0-Hecke Algebras of the Symmetric Groups

Centers and Modules Associated to Quasisymmetric Schur Functions

Von der Fakultät für Mathematik und Physik

der Gottfried Wilhelm Leibniz Universität Hannover

zur Erlangung des akademischen Grades

Doktor der Naturwissenschaften

Dr. rer. nat.

genehmigte Dissertation von

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2021

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Tag der Promotion	15. September 2021

Abstract

The 0-Hecke algebra $H_n(0)$ over a field $\mathbb{K}$ is a deformation of the group algebra $\mathbb{K}\mathfrak{S}_n$ of the symmetric group $\mathfrak{S}_n$. In this thesis we study the center of $H_n(0)$ and $H_n(0)$-modules that are associated to quasisymmetric Schur functions.

The quasisymmetric Schur functions $\mathcal{S}_\alpha$ are analogues of the Schur functions in the algebra of quasisymmetric functions *QSym*. An algebra isomorphism from the Grothendieck groups of the finitely generated modules of the 0-Hecke algebras $H_n(0)$ to *QSym* is given by the quasisymmetric characteristic *Ch*. Tewari and van Willigenburg constructed $H_n(0)$-modules $\boldsymbol{S}_\alpha$ that are mapped to the quasisymmetric Schur functions $\mathcal{S}_\alpha$ by *Ch*. Moreover, they used an equivalence relation in order to decompose $\boldsymbol{S}_\alpha$ into submodules $\boldsymbol{S}_\alpha = \bigoplus_E \boldsymbol{S}_{\alpha,E}$. Analogously, they defined and decomposed skew modules $\boldsymbol{S}_{\alpha /\!\!/ \beta}$ corresponding to skew quasisymmetric Schur functions $\mathcal{S}_{\alpha /\!\!/ \beta}$.

In Chapter 3 we consider these modules. We show that the modules $\boldsymbol{S}_{\alpha,E}$ are indecomposable. The skew modules $\boldsymbol{S}_{\alpha /\!\!/ \beta,E}$ on the other hand can be decomposable. For a certain family of skew modules $\boldsymbol{S}_{\alpha /\!\!/ \beta}$, which we call *pacific*, we describe a decomposition into indecomposable submodules. From this we obtain combinatorial formulas for top and socle of the pacific modules. These formulas are then generalized to all skew modules $\boldsymbol{S}_{\alpha /\!\!/ \beta,E}$. This includes the straight modules $\boldsymbol{S}_{\alpha,E}$. We close the chapter by discussing how the results on the modules $\boldsymbol{S}_{\alpha,E}$ can be transferred to permuted versions of them which were also introduced by Tewari and van Willigenburg.

Chapter 4 is concerned with the center $Z(H_n(0))$ of $H_n(0)$. A $\mathbb{K}$-basis of $Z(H_n(0))$ was defined by He. This basis is given by certain equivalence classes ${}^{(\mathfrak{S}_n)_{\max}}\!/_{\approx}$ of $\mathfrak{S}_n$. For $\Sigma \in {}^{(\mathfrak{S}_n)_{\max}}\!/_{\approx}$ the basis element $\bar{\pi}_{\leq \Sigma}$ indexed by Σ corresponds to the order ideal in Bruhat order generated by Σ. We provide two sets of representatives of ${}^{(\mathfrak{S}_n)_{\max}}\!/_{\approx}$ and obtain a parametrization of the elements of ${}^{(\mathfrak{S}_n)_{\max}}\!/_{\approx}$ by certain kinds of compositions called *maximal*. These compositions have the property that their odd parts are weakly decreasing. We give a combinatorial characterization of the $\Sigma_\alpha \in {}^{(\mathfrak{S}_n)_{\max}}\!/_{\approx}$ in the case where α is a hook and a recursion rule for Σ_α that allows us to deal with the even parts of α. As a consequence, we obtain a description of Σ_α for all maximal compositions α whose odd parts form a hook.

In Chapter 5 we study the action of the elements of He's basis $\bar{\pi}_{\leq \Sigma_\alpha}$ on the simple $H_n(0)$-modules. For $n \geq 3$ the 0-Hecke algebra $H_n(0)$ has three blocks: one nontrivial block B and two blocks of dimension one. Based on computer experiments, we conjecture that if $\bar{\pi}_{\leq \Sigma_\alpha} \neq 1$ then $\bar{\pi}_{\leq \Sigma_\alpha}$ annihilates all simple $H_n(0)$-modules belonging to the block B. Using the results of Chapter 4, we confirm this conjecture in the case where the odd parts of α form a hook.

Keywords: 0-Hecke algebra, center, quasisymmetric Schur function

Contents

1 Introduction

For a finite Coxeter group W with Coxeter generators S, the 0-Hecke algebra $H_W(0)$ over a field $\mathbb{K}$ is a deformation of the group algebra $\mathbb{K}W$ which can be obtained by replacing the involutions $s \in S$ by projections π_s satisfying the same homogeneous relations as the $s \in S$. These algebras appear in the modular representation theory of finite groups of Lie type [Nor79, CL76]. The adjacent transpositions $(i, i+1)$ for $i = 1, \ldots, n-1$ generate the symmetric group $\mathfrak{S}_n$ as a Coxeter group. We write $H_n(0)$ for the 0-Hecke algebra of the symmetric group $\mathfrak{S}_n$.

This thesis is concerned with $H_n(0)$-modules that are associated to quasisymmetric Schur functions and the center of $H_n(0)$. Chapter 2 contains the background material on combinatorial concepts, Coxeter groups and 0-Hecke algebras.

The representation theory of $H_W(0)$ was first considered by Norton [Nor79]. Further results on $H_n(0)$ were obtained by Carter [Car86]. Duchamp, Hivert and Thibon showed that $H_n(0)$ has infinite representation type for $n \geq 4$ [DHT02]. Deng and Yang determined the representation type of $H_W(0)$ for irreducible finite Coxeter groups W [DY11]. They showed that in most cases $H_W(0)$ has infinite and (if $\mathbb{K}$ is algebraically closed) wild representation type. In particular, the latter is true for $H_n(0)$ with $n \geq 5$.

Let $\mathcal{G} := \bigoplus_{n \geq 0} \mathcal{G}_0(H_n(0))$ where $\mathcal{G}_0(H_n(0))$ denotes the Grothendieck group of the finitely generated $H_n(0)$-modules. Duchamp, Krob, Leclerc and Thibon introduced an algebra isomorphism Ch from $\mathcal{G}$ to the algebra of quasisymmetric functions QSym called *quasisymmetric characteristic* [DKLT96, KT97]. This mirrors the connection between the representation theory of the symmetric groups over $\mathbb{C}$ and the algebra Sym of symmetric functions given by the characteristic map ch which sends the irreducible character χ^λ to the Schur function s_λ [Sag01, Sta99]. The algebra QSym is a generalization of Sym that was defined by Gessel [Ges84]. For an introduction to QSym refer to [Sta99, GR14].

Haglund, Luoto, Mason and van Willigenburg defined the *quasisymmetric Schur functions* $\mathcal{S}_\alpha$ [HLMvW11]. The $\mathcal{S}_\alpha$ form a basis of QSym and share many properties with the Schur functions s_λ. Bessenrodt, Luoto and van Willigenburg generalized them to skew quasisymmetric Schur functions $\mathcal{S}_{\alpha /\!/ \beta}$ [BLvW11]. For each quasisymmetric Schur function $\mathcal{S}_\alpha$, Tewari and van Willigenburg constructed a 0-Hecke module $\boldsymbol{S}_\alpha$ that is mapped to $\mathcal{S}_\alpha$ by Ch [TvW15]. Furthermore, they showed that the module $\boldsymbol{S}_\alpha$ admits a natural decomposition into submodules $\boldsymbol{S}_\alpha = \bigoplus_E \boldsymbol{S}_{\alpha,E}$ given by an equivalence relation on its defining $\mathbb{K}$-basis. Similarly, they constructed and decomposed skew modules $\boldsymbol{S}_{\alpha /\!/ \beta}$ that are preimages of the skew quasisymmetric Schur functions $\mathcal{S}_{\alpha /\!/ \beta}$ under Ch.

In Chapter 3 we consider these modules. We first show in Theorem 3.3.11 that the modules $\boldsymbol{S}_{\alpha,E}$ are indecomposable. This part of the author's PhD research has already been published in [Kön19]. Skew modules $\boldsymbol{S}_{\alpha /\!/ \beta, E}$ however can be decomposable. In Theorem 3.4.17 we give a decomposition of certain skew modules $\boldsymbol{S}_{\alpha /\!/ \beta}$, which we call *pacific*, into indecomposable submodules. From this we obtain combinatorial rules for the top and the socle of pacific modules $\boldsymbol{S}_{\alpha/\!\!/\beta}$ in Corollary 3.4.21. The rules are then generalized to all skew modules $\boldsymbol{S}_{\alpha/\!\!/\beta,E}$ in Theorem 3.5.42 for the top and in Corollary 3.6.41 for the socle. Via the direct sum decomposition, this also yields formulas for the top and the socle of $\boldsymbol{S}_{\alpha/\!\!/\beta}$. These results hold in particular for the straight modules $\boldsymbol{S}_{\alpha}$. Tewari and van Willigenburg also introduced permuted versions $\boldsymbol{S}^{\sigma}_{\alpha}$ and $\boldsymbol{S}^{\sigma}_{\alpha,E}$ of the straight modules [TvW19]. At the end of Chapter 3 we briefly discuss how our results on the straight modules can be transferred to the permuted ones. For the indecomposability of $\boldsymbol{S}^{\sigma}_{\alpha,E}$ this already has be done in a slightly different way by Choi, Kim, Nam and Oh [CKNO21].

Let $Z(H_W(0))$ denote the center of $H_W(0)$. Brichard determined the dimension of $Z(H_n(0))$ [Bri08]. Yang and Li obtained a lower bound for the dimension of $Z(H_W(0))$ in several types other than A [YL15]. A $\mathbb{K}$-basis of $Z(H_W(0))$ for arbitrary W depending on certain equivalence classes $W_{\max}/\!\approx$ of W was defined by He [He15]. For $\Sigma \in W_{\max}/\!\approx$ the basis element $\bar{\pi}_{\leq\Sigma}$ indexed by Σ corresponds to the order ideal in Bruhat order generated by Σ.

In Chapter 4 we study $Z(H_W(0))$ and its basis given by He with focus on the case $W = \mathfrak{S}_n$. We give two sets of representatives of $(\mathfrak{S}_n)_{\max}/\!\approx$ in Proposition 4.2.10 and Proposition 4.2.14. The second set consists of *elements in stair form*, which were defined by Kim [Kim98]. Both sets are indexed by certain kinds of compositions called *maximal*. The defining property of these compositions is that their odd parts are weakly decreasing and appear after the even parts (see Definition 4.2.4). Using the elements in stair form, we parametrize the elements of $(\mathfrak{S}_n)_{\max}/\!\approx$ by maximal compositions. In addition, we use results of Gill [Gil00] in order to determine the dimension of $Z(H_W(0))$ in types B_n and D_{2n} in Subsection 4.2.4.

We proceed by giving a combinatorial characterization of the elements of the equivalence class $\Sigma_\alpha \in (\mathfrak{S}_n)_{\max}/\!\approx$ in the cases where the maximal composition α has only one part (Theorem 4.3.20) or is a hook $(k, 1^{n-k})$ with odd k (Theorem 4.3.40). Moreover, we obtain a recursive rule for Σ_α which allows us to deal with the even parts of α in Corollary 4.3.56. This results in a description of the elements of Σ_α for each maximal composition α whose odd parts form a hook.

In Chapter 5 we consider the action of He's basis of $Z(H_n(0))$ on the simple $H_n(0)$-modules. For $n \geq 3$ the 0-Hecke algebra $H_n(0)$ has exactly three blocks: Two blocks of dimension 1 and one nontrivial block B. Computer experiments suggest that apart from the identity element of $H_n(0)$, the basis elements annihilate all the simple modules belonging to the block B. Building on our results from Chapter 4, we confirm this in Corollary 5.4.10 for the basis elements corresponding to the maximal compositions whose odd parts form a hook.

At the beginning of each chapter, we give a more detailed introduction of its content.

2 Background

In this chapter we introduce the basic definitions relevant to all parts of the thesis. The first topics are compositions and composition diagrams in Section 2.1. Section 2.2 deals with finite Coxeter groups and related concepts. This will mostly be applied to the symmetric group $\mathfrak{S}_n$. Other Coxeter groups will only appear in Section 4.1 and Subsection 4.2.4.

In Section 2.3 we define the 0-Hecke algebras of finite Coxeter groups. Moreover, we describe the simple and indecomposable projective modules as well as the block decomposition of the 0-Hecke algebras. This includes the central object of this thesis: the 0-Hecke algebra $H_n(0)$ of the symmetric group $\mathfrak{S}_n$.

Throughout the thesis $\mathbb{K}$ denotes an arbitrary field. We set $\mathbb{N} := \{1, 2, \dots\}$ and always assume that $n \in \mathbb{N}$. For $a, b \in \mathbb{Z}$ we define the *discrete interval* $[a, b] := \{c \in \mathbb{Z} \mid a \leq c \leq b\}$ and use the shorthand $[a] := [1, a]$. For a set X, $\operatorname{span}_{\mathbb{K}} X$ is the formal $\mathbb{K}$-vector space with basis X.

Let A be a ring and M be a (left) A-module. With $\operatorname{rad}(M)$ we denote the *radical* of M which is the intersection of all maximal submodules of M. The *top* of M is the factor module $\operatorname{top}(M) := M \,/\operatorname{rad}(M)$. The *socle* of M is the sum of all simple submodules of M and denoted by $\operatorname{soc}(M)$. We call M *projective* if M is a direct summand of a free A-module.

We recall some notions related to partially ordered sets. For an introduction to the subject refer to [Sta12]. Let $(P, \leq)$ be a poset. For $x, y \in P$ we say that y *covers* x and write $x \lessdot y$ if all $z \in P$ with $x \leq z \leq y$ are either equal to x or y. A subset O of P is called *order ideal* of P if for all $x \in O$ and $y \in P$ we have that $x \geq y$ implies $y \in O$. Dually, a subset F of P is called *filter* of P if for all $x \in F$ and $y \in P$ we have that $x \leq y$ implies $y \in F$. For two subsets X and Y of P we write $X \leq Y$ (resp. $X < Y$) if $x \leq y$ (resp. $x < y$) for all $x \in X$ and $y \in Y$.

Let $x, y \in P$. Then $z \in P$ is called a *lower bound* of x and y if $z \leq x$ and $z \leq y$. We call $z \in P$ the *meet* (or *greatest lower bound*) of x and y if z is a lower bound of x and y and $w \leq z$ for all lower bounds w of x and y. If there is a meet of x and y, it is denoted by $x \wedge y$.

2.1 Compositions and diagrams

A *composition* $\alpha = (\alpha_1, \dots, \alpha_l)$ is a finite sequence of positive integers. The *length* and the *size* of α are given by $\ell(\alpha) := l$ and $|\alpha| := \sum_{i=1}^{l} \alpha_i$, respectively. The α_i are called *parts* of α. If α has size n, α is called *composition of* n and we write $\alpha \vDash n$. A *weak* composition of n is a finite sequence of nonnegative integers that sum up to n. We write $\alpha \vDash_0 n$ if α is a weak composition of n. The *empty composition* $\emptyset$ is the unique composition of length and size 0. A *partition* is a composition whose parts are weakly decreasing. We write $\lambda \vdash n$ if λ is a partition of size n. Partitions of n of the form $(k, 1^{n-k})$ with $k \in [n]$ are called *hooks*. For a composition α we denote the partition obtained by sorting the parts of α in decreasing order by $\widetilde{\alpha}$.

Example 2.1.1. For $\alpha = (1,4,3) \vDash 8$, we have $\widetilde{\alpha} = (4,3,1) \vdash 8$.

For $\alpha = (\alpha_1, \ldots, \alpha_l) \vDash n$ define the *set associated to* α as the subset of $[n-1]$

$$\mathrm{Set}(\alpha) := \{d_1, d_2, \ldots, d_{l-1}\}$$

where $d_k := \sum_{j=1}^{k} \alpha_j$. Conversely, for $D = \{d_1 < d_2 < \cdots < d_m\} \subseteq [n-1]$ define

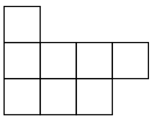
$$\mathrm{comp}(D) := (d_1, d_2 - d_1, \ldots, d_m - d_{m-1}, n - d_m)$$

the *composition of* n *associated to* D. Then $\alpha \mapsto \mathrm{Set}(\alpha)$ is a bijection from the compositions of n to the subsets of $[n-1]$ with inverse map given by $D \mapsto \mathrm{comp}(D)$. For $\alpha \vDash n$ define the *complementary composition of* α as $\alpha^c := \mathrm{comp}([n-1] \setminus \mathrm{Set}(\alpha))$.

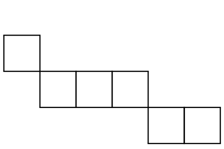
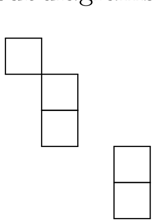
Example 2.1.2. For $\alpha = (1,4,3) \vDash 8$ we have $D(\alpha) = \{1,5\}$ and $\alpha^c = (2,1,1,2,1,1)$.

A *cell* (i,j) is an element of $\mathbb{N} \times \mathbb{N}$. A finite set of cells is called *diagram*. Diagrams are visualized in English notation. That is, for each cell (i,j) of a diagram we draw a box at position (i,j) in matrix coordinates. The *diagram* of $\alpha \vDash n$ is the set $\{(i,j) \in \mathbb{N} \times \mathbb{N} \mid i \le \ell(\alpha), j \le \alpha_i\}$. We display the diagram of α by putting α_i boxes in row i where the top row has index 1. We often identify α with its diagram.

Example 2.1.3.

$(1,4,3) =$ 

Let D be a diagram. We call D a *horizontal strip* if it has at most one cell per column. The diagram D is a *vertical strip* if it has at most one cell per row. We say that D is *connected* if the interior of D viewed as a union of solid squares is a connected open set. The *components* of D are the maximal connected subdiagrams of D. The two diagrams

 and 

are examples for a horizontal and a vertical strip, respectively. Both diagrams are not connected. Note that a connected horizontal strip is a one-row diagram which contains all cells between its leftmost and rightmost cell, i.e. it looks like

$$\square\square\cdots\square\,.$$

Let D be a diagram. A *tableau* T of shape D is a map $T\colon D \to \mathbb{N}$. It is visualized by filling each $(i,j) \in D$ with $T(i,j)$.

In Section 3.1 we will define standard composition tableaux as fillings of composition diagrams. In Section 5.1 we will associate a tableau of size n to each element of the symmetric group $\mathfrak{S}_n$. These tableaux are used in Theorem 5.1.5 for a characterization of the Bruhat order of $\mathfrak{S}_n$ which we define in the next section.

2.2 Coxeter groups

Basic definitions

We review basic concepts of Coxeter groups. This includes Bruhat and left weak order, descent sets, parabolic subgroups and the longest element. Our main motivation is the application to the symmetric groups. Comprehensive treatments of the subject can be found in [BB05, Hum90]. We mainly follow [BB05].

Let S be a set. A *Coxeter matrix* is a map $m\colon S \times S \to \mathbb{N} \cup \{\infty\}$ such that for all $s, s' \in S$

(1) $m(s,s') = 1$ if and only if $s' = s$,

(2) $m(s,s') = m(s',s)$.

The corresponding *Coxeter graph* is the undirected graph with vertex set S containing the edge $\{s,s'\}$ if and only if $m(s,s') \geq 3$. If $m(s,s') \geq 4$ then the edge $\{s,s'\}$ is labeled with $m(s,s')$.

A group W is called *Coxeter group* with *Coxeter generators* S if W is generated by S subject to the relations

$$(ss')^{m(s,s')} = 1 \text{ for all } s, s' \in S \text{ with } m(s,s') < \infty$$

where m is a Coxeter matrix with domain $S \times S$ and 1 denotes the identity element. The relations can be rephrased as

(1) $s^2 = 1$ for all $s \in S$,

(2) $(ss's\cdots)_{m(s,s')} = (s'ss'\cdots)_{m(s,s')}$ for all $s, s' \in S$ with $s \neq s'$ and $m(s,s') < \infty$

where $(ss's\cdots)_p$ denotes the the alternating product of s and s' with p factors. The relations (2) are called *braid relations* or *homogeneous relations*. A Coxeter group W with Coxeter generators S is called *irreducible* if its Coxeter graph is connected. The irreducible finite Coxeter groups are classified and we use the notation from [BB05, Appendix A1] in order to reference their types.

For a finite set X we define $\mathfrak{S}(X)$ to be the group formed by all bijections from X to itself. The *symmetric group* $\mathfrak{S}_n$ is the group $\mathfrak{S}([n])$. Its elements are called *permutations*. A permutation $\sigma \in \mathfrak{S}_n$ can be represented in cycle notation where cycles of length one

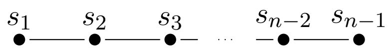


Figure 2.1: The Coxeter graph of $\mathfrak{S}_n$.

may be omitted. The *cycle type* (or simply *type*) of a permutation $\sigma \in \mathfrak{S}_n$ is the partition of n whose parts are the sizes of all the cycles of σ. If σ has cycle type $(k, 1^{n-k})$ for a $k \in [n]$ we also call it a k-cycle. A k-cycle is *trivial* if $k = 1$. Writing σ in cycle notation is the same as expanding σ into a product $\sigma_1 \cdots \sigma_r$ of disjoint cycles where the trivial cycles may be omitted in the expansion. On the other hand, in order to describe the cycle notation of a permutation combinatorially, it can be useful to include them. In Section 4.3 we will characterize the elements of certain equivalence classes of $\mathfrak{S}_n$ by considering them in cycle notation.

Let S be the set of adjacent transpositions $s_i := (i, i+1) \in \mathfrak{S}_n$ for $i = 1, \dots, n-1$. The elements of S satisfy the relations

$$\begin{aligned} s_i^2 &= 1, \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}, \\ s_i s_j &= s_j s_i \text{ if } |i-j| \geq 2. \end{aligned}$$

Then $\mathfrak{S}_n$ together with the generators S is a Coxeter group [BB05, Proposition 1.5.4]. The Coxeter graph of $\mathfrak{S}_n$ is shown in Figure 2.1. For $n \geq 2$, $\mathfrak{S}_n$ is an irreducible Coxeter group of type A_{n-1}. While considering the symmetric group $\mathfrak{S}_n$ as a Coxeter group, we always assume that S is the corresponding set of adjacent transpositions.

Words and partial orders

For the remainder of the section let W be a finite Coxeter group with set of Coxeter generators S. In this thesis we only encounter finite Coxeter groups.

Each $w \in W$ can be written as a product $w = s_1 \cdots s_k$ with $s_i \in S$. Then $s_1 \cdots s_k$ is called a *word* for w. If k is minimal among all words for w, $s_1 \cdots s_k$ is a *reduced word* for w and $\ell(w) := k$ is the *length* of w. One assertion of the *word property* of Coxeter groups [BB05, Theorem 3.3.1] is that a reduced word for w can be transformed into any other reduced word for w by applying a sequence of braid relations.

We now introduce two partial orders on W: the Bruhat order $\leq$ and the left weak order $\leq_L$. Let $s_1 \cdots s_k$ be a word over the alphabet S. A *subword* of $s_1 \cdots s_k$ is a word $s_{i_1} \cdots s_{i_r}$ with $1 \leq i_1 < i_2 < \cdots < i_r \leq k$. A *suffix* of $s_1 \cdots s_k$ is a word of the form $s_j s_{j+1} \cdots s_k$ with $j \geq 1$.

Let $u, w \in W$. The *Bruhat order* $\leq$ is the partial order on W given by $u \leq w$ if and only if there exists a reduced word for w which contains a reduced word of u as a subword. Equivalently, one can demand that each reduced word for w contains a reduced word for u as a subword [BB05, Corollary 2.2.3]. The *left weak order* $\leq_L$ is the partial order on W given by $u \leq_L w$ if and only if there are $s_1, \dots, s_k \in S$ such that

(1) $w = s_k \cdots s_1 u$,

(2) $\ell(s_r \cdots s_1 u) = \ell(u) + r$ for $r = 1, \ldots, k$.

Equivalently, we have $u \leq_L w$ if and only if a reduced word of w contains a reduced word of u as a suffix. As a consequence,

$$u \leq_L w \implies u \leq w.$$

Since $1 \in W$ is the unique element of length 0, it is the least element in Bruhat and in left weak order. The *interval in Bruhat order* between u and w is given by

$$[u, w] := \{x \in W \mid u \leq x \leq w\}.$$

Analogously, we define the *interval in left weak order* $[u, w]_L$.

The following proposition gathers some immediate consequences of the definition of the left weak order. It is used in Theorem 3.1.18. Recall that we use the notation $\lessdot_L$ to indicate covering relations.

Proposition 2.2.1 ([BB05, Proposition 3.1.2])**.** *Let $u, w \in W$.*

(*1*) *We have $u \leq_L w$ if and only if $\ell(wu^{-1}) = \ell(w) - \ell(u)$.*

(*2*) *If $u \leq_L w$ then the reduced words for wu^{-1} are in bijection with saturated chains in the left weak order poset $(W, \leq_L)$ from u to w via*

$$s_k \cdots s_1 \quad \longleftrightarrow \quad u \lessdot_L s_1 u \lessdot_L s_2 s_1 u \lessdot_L \cdots \lessdot_L s_k \cdots s_1 u = w.$$

(*3*) *The poset $(W, \leq_L)$ is graded by the length function.*

Theorem 2.2.2 ([BB05, Corollary 3.2.2])**.** *Let $u, w \in W$. The interval in left weak order $[u, w]_L$ is a graded lattice with rank function $x \mapsto \ell(xu^{-1})$.*

Each interval in Bruhat order $[u, w]$ is also graded by the length function. However, in general it is not a lattice. For example, consider the Bruhat order on $\mathfrak{S}_3$. Then $s_1 s_2$ and $s_2 s_1$ have no meet since $s_1, s_2 \lessdot s_1 s_2$ and $s_1, s_2 \lessdot s_2 s_1$.

Descents and parabolic subgroups

Let $w \in W$. The *left* and the *right descent set* of w are given by

$$D_L(w) := \{s \in S \mid \ell(sw) < \ell(w)\}$$

and

$$D_R(w) := \{s \in S \mid \ell(ws) < \ell(w)\},$$

respectively. It follows that $D_L(w) = D_R(w^{-1})$. Moreover, we have for $s \in S$ that $s \in D_R(w)$ if and only if w has a reduced word ending with s. The analogous statement is true for D_L. Given $\sigma \in \mathfrak{S}_n$ we have

$$\begin{aligned} D_L(\sigma) &= \left\{ s_i \in S \mid \sigma^{-1}(i) > \sigma^{-1}(i+1) \right\}, \\ D_R(\sigma) &= \{ s_i \in S \mid \sigma(i) > \sigma(i+1) \} \end{aligned} \tag{2.1}$$

by [BB05, Proposition 1.5.3]. For $I \subseteq J \subseteq S$ we define the (right) *descent class* $\mathcal{D}_I^J$ as

$$\mathcal{D}_I^J := \{ w \in W \mid I \subseteq D_R(w) \subseteq J \}$$

and set $\mathcal{D}_I := \mathcal{D}_I^I$. We will use descent classes as index sets of bases of projective modules of 0-Hecke algebras.

Let $I \subseteq S$. We write I^c for the complement $S \setminus I$. The *parabolic subgroup* W_I is the subgroup of W generated by I. It is a Coxeter group with Coxeter generators I. The associated set of *quotients* is given by $W^I := \mathcal{D}_\emptyset^{I^c}$. By the following result, each element of W has a unique factorization as a product of elements of W^I and W_I.

Proposition 2.2.3 ([BB05, Proposition 2.4.4])**.** *Let $I \subseteq S$ and $w \in W$. Then there are unique $w^I \in W^I$ and $w_I \in W_I$ such that $w = w^I \cdot w_I$. Moreover, $\ell(w) = \ell(w^I) + \ell(w_I)$.*

The parabolic subgroups of $\mathfrak{S}_n$ are often called *Young subgroups* [Sag01, Sta99]. Commonly, they are indexed by compositions and defined as follows. For $\alpha = (\alpha_1, \ldots, \alpha_l) \vDash n$ let the *Young subgroup* $\mathfrak{S}_\alpha$ be given by

$$\mathfrak{S}_\alpha := \mathfrak{S}([1, d_1]) \times \mathfrak{S}([d_1 + 1, d_2]) \times \cdots \times \mathfrak{S}([d_{l-1}, n])$$

where $d_k := \sum_{j=1}^{k} \alpha_j$. Then $\mathfrak{S}_\alpha$ is isomorphic to

$$\mathfrak{S}_{\alpha_1} \times \mathfrak{S}_{\alpha_2} \times \cdots \times \mathfrak{S}_{\alpha_l}.$$

For $\alpha \vDash n$ we have that $\mathfrak{S}_\alpha = (\mathfrak{S}_n)_I$ where $I = \{ s_i \in S \mid i \notin \mathrm{Set}(\alpha) \}$.

In this thesis we will usually index parabolic subgroups with subsets of S. Given $I \subseteq S$, we may use the shorthand $\mathfrak{S}_I$ for the parabolic subgroup $(\mathfrak{S}_n)_I$ if n is clear from the context.

The next result describes the maximal parabolic subgroups of $\mathfrak{S}_n$ as stabilizer of subsets of $[n]$. For a group G acting on a set X and $Y \subseteq X$ we denote the *stabilizer* of Y by $\mathrm{Stab}(Y)$.

Lemma 2.2.4 ([BB05, Lemma 2.4.7])**.** *Let S be the set of adjacent transpositions of $\mathfrak{S}_n$, $k \in [n-1]$ and $I = S \setminus \{s_k\}$. Then $(\mathfrak{S}_n)_I = \mathrm{Stab}([k])$.*

The longest element

Recall that we assumed that W is a finite Coxeter group with Coxeter generators S. Since W is finite, there exists a greatest element in Bruhat order on W [BB05, Proposition 2.2.9]. This element is called the *longest element* of W and is denoted by w_0. It is the unique element of maximal length in W. Proposition 2.3.2 and Corollary 2.3.3 of [BB05] prove the following.

Proposition 2.2.5. *Let w_0 be the longest element of W. Then we have*
(1) $w_0^2 = 1$,
(2) $\ell(ww_0) = \ell(w_0w) = \ell(w_0) - \ell(w)$ *for all* $w \in W$,
(3) $\ell(w_0ww_0) = \ell(w)$ *for all* $w \in W$.

It follows by Proposition 2.2.1 (1) that w_0 is also the greatest element of W in left weak order.

In the upcoming proposition we consider the maps from W to itself given by multiplication and conjugation with w_0. The main ingredient of its proof is Proposition 2.2.5. See Propositions 2.3.4 and 3.1.5 of [BB05] for details.

Proposition 2.2.6. *For the Bruhat order and the left weak order on W, we have the following:*
(1) $w \mapsto ww_0$ *and* $w \mapsto w_0w$ *are antiautomorphisms,*
(2) $w \mapsto w_0ww_0$ *is an automorphism.*

We continue with an application of Proposition 2.2.6 on descent classes which we prepare for the proof of Theorem 2.3.5.

Lemma 2.2.7. *For $I \subseteq S$ we have $|\mathcal{D}_I| = |\mathcal{D}_{I^c}|$.*

Proof. From Proposition 2.2.6 we know that $\varphi\colon W \to W$, $w \mapsto w_0w$ is an antiautomorphism in Bruhat order. For all $w \in W$ we have

$$\begin{aligned} s \in D_R(w) &\iff \ell(ws) < \ell(w) \\ &\iff \ell(w_0ws) > \ell(w_0w) \iff s \in S \setminus D_R(w_0w). \end{aligned}$$

Now restrict φ to $\mathcal{D}_I$. □

For $I \subseteq S$ we denote the longest element of the parabolic subgroup W_I by $w_0(I)$. The next proposition characterizes $w_0(I)$ in W_I in terms of descent sets.

Proposition 2.2.8. *Let $I \subseteq S$ and $w \in W_I$. The following are equivalent.*
(1) $w = w_0(I)$.
(2) $D_L(w) = I$.
(3) $D_R(w) = I$.

Proof. In [BB05, Proposition 2.3.1] the equivalence of (1) and (2) is shown. From Proposition 2.2.5 we obtain that $w_0(I)^{-1} = w_0(I)$. Moreover, for $w \in W$ we have that $D_R(w) = I$ if and only if $D_L(w^{-1}) = I$ because $D_R(w) = D_L(w^{-1})$. Hence, the equivalence of (1) and (2) implies the claim. □

Example 2.2.9. We determine the longest element w_0 of $\mathfrak{S}_n$. From Proposition 2.2.8 it follows that $D_R(w_0) = S$. Thus, the description of D_R for elements of $\mathfrak{S}_n$ from (2.1) yields that $w_0(i) > w_0(i+1)$ for all $i \in [n-1]$. Hence,

$$w_0(i) = n - i + 1 \text{ for all } i \in [n].$$

The next result shows that the parabolic subgroup W_I and the interval $[1, w_0(I)]$ coincide. This will be important in Section 5.1.

Lemma 2.2.10. *For $I \subseteq S$ and $w \in W$ we have $w \leq w_0(I)$ if and only if $w \in W_I$.*

Proof. The implication from left to right is easy to see. Since $w_0(I)$ is the greatest element of W_I in Bruhat order, we also have the other direction. □

Let $I \subseteq J \subseteq S$. We now express the descent class $\mathcal{D}_I^J$ as an interval in left weak order. We are mostly interested in the descent class $\mathcal{D}_I^S$. It will be important in Theorem 3.4.17, the main result of Section 3.4. Note that since W is finite, each quotient W^I has a greatest element in Bruhat order [BB05, Corollary 2.5.3].

Theorem 2.2.11 ([BW88, Theorem 6.2])**.** *For $I \subseteq J \subseteq S$ we have $\mathcal{D}_I^J = [w_0(I), w_0^{J^c}]_L$ where $w_0^{J^c}$ is the greatest element of W^{J^c}.*

Corollary 2.2.12. *Let $I \subseteq S$. Then $\mathcal{D}_I^S = [w_0(I), w_0]_L$ where w_0 is the longest element of W.*

Proof. By definition $W^{\emptyset} = W$. Hence, w_0 is the greatest elements of $W^{\emptyset}$. Now use Theorem 2.2.11. □

2.3 0-Hecke algebras

In this section we introduce the main object of this thesis, the 0-Hecke algebra $H_n(0)$ of the symmetric group $\mathfrak{S}_n$. Chapter 3 deals with modules of $H_n(0)$ associated to quasisymmetric Schur functions. In Chapter 4 we study the center of $H_n(0)$ and finally in Chapter 5 the action of the center on the simple $H_n(0)$-modules. Therefore, we also consider the representation theory of $H_n(0)$ in this section.

As before let W be a finite Coxeter group with Coxeter generators S and Coxeter matrix m. Norton introduces the 0-Hecke algebra $H_W(0)$ and studies its representation theory in [Nor79]. Most of the results of the section go back to this source. The textbook [Mat99] provides some background on the 0-Hecke algebras in its first chapter.

We now define the 0-Hecke algebra $H_W(0)$ of W. We use the presentation as in [Fay05].

Definition 2.3.1. *The* 0-Hecke algebra $H_W(0)$ *of W is the unital associative $\mathbb{K}$-algebra generated by the elements π_s for $s \in S$ subject to the relations*

(1) $\pi_s^2 = \pi_s$,

(2) $(\pi_s \pi_{s'} \pi_s \cdots)_{m(s,s')} = (\pi_{s'} \pi_s \pi_{s'} \cdots)_{m(s,s')}$ *for all $s, s' \in S$ with $s \neq s'$.*

Note that the π_s for $s \in S$ are projections satisfying the same braid relations as the $s \in S$ themselves. Another set of generators is given by $\bar{\pi}_s := \pi_s - 1$ for $s \in S$. Then $\bar{\pi}_s^2 = -\bar{\pi}_s$ and in [Fay05, Lemma 3.1] it is shown that the $\bar{\pi}_s$ satisfy the same braid relations as the π_s. Note that $\bar{\pi}_s \pi_s = \pi_s \bar{\pi}_s = 0$ for all $s \in S$.

As in [TvW15] we denote the 0-Hecke algebra of the symmetric group $\mathfrak{S}_n$ with $H_n(0) := H_{\mathfrak{S}_n}(0)$. For $i \in [n-1]$ we use the shorthands π_i and $\bar{\pi}_i$ for the generators π_{s_i} and $\bar{\pi}_{s_i}$ of $H_n(0)$.

Let $w \in W$. We define $\pi_w := \pi_{s_1} \cdots \pi_{s_k}$ where $s_1 \cdots s_k$ is a reduced word for w. The word property ensures that this is well defined. Multiplication is given by

$$\pi_s \pi_w = \begin{cases} \pi_{sw} & \text{if } \ell(sw) > \ell(w) \\ \pi_w & \text{if } \ell(sw) < \ell(w) \end{cases}$$

for $s \in S$. As a consequence, $\{\pi_w \mid w \in W\}$ spans $H_W(0)$ over $\mathbb{K}$. We will see in a moment that this set is a $\mathbb{K}$-basis of $H_W(0)$. The elements $\bar{\pi}_w$ for $w \in W$ can be defined analogously. Their multiplication rule is

$$\bar{\pi}_s \bar{\pi}_w = \begin{cases} \bar{\pi}_{sw} & \text{if } \ell(sw) > \ell(w) \\ -\bar{\pi}_w & \text{if } \ell(sw) < \ell(w) \end{cases}$$

for $s \in S$. Thus, they span $H_W(0)$ over $\mathbb{K}$ as well. By [Mat99, Theorem 1.13], $\{\bar{\pi}_w \mid w \in W\}$ is a $\mathbb{K}$-basis of $H_W(0)$.

We now consider the expansion of the π_w in terms of the $\bar{\pi}_w$ and vice versa. Lascoux proved the following result in the case $W = \mathfrak{S}_n$. The proof, however, works for all finite Coxeter groups. From this we obtain bases of the projective 0-Hecke modules which are expressed entirely by the elements π_w in Corollary 2.3.8.

