + x_2^{d_3} & g_{33} &= \mu^{t-d+2r_3} \end{aligned}$$

Proof of the claim. — The claim is obvious for $d_2 = 0$, so we may assume $d_2 > 0$ in the following. As in the case of the ideal I, we consider instead the larger homogeneous ideal

$$L = \bigoplus_{m,n\in\mathbb{Z}} \{r \in U(m,n) \mid rU \cap U(t-d+2r_2,1) \subset K\}$$

and we want to show that $U(t-d+2r_2,1) \subset L$ (or equivalently, $L = U$). This will be done by proving first $z_1 \in L$ and then $z_3 \in L$. Since $K \subset L$, we have

$$0 \equiv g_{11}z_1 + g_{33}z_3 = g_{11}z_1 + \mu^{r_3-r_2} g_{23}z_3 \equiv g_{11}z_1 \pmod{L}.$$

By Proposition 4.5, the complete intersection f_1, f_2, g_{11} in $\mathbb{C}[x_0,x_1,x_2]$ possesses the strong Lefschetz property. We may thus assume that ν is a strong Lefschetz element for f_1, f_2, g_{11}. Lemma 4.3 then implies

$$z_1U(m,0) \subset f_1z_1U + f_2z_1U + g_{11}z_1U + g_{12}z_1U \subset L$$

for all $m \geqslant \frac{1}{2}(d_1+d_2+t-d+2r_1+t-d+r_1+r_2-3)$. In order to show $z_1 \in L$, we thus need to check that

$$2(t-d+2r_2+d_1) \geqslant d_1+d_2+t-d+2r_1+t-d+r_1+r_2-3\,.$$

This is equivalent to

$$4r_2 + 2d_1 \geqslant d_1 + d_2 + 3r_1 + r_2 - 3\,,$$

which simplifies to $r_2 \geqslant r_1 - 3$. The last inequality is obviously true.

Next we show $z_3 \in L$. If $d_0 > 0$, we have

$$\det \widehat{A}_1 = \det \begin{pmatrix} d_0x_0^{d_0-1} & d_2x_0^{d_2-1} & d_3x_0^{d_3-1} \\ 0 & d_2x_1^{d_2-1} & 0 \\ 0 & 0 & d_3x_2^{d_3-1} \end{pmatrix} = d_0d_2d_3x_0^{d_0-1}x_1^{d_2-1}x_2^{d_3-1}\,.$$

Together with $K \subset L$ and $z_1 \in L$, this implies

$$\begin{aligned} 0 &\equiv (d_0 d_2 d_3)^{-1} x_1 x_2 \big(\det \widehat{A}_1\big) z_3 \\ &= x_0^{d_0-1} x_1^{d_2} x_2^{d_3} z_3 \\ &\equiv x_0^{d_0+d_2+d_3-1} z_3 \pmod{L}. \end{aligned}$$

Similarly, for $d_0 = 0$ we have

$$\det \widehat{A}_1 = \det \begin{pmatrix} d_2 x_0^{d_2-1} & d_3 x_0^{d_3-1} \\ d_2 x_1^{d_2-1} & 0 \end{pmatrix} = -d_2 d_3 x_0^{d_3-1} x_1^{d_2-1}$$

and thus

$$0 \equiv (d_2 d_3)^{-1} x_1 \big(\det \widehat{A}_1\big) z_3 = -x_0^{d_3-1} x_1^{d_2} z_3 \equiv x_0^{d_0+d_2+d_3-1} z_3 \pmod{L}$$

as well. By Proposition 4.5, the complete intersection $x_0^{d_0+d_2+d_3-1}, f_2, f_3$ has the strong Lefschetz property. Hence, we may assume that μ is a strong Lefschetz element for $x_0^{d_0+d_2+d_3-1}, f_2, f_3$. Lemma 4.3 implies

$$z_3 U(m,0) \subset x_0^{d_0+d_2+d_3-1} z_3 U + f_2 z_3 U + f_3 z_3 U + g_{23} z_3 U \subset L$$

for all $m \geqslant \frac{1}{2}(d_0+d_2+d_3-1+d_2+d_3+t-d+r_2+r_3-3)$. It thus remains to check

$$2(t-d+2r_2+d_3) \geqslant d_0+d_2+d_3-1+d_2+d_3+t-d+r_2+r_3-3$$

or

$$2r_0+2r_1+6r_2+6r_3+8d-6 \geqslant 3r_0+r_1+6r_2+6r_3+8d-7.$$

This reduces to $r_1 \geqslant r_0 - 1$, which is clearly true. This finishes the proof of (5.11). □