Lemma 2.3.2 ([Las90, Lemma 1.13])**.** *Let $w \in W$. Then*

$$\pi_w = \sum_{u \leq w} \bar{\pi}_u \quad \textit{and} \quad \bar{\pi}_w = \sum_{u \leq w} (-1)^{\ell(w)-\ell(u)} \pi_u.$$

Since $\{\bar{\pi}_w \mid w \in W\}$ is a $\mathbb{K}$-basis of $H_W(0)$, Lemma 2.3.2 implies that $\{\pi_w \mid w \in W\}$ is a $\mathbb{K}$-basis of $H_W(0)$ too.

Remark 2.3.3. We give some background information on the relation between the 0-Heck algebras and the Iwahori-Hecke algebras which were introduced by Iwahori [Iwa64]. Define the *Iwahori-Hecke algebra* $H_W(q_s, s \in S)$ of the finite Coxeter group W as the associative and unitary $\mathbb{K}$-algebra generated by the elements $\bar{\pi}_s$ for $s \in S$ subject to the same homogeneous relations as the $s \in S$ and the quadratic relations

$$\bar{\pi}_s^2 = (q_s - 1)\bar{\pi}_s + q_s$$

where $q_s \in \mathbb{K}$ for $s \in S$ are parameters with $q_s = q_{s'}$ whenever $s, s' \in S$ are conjugate in W. If we choose $q_s = 0$ for all $s \in S$, the generators satisfy $\bar{\pi}_s^2 = -\bar{\pi}_s$ so that we obtain the 0-Hecke algebra $H_W(0)$. We recover the group algebra $\mathbb{K}W$ by setting $q_s = 1$ for all $s \in S$. In this way, $H_W(q_s, s \in S)$ is a deformation of $\mathbb{K}W$. This can be described more formally in terms of generic algebras (see e.g. [Car86, CR87, GP00]).

The Iwahori-Hecke algebras $H_W(q_s, s \in S)$ arise as follows in the representation theory of finite groups of Lie type (cf. [CR87, GP00]). Suppose that W is the Weyl group of a finite group G with BN-pair and the q_s are the corresponding index parameters. Then by Iwahoris theorem $H_W(q_s, s \in S)$ is isomorphic to the Hecke algebra $H(G, B)$, the endomorphism ring of the $\mathbb{K}G$-module affording the representation of G induced from the trivial representation of B. If $\mathbb{K} = \mathbb{C}$ then the index parameters are invertible and $H_W(q_s, s \in S)$ is isomorphic to the group algebra $\mathbb{K}W$ and semi-simple. This is the case in which the Iwahori-Hecke first appeared in [Iwa64]. If the characteristic of $\mathbb{K}$ divides all the index parameters then $H_W(q_s, s \in S)$ is the 0-Hecke algebra $H_W(0)$ [Nor79, Example 1.2].

Modules of the 0-Hecke algebras

In the following we describe the simple and the indecomposable projective modules as well as the block decomposition of $H_W(0)$. These results are due to [Nor79]. We merely rephrase them in a way suitable for this thesis and add an expansion of a basis for the projective modules. For algebraically closed $\mathbb{K}$ and irreducible W, Deng and Yang show in [DY11] that $H_W(0)$ has wild representation type if and only if the type of W is different from A_1, A_2, A_3, B_2 and $I_2(m)$. Therefore, we do not consider indecomposable $H_W(0)$-modules in general.

For $I \subseteq S$ we define $\boldsymbol{F}_I$ to be the one dimensional $H_W(0)$-module generated by the vector v_I equipped with the 0-Hecke action given by

$$\pi_s v_I = \begin{cases} 0 & \text{if } s \in I \\ v_I & \text{if } s \notin I \end{cases}$$

for $s \in S$. By [Nor79, Section 3], the modules $\boldsymbol{F}_I$ for $I \subseteq S$ form a complete list of pairwise non-isomorphic representatives of the isomorphism classes of the simple modules of $H_W(0)$.

In the case of $H_n(0)$ we also use an alternative notation for the simple modules. For $D \subseteq [n-1]$ set $\boldsymbol{F}_D := \boldsymbol{F}_I$ and $v_D := v_I$ where $I = \{s_i \in S \mid i \in D\}$. Then

$$\pi_i v_D = \begin{cases} 0 & \text{if } i \in D \\ v_D & \text{if } i \notin D \end{cases}$$

for all $i \in [n-1]$.

Remark 2.3.4. For a Coxeter group W and $I \subseteq S$, $\boldsymbol{F}_I$ corresponds to the representation $\lambda_{S \setminus I}$ used in [Nor79]. For $D \subseteq [n-1]$, the $H_n(0)$-module $\boldsymbol{F}_D$ coincides with $\mathbf{F}_{\mathrm{comp}(D)}$ from [TvW15].

We now decompose $H_W(0)$ into indecomposable submodules, i.e. we classify the finite dimensional indecomposable projective $H_W(0)$-modules. The decomposition will be applied in Section 3.4. Moreover, we use the summands in order to describe the block structure of $H_W(0)$ at the end of this section.

Recall that for $I \subseteq S$, $w_0(I)$ denotes the longest element of the parabolic subgroup W_I of W. Define $\pi_I := \pi_{w_0(I)}$ and $\bar{\pi}_I := \bar{\pi}_{w_0(I)}$ for $I \subseteq S$. Decompositions of $H_W(0)$ into indecomposable submodules were given by Norton [Nor79, Section 4]. The following theorem rephrases some of her results. For $I \subseteq S$ the module $\boldsymbol{P}_I$ defined below is denoted by $He_I o_{\hat{I}}$ in [Nor79]. Huang describes a more combinatorial approach to the finite dimensional projective modules of $H_W(0)$ in types A, B and D [Hua16].

Let w_0 be the longest element of W. From Propositions 2.2.5 and 2.2.6 it follows that $\nu\colon W \to W$, $w \mapsto w_0 w w_0^{-1}$ is an automorphism of Bruhat order. In particular, $\nu(S) = S$. Below we use this map in order to describe the socles of the $\boldsymbol{P}_I$.

Theorem 2.3.5. *(1) Let $I \subseteq S$. The $H_W(0)$-module $\boldsymbol{P}_I := H_W(0)\pi_I\bar{\pi}_{I^c}$ has a $\mathbb{K}$-basis*

$$\{\pi_w\bar{\pi}_{I^c} \mid w \in \mathcal{D}_I\}\,.$$

In particular, $\dim \boldsymbol{P}_I = |\mathcal{D}_I|$. *Moreover,*

$$\operatorname{top}(\boldsymbol{P}_I) \cong \boldsymbol{F}_{I^c} \quad \textit{and} \quad \operatorname{soc}(\boldsymbol{P}_I) \cong \boldsymbol{F}_{\nu(I^c)}$$

as $H_W(0)$-modules where $\nu\colon W \to W, w \mapsto w_0 w w_0^{-1}$ with the longest element w_0 of W.

(2) The modules $\boldsymbol{P}_I$ for $I \subseteq S$ form a complete list of non-isomorphic projective indecomposable $H_W(0)$-modules. They decompose $H_W(0)$ as an $H_W(0)$-module as $H_W(0) = \bigoplus_{I\subseteq S} \boldsymbol{P}_I$.

Proof. Let $I \subseteq S$ and $w \in W$. Recall that $\pi_s\bar{\pi}_s = 0$ for all $s \in S$. Moreover, $D_L(w_0(I^c)) = I^c$ by Proposition 2.2.8. Hence,

$$\begin{aligned}\pi_w\bar{\pi}_{I^c} = 0 &\iff D_R(w) \cap D_L(w_0(I^c)) \neq \emptyset\\ &\iff D_R(w) \cap I^c \neq \emptyset\\ &\iff D_R(w) \not\subseteq I.\end{aligned}$$

Let $s \in S$ and assume $D_R(w) = I$. Then

$$\pi_s\pi_w\bar{\pi}_{I^c} = \begin{cases}\pi_w\bar{\pi}_{I^c} & \text{if } \ell(sw) < \ell(w),\\ 0 & \text{if } \ell(sw) > \ell(w) \text{ and } D_R(sw) \not\subseteq I,\\ \pi_{sw}\bar{\pi}_{I^c} & \text{if } \ell(sw) > \ell(w) \text{ and } D_R(sw) \subseteq I.\end{cases}$$

In the third case, prefixing s to a reduced word for w yields a reduced word for sw. Thus $D_R(w) \subseteq D_R(sw)$ and hence $D_R(sw) = I$.

It follows that $\boldsymbol{P}_I$ is the $\mathbb{K}$-span of $B := \{\pi_w\bar{\pi}_{I^c} \mid w \in \mathcal{D}_I\}$. In [Nor79, Theorem 4.12] it is shown that the dimension of $\boldsymbol{P}_I$ is given by $|\mathcal{D}_{I^c}|$. Since $|\mathcal{D}_{I^c}| = |\mathcal{D}_I|$ by Lemma 2.2.7, B is a basis.

The top and the socle of $\boldsymbol{P}_I$ are determined in Theorem 4.22 and Lemma 4.23 of [Nor79], respectively. Since the top is simple, $\boldsymbol{P}_I$ is indecomposable.

In [Nor79, Theorem 4.12] the decomposition $H_W(0) = \bigoplus_{I\subseteq S} \boldsymbol{P}_I$ is shown. Because the tops of the $\boldsymbol{P}_I$ for $I \subseteq S$ are pairwise non-isomorphic, it follows that the $\boldsymbol{P}_I$ form a complete list of non-isomorphic projective indecomposable $H_W(0)$-modules. ☐

Remark 2.3.6. In Section 4.1 we will see that $H_W(0)$ is a Frobenius algebra with Nakayama automorphism ν. The correspondence between $\mathrm{top}(\boldsymbol{P}_I)$ and $\mathrm{soc}(\boldsymbol{P}_I)$ given by ν from Theorem 2.3.5 can be traced back to a general property of Frobenius algebras: Given an indecomposable projective module P of a Frobenius algebra A, $\mathrm{soc}(P)$ is isomorphic to $\mathrm{top}(P)$ twisted by the Nakayama automorphism of A. See for instance Lemma III.5.1 and Proposition IV.3.13 of [SY11].

In Section 3.4 we consider $H_n(0)$-modules on which the 0-Hecke action is defined in terms of the generators π_i. Therefore, we want to expand the basis elements $\pi_w\bar{\pi}_{I^c}$ of $\boldsymbol{P}_I$ in terms of the basis $\{\pi_u \mid u \in W\}$ of $H_W(0)$.

Lemma 2.3.7. *Let $I \subseteq S$ and $w \in W^I$. Then*

$$\pi_w\bar{\pi}_I = \sum_{u \in wW_I} (-1)^{\ell(ww_0(I))-\ell(u)}\pi_u.$$

Proof. Let $w \in W^I$. By Proposition 2.2.3 we have $\ell(wu) = \ell(w) + \ell(u)$ and $wu \neq wu'$ for all $u, u' \in W_I$ with $u \neq u'$. Thus, $\pi_w\pi_u = \pi_{wu}$ and $\pi_w\pi_{u'} \neq \pi_{wu}$ for all $u, u' \in W_I$ with $u \neq u'$. We conclude

$$\begin{aligned}\pi_w\bar{\pi}_I &= \sum_{u \in W_I} (-1)^{\ell(w_0(I))-\ell(u)}\pi_w\pi_u \\ &= \sum_{u \in W_I} (-1)^{\ell(w_0(I))-\ell(u)}\pi_{wu} \\ &= \sum_{u \in wW_I} (-1)^{\ell(w_0(I))-\ell(w^{-1}u)}\pi_u, \end{aligned} \tag{2.2}$$

where the first equality uses Lemma 2.3.2 combined with Lemma 2.2.10 and the second equality follows from the discussion above.

Given $u \in wW_I$ we have $u = w \cdot w^{-1}u$ with $w^{-1}u \in W_I$ and $\ell(u) = \ell(w) + \ell(w^{-1}u)$. In particular, this is true for $ww_0(I) \in wW_I$. Therefore,

$$\ell(w_0(I)) - \ell(w^{-1}u) = \ell(w_0(I)) - \ell(u) + \ell(w) = \ell(ww_0(I)) - \ell(u).$$

Hence, (2.2) yields the claim. $\square$

Let $I \subseteq S$. Then $\mathcal{D}_I \subseteq W^{I^c}$ by definition. Thus, we can use Lemma 2.3.7 in order to expand the elements of the basis $\{\pi_w\bar{\pi}_{I^c} \mid w \in \mathcal{D}_I\}$ of $\boldsymbol{P}_I$ from Theorem 2.3.5. Therefore we have the following.

Corollary 2.3.8. *Let $I \subseteq S$. The $H_W(0)$-module $\boldsymbol{P}_I$ has a $\mathbb{K}$-basis*

$$\left\{ \sum_{u \in wW_{I^c}} (-1)^{\ell(ww_0(I^c))-\ell(u)}\pi_u \mid w \in \mathcal{D}_I \right\}.$$

Example 2.3.9. We use Corollary 2.3.8 in order to compute $\mathbb{K}$-bases of the modules $\boldsymbol{P}_I$ of $H_3(0)$.

$I \subset S$	I^c	$\mathcal{D}_I$	$(\mathfrak{S}_3)_{I^c}$	Basis of $\boldsymbol{P}_I$
$\emptyset$	s_1, s_2	1	$\mathfrak{S}_3$	$-\sum_{\sigma\in\mathfrak{S}_3}(-1)^{\ell(\sigma)}\pi_\sigma$
s_1	s_2	s_1, s_2s_1	$1, s_2$	$\pi_1\pi_2 - \pi_1, \pi_1\pi_2\pi_1 - \pi_2\pi_1$
s_2	s_1	s_2, s_1s_2	$1, s_1$	$\pi_2\pi_1 - \pi_2, \pi_1\pi_2\pi_1 - \pi_1\pi_2$
s_1, s_2	$\emptyset$	$s_1s_2s_1$	1	$\pi_1\pi_2\pi_1$

We end this Section with the block decomposition of $H_W(0)$ for irreducible W. It will be relevant in Chapter 5.

Theorem 2.3.10 ([Nor79, Theorem 5.2])**.** *Let W be an irreducible Coxeter group with Coxeter generators $S \neq \emptyset$. Then the block decomposition of $H_W(0)$ is given by*

$$H_W(0) = \boldsymbol{P}_\emptyset \oplus \boldsymbol{P}_S \oplus B$$

where B is the direct sum of $H_W(0)$-submodules $\bigoplus_{\emptyset\subsetneq I\subsetneq S}\boldsymbol{P}_I$.

Let W be irreducible. We consider the block decomposition from Theorem 2.3.10. From Theorem 2.3.5 it follows that $\boldsymbol{P}_\emptyset \cong \boldsymbol{F}_S$ and $\boldsymbol{P}_S \cong \boldsymbol{F}_\emptyset$ as $H_W(0)$-modules. Hence, if $|S| = 1$ then $H_W(0)$ has only these two one-dimensional blocks. If $|S| > 1$ then $H_W(0)$ has an additional nontrivial block B. Note that then all the simple modules $\boldsymbol{F}_I$ with $I \neq \emptyset, S$ belong to the block B.

3 0-Hecke modules associated to quasisymmetric Schur functions

Since the 19th century mathematicians have been interested in the Schur functions s_λ and their various properties. For example, the s_λ form an orthonormal basis of *Sym*, the algebra of symmetric functions, are the images of the irreducible complex characters of the symmetric groups under the characteristic map and play an important role in Schubert calculus [Sta99].

As *Sym* is contained in the algebra of quasisymmetric functions *QSym*, it is interesting to find bases of *QSym* that share properties with the Schur functions. A classical one is given by the *fundametal quasisymmetric functions* of Gessel [Ges84]. More recently, other *Schur-like* families of quasisymmetric functions that form bases of *QSym* have been discovered: the *quasisymmetric Schur functions* of Haglund, Luoto, Mason and van Willigenburg [HLMvW11], the *dual immaculate functions* of Berg, Bergeron, Saliola, Serrano and Zabrocki [BBS$^+$14] and the *dual Shin functions* of Campbell, Feldman, Light, and Xu [CFL$^+$14]. The dual Shin functions are also called *extended Schur functions* [AS19] since the dual Shin basis contains the Schur functions [CFL$^+$14].

This chapter is related to the quasisymmetric Schur functions $\mathcal{S}_\alpha$. The following properties of $\mathcal{S}_\alpha$ go back to [HLMvW11]. While the Schur functions s_λ are naturally indexed by partitions, the quasisymmetric Schur functions $\mathcal{S}_\alpha$ are indexed by compositions (see Section 2.1 for definitions). Haglund et al. define composition tableaux as a composition shaped analogue of semistandard Young tableaux. In the same way as the Schur function s_λ is the generating function of the semistandard Young tableaux of shape λ, the quasisymmetric Schur function $\mathcal{S}_\alpha$ is the generating function of the composition tableaux of shape α. The $\mathcal{S}_\alpha$ also refine the expansion into fundamental quasisymmetric functions and the Pieri rule of the Schur functions. Finally, the Schur functions expand nicely in the quasisymmetric Schur basis via

$$s_\lambda = \sum_{\widetilde{\alpha}=\lambda} \mathcal{S}_\alpha$$

where the sum runs over all compositions α that rearrange the partition λ.

Bessenrodt, Luoto and van Willigenburg define *skew quasisymmetric Schur functions* $\mathcal{S}_{\alpha /\!\!/ \beta}$ and prove a Littlewood–Richardson rule for expressing them in the basis of quasisymmetric Schur functions in [BLvW11]. Other variants of the $\mathcal{S}_\alpha$ such as *Young quasisymmetric Schur functions* [LMvW13] and *row-strict quasisymmetric Schur functions* [MR14] have also been considered.

Allen, Hallem and Mason show that the dual immaculate functions expand positively into Young quasisymmetric functions and interpret the expansion coefficients as the number of certain tableaux [AHM18]. Mason and Searles obtain a similar result for the transition from the reversed dual immaculate functions to the quasisymmetric Schur functions in [MS21] which involves a variant of the dual immaculate functions. It relies on lifts of quasisymmetric bases to the polynomial ring. Such lifts are also motivated by Schubert calculus and were constructed for all Schur-like bases [AS17, AS18, AS19, MS21].

Recall that the quasisymmetric characteristic *Ch* is an isomorphism from the direct sum of the Grothendieck groups of the 0-Hecke algebras of the symmetric groups $\mathcal{G} = \bigoplus_{n\geq 0} \mathcal{G}_0\left(H_n(0)\right)$ to *QSym*. As the Schur functions are the images of simple modules of the complex group algebras of the symmetric groups, it is natural to ask for $H_n(0)$-modules that are mapped to the Schur-like bases of *QSym* by *Ch*. From this viewpoint, the fundamental quasisymmetric functions are the best analogue of the Schur functions because they are the images of the simple $H_n(0)$-modules under *Ch* [DKLT96]. Nevertheless, modules that are preimages of the other Schur-like bases have also been found: for the dual immaculate functions by Berg et al. [BBS$^+$15], for the quasisymmetric Schur functions by Tewari and van Willigenburg [TvW15] and for the extended Schur functions by Searles [Sea20]. In addition, Bardwell and Searles define modules that are mapped to Young row-strict quasisymmetric Schur functions in [BS20].

We denote the module corresponding to the quasisymmetric Schur function $\mathcal{S}_\alpha$ by $\boldsymbol{S}_\alpha$. By definition it has a $\mathbb{K}$-basis formed by the standard composition tableaux of shape α (see Definition 3.1.4). These tableaux have the property that the entries in the first column increase from top to bottom. Tewari and van Willigenburg generalize the modules $\boldsymbol{S}_\alpha$ in two ways by altering the underlying combinatorics. First, they use skew composition tableaux of shape $\alpha /\!\!/ \beta$ in order to define skew modules $\boldsymbol{S}_{\alpha/\!\!/\beta}$ with characteristic $\mathcal{S}_{\alpha/\!\!/\beta}$ [TvW15]. Second, they define standard permuted composition tableaux of shape α and type σ by letting the relative order of the entries in the first column be given by an arbitrary permutation σ [TvW19] (see Definition 3.7.1). With these tableaux as $\mathbb{K}$-basis, they define $H_n(0)$-modules which we denote with $\boldsymbol{S}_\alpha^\sigma$ and call *permuted*. The modules $\boldsymbol{S}_\alpha^\sigma$ are also studied in [CKNO21].

This chapter is mainly concerned with the modules $\boldsymbol{S}_\alpha$ and $\boldsymbol{S}_{\alpha/\!\!/\beta}$. However, many of our results also hold for the permuted modules $\boldsymbol{S}_\alpha^\sigma$. We discuss the necessary adjustments in the argumentation at the end of the chapter.

By the Krull–Schmidt theorem, each of the aforementioned $H_n(0)$-modules decomposes as a direct sum of indecomposable submodules. For the modules of the dual immaculate and the extended Schur functions, the decomposition is trivial since the modules themselves are indecomposable [BBS$^+$15, Sea20]. The modules $\boldsymbol{S}_\alpha$ however can be decomposable. Tewari and van Willigenburg give a decomposition as follows. By using an equivalence relation, they divide the $\mathbb{K}$-basis of standard composition tableaux of $\boldsymbol{S}_\alpha$ into equivalence classes, obtain a submodule $\boldsymbol{S}_{\alpha,E}$ of $\boldsymbol{S}_\alpha$ for each such equivalence class E and decompose $\boldsymbol{S}_\alpha$ as $\boldsymbol{S}_\alpha = \bigoplus_E \boldsymbol{S}_{\alpha,E}$ [TvW15]. In the same vein, the modules $\boldsymbol{S}_{\alpha/\!\!/\beta}$ and $\boldsymbol{S}_\alpha^\sigma$ as well as those corresponding to the Young row-strict quasisymmetric Schur functions can be decomposed (see [TvW15], [TvW19] and [BS20], respectively).

In [TvW15] Tewari and van Willigenburg characterize the case where $\boldsymbol{S}_\alpha$ is indecomposable. Moreover, they show for a special *canonical* equivalence class E_α that $\boldsymbol{S}_{\alpha,E_\alpha}$

is indecomposable. Yet, the question of the indecomposability of the modules $\boldsymbol{S}_{\alpha,E}$ in general remained open. The first goal of the chapter is to answer this question. For each $\boldsymbol{S}_{\alpha,E}$ we consider the $H_n(0)$-endomorphisms of $\boldsymbol{S}_{\alpha,E}$ and show in Theorem 3.3.11 that $\operatorname{End}_{H_n(0)}(\boldsymbol{S}_{\alpha,E}) = \mathbb{K}\,\mathrm{id}$ which implies that $\boldsymbol{S}_{\alpha,E}$ is indecomposable.

This result is a part of the author's PhD research that has already been published in [Kön19]. Choi, Kim, Nam and Oh show that the proof can easily be adapted to the permuted modules $\boldsymbol{S}^{\sigma}_{\alpha,E}$ [CKNO21]. Bardwell and Searles employ similar techniques in order to obtain the analogue result for the modules corresponding to the Young row-strict quasisymmetric Schur functions [BS20].

The skew modules $\boldsymbol{S}_{\alpha/\!\!/\beta,E}$ on the other hand can be decomposable. In Section 3.4 we consider a certain class of skew modules $\boldsymbol{S}_{\alpha/\!\!/\beta}$ which we call *pacific*. For these modules we give a decomposition into indecomposable submodules in Theorem 3.4.17. It turns out that the submodules, and thus the $\boldsymbol{S}_{\alpha/\!\!/\beta}$ themselves, are projective. In particular, they are their own projective covers. Choi et al. describe projective covers of the modules of the dual immaculate and the extended Schur functions as well as for the permuted modules $\boldsymbol{S}^{\sigma}_{\alpha,E}$ [CKNO20]. We exploit the projectivity of the pacific modules $\boldsymbol{S}_{\alpha/\!\!/\beta}$ in order to obtain combinatorial formulas for their tops and socles in Corollary 3.4.21.

We then generalize the formulas for top and socle to all skew modules $\boldsymbol{S}_{\alpha/\!\!/\beta,E}$ in Theorem 3.5.42 and Corollary 3.6.41, respectively. On the way, we construct a $\mathbb{K}$-basis of the radical (see Proposition 3.5.41) and the simple submodules (see Theorem 3.6.39) of $\boldsymbol{S}_{\alpha/\!\!/\beta,E}$. Via the direct sum decomposition, we then obtain formulas for the top and the socle of $\boldsymbol{S}_{\alpha/\!\!/\beta}$ in Corollary 3.5.46 and Corollary 3.6.45, respectively. The results hold in particular for the straight modules.

Finally, we briefly discuss how the results of the chapter pertaining the modules $\boldsymbol{S}_{\alpha}$ can be generalized to the permuted modules $\boldsymbol{S}^{\sigma}_{\alpha}$. This includes the indecomposability of $\boldsymbol{S}^{\sigma}_{\alpha,E}$ and the formulas for top and socle. Our approach is slightly different to that Choi et al. use in [CKNO21] in order to prove the indecomposability of $\boldsymbol{S}^{\sigma}_{\alpha,E}$.

The chapter is structured as follows. Section 3.1 contains the necessary background material on the modules $\boldsymbol{S}_{\alpha/\!\!/\beta}$. Let $T_1, T_2 \in \boldsymbol{S}_{\alpha/\!\!/\beta}$ be two skew standard composition tableaux such that T_1 can be transformed into T_2 via the 0-Hecke action on $\boldsymbol{S}_{\alpha/\!\!/\beta}$. The purpose of Section 3.2 is to give a characterization of the set of elementary $H_n(0)$-operators involved in this transformation by comparing the shapes of certain sub tableaux of T_1 and T_2 in Proposition 3.2.9. This is a valuable tool which we apply in all subsequent sections. In Section 3.3 we show that $\boldsymbol{S}_{\alpha,E}$ is indecomposable. The decomposition of the pacific skew modules $\boldsymbol{S}_{\alpha/\!\!/\beta}$ is the topic of Section 3.4. We consider the top and the socle of arbitrary skew modules $\boldsymbol{S}_{\alpha/\!\!/\beta}$ in Section 3.5 and Section 3.6, respectively. Section 3.7 deals with the permuted modules $\boldsymbol{S}^{\sigma}_{\alpha}$.

3.1 0-Hecke modules of standard composition tableaux

In this section we introduce the modules $\boldsymbol{S}_{\alpha/\!/\beta}$ and $\boldsymbol{S}_{\alpha/\!/\beta,E}$, the related combinatorics of standard composition tableaux and further preliminary results which are used throughout the chapter.

Standard composition tableaux

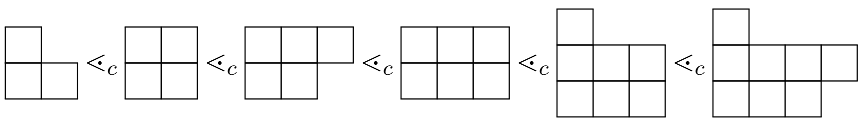
We begin with a poset of compositions which is related to standard composition tableaux and first arose in [BLvW11].

Definition 3.1.1. *The* composition poset $\mathcal{L}_c$ *is the set of all compositions together with the partial order $\leq_c$ given as the transitive closure of the following covering relation. For compositions α and $\beta = (\beta_1, \ldots, \beta_l)$*

$$\beta \lessdot_c \alpha \iff \begin{array}{l} \alpha = (1, \beta_1, \ldots, \beta_l) \text{ or} \\ \alpha = (\beta_1, \ldots, \beta_k + 1, \ldots, \beta_l) \text{ and } \beta_i \neq \beta_k \text{ for all } i < k. \end{array}$$

In other words, β is covered by α in $\mathcal{L}_c$ if and only if the diagram of α can be obtained from the diagram of β by adding a box as the new first row or appending a box to a row which is the topmost row of its length in β.

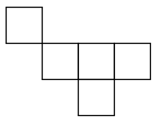
Example 3.1.2.



Let α and β be two compositions such that $\beta \leq_c \alpha$. In this situation we always assume that the diagram of β is moved to the bottom of the diagram of α, and we define the *skew composition diagram* (or *skew shape*) $\alpha/\!/\beta$ to consist of all cells of α which are not contained in β. Moreover, we define $\operatorname{osh}(\alpha/\!/\beta) := \alpha$ and $\operatorname{ish}(\alpha/\!/\beta) := \beta$ as the *outer* and the *inner shape* of $\alpha/\!/\beta$, respectively.

The size of a skew shape is $|\alpha/\!/\beta| := |\alpha| - |\beta|$. We call $\alpha/\!/\beta$ *straight* if $\beta = \emptyset$. In this case the skew composition diagram $\alpha/\!/\beta$ is nothing but the ordinary composition diagram α.

Example 3.1.3. The skew composition diagram $(1, 4, 3)/\!/(1, 2)$ looks as follows.



Note that $\beta \leq_c \alpha$ implies $\beta_{\ell(\beta)-i} \leq \alpha_{\ell(\alpha)-i}$ for $i = 0, \ldots, \ell(\beta) - 1$. One could define skew shapes for all pairs of compositions fulfilling this containment condition. Anyway, we demand $\leq_c$ rather than containment since with the latter one allows skew shapes for which standard composition tableaux (which we will define next) do not exist. For instance, the compositions $\beta = (1,1)$ and $\alpha = (1,2)$ satisfy the containment condition but $\beta \not\leq_c \alpha$. Even if $\alpha /\!\!/ \beta$ were a skew shape, there would be no standard composition tableau of this shape (because of the triple rule stated below).

Definition 3.1.4. *Let $\alpha /\!\!/ \beta$ be a skew shape of size n. A* standard composition tableau *(SCT) of shape $\alpha /\!\!/ \beta$ is a bijective filling $T\colon \alpha /\!\!/ \beta \to [n]$ satisfying the following conditions:*

(*1*) *The entries are decreasing in each row from left to right.*

(*2*) *The entries are increasing in the first column from top to bottom.*

(*3*) (*Triple rule*). *Set $T(i,j) := \infty$ for all $(i,j) \in \beta$. If $(j,k) \in \alpha /\!\!/ \beta$ and $(i,k-1) \in \alpha$ such that $j > i$ and $T(j,k) < T(i,k-1)$ then $(i,k) \in \alpha$ and $T(j,k) < T(i,k)$.*

The plural form of the acronym SCT is SCTx. Let $a := T(j,k)$, $b := T(i,k-1)$ be two entries of an SCT T occurring in adjacent columns. Then the triple rule can be visualized as follows by considering the positions of entries in T:

$$\begin{array}{cc} \boxed{b} & \\ & \\ & \boxed{a} \end{array} \quad \text{and } a < b \quad \overset{\text{triple rule}}{\Longrightarrow} \quad \exists c \in T\colon \quad \begin{array}{cc} \boxed{b} & \boxed{c} \\ & \\ & \boxed{a} \end{array} \quad \text{and } a < c.$$

Let $\mathrm{SCT}(\alpha /\!\!/ \beta)$ denote the set of SCTx of shape $\alpha /\!\!/ \beta$. For an SCT T we write $\mathrm{sh}(T)$ for its shape. The notions of outer and inner shape are carried over from $\mathrm{sh}(T)$ to T. We call T *straight* if its shape is straight.

Example 3.1.5. An SCT is shown below.

$$T = \begin{array}{cccc} \boxed{2} & & & \\ & \boxed{5} & \boxed{4} & \boxed{1} \\ & & \boxed{3} & \end{array}$$

We have $\mathrm{osh}(T) = (1,4,3)$ and $\mathrm{ish}(T) = (1,2)$.

Standard composition tableaux encode saturated chains of $\mathcal{L}_c$ in the following way.

Proposition 3.1.6 ([BLvW11, Proposition 2.11])**.** *Let $\alpha /\!\!/ \beta$ be a skew composition of size n. For $T \in \mathrm{SCT}(\alpha /\!\!/ \beta)$,*

$$\beta = \alpha^n \lessdot_c \alpha^{n-1} \lessdot_c \cdots \lessdot_c \alpha^0 = \alpha$$

given by

$$\alpha^n = \beta, \quad \alpha^{k-1} = \alpha^k \cup T^{-1}(k) \quad \textit{for} \quad k = 1, \ldots, n \tag{3.1}$$

is a saturated chain in $\mathcal{L}_c$. Moreover, we obtain a bijection from $\mathrm{SCT}(\alpha /\!\!/ \beta)$ to the set of saturated chains in $\mathcal{L}_c$ from β to α by mapping each tableau of $\mathrm{SCT}(\alpha /\!\!/ \beta)$ to its corresponding chain given by (3.1).

Example 3.1.7. The SCT from Example 3.1.5 corresponds to the chain from Example 3.1.2.

From the perspective of Proposition 3.1.6, the triple rule reflects the fact that by adding a cell to a row of a composition diagram, a covering relation in $\mathcal{L}_c$ is established if and only if the row in question is the topmost row of its length.

Some of the upcoming notions already played a role in [TvW15]. Let (i, j) and (i', j') be two cells. Define $r(i, j) := i$ and $c(i, j) := j$ the *row* and the *column* of (i, j), respectively. We say that (i, j) *attacks* (i', j') and write $(i, j) \rightsquigarrow (i', j')$ if $j = j'$ and $i \neq i'$ or $j = j'-1$ and $i < i'$. That is, the two cells are distinct and either they appear in the same column or they appear in adjacent columns such that (i', j') is located strictly below and right of (i, j). We call (i, j) the *left neighbor* of (i', j') and write $(i, j) \wr (i', j')$ if $i = i'$ and $j = j' - 1$.

Let T be an SCT and $i, j \in T$ be two entries. We refer to the *row* and the *column* of i in T by $r_T(i) := r(T^{-1}(i))$ and $c_T(i) := c(T^{-1}(i))$, respectively. We say that i *attacks* j in T and write $i \rightsquigarrow_T j$ if $T^{-1}(i) \rightsquigarrow T^{-1}(j)$. Note that $i \rightsquigarrow_T j$ implies $i \neq j$. If $T^{-1}(i)$ is the left neighbor of $T^{-1}(j)$ then we also call i the *left neighbor* of j in T and write $i \wr_T j$. The index T may be omitted if it is clear from the context.

For two sets of cells $C_1, C_2 \subseteq \mathbb{N}^2$ we say C_1 attacks C_2 and write $C_1 \rightsquigarrow C_2$ if there are cells $c_1 \in C_1$ and $c_2 \in C_2$ such that $c_1 \rightsquigarrow c_2$. If $c(c_1) \leq c(c_2)$ for all $c_1 \in C_1, c_2 \in C_2$ then C_1 is called *left* of C_2. If $c(c_1) < c(c_2)$ for all $c_1 \in C_1, c_2 \in c_2$, C_1 is *strictly left* of C_2. To simplify notation we may replace singletons by their respective element. For instance, given a cell c_1 we may write $c_1 \rightsquigarrow C_2$ instead of $\{c_1\} \rightsquigarrow C_2$. In the same way we use these notions for sets of entries of an SCT and $\wr$.

Example 3.1.8. Consider the standard composition tableau T from Example 3.1.5. We have $2 \rightsquigarrow_T 5, 3 \rightsquigarrow_T 4, 4 \rightsquigarrow_T 3$, $5 \rightsquigarrow_T 3$ and $i \not\rightsquigarrow_T j$ for all other pairs of entries. Moreover, 3 is left of $\{1, 4\}$ in T, $2 \rightsquigarrow_T \{3, 5\}$ and $5 \wr_T 4$.

Let T be an SCT of size n. An entry i of T is called *descent* if i appears weakly left of $i+1$ in T. We distinguish between attacking and non-attacking descents. The entry i is called *ascent* of T if it appears strictly right of $i+1$ in T. If i is an ascent of T which has $i+1$ as a neighbor then $i+1$ must be the left neighbor of i. We distinguish between ascents i that have $i+1$ as left neighbor and those which have not. More formally, we have the following.

Definition 3.1.9. *Let T be an SCT of size n.*

(1) $D(T) := \{i \in [n-1] \mid c_T(i) \leq c_T(i+1)\}$ *is the* descent set *of T.*
(2) $AD(T) := \{i \in D(T) \mid i \rightsquigarrow_T i+1\}$ *is the* set of attacking descents *of T.*
(3) $nAD(T) := \{i \in D(T) \mid i \notin AD(T)\}$ *is the* set of non-attacking descents *of T.*
(1') $D^c(T) := \{i \in [n-1] \mid c_T(i+1) < c_T(i)\} = [n-1] \setminus D(T)$ *is the* ascent set *of T.*
(2') $ND^c(T) := \{i \in D^c(T) \mid i+1 \wr_T i\}$ *is the* set of neighborly ascents *of T.*
(3') $nND^c(T) := \{i \in D^c(T) \mid i \notin ND^c(T)\}$ *is the* set of non-neighborly ascents *of T.*

Example 3.1.10. Let T be the tableau from Example 3.1.5. Then $D(T) = \{2,3\}$, $AD(T) = \{3\}$, $D^c(T) = \{1,4\}$ and $ND^c(T) = \{4\}$.

0-Hecke modules of standard composition tableaux

We now come to the 0-Hecke modules $\boldsymbol{S}_{\alpha/\!/\beta}$ and $\boldsymbol{S}_{\alpha/\!/\beta,E}$.

Theorem 3.1.11 ([TvW15, Theorem 9.8])**.** *Let $\alpha/\!/\beta$ be a skew composition of size n. Then $\boldsymbol{S}_{\alpha/\!/\beta} := \operatorname{span}_{\mathbb{K}} \mathrm{SCT}(\alpha/\!/\beta)$ is an $H_n(0)$-module with respect to the following action. For $T \in \mathrm{SCT}(\alpha/\!/\beta)$ and $i = 1, \ldots, n-1$,*

$$\pi_i T = \begin{cases} T & \text{if } i \notin D(T) \\ 0 & \text{if } i \in AD(T) \\ s_i T & \text{if } i \in nAD(T) \end{cases}$$

where $s_i T$ is the tableau obtained from T by interchanging i and $i+1$.

The module $\boldsymbol{S}_\alpha$ is called *straight* if $\alpha = \alpha/\!/\beta$ is a composition.