Now let us take a monomial $hy_1^2 y_2^2 \in S(t,4)$ where $h \in S(t+2r_1+2r_2, 0)$. We have $hz_1 \in U(t-d+2r_2, 1)$ and thus

$$\begin{aligned} hz_1 = {} & h_1 f_1 z_1 + h_2 f_2 + h_3 f_3 z_3 + h_4 g_{12} z_1 + h_5 g_{23} z_3 \\ & + h_6 (g_{11} z_1 + g_{33} z_3) + h_7 \big(\big(\det \widehat{A}_3\big) z_1 - \varepsilon_{13} \big(\det \widehat{A}_1\big) z_3\big) \end{aligned}$$

for certain homogeneous polynomials $h_1, \ldots, h_7 \in U$. Substituting z_j by y_j^2 for $j \in \{1,3\}$ and multiplying with y_2^2, we get

$$\begin{aligned} hy_1^2 y_2^2 &= h_1 f_1 y_1^2 y_2^2 + \widetilde{h}_2 f_2 y_2^2 + h_3 f_3 y_2^2 y_3^2 + h_4 g_{12} y_1^2 y_2^2 + h_5 g_{23} y_2^2 y_3^2 \\ &\quad + h_6 \left(g_{11} y_1^2 y_2^2 + g_{33} y_2^2 y_3^2\right) + h_7 \big(\big(\det \widehat{A}_3\big) y_1^2 - \varepsilon_{13} \big(\det \widehat{A}_1\big) y_3^2\big) y_2^2 \\ &\equiv h_1 y_1 y_2^2 \cdot 0 + \widetilde{h}_2 y_2 \cdot 0 + h_3 y_2^2 y_3 \cdot 0 + h_4 \cdot 0 + h_5 \cdot 0 + h_6 \cdot 0 + h_7 y_2^2 \cdot 0 \\ &\equiv 0 \pmod{J} \end{aligned}$$

where we used the congruences (5.1), (5.7), (5.8), (5.9), and (5.10), and where $\widetilde{h}_2$ denotes the result of the substitution inside h_2. This concludes the proof of $y_1^2 y_2^2 \in J$.

At this point, we are ready to handle the general case of $\{\sigma(0), \sigma(1)\}$. For this, we show the following claim:

Claim. — If τ is a permutation of $\{0,1,2,3\}$ such that $\tau(3) < \tau(2)$, then any multiple of $y_{\tau(0)}^2 y_{\tau(1)}^2$ in $S(t,4)$ can be replaced modulo J by a multiple of $y_{\tau(0)}^2 y_{\tau(2)}^2$ in $S(t,4)$.

Proof of the claim. — In view of (5.1), (5.7), (5.8), and (5.9), it suffices to show that

$$S(t + 2r_{\tau(0)} + 2r_{\tau(1)}, 0) \subset f_{\tau(0)} S + f_{\tau(1)} S + g_{\tau(0)\tau(1)} S + \big(\det \widehat{A}_{\tau(2)}\big) S\,.$$

This will follow from Lemma 4.1 once we provide a special choice for the general polynomials $f_{\tau(0)}, f_{\tau(1)}, f_{\tau(3)}, g_{\tau(0)\tau(1)}$ satisfying this property. Let $a = d_{\tau(0)}$, $b = d_{\tau(1)}$, and $c = d_{\tau(3)}$. We may assume $a, b > 0$ because otherwise we would already have $y_{\tau(0)}^2 y_{\tau(1)}^2 \equiv 0 \pmod{J}$ by (5.1). We take

$$f_{\tau(0)} = x_0^a + x_1^a\,, \quad f_{\tau(1)} = x_0^b + x_2^b\,, \quad f_{\tau(3)} = x_0^c\,.$$