Example 3.1.12. Consider the SCT $T = \begin{array}{|c|c|c|c|} \cline{1-1} 1 \\ \hline 6 & 5 & 4 & 3 \\ \hline 8 & 7 & 2 \\ \cline{1-3} \end{array}$. Then $D(T) = \{1,2,6\}$,

$$\pi_i T = \begin{cases} T & \text{for } i = 3,4,5,7 \\ 0 & \text{for } i = 6 \\ s_i T & \text{for } i = 1,2, \end{cases} \qquad s_1 T = \begin{array}{|c|c|c|c|} \cline{1-1} \mathbf{2} \\ \hline 6 & 5 & 4 & 3 \\ \hline 8 & 7 & \mathbf{1} \\ \cline{1-3} \end{array} \quad \text{and} \quad s_2 T = \begin{array}{|c|c|c|c|} \cline{1-1} 1 \\ \hline 6 & 5 & 4 & \mathbf{2} \\ \hline 8 & 7 & \mathbf{3} \\ \cline{1-3} \end{array}.$$

We now decompose $\boldsymbol{S}_{\alpha/\!/\beta}$ as in [TvW15]. To do this we use an equivalence relation. Let $\alpha/\!/\beta$ be a skew composition of size n and $T_1, T_2 \in \mathrm{SCT}(\alpha/\!/\beta)$. The equivalence relation $\sim$ on $\mathrm{SCT}(\alpha/\!/\beta)$ is given by

$$T_1 \sim T_2 \iff \text{in each column the relative orders of entries in } T_1 \text{ and } T_2 \text{ coincide.}$$

For example, the straight tableaux shown in Figure 3.1 form an equivalence class under $\sim$. The same is true for the skew tableaux from Figure 3.3. We denote the *set of equivalence classes* under $\sim$ on $\mathrm{SCT}(\alpha/\!/\beta)$ by $\mathcal{E}(\alpha/\!/\beta)$.

For $E \in \mathcal{E}(\alpha/\!/\beta)$ define $\boldsymbol{S}_{\alpha/\!/\beta,E} := \operatorname{span}_{\mathbb{K}} E$. It is easy to see that the definition of the 0-Hecke action on standard composition tableaux in Theorem 3.1.11 implies that $\boldsymbol{S}_{\alpha/\!/\beta,E}$ is an $H_n(0)$-submodule of $\boldsymbol{S}_{\alpha/\!/\beta}$. Thus, we have the following.

Proposition 3.1.13 ([TvW15, Lemma 6.6])**.** *Let $\alpha/\!/\beta$ be a skew composition. Then we have that $\boldsymbol{S}_{\alpha/\!/\beta} = \bigoplus_{E \in \mathcal{E}(\alpha/\!/\beta)} \boldsymbol{S}_{\alpha/\!/\beta,E}$ as $H_n(0)$-modules.*

In this chapter we will work mostly with the modules $\boldsymbol{S}_{\alpha/\!/\beta,E}$ and transfer the results to $\boldsymbol{S}_{\alpha/\!/\beta}$ via the above decomposition. For example, the main result of Section 3.3 is that

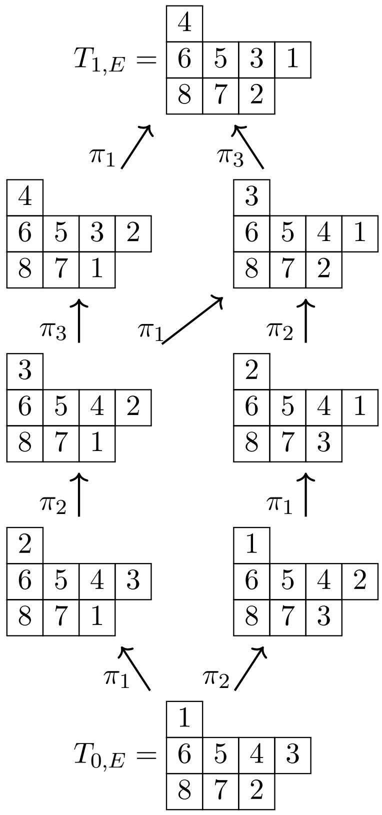


Figure 3.1: A poset given by an equivalence class of standard composition tableaux and the corresponding partial order $\preceq$. Each covering relation is labeled with the 0-Hecke generator π_i realizing it.

the $H_n(0)$-endomorphism ring of each straight module $\boldsymbol{S}_{\alpha,E}$ is $\mathbb{K}\,\mathrm{id}$ and, therefore, we obtain a decomposition of $\boldsymbol{S}_\alpha$ into indecomposable submodules from Proposition 3.1.13.

Let $\alpha /\!\!/ \beta$ be a skew composition of size n and $E \in \mathcal{E}(\alpha /\!\!/ \beta)$. We continue by studying E and its module $\boldsymbol{S}_{\alpha /\!\!/ \beta,E}$ more deeply. First, we consider a partial order $\preceq$ on E. It will turn out that $(E, \preceq)$ is a graded lattice. Afterwards, we prepare two technical results, Corollary 3.1.19 and Proposition 3.1.20, on the 0-Hecke action on standard composition tableaux for later use.

Suppose $T_1, T_2 \in E$. In [TvW15, Section 4] it is shown that a partial order $\preceq$ on E is given by

$$T_1 \preceq T_2 \iff \exists \sigma \in \mathfrak{S}_n \text{ such that } \pi_\sigma T_1 = T_2.$$

We refer to the poset $(E, \preceq)$ simply by E. Two examples are shown in Figure 3.1 and Figure 3.3. The following theorem summarizes results of [TvW15, Section 6].

Theorem 3.1.14. *Let $\alpha/\!\!/\beta$ be a skew composition, $E \in \mathcal{E}(\alpha/\!\!/\beta)$ and $T \in E$.*

(1) *The tableau T is minimal according to $\preceq$ if and only if $D^c(T) = ND^c(T)$. There is a unique tableau $T_{0,E} \in E$ which satisfies these conditions called* source tableau *of E.*

(2) *The tableau T is maximal according to $\preceq$ if and only if $D(T) = AD(T)$. There is a unique tableau $T_{1,E} \in E$ which satisfies these conditions called* sink tableau *of E.*

In particular, $\boldsymbol{S}_{\alpha/\!\!/\beta,E}$ is a cyclic module generated by $T_{0,E}$.

A source and a sink tableau can be observed in Figure 3.1. We now establish a connection between E and an interval of the left weak order. To do this we use the notion of *column words*. Given $T \in \mathrm{SCT}(\alpha/\!\!/\beta)$ and $j \geq 1$, let w_j be the word obtained by reading the entries in the jth column of T from top to bottom. Then $\mathrm{col}_T := w_1 w_2 \cdots$ is the *column word* of T. Clearly, col_T can be regarded as an element of $\mathfrak{S}_n$ (in one-line notation).

Example 3.1.15. The column word of the tableau $T_{0,E}$ from Figure 3.1 is given by $\mathrm{col}_{T_{0,E}} = 16857423 \in \mathfrak{S}_8$.

Lemma 3.1.16 ([TvW15, Lemma 4.4])**.** *Let T_1 be an SCT, $i \in nAD(T_1)$ and $T_2 := \pi_i T_1$. Then $\mathrm{col}_{T_2} = s_i \,\mathrm{col}_{T_1}$ and $\ell(\mathrm{col}_{T_2}) = \ell(\mathrm{col}_{T_1}) + 1$. That is, col_{T_2} covers col_{T_1} in left weak order.*

The following statement is similar to [TvW15, Lemma 4.3].

Lemma 3.1.17. *Let T_1 and T_2 be two standard composition tableaux and $i_p, \dots, i_1 \in [n-1]$ such that $\pi_{i_p} \cdots \pi_{i_1} T_1 = T_2$. Then there is a subsequence $j_q, \dots, j_1$ of $i_p, \dots, i_1$ such that*

(1) $T_2 = \pi_{j_q} \cdots \pi_{j_1} T_1$,

(2) $s_{j_q} \cdots s_{j_1}$ *is a reduced word for* $\mathrm{col}_{T_2} \mathrm{col}_{T_1}^{-1}$.

In particular, $T_2 = \pi_{\mathrm{col}_{T_2} \mathrm{col}_{T_1}^{-1}} T_1$.

Proof. It follows from the definition of the 0-Hecke operation that we can find a subsequence $j_q, \dots, j_1$ of $i_p, \dots, i_1$ of minimal length such that $T_2 = \pi_{j_q} \cdots \pi_{j_1} T_1$. If $q = 0$ then $T_2 = T_1$ and the result is trivial. If $q = 1$ set $i := j_1$. Then by the minimality of q, $T_2 \neq T_1$ and thus $i \in nAD(T_1)$. Now Lemma 3.1.16 shows that s_i is a reduced word for $\mathrm{col}_{T_2} \mathrm{col}_{T_1}^{-1}$. If $q > 1$ use the case $q = 1$ iteratively. $\square$

Theorem 3.1.18 ([TvW15, Theorem 6.18])**.** *Let $\alpha/\!\!/\beta$ be a skew composition, $E \in \mathcal{E}(\alpha/\!\!/\beta)$ and $I := [\mathrm{col}_{T_{0,E}}, \mathrm{col}_{T_{1,E}}]_L$ be an interval in left weak order. Then the map $\mathrm{col}\colon E \to I$, $T \mapsto \mathrm{col}_T$ is a poset isomorphism. In particular, E is a graded lattice with rank function $\delta\colon T \mapsto \ell(\mathrm{col}_T \mathrm{col}_{T_{0,E}}^{-1})$.*

Actually, Theorem 3.1.14, Lemma 3.1.16 and Lemma 3.1.17 are everything needed to prove Theorem 3.1.18 as in [TvW15]. They imply that col (and its inverse) map maximal chains to maximal chains. Note that it follows from Theorem 3.1.18 and Proposition 2.2.1 that for $T_1 \preceq T_2$ saturated chains from T_1 to T_2 correspond to reduced words for $\mathrm{col}_{T_2} \mathrm{col}_{T_1}^{-1}$.

Corollary 3.1.19. *Let T_1 and T_2 be two standard composition tableaux of size n and $\sigma \in \mathfrak{S}_n$ such that $T_2 = \pi_\sigma T_1$. Then T_1 and T_2 belong to the same equivalence class under $\sim$. Let δ be the rank function of that class. Then*

$$\delta(T_2) - \delta(T_1) \leq \ell(\sigma)$$

and we have equality if and only if $\sigma = \mathrm{col}_{T_2} \mathrm{col}_{T_1}^{-1}$.

Proof. Since $T_2 = \pi_\sigma T_1$, $T_2 \sim T_1$. As saturated chains from T_1 to T_2 correspond to reduced words for $\mathrm{col}_{T_2} \mathrm{col}_{T_1}^{-1}$, we have

$$\delta(T_2) - \delta(T_1) = \ell(\mathrm{col}_{T_2} \mathrm{col}_{T_1}^{-1}).$$

Now Lemma 3.1.17 implies the claim. $\square$

We finish this section by preparing another consequence of Lemma 3.1.16 for Sections 3.3 and 3.5.

Proposition 3.1.20. *Let T be an SCT, $i, j \in T$ be such that $i < j$, $\square := T^{-1}(i)$ and $C := T^{-1}([i+1, j])$. If i is located left of $[i+1, j]$ and does not attack $[i+1, j]$ in T then*

(*1*) *$T' := \pi_{j-1} \cdots \pi_{i+1}\pi_i T$ is an SCT,*

(*2*) *$s_{j-1} \cdots s_{i+1}s_i$ is a reduced word for $\mathrm{col}_{T'} \mathrm{col}_T^{-1}$,*

(*3*) *$T'(\square) = j$.*

(*4*) *$T'(C) = [i, j-1]$.*

Proof. Assume that i is located left of $[i+1, j]$ and $i \not\rightsquigarrow [i+1, j]$ and set $T' := \pi_{j-1} \cdots \pi_{i+1}\pi_i T$.

We first show (1) – (3) by induction on $m := j - i$. If $m = 1$ then $i \in nAD(T)$ and $T' = \pi_i T$. Thus, (1) and (3) hold by the definition of the 0-Hecke action and (2) is a consequence of Lemma 3.1.16.

Now, let $m > 1$. Since by assumption i is located left of $[i+1, j]$ and $i \not\rightsquigarrow [i+1, j]$, we can apply the induction hypothesis on i and $j-1$ and obtain that $T'' := \pi_{j-2} \cdots \pi_{i+1}\pi_i T$ is an SCT, $s_{j-2} \cdots s_{i+1}s_i$ is a reduced word for $\mathrm{col}_{T''} \mathrm{col}_T^{-1}$ and $T''(\square) = j-1$. Since the operators $\pi_{j-2}, \ldots, \pi_{i+1}, \pi_i$ are unable to move j, we have $T''^{-1}(j) = T^{-1}(j)$. By choice of i and j, $\square \not\rightsquigarrow T^{-1}(j) = T''^{-1}(j)$ and $\square$ is left of $T''^{-1}(j)$. Thus, $j - 1 \in nAD(T'')$ so that $T' = \pi_{j-1}\pi_{j-2} \cdots \pi_i T = \pi_{j-1}(T'')$ is an SCT and $T'(\square) = j$. It follows from Lemma 3.1.16 that $\mathrm{col}_{T'} \mathrm{col}_T^{-1} = s_{j-1} \mathrm{col}_{T''} \mathrm{col}_T^{-1} = s_{j-1}s_{j-2} \cdots s_i$ and that $s_{j-1}s_{j-2} \cdots s_i$ is a reduced word. This finishes the induction.

Now we show (4). In T the elements of $[i, j]$ occupy $\square$ and the cells of C. As the operators $\pi_i, \ldots, \pi_{j-1}$ only move the elements of $[i, j]$, it follows that these elements occupy the same set of cells in T'. Moreover, $T'(\square) = j$. Thus, $T'(C) = [i, j-1]$. $\square$

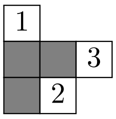
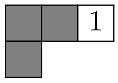
3.2 A 0-Hecke action on chains of the composition poset

In Proposition 3.1.6 a bijection between saturated chains in the composition poset $\mathcal{L}_c$ and standard composition tableaux was given. In this section we study the 0-Hecke action on these chains induced by this bijection. The goal is to provide in Proposition 3.2.9 a characterization of the operators π_i appearing in the saturated chains of SCTx connecting two SCTx T_1 and T_2. This result is essential for the argumentation in Sections 3.3, 3.5 and 3.6.

Definition 3.2.1. *Let T be an SCT of shape $\alpha /\!\!/ \beta$ and size n, $m \in [0, n]$ and*

$$\beta = \alpha^n \lessdot_c \alpha^{n-1} \lessdot_c \cdots \lessdot_c \alpha^0 = \alpha$$

be the chain in $\mathcal{L}_c$ corresponding to T. The SCT of shape $\alpha^m /\!\!/ \beta$ corresponding to the chain $\alpha^n \lessdot_c \alpha^{n-1} \lessdot_c \cdots \lessdot_c \alpha^m$ is denoted by $T^{>m}$.

Example 3.2.2. For $T =$  we have $T^{>2} =$  where the cells of the inner shape are shaded.

The following lemma shows how we can obtain $T^{>m}$ directly from T.

Lemma 3.2.3. *Let T be an SCT of size n and shape $\alpha /\!\!/ \beta$, $\beta = \alpha^n \lessdot_c \alpha^{n-1} \lessdot_c \cdots \lessdot_c \alpha^0 = \alpha$ the chain in $\mathcal{L}_c$ corresponding to T and $m \in [0, n]$.*

(*1*) $\alpha^m = \operatorname{osh}(T^{>m})$.

(*2*) *We obtain $T^{>m}$ from T by removing the cells containing $1, \ldots, m$ and subtracting m from the remaining entries.*

Proof. Part (1) is a immediate consequence of Definition 3.2.1. By Proposition 3.1.6, we obtain $T^{>m}$ by successively adding cells with entries $n - m, \ldots, 1$ to the inner shape β at exactly the same positions where we would add $n, \ldots, m+1$ to β in order to obtain T from its corresponding chain. This implies Part (2). ☐

With the first part of Lemma 3.2.3 we can access the compositions within a chain of a given SCT. We use the following preorder to describe how the 0-Hecke action affects these compositions.

Definition 3.2.4. (*1*) *For a composition $\alpha = (\alpha_1, \ldots, \alpha_l)$ of n and $j \in \mathbb{N}$ we define $|\alpha|_j := |\{i \in [l] \mid \alpha_i \geq j\}|$.*

(*2*) *On the set of compositions of size n we define the preorder $\trianglelefteq$ by*

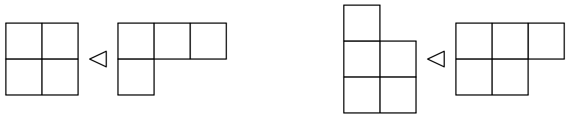
$$\alpha \trianglelefteq \beta \iff \sum_{j=1}^{k} |\beta|_j \leq \sum_{j=1}^{k} |\alpha|_j \text{ for all } k \geq 1.$$

Moreover, set $\alpha \triangleleft \beta \iff \alpha \trianglelefteq \beta$ and $\alpha \neq \beta$.

Note that $|\alpha|_j$ is the number of cells in the jth column of the diagram of α. Obviously $\trianglelefteq$ is reflexive and transitive. It is not antisymmetric since for example $(2, 1) \trianglelefteq (1, 2)$ and $(1, 2) \trianglelefteq (2, 1)$. In general, for $\alpha, \beta \vDash n$ we have

$$\alpha \trianglelefteq \beta \text{ and } \beta \trianglelefteq \alpha \iff |\alpha|_j = |\beta|_j \text{ for all } j = 1, 2, \ldots \iff \widetilde{\alpha} = \widetilde{\beta}.$$

Example 3.2.5.



If we restrict $\trianglelefteq$ to partitions, we obtain the well known dominance order appearing, for example, in [Sta99]. However, $\trianglelefteq$ on partitions may seem to be reversed to the dominance order. This is because in the definition above we are considering the number of cells in columns rather then in rows as usual.

Lemma 3.2.6. *Let $\alpha /\!\!/ \beta$ be a skew composition of size n and $T_1, T_2 \in \mathrm{SCT}(\alpha /\!\!/ \beta)$ be such that $T_2 = \pi_i T_1$ for an $i \in nAD(T_1)$. Then*

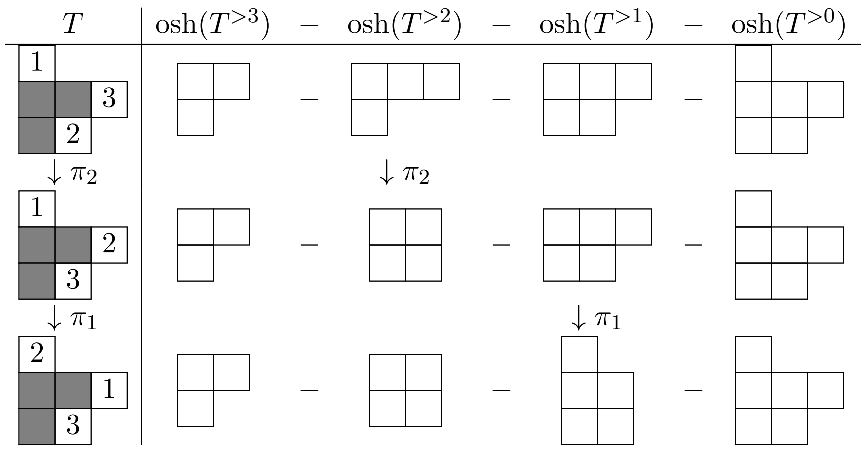
$$\begin{aligned} \operatorname{osh}(T_2^{>i}) &\lhd \operatorname{osh}(T_1^{>i}), \\ \operatorname{osh}(T_2^{>m}) &= \operatorname{osh}(T_1^{>m}) \text{ for all } m \in [0,n] \text{ with } m \neq i. \end{aligned}$$

Proof. We obtain T_2 from T_1 by swapping the entries i and $i+1$ of T_1. Let $m \in [0, n]$.

If $m \neq i$ then either $\{i, i+1\} \subseteq [1, m]$ or $\{i, i+1\} \cap [1, m] = \emptyset$. Hence, $T_1^{-1}([1, m]) = T_2^{-1}([1, m])$, i.e. from the perspective of Lemma 3.2.3 we remove the same set of cells from T_1 to obtain $T_1^{>m}$ as we remove from T_2 to obtain $T_2^{>m}$. That is, $\operatorname{sh}(T_1^{>m}) = \operatorname{sh}(T_2^{>m})$.

If $m = i$, set $(r_k, c_k) := T_1^{-1}(k)$ for $k = i, i+1$, $\gamma_1 := \operatorname{osh}(T_1^{>i})$ and $\gamma_2 := \operatorname{osh}(T_2^{>i})$. We assume that all composition diagrams appearing here are moved to the bottom of α. Observe that as $T_2 = s_i T_1$, one obtains $\operatorname{sh}(T_2^{>i})$ from $\operatorname{sh}(T_1^{>i})$ by moving the cell (r_{i+1}, c_{i+1}) to the position (r_i, c_i). Since $\operatorname{ish}(T_2^{>i}) = \beta = \operatorname{ish}(T_1^{>i})$, we obtain γ_2 from γ_1 by this movement. Moreover, $i \in nAD(T_1)$ implies $c_i < c_{i+1}$. That is, we obtain γ_2 from γ_1 by moving a cell strictly to the left. By the definition of $\trianglelefteq$, this means that $\gamma_2 \lhd \gamma_1$. $\square$

Example 3.2.7. The $H_n(0)$-action on tableaux and the corresponding chains of the composition poset is shown below.



Definition 3.2.8. *Let $\sigma \in \mathfrak{S}_n$. We define the* content *of σ as*

$$\operatorname{cont}(\sigma) := \{i_1, \ldots, i_k\}$$

where $s_{i_1} \cdots s_{i_k}$ is a reduced word for σ.

Let $\sigma \in \mathfrak{S}_n$. Note that the word property ensures that $\operatorname{cont}(\sigma)$ is well defined. Moreover, $i \in \operatorname{cont}(\sigma)$ if and only if $s_i \leq \sigma$.

Let $\alpha /\!\!/ \beta$ be a skew composition, $E \in \mathcal{E}(\alpha /\!\!/ \beta)$ and $T_1, T_2 \in E$ be such that $T_1 \preceq T_2$. From Theorem 3.1.18 it follows that for each saturated chain from T_1 to T_2 in E the index set of the 0-Hecke operators establishing the covering relations within the chain is $\operatorname{cont}(\operatorname{col}_{T_2} \operatorname{col}_{T_1}^{-1})$. As a consequence of Lemma 3.2.6 we obtain a criterion for determining whether an operator π_i appears in the saturated chains from T_1 to T_2 or not.

Proposition 3.2.9. *Let $\alpha /\!\!/ \beta$ be a skew composition of size n, $i \in [n-1], E \in \mathcal{E}(\alpha /\!\!/ \beta)$ and $T_1, T_2 \in E$ be such that $T_1 \preceq T_2$. Then*

$$i \in \operatorname{cont}(\operatorname{col}_{T_2} \operatorname{col}_{T_1}^{-1}) \textit{ if and only if } \operatorname{sh}(T_2^{>i}) \neq \operatorname{sh}(T_1^{>i}).$$

Proof. Lemma 3.2.6 applied to each covering relation in a saturated chain from T_1 to T_2 in E and the fact that $\trianglelefteq$ is a preorder imply

$$i \in \operatorname{cont}(\operatorname{col}_{T_2} \operatorname{col}_{T_1}^{-1}) \text{ if and only if } \operatorname{osh}(T_2^{>i}) \neq \operatorname{osh}(T_1^{>i}).$$

From this we obtain the claim since $\operatorname{ish}(T_1^{>i}) = \beta = \operatorname{ish}(T_2^{>i})$. $\square$

3.3 The decomposition of straight modules

For each $\alpha \vDash n$ there is an equivalence class $E_\alpha \in \mathcal{E}(\alpha)$ such that for all $T \in E_\alpha$ the entries increase in each column from top to bottom [TvW15, Section 8]. In [TvW15] Tewari and van Willigenburg show that $\boldsymbol{S}_{\alpha, E_\alpha}$ is indecomposable.

The objective of this section is to show for all $E \in \mathcal{E}(\alpha)$ that $\operatorname{End}_{H_n(0)}(\boldsymbol{S}_{\alpha,E}) = \mathbb{K}\operatorname{id}$ and hence $\boldsymbol{S}_{\alpha,E}$ is indecomposable; this extends the result of Tewari and van Willigenburg to the general case. As a consequence, Proposition 3.1.13 yields a decomposition of $\boldsymbol{S}_\alpha$ with indecomposable summands. In contrast, skew modules $\boldsymbol{S}_{\alpha/\!\!/\beta,E}$ can be decomposable (see Example 3.3.13). Section 3.4 is concerned with the decomposition of skew modules $\boldsymbol{S}_{\alpha/\!\!/\beta,E}$ of a certain type.

We fix some notation that we use in the entire section unless otherwise stated. Let $\alpha \vDash n$, $E \in \mathcal{E}(\alpha)$ and $T_0 := T_{0,E}$ be the source tableau of E. Moreover, let $f \in \operatorname{End}_{H_n(0)}(\boldsymbol{S}_{\alpha,E})$, $v := f(T_0)$ and $v = \sum_{T \in E} a_T T$ be the expansion of v in the $\mathbb{K}$-basis E. Since $\boldsymbol{S}_{\alpha,E}$ is cyclically generated by T_0, f is determined by v. The *support* of v is given by $\operatorname{supp}(v) := \{T \in E \mid a_T \neq 0\}$. Our goal is to show that T_0 is the only tableau that may occur in $\operatorname{supp}(v)$ since then $f = a_{T_0} \operatorname{id} \in \mathbb{K}\operatorname{id}$. We begin with a property holding for $\operatorname{supp}(v)$ that also appeared in the proof of [TvW15, Theorem 7.8].

Lemma 3.3.1. *If $T \in \operatorname{supp}(v)$ then $D(T) \subseteq D(T_0)$.*

Proof. Let $T_* \in E$ be such that $D(T_*) \not\subseteq D(T_0)$. Then there is an $i \in D(T_*) \cap D^c(T_0)$. Because $i \in D^c(T_0)$, $\pi_i v = f(\pi_i T_0) = v$. Thus, a_{T_*} is the coefficient of T_* in $\pi_i v = \sum_{T\in E} a_T \pi_i T$. But this coefficient is 0 since $\pi_i T \neq T_*$ for all $T \in E$. To see this, assume that there is a $T \in E$ such that $\pi_i T = T_*$. Then we obtain a contradiction as

$$T_* \neq \pi_i T_* = \pi_i^2 T = \pi_i T = T_*. \qquad \square$$

Thanks to Lemma 3.3.1 it remains to show $a_T = 0$ for all $T \in E$ such that $T \neq T_0$ and $D(T) \subseteq D(T_0)$. Thus, fix such a tableau T. In order to determine a_T, we will use a 0-Hecke operator π_σ where $\sigma := s_{j-1} \cdots s_i$ and i and j are given by

$$\begin{aligned} i &:= \max\{k \in [n] \mid T^{-1}(k) \neq T_0^{-1}(k)\}, \\ j &:= \min\{k \in [n] \mid k > i \text{ and } i \rightsquigarrow_{T_0} k\}. \end{aligned} \tag{3.2}$$

That is, i is the greatest entry whose position in T differs from that in T_0 and j is the smallest entry in T_0 which is greater than i and attacked by i in T_0. At this point it is not clear that j is well defined since the defining set could be empty. However, the following two lemmas will show that there always exists an element in this set.

Example 3.3.2. Consider the equivalence class E from Figure 3.1. Then $T_0 = T_{0,E}$ and there is exactly one other tableau T in E with $D(T) \subseteq D(T_0)$:

$$T_0 = \begin{array}{|c|c|c|c|}\hline 1 \\ \cline{1-1} 6 & 5 & \mathbf{4} & 3 \\ \hline 8 & 7 & \mathbf{2} \\ \cline{1-3}\end{array} \xrightarrow{\pi_1} T = \begin{array}{|c|c|c|c|}\hline \mathbf{2} \\ \cline{1-1} 6 & 5 & \mathbf{4} & 3 \\ \hline 8 & 7 & 1 \\ \cline{1-3}\end{array}$$

Defining i and j for T as in (3.2), we obtain $i = 2$ and $j = 4$. Note that $2 \in D(T_0)$. This property holds in general by the following result.

Lemma 3.3.3. *Let $T \in E$ be such that $T \neq T_0$ and $D(T) \subseteq D(T_0)$ and set*

$$i := \max\{k \in [n] \mid T^{-1}(k) \neq T_0^{-1}(k)\}.$$

Then $i \in D(T_0)$.

Proof. We introduce integers d_j such that $D(T_0) = \{d_1 < d_2 < \cdots < d_m\}$, $d_0 = 0$ and $d_{m+1} = n$. Moreover, define $I_k := [d_{k-1}+1, d_k]$ for $k = 1, \ldots, m+1$. Recall that since T_0 is a source tableau, $D^c(T_0) = ND^c(T_0)$ by Theorem 3.1.14. That is, $a+1$ is the left neighbor of a for each ascent a of T_0. Therefore, we have $I_k \setminus \{d_k\} \subseteq ND^c(T_0)$ and conclude that $T_0^{-1}(I_k)$ is a connected horizontal strip for $k = 1, \ldots, m+1$.

Set $\square_k := T_0^{-1}(k)$ for $k = 1, \ldots, n$ and let x be the index such that $T(\square_x) = i$. Since T_0 and T are straight, the ordering conditions of standard composition tableaux imply $T^{-1}(n) = (\ell(\alpha), 1) = T_0^{-1}(n)$. Therefore $i \neq n$ and we now show $i \notin D^c(T_0)$. Assume for the sake of contradiction that $i \in D^c(T_0)$. Let $l \in [m+1]$ be such that $i \in I_l$.

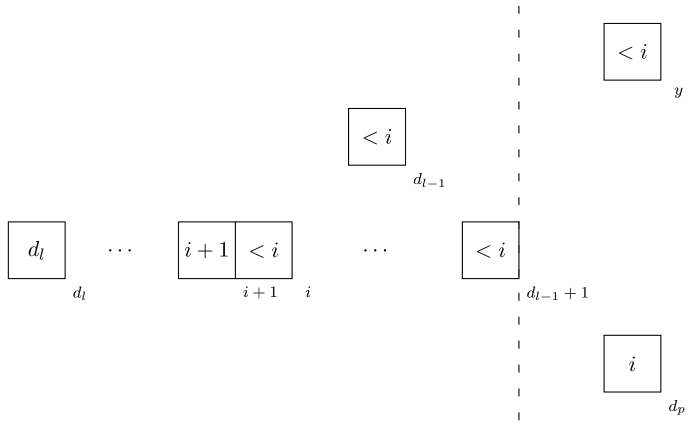


Figure 3.2: An example for the positions of cells and entries in the tableau T from Case 2 of the proof of Lemma 3.3.3.

Since $i \in D^c(T_0)$, we have $i < d_l$ and $i+1 \in I_l$. The horizontal strip $T_0^{-1}(I_l)$ looks as follows:

$$\square_{d_l}\square_{d_l-1}\cdots\square_{i+1}\square_i\cdots\square_{d_{l-1}+1}. \tag{3.3}$$

By choice of i, we have

$$T(\square_k) = k \text{ for } k = i+1,\dots,n \text{ and } T(\square_i) < i. \tag{3.4}$$

Since entries decrease in rows of T, (3.3) implies

$$T(\square_k) < i \text{ for } k = d_{l-1}+1,\dots,i. \tag{3.5}$$

By combining (3.4) and (3.5), we obtain

$$x \leq d_{l-1}. \tag{3.6}$$

We deal with two cases depending on $c_T(i)$. In both cases we will end up with a contradiction.

Case 1 $c_T(i) \leq c_{T_0}(d_{l-1}+1)$. It follows from $D(T) \subseteq D(T_0)$ and $i \in D^c(T_0)$ that $i \in D^c(T)$ and thus $c_T(i+1) < c_T(i)$. Using $c_{T_0}(i) = c_{T_0}(i+1)+1 = c_T(i+1)+1$, we obtain that $c_{T_0}(i) \leq c_T(i) \leq c_{T_0}(d_{l-1}+1)$. Then there is a $y \in [d_{l-1}+1, i]$ such that $\square_x$ and $\square_y$ are in the same column. On the one hand, we obtain from (3.5) that $T(\square_y) < i = T(\square_x)$. On the other hand, the choice of y and (3.6) imply $y > d_{l-1} \geq x$ and hence $T_0(\square_y) = y > x = T_0(\square_x)$. That is, in the column of $\square_x$ and $\square_y$ the relative order of entries in T differs from that in T_0. Hence, $T \not\sim T_0$ which contradicts the assumption $T, T_0 \in E$.

Case 2 $c_T(i) > c_{T_0}(d_{l-1}+1)$. This case is illustrated in Figure 3.2. Since by (3.6) $x \leq d_{l-1}$, there is a $1 \leq p \leq l-1$ such that $x \in I_p$. The leftmost cell of the connected horizontal strip $T_0^{-1}(I_p)$ is $\square_{d_p}$. As entries decrease in rows of T from left to right, we have $T(\square_{d_p}) \geq T(\square_x) = i$. In addition, the choice of p and (3.4) imply that $T(\square_{d_p}) \leq i$. Thus, $d_p = x$.

From $d_p = x$ we obtain $d_p \neq d_{l-1}$ since

$$c_{T_0}(d_{l-1}) \leq c_{T_0}(d_{l-1}+1) < c_T(i) = c_{T_0}(d_p)$$

where we use $d_{l-1} \in D(T_0)$ for the first inequality.

We claim that there exists an index $y \in [d_p+1, d_{l-1}-1]$ such that $\square_y$ and $\square_{d_p}$ are located in the same column. To prove the claim, assume for the sake of contradiction that this is not the case. We show by induction that

$$c_{T_0}(d_p) < c_{T_0}(z) \text{ for all } z \in [d_p+1, d_{l-1}-1]. \tag{3.7}$$

First, $d_p \in D(T_0)$ implies $c_{T_0}(d_p) \leq c_{T_0}(d_p+1)$. Since $c_{T_0}(d_p) \neq c_{T_0}(d_p+1)$ by assumption, it follows that $c_{T_0}(d_p) < c_{T_0}(d_p+1)$. Let $z \in [d_p+2, d_{l-1}-1]$ and assume that (3.7) is true for $z-1$. If $z-1 \in D(T_0)$ then

$$c_{T_0}(d_p) < c_{T_0}(z-1) \leq c_{T_0}(z).$$

If $z-1 \in D^c(T_0)$ then $z-1 \in ND^c(T_0)$ so that

$$c_{T_0}(d_p) \leq c_{T_0}(z-1) - 1 = c_{T_0}(z)$$

and hence $c_{T_0}(d_p) < c_{T_0}(z)$ since $c_{T_0}(d_p) \neq c_{T_0}(z)$ by assumption. This proves (3.7).

As a consequence,

$$c_{T_0}(d_{l-1}) < c_{T_0}(d_p) < c_{T_0}(d_{l-1}-1).$$

In other words, $d_{l-1}-1$ is an ascent of T_0 but d_{l-1} is not the left neighbor of $d_{l-1}-1$. This is a contradiction to the fact that T_0 is a source tableau and finishes the proof of the claim.

Now, let y be as claimed above. Then $y \in [d_p+1, d_{l-1}-1]$ and in particular $y \neq d_p = x$. Hence, (3.4) implies $T(\square_y) < i$ and therefore $T(\square_y) < i = T(\square_{d_p})$. On the other hand, $y \in [d_p+1, d_{l-1}-1]$ yields $T_0(\square_y) = y > d_p = T_0(\square_{d_p})$. As in Case 1, this is a contradiction to $T, T_0 \in E$. $\square$

Note that the i appearing in the following lemma is not the same as in (3.2).

Lemma 3.3.4. *For all $i \in D(T_0)$ there exists $k \in T_0$ such that $k > i$ and $i \rightsquigarrow_{T_0} k$.*

Proof. Let $i \in D(T_0)$. Then $c_{T_0}(i) \leq c_{T_0}(i+1)$ and thus $r_{T_0}(i) \neq r_{T_0}(i+1)$. Since T_0 is straight by assumption, the cell $(r_{T_0}(i+1), c_{T_0}(i))$ is contained in the shape of T_0. Let k be the entry of T_0 in that cell. Then $i \rightsquigarrow_{T_0} k$ and $k \geq i+1$ because entries decrease in rows. $\square$

Let T, i and j be as in (3.2). Lemma 3.3.3 and Lemma 3.3.4 show that j is well defined. We proceed by considering the relative positions of i and $[i+1, j]$ first in T_0

and then in T. This will allow us to deduce useful properties of the operator π_σ to be defined in Lemma 3.3.9. In the following lemma, i is slightly more general than in (3.2).

Lemma 3.3.5. *Let $i \in D(T_0)$ and set $j := \min\{k \in [n] \mid k > i$ and $i \rightsquigarrow_{T_0} k\}$. Then j is well defined and i is located strictly left of $[i+1, j-1]$ and does not attack $[i+1, j-1]$ in T_0.*

We illustrate Lemma 3.3.5 before we prove it.

Example 3.3.6. For the source tableau from above

$$T_0 = \begin{array}{|c|c|c|c|}\cline{1-1} 1 \\ \hline 6 & 5 & \mathbf{4} & \mathbf{3} \\ \hline 8 & 7 & \mathbf{2} \\ \cline{1-3}\end{array}$$

and $i = 2 \in D(T_0)$ we have $j = 4 = \min\{k \in [n] \mid k > i$ and $i \rightsquigarrow_{T_0} k\}$ and $\{3\} = [i+1, j-1]$. Note $2 \rightsquigarrow_{T_0} 4$ but $2 \not\rightsquigarrow_{T_0} 3$.

Proof of Lemma 3.3.5. It follows from Lemma 3.3.4 that j is well defined. We set $I := [i+1, j-1]$ and $c_l := c_{T_0}(l)$ for $l \in T_0$. By the minimality of j, we have $i \not\rightsquigarrow_{T_0} I$. It remains to show that i is strictly left of I or equivalently that $c_i < c_l$ for all $l \in I$. We may assume $I \neq \emptyset$ and use an induction argument to show this.

We begin with $i+1$, the minimum of I. Since $i \in D(T_0)$, $c_i \leq c_{i+1}$. Moreover, $i+1 \in I$ implies $i \not\rightsquigarrow_{T_0} i+1$ and consequently $c_i < c_{i+1}$.

Now, let $l \in I$ be such that $l > i+1$ and $c_i < c_{l-1}$. If $l-1 \in D(T_0)$ then $c_i < c_{l-1} \leq c_l$. If $l-1 \in D^c(T_0)$ then $l-1 \in ND^c(T_0)$ as T_0 is a source tableau. Thus $c_l = c_{l-1} - 1$ and $c_i \leq c_l$. Furthermore $c_i \neq c_l$ since $i \not\rightsquigarrow_{T_0} I \ni l$. Hence, $c_i < c_l$. ☐

Let T, i and j be as in (3.2). By definition, i attacks j in T_0. In contrast, the next lemma shows that i does not attack j in T. Here, i and j are defined as in (3.2).

Lemma 3.3.7. *Let $T \in E$ be such that $T \neq T_0$ and $D(T) \subseteq D(T_0)$. Define*

$$i := \max\{k \in [n] \mid T^{-1}(k) \neq T_0^{-1}(k)\},$$
$$j := \min\{k \in [n] \mid k > i \text{ and } i \rightsquigarrow_{T_0} k\}.$$

Then i and j are well defined and i appears strictly left of $[i+1, j]$ and does not attack $[i+1, j]$ in T.

We first give an example and then the proof of Lemma 3.3.7.

Example 3.3.8. Recall that in our running example $i = 2$ and $j = 4$ when defined for

$$T = \begin{array}{|c|c|c|c|}\cline{1-1} \mathbf{2} \\ \hline 6 & 5 & \mathbf{4} & \mathbf{3} \\ \hline 8 & 7 & 1 \\ \cline{1-3}\end{array}$$

as in Lemma 3.3.7. Then $[i+1, j] = \{3, 4\}$ and $2 \not\rightsquigarrow_T \{3, 4\}$.

Proof of Lemma 3.3.7. Lemma 3.3.3 yields $i \in D(T_0)$. Therefore, Lemma 3.3.4 implies that j is well defined. Set $\sigma := \mathrm{col}_T \mathrm{col}_{T_0}^{-1}$, $\Box_k := T_0^{-1}(k)$ for $k = 1, \dots, n$ and let x be the index such that $T(\Box_x) = i$.

By choice of i, we have $T^{>i} = T_0^{>i}$. Thus, $\mathrm{sh}(T^{>k}) = \mathrm{sh}(T_0^{>k})$ for $k = i, \dots, n$. Hence, Proposition 3.2.9 yields

$$\mathrm{cont}(\sigma) \subseteq [i-1]. \tag{3.8}$$

Let $s_{i_p} \cdots s_{i_1}$ be a reduced word for σ. Then $T = \pi_{i_p} \cdots \pi_{i_1} T_0$. From (3.8) it follows that $i_q \neq i$ for $q = 1, \dots, p$. Moreover, at least one π_{i_q} has to move i because the position of i in T differs from its position in T_0. Hence, there is a q such that $i_q = i-1$ since π_{i-1} and π_i are the only operators that are able to move i. For two standard composition tableaux T_1 and T_2 such that $T_2 = \pi_{i-1} T_1 = s_{i-1} T_1$ we have that $i - 1 \in nAD(T_1)$ and thus $T_2^{-1}(i)$ is left of $T_1^{-1}(i)$ and $T_2^{-1}(i) \not\rightsquigarrow T_1^{-1}(i)$. Hence, by applying $\pi_{i_p} \cdots \pi_{i_1}$ to T_0, i is moved (possibly multiple times) strictly to the left into a cell that does not attack $\Box_i$. That is,

$$\Box_x \text{ is located strictly left of } \Box_i \text{ and } \Box_x \not\rightsquigarrow \Box_i. \tag{3.9}$$

It follows from the choice of i that the elements of $[i+1, j-1]$ have the same position in T and T_0. By combining (3.9) and Lemma 3.3.5 we obtain:

$$i \text{ is located strictly left of } [i+1, j-1] \text{ in } T \text{ and } i \not\rightsquigarrow_T [i+1, j-1]. \tag{3.10}$$

Recall that j has the same position in T and T_0. It follows from (3.9) and $i \rightsquigarrow_{T_0} j$ that $c_T(i) < c_{T_0}(i) \leq c_{T_0}(j)$. Thus, i is strictly left of j in T.

It remains to show $i \not\rightsquigarrow_T j$. We have either $c_{T_0}(j) = c_{T_0}(i) + 1$ or $c_{T_0}(j) = c_{T_0}(i)$ since $i \rightsquigarrow_{T_0} j$.

Case 1 $c_{T_0}(j) = c_{T_0}(i) + 1$. Then (3.9) implies $c_T(i) < c_{T_0}(i) < c_{T_0}(j) = c_T(j)$ and hence $i \not\rightsquigarrow_T j$.

Case 2 $c_{T_0}(j) = c_{T_0}(i)$. If $c_T(i) < c_{T_0}(i) - 1$ then $c_T(i) < c_T(j) - 1$ and thus $i \not\rightsquigarrow {}_T j$. If $c_T(i) = c_{T_0}(i) - 1$ then i and j appear in adjacent columns of T and for $i \not\rightsquigarrow {}_T j$ we have to show that $r_T(j) < r_T(i)$. On the one hand, we have $1 \leq c_T(i) < c_{T_0}(i)$ so that i has a left neighbor $t > i$ in T_0. In addition, i being strictly left of $[i+1, j-1]$ in T_0 by Lemma 3.3.5 and $c_{T_0}(j) = c_{T_0}(i)$ imply that i is weakly left of $[i+1, j]$ in T_0. Thus, $t > j$ and hence $r_{T_0}(j) < r_{T_0}(i)$ because otherwise t, i and j would violate the triple rule in T_0. On the other hand, $c_T(i) = c_{T_0}(i) - 1$ and $i \not\rightsquigarrow_T \Box_i$ imply $r_{T_0}(i) < r_T(i)$. Therefore, $r_T(j) = r_{T_0}(j) < r_{T_0}(i) < r_T(i)$ and thus $i \not\rightsquigarrow_T j$. $\Box$

We now come to the useful properties of the operators π_σ mentioned in (3.2).

Lemma 3.3.9. *Keep the notation of Lemma 3.3.7 and set $\sigma := s_{j-1} \cdots s_{i+1} s_i$. Then*

(1) $\pi_\sigma T_0 = 0$,

(2) $\pi_\sigma T \in E$,

(3) $\sigma = \mathrm{col}_{\pi_\sigma T} \mathrm{col}_T^{-1}$.

Proof. First observe that $s_{j-1} \cdots s_{i+1} s_i$ is a reduced word, i.e.

$$\pi_\sigma = \pi_{j-1} \cdots \pi_{i+1} \pi_i.$$

Set $\Box_k := T_0^{-1}(k)$ for $k = 1, \dots, n$.

We consider T_0. Set $T' := \pi_{j-2} \cdots \pi_{i+1} \pi_i T_0$. We can apply Proposition 3.1.20 in T_0 to i and $[i+1, j-1]$ because of Lemma 3.3.3 and Lemma 3.3.5. By doing this, we obtain that $T' \in E$ and $T'(\Box_i) = j-1$. In addition, $T'(\Box_j) = T_0(\Box_j) = j$ as none of the operators $\pi_{j-2}, \dots, \pi_{i+1}$ moves j. Recall that j is defined such that $\Box_i \rightsquigarrow \Box_j$. Thus $j-1 \in AD(T')$ and $\pi_\sigma T_0 = \pi_{j-1} T' = 0$.

Now consider T. Because of Lemma 3.3.7, we can apply Proposition 3.1.20 in T to i and $[i+1, j]$. This immediately gives us (2) and (3). $\Box$

Example 3.3.10. Continuing our running example, we have $i = 2$, $j = 4$ and $\pi_\sigma = \pi_3 \pi_2$. Moreover,

$$T_0 = \begin{array}{|c|c|c|c|} \cline{1-1} 1 \\ \hline 6 & 5 & 4 & 3 \\ \hline 8 & 7 & 2 \\ \cline{1-3} \end{array} \xrightarrow{\pi_2} \begin{array}{|c|c|c|c|} \cline{1-1} 1 \\ \hline 6 & 5 & 4 & 2 \\ \hline 8 & 7 & 3 \\ \cline{1-3} \end{array} \xrightarrow{\pi_3} 0,$$

$$T = \begin{array}{|c|c|c|c|} \cline{1-1} 2 \\ \hline 6 & 5 & 4 & 3 \\ \hline 8 & 7 & 1 \\ \cline{1-3} \end{array} \xrightarrow{\pi_2} \begin{array}{|c|c|c|c|} \cline{1-1} 3 \\ \hline 6 & 5 & 4 & 2 \\ \hline 8 & 7 & 1 \\ \cline{1-3} \end{array} \xrightarrow{\pi_3} \begin{array}{|c|c|c|c|} \cline{1-1} 4 \\ \hline 6 & 5 & 3 & 2 \\ \hline 8 & 7 & 1 \\ \cline{1-3} \end{array}.$$

We are ready to prove the main result of this section now.

Theorem 3.3.11. *Let $\alpha \vDash n$ and $E \in \mathcal{E}(\alpha)$. Then $\mathrm{End}_{H_n(0)}(\boldsymbol{S}_{\alpha,E}) = \mathbb{K}\,\mathrm{id}$. In particular, $\boldsymbol{S}_{\alpha,E}$ is an indecomposable $H_n(0)$-module.*

Proof. For the second part, note that if $\mathrm{End}_{H_n(0)}(\boldsymbol{S}_{\alpha,E}) = \mathbb{K}\,\mathrm{id}$ then $\boldsymbol{S}_{\alpha,E}$ is indecomposable.

To prove the first part, let $f \in \mathrm{End}_{H_n(0)}(\boldsymbol{S}_{\alpha,E})$, $v := f(T_0)$ and $v = \sum_{T \in E} a_T T$ as at the beginning of Section 3.3. We show $\mathrm{supp}(v) \subseteq \{T_0\}$ since this and the fact that $\boldsymbol{S}_{\alpha,E}$ is cyclically generated by T_0 imply $f = a_{T_0}\,\mathrm{id} \in \mathbb{K}\,\mathrm{id}$.

If $v = 0$, this is clear. Hence, we can assume $v \neq 0$. Recall that E is a graded poset by Theorem 3.1.18. We denote its rank function with δ. Let $T_* \in \mathrm{supp}(v)$ be of maximal rank in $\mathrm{supp}(v)$. Assume for the sake of contradiction that $T_* \neq T_0$. Lemma 3.3.1 yields $D(T_*) \subseteq D(T_0)$. Thus, Lemma 3.3.9 provides the existence of a $\sigma \in \mathfrak{S}_n$ such that $\pi_\sigma T_* \in E$, $\pi_\sigma T_0 = 0$ and $\sigma = \mathrm{col}_{\pi_\sigma T_*} \mathrm{col}_{T_*}^{-1}$.

We claim that if $T \in \mathrm{supp}(v)$ and $\pi_\sigma T = \pi_\sigma T_*$ then $T = T_*$. To see this, let $T \in \mathrm{supp}(v)$ be such that $\pi_\sigma T = \pi_\sigma T_*$. Then

$$\ell(\sigma) \geq \delta(\pi_\sigma T) - \delta(T) = \delta(\pi_\sigma T_*) - \delta(T) \geq \delta(\pi_\sigma T_*) - \delta(T_*) = \ell(\sigma),$$

where Corollary 3.1.19 is used to establish the first and the last equality. Hence, $\ell(\sigma) = \delta(\pi_\sigma T) - \delta(T)$ and another application of Corollary 3.1.19 yields that $\mathrm{col}_{\pi_\sigma T_*} \mathrm{col}_T^{-1} = \sigma$.

But then

$$\mathrm{col}_{\pi_\sigma T_*} \mathrm{col}_T^{-1} = \sigma = \mathrm{col}_{\pi_\sigma T_*} \mathrm{col}_{T_*}^{-1}$$

so that $\mathrm{col}_T = \mathrm{col}_{T_*}$ and thus $T = T_*$ as claimed.

The claim implies that the coefficient of $\pi_\sigma T_*$ in $\pi_\sigma v = \sum_{T \in \mathrm{supp}(v)} a_T \pi_\sigma T$ is a_{T_*}. Yet, $\pi_\sigma v = f(\pi_\sigma T_0) = 0$ and hence $a_{T_*} = 0$ which contradicts the assumption $T_* \in \mathrm{supp}(v)$ and therefore completes the proof of $\mathrm{supp}(v) \subseteq \{T_0\}$. $\square$

Combining Theorem 3.3.11 with Proposition 3.1.13, we obtain the desired decomposition of $\boldsymbol{S}_\alpha$.

Corollary 3.3.12. *Let $\alpha \vDash n$. Then $\boldsymbol{S}_\alpha = \bigoplus_{E \in \mathcal{E}(\alpha)} \boldsymbol{S}_{\alpha,E}$ is a decomposition into indecomposable submodules.*

Example 3.3.13. In general, Theorem 3.3.11 does not hold for skew modules $\boldsymbol{S}_{\alpha /\!\!/ \beta, E}$. The two SCTx of shape $\alpha /\!\!/ \beta = (1,3) /\!\!/ (2)$ and size $n = 2$

$$T_0 = \begin{array}{ccc} \boxed{1} & & \\ & & \boxed{2} \end{array} \xrightarrow{\pi_1} T_1 = \begin{array}{ccc} \boxed{2} & & \\ & & \boxed{1} \end{array}$$

form an equivalence class E. We obtain an idempotent $H_n(0)$-endomorphism φ by setting $\varphi(T_0) := \varphi(T_1) := T_1$. Clearly, φ is none of the trivial idempotents $0, \mathrm{id} \in \mathrm{End}_{H_n(0)}(\boldsymbol{S}_{\alpha /\!\!/ \beta, E})$. Thus, $\mathrm{End}_{H_n(0)}(\boldsymbol{S}_{\alpha /\!\!/ \beta, E}) \neq \mathbb{K}\,\mathrm{id}$. Moreover, we obtain a decomposition

$$\boldsymbol{S}_{\alpha /\!\!/ \beta, E} = \varphi(\boldsymbol{S}_{\alpha /\!\!/ \beta, E}) \oplus (\mathrm{id} - \varphi)(\boldsymbol{S}_{\alpha /\!\!/ \beta, E}) = \mathrm{span}_{\mathbb{K}}(T_1) \oplus \mathrm{span}_{\mathbb{K}}(T_1 - T_0)$$

into two submodules of dimension 1. The module $\boldsymbol{S}_{\alpha /\!\!/ \beta, E}$ is an example of a type of skew modules which we call pacific and decompose in Section 3.4. It also illustrates how the argumentation of this section can fail when it is applied to skew modules. Note that $D(T_1) \subseteq D(T_0)$. We may try to set

$$\begin{aligned} i &:= \max \{k \in [n] \mid T_1^{-1}(k) \neq T_0^{-1}(k)\}, \\ j &:= \min \{k \in [n] \mid k > i \text{ and } i \rightsquigarrow_{T_0} k\}. \end{aligned}$$

as before. But then $i = 2$ so that j does not exist.

3.4 The decomposition of pacific modules

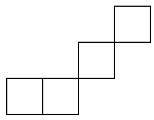
In the last section we decomposed the straight $H_n(0)$-modules $\boldsymbol{S}_\alpha$ into a direct sum of indecomposable submodules. In this section we determine such a decomposition for a certain class of skew $H_n(0)$-modules $\boldsymbol{S}_{\alpha /\!\!/ \beta}$ which we call *pacific*. This is done in Theorem 3.4.17. The summands of the decomposition are isomorphic to projective indecomposable $H_n(0)$-modules $\boldsymbol{P}_I$. Thus, the pacific skew modules $\boldsymbol{S}_{\alpha /\!\!/ \beta}$ are projective. From the decomposition we also obtain combinatorial formulas for the top and the socle of the pacific skew module $\boldsymbol{S}_{\alpha/\!\!/\beta}$ in Corollary 3.4.21. These rules for top and socle will be generalized to all skew modules $\boldsymbol{S}_{\alpha/\!\!/\beta}$ in Section 3.5 and Section 3.6, respectively.

As the pacific skew module $\boldsymbol{S}_{\alpha/\!\!/\beta}$ is projective, it is its own projective cover. In [CKNO20] Choi, Kim, Nam and Oh construct projective covers for $H_n(0)$-modules $\boldsymbol{S}^{\sigma}_{\alpha,E}$ formed by standard permuted composition tableaux of shape α. This includes the straight modules $\boldsymbol{S}_{\alpha,E}$.

The notation related to Coxeter groups from Sections 2.2 and 2.3 used in this section always refers to the symmetric group $\mathfrak{S}_n$ where n is the size of the skew composition or tableaux in question. For instance, S denotes the set of simple reflections of $\mathfrak{S}_n$.

Definition 3.4.1. *A skew composition $\alpha/\!\!/\beta$ is called* pacific *if for each pair of cells $\square_1, \square_2 \in \alpha/\!\!/\beta$ we have $\square_1 \not\rightharpoonup \square_2$. Likewise, a tableau T is* pacific *if for all pairs of entries $i, j \in T$ we have $i \not\rightharpoonup j$. The module $\boldsymbol{S}_{\alpha/\!\!/\beta}$ is called* pacific *if $\alpha/\!\!/\beta$ is pacific.*

Example 3.4.2. The skew composition $\alpha/\!\!/\beta = (5,4,3)/\!\!/(4,3,1)$ is pacific. Its diagram looks as follows:



Note that the first column is empty. The SCTx of shape $\alpha/\!\!/\beta$ are shown in Figure 3.3.

Example 3.4.3. Let T_0 and T_1 be the tableaux of shape $\alpha/\!\!/\beta = (1,3)/\!\!/(2)$ from Example 3.3.13 and E be the equivalence class formed by them. Then T_0, T_1, $\alpha/\!\!/\beta$ and $\boldsymbol{S}_{\alpha/\!\!/\beta}$ are pacific. Moreover, $\alpha/\!\!/\beta$ has at most one cell per column, so that E is the only equivalence class of $\mathrm{SCT}(\alpha/\!\!/\beta)$ under $\sim$. Hence, $E = \mathrm{SCT}(\alpha/\!\!/\beta)$ and $\boldsymbol{S}_{\alpha/\!\!/\beta} = \boldsymbol{S}_{\alpha/\!\!/\beta,E}$. From Example 3.3.13 we have the decomposition

$$\boldsymbol{S}_{\alpha/\!\!/\beta} = \mathrm{span}_{\mathbb{K}}(T_1) \oplus \mathrm{span}_{\mathbb{K}}(T_1 - T_0).$$

An application of Corollary 2.3.8 yields that the indecomposable projective $H_2(0)$-modules $\boldsymbol{P}_\emptyset$ and $\boldsymbol{P}_S$ have $\mathbb{K}$-bases given by $\pi_1 - 1$ and π_1, respectively. As a consequence,

$$\boldsymbol{P}_\emptyset T_0 = \mathrm{span}_{\mathbb{K}}(T_1 - T_0), \quad \boldsymbol{P}_S T_0 = \mathrm{span}_{\mathbb{K}}(T_1)$$

and

$$\boldsymbol{S}_{\alpha/\!\!/\beta} = \boldsymbol{P}_\emptyset T_0 \oplus \boldsymbol{P}_S T_0. \tag{3.11}$$

Moreover, we have $\boldsymbol{P}_\emptyset T_0 \cong \boldsymbol{P}_\emptyset$ and $\boldsymbol{P}_S T_0 \cong \boldsymbol{P}_S$.

In this section we will generalize the decomposition from (3.11) to arbitrary pacific modules $\boldsymbol{S}_{\alpha/\!\!/\beta}$. The method for obtaining it, however, will be different.

We proceed as follows. Let $\alpha/\!\!/\beta$ be a pacific skew composition of size n. First, we consider basic properties of $\alpha/\!\!/\beta$ and $\mathrm{SCT}(\alpha/\!\!/\beta)$. In particular, we show that $\mathrm{SCT}(\alpha/\!\!/\beta)$ is a single equivalence class under $\sim$ so that there is only one source tableau T_0 and one sink tableau T_1 of shape $\alpha /\!\!/ \beta$. Then we describe T_0 and thereby the diagram of $\alpha /\!\!/ \beta$. Recall that the components of a diagram are its maximal connected subdiagrams. It will turn out that the components of the diagram of $\alpha /\!\!/ \beta$ are horizontal strips which do not attack each other. With this we can identify col_{T_0} and col_{T_1} as maximal elements of certain parabolic subgroups of $\mathfrak{S}_n$. This is a central argument in the poof of the decomposition of $\boldsymbol{S}_{\alpha /\!\!/ \beta}$ in the main result, Theorem 3.4.17.

We will see in Proposition 3.4.23 that most of the pacific modules $\boldsymbol{S}_{\alpha /\!\!/ \beta}$ are decomposable. Since $\boldsymbol{S}_{\alpha /\!\!/ \beta} = \boldsymbol{S}_{\alpha /\!\!/ \beta, E}$ for $E = \mathrm{SCT}(\alpha /\!\!/ \beta)$ if $\alpha /\!\!/ \beta$ is pacific, we obtain a class of decomposable modules $\boldsymbol{S}_{\alpha /\!\!/ \beta, E}$. This is a difference to the case of straight modules $\boldsymbol{S}_{\alpha, E}$ which are always indecomposable by Theorem 3.3.11.

Let $\alpha /\!\!/ \beta$ be a skew composition and T be a SCT of shape $\alpha /\!\!/ \beta$. Recall that for $i, j \in T$ we have by definition that $i \rightsquigarrow_T j$ if and only if $T^{-1}(i) \rightsquigarrow T^{-1}(j)$. With this in mind we can directly deduce the next lemma from Definition 3.4.1.

Lemma 3.4.4. *Let $\alpha /\!\!/ \beta$ be a skew composition. Then the following are equivalent.*

(*1*) *$\alpha /\!\!/ \beta$ is pacific.*

(*2*) *Each $T \in \mathrm{SCT}(\alpha /\!\!/ \beta)$ is pacific.*

(*3*) *There is a $T \in \mathrm{SCT}(\alpha /\!\!/ \beta)$ which is pacific.*

By Figure 3.3, all SCTx of pacific shape $(5,4,3) /\!\!/ (4,3,1)$ are equivalent with respect to $\sim$. Now we show that this is true for all pacific skew compositions.

Lemma 3.4.5. *Let $\alpha /\!\!/ \beta$ be a pacific skew composition. Then* $\mathrm{SCT}(\alpha /\!\!/ \beta)$ *is the only element of $\mathcal{E}(\alpha /\!\!/ \beta)$.*

Proof. Since two distinct cells in the same column attack each other, the pacific skew composition $\alpha /\!\!/ \beta$ has at most one cell per column. Hence, each $T \in \mathrm{SCT}(\alpha /\!\!/ \beta)$ has at most one entry per column. By the definition of the equivalence relation $\sim$, this means that all elements of $\mathrm{SCT}(\alpha /\!\!/ \beta)$ are equivalent with respect to $\sim$. □

Let $\alpha /\!\!/ \beta$ be a pacific skew composition. Because of Lemma 3.4.5, we do not obtain a decomposition of $\boldsymbol{S}_{\alpha /\!\!/ \beta}$ from Proposition 3.1.13. It only yields that $\boldsymbol{S}_{\alpha /\!\!/ \beta} = \boldsymbol{S}_{\alpha /\!\!/ \beta, E}$ for $E = \mathrm{SCT}(\alpha /\!\!/ \beta)$. Nonetheless, we can exploit the fact that $\mathrm{SCT}(\alpha /\!\!/ \beta)$ is an equivalence class under $\sim$. For example, this means that there is only one source tableau and only one sink of shape $\alpha /\!\!/ \beta$. Thus, we can speak of *the* source tableau and *the* sink tableau of shape $\alpha /\!\!/ \beta$. Our next goal is to describe the source tableau of $\alpha /\!\!/ \beta$.

Given an arbitrary source tableau T_0 of size n, we use the following notation which already appeared in the proof of Lemma 3.3.3. Recall that by Theorem 3.1.14, T_0 being a source tableau means that T_0 is a SCT with $D^c(T_0) = ND^c(T_0)$. We introduce integers $m := |D(T_0)|$ and $d_k \in [0, n]$ for $k = 0, \dots, m+1$ such that $d_0 = 0$, $D(T_0) = \{d_1 < d_2 < \cdots < d_m\}$ and $d_{m+1} = n$. Define $I_k := [d_{k-1}+1, d_k]$ for $k = 1, \dots, m+1$. Then for each $k \in [m+1]$ and $i \in I_k \setminus \{d_k\}$, $i+1$ is the left neighbor of i. That is, I_k forms a connected horizontal strip in T_0 that looks as follows:

d_k	$d_k - 1$	$\cdots$	$d_{k-1}+1$

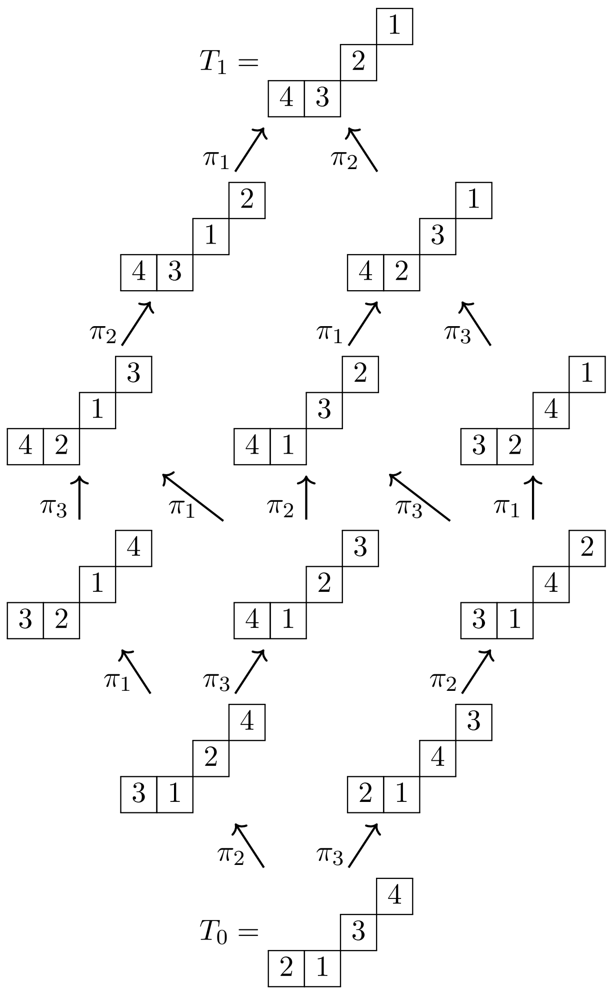


Figure 3.3: The 0-Hecke action on $\mathrm{SCT}(\alpha /\!\!/ \beta)$ for the pacific skew composition $\alpha /\!\!/ \beta = (5,4,3) /\!\!/ (4,3,1)$. The tableaux T_0 and T_1 are the only source and sink tableau of shape $\alpha /\!\!/ \beta$, respectively.

Example 3.4.6. Let T_0 be the source tableau from Figure 3.3. Then we have $D(T_0) = \{d_1 = 2, d_2 = 3\}$ and $d_3 = 4$, i.e. $I_1 = \{1, 2\}, I_2 = \{3\}$ and $I_3 = \{4\}$.

Lemma 3.4.7. *Let T_0 be a source tableau of size n and $k \in [m]$. If $I_k \not\rightsquigarrow I_{k+1}$ in T_0 then I_k is strictly left of I_{k+1} in T_0.*

Proof. Assume $I_k \not\rightsquigarrow_{T_0} I_{k+1}$. We consider the positioning of entries in T_0. Let $i \in I_k$ and $j \in I_{k+1}$. Then i and j appear in different columns since otherwise $i \rightsquigarrow j$. Thus, I_k is positioned either strictly right or strictly left of I_{k+1}. Moreover, $d_k \in I_k$ must be left of $d_k + 1 \in I_{k+1}$ since $d_k \in D(T_0)$. This implies the claim. □

The next lemma characterizes pacific source tableaux.

Lemma 3.4.8. *Let T_0 be a source tableau. Then T_0 is pacific if and only if $I_k \not\rightsquigarrow_{T_0} I_{k+1}$ and I_k is strictly left of I_{k+1} for $k = 1, \ldots, m$.*

Proof. The implication from left to right is an application of Lemma 3.4.7.

For the converse direction assume that the intervals I_k of T_0 satisfy the condition on the right hand side. Let $k \in [m+1]$. Since the entries of I_k form a connected horizontal strip in T_0 they do not attack each other. Hence, consider another interval I_l with $l \in [m+1]$ and $l \neq k$. Without loss of generality suppose $k < l$. Then I_k is strictly left of I_l. If $l > k+1$ then I_{k+1} is located between I_k and I_l. Hence, I_k and I_l are separated by at least one column and thus $I_k \not\rightsquigarrow_{T_0} I_l$ in that case. If $l = k+1$ then $I_k \not\rightsquigarrow_{T_0} I_l$ by assumption. Hence, the entries of I_k do not attack the entries of I_l in T_0. Therefore, T_0 is pacific. □

Let $\alpha /\!\!/ \beta$ be a pacific skew composition and T_0 be the source tableau of shape $\alpha /\!\!/ \beta$. From Lemma 3.4.8 it follows that there is an $m \in \mathbb{N}_0$ and connected horizontal strips $B_1, \ldots, B_{m+1}$ such that $B_k \not\rightsquigarrow B_{k+1}$, B_k is strictly left of B_{k+1} for $k = 1, \ldots, m$ and the B_k are the components of the diagram of $\alpha /\!\!/ \beta$. The B_k are nothing but the preimages of the intervals I_k associated with T_0 under T_0.

That is, we obtain T_0 from $\alpha /\!\!/ \beta$ by setting $d_0 := 0$ and $d_k = \sum_{i=1}^{k} |B_k|$ for $k = 1, \ldots, m+1$ and then filling B_k from left to right with $d_k, d_k - 1, \ldots, d_{k-1} + 1$ for $k = 1, \ldots, m+1$. One may check that we obtain T_0 from Figure 3.3 in this way.

Let T_1 be the sink tableau of $\alpha /\!\!/ \beta$. From Theorem 3.1.18 it follows that $\mathrm{SCT}(\alpha /\!\!/ \beta)$ is isomorphic to the interval in left weak order $[\mathrm{col}_{T_0}, \mathrm{col}_{T_1}]_L$ via the map $T \mapsto \mathrm{col}_T$. We now want to determine this interval. To do this, we use the following definition.

Definition 3.4.9. *Let T be a SCT of size n and S be the set simple reflections of $\mathfrak{S}_n$. Define the* set of simple reflections associated to T *as*

$$J_T := \{s_i \in S \mid i \in D^c(T)\}.$$

Example 3.4.10. Consider the pacific skew composition $\alpha /\!\!/ \beta = (5, 4, 3) /\!\!/ (4, 3, 1)$ of size 4, T_0 the source and T_1 the sink tableau of shape $\alpha /\!\!/ \beta$. The tableaux are shown in

Figure 3.3. We have $D^c(T_0) = \{1\}$. Thus, $J_{T_0} = \{s_1\}$. Moreover, we read $\mathrm{col}_{T_0} = 2134$ and $\mathrm{col}_{T_1} = 4321$. That is

$$\mathrm{col}_{T_0} = s_1 = w_0(J_{T_0}) \quad \text{and} \quad \mathrm{col}_{T_1} = w_0$$

where $w_0(J_{T_0})$ and w_0 are the longest elements of $(\mathfrak{S}_4)_{J_{T_0}}$ and $\mathfrak{S}_4$, respectively. In other words,

$$[\mathrm{col}_{T_0}, \mathrm{col}_{T_1}]_L = [w_0(J_{T_0}), w_0]_L.$$

We need the following lemma in order to generalize Example 3.4.10.

Lemma 3.4.11. *Let T be a SCT and $i \in T$.*

(1) If $i \in nAD(T)$ then $s_i \notin D_L(\mathrm{col}_T)$.

(2) If $i \in D^c(T)$ then $s_i \in D_L(\mathrm{col}_T)$.

Proof. In both cases i and $i+1$ appear in different columns of T. If $i \in nAD(T)$ then i is located in a column strictly left of $i+1$ in T. Thus, i appears left of $i+1$ in col_T, i.e. $\mathrm{col}_T^{-1}(i) < \mathrm{col}_T^{-1}(i+1)$. Hence, $s_i \notin D_L(\mathrm{col}_T)$ by Equation (2.1). If $i \in D^c(T)$ then $i+1$ is located in a column strictly left of i in T and thus $i+1$ appears left of i in col_T. Hence, (2.1) implies $s_i \in D_L(\mathrm{col}_T)$. □

Now we determine $[\mathrm{col}_{T_0}, \mathrm{col}_{T_1}]_L$ for the source and the sink tableau T_0 and T_1 of a pacific shape. Note that by Corollary 2.2.12 the interval $[w_0(J_{T_0}), w_0]_L$ is the descent class $\mathcal{D}^S_{J_{T_0}}$.

Proposition 3.4.12. *Let $\alpha /\!\!/ \beta$ be a pacific skew composition of size n, T_0 the source and T_1 the sink tableau of shape $\alpha /\!\!/ \beta$. Then*

(1) $\mathrm{col}_{T_0} = w_0(J_{T_0})$,

(2) $\mathrm{col}_{T_1} = w_0$

where $w_0(J_{T_0})$ and w_0 refer to elements of $\mathfrak{S}_n$. That is, $\mathrm{SCT}(\alpha /\!\!/ \beta)$ *is isomorphic as a poset to the interval $[w_0(J_{T_0}), w_0]_L$.*

Proof. Set $J := J_{T_0}$.

(1) First we show that col_{T_0} is an element of the parabolic subgroup $(\mathfrak{S}_n)_J$. For $1 \leq k \leq m+1$ the entries of I_k in T_0 form a horizontal strip which looks as follows:

d_k	$d_k - 1$	$\cdots$	$d_{k-1}+1$

Moreover, since $\alpha /\!\!/ \beta$ is pacific, T_0 is. Thus, Lemma 3.4.8 implies that I_k is located strictly left of I_{k+1} for $1 \leq k \leq m$. As a consequence,

$$\mathrm{col}_{T_0} = d_1 \; d_1 - 1 \; \cdots \; 1 \;\; d_2 \; d_2 - 1 \; \cdots \; d_1 + 1 \;\; \cdots \;\; n \; n-1 \; \cdots \; d_m + 1.$$

In particular, $\mathrm{col}_{T_0}([d_k]) = [d_k]$ for $k = 1, \dots, m$. Consider the natural action of $\mathfrak{S}_n$ on $[n]$. Then it follows that $\mathrm{col}_{T_0} \in \mathrm{Stab}([d_k])$ for $k = 1, \dots, m$. Hence

$$\mathrm{col}_{T_0} \in \bigcap_{k=1}^{m} \mathrm{Stab}([d_k]) = \bigcap_{k=1}^{m} (\mathfrak{S}_n)_{S\setminus\{s_{d_k}\}} = (\mathfrak{S}_n)_{\bigcap_{k=1}^{m} S\setminus\{s_{d_k}\}} = (\mathfrak{S}_n)_J$$

where the first equality is an application of Lemma 2.2.4 and the second equality is a consequence of [BB05, Proposition 2.4.1].

As T_0 is pacific, Lemma 3.4.11 implies $D_L(\mathrm{col}_{T_0}) = J$. Furthermore, Proposition 2.2.8 states that the only element of $(\mathfrak{S}_n)_J$ which has left descent set J is $w_0(J)$. Thus $\mathrm{col}_{T_0} = w_0(J)$.

(2) We have $D(T_1) = nAD(T_1)$ since $\alpha /\!\!/ \beta$ is pacific. But T_1 is a sink tableau and hence it follows from Theorem 3.1.14 that $D(T_1) = \emptyset$. Then Lemma 3.4.11 yields that $D_L(\mathrm{col}_{T_1}) = S$. Therefore, Proposition 2.2.8 implies $\mathrm{col}_{T_1} = w_0$.

(3) Because $\alpha /\!\!/ \beta$ is pacific, we can apply Lemma 3.4.5 and obtain that $\mathrm{SCT}(\alpha /\!\!/ \beta)$ is an equivalence class under $\sim$. Then Theorem 3.1.18 yields that $\mathrm{SCT}(\alpha /\!\!/ \beta)$ is as a poset isomorphic to the interval $[\mathrm{col}_{T_0}, \mathrm{col}_{T_1}]_L = [w_0(J), w_0]_L$. □

It is interesting that pacific modules are in fact characterized by Proposition 3.4.12. Although this is not important for their decomposition, we prove it in the next lemma.

Lemma 3.4.13. *Let $\alpha /\!\!/ \beta$ be a skew composition of size n, $E \in \mathcal{E}(\alpha /\!\!/ \beta)$, T_0 be the source and T_1 the sink tableau of E. If $\mathrm{col}_{T_0} = w_0(J_{T_0})$ and $\mathrm{col}_{T_1} = w_0$ then $\alpha /\!\!/ \beta$ is pacific and $E = \mathrm{SCT}(\alpha /\!\!/ \beta)$.*

Proof. Assume that $\mathrm{col}_{T_0} = w_0(J_{T_0})$ and $\mathrm{col}_{T_1} = w_0$. We want to show that T_0 is pacific. Then $\alpha /\!\!/ \beta$ is pacific by Lemma 3.4.4 and therefore $E = \mathrm{SCT}(\alpha /\!\!/ \beta)$ by Lemma 3.4.5.

Define m, the d_k and the I_k according to T_0 as before. Because of Lemma 3.4.8, we have to show that $I_k \not\rightsquigarrow_{T_0} I_{k+1}$ for $k = 1, \dots, m$. Recall that since T_0 is a source tableau, the elements $d_k, d_k - 1, \dots, d_{k-1} + 1$ of I_k form a connected horizontal strip in T_0. For the sake of contradiction assume that there exists an index k such that $I_k \rightsquigarrow_{T_0} I_{k+1}$. In addition, suppose that k is the smallest index with this property. Set $a_0 := d_{k-1} + 1$ and $b_0 := d_{k+1}$. Moreover let a_1 and b_1 be the entries of T_1 in the cells filled with a_0 and b_0 in T_0, respectively. An application of Lemma 3.4.7 yields that I_l is strictly left of I_k in T_0 for all $l < k$. By assumption, we have

$$\begin{aligned} \mathrm{col}_{T_0} &= w_0(J) \\ &= d_1\ d_1 - 1\ \cdots\ 1\ \ \cdots\ \ d_k\ d_k - 1\ \cdots\ d_{k-1} + 1\ \ d_{k+1}\ d_{k+1} - 1\ \cdots\ d_k + 1\ \cdots. \end{aligned}$$

Recall that for obtaining the column word, we read each column from top to bottom starting with the leftmost column. Hence, it follows that I_k is weakly left of I_{k+1} in T_0. Therefore, $a_0 \rightsquigarrow_{T_0} b_0$ since in T_0 a_0 is the rightmost entry of I_k, b_0 is the leftmost entry of I_{k+1} and I_k attacks I_{k+1}. Then in T_0 either a_0 and b_0 are in the same column or a_0 and b_0 are in adjacent columns with a_0 strictly above and left of b_0. We distinguish these two cases.

Assume that the first one is true. Because a_0 precedes b_0 in col_{T_0}, we then have that a_0 is above of b_0 in T_0. Moreover, $\mathrm{col}_{T_1} = w_0 = n\ n-1\ \cdots\ 1$. Since columns are read from top to bottom in the column word, it follows that $a_1 > b_1$. Yet, $a_0 < b_0$ so that we obtain the contradiction $T_0 \not\sim T_1$.

Assume that the second case is true. Then a_1 is left of b_1 in col_{T_1}. Hence, $\mathrm{col}_{T_1} = w_0$ implies $a_1 > b_1$. Since a_1 and b_1 appear in T_1 in adjacent columns with a_1 strictly above and left of b_1, we can apply the triple rule which demands the existence of a $c_1 \in T_1$ which is the right neighbor of a_1. Then a_0 has a right neighbor c_0 in T_0 too. Since entries decrease in rows of SCTx, it follows that $a_0 > c_0$. But all entries which are smaller than $a_0 = d_{k-1}+1$ are elements of $\bigcup_{i=1}^{k-1} I_k$ and these entries are strictly left of a_0 in T_0. That is, we end up with a contradiction again. $\square$

The next two lemmas are the last ingredients needed for the proof of our main result.

Lemma 3.4.14. *Let T be a standard composition tableau of size n, S be the set of simple reflections of $\mathfrak{S}_n$ and $J \subseteq S$. If there is an $i \in D^c(T)$ such that $s_i \in J^c$ then $\bar{\pi}_{J^c}T = 0$.*

Proof. Assume that there is an $i \in D^c(T)$ with $s_i \in J^c$. Then Proposition 2.2.8 provides the existence of a reduced word $s_{i_p} \cdots s_{i_2} s_i$ of $w_0(J^c)$. Furthermore, $\bar{\pi}_i T = (\pi_i - 1)T = 0$ as $i \in D^c(T)$. Thus, $\bar{\pi}_{J^c}T = \bar{\pi}_{i_p} \cdots \bar{\pi}_{i_2}\bar{\pi}_i T = 0$. $\square$

Lemma 3.4.15. *Let $\alpha /\!\!/ \beta$ be a skew composition of size n, S be the simple reflections of $\mathfrak{S}_n$ and $E \in \mathcal{E}(\alpha/\!\!/\beta)$ with source tableau T_0. Then $\boldsymbol{S}_{\alpha/\!\!/\beta,E} = \sum_{J_{T_0} \subseteq J \subseteq S} \boldsymbol{P}_J T_0$.*

Proof. From Theorem 2.3.5 we have the decomposition $H_n(0) = \bigoplus_{J\subseteq S} \boldsymbol{P}_J$. Thus,

$$\boldsymbol{S}_{\alpha/\!\!/\beta,E} = H_n(0)T_0 = \sum_{J\subseteq S} \boldsymbol{P}_J T_0.$$

Let $J \subseteq S$ such that $J_{T_0} \not\subseteq J$. Then there is an $i \in D^c(T_0)$ such that $s_i \in J^c$ and from Lemma 3.4.14 we obtain

$$\boldsymbol{P}_J T_0 = H_n(0)\pi_J \bar{\pi}_{J^c} T_0 = 0. \qquad \square$$

We will see in the proof of Theorem 3.4.17 that the sum from Lemma 3.4.15 is direct if $\alpha/\!\!/\beta$ is pacific. The following example shows that in general this is not the case and that summands can be zero.

Example 3.4.16. Consider the equivalence class E

$$T_0 = \begin{array}{|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & 3 \\ \cline{2-2} \multicolumn{1}{c|}{} & 2 \\ \hline 1 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array} \xrightarrow{\pi_1} \begin{array}{|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & 3 \\ \cline{2-2} \multicolumn{1}{c|}{} & 1 \\ \hline 2 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array} \xrightarrow{\pi_2} T_1 = \begin{array}{|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & 2 \\ \cline{2-2} \multicolumn{1}{c|}{} & 1 \\ \hline 3 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array}$$

of SCTx of size 3 and shape $\alpha/\!\!/\beta = (3,3,2)/\!\!/(2,2,1)$ which are not pacific. Then $J_{T_0} = \emptyset$. Lemma 3.4.15 yields that $\boldsymbol{S}_{\alpha/\!\!/\beta,E} = \sum_{J\subseteq S} \boldsymbol{P}_J T_0$ where S is the set of simple reflections of $\mathfrak{S}_3$ and the $\boldsymbol{P}_J$ are the indecomposable projective modules of $H_3(0)$. On the other hand, from Example 2.3.9 we have that

$$-\pi_2\pi_1\pi_2 + \pi_2\pi_1 \in \boldsymbol{P}_{\{s_1\}} \quad \text{and} \quad \pi_2\pi_1 - \pi_2 \in \boldsymbol{P}_{\{s_2\}}.$$

Thus,

$$T_1 = (-\pi_2\pi_1\pi_2 + \pi_2\pi_1)T_0 \in \boldsymbol{P}_{\{s_1\}}T_0 \quad \text{and} \quad T_1 = (\pi_2\pi_1 - \pi_2)T_0 \in \boldsymbol{P}_{\{s_2\}}T_0.$$

Hence, the intersection of the modules $\boldsymbol{P}_{\{s_1\}}T_0$ and $\boldsymbol{P}_{\{s_2\}}T_0$ is not trivial which means that $\sum_{J\subseteq S} \boldsymbol{P}_J T_0$ is not a direct sum. Moreover, $\boldsymbol{P}_{\{s_1,s_2\}}T_0 = H_n(0)\pi_1\pi_2\pi_1 T_0 = 0$.

We now come to the main result of the section: The decomposition of pacific modules $\boldsymbol{S}_{\alpha/\!\!/\beta}$ into indecomposable submodules. Recall that $\mathcal{D}_J^S = \{\sigma \in \mathfrak{S}_n \mid J \subseteq D_R(\sigma)\}$ and that $\mathcal{D}_J^S = [w_0(J), w_0]_L$ by Corollary 2.2.12 for $J \subseteq S$.

Theorem 3.4.17. *Let $\alpha/\!\!/\beta$ be a pacific skew composition of size n with source tableau T_0, S be the set of simple reflections of $\mathfrak{S}_n$ and $\boldsymbol{P}_J = H_n(0)\pi_J\bar{\pi}_{J^c}$ for $J \subseteq S$.*

(1) *Let $J_{T_0} \subseteq J \subseteq S$. The $H_n(0)$-modules $\boldsymbol{P}_J$ and $\boldsymbol{P}_J T_0$ are isomorphic via the map $a \mapsto aT_0$. The module $\boldsymbol{P}_J T_0$ is generated by $\pi_J\bar{\pi}_{J^c}T_0$ and it has a $\mathbb{K}$-basis*

$$\left\{ \sum_{\sigma \in \rho(\mathfrak{S}_n)_{J^c}} (-1)^{\ell(\rho w_0(J^c)) - \ell(\sigma)} \pi_\sigma T_0 \mid \rho \in \mathcal{D}_J \right\}$$

In particular, $\dim \boldsymbol{P}_J T_0 = |\mathcal{D}_J|$. *Moreover,*

$$\operatorname{top}(\boldsymbol{P}_J T_0) \cong \boldsymbol{F}_{J^c} \quad \textit{and} \quad \operatorname{soc}(\boldsymbol{P}_J T_0) \cong \boldsymbol{F}_{\nu(J^c)}$$

where w_0 is the longest element of $\mathfrak{S}_n$ and $\nu\colon \mathfrak{S}_n \to \mathfrak{S}_n, \sigma \mapsto w_0\sigma w_0^{-1}$.

(2) *$\boldsymbol{S}_{\alpha/\!\!/\beta} = \bigoplus_{J_{T_0}\subseteq J\subseteq S} \boldsymbol{P}_J T_0$ is a decomposition into indecomposable submodules.*

(3) *$\boldsymbol{S}_{\alpha/\!\!/\beta}$ is projective and has dimension $\left|\mathcal{D}_{J_{T_0}}^S\right|$.*

Proof. Assume that $\boldsymbol{S}_{\alpha/\!\!/\beta}$ is pacific. Throughout this proof $\bigoplus$ refers to the outer direct sum, all homomorphisms are $H_n(0)$-homomorphisms and all (direct) sums indexed by J run over the set $\{J \mid J_{T_0} \subseteq J \subseteq S\}$.

There are natural epimorphisms

$$\phi_J\colon \boldsymbol{P}_J \to \boldsymbol{P}_J T_0, \quad a \mapsto aT_0$$

for $J_{T_0} \subseteq J \subseteq S$. Let

$$\phi\colon \bigoplus_J \boldsymbol{P}_J \to \bigoplus_J \boldsymbol{P}_J T_0, \quad (a_J)_J \mapsto (\phi_J(a_J))_J$$

be the corresponding epimorphism of direct sums. Since $\alpha/\!\!/\beta$ is pacific, Lemma 3.4.5 yields that $\operatorname{SCT}(\alpha/\!\!/\beta)$ is an equivalence class under $\sim$. Thus, Lemma 3.4.15 implies that

Table 3.1: Dimensions and generators of the modules $\boldsymbol{P}_J T_0$ decomposing the pacific module $\boldsymbol{S}_{\alpha/\!\!/\beta}$ for $\alpha/\!\!/\beta$ from Figure 3.3.

J	$\dim \boldsymbol{P}_J T_0$	Generator $\pi_J \bar{\pi}_{J^c} T_0$ of $\boldsymbol{P}_J T_0$
$\{s_1\}$	3	$\begin{smallmatrix} &&&1\\ &&3&\\ 4&2&& \end{smallmatrix} - \begin{smallmatrix} &&&1\\ &&4&\\ 3&2&& \end{smallmatrix} - \begin{smallmatrix} &&&3\\ &&1&\\ 4&2&& \end{smallmatrix} + \begin{smallmatrix} &&&3\\ &&4&\\ 2&1&& \end{smallmatrix} + \begin{smallmatrix} &&&4\\ &&1&\\ 3&2&& \end{smallmatrix} - \begin{smallmatrix} &&&4\\ &&3&\\ 2&1&& \end{smallmatrix}$
$\{s_1, s_2\}$	3	$\begin{smallmatrix} &&&1\\ &&4&\\ 3&2&& \end{smallmatrix} - \begin{smallmatrix} &&&4\\ &&1&\\ 3&2&& \end{smallmatrix}$
$\{s_1, s_3\}$	5	$\begin{smallmatrix} &&&3\\ &&1&\\ 4&2&& \end{smallmatrix} - \begin{smallmatrix} &&&3\\ &&4&\\ 2&1&& \end{smallmatrix}$
$\{s_1, s_2, s_3\}$	1	$\begin{smallmatrix} &&&1\\ &&2&\\ 4&3&& \end{smallmatrix}$

$\boldsymbol{S}_{\alpha/\!\!/\beta} = \sum_{J_{T_0} \subseteq J \subseteq S} \boldsymbol{P}_J T_0$. Therefore, $\psi\colon \bigoplus_J \boldsymbol{P}_J T_0 \to \boldsymbol{S}_{\alpha/\!\!/\beta}$, $(x_J)_J \mapsto \sum_J x_J$ is another epimorphism. We have

$$\begin{aligned} \dim \boldsymbol{S}_{\alpha/\!\!/\beta} &\leq \dim \bigoplus_J \boldsymbol{P}_J T_0 && (\psi \text{ is an epimorphism}) \\ &\leq \dim \bigoplus_J \boldsymbol{P}_J && (\phi \text{ is an epimorphism}) \\ &= \sum_J \dim \boldsymbol{P}_J \\ &= \sum_J |\mathcal{D}_J| && (\text{Theorem 2.3.5}) \\ &= \left|\mathcal{D}^S_{J_{T_0}}\right| \\ &= |[w_0(J_{T_0}), w_0]_L| && (\text{Corollary 2.2.12}) \\ &= |\mathrm{SCT}(\alpha/\!\!/\beta)| && (\text{Proposition 3.4.12}) \\ &= \dim \boldsymbol{S}_{\alpha/\!\!/\beta}. \end{aligned}$$

That is, $\dim \boldsymbol{S}_{\alpha/\!\!/\beta} = \dim \bigoplus_J \boldsymbol{P}_J T_0 = \dim \bigoplus_J \boldsymbol{P}_J$. Consequently ψ, ϕ and all the ϕ_J are isomorphisms.

Let $J_{T_0} \subseteq J \subseteq S$. We obtain the claimed basis of $\boldsymbol{P}_J T_0$ by applying ϕ_J on the basis of $\boldsymbol{P}_J$ from Corollary 2.3.8. The statements about top, socle and indecomposability are transferred from Theorem 2.3.5 by ϕ_J as well. As $\boldsymbol{S}_{\alpha/\!\!/\beta}$ is the direct sum of projective modules, it is projective too. We have seen above that $\dim \boldsymbol{S}_{\alpha/\!\!/\beta} = \left|\mathcal{D}^S_{J_{T_0}}\right|$. $\square$

Remark 3.4.18. Let $\boldsymbol{S}_{\alpha/\!\!/\beta}$ be pacific. Then it is projective and by [KT97, Proposition 5.9] the quasisymmetric characteristic of $\boldsymbol{S}_{\alpha/\!\!/\beta}$, $Ch(\boldsymbol{S}_{\alpha/\!\!/\beta})$, is a symmetric function. That is, the quasisymmetric Schur function $\mathcal{S}_{\alpha/\!\!/\beta}$ is a symmetric function if $\alpha/\!\!/\beta$ is pacific.

Example 3.4.19. Consider the pacific skew composition $\alpha/\!\!/\beta = (5, 4, 3)/\!\!/(4, 3, 1)$ and its source tableau T_0. The SCTx of shape $\alpha/\!\!/\beta$ are shown in Figure 3.3. Then $J_{T_0} = \{s_1\}$

and Theorem 3.4.17 yields that

$$\boldsymbol{S}_{\alpha/\!\!/\beta} = \boldsymbol{P}_{\{s_1\}}T_0 \oplus \boldsymbol{P}_{\{s_1,s_2\}}T_0 \oplus \boldsymbol{P}_{\{s_1,s_3\}}T_0 \oplus \boldsymbol{P}_{\{s_1,s_2,s_3\}}T_0$$

is a decomposition into indecomposable $H_4(0)$-modules. The dimension and a generator for each of them is shown in Table 3.1.

Let $J = \{s_1, s_2\}$. We determine a basis of $\boldsymbol{P}_J T_0$. We have $J^c = \{s_3\}$,

$$\mathcal{D}_J = \{\rho_1 = s_1s_2s_1, \rho_2 = s_3s_1s_2s_1, \rho_3 = s_2s_3s_1s_2s_1\} \quad \text{and} \quad (\mathfrak{S}_4)_{J^c} = \{1, s_3\}\,.$$

From Corollary 2.3.8 it follows that the elements

$$\begin{aligned} b_1 &:= \pi_{\rho_1}\bar{\pi}_{J^c} = \pi_1\pi_2\pi_1\pi_3 - \pi_1\pi_2\pi_1, \\ b_2 &:= \pi_{\rho_2}\bar{\pi}_{J^c} = \pi_3\pi_1\pi_2\pi_1\pi_3 - \pi_3\pi_1\pi_2\pi_1, \\ b_3 &:= \pi_{\rho_3}\bar{\pi}_{J^c} = \pi_2\pi_3\pi_1\pi_2\pi_1\pi_3 - \pi_2\pi_3\pi_1\pi_2\pi_1 \end{aligned}$$

form a basis of $\boldsymbol{P}_J$. By Theorem 3.4.17, a basis of $\boldsymbol{P}_J T_0$ is given by the elements

$$b_1T_0 = \begin{array}{cccc} & & & \boxed{1} \\ & & \boxed{4} & \\ \boxed{3} & \boxed{2} & & \end{array} - \begin{array}{cccc} & & & \boxed{4} \\ & & \boxed{1} & \\ \boxed{3} & \boxed{2} & & \end{array},$$

$$b_2T_0 = \begin{array}{cccc} & & & \boxed{1} \\ & & \boxed{3} & \\ \boxed{4} & \boxed{2} & & \end{array} - \begin{array}{cccc} & & & \boxed{3} \\ & & \boxed{1} & \\ \boxed{4} & \boxed{2} & & \end{array},$$

$$b_3T_0 = \begin{array}{cccc} & & & \boxed{1} \\ & & \boxed{2} & \\ \boxed{4} & \boxed{3} & & \end{array} - \begin{array}{cccc} & & & \boxed{2} \\ & & \boxed{1} & \\ \boxed{4} & \boxed{3} & & \end{array}.$$

Note that

$$b_1T_0 \xrightarrow{\pi_3} b_2T_0 \xrightarrow{\pi_2} b_3T_0.$$

One may check that we obtain the decomposition of $\boldsymbol{S}_{(3,1)/\!\!/(2)}$ given in (3.11) also from an application of Theorem 3.4.17.

We can reformulate the decomposition of the pacific module $\boldsymbol{S}_{\alpha/\!\!/\beta}$ from Theorem 3.4.17 in a more combinatorial fashion. We call $i \in [n-1]$ a *descent* of $\sigma \in \mathfrak{S}_n$ if $\sigma(i) > \sigma(i+1)$.

Corollary 3.4.20. *Let $\alpha/\!\!/\beta$ be a pacific skew composition of size n with source tableau T_0. For $D \subseteq [n-1]$ we set $\boldsymbol{P}_D := \boldsymbol{P}_J$ where $J = \{s_i \in S \mid i \notin D\}$.*

(1) For $D \subseteq D(T_0)$ the $H_n(0)$-modules $\boldsymbol{P}_D T_0$ is isomorphic to $\boldsymbol{P}_D$. Its dimension is the number of $\sigma \in \mathfrak{S}_n$ with descent set $[n-1] \setminus D$. Furthermore,

$$\operatorname{top}(P_D T_0) \cong \boldsymbol{F}_D \quad \textit{and} \quad \operatorname{soc}(P_D T_0) \cong \boldsymbol{F}_{n-D}$$

as $H_n(0)$-modules where $\boldsymbol{F}_{n-D}$ is the simple $H_n(0)$-module indexed by the set

$$n - D := \{n - d \mid d \in D\}\,.$$

(2) $\boldsymbol{S}_{\alpha/\!\!/\beta} = \bigoplus_{D \subseteq D(T_0)} \boldsymbol{P}_D T_0$ *is a decomposition into indecomposable submodules. The dimension of* $\boldsymbol{S}_{\alpha/\!\!/\beta}$ *is the number of* $\sigma \in \mathfrak{S}_n$ *whose descent set contains* $D^c(T_0)$*.*

Proof. Recall that $J_{T_0} = \{s_i \in S \mid i \in D^c(T_0)\}$. Hence,

$$D \subseteq D(T_0) \text{ if an only if } J_{T_0} \subseteq \{s_i \in S \mid i \notin D\}. \tag{3.12}$$

Now fix an $D \subseteq D(T_0)$ and let $J := \{s_i \in S \mid i \notin D\}$, w_0 be the longest element of $\mathfrak{S}_n$ and $\nu\colon \mathfrak{S}_n \to \mathfrak{S}_n$, $\sigma \mapsto w_0 \sigma w_0^{-1}$. Then $\boldsymbol{P}_D = \boldsymbol{P}_J$ and Theorem 3.4.17 yields that $\boldsymbol{P}_D T_0$ is isomorphic to $\boldsymbol{P}_D$, has dimension $|\mathcal{D}_J|$, top isomorphic to $\boldsymbol{F}_{J^c}$ and socle isomorphic to $\boldsymbol{F}_{\nu(J^c)}$. Recall from Section 2.2 that $D_R(\sigma) = \{s_i \in S \mid \sigma(i) > \sigma(i+1)\}$ for $\sigma \in \mathfrak{S}_n$. Therefore,

$$\begin{aligned}\mathcal{D}_J &= \{\sigma \in \mathfrak{S}_n \mid J = D_R(\sigma)\}\\ &= \{\sigma \in \mathfrak{S}_n \mid \sigma(i) > \sigma(i+1) \text{ if and only if } i \in [n-1] \setminus D\}\,.\end{aligned}$$

This yields the claim on the dimension of $\boldsymbol{P}_D T_0$.

By the definition of $\boldsymbol{F}_D$ in Section 2.3, we have $\boldsymbol{F}_{J^c} = \boldsymbol{F}_D$ and therefore $\operatorname{top}(\boldsymbol{P}_D T_0) = \boldsymbol{F}_D$. Regarding $\operatorname{soc}(\boldsymbol{P}_D T_0)$, it remains to show that $\boldsymbol{F}_{\nu(J^c)} = \boldsymbol{F}_{n-D}$. Recall that $w_0(i) = n - i + 1$ for each $i \in [n]$. Thus,

$$\nu(s_i) = w_0(i, i+1) w_0^{-1} = (n-i+1, n-i) = s_{n-i}$$

for all $i \in [n-1]$. As a consequence,

$$\nu(J^c) = \{s_{n-i} \mid s_i \in J^c\} = \{s_{n-i} \mid i \in D\} = \{s_i \mid i \in n - D\}$$

Therefore, $\boldsymbol{F}_{\nu(J^c)} = \boldsymbol{F}_{n-D}$ as desired.

From (3.12) and Theorem 3.4.17 we obtain the decomposition

$$\boldsymbol{S}_{\alpha/\!\!/\beta} = \bigoplus_{D \subseteq D(T_0)} \boldsymbol{P}_D T_0$$

and that $\dim \boldsymbol{S}_{\alpha/\!\!/\beta} = \left|\mathcal{D}^S_{J_{T_0}}\right|$. In addition,

$$\begin{aligned}\mathcal{D}^S_{J_{T_0}} &= \{\sigma \in \mathfrak{S}_n \mid J_{T_0} \subseteq D_R(\sigma)\}\\ &= \{\sigma \in \mathfrak{S}_n \mid \sigma(i) > \sigma(i+1) \text{ for all } i \in D^c(T_0)\}\,.\end{aligned}$$

Thus, we also get the statement on $\dim \boldsymbol{S}_{\alpha/\!\!/\beta}$. $\square$

From Corollary 3.4.20 we obtain combinatorial rules for top and socle of pacific modules $\boldsymbol{S}_{\alpha/\!\!/\beta}$.

Corollary 3.4.21. *Let $\boldsymbol{S}_{\alpha /\!\!/ \beta}$ be pacific with source tableau T_0 and $n = |\alpha /\!\!/ \beta|$. Then*

$$\operatorname{top}(\boldsymbol{S}_{\alpha /\!\!/ \beta}) \cong \bigoplus_{D \subseteq D(T_0)} \boldsymbol{F}_D \quad \textit{and} \quad \operatorname{soc}(\boldsymbol{S}_{\alpha /\!\!/ \beta}) \cong \bigoplus_{D \subseteq D(T_0)} \boldsymbol{F}_{n-D}$$

as $H_n(0)$-modules where $n - D = \{n - d \mid d \in D\}$.

Proof. From Corollary 3.4.20 it follows that

$$\operatorname{top}(\boldsymbol{S}_{\alpha /\!\!/ \beta}) = \operatorname{top}(\bigoplus_{D \subseteq D(T_0)} \boldsymbol{P}_D T_0) = \bigoplus_{D \subseteq D(T_0)} \operatorname{top}(\boldsymbol{P}_D T_0) \cong \bigoplus_{D \subseteq D(T_0)} \boldsymbol{F}_D,$$

where we use for the second equality that the operator top is compatible with direct sums. In the same way we obtain the formula for the socle. $\square$

The topic of Section 3.5 is to generalize the formula for the top from Corollary 3.4.21 to arbitrary modules $\boldsymbol{S}_{\alpha /\!\!/ \beta}$. In Section 3.6 we do the same for the socle.

Example 3.4.22. Consider the pacific $H_4(0)$-module $\boldsymbol{S}_{\alpha /\!\!/ \beta}$ with source tableau

$$T_0 = \begin{array}{cccc} & & & \boxed{4} \\ & & \boxed{3} & \\ \boxed{2} & \boxed{1} & & \end{array}.$$

Then $D(T_0) = \{2, 3\}$ and Corollary 3.4.21 yields

$$\begin{aligned} \operatorname{top}(\boldsymbol{S}_{\alpha /\!\!/ \beta}) &\cong \boldsymbol{F}_\emptyset \oplus \boldsymbol{F}_{\{2\}} \oplus \boldsymbol{F}_{\{3\}} \oplus \boldsymbol{F}_{\{2,3\}}, \\ \operatorname{soc}(\boldsymbol{S}_{\alpha /\!\!/ \beta}) &\cong \boldsymbol{F}_\emptyset \oplus \boldsymbol{F}_{\{1\}} \oplus \boldsymbol{F}_{\{2\}} \oplus \boldsymbol{F}_{\{1,2\}}. \end{aligned}$$

We end this section with two consequences of Theorem 3.4.17. First we characterize the indecomposable pacific modules $\boldsymbol{S}_{\alpha /\!\!/ \beta}$ and second the modules $\boldsymbol{S}_{\alpha /\!\!/ \beta, E}$ which are isomorphic to $H_n(0)$. Recall that the components of the diagram of a pacific skew composition are connected horizontal strips that do not attack each other.

Proposition 3.4.23. *Let $\alpha /\!\!/ \beta$ be a pacific skew composition of size n. Then the following are equivalent.*

(1) $\alpha /\!\!/ \beta$ is a single connected horizontal strip.

(2) $\boldsymbol{S}_{\alpha /\!\!/ \beta}$ is indecomposable.

(3) $\boldsymbol{S}_{\alpha /\!\!/ \beta}$ is isomorphic to the simple $H_n(0)$-module $\boldsymbol{F}_\emptyset$.

Proof. Let T_0 be the source tableau of pacific shape $\alpha /\!\!/ \beta$, $m := |D(T_0)|$ and I_k for $k = 1, \dots, m+1$ be the intervals associated to T_0. Then

$$\begin{aligned} \boldsymbol{S}_{\alpha /\!\!/ \beta} \text{ is indecomposable} &\iff D(T_0) = \emptyset && \text{(Corollary 3.4.20)} \\ &\iff m = 0 \text{ and } I_1 = [n] \\ &\iff \alpha /\!\!/ \beta \text{ is a single} && (\alpha /\!\!/ \beta = \operatorname{sh}(T_0)) \\ &\qquad \text{connected horizontal strip.} \end{aligned}$$

That is, (1) is equivalent to (2). Moreover, if $D(T_0) = \emptyset$ then $\pi_i T_0 = T_0$ for $i = 1, \ldots, n-1$. Hence, $\boldsymbol{S}_{\alpha/\!\!/\beta}$ is isomorphic to $\boldsymbol{F}_\emptyset$ in this case. Thus (2) implies (3). Clearly, (3) also implies (2). $\square$

Let $\alpha/\!\!/\beta$ be a pacific skew composition and $E := \mathrm{SCT}(\alpha/\!\!/\beta)$. By Lemma 3.4.5, $\boldsymbol{S}_{\alpha/\!\!/\beta} = \boldsymbol{S}_{\alpha/\!\!/\beta,E}$. From Proposition 3.4.23 we know that apart from the case where $\alpha/\!\!/\beta$ is a single connected horizontal strip, $\boldsymbol{S}_{\alpha/\!\!/\beta,E}$ is decomposable. In contrast, the straight modules $\boldsymbol{S}_{\alpha,E}$ for $\alpha \vDash n$ and $E \in \mathcal{E}(\alpha)$ are always indecomposable by Theorem 3.3.11.

From Theorem 3.4.17 we obtain a characterization of the modules $\boldsymbol{S}_{\alpha/\!\!/\beta,E}$ which are isomorphic to $H_n(0)$.

Proposition 3.4.24. *Let $\alpha/\!\!/\beta$ be a skew composition of size n and $E \in \mathcal{E}(\alpha/\!\!/\beta)$ with source tableau T_0. Then $\boldsymbol{S}_{\alpha/\!\!/\beta,E} \cong H_n(0)$ as $H_n(0)$-modules if and only if $nAD(T_0) = [n-1]$.*

Proof. Let T_1 be the sink tableau of E.

Assume first that $H_n(0) \cong \boldsymbol{S}_{\alpha/\!\!/\beta,E}$. Since E is a basis of $S_{\alpha/\!\!/\beta,E}$ and $H_n(0)$ has a $\mathbb{K}$-basis indexed by $\mathfrak{S}_n$, it follows that $|E| = |\mathfrak{S}_n|$. Thus, Theorem 3.1.18 implies $[\mathrm{col}_{T_0}, \mathrm{col}_{T_1}]_L = \mathfrak{S}_n$ and that the map $\mathfrak{S}_n \to E$, $\sigma \mapsto \pi_\sigma T_0$ is well defined and injective. In particular, $\pi_i T_0 \in E \setminus \{T_0\}$ for all $i \in [n-1]$, i.e. $nAD(T_0) = [n-1]$.

Assume now that $nAD(T_0) = [n-1]$. Then we have that $I_k = \{d_k\}$ for the interval I_k of T_0 and $k = 1, \ldots, n$. Thus, $nAD(T_0) = [n-1]$ implies $I_k \not\rightsquigarrow I_{k+1}$ for $k = 1, \ldots, n-1$. Hence, it follows from Lemma 3.4.7 and Lemma 3.4.8 that T_0 is pacific. As a consequence, $\alpha/\!\!/\beta$ is pacific and $\boldsymbol{S}_{\alpha/\!\!/\beta,E} = \boldsymbol{S}_{\alpha/\!\!/\beta}$. In addition, $nAD(T_0) = [n-1]$ implies $J_{T_0} = \emptyset$. Thus, Theorem 3.4.17 yields

$$\boldsymbol{S}_{\alpha/\!\!/\beta,E} \cong \bigoplus_{J \subseteq S} \boldsymbol{P}_J.$$

Lastly, $\bigoplus_{J \subseteq S} \boldsymbol{P}_J = H_n(0)$ by Theorem 2.3.5. $\square$

Example 3.4.25. Let $\alpha/\!\!/\beta$ be the shape of size 3 of the source tableau

$$T_0 = \begin{array}{ccc} & & \boxed{3} \\ & \boxed{2} & \\ \boxed{1} & & \end{array}.$$

Then $nAD(T_0) = \{1, 2\}$ and the $H_3(0)$-modules $\boldsymbol{S}_{\alpha/\!\!/\beta}$ and $H_3(0)$ are isomorphic.

3.5 The top of skew modules

Let $\alpha/\!\!/\beta$ be a skew composition of size n. In this section we seek a combinatorial formula for $\mathrm{top}(\boldsymbol{S}_{\alpha/\!\!/\beta})$. This formula is stated in Corollary 3.5.46. It generalizes the one for pacific modules from Corollary 3.4.21.

From Proposition 3.1.13 we have that $\boldsymbol{S}_{\alpha/\!/\beta} = \bigoplus_{E\in\mathcal{E}(\alpha/\!/\beta)} \boldsymbol{S}_{\alpha/\!/\beta,E}$. It follows that $\operatorname{top}(\boldsymbol{S}_{\alpha/\!/\beta}) = \bigoplus_{E\in\mathcal{E}(\alpha/\!/\beta)} \operatorname{top}(\boldsymbol{S}_{\alpha/\!/\beta,E})$. Therefore, our main objective is to determine $\operatorname{top}(\boldsymbol{S}_{\alpha/\!/\beta,E})$ for $E \in \mathcal{E}(\alpha/\!/\beta)$. This is done in Theorem 3.5.42.

The section can be divided in two parts. In the first part we develop the combinatorics that we use in the second part in order to determine the radical and the top of $\boldsymbol{S}_{\alpha/\!/\beta,E}$.

Note that a fraction of the new terms of this section is sufficient for only formulating Theorem 3.5.42: the horizontal strips B_k which we will introduce below and Definition 3.5.10.

Let $E \in \mathcal{E}(\alpha/\!/\beta)$. As in Section 3.4, we will use the descents of the source tableau T_0 of E in order to decompose $[n]$ into intervals $I_1, \dots, I_{m+1}$. The preimages under T_0 of these intervals $B_1, \dots, B_{m+1}$ then form a set partition of the diagram of $\alpha/\!/\beta$. The relative positions of the B_k play an crucial role in the determination of $\operatorname{top}(\boldsymbol{S}_{\alpha/\!/\beta,E})$. In particular, for $k \in [m]$ it is important whether $B_k \rightsquigarrow B_{k+1}$ or not. We fix some notation for the entire section.

Notation 3.5.1. Let $\alpha/\!/\beta$ be a skew composition of size n, $E \in \mathcal{E}(\alpha/\!/\beta)$, T_0 be the source tableau of E and $d_0 = 0 < d_1 < \cdots < d_{m+1} = n$ be integers such that

$$D(T_0) = \{d_1, d_2, \dots, d_m\}\,.$$

For $k, l \in [m+1]$ with $k \le l$ define integer intervals

$$I_{k,l} := [d_{k-1}+1, d_l], \quad \mathring{I}_{k,l} := I_{k,l} \setminus \{d_l\} \quad \text{and} \quad I_k := I_{k,k}.$$

Then $I_{k,l} = \bigcup_{j=k}^{l} I_j$ and $I_k = [d_{k-1}+1, d_k]$. Set $B_{k,l} := T_0^{-1}(I_{k,l})$ and $B_k := T_0^{-1}(I_k)$. Note that m, the d_k and the I_k are defined as in Section 3.4. Recall that because T_0 is a source tableau, we have $D^c(T_0) = ND^c(T_0)$ and therefore B_k is a connected horizontal strip. As $B_{k,l}$ is the union of the connected horizontal strips $B_k, B_{k+1} \dots, B_l$, we call it (*horizontal*) *strip sequence*. Note that $B_{k,l}$ can be realized as the diagram of a skew composition. Accordingly, we call $B_{k,l}$ *pacific* if no cell of $B_{k,l}$ attacks another cell of $B_{k,l}$.

Example 3.5.2. Consider

$$T_0 = \begin{array}{|c|c|c|c|}\cline{1-1} \mathbf{1} & \multicolumn{3}{c}{} \\ \hline \mathbf{6} & 5 & 4 & 3 \\ \hline 8 & 7 & \mathbf{2} & \multicolumn{1}{c}{} \\ \cline{1-3} \end{array}$$

where the descents of T_0 are printed boldface. This is the source tableau from Figure 3.1. Then $d_0 = 0$, $d_4 = 8$ and $D(T_0) = \{d_1 = 1, d_2 = 2, d_3 = 6\}$. Moreover, $I_1 = \{1\}$, $I_2 = \{2\}$, $I_3 = \{3, 4, 5, 6\}$ and $I_4 = \{7, 8\}$. The cells of the same connected horizontal strip B_k are filled with the same shade of gray for $k = 1, \dots, 4$. Observe $B_1 \not\rightsquigarrow B_2$, $B_2 \rightsquigarrow B_3$ and $B_3 \rightsquigarrow B_4$. Hence, $B_{1,2}$ is the only pacific strip sequence $B_{k,l}$ with $k < l$.

Horizontal strip sequences

We begin with some basic lemmas on pacific strip sequences $B_{k,l}$. The first is an immediate consequence of the definitions.

Lemma 3.5.3. *Let $B_{k,l}$ be a pacific strip sequence and $T \in E$ with $T(B_{k,l}) = I_{k,l}$. Then each descent of T contained in $\mathring{I}_{k,l}$ is non-attacking.*

In Example 3.5.2 we have that $B_{1,2}$ is pacific, $T_0(B_{1,2}) = I_{1,2}$, $1 \in D(T_0)$ and $1 \in \mathring{I}_{1,2}$. Indeed, 1 is non-attacking. Moreover, B_1 is strictly left of B_2 which is a consequence of the next result.

Lemma 3.5.4. *Let $B_{k,l}$ be a pacific strip sequence. For $j = k, \dots, l-1$ we have that $B_j \not\rightharpoonup B_{j+1}$ and B_j is strictly left of B_{j+1}.*

Proof. Let $j \in [k, l-1]$. As $B_{k,l}$ is pacific, we have $B_j \not\rightharpoonup B_{j+1}$. Now, Lemma 3.4.7 yields that B_j is strictly left of B_{j+1}. $\square$

Definition 3.5.5. *For $T \in E$ and a strip sequence $B_{k,l}$, define $\mathrm{col}_{B_{k,l},T}$ to be the column word of the tableau obtained by restricting T to $B_{k,l}$. We say that T is $B_{k,l}$-*sorted *if*

$$\mathrm{col}_{B_{k,l},T} = d_l \; d_l - 1 \; \cdots \; d_{k-1} + 1.$$

Example 3.5.6. (1) We show that T_0 is B_k-sorted for each $k \in [m+1]$. Let $k \in [m+1]$. By definition, B_k is a single connected horizontal strip. In T_0 this strip is filled from left to right with d_k, $d_k - 1, \dots, d_{k-1} + 1$. Thus, $\mathrm{col}_{B_k,T_0} = d_k d_k - 1 \dots d_{k-1} + 1$, i.e. T_0 is B_k-sorted.

(2) In the situation of Example 3.5.2, we have

$$\mathrm{col}_{B_1,T_0} = 1, \quad \mathrm{col}_{B_2,T_0} = 2 \quad \text{and} \quad \mathrm{col}_{B_{1,2},T_0} = 12.$$

That is, T_0 is B_1- and B_2-sorted but not $B_{1,2}$-sorted.

(3) Let E be the equivalence class from Figure 3.3. Its source and sink tableau are

$$T_0 = \begin{array}{cccc} & & & \boxed{4} \\ & & \boxed{3} & \\ \boxed{2} & \boxed{1} & & \end{array} \quad \text{and} \quad T_1 = \begin{array}{cccc} & & & \boxed{1} \\ & & \boxed{2} & \\ \boxed{4} & \boxed{3} & & \end{array},$$

respectively. We have $I_1 = \{1, 2\}$, $I_2 = \{3\}$ and $I_3 = \{4\}$. The cells of B_k have the same shade for $k = 1, 2, 3$. Then

$$\mathrm{col}_{B_{1,3},T_0} = 2134,$$
$$\mathrm{col}_{B_{1,3},T_1} = 4321.$$

Thus, T_1 is $B_{1,3}$-sorted but T_0 is not.

If $B_{k,l}$ is pacific then it has at most one cell per column. Therefore, Lemma 3.5.4 implies the following.

Lemma 3.5.7. *Let $B_{k,l}$ be a pacific strip sequence and $T \in E$. The column word $\mathrm{col}_{B_{k,l},T}$ is the concatenation of the words $\mathrm{col}_{B_k,T}, \mathrm{col}_{B_{k+1},T}, \dots, \mathrm{col}_{B_l,T}$. That is, we obtain $\mathrm{col}_{B_{k,l},T}$ by reading the entries of T in $B_{k,l}$ from left to right.*

We characterize the property of being $B_{k,l}$ sorted for a pacific strip sequence $B_{k,l}$. Recall that for two sets of integers A and B we write $A < B$ if $a < b$ for all $a \in A$ and $b \in B$.

Lemma 3.5.8. *Let $T \in E$ and $B_{k,l}$ be a pacific strip sequence such that $T(B_{k,l}) = I_{k,l}$. Then the following are equivalent.*

(*1*) *T is $B_{k,l}$-sorted.*

(*2*) *$D(T) \cap \mathring{I}_{k,l} = \emptyset$.*

(*3*) *$T(B_k) > T(B_{k+1}) > \cdots > T(B_l)$.*

Proof. We show (1) $\implies$ (3) $\implies$ (2) $\implies$ (1).

The first implication is a consequence of Lemma 3.5.7.

In order to show the implication from (3) to (2), suppose that (3) holds and let $i \in \mathring{I}_{k,l}$. Then $i+1 \in I_{k,l}$. Thus, both i and $i+1$ are entries of T in $B_{k,l}$. If i and $i+1$ appear in the same connected horizontal strip then $i+1$ is strictly left of i since entries decrease in rows of SCTx from left to right. Thus, i is an ascent of T.

If i and $i+1$ appear in different connected horizontal strips B_r and B_t, respectively then we have $t < r$ by assumption. An application of Lemma 3.5.4 now yields that B_t is strictly left of B_r. Hence, i is an ascent again.

Lastly, we show that (2) implies (1). Assume that (2) holds. Then $\mathring{I}_{k,l} \subseteq D^c(T)$. Thus $i+1 \in T(B_{k,l})$ and $i+1$ is strictly left of i for each $i \in \mathring{I}_{k,l}$. Since $B_{k,l}$ is pacific, it has at most one cell per column. Therefore, it follows that the entries of T in $B_{k,l}$ read from left to right are

$$d_l \; d_{l-1} \; \cdots \; d_{k-1}+1.$$

By Lemma 3.5.7 this is $\mathrm{col}_{B_{k,l},T}$. Thus, T is $B_{k,l}$-sorted. $\square$

Example 3.5.9. Let

$$T_0 = \begin{array}{cccc} & & & \boxed{4} \\ & & \boxed{3} & \\ \boxed{2} & \boxed{1} & & \end{array} \quad \text{and} \quad T_1 = \begin{array}{cccc} & & & \boxed{1} \\ & & \boxed{2} & \\ \boxed{4} & \boxed{3} & & \end{array}$$

be the skew tableaux from Example 3.5.6. The cells of B_k have the same shade for $k = 1, 2, 3$. Note that $\mathring{I}_{1,3} = \{1, 2, 3\}$. We have already checked that T_1 is $B_{1,3}$-sorted. Moreover,

$$D(T_1) \cap \mathring{I}_{1,3} = \emptyset \quad \text{and} \quad T_1(B_1) > T_1(B_2) > T_1(B_3)$$

in accordance with Lemma 3.5.8.

Offensive descents and D-sortable tableaux

We generalize the concept of attacking descents of T_0.

Definition 3.5.10. *The set of* offensive descents *of T_0 is given by*

$$OD(T_0) := \{d_k \in D(T_0) \mid B_k \rightsquigarrow B_{k+1}\}.$$

We write

$$\mathcal{OD} := \{D \subseteq [n-1] \mid OD(T_0) \subseteq D \subseteq D(T_0)\}$$

for the subsets of $D(T_0)$ containing $OD(T_0)$.

If we have $d_k \in AD(T_0)$ then $d_k \rightsquigarrow_{T_0} d_k + 1$ so that $B_k \rightsquigarrow B_{k+1}$. That is, each attacking descent of T_0 indeed is an offensive descent of T_0. The set $\mathcal{OD}$ is the main datum in the formula for $\mathrm{top}(\boldsymbol{S}_{\alpha/\!\!/\beta,E})$ in Theorem 3.5.42. We emphasize that the set $\mathcal{OD}$ has nothing to do with the right descent classes $\mathcal{D}_I^J$ of a Coxeter group W defined in Section 2.2.

Example 3.5.11. Let T_0 be the straight source tableau from Example 3.5.2. There we have already noted that $D(T_0) = \{1, 2, 6\}$, $B_1 \not\rightsquigarrow B_2$, $B_2 \rightsquigarrow B_3$ and $B_3 \rightsquigarrow B_4$. Therefore $OD(T_0) = \{2, 6\}$ and $\mathcal{OD} = \{\{2, 6\}, \{1, 2, 6\}\}$.

Recall that for the set partitions $I_1, \ldots I_{m+1}$ of $[n]$ and $B_1, \ldots, B_{m+1}$ of the diagram of $\alpha/\!\!/\beta$ we have that $T_0(B_r) = I_r$ for each $r \in [m+1]$. This set partitions are associated to $D(T_0)$ as we divided $[n]$ according to the elements of $D(T_0)$. We now consider pairs of coarser set partitions I_{k_r,l_r} and B_{k_r,l_r} for $r \in [p]$ with $p \in [m+1]$ and pacific strip sequences B_{k_r,l_r} each given by a $D \in \mathcal{OD}$. For each such pair, we are interested in the $T \in E$ satisfying $T(B_{k_r,l_r}) = I_{k_r,l_r}$ for all $r \in [p]$.

Notation 3.5.12. Let $D \in \mathcal{OD}$. We associate the following notation to D. Let $p \in [m+1]$ and indices $l_0 < l_1 < \cdots < l_p$ be such that

$$d_{l_0} = 0, \quad D = \left\{d_{l_1}, d_{l_2}, \ldots, d_{l_{p-1}}\right\} \quad \text{and} \quad d_{l_p} = n.$$

In addition, set $k_r := l_{r-1} + 1$ for $r \in [p]$.

Then for $r \in [p]$ we have $I_{k_r,l_r} = [d_{l_{r-1}} + 1, d_{l_r}]$. Thus, the I_{k_r,l_r} for $r \in [p]$ form a set partition of $[n]$. That is, the strip sequences B_{k_r,l_r} for $r \in [p]$ form a set partition of the diagram of $\alpha/\!\!/\beta$. Moreover, the B_{k_r,l_r} are pacific since $OD(T_0) \subseteq D$.

Example 3.5.13. Let T_0 be the source tableau from Example 3.5.11 and E be its equivalence class. Recall $d_0 = 0$, $d_1 = 1$, $d_2 = 2$, $d_3 = 6$, $d_4 = 8$, $D(T_0) = \{1, 2, 6\}$, $OD(T_0) = \{2, 6\}$ and $\mathcal{OD} = \{OD(T_0), D(T_0)\}$.

We illustrate Notation 3.5.12 regarding $D = OD(T_0)$. Then $p = 3$. The other parameters are shown in the table below.

r	0	1	2	3
l_r	0	2	3	4
k_r	–	1	3	4
d_{l_r}	–	2	6	8
I_{k_r,l_r}	–	$\{1,2\}$	$\{3,4,5,6\}$	$\{7,8\}$
B_{k_r,l_r}	–	$B_{1,2}$	B_3	B_4

Note that we obtain the set partition $\{I_{k_r,l_r} \mid r \in [3]\}$ of $[8]$ by splitting the list $1, 2, \dots, 8$ behind the elements of D, $d_{l_1} = 2$ and $d_{l_2} = 6$. The strip sequences B_{k_r,l_r} are depicted in Example 3.5.16.

Definition 3.5.14. *For $D \in \mathcal{OD}$ define*

$$E_D := \{T \in E \mid T(B_{k_r,l_r}) = I_{k_r,l_r} \textit{ for all } r \in [p]\}$$

*the set of D-*sortable tableaux *of E.*

Let $D, D' \in \mathcal{OD}$. We may consider E_D as a poset with the partial order $\preceq$ inherited from E. Note that T_0 is always and element of E_D because $T_0(B_{k,l}) = I_{k,l}$ for all $k \leq l$. From the definition it also follows that if $D \subseteq D'$ then $E_{D'} \subseteq E_D$. In particular, $E_{D(T_0)} \subseteq E_D \subseteq E_{OD(T_0)}$.

The purpose of the next lemma merely is to illustrate the new notation.

Lemma 3.5.15. *The only element of $E_{D(T_0)}$ is T_0.*

Proof. Let $D = D(T_0)$. Then $p = m + 1$ and $l_r = k_r = r$ for $r = 1, \dots, p$. That is $B_{k_r,l_r} = B_r$ and $I_{k_r,l_r} = I_r$ for $r = 1, \dots, p$. Let $T \in E_D$. Then $T(B_r) = I_r = T_0(B_r)$ for $r = 0, \dots, m+1$. Moreover, B_r is a connected horizontal strip. Thus, there is only one way to fill I_r into B_r in a SCT. Hence $T = T_0$. □

Example 3.5.16. Let T_0 be the source tableau from Example 3.5.11 and E be its equivalence class. Recall $D(T_0) = \{1, 2, 6\}$, $OD(T_0) = \{2, 6\}$ and $\mathcal{OD} = \{OD(T_0), D(T_0)\}$. We determine E_D for each $D \in \mathcal{OD}$.

Regarding $D(T_0)$, Lemma 3.5.15 implies that $E_{D(T_0)} = \{T_0\}$.

For $OD(T_0)$ the I_{k_r,l_r} and B_{k_r,l_r} are given in the table from Example 3.5.13. It follows that $E_{OD(T_0)}$ consists of the elements $T \in E$ with $T(B_{1,2}) = \{1, 2\}$, $T(B_3) = \{3, 4, 5, 6\}$ and $T(B_4) = \{7, 8\}$. Thus, $E_{OD(T_0)}$ consists of the following two tableaux

$$T_0 = \begin{array}{|c|c|c|c|} \cline{1-1} 1 \\ \hline 6 & 5 & 4 & 3 \\ \hline 8 & 7 & 2 \\ \cline{1-3} \end{array} \quad \text{and} \quad \pi_1 T_0 = \begin{array}{|c|c|c|c|} \cline{1-1} 2 \\ \hline 6 & 5 & 4 & 3 \\ \hline 8 & 7 & 1 \\ \cline{1-3} \end{array}$$

where we draw the cells of $B_{1,2}$, B_3 and B_4 with the same shade, respectively.

Example 3.5.17. For the equivalence class E of skew tableaux from Figure 3.3 the sets E_D and elements T_D are given in the table in Example 3.5.28.

Let $D \in \mathcal{OD}$. We will see in Lemma 3.5.26 that there is a unique tableau $T_D \in E$ such that T_D is B_{k_r,l_r}-sorted for each $r \in [p]$. Moreover, it will turn out that T_D is the greatest element of E_D. Thus, for each $T \in E_D$ there are operators $\pi_{i_j}, \ldots, \pi_{i_1}$ such that $T_D = \pi_{i_j} \cdots \pi_{i_1} T$. The operators can be thought of *sorting* the entries in B_{k_r,l_r} of T for each $r \in [p]$ in order to obtain T_D. Therefore, the naming of the set E_D.

We now come to a characterization of the elements of E_D in terms of the content of column words. Recall that for $T \in E$, $\operatorname{cont}(\operatorname{col}_T \operatorname{col}_{T_0}^{-1})$ is the index set of operators π_i establishing the covering relations in each saturated chain from T_0 to T in E.

For $D \in \mathcal{OD}$, define $D^c := [n-1] \setminus D$. Then $D^c = \bigcup_{r=1}^{p} \mathring{I}_{k_r,l_r}$.

Lemma 3.5.18. *Let $T \in E$ and $D \in \mathcal{OD}$. Then the following are equivalent.*

(*1*) $T(B_{k_r,l_r}) = I_{k_r,l_r}$ *for all* $r \in [p]$.

(*2*) $\operatorname{cont}(\operatorname{col}_T \operatorname{col}_{T_0}^{-1}) \subseteq D^c$.

That is,

$$E_D = \left\{ T \in E \mid \operatorname{cont}(\operatorname{col}_T \operatorname{col}_{T_0}^{-1}) \subseteq D^c \right\}.$$

Proof. Let $\sigma := \operatorname{col}_T \operatorname{col}_{T_0}^{-1}$. Since

$$D^c = [n-1] \setminus \left\{ d_{l_1}, \ldots, d_{l_{p-1}} \right\},$$

we have that $\operatorname{cont}(\sigma) \subseteq D^c$ if and only if

$$\operatorname{cont}(\sigma) \subseteq [n-1] \setminus \left\{ d_{l_1}, \ldots, d_{l_{p-1}} \right\}.$$

From Proposition 3.2.9 it follows that this is equivalent to

$$\operatorname{sh}(T^{>d_{l_r}}) = \operatorname{sh}(T_0^{>d_{l_r}}) \text{ for all } r \in [p-1]. \tag{3.13}$$

Let $r \in [p-1]$. By definition $l_r = k_{r+1} - 1$. Therefore,

$$[d_{l_r} + 1, n] = [d_{k_{r+1}-1} + 1, n] = I_{k_{r+1},m+1}.$$

Hence,

$$\operatorname{sh}(T^{>d_{l_r}}) = T^{-1}(I_{k_{r+1},m+1}).$$

Moreover, $I_{k_1,m+1} = I_{1,m+1} = [n]$ so that $T^{-1}(I_{k_1,m+1}) = \alpha /\!\!/ \beta$. Since this also holds for T_0, it follows that (3.13) is equivalent to

$$T^{-1}(I_{k_r,m+1}) = T_0^{-1}(I_{k_r,m+1}) \text{ for all } r \in [p]. \tag{3.14}$$

For $r \in [p-1]$ we have

$$I_{k_r,l_r} = I_{k_r,m+1} \setminus I_{l_r+1,m+1} = I_{k_r,m+1} \setminus I_{k_{r+1},m+1}$$

so that

$$T^{-1}(I_{k_r,l_r}) = T^{-1}(I_{k_r,m+1}) \setminus T^{-1}(I_{k_{r+1},m+1}).$$

In addition,

$$I_{k_p,l_p} = I_{l_{p-1}+1,l_p} = [n] \setminus \bigcup_{r=1}^{p-1} I_{k_r,l_r}.$$

Therefore, (3.14) is equivalent to

$$T^{-1}(I_{k_r,l_r}) = T_0^{-1}(I_{k_r,l_r}) \text{ for all } r \in [p]. \tag{3.15}$$

As $B_{k_r,l_r} = T_0^{-1}(I_{k_r,l_r})$, it follows that (3.15) is equivalent to

$$T(B_{k_r,l_r}) = I_{k_r,l_r} \text{ for all } r \in [p].$$

By combining all the equivalences, we get the equivalence from the claim. The set equality is a reformulation of this equivalence. □

Example 3.5.19. In Example 3.5.16 we have $\mathcal{OD} = \{OD(T_0), D(T_0)\}$ with $OD(T_0) = \{2, 6\}$ and $D(T_0) = \{1, 2, 6\}$.

Consider $D = D(T_0)$. Then $D^c = D^c(T_0)$. The only tableau $T \in E$ which satisfies $\operatorname{cont}(\operatorname{col}_T \operatorname{col}_{T_0}^{-1}) \subseteq D^c$ is T_0. This is also the only element of E_D.

Now consider $D = OD(T_0)$. Then $D^c = \{1, 3, 4, 5, 7\}$. We can see in Figure 3.1 that the elements of $T \in E$ with $\operatorname{cont}(\operatorname{col}_T \operatorname{col}_{T_0}^{-1}) \subseteq \{1, 3, 4, 5, 7\}$ are T_0 and $\pi_1 T_0$. These are the elements of E_D.

We obtain the following properties of the subposet E_D of E from Lemma 3.5.18. Order ideals and filters were defined at the beginning of Chapter 2.

Lemma 3.5.20. *Let $D \in \mathcal{OD}$.*

(1) E_D is an order ideal of E,

(2) $E \setminus E_D$ is a filter of E.

Proof. For $T \in E$ set $\sigma_T := \operatorname{col}_T \operatorname{col}_{T_0}^{-1}$. We show Part (1). Part (2) is a direct consequence of Part (1).

Let $T \in E_D$ and $T' \in E$ such that $T' \preceq T$. Then by Theorem 3.1.18 we have

$$\operatorname{col}_{T_0} \leq_L \operatorname{col}_{T'} \leq_L \operatorname{col}_T .$$

Therefore, we have $\sigma_{T'} \leq_L \sigma_T$. Hence,

$$\operatorname{cont}(\sigma_{T'}) \subseteq \operatorname{cont}(\sigma_T) \subseteq D^c$$

where the left inclusion is a consequence of the definition of the left weak order and the second inclusion an application of Lemma 3.5.18. Hence $T' \in E_D$ by Lemma 3.5.18 again. □

Let $D \in \mathcal{OD}$ and $T \in E_D$. We want to show that $AD(T) \subseteq D \subseteq D(T)$. It its easy to see that these inclusions hold if $T = T_0$. In order to prove them in general, we compare the positions of certain entries in T_0 with their positions in T. The main idea is that for $i \in D$ the operators π_j used to go up from T_0 to T in E_D are only able of moving i to the left and $i+1$ to the right.

Lemma 3.5.21. *Let $D \in \mathcal{OD}$ and $T \in E_D$.*

(*1*) *The cell $T^{-1}(i)$ is weakly left of the cell $T_0^{-1}(i)$ for all $i \in D \cup \{n\}$.*

(*2*) *The cell $T_0^{-1}(i+1)$ is weakly left of the cell $T^{-1}(i+1)$ for all $i \in D$.*

Proof. Set $\sigma := \operatorname{col}_T \operatorname{col}_{T_0}^{-1}$. For Part (1) let $i \in D \cup \{n\}$. If $T^{-1}(i) = T_0^{-1}(i)$ the statement is clear. Thus, assume $T^{-1}(i) \neq T_0^{-1}(i)$. Let $s_{i_k} \cdots s_{i_1}$ be a reduced word for σ. Then $T = \pi_{i_k} \cdots \pi_{i_1} T_0$. By Lemma 3.5.18, $\operatorname{cont}(\sigma) \subseteq D^c$. Thus, $i \neq i_j$ for all $j \in [k]$. On the other hand, $T^{-1}(i) \neq T_0^{-1}(i)$ so that at least one of the π_{i_j} has to move i. Since π_{i-1} and π_i are the only operators among the π_r with $r \in [n-1]$ that are capable of moving i, it follows that at least one of the i_j equals $i-1$ (of course π_{i-1} and π_i are only defined for $1 < i$ and $i < n$, respectively). That is, in order to obtain T from T_0 by applying π_σ, i is moved strictly to the left. In other words, $T^{-1}(i)$ is strictly left of $T_0^{-1}(i)$.

Part (2) is proven similarly. $\square$

We now consider the descents of the elements of E_D for $D \in \mathcal{OD}$.

Lemma 3.5.22. *Let $D \in \mathcal{OD}$ and $T \in E_D$. Then*

$$AD(T) \subseteq D \subseteq D(T).$$

Proof. Let $\sigma := \operatorname{col}_T \operatorname{col}_{T_0}^{-1}$. Recall that we denote the index of the column of the entry i in T by $c_T(i)$. Thus, we have $i \in D(T)$ if and only if $c_T(i) \leq c_T(i+1)$.

First we show $D \subseteq D(T)$. Let $i \in D$. Since $D \in \mathcal{OD}$, we have $i \in D(T_0)$. Therefore $c_{T_0}(i) \leq c_{T_0}(i+1)$. Furthermore, Lemma 3.5.21 yields

$$c_T(i) \leq c_{T_0}(i) \text{ and } c_{T_0}(i+1) \leq c_T(i+1)$$

for all $i \in D$. Hence,

$$c_T(i) \leq c_{T_0}(i) \leq c_{T_0}(i+1) \leq c_T(i+1),$$

i.e. $i \in D(T)$.

We now show $AD(T) \subseteq D$. Because $D \subseteq D(T)$, this is equivalent to

$$D(T) \setminus D \subseteq nAD(T).$$

We prove the latter. Let $i \in D(T) \setminus D$. Then $i \in D^c = \bigcup_{r=1}^{p} \mathring{I}_{k_r,l_r}$. Thus, there is an $r \in [p]$ such that $i, i+1 \in I_{k,l}$ where $k := k_r$ and $l{=}l_r$. Since $OD(T_0) \subseteq D$, $B_{k,l}$ is pacific. Moreover, $I_{k,l} = T(B_{k,l})$ as $T \in E_D$. Therefore $i \not\rightsquigarrow_T i+1$, i.e. $i \in nAD(T)$. $\square$

Example 3.5.23. We consider $D := OD(T_0) = \{2, 6\}$ and $T := \pi_1 T_0$ from Example 3.5.16. Then $T \in E_D$, $AD(T) = \{6\}$ and $D(T) = \{2, 6\}$. Hence,

$$AD(T) \subseteq D \subseteq D(T).$$

Let $D \in \mathcal{OD}$. We now show that E_D has a greatest element which we call the *D-sorted tableau* T_D. We begin with defining T_D in terms of sorted horizontal strip sequences.

Definition 3.5.24. *Let $D \in \mathcal{OD}$. Define the D-*sorted tableau *T_D to be the $\alpha /\!\!/ \beta$-tableau such that T_D is B_{k_r,l_r}-sorted for all $r \in [p]$.*

We show in Lemma 3.5.26 that $T_D \in E_D$. Clearly, $T_D(B_{k_r,l_r}) = I_{k_r,l_r}$ for each $r \in [p]$. However, $T_D \in E$ or even $T_D \in \mathrm{SCT}(\alpha /\!\!/ \beta)$ is less obvious.

From $OD(T_0) \subseteq D$ it follows that each B_{k_r,l_r} is pacific. Hence, Lemma 3.5.7 implies that we obtain T_D by filling B_{k_r,l_r} from left to right with

$$d_{l_r}, d_{l_r} - 1, \ldots, d_{l_{r-1}} + 1$$

for all $r \in [p]$.

Note that if $D = D(T_0)$ then $p = m + 1$ and $B_{k_r,l_r} = B_r$ for all $r \in [p]$. Therefore $T_{D(T_0)} = T_0$.

Example 3.5.25. We continue Example 3.5.16. Recall that we have $D(T_0) = \{1, 2, 6\}$, $OD(T_0) = \{2, 6\}$, $\mathcal{OD} = \{OD(T_0), D(T_0)\}$, $E_{D(T_0)} = \{T_0\}$ and $E_{OD(T_0)} = \{T_0, \pi_1 T_0\}$. Then, by definition,

$$T_{D(T_0)} = T_0 = \begin{array}{|c|c|c|c|} \hline 1 \\ \cline{1-1} 6 & 5 & 4 & 3 \\ \hline 8 & 7 & 2 \\ \cline{1-3} \end{array} \quad \text{and} \quad T_{OD(T_0)} = \pi_1 T_0 = \begin{array}{|c|c|c|c|} \hline 2 \\ \cline{1-1} 6 & 5 & 4 & 3 \\ \hline 8 & 7 & 1 \\ \cline{1-3} \end{array},$$

where for $D \in \mathcal{OD}$ and $r \in [p]$ the cells of B_{k_r,l_r} in T_D are equally shaded. For both $D \in \mathcal{OD}$ we have that $D(T_D) = D$ and that T_D is the greatest element of E_D. The following result proves this in general.

Lemma 3.5.26. *Let $D \in \mathcal{OD}$. Then T_D is the greatest element of E_D. Moreover, T_D is the unique element of E_D with descent set D.*

Proof. Let $T \in E_D$. We consider the following statements.

(1) T is maximal in E_D.

(2) $D(T) = D$.

(3) $T = T_D$.

We show (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3). In addition, because $T_0 \in E_D$, there exists a maximal element in E_D. Thus, it follows that T_D is the greatest element of E_D and $D(T_D) = D$.

We begin by proving that (1) implies (2). Assume that T is maximal. Lemma 3.5.22 yields that

$$AD(T) \subseteq D \subseteq D(T).$$

Therefore, in order to show that $D(T) = D$ it suffices to show that $nAD(T) \subseteq D$. Assume instead that there is an $i \in nAD(T) \cap D^c$. Then $\pi_i T \in E_D$ by Lemma 3.5.18. This contradicts the maximality of T since $T \prec \pi_i T$.

For the proof of the implication from (2) to (3) assume that $D(T) = D$. We have to show that T is B_{k_r,l_r} sorted for each $r \in [p]$. Let $r \in [p]$, $k := k_r$ and $l := l_r$. As $\mathring{I}_{k,l} \subseteq D^c$, we have that

$$D(T) \cap \mathring{I}_{k,l} = \emptyset.$$

Now, we can apply Lemma 3.5.8 and obtain that T is $B_{k,l}$-sorted. $\square$

Note that from Lemma 3.5.26, Lemma 3.5.20 and the fact that T_0 is the least element of E it follows that E_D is the $\preceq$-interval $[T_0, T_D]$ for all $D \in \mathcal{OD}$.

Let $D \in \mathcal{OD}$. Because the D-sorted tableau T_D is the greatest element of E_D, there is an operator π_σ with $\sigma \in \mathfrak{S}_n$ so that $\pi_\sigma T = T_D$ for each D-sortable tableau T. One can think of π_σ as sorting the entries in T.

Definition 3.5.27. *The set of* horizontally sorted tableaux *of E is given by*

$$E_{\text{hsort}} := \{T_D \mid D \in \mathcal{OD}\}.$$

Note that $E_{\text{hsort}} \subseteq E_{OD(T_0)}$. For the equivalence class of straight tableaux E from Example 3.5.25 we even have $E_{\text{hsort}} = E_{OD(T_0)}$. But this is merely a coincidence as can be seen in the next example.

Example 3.5.28. Let E be the equivalence class of skew tableaux from Figure 3.3 and Example 3.5.9 with source tableau

$$T_0 = \begin{array}{cccc} &&&\boxed{4}\\ &&\boxed{3}& \\ \boxed{2}&\boxed{1}&& \end{array}.$$

We have $OD(T_0) = \emptyset$ and $D(T_0) = \{2, 3\}$. For $D \in \mathcal{OD}$ we write E_D and T_D in the table below. For each $D \in \mathcal{OD}$ and each $r \in [p]$ the cells of the strip sequence B_{k_r,l_r} associated to D are equally shaded in T_D.

As $T_{OD(T_0)}$ is the sink tableau of E, we have that $E = E_{OD(T_0)}$. We can see in Figure 3.3 that $|E| = 12$. Hence $E_{\text{hsort}} \subsetneq E_{OD(T_0)}$.

D	$\emptyset$	$\{2\}$	$\{3\}$	$\{2,3\}$
E_D	E	$T_0, \pi_3 T_0$	$T_0, \pi_2 T_0, \pi_1\pi_2 T_0$	T_0
T_D	T_1	$\pi_3 T_0$	$\pi_1\pi_2 T_0$	T_0
T_D	$\begin{array}{cccc} &&&\boxed{1}\\ &&\boxed{2}& \\ \boxed{4}&\boxed{3}&& \end{array}$	$\begin{array}{cccc} &&&\boxed{3}\\ &&\boxed{4}& \\ \boxed{2}&\boxed{1}&& \end{array}$	$\begin{array}{cccc} &&&\boxed{4}\\ &&\boxed{1}& \\ \boxed{3}&\boxed{2}&& \end{array}$	$\begin{array}{cccc} &&&\boxed{4}\\ &&\boxed{3}& \\ \boxed{2}&\boxed{1}&& \end{array}$

Observe that for each $D \in \mathcal{OD}$ we have $D(T_D) = D$ and $D(T_D) \cap \operatorname{cont}(\operatorname{col}_{T_D} \operatorname{col}_{T_0}^{-1}) = \emptyset$. By the next result, these properties characterize E_{hsort} in E.

Corollary 3.5.29. *Let $T \in E$. Then $T \in E_{\text{hsort}}$ if and only if*

$$D(T) \in \mathcal{OD} \quad \text{and} \quad D(T) \cap \operatorname{cont}(\operatorname{col}_T \operatorname{col}_{T_0}^{-1}) = \emptyset.$$

Proof. Let $\sigma_T := \operatorname{col}_T \operatorname{col}_{T_0}^{-1}$.

For the implication from left to right assume that $T = T_D$ for some $D \in \mathcal{OD}$. Then $T \in E_D$ and $D(T) = D$ by Lemma 3.5.26. As $T \in E_D$, Lemma 3.5.18 implies $\operatorname{cont}(\sigma_T) \subseteq D^c$. Hence $D(T) \cap \operatorname{cont}(\sigma_T) = \emptyset$.

For the converse direction assume that $D := D(T) \in \mathcal{OD}$ and $D(T) \cap \operatorname{cont}(\sigma_T) = \emptyset$. Then $\operatorname{cont}(\sigma_T) \subseteq D^c$ so that $T \in E_D$. But by Lemma 3.5.26, T_D is the only element of E_D with descent set D. Thus $T = T_D$, i.e. $T \in E_{\text{hsort}}$. □

Example 3.5.30. We continue Example 3.5.28. The element of E

$$T := \pi_3\pi_1\pi_2 T_0 = \begin{array}{cccc} & & & \boxed{3} \\ & & \boxed{1} & \\ \boxed{4} & \boxed{2} & & \end{array}$$

has descent set $\{2\}$ so that $D(T) \in \mathcal{OD}$. Moreover, $T \notin E_{\text{hsort}}$ by Example 3.5.28. Therefore, Corollary 3.5.29 demands that $D(T) \cap \operatorname{cont}(\operatorname{col}_T \operatorname{col}_{T_0}^{-1}) \neq \emptyset$ which is true since 2 is an element of this intersection.

Radical and top

So far, we focused on combinatorics related to D-sortable tableaux. Now we use our previous results in order to describe the radical and the top of $\boldsymbol{S}_{\alpha /\!\!/ \beta, E}$. We begin with defining an $H_n(0)$-epimorphism from $\boldsymbol{S}_{\alpha /\!\!/ \beta, E}$ to the simple module $\boldsymbol{F}_D$ for each $D \in \mathcal{OD}$.

Proposition 3.5.31. *Let $D \in \mathcal{OD}$. The $\mathbb{K}$-linear map given by*

$$\begin{aligned} \varphi_D \colon \boldsymbol{S}_{\alpha /\!\!/ \beta, E} &\to \boldsymbol{F}_D \\ T &\mapsto \begin{cases} v_D & \text{if } T \in E_D \\ 0 & \text{if } T \notin E_D \end{cases} \end{aligned}$$

for $T \in E$ is an $H_n(0)$-epimorphism.

Proof. Let $\varphi := \varphi_D$. For each $T \in E$ set $\sigma_T := \operatorname{col}_T \operatorname{col}_{T_0}^{-1}$. Since $T_0 \in E_D$, φ is a surjective map. It remains to show that φ is a homomorphism of $H_n(0)$-modules. Let $T \in E$ and $i \in [n-1]$.

We consider the case where $T \notin E_D$ first. Then $\pi_i \varphi(T) = \pi_i 0 = 0$. Thus, we have to show that $\varphi(\pi_i T) = 0$. Since $E \setminus E_D$ is a filter of E by Lemma 3.5.20, $\pi_i T \notin E_D$ if

$i \in nAD(T)$. It follows that

$$\varphi(\pi_i T) = \begin{cases} \varphi(\pi_i T) = 0 & \text{if } i \in nAD(T) \\ \varphi(0) = 0 & \text{if } i \in AD(T) \\ \varphi(T) = 0 & \text{if } i \in D^c(T) \end{cases}$$

as desired.

We now suppose that $T \in E_D$. Then $\varphi(T) = v_D$ and by Lemma 3.5.22 we have that $D \subseteq D(T)$.

Assume first that $i \in D^c(T)$. Then $D \subseteq D(T)$ implies $i \notin D$ so that $\pi_i v_D = v_D$. Hence,

$$\varphi(\pi_i T) = \varphi(T) = v_D = \pi_i \varphi(T).$$

Assume now that $i \in D(T)$. We distinguish two cases.

Case 1. Suppose $i \in D$. Then $\pi_i v_D = 0$ and we have to show that $\varphi(\pi_i T) = 0$. If $i \in AD(T)$ then $\varphi(\pi_i T) = \varphi(0) = 0$. Hence, assume $i \in nAD(T)$. Then $\pi_i T \in E$ and $i \in \mathrm{cont}(\sigma_{\pi_i T})$. Thus, Lemma 3.5.18 yields $\pi_i T \notin E_D$. Therefore, we also have $\varphi(\pi_i T) = 0$.

Case 2. Suppose $i \notin D$. Then $\pi_i v_D = v_D$ and we have to show that $\varphi(\pi_i T) = v_D$. As $i \in D(T) \setminus D$, Lemma 3.5.22 yields that $i \in nAD(T)$. Thus, $\pi_i T \in E$ and

$$\mathrm{cont}(\mathrm{col}_{\pi_i T}) = \{i\} \cup \mathrm{cont}(\mathrm{col}_T) \subseteq D^c$$

where the inclusion is a consequence of $i \in D^c$, $T \in E_D$ and the description of E_D from Lemma 3.5.18. That is, $\pi_i T \in E_D$ by the same result. Hence $\varphi(\pi_i T) = v_D$. □

Example 3.5.32. We continue Example 3.5.25. Recall that the elements of $\mathcal{OD}$ are $OD(T_0) = \{2, 6\}$ and $D(T_0) = \{1, 2, 6\}$. Moreover, we have seen that $E_{\{1,2,6\}} = \{T_0\}$ and $E_{\{2,6\}} = \{T_0, \pi_1 T_0\}$. From Proposition 3.5.31 we obtain that the $\mathbb{K}$-linear maps given by

$$\varphi_{\{1,2,6\}}(T) = \begin{cases} v_{\{1,2,6\}} & \text{if } T = T_0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \varphi_{\{2,6\}}(T) = \begin{cases} v_{\{2,6\}} & \text{if } T \in \{T_0, \pi_1 T_0\} \\ 0 & \text{otherwise} \end{cases}$$

for $T \in E$ are $H_n(0)$-epimorphisms from $\boldsymbol{S}_{\alpha /\!\!/ \beta, E}$ to $\boldsymbol{F}_{\{1,2,6\}}$ and $\boldsymbol{F}_{\{2,6\}}$, respectively.

The radical of a module M over a ring A is given by the intersection $\bigcap_U \bigcap_\varepsilon \ker \varepsilon$ where U runs over all simple A-modules and ε runs over all A-epimorphisms from M to U [AF92, Proposition 9.13].

Let $\varphi \colon \boldsymbol{S}_{\alpha /\!\!/ \beta, E} \to \bigoplus_{D \in \mathcal{OD}} \boldsymbol{F}_D, x \mapsto (\varphi_D(x))_{D \in \mathcal{OD}}$ be the direct sum of the $H_n(0)$-epimorphisms φ_D from Proposition 3.5.31. Then $\bigcap_{D \in \mathcal{OD}} \ker \varphi_D = \ker \varphi$. In addition, from the above description of the radical, we obtain that $\mathrm{rad}(\boldsymbol{S}_{\alpha /\!\!/ \beta, E}) \subseteq \bigcap_{D \in \mathcal{OD}} \ker \varphi_D$.

Hence,

$$\operatorname{rad}(\boldsymbol{S}_{\alpha/\!\!/\beta,E}) \subseteq \ker\varphi. \tag{3.16}$$

We will show that $\operatorname{rad}(\boldsymbol{S}_{\alpha/\!\!/\beta,E}) = \ker\varphi$ with a dimension argument. It then follows that $\operatorname{top}(\boldsymbol{S}_{\alpha/\!\!/\beta,E}) \cong \bigoplus_{D\in\mathcal{OD}} \boldsymbol{F}_D$ by factoring φ through $\operatorname{rad}(\boldsymbol{S}_{\alpha/\!\!/\beta,E})$. This is our main result on the top of $\boldsymbol{S}_{\alpha/\!\!/\beta,E}$, Theorem 3.5.42. As φ is a $\mathbb{K}$-linear map, we have

$$\dim\ker\varphi = \dim \boldsymbol{S}_{\alpha/\!\!/\beta,E} - \dim \bigoplus_{D\in\mathcal{OD}} \boldsymbol{F}_D = |E| - |\,\mathcal{OD}\,|. \tag{3.17}$$

Therefore, $\dim\operatorname{rad}(\boldsymbol{S}_{\alpha/\!\!/\beta,E}) \le |E| - |\,\mathcal{OD}\,|$ and our aim is to show that we actually have equality. We do this by constructing a $\mathbb{K}$-linear independent subset of $\operatorname{rad}(\boldsymbol{S}_{\alpha/\!\!/\beta,E})$ of size $|E| - |\,\mathcal{OD}\,|$.

This is based on a description of the radical of $H_n(0)$ due to Schocker [Sch08]. To state it, we define cont on the basis $\{\pi_\sigma \mid \sigma \in \mathfrak{S}_n\}$ of $H_n(0)$. For $i_1, \ldots, i_k \in [n-1]$ define

$$\operatorname{cont}(\pi_{i_1}\pi_{i_2}\cdots\pi_{i_k}) = \{i_1, i_2, \ldots i_k\}\,.$$

Since applying the braid relations or the relation $\pi_i^2 = \pi_i$ on an element $\pi_{i_1}\pi_{i_2}\cdots\pi_{i_k}$ does not change the set of indices, this map is well defined. Recall that for each product $\pi_{i_1}\pi_{i_2}\cdots\pi_{i_k}$ there exists a unique $\sigma \in \mathfrak{S}_n$ such that $\pi_{i_1}\pi_{i_2}\cdots\pi_{i_k} = \pi_\sigma$. Then $\operatorname{cont}(\pi_{i_1}\cdots\pi_{i_k}) = \operatorname{cont}(\pi_\sigma)$. It is not hard to see that

(1) $\operatorname{cont}(\pi_\sigma) = \operatorname{cont}(\sigma)$ for all $\sigma \in \mathfrak{S}_n$,

(2) $\operatorname{cont}(\pi_\sigma\pi_\tau) = \operatorname{cont}(\pi_\sigma) \cup \operatorname{cont}(\pi_\tau)$ for all $\sigma, \tau \in \mathfrak{S}_n$.

Note that [Sch08] considers a map from $H_n(0)$ to the $\mathbb{K}$-vector space spanned by the subsets of $[n-1]$ that linearly extends cont. This extension is not necessary for our purposes.

Theorem 3.5.33 ([Sch08, Theorem 3.2])**.** *The radical of $H_n(0)$ is given by*

$$\operatorname{rad}(H_n(0)) = \operatorname{span}_{\mathbb{K}}\{\pi_{\sigma_1} - \pi_{\sigma_2} \mid \sigma_1, \sigma_2 \in \mathfrak{S}_n \textit{ and } \operatorname{cont}(\pi_{\sigma_1}) = \operatorname{cont}(\pi_{\sigma_2})\}\,.$$

For example, $\pi_2\pi_1 - \pi_1\pi_2\pi_1$ is an element of $\operatorname{rad}(H_3(0))$ by Theorem 3.5.33. We exploit the theorem in the following way.

Lemma 3.5.34. *Let $\alpha/\!\!/\beta$ be a skew composition of size n and $E \in \mathcal{E}(\alpha/\!\!/\beta)$ with source tableau T_0. Then*

$$\operatorname{rad}(\boldsymbol{S}_{\alpha/\!\!/\beta,E}) = \operatorname{rad}(H_n(0))T_0.$$

As a consequence,

$$\operatorname{rad}(\boldsymbol{S}_{\alpha/\!\!/\beta,E}) = \operatorname{span}_{\mathbb{K}}\{(\pi_{\sigma_1} - \pi_{\sigma_2})T_0 \mid \sigma_1, \sigma_2 \in \mathfrak{S}_n \textit{ and } \operatorname{cont}(\pi_{\sigma_1}) = \operatorname{cont}(\pi_{\sigma_2})\}\,.$$

Proof. As $H_n(0)$ is artinian, we have that $\operatorname{rad}(\boldsymbol{S}_{\alpha/\!\!/\beta,E}) = \operatorname{rad}(H_n(0))\boldsymbol{S}_{\alpha/\!\!/\beta,E}$. Moreover, $\boldsymbol{S}_{\alpha/\!\!/\beta,E} = H_n(0)T_0$. Hence,

$$\operatorname{rad}(\boldsymbol{S}_{\alpha/\!\!/\beta,E}) = \operatorname{rad}(H_n(0))H_n(0)T_0 = \operatorname{rad}(H_n(0))T_0,$$

where we use that $\operatorname{rad}(H_n(0))$ is a two sided ideal of $H_n(0)$. Theorem 3.5.33 now implies the second statement. $\square$

Since E is a $\mathbb{K}$-basis of $\boldsymbol{S}_{\alpha/\!\!/\beta,E}$, $E \setminus E_{OD(T_0)}$ is a $\mathbb{K}$-linear independent subset of $\boldsymbol{S}_{\alpha/\!\!/\beta,E}$. We now show that $E \setminus E_{OD(T_0)}$ is contained in $\operatorname{rad}(\boldsymbol{S}_{\alpha/\!\!/\beta,E})$. To do this, we write certain $T \in E \setminus E_{OD(T_0)}$ as $T = (\pi_{\sigma_1} - \pi_{\sigma_2})T_0$ with $\sigma_1, \sigma_2 \in \mathfrak{S}_n$ and $\operatorname{cont}(\pi_{\sigma_1}) = \operatorname{cont}(\pi_{\sigma_2})$. In order to obtain the elements π_{σ_1} and π_{σ_2} we use the following result, which is illustrated in Example 3.5.36.

Lemma 3.5.35. *Let $k \in [m]$ be such that $d_k \in OD(T_0)$. Then there is a $\sigma \in \mathfrak{S}_n$ such that*

(1) $d_k \in \operatorname{cont}(\sigma)$,

(2) $\operatorname{cont}(\sigma) \subseteq \mathring{I}_{k,k+1}$,

(3) $\pi_\sigma(T_0) = 0$.

Proof. Set $\square_i := T_0^{-1}(i)$ for $i = 1, \ldots, n$ and $d := d_k$. By assumption, $I_k \rightsquigarrow_{T_0} I_{k+1}$. Define

$$\begin{aligned} a &:= \max\left\{i \in I_k \mid i \rightsquigarrow I_{k+1}\right\}, \\ b &:= \min\left\{i \in I_{k+1} \mid a \rightsquigarrow i\right\} \end{aligned}$$

and $B := T_0^{-1}([d+1, b])$. Moreover for $j = 0, 1, \ldots, d-a-1$ we set

$$\begin{aligned} \sigma_j &:= s_{b-j-1}s_{b-j-2}\cdots s_{d-j}, \\ T_{j+1} &:= \pi_{\sigma_j}T_j. \end{aligned}$$

We claim that for $j = 0, \ldots, d-a$ we have that

(*i*) $T_j \in E$,

(*ii*) $\operatorname{col}_{T_j} \operatorname{col}_{T_0}^{-1} = \sigma_{j-1}\sigma_{j-2}\cdots\sigma_0$.

(*iii*) $T_j(\square_i) = i$ for all $i \le d-j$,

(*iv*) $T_j(B) = [d-j+1, b-j]$.

We prove the claim by induction on j. For $j = 0$ we are dealing with the source tableau T_0 which satisfies (*i*) – (*iv*). Thus, assume that the claim holds for a j such that $0 \le j < d-a$. Then $a < d-j \le d$ so that $d-j \in I_k$. First, as the entries of I_k in T_0 form a connected horizontal strip, $d-j$ is strictly left of a in T_0. Second, from the choice of a and b it follows that a is weakly left of B in T_0. Third, $d-j > a$ and the choice of a imply that $d-j$ does not attack B in T_0. Therefore, $\square_{d-j}$ is strictly left of B and does not attack B. Moreover, from (*iii*) we obtain that $T_j(\square_{d-j}) = d-j$ so that (*iv*) implies $d-j \not\rightsquigarrow_{T_j} [d-j+1, b-j]$. Hence, we can apply Proposition 3.1.20 and obtain that $T_{j+1} \in E$, $T_{j+1}(B) = [d-(j+1)+1, b-(j+1)]$ and $\mathrm{col}_{T_{j+1}} \mathrm{col}_{T_j}^{-1} = \sigma_j$. From the latter it follows that we obtain T_{j+1} from T_j without moving any of the elements of $[1, d-(j+1)]$. That is, $T_{j+1}^{-1}(i) = T_j^{-1}(i) = \square_i$ for all $i \leq d-(j+1)$. We have that

$$\mathrm{col}_{T_{j+1}} \mathrm{col}_{T_0}^{-1} = \mathrm{col}_{T_{j+1}} \mathrm{col}_{T_j}^{-1} \mathrm{col}_{T_j} \mathrm{col}_{T_0}^{-1} = \sigma_j \sigma_{j-1} \cdots \sigma_0$$

since (ii) holds for T_j. This finishes the proof of the claim.

Now consider T_{d-a} and set $c := a+b-d$. The claim yields that

$$T_{d-a} \in E, \quad T_{d-a}(\square_a) = a \quad \text{and} \quad T_{d-a}(B) = [a+1, c].$$

As $B \subseteq B_{k+1}$, B is an connected horizontal strip. Since the entries decrease in the rows of T_0, B looks like

$$\square_b \square_{b-1} \cdots \square_{d+1}.$$

Hence, $T_{d-a}(\square_b) = c$ since $T_{d-a}(B) = [a+1, c]$ and the entries decrease in the rows of T_{d-a} too. Moreover, by choice of b, $\square_b$ is the only element of B that is attacked by $\square_a$. From $T_{d-a}(\square_a) = a$, $T_{d-a}(\square_b) = c$ and $T_{d-a}(B) = [a+1, c]$ it now follows that $a \not\rightsquigarrow_{T_{d-a}} [a+1, c-1]$. In addition, recall that $\square_a$ is weakly left of B. Thus, from Proposition 3.1.20 we obtain a $\sigma' := s_{c-2} s_{c-3} \cdots s_a$ and a $T \in E$ such that

(i) $T = \pi_{\sigma'} T_{d-a}$ and $\sigma' = \mathrm{col}_T \mathrm{col}_{T_{d-a}}^{-1}$,

(ii) $T(\square_a) = c-1$,

(iii) $T(\square_b) = c$.

From $\square_a \rightsquigarrow \square_b$ it follows that $c-1 \in AD(T)$ and thus $\pi_{c-1} T = 0$. Set

$$\sigma := s_{c-1} \sigma' \sigma_{d-a-1} \sigma_{d-a-2} \cdots \sigma_0.$$

Note that from $\pi_{c-1} T = 0$ it follows that $\ell(\sigma) > \ell(\sigma' \sigma_{d-a-1} \cdots \sigma_0)$. By construction we have $\mathrm{cont}(\sigma) \subseteq [a, b-1] \subseteq \mathring{I}_{k,k+1}$ and $\pi_\sigma T_0 = 0$. Hence, σ satisfies Properties (2) and (3). In order to show that σ has Property (1), assume for the sake of contradiction that $d \notin \mathrm{cont}(\sigma)$. Then $\mathrm{cont}(\sigma) \subseteq \mathring{I}_{k,k+1} \setminus \{d_k\} \subseteq D^c(T_0)$. Hence, $\pi_\sigma T_0 = T_0$. But this contradicts $\pi_\sigma T_0 = 0$ which we have by Property (3). $\square$

Example 3.5.36. We illustrate Lemma 3.5.35 including the notation used in its proof.

(1) For the source tableau from our running example

$$T_0 = \begin{array}{|c|c|c|c|} \cline{1-1} 1 & \multicolumn{3}{c}{} \\ \hline 6 & 5 & 4 & 3 \\ \hline 8 & 7 & 2 & \multicolumn{1}{c}{} \\ \cline{1-3} \end{array}$$

and its offensive descent $d = d_2 = 2$ we have $a = 2$, $b = 4$, $\sigma' = s_2$ and $\sigma = s_3 s_2$. Hence, $d \in \mathrm{cont}(\sigma)$ and $\mathrm{cont}(\sigma) \subseteq \mathring{I}_{2,3} = \{2, 3, 4, 5\}$. In Example 3.3.10 we applied π_σ on T_0 with the result $\pi_\sigma T_0 = 0$.

(2) Consider the source tableau

$$T_0 = \begin{array}{cccc} \boxed{3} & \boxed{2} & \boxed{1} & \\ & & \boxed{5} & \boxed{4} \end{array}$$

and its offensive descent $d = 3$.
Then $a = 2$ and $b = 5$. We have $\sigma_0 = s_4 s_3$, $\sigma' = s_2$ and $\sigma = s_3\sigma'\sigma_0 = s_3 s_2 s_4 s_3$. That is, $d \in \mathrm{cont}(\sigma)$ and $\mathrm{cont}(\sigma) \subseteq \mathring{I}_{1,2} = \{1,2,3,4\}$. Moreover,

$$T_0 = \begin{array}{cccc} \boxed{3} & \boxed{2} & \boxed{1} & \\ & & \boxed{5} & \boxed{4} \end{array} \xrightarrow{\pi_3} \begin{array}{cccc} \boxed{4} & \boxed{2} & \boxed{1} & \\ & & \boxed{5} & \boxed{3} \end{array} \xrightarrow{\pi_4} T_1 = \begin{array}{cccc} \boxed{5} & \boxed{2} & \boxed{1} & \\ & & \boxed{4} & \boxed{3} \end{array} \xrightarrow{\pi_2} T = \begin{array}{cccc} \boxed{5} & \boxed{3} & \boxed{1} & \\ & & \boxed{4} & \boxed{2} \end{array}$$

and $\pi_3 T = 0$ so that $\pi_\sigma T_0 = 0$ as well.

Lemma 3.5.37. *The set $E \setminus E_{OD(T_0)}$ is contained in* $\mathrm{rad}(\boldsymbol{S}_{\alpha /\!\!/ \beta, E})$.

Proof. Let $T \in E \setminus E_{OD(T_0)}$. If $T \in \mathrm{rad}(\boldsymbol{S}_{\alpha /\!\!/ \beta, E})$ then $\pi_\sigma T \in \mathrm{rad}(\boldsymbol{S}_{\alpha /\!\!/ \beta, E})$ for all $\sigma \in \mathfrak{S}_n$ since $\mathrm{rad}(\boldsymbol{S}_{\alpha /\!\!/ \beta, E})$ is an $H_n(0)$-module. Hence, we can assume that T is minimal in $E \setminus E_{OD(T_0)}$ according to $\preceq$. Since $T_0 \in E_{OD(T_0)}$ and therefore $T \neq T_0$, there exists a $T' \in E$ and an $i \in nAD(T')$ such that $\pi_i T' = T$. By the minimality of T, $T' \in E_{OD(T_0)}$. As $\pi_i T' \notin E_{OD(T_0)}$, Lemma 3.5.18 implies that $i \in OD(T_0)$.

Let $k \in [m]$ be such that $i = d_k$. Then Lemma 3.5.35 provides a $\sigma \in \mathfrak{S}_n$ such that $i \in \mathrm{cont}(\sigma) \subseteq \mathring{I}_{k,k+1}$ and $\pi_\sigma T_0 = 0$. Set $\sigma_T := \mathrm{col}_T \, \mathrm{col}_{T_0}^{-1}$ and $\sigma' := s_{j_1} s_{j_2} \cdots s_{j_r}$ where $j_1 < j_2 < \cdots < j_r$ are the elements of $\mathrm{cont}(\sigma)$ unequal to i. Since the j_q are distinct, $s_{j_1} s_{j_2} \cdots s_{j_r}$ is a reduced word. Hence,

$$\mathrm{cont}(\sigma') = \mathrm{cont}(\sigma) \setminus \{i\} \subseteq \mathring{I}_{k,k+1} \setminus \{d_k\} \subseteq D^c(T_0). \tag{3.18}$$

We show that $T = (\pi_{\sigma_T}\pi_{\sigma'} - \pi_{\sigma_T}\pi_\sigma)T_0$ and $\mathrm{cont}(\pi_{\sigma_T}\pi_{\sigma'}) = \mathrm{cont}(\pi_{\sigma_T}\pi_\sigma)$. Then Lemma 3.5.34 implies that $T \in \mathrm{rad}(\boldsymbol{S}_{\alpha /\!\!/ \beta, E})$. By (3.18), $\mathrm{cont}(\sigma') \subseteq D^c(T_0)$ and hence $\pi_{\sigma'} T_0 = T_0$. Thus,

$$\pi_{\sigma_T}\pi_{\sigma'} T_0 = \pi_{\sigma_T} T_0 = T.$$

Furthermore, $\pi_{\sigma_T}\pi_\sigma T_0 = \pi_{\sigma_T} 0 = 0$. That is,

$$T = (\pi_{\sigma_T}\pi_{\sigma'} - \pi_{\sigma_T}\pi_\sigma)T_0.$$

We have $\mathrm{cont}(\sigma') = \mathrm{cont}(\sigma) \setminus \{i\}$ from (3.18). Moreover, $i \in \mathrm{cont}(\sigma_T)$ because $\pi_i T' = T$. Therefore,

$$\mathrm{cont}(\sigma_T) \cup \mathrm{cont}(\sigma') = \mathrm{cont}(\sigma_T) \cup \mathrm{cont}(\sigma).$$

In addition,

$$\mathrm{cont}(\pi_{\sigma_T}\pi_{\sigma'}) = \mathrm{cont}(\sigma_T) \cup \mathrm{cont}(\sigma') \quad \text{and} \quad \mathrm{cont}(\pi_{\sigma_T}\pi_\sigma) = \mathrm{cont}(\sigma_T) \cup \mathrm{cont}(\sigma).$$

Hence,

$$\mathrm{cont}(\pi_{\sigma_T}\pi_{\sigma'}) = \mathrm{cont}(\pi_{\sigma_T}\pi_\sigma)$$

as desired. $\square$

Example 3.5.38. Let T_0 be the straight source tableau from Example 3.5.36, E its equivalence class and α its shape. We consider the element of $E \setminus E_{OD(T_0)}$

$$T := \pi_2 T_0 = \begin{array}{|c|c|c|c|}\cline{1-1} 1 \\ \hline 6 & 5 & 4 & 2 \\ \hline 8 & 7 & 3 \\ \cline{1-3}\end{array}$$

and show $T \in \mathrm{rad}(\boldsymbol{S}_{\alpha,E})$ by using the argumentation and notation of the the proof of Lemma 3.5.37. We have $\sigma_T = s_2$. For the offensive descent 2 of T_0 we obtain the permutation $\sigma = s_3 s_2$ from Example 3.5.36 and thus $\sigma' = s_3$. Then

$$(\pi_{\sigma_T}\pi_{\sigma'} - \pi_{\sigma_T}\pi_\sigma)T_0 = \pi_2\pi_3 T_0 - \pi_2\pi_3\pi_2 T_0 = T - \pi_2\pi_3 T = T$$

where we use that $3 \in D^c(T_0)$ and $3 \in AD(T)$. Moreover,

$$\mathrm{cont}(\pi_2\pi_3) = \mathrm{cont}(\pi_2\pi_3\pi_2).$$

Therefore, $T \in \mathrm{rad}(\boldsymbol{S}_{\alpha,E})$ by Lemma 3.5.34.

Lemma 3.5.37 provides us with a $\mathbb{K}$-linear independent subset of $\mathrm{rad}(\boldsymbol{S}_{\alpha/\!\!/\beta,E})$ with cardinality $|E| - |E_{OD(T_0)}|$. We will use the following lemma to extend this set by additional $|E_{OD(T_0)}| - |\,\mathcal{OD}\,|$ elements of $\mathrm{rad}(\boldsymbol{S}_{\alpha/\!\!/\beta,E})$ preserving the linear independence.

Lemma 3.5.39. *Let $T \in E_{OD(T_0)} \setminus E_{\mathrm{hsort}}$. Then there exists $i \in nAD(T)$ such that $\pi_i T \in E_{OD(T_0)}$ and $T - \pi_i T \in \mathrm{rad}(\boldsymbol{S}_{\alpha/\!\!/\beta,E})$.*

Proof. For $T \in E$ set $\sigma_T := \mathrm{col}_T\,\mathrm{col}_{T_0}^{-1}$. Let $D := OD(T_0)$ and fix a $T \in E_D \setminus E_{\mathrm{hsort}}$. We distinguish two cases.

Case 1. $D(T) \not\subseteq D(T_0)$. Then there exists an $i \in D(T)$ such that $i \notin D(T_0)$. As $D \subseteq D(T_0)$, it follows that $i \notin D$. Moreover, from $T \in E_D$ and Lemma 3.5.22 we obtain that $AD(T) \subseteq D$. Hence, $i \in nAD(T)$. Thus, $\pi_i T \in E$ and $\mathrm{cont}(\sigma_{\pi_i T}) = \{i\} \cup \mathrm{cont}(\sigma_T)$. As $T \in E_D$, Lemma 3.5.18 yields that $\mathrm{cont}(\sigma_T) \subseteq D^c$. Further $i \in D^c$ so that $\mathrm{cont}(\sigma_{\pi_i T}) \subseteq D^c$ and thus $\pi_i T \in E_D$ by the same lemma.

We have

$$(\pi_{\sigma_T}\pi_i - \pi_i\pi_{\sigma_T})T_0 = \pi_{\sigma_T}\pi_i T_0 - \pi_i\pi_{\sigma_T} T_0 = \pi_{\sigma_T} T_0 - \pi_i\pi_{\sigma_T} T_0 = T - \pi_i T$$

where the second equality holds since $i \in D^c(T_0)$. Moreover, $\mathrm{cont}(\pi_{\sigma_T}\pi_i) = \mathrm{cont}(\pi_i\pi_{\sigma_T})$. Thus, Lemma 3.5.34 implies $T - \pi_i T \in \mathrm{rad}(\boldsymbol{S}_{\alpha/\!\!/\beta,E})$.

Case 2. $D(T) \subseteq D(T_0)$. Since $T \in E_D$ we have $D \subseteq D(T)$ by Lemma 3.5.22. Thus, $D(T) \in \mathcal{OD}$. As $T \notin E_{\mathrm{hsort}}$, Corollary 3.5.29 yields that there is $i \in D(T) \cap \mathrm{cont}(\sigma_T)$.

On the one hand, from $T \in E_D$ and Lemma 3.5.18 it follows that $\operatorname{cont}(\sigma_T) \subseteq D^c$. On the other hand, $AD(T) \subseteq D$ by Lemma 3.5.22. Hence, $i \in nAD(T)$, $\pi_i T \in E$ and

$$\operatorname{cont}(\sigma_{\pi_i T}) = \{i\} \cup \operatorname{cont}(\sigma_T) = \operatorname{cont}(\sigma_T).$$

Thus, $\pi_i T \in E_D$ by Lemma 3.5.18. Further

$$(\pi_{\sigma_T} - \pi_i \pi_{\sigma_T}) T_0 = T - \pi_i T$$

is an element of $\operatorname{rad}(\boldsymbol{S}_{\alpha /\!\!/ \beta, E})$ by Lemma 3.5.34. $\square$

Example 3.5.40. We illustrate Lemma 3.5.39. Let E be the equivalence class from Example 3.5.28, T_0 be its source tableau and $\alpha /\!\!/ \beta = \operatorname{sh}(T_0)$. Recall $D(T_0) = \{2, 3\}$.

(1) Consider the elements of E

$$T = \pi_2 T_0 = \begin{ytableau} \none & \none & \none & 4 \\ \none & \none & 2 \\ 3 & 1 \end{ytableau} \quad \text{and} \quad \pi_1 T = \begin{ytableau} \none & \none & \none & 4 \\ \none & \none & 1 \\ 3 & 2 \end{ytableau}\ .$$

Then $T \in E_{OD(T_0)} \setminus E_{\text{hsort}}$, $1 \in nAD(T)$ and $1 \notin D(T_0)$. This is the situation in Case 1 of the proof of Lemma 3.5.39. Then

$$T - \pi_1 T = \pi_2 \pi_1 T_0 - \pi_1 \pi_2 T_0.$$

Hence, Lemma 3.5.34 yields $T - \pi_1 T \in \operatorname{rad}(\boldsymbol{S}_{\alpha /\!\!/ \beta, E})$.

(2) Consider the elements of E

$$T = \pi_3 \pi_1 \pi_2 T_0 = \begin{ytableau} \none & \none & \none & 3 \\ \none & \none & 1 \\ 4 & 2 \end{ytableau} \quad \text{and} \quad \pi_2 T = \begin{ytableau} \none & \none & \none & 2 \\ \none & \none & 1 \\ 4 & 3 \end{ytableau}\ .$$

In Example 3.5.30 we have seen that $T \in E_{OD(T_0)} \setminus E_{\text{hsort}}$ and $D(T) = \{2\} \subseteq D(T_0)$. Hence, we are in the situation of Case 2 of the proof of Lemma 3.5.39 and

$$T - \pi_2 T = (\pi_3 \pi_1 \pi_2 - \pi_2 \pi_3 \pi_1 \pi_2) T_0.$$

Thus, $T - \pi_2 T \in \operatorname{rad}(\boldsymbol{S}_{\alpha /\!\!/ \beta, E})$ by Lemma 3.5.34. We have taken advantage of the fact that $2 \in D(T) \cap \operatorname{cont}(\operatorname{col}_T \operatorname{col}_{T_0}^{-1})$. Corollary 3.5.29 ensures that such an element exists.

We now determine the dimension of $\operatorname{rad}(\boldsymbol{S}_{\alpha /\!\!/ \beta, E})$. We do this by constructing a basis from Lemma 3.5.37 and Lemma 3.5.39.

Proposition 3.5.41. *A $\mathbb{K}$-basis of $\operatorname{rad}(\boldsymbol{S}_{\alpha /\!\!/ \beta, E})$ is given by*

$$(E \setminus E_{OD(T_0)}) \cup \left\{ T - \pi_{i_T} T \mid T \in E_{OD(T_0)} \setminus E_{\text{hsort}} \right\}$$

where i_T is the integer provided by Lemma 3.5.39 for each $T \in E_{OD(T_0)} \setminus E_{\text{hsort}}$. In *particular, we have*

$$\dim \operatorname{rad}(\boldsymbol{S}_{\alpha /\!/ \beta, E}) = |E| - |\,\mathcal{OD}\,|.$$

Proof. Let $F := \left\{T - \pi_{i_T} T \mid T \in E_{OD(T_0)} \setminus E_{\mathrm{hsort}}\right\}$ and $B := (E \setminus E_{OD(T_0)}) \cup F$. Then

$$|B| = |E| - |E_{OD(T_0)}| + |E_{OD(T_0)}| - |E_{\mathrm{hsort}}| = |E| - |\,\mathcal{OD}\,|.$$

We have to show that B is a $\mathbb{K}$-basis of $\operatorname{rad}(\boldsymbol{S}_{\alpha /\!/ \beta, E})$. From (3.16) and (3.17) it follows that

$$\dim \operatorname{rad}(\boldsymbol{S}_{\alpha /\!/ \beta, E}) \leq |E| - |\mathcal{OD}|\,.$$

Hence, it remains to check that B is a $\mathbb{K}$-linear independent subset of $\operatorname{rad}(\boldsymbol{S}_{\alpha /\!/ \beta, E})$.

That $E \setminus E_{OD(T_0)}$ and F are subsets of $\operatorname{rad}(\boldsymbol{S}_{\alpha /\!/ \beta, E})$ was shown in Lemmas 3.5.37 and 3.5.39, respectively. In order to prove the linear independence of B, consider the $\mathbb{K}$-endomorphism ψ of $\boldsymbol{S}_{\alpha /\!/ \beta, E}$ given by

$$\psi(T) = \begin{cases} T - \pi_{i_T} T & \text{if } T \in E_{OD(T_0)} \setminus E_{\mathrm{hsort}} \\ T & \text{otherwise} \end{cases}$$

for $T \in E$. Let M be the transition matrix of ψ associated to the basis E ordered by a total order extending $\preceq$. Then M is an unitriangular matrix and therefore ψ a $\mathbb{K}$-automorphism. As B is a subset of the image of ψ, it follows that B is $\mathbb{K}$-linear independent. $\square$

We now combine Propositions 3.5.31 and 3.5.41 to obtain the main result of this section. Recall that given an equivalence class E with source tableau T_0 we have

$$\mathcal{OD} = \{D \subseteq [n-1] \mid OD(T_0) \subseteq D \subseteq D(T_0)\}$$

with the set of offensive descents $OD(T_0)$ from Definition 3.5.10.

Theorem 3.5.42. *Let $\alpha /\!/ \beta$ be a skew composition of size n and $E \in \mathcal{E}(\alpha /\!/ \beta)$. Then*

$$\operatorname{top}(\boldsymbol{S}_{\alpha /\!/ \beta, E}) \cong \bigoplus_{D \in \mathcal{OD}} \boldsymbol{F}_D$$

as $H_n(0)$-modules.

Proof. Let T_0 be the source tableau of E. For $D \in \mathcal{OD}$ let $\varphi_D \colon \boldsymbol{S}_{\alpha /\!/ \beta, E} \to \boldsymbol{F}_D$ be the $H_n(0)$-epimorphism from Proposition 3.5.31. Define the $H_n(0)$-epimorphism

$$\varphi \colon \boldsymbol{S}_{\alpha /\!/ \beta, E} \to \bigoplus_{D \in \mathcal{OD}} \boldsymbol{F}_D, \quad x \mapsto (\varphi_D(x))_{D \in \mathcal{OD}}.$$

as in the discussion after the proposition. On the one hand, we have

$$\operatorname{rad}(\boldsymbol{S}_{\alpha/\!\!/\beta,E}) \subseteq \ker\varphi \quad \text{and} \quad \dim\ker\varphi = |E| - |\,\mathcal{OD}\,|$$

by (3.16) and (3.17), respectively. On the other hand, $\dim\operatorname{rad}(\boldsymbol{S}_{\alpha/\!\!/\beta,E}) = |E| - |\,\mathcal{OD}\,|$ by Proposition 3.5.41. Therefore, $\operatorname{rad}(\boldsymbol{S}_{\alpha/\!\!/\beta,E}) = \ker\varphi$. This means that by factoring φ through $\operatorname{rad}(\boldsymbol{S}_{\alpha/\!\!/\beta,E})$ we get an $H_n(0)$-isomorphism between $\operatorname{top}(\boldsymbol{S}_{\alpha/\!\!/\beta,E})$ and $\bigoplus_{D\in\mathcal{OD}} \boldsymbol{F}_D$. □

Example 3.5.43. Let $\alpha = (1,4,3)$ and $E \in \mathcal{E}(\alpha)$ be the equivalence class from our running example with source tableau

$$T_0 = \begin{array}{|c|c|c|c|} \hline 1 \\ \cline{1-1} 6 & 5 & 4 & 3 \\ \hline 8 & 7 & 2 \\ \cline{1-3} \end{array}\,.$$

We have $\mathcal{OD} = \{\{2,6\},\{1,2,6\}\}$ so that

$$\operatorname{top}(\boldsymbol{S}_{\alpha,E}) \cong \boldsymbol{F}_{\{2,6\}} \oplus \boldsymbol{F}_{\{1,2,6\}}$$

by Theorem 3.5.42.

We directly obtain the following from Corollary 3.5.46. Note that for the last part we use that a module with simple top is always indecomposable.

Corollary 3.5.44. *Let $\alpha/\!\!/\beta$ be a skew composition of size n and $E \in \mathcal{E}(\alpha/\!\!/\beta)$ and T_0 be the source tableau of E. Then we have the following.*

(*1*) $\dim\operatorname{top}(\boldsymbol{S}_{\alpha/\!\!/\beta,E}) = |\mathcal{OD}|$.
(*2*) $\operatorname{top}(\boldsymbol{S}_{\alpha/\!\!/\beta,E})$ *is simple if and only if $OD(T_0) = D(T_0)$.*
(*3*) *If $OD(T_0) = D(T_0)$ then $\boldsymbol{S}_{\alpha/\!\!/\beta,E}$ is indecomposable.*

The sufficient condition for the indecomposability of $\boldsymbol{S}_{\alpha/\!\!/\beta,E}$ from Corollary 3.5.44 is not a necessary condition: Let E be the equivalence class of tableaux of straight shape α from our running example. Then we have $OD(T_0) \subsetneq D(T_0)$. Nevertheless, $\boldsymbol{S}_{\alpha,E}$ is indecomposable by Theorem 3.3.11 as it is a straight module.

Remark 3.5.45. Let $\alpha \vDash n$ and E_α be the equivalence class of $\mathrm{SCT}(\alpha)$ in which the entries of each column of each tableau increase from top to bottom. As mentioned before, in [TvW15] Tewari and van Willigenburg show that S_{α,E_α} is indecomposable. From Part (3) of Corollary 3.5.44 we get an alternative proof for this result as follows.

Let $e_0 < \cdots < e_l$ be such that $e_0 = 0$, $\{e_1, e_2, \ldots, e_{l-1}\} = \operatorname{Set}(\alpha)$ and $e_l = n$. We obtain the source tableau T_0 of E_α by filling row i of the diagram of α from left to right with

$$e_i, e_i - 1, \ldots, e_{i-1} + 1.$$

This tableau is called the *canonical tableau* of shape α. For example,

$$\begin{array}{|c|c|c|c|}\cline{1-1} 1 \\ \hline 5 & 4 & 3 & 2 \\ \hline 8 & 7 & 6 \\ \cline{1-3}\end{array}$$

is the canonical tableau of shape $(1,4,3)$. It follows that $e_1,\ldots,e_{l-1}$ are the descents of T_0 and they all appear in the first column of T_0. Thus, each descent of T_0 is an offensive descent and therefore $\boldsymbol{S}_{\alpha,E_\alpha}$ is indecomposable by Corollary 3.5.44.

Putting together the tops of the $\boldsymbol{S}_{\alpha/\!\!/\beta,E}$ for $E \in \mathcal{E}(\alpha/\!\!/\beta)$ yields the formula for $\mathrm{top}(\boldsymbol{S}_{\alpha/\!\!/\beta})$. Obtaining this formula was the initial motivation of the section.

Corollary 3.5.46. *Let $\alpha/\!\!/\beta$ be a skew composition of size n. Then*

$$\mathrm{top}(\boldsymbol{S}_{\alpha/\!\!/\beta}) \cong \bigoplus_{E\in\mathcal{E}(\alpha/\!\!/\beta)} \bigoplus_{D\in\mathcal{OD}_E} \boldsymbol{F}_D$$

as $H_n(0)$-modules where $\mathcal{OD}_E = \{D \subseteq [n-1] \mid OD(T_{0,E}) \subseteq D \subseteq D(T_{0,E})\}$.

Proof. Proposition 3.1.13 yields that $\mathrm{top}(\boldsymbol{S}_{\alpha/\!\!/\beta}) = \mathrm{top}\left(\bigoplus_{E\in\mathcal{E}(\alpha/\!\!/\beta)} \boldsymbol{S}_{\alpha/\!\!/\beta,E}\right)$. As the top is compatible with direct sums we additionally have that

$$\mathrm{top}\left(\bigoplus_{E\in\mathcal{E}(\alpha/\!\!/\beta)} \boldsymbol{S}_{\alpha/\!\!/\beta,E}\right) \cong \bigoplus_{E\in\mathcal{E}(\alpha/\!\!/\beta)} \mathrm{top}(\boldsymbol{S}_{\alpha/\!\!/\beta,E}).$$

Now apply Theorem 3.5.42. □

We conclude the section by showing how the formula for the top of pacific modules from Corollary 3.4.21 can be derived from Corollary 3.5.46.

Let $\alpha/\!\!/\beta$ be a pacific skew composition. Then Lemma 3.4.5 yields that $\mathrm{SCT}(\alpha/\!\!/\beta)$ is the only element of $\mathcal{E}(\alpha/\!\!/\beta)$. Let T_0 be the source tableau of $\mathrm{SCT}(\alpha/\!\!/\beta)$. Since $\alpha/\!\!/\beta$ is pacific $OD(T_0) = \emptyset$. Hence, Corollary 3.5.46 yields

$$\mathrm{top}(\boldsymbol{S}_{\alpha/\!\!/\beta}) \cong \bigoplus_{D\subseteq D(T_0)} \boldsymbol{F}_D.$$

Indeed, this is the formula from Corollary 3.4.21. The top of our running example of skew modules was determined by this formula in Example 3.4.22.

3.6 The socle of skew modules

Let $\alpha/\!\!/\beta$ be a skew composition of size n. In Corollary 3.5.46 of Section 3.5 we gave a combinatorial formula for the top of $\boldsymbol{S}_{\alpha/\!\!/\beta}$. The aim of this section is to provide a similar formula for the socle of $\boldsymbol{S}_{\alpha/\!\!/\beta}$ in Corollary 3.6.45. Moreover, we show that this formula generalizes the one for the socle of pacific modules from Corollary 3.4.21. Again we concentrate on the modules $\boldsymbol{S}_{\alpha/\!\!/\beta,E}$ for $E \in \mathcal{E}(\alpha/\!\!/\beta)$ since from the decomposition of $\boldsymbol{S}_{\alpha/\!\!/\beta}$ in Proposition 3.1.13 it follows that $\mathrm{soc}(\boldsymbol{S}_{\alpha/\!\!/\beta}) = \bigoplus_{E\in\mathcal{E}(\alpha/\!\!/\beta)} \mathrm{soc}(\boldsymbol{S}_{\alpha/\!\!/\beta,E})$. Theorem 3.6.39 determines $\mathrm{soc}(\boldsymbol{S}_{\alpha/\!\!/\beta,E})$.

Let $E \in \mathcal{E}(\alpha/\!\!/\beta)$. The socle of $\boldsymbol{S}_{\alpha/\!\!/\beta,E}$ is the direct sum of the simple submodules of $\boldsymbol{S}_{\alpha/\!\!/\beta,E}$. One of them is easy to identify. Let T_1 be the sink tableau of E. From Theorem 3.1.14 it follows that each descent of T_1 is attacking. Therefore,

$$\pi_i T_1 = \begin{cases} 0 & \text{if } i \in D(T_1) \\ T_1 & \text{if } i \notin D(T_1) \end{cases}$$

for each $i \in [n-1]$. That is, $\mathbb{K}T_1$ is a simple $H_n(0)$-submodule of $\boldsymbol{S}_{\alpha/\!\!/\beta,E}$ isomorphic to $\boldsymbol{F}_{D(T_1)}$. From Example 3.4.22 we know that there can be other simple submodules as well. We will construct them explicitly using an approach similar to that of Section 3.5. There we divided the diagram of $\alpha/\!\!/\beta$ into horizontal strips using the descents of the source tableau of E. Here we divide the diagram of $\alpha/\!\!/\beta$ into vertical strips according to the the ascents of the sink tableau T_1 of E. The relative positions of these vertical strips will again be crucial. As before, the section can be roughly divided in two parts. First we develop the necessary combinatorics and then consider the simple modules and the socle of $\boldsymbol{S}_{\alpha/\!\!/\beta,E}$. We begin with fixing notation for the entire section.

Notation 3.6.1. Let $\alpha/\!\!/\beta$ be a skew composition of size n, $E \in \mathcal{E}(\alpha/\!\!/\beta)$, T_1 be the corresponding sink tableau and $a_0 = 0 < a_1 < \cdots < a_{m+1} = n$ be integers such that the ascent set of T_1 is given by

$$D^c(T_1) = \{a_1, a_2, \ldots, a_m\}\,.$$

For $k, l \in [m+1]$ with $k \leq l$ define the integer intervals

$$J_{k,l} := [a_{k-1}+1, a_l], \quad \mathring{J}_{k,l} := J_{k,l} \setminus \{a_l\} \quad \text{and} \quad J_k := J_{k,k}.$$

Then $J_{k,l} = \bigcup_{j=k}^{l} J_j$ and $J_k = [a_{k-1}+1, a_k]$. Note that since T_1 is a sink tableau, we have $D(T_1) = AD(T_1)$ by Theorem 3.1.14. Hence, each element of $\mathring{J}_k$ is an attacking descent.

Define $C_{k,l} := T_0^{-1}(J_{k,l})$ and $C_k := T_0^{-1}(J_k)$. From Lemma 3.6.3 below it follows that each C_k has at most one cell per row, i.e. it is a vertical strip. The $C_{k,l}$ are therefore called *vertical strip sequences*.

We often use the notation introduced in Section 3.1 before Definition 3.1.9. In particular, recall that for two sets of cells A and B we write $A \wr B$ if there are $a \in A$ and $b \in B$ such that $a \wr b$, i.e. a is the left neighbor of b. We call the vertical strip sequence $C_{k,l}$ *separated* if $C_{j+1} \not\wr C_j$ for all $j = k, \ldots, l-1$.

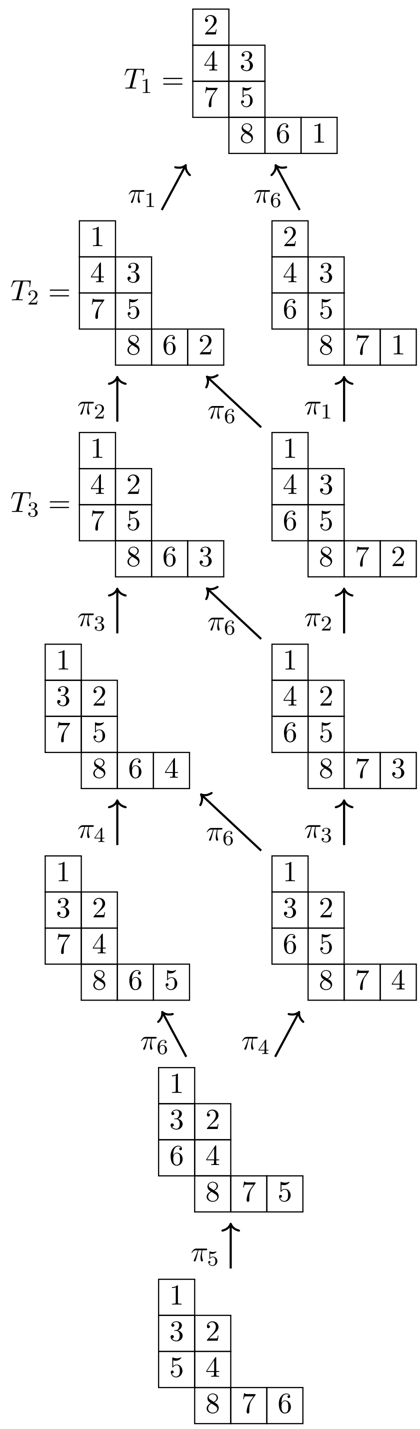


Figure 3.4: An equivalence class of SCTx with sink tableau T_1.

Example 3.6.2. (1) For the sink tableau

$$T_1 = \begin{array}{|c|c|c|c|}\hline 2 & \multicolumn{3}{c}{} \\ \cline{1-2} 4 & 3 & \multicolumn{2}{c}{} \\ \cline{1-2} 7 & 5 & \multicolumn{2}{c}{} \\ \hline \multicolumn{1}{c|}{} & 8 & 6 & 1 \\ \cline{2-4} \end{array} .$$

of the equivalence class from Figure 3.4 we have

$$D^c(T_1) = \{a_1 = 1, a_2 = 3, a_3 = 6\} ,$$

$J_1 = \{1\}$, $J_2 = \{2, 3\}$, $J_3 = \{4, 5, 6\}$ and $J_4 = \{7, 8\}$. The cells of the vertical strip C_k are filled with the same shade of gray for $k = 1, \ldots, 4$. Observe $C_2 \not\wr C_1$ and $C_{k+1} \wr C_k$ for $k = 2, 3$. Hence, $C_{1,2}$ is the only separated strip sequence among the strip sequences $C_{k,l}$ with $k < l$ associated to T_1.

(2) The vertical strips C_k can be more complicated as those of Part (1). For instance, all the cells occupied by entries of the sink tableau

$$T_1 = \begin{array}{ccc} \boxed{1} & & \\ \boxed{2} & & \\ & & \boxed{7} \\ & \boxed{5} & \\ & \boxed{4} & \\ & & \boxed{6} \\ & \boxed{3} & \end{array}$$

belong to the one vertical strip C_1 associated to T_1.

Vertical strip sequences

We consider the vertical strip sequences $C_{k,l}$ associated to the sink tableau T_1. Our main goal is to show in Lemma 3.6.5 that if $C_{k,l}$ is a separated vertical strip sequence then C_j is strictly left of C_i and $C_j \not\wr C_i$ for all $k \leq i < j \leq l$.

We begin describing the geometry of a vertical strip C_k. In the case of the horizontal strips B_k assoiciated to source tableaux from Section 3.6 this was easy: Each B_k is a connected horizontal strip, that is, a horizontal line of cells. As we have seen in Example 3.6.2, the C_k are more complicated and in general not connected.

In the lemma below we describe the vertical strip C_k in terms of the entries of J_k in T_1. Given two entries $i, j \in J_k$ of T_1 with $i < j$ we show that i and j appear in different rows of T_1 and that i is weakly left of j. This implies that C_k indeed is a vertical strip. Moreover, we show that $c(C_k)$, the set of indices of the columns containing a cell of C_k, is an integer interval.

Lemma 3.6.3. *Let $k \in [m + 1]$. For $i \in [n]$ we write $c(i) := c_{T_1}(i)$ and $r(i) := r_{T_1}(i)$ for the column and row of i in T_1, respectively.*

(*1*) *Let* $i, j \in J_k$ *with* $i < j$. *Then*

$$c(i) \leq c(j) \quad \textit{and} \quad r(i) \neq r(j).$$

In particular, C_k *is a vertical strip.*

(*2*) *We have*

$$c(J_k) = [c(a_{k-1}+1), c(a_k)]\,.$$

Proof. Let $i \in \mathring{J}_k$. Then $i \in D(T_1) = AD(T_1)$, i.e. $i \rightsquigarrow i+1$. Thus, $c(i+1) = c(i)$ or $c(i+1) = c(i)+1$. Using this argument iteratively yields that

$$c(a_{k-1}+1) \leq c(a_{k-1}+2) \leq \cdots \leq c(a_k)$$

and $c(J_k) = [c(a_{k-1}+1), c(a_k)]\,.$

Now let $i, j \in J_k$ with $i < j$. We have already shown that then i is weakly left of j in T_1, i.e. $c(i) \leq c(j)$. Since the entries of SCTx decrease in rows from left to right, it follows that j cannot appear in the same row as i. In other words, $r(i) \neq r(j)$. □

One may check the statements of Lemma 3.6.3 in Example 3.6.2.

We now want to show that for two vertical strips C_i and C_j with $i < j$ of a separated vertical strip sequence $C_{k,l}$, we have $C_j \not\wr C_i$. That is, there are no two cells $\square_i \in C_i$ and $\square_j \in C_j$ such that $\square_j$ is the left neighbor of $\square_i$. As a first step, we consider consecutive vertical strips and show that C_{k+1} attacking C_k is sufficient for C_{k+1} neighboring C_k.

Lemma 3.6.4. *Let* $k \in [m]$. *If* $C_{k+1} \rightsquigarrow C_k$ *then* $C_{k+1} \wr C_k$.

Proof. We prove the corresponding statement for entries of T_1. That is, we assume $J_{k+1} \rightsquigarrow J_k$ and have to show that $J_{k+1} \wr J_k$ (in T_1). By assumption there are $i \in J_k$ and $j \in J_{k+1}$ such that $j \rightsquigarrow i$. Then i and j are located either in the same column or in two adjacent columns. We distinguish three cases.

Case 1. Assume that i and j occupy different columns. Then $j \rightsquigarrow i$ implies that i is located in the column to the immediate right of j and in a row strictly below j, Therefore, the triple rule applied on i and j yields that j has a right neighbor t such that $j > t > i$. That is, $t \in J_k \cup J_{k+1}$. By Lemma 3.6.3, T_1 has at most one entry of J_{k+1} per row. Therefore, $t \in J_k$. Since $j \wr t$, it follows that $J_{k+1} \wr J_k$.

Case 2. Assume that i and j occupy the same but not the first column. Since by Lemma 3.6.3 $c(i) \leq c(i')$ for all $i' \in J_k$ with $i < i'$, we can assume without loss of generality that i is the greatest element of J_k sharing a column with an element of J_{k+1} and j is the minimal element of J_{k+1} in the column of i.

First assume that i is above of j. Let l be the left neighbor of i in T_1 (setting $l := \infty$ if the cell left of i is part of the inner shape of T_1). Then $l < j$ since otherwise the triple rule applied to i, j and l would yield the contradiction $i > j$. Thus, $i < l < j$, i.e. $l \in J_k \cup J_{k+1}$. In addition, $l \notin J_k$ by Lemma 3.6.3. Hence, $l \in J_{k+1}$ and therefore $J_{k+1} \wr J_k$.

Assume now that i is below of j, i.e. $r(i) > r(j)$.

- Suppose $i = a_k$. Then $a_k \in D^c(T_1)$ implies $c(a_k + 1) < c(a_k)$. By definition $J_{k+1} = [a_k + 1, a_{k+1}]$. Since $c(a_k + 1) \neq c(j)$, it follows that $j - 1 \in J_{k+1}$ and thus $j - 1 \rightsquigarrow j$. As j is by assumption the smallest element on J_{k+1} in its column, it follows that $c(j-1) = c(j) - 1$ and $r(j-1) < r(j)$. Moreover, we supposed $r(j) < r(i)$. Therefore, $r(j-1) < r(i)$ and $c(i) = c(j-1) + 1$. That is, $j - 1 \rightsquigarrow i$ and $j - 1$ and i occupy different columns. Case 1 now implies that $J_{k+1} \wr J_k$.
- Suppose $i < a_k$. Then $i+1 \in J_k$ and by the maximality of i we have $c(i+1) \neq c(i)$. Moreover, $i \rightsquigarrow i+1$ so that $c(i+1) = c(i) + 1$ and $r(i) < r(i+1)$. Now $c(j) = c(i)$ and $r(j) < r(i)$ imply $c(i+1) = c(j) + 1$ and $r(j) < r(i+1)$. In other words $j \rightsquigarrow i+1$ and j and $i+1$ are located in different columns. By Case 1, we then have $J_{k+1} \wr J_k$.

Case 3. Assume that i and j appear in the first column. Let both i and j be maximal with this property in J_k and J_{k+1}, respectively. As a_k and a_{k+1} are ascents of T_1, they cannot appear in the first column. Hence $i \in \mathring{J}_k$ and $j \in \mathring{J}_{k+1}$. Consequently, $i+1 \in J_k$ and $i \rightsquigarrow i+1$ as well as $j+1 \in J_{k+1}$ and $j \rightsquigarrow j+1$. Now the maximality of i and j implies that $i+1$ and $j+1$ are located in the second column. Then we also have $j+1 \rightsquigarrow i+1$. That is, $j+1$ and $i+1$ meet the prerequisites of Case 2 which in turn yields $J_{k+1} \wr J_k$. □

The next lemma shows that in a separated strip sequence $C_{k,l}$ no two cells are horizontal neighbors.

Lemma 3.6.5. *Let $C_{k,l}$ be a separated strip sequence. For $i, j \in [k, l]$ with $i < j$ we have that C_j is strictly left of C_i, $C_j \not\wr C_i$ and $C_j \not\rightsquigarrow C_i$.*

Proof. Let $C_{k,l}$ be a separated strip sequence and i, j with $k \leq i < j \leq l$ be as above.

First, we consider the case where $j = i+1$. As $C_{k,l}$ is separated, we have $C_{i+1} \not\wr C_i$ and therefore can apply Lemma 3.6.4 in order to obtain $C_{i+1} \not\rightsquigarrow C_i$. Because two cells in the same column attack each other, it follows that $c(C_{i+1})$ and $c(C_i)$ are disjoint. Since $c_{T_1}(a_i + 1) < c_{T_1}(a_i)$ as $a_i \in D^c(T_1)$ and

$$c(C_r) = [c_{T_1}(a_{r-1}) + 1, c_{T_1}(a_r)] \quad \text{for } r = i, i+1$$

by Lemma 3.6.3, we then have that $\max c(C_{i+1}) < \min c(C_i)$, i.e. C_{i+1} is strictly left of C_i. This settles the case where $j = i+1$.

Suppose now $j > i+1$. Recall that for two sets of integers we write $A < B$ if $a < b$ for all $a \in A$ and $b \in B$. Using the first case iteratively yields

$$c(C_j) < c(C_{i+1}) < c(C_i).$$

Thus, C_j is strictly left of C_i. Moreover, $c(C_j) < c(C_i) - 1$ ensures that $C_j \not\wr C_i$ and $C_j \not\rightsquigarrow C_i$. □

Flanking ascents and A-sortable tableaux

We introduce the set of *flanking ascents* of T_1.

Definition 3.6.6. *The set of* flanking ascents *of T_1 is given by*

$$FD^c(T_1) := \{a_k \in D^c(T_1) \mid C_{k+1} \wr C_k\}$$

We write

$$\mathcal{FD}^{\mathrm{c}} := \{A \subseteq [n-1] \mid FD^c(T_1) \subseteq A \subseteq D^c(T_1)\}$$

for the subsets of $D(T_1)$ containing $FD^c(T_1)$.

We proceed as follows. Let $A \in \mathcal{FD}^{\mathrm{c}}$. To A we associate set partitions $\{J_{k_r,l_r} \mid r \in [p]\}$ of $[n]$ and $\{C_{k_r,l_r} \mid r \in [p]\}$ of $\alpha /\!\!/ \beta$. With them we define the set of *A-sortable tableaux* E_A. Then we consider two properties of E_A. We first give in Lemma 3.6.13 a characterization of the elements of E_A in terms of contents of column words . We then show that $ND^c(T) \subseteq A \subseteq D^c(T)$ for all $T \in E_A$ in Lemma 3.6.19. These results will then be used in order to construct a simple submodule of $\boldsymbol{S}_{\alpha/\!\!/\beta,E}$ from E_A. Moreover, it will turn out that up to isomorphism $\mathrm{soc}(\boldsymbol{S}_{\alpha/\!\!/\beta,E})$ is determined by $\mathcal{FD}^{\mathrm{c}}$.

Note that the concept of flanking ascents of T_1 generalizes that of neighborly ascents of T_1 since $ND^c(T_1) \subseteq FD^c(T_1)$. To see this let $a_k \in ND^c(T_1)$. Then $a_k + 1 \wr_{T_1} a_k$. Because $a_k \in T_1(C_k)$ and $a_k + 1 \in T_1(C_{k+1})$, it follows that $C_{k+1} \wr C_k$ so that $a_k \in FD^c(T_1)$ as desired.

Example 3.6.7. Let T_1 be the sink tableau from Example 3.6.2. Recall

$$D^c(T_1) = \{a_1 = 1, a_2 = 3, a_3 = 6\} \quad \text{and} \quad C_4 \wr C_3 \wr C_2 \not\wr C_1.$$

Thus, $FD^c(T_1) = \{3, 6\}$ and $\mathcal{FD}^{\mathrm{c}} = \{\{3,6\}, \{1,3,6\}\}$.

We now relate each $A \in \mathcal{FD}^{\mathrm{c}}$ to a partition of $[n]$ and a corresponding partition of $\alpha/\!\!/\beta$ into separated vertical strip sequences. The idea is that we separate $[n]$ according to the elements of A. We fix some more notation for the remainder of the section.

Notation 3.6.8. Let $A \in \mathcal{FD}^{\mathrm{c}}$. We associate the following objects to A. Since $A \subseteq D^c(T_1)$ there are $p \in [m+1]$ and indices $l_0 < l_1 < l_2 < \cdots < l_p$ such that

$$a_{l_0} = 0, \quad A = \left\{a_{l_1}, a_{l_2}, \ldots, a_{l_{p-1}}\right\} \quad \text{and} \quad a_{l_p} = n.$$

In addition, set $k_r := l_{r-1} + 1$ for $r \in [p]$. Then $J_{k_r,l_r} = [a_{l_{r-1}} + 1, a_{l_r}]$ for $r \in [p]$. That is, the sets J_{k_r,l_r} for $r \in [p]$ form a set partition of $[n]$. Since $C_{k_r,l_r} = T_1^{-1}(J_{k_r,l_r})$, the C_{k_r,l_r} form a set partition of the diagram of $\alpha/\!\!/\beta$. As $FD^c(T_1) \subseteq A$, each C_{k_r,l_r} is a separated vertical strip sequence. Lastly we define $A^c := [n-1] \setminus A$. Then $\bigcup_{r=1}^{p} \mathring{J}_{k_r,l_r} = A^c$.

Example 3.6.9. We continue Example 3.6.7 considering the sink tableau

$$T_1 = \begin{array}{|c|c|c|c|} \cline{1-1} 2 \\ \cline{1-2} 4 & 3 \\ \cline{1-2} 7 & 5 \\ \cline{1-4} \multicolumn{1}{c|}{} & 8 & 6 & 1 \\ \cline{2-4} \end{array} .$$

Recall that

$$D^c(T_1) = \{a_1 = 1, a_2 = 3, a_3 = 6\},\ FD^c(T_1) = \{3, 6\} \text{ and } \mathcal{FD}^{\mathrm{c}} = \{D^c(T_1), FD^c(T_1)\}.$$

We illustrate Notation 3.6.8 by regarding $A = FD^c(T_1)$. Then $p = 3$. The other parameters are shown below.

r	1	2	3
l_r	2	3	4
k_r	1	3	4
a_{l_r}	3	6	8
J_{k_r,l_r}	$\{1,2,3\}$	$\{4,5,6\}$	$\{7,8\}$
C_{k_r,l_r}	$C_{1,2}$	C_3	C_4

Note that we obtain the partition $\{J_{k_r,l_r} \mid r \in [3]\}$ of $[8]$ by splitting the list $1, 2, \ldots, 8$ behind the elements of A. In the picture above, the cells of C_{k_r,l_r} have the same shade of gray for $r = 1, 2, 3$.

Definition 3.6.10. *For $A \in \mathcal{FD}^{\mathrm{c}}$ define*

$$E_A := \{T \in E \mid T(C_{k_r,l_r}) = J_{k_r,l_r} \text{ for all } r \in [p]\}$$

the set of A-sortable tableaux *of E.*

The definition implies that $E_{A'} \subseteq E_A$ for all $A, A' \in \mathcal{FD}^{\mathrm{c}}$ with $A \subseteq A'$. In particular, $E_{D^c(T_1)} \subseteq E_A \subseteq E_{FD^c(T_1)}$ for each $A \in \mathcal{FD}^{\mathrm{c}}$.

We have $T_1 \in E_A$ for all $A \in \mathcal{FD}^{\mathrm{c}}$ since $T_1(C_{k,l}) = J_{k,l}$ for all $k \le l$. In Corollary 3.6.16 we will show that the only element of $E_{D^c(T_1)}$ is T_1.

Remark 3.6.11. Let $A \in \mathcal{FD}^{\mathrm{c}}$. The definition of the set of A-sortable tableaux E_A is dual to that of the set of D-sortable tableaux E_D for $D \in \mathcal{OD}$ from Definition 3.5.14. They are therefore called *A-sortable*. In fact, one can show that E_A has a least element T_A which also can be regarded as being dual to the D-sorted tableau T_D for $D \in \mathcal{OD}$. The idea of the proof the same as the one for T_D from Lemma 3.5.26. However, it relies on the combinatorics of separated vertical strip sequences instead of pacific horizontal strip sequences which is more tedious. In order to construct the socle of $\boldsymbol{S}_{\alpha/\!\!/\beta,E}$ this is not necessary and therefore not carried out in this thesis.

Example 3.6.12. We consider the tableaux

$$T_3 = \begin{array}{cccc} 1 & & & \\ 4 & 2 & & \\ 7 & 5 & & \\ & 8 & 6 & 3 \end{array} \xrightarrow{\pi_2} T_2 = \begin{array}{cccc} 1 & & & \\ 4 & 3 & & \\ 7 & 5 & & \\ & 8 & 6 & 2 \end{array} \xrightarrow{\pi_1} T_1 = \begin{array}{cccc} 2 & & & \\ 4 & 3 & & \\ 7 & 5 & & \\ & 8 & 6 & 1 \end{array}$$

from Figure 3.4 and denote their equivalence class with E. We emphasize that T_1 is the sink tableau of E. Recall from Example 3.6.9 that $D^c(T_1) = \{1, 3, 6\}$, $FD^c(T_1) = \{3, 6\}$ and $\mathcal{FD}^{\mathrm{c}} = \{D^c(T_1), FD^c(T_1)\}$. We determine E_A for $A \in \mathcal{FD}^{\mathrm{c}}$.

For $A = D^c(T_1)$ we have $E_A = \{T_1\}$ by Corollary 3.6.16.

Now consider $A = FD^c(T_1)$. In this case the corresponding sets I_{k_r,l_r} and C_{k_r,l_r} for $r \in [3]$ have been determined in Example 3.6.9. From this it follows that E_A consists of the elements $T \in E$ with

$$T(C_{1,2}) = \{1,2,3\}\,, \quad T(C_3) = \{4,5,6\} \quad \text{and} \qquad T(C_4) = \{7,8\}\,.$$

The cells of T_1, T_2 and T_3 above are shaded accordingly to the partition of the diagram of $\alpha /\!\!/ \beta$ given by the C_{k_r,l_r}. One can check in Figure 3.4 that T_1, T_2 and T_3 are the only elements of E that satisfy the above conditions. Hence, $E_A = \{T_1, T_2, T_3\}$.

Recall from Notation 3.6.8 that given $A \in \mathcal{FD}^{\mathrm{c}}$ we use the shorthand $A^c = [n-1] \setminus A$. In the following result we characterize the elements $T \in E_A$ in terms of $\mathrm{cont}(\mathrm{col}_{T_1} \mathrm{col}_T^{-1})$. Recall that this is the index set of the operators π_i establishing the covering relations in the saturated chains from T to T_1 in E. The result will play an important role in the identification of the simple submodules of $\boldsymbol{S}_{\alpha /\!\!/ \beta, E}$. It is dual to Lemma 3.5.18. The proof is completely analogous and therefore omitted. It is mainly an application of Proposition 3.2.9.

Lemma 3.6.13. *Let $T \in E$ and $A \in \mathcal{FD}^{\mathrm{c}}$. Then the following are equivalent.*

(1) $T(C_{k_r,l_r}) = J_{k_r,l_r}$ *for all* $r \in [p]$.

(2) $\mathrm{cont}(\mathrm{col}_{T_1} \mathrm{col}_T^{-1}) \subseteq A^c$.

In other words,

$$E_A = \left\{ T \in E \mid \mathrm{cont}(\mathrm{col}_{T_1} \mathrm{col}_T^{-1}) \subseteq A^c \right\}.$$

Example 3.6.14. Let T_1, T_2 and T_3 be the tableaux with corresponding equivalence class E from Figure 3.4 and Example 3.6.12. Recall

$$D^c(T_1) = \{1,3,6\}\,, \quad FD^c(T_1) = \{3,6\} \quad \text{and} \quad \mathcal{FD}^{\mathrm{c}} = \{D^c(T_1), FD^c(T_1)\}\,.$$

By Lemma 3.6.13, we have

$$\begin{aligned} E_{D^c(T_1)} &= \left\{ T \in E \mid \mathrm{cont}(\mathrm{col}_{T_1} \mathrm{col}_T^{-1}) \subseteq \{2,4,5,7\} \right\}, \\ E_{\mathcal{FD}^c(T_1)} &= \left\{ T \in E \mid \mathrm{cont}(\mathrm{col}_{T_1} \mathrm{col}_T^{-1}) \subseteq \{1,2,4,5,7\} \right\}. \end{aligned}$$

Using the fact that $\mathrm{cont}(\mathrm{col}_{T_1} \mathrm{col}_T^{-1})$ is the index set of the operators π_i in the saturated chains from T to T_1 in E, we can read from Figure 3.4 that $E_{D^c(T_1)} = \{T_1\}$ and $E_{FD^c(T_1)} = \{T_1, T_2, T_3\}$ in accordance with Example 3.6.12.

The following result on the poset structure of E_A is dual to Lemma 3.5.20. It is a consequence of Lemma 3.6.13. The proof is left out as it is almost literally the one of Lemma 3.5.20.

Lemma 3.6.15. *For each $A \in \mathcal{FD}^{\mathrm{c}}$, E_A is a filter of E*

As a consequence of Lemma 3.6.13 and Lemma 3.6.15, we can determine $E_{D^c(T_1)}$.

Corollary 3.6.16. *The sink tableau T_1 is the only element of $E_{D^c(T_1)}$.*

Proof. Clearly $D^c(T_1) \in \mathcal{FD}^c$. Then from Lemma 3.6.13 it follows that

$$E_{D^c(T_1)} = \left\{ T \in E \mid \operatorname{cont}(\operatorname{col}_{T_1} \operatorname{col}_T^{-1}) \subseteq D(T_1) \right\}.$$

Thus, $\operatorname{col}_{T_1} \operatorname{col}_{T_1}^{-1} = 1$ implies $T_1 \in E_{D^c(T_1)}$.

Suppose that $T \in E$ is an element covered by T_1. Then there is an $i \in nAD(T)$ such that $\pi_i T = T_1$ and thus $\operatorname{cont}(\operatorname{col}_{T_1} \operatorname{col}_T) = \{i\}$. Since $i \in nAD(T)$, i is strictly left of $i+1$ in T. Because we obtain T_1 from T by swapping i and $i+1$, it follows that $i \in D^c(T_1)$. That is, $\operatorname{cont}(\operatorname{col}_{T_1} \operatorname{col}_T^{-1}) \not\subseteq D(T_1)$ and hence $T \notin E_{D^c(T_1)}$. Now we can use that $E_{D^c(T_1)}$ is a filter by Lemma 3.6.15 and obtain that $E_{D^c(T_1)} = \{T_1\}$. □

Let $A \in \mathcal{FD}^c$ and $T \in E_A$. Our next goal is to show that $ND^c(T) \subseteq A \subseteq D^c(T)$. In order to prove this, we consider the position of the elements of A in T_1 relative to their positions in T in the following lemma. This result is dual to Lemma 3.5.21 and again the proof can easily be adapted. The main idea is the same: for $i \in A$ the operators π_j corresponding to an arbitrary saturated chain in E_A from T to T_1 are only capable of moving i to the left and $i+1$ to the right.

Lemma 3.6.17. *Let $A \in \mathcal{FD}^c$ and $T \in E_A$.*

(*1*) *The cell $T_1^{-1}(i)$ is weakly left of the cell $T^{-1}(i)$ for all $i \in A \cup \{n\}$.*

(*2*) *The cell $T^{-1}(i+1)$ is weakly left of the cell $T_1^{-1}(i+1)$ for all $i \in A$.*

Example 3.6.18. Let again

$$T_3 = \begin{array}{cccc} 1 & & & \\ 4 & 2 & & \\ 7 & 5 & & \\ & 8 & 6 & 3 \end{array} \xrightarrow{\pi_2} T_2 = \begin{array}{cccc} 1 & & & \\ 4 & 3 & & \\ 7 & 5 & & \\ & 8 & 6 & 2 \end{array} \xrightarrow{\pi_1} T_1 = \begin{array}{cccc} 2 & & & \\ 4 & 3 & & \\ 7 & 5 & & \\ & 8 & 6 & 1 \end{array}.$$

From Example 3.6.12 we have that these tableaux form E_A for $A = \mathcal{FD}^c(T_1) = \{3, 6\}$.

(1) Observe that $3 \in A$ and $T_1^{-1}(3)$ is located weakly left of $T^{-1}(3)$ for each $T \in E_A$ as predicted by Lemma 3.6.17.

(2) We consider the ascents and neighborly ascents of the tableaux.

T	$ND^c(T)$	$D^c(T)$
T_1	3	$1, 3, 6$
T_2	3	$2, 3, 6$
T_3		$3, 6$

Hence, $ND^c(T) \subseteq A \subseteq D^c(T)$ for each $T \in E_A$. This property is generalized in the next lemma.

We now come to the second important result on the A-sortable tableaux. It is dual to Lemma 3.5.22.

Lemma 3.6.19. *Let $A \in \mathcal{FD}^{\mathrm{c}}$ and $T \in E_A$. Then*

$$ND^c(T) \subseteq A \subseteq D^c(T).$$

Proof. Let $\sigma := \mathrm{col}_{T_1} \mathrm{col}_T^{-1}$. Recall that $c_T(i)$ denotes the column of the entry i in T and that $i \in D^c(T)$ if and only if $c_T(i+1) < c_T(i)$.

We begin with showing $A \subseteq D^c(T)$. Let $i \in A$. Since $A \in \mathcal{FD}^{\mathrm{c}}$, we have $i \in D^c(T_1)$ and thus $c_{T_1}(i+1) < c_{T_1}(i)$. In addition, since $i \in A$, we can apply Lemma 3.6.17 and obtain

$$c_{T_1}(i) \leq c_T(i) \quad \text{and} \quad c_T(i+1) \leq c_{T_1}(i+1).$$

Therefore,

$$c_T(i+1) \leq c_{T_1}(i+1) < c_{T_1}(i) \leq c_T(i),$$

that is, $i \in D^c(T)$. This proves $A \subseteq D^c(T)$.

Now we show $ND^c(T) \subseteq A$. As $A \subseteq D^c(T)$, we have $ND^c(T) \subseteq A$ if and only if

$$D^c(T) \setminus A \subseteq nND^c(T).$$

That is, we have to show that each ascent of T that is not contained in A is not neighborly.

Let $i \in D^c(T) \setminus A$. We have $[n-1] \setminus A = \bigcup_{r=1}^{p} \mathring{J}_{k_r,l_r}$. Hence, there is $r \in [p]$ such that $i, i+1 \in J_{k,l}$ for $k := k_r$ and $l := l_r$.

Since $T \in E_A$, we have $J_{k,l} = T(C_{k,l})$. Thus, we can define $v(j) \in [k,l]$ such that $j \in T(C_{v(j)})$ for $j = i, i+1$, the index of the vertical strip containing j in T. If $v(i) = v(i+1)$ then $i+1 \not\nearrow_T i$ because $C_{v(i)}$ is a vertical strip by Lemma 3.6.3. Hence, $i \in nND^c(T)$ in this case.

Assume $v(i) \neq v(i+1)$. As $i \in D^c(T)$, $i+1$ is strictly left of i in T. In addition, $C_{k,l}$ is a separated vertical strip sequence because $A \in \mathcal{FD}^{\mathrm{c}}$. Thus, we can apply Lemma 3.6.5 on $C_{k,l}$ and obtain that $v(i+1) > v(i)$. But then the same lemma yields $C_{v(i+1)} \not\nearrow C_{v(i)}$. Hence, $i+1 \not\nearrow_T i$. $\square$

Simple submodules

The socle of $\boldsymbol{S}_{\alpha /\!\!/ \beta, E}$ is the direct sum of the simple submodules of $\boldsymbol{S}_{\alpha /\!\!/ \beta, E}$. For each $A \in \mathcal{FD}^{\mathrm{c}}$ we now define a $\mathbb{K}$-subspace U_A of $\boldsymbol{S}_{\alpha /\!\!/ \beta, E}$. It will turn out that each U_A is a simple submodule of $\boldsymbol{S}_{\alpha /\!\!/ \beta, E}$. We will also see that we obtain all simple submodules of $\boldsymbol{S}_{\alpha /\!\!/ \beta, E}$ in this way.

Definition 3.6.20. (*1*) *Let $\mathcal{U}$ denote the set of simple $H_n(0)$-submodules of $\boldsymbol{S}_{\alpha /\!\!/ \beta, E}$.*

(*2*) *For $A \in \mathcal{FD}^{\mathrm{c}}$ define $u_A := \sum_{T \in E_A} (-1)^{\delta(T)} T$ where δ is the rank function of E and $U_A := \mathbb{K} u_A$.*

We proceed as follows. First, we show that U_A is a simple $H_n(0)$-submodule of $\boldsymbol{S}_{\alpha/\!\!/\beta,E}$ for each $A \in \mathcal{FD}^{\mathrm{c}}$. The proof is based on the combinatorics of A-sorted tableaux we developed so far. Second, we show that for each $U \in \mathcal{U}$ there is a $A \in \mathcal{FD}^{\mathrm{c}}$ such that $U = U_A$. This will require most of the remaining work. Third, we conclude that $\mathrm{soc}(\boldsymbol{S}_{\alpha/\!\!/\beta,E}) = \bigoplus_{A\in\mathcal{FD}^{\mathrm{c}}} U_A$ in Theorem 3.6.39. From this we obtain in Corollary 3.6.41 a combinatorial formula that determines $\mathrm{soc}(\boldsymbol{S}_{\alpha/\!\!/\beta,E})$ up to isomorphism. This formula only depends on $\mathcal{FD}^{\mathrm{c}}$. Lastly, we derive such a rule for $\mathrm{soc}(\boldsymbol{S}_{\alpha/\!\!/\beta})$ in Corollary 3.6.45.

Example 3.6.21. (1) Consider $D^c(T_1) \in \mathcal{FD}^{\mathrm{c}}$. Then $E_{D^c(T_1)} = \{T_1\}$ by Corollary 3.6.16. Thus, $U_{D^c(T_1)}$ is the simple submodule $\mathbb{K}T_1$ of $\boldsymbol{S}_{\alpha/\!\!/\beta,E}$ isomorphic to $\boldsymbol{F}_{D(T_1)}$ mentioned in the introduction of the section.

(2) Let

$$T_3 = \begin{array}{|c|c|c|c|} \hline 1 & & & \\ \hline 4 & 2 & & \\ \hline 7 & 5 & & \\ \hline & 8 & 6 & 3 \\ \hline \end{array} \xrightarrow{\pi_2} T_2 = \begin{array}{|c|c|c|c|} \hline 1 & & & \\ \hline 4 & 3 & & \\ \hline 7 & 5 & & \\ \hline & 8 & 6 & 2 \\ \hline \end{array} \xrightarrow{\pi_1} T_1 = \begin{array}{|c|c|c|c|} \hline 2 & & & \\ \hline 4 & 3 & & \\ \hline 7 & 5 & & \\ \hline & 8 & 6 & 1 \\ \hline \end{array}$$

be the tableaux and E be the equivalence class from the running example. In Example 3.6.7 we have seen that the set $\mathcal{FD}^{\mathrm{c}}$ associated to the sink tableau T_1 of E is formed by the two elements $D^c(T_1) = \{1,3,6\}$ and $FD^c(T_1) = \{3,6\}$. From Example 3.6.12 we have $E_{D^c(T_1)} = \{T_1\}$ and $E_{FD^c(T_1)} = \{T_1, T_2, T_3\}$. Moreover, we obtain from Figure 3.4 that $(-1)^{\delta(T_1)} = 1$. Hence,

$$u_{D^c(T_1)} = T_1 \quad \text{and} \quad u_{FD^c(T_1)} = T_1 - T_2 + T_3.$$

Observe that for $i \in [7]$

$$\pi_i u_{D^c(T_1)} = \begin{cases} 0 & \text{if } i \notin D^c(T_1) \\ u_{D^c(T_1)} & \text{if } i \in D^c(T_1) \end{cases} \quad \text{and} \quad \pi_i u_{FD^c(T_1)} = \begin{cases} 0 & \text{if } i \notin FD^c(T_1) \\ u_{FD^c(T_1)} & \text{if } i \in FD^c(T_1). \end{cases}$$

Thus for each $A \in \mathcal{FD}^{\mathrm{c}}$, the $\mathbb{K}$-vector space $U_A = \mathbb{K}u_A$ is in fact an $H_n(0)$-submodule of $\boldsymbol{S}_{\alpha/\!\!/\beta,E}$ isomorphic to $\boldsymbol{F}_{A^c}$. Showing this in general is the purpose of Proposition 3.6.24.

In order to show that U_A is a simple $H_n(0)$-submodule of $\boldsymbol{S}_{\alpha/\!\!/\beta,E}$ we need two basic results on the $H_n(0)$-operation on $\boldsymbol{S}_{\alpha/\!\!/\beta,E}$.

Lemma 3.6.22 ([TvW15, Lemma 3.7])**.** *Let T be an SCT. If $i \in nND^c(T)$ then there exists an SCT T' such that $T \neq T'$ and $\pi_i T' = T$.*

Lemma 3.6.23. *Let $T \in E$ and $i \in D(T)$. Then $\pi_i T' \neq T$ for all $T' \in E$.*

Proof. As $i \in D(T)$, we have $\pi_i T \in \{0, s_i T\}$. In any case $\pi_i T \neq T$. Assume that there is an $T' \in E$ with $\pi_i T' = T$. Then we obtain the contradiction

$$T \neq \pi_i T = \pi_i \pi_i T' = \pi_i T' = T$$

using $\pi_i^2 = \pi_i$. $\square$

We now show that $U_A \in \mathcal{U}$ for all $A \in \mathcal{FD}^{\mathrm{c}}$.

Proposition 3.6.24. *Let $A \in \mathcal{FD}^{\mathrm{c}}$. Then U_A is a simple $H_n(0)$-submodule of $\boldsymbol{S}_{\alpha /\!/ \beta, E}$ isomorphic to $\boldsymbol{F}_{A^c}$.*

Proof. Let $U := U_A$ and $u := u_A$. We show for $i \in [n-1]$ that $\pi_i u = 0$ if $i \notin A$ and $\pi_i u = u$ if $i \in A$. Then it follows that U is a simple submodule of $\boldsymbol{S}_{\alpha /\!/ \beta, E}$ isomorphic to $\boldsymbol{F}_{A^c}$.

Fix an $i \in [n-1]$. We deal with two cases. If $i \in A$ then Lemma 3.6.19 implies that $i \in D^c(T)$ for each $T \in E_A$ and thus

$$\pi_i u = \sum_{T \in E_A} (-1)^{\delta(T)} \pi_i T = \sum_{T \in E_A} (-1)^{\delta(T)} T = u$$

as desired.

Now suppose $i \notin A$. We have to show $\pi_i u = 0$. As $A^c = \bigcup_{r=1}^{p} \mathring{J}_{k_r, l_r}$ there exists an $r \in [p]$ such that $i, i+1 \in J_{k,l}$ for $k := k_r$ and $l := l_r$. Let $T \in E_A$. We show that $[T]\pi_i u$, the coefficient of T in $\pi_i u$, is zero. Again we have two cases.

Assume first that $i \in D(T)$. Then Lemma 3.6.23 yields that $T' \neq T$ for all $T' \in E$. Hence, $[T]\pi_i u = 0$.

Now assume that $i \in D^c(T)$. Since $i \in A^c$ and $ND^c(T) \subseteq A$ by Lemma 3.6.19, we