If also $c > 0$, we have

$$\det \widehat{A}_{\tau(2)} = \pm \det \begin{pmatrix} a x_0^{a-1} & b x_0^{b-1} & c x_0^{c-1} \\ a x_1^{a-1} & 0 & 0 \\ 0 & b x_2^{b-1} & 0 \end{pmatrix} = \pm abc x_0^{c-1} x_1^{a-1} x_2^{b-1}\,.$$

Therefore, we get

$$x_0^{a+b+c-1} \in f_{\tau(0)} S + f_{\tau(1)} S + \big(\det \widehat{A}_{\tau(2)}\big) S\,.$$

If $c = 0$, it follows that $d_0 = 0$. Since $a, b > 0$ and $\tau(3) < \tau(2)$, only $\tau(3) = 0$ is possible. Then we have

$$\det \widehat{A}_{\tau(2)} = \pm \det \begin{pmatrix} a x_0^{a-1} & b x_0^{b-1} \\ a x_1^{a-1} & 0 \end{pmatrix} = \mp ab x_0^{b-1} x_1^{a-1}$$

und thus again

$$x_0^{a+b+c-1} = x_0^{a+b-1} \in f_{\tau(0)} S + f_{\tau(1)} S + \big(\det \widehat{A}_{\tau(2)}\big) S\,.$$

In either case, the complete intersection $x_0^{a+b+c-1}, f_{\tau(0)}, f_{\tau(1)}$ has the strong Lefschetz property by Proposition 4.5, so we may pick for $g_{\tau(0)\tau(1)}$ an adequate power of a strong Lefschetz element and obtain via Lemma 4.3

$$S(m, 0) \subset f_{\tau(0)} S + f_{\tau(1)} S + g_{\tau(0)\tau(1)} S + \big(\det \widehat{A}_{\tau(2)}\big) S$$

for all $m \geqslant \frac{1}{2}(a+b+c-1+a+b+t-d+r_{\tau(0)}+r_{\tau(1)}-3)$. Therefore, it remains to prove that

$$2(t+2r_{\tau(0)}+2r_{\tau(1)}) \geqslant a+b+c-1+a+b+t-d+r_{\tau(0)}+r_{\tau(1)}-3\,.$$

This simplifies to

$$6r_{\tau(0)}+6r_{\tau(1)}+2r_{\tau(2)}+2r_{\tau(3)}+8d-6 \geqslant 6r_{\tau(0)}+6r_{\tau(1)}+r_{\tau(2)}+3r_{\tau(3)}+8d-7$$

or just $r_{\tau(2)} \geqslant r_{\tau(3)}-1$, which holds because $\tau(3)<\tau(2)$. □

With this result at hand, we proceed as follows: We start with a monomial of degree $(t,4)$ divisible by $y_{\sigma(0)}^2 y_{\sigma(1)}^2$ and repeatedly apply transitions of the form

$$y_{\tau(0)}^2 y_{\tau(1)}^2 \rightsquigarrow y_{\tau(0)}^2 y_{\tau(2)}^2$$

with $\tau(3)<\tau(2)$ for a suitable permutation τ until we arrive at a polynomial divisible by $y_1^2 y_2^2$, for which we have already shown that it vanishes modulo J. The fact that such a sequence of transitions always exists can be most easily seen from the following diagram:

$$\begin{tikzcd}
\{0,1\} \arrow[r, bend left, "2<3"] \arrow[d, bend right, "2<3"] & \{1,3\} \arrow[d, bend right, "0<2"] \arrow[dr, bend left, "0<2"] & \\
\{0,3\} \arrow[r, bend left, "1<2"] & \{2,3\} \arrow[r, bend left, "0<1"] & \{1,2\} \\
\{0,2\} \arrow[u, bend left, "1<3"] \arrow[ur, bend left, "1<3"'] & &
\end{tikzcd}$$

The arrows are labeled with the inequalities $\tau(3)<\tau(2)$ which hold for the employed permutations τ. For every possible subset $\{\sigma(0),\sigma(1)\} \subset \{0,1,2,3\}$, there exists at least one directed path ending in $\{1,2\}$. This completes the proof of step 5.3. □

5.4. Fourth step

We have $y_j^4 \in J$ for all $j \in \{0,1,2,3\}$.

Proof. — Let us take a monomial hy_j^4 where $h \in S(t+4r_j,0)$. If $d_j=0$, we are done by (5.1). Otherwise, multiplying (5.2) with y_j^2 and using step 5.3 produces

$$\frac{\partial f_j}{\partial x_k} y_j^4 \equiv 0 \pmod{J}\,, \quad k \in \{0,1,2\}\,.$$

First suppose $j < 3$. By Lemmas 4.1 and 4.2, we have

$$S(3d_j - 5, 0) \subset \frac{\partial f_j}{\partial x_0} S + \frac{\partial f_j}{\partial x_1} S + \frac{\partial f_j}{\partial x_2} S$$

since the partial derivatives of $f_j = x_0^{d_j} + x_1^{d_j} + x_2^{d_j}$ form a complete intersection. Therefore, it remains to show that $t + 4r_j \geqslant 3d_j - 5$. This is equivalent to

$$r_0 + r_1 + r_2 + r_3 + 4r_j + 4d - 3 \geqslant 6r_j + 3d - 5\,,$$

which in turn is equivalent to

$$r_0 + r_1 + r_2 + r_3 + d + 2 \geqslant 2r_j\,.$$

The last inequality is true because $j \leqslant 2$ implies $r_2 + r_3 \geqslant r_j + r_j$.

Now let $j = 3$. If we multiply (5.3) with y_3^2 and use all previous steps, we obtain

$$g_{33} y_3^4 \equiv 0 \pmod{J}\,.$$

We claim that

$$S(t + 4r_3, 0) \subset \frac{\partial f_3}{\partial x_0} S + \frac{\partial f_3}{\partial x_1} S + \frac{\partial f_3}{\partial x_2} S + g_{33} S\,.$$

By Lemma 4.1, it is enough to give one working example for f_3 and g_{33}. If we take again $f_3 = x_0^{d_3} + x_1^{d_3} + x_2^{d_3}$, the complete intersection given by the partial derivatives of f_3 has the strong Lefschetz property by Proposition 4.5, so we may choose for g_{33} a power of a strong Lefschetz element and obtain via Lemma 4.3 that

$$S(m, 0) \subset \frac{\partial f_3}{\partial x_0} S + \frac{\partial f_3}{\partial x_1} S + \frac{\partial f_3}{\partial x_2} S + g_{33} S$$

for all $m \geqslant \frac{1}{2}(3d_3 - 3 + t - d + 2r_3 - 3)$. Therefore, we are finished if

$$2(t + 4r_3) \geqslant 3d_3 - 3 + t - d + 2r_3 - 3\,.$$

This simplifies to

$$2r_0 + 2r_1 + 2r_2 + 10r_3 + 8d - 6 \geqslant r_0 + r_1 + r_2 + 9r_3 + 6d - 9\,,$$

or equivalently,

$$r_0 + r_1 + r_2 + r_3 + 2d + 3 \geqslant 0\,.$$

The last statement is clearly true. □

Since every monomial in $S(t, 4)$ is divisible by an element handled in one of the four steps above, we obtain $S(t, 4) \subset J$ as desired. This finally ends the proof of Proposition 3.1.

Remark 5.1. — It was crucial in the choice of g to leave out the terms g_{00} and g_{22}, i.e. the ones belonging to the smallest and second-largest values among the degrees d_0, d_1, d_2, d_3. With any other two indices, the above proof would not work. Furthermore, if we would also set $g_{33} = 0$, the proof of step 5.3 would be much simpler, but then step 5.4 would work out only if $d_3 \leqslant d_0 + d_1 + d_2 + 4$. And if we would instead set $g_{11} = 0$, step 5.4 could be left untouched, but step 5.3, though it would be simpler, would turn out right only if $d_3 \leqslant d_2 + 6$. It is also worth to mention that the properties of J we are proving in each of the four steps are in general *not* open on the polynomials f_j and g_{ij}, thus an argument where one specializes to $g_{33} = 0$ in one step but not in another one does not succeed.

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Matthias PAULSEN
Institut für Algebraische Geometrie
Leibniz Universität Hannover
Welfengarten 1
D-30167 Hannover (Germany)
paulsen@math.uni-hannover.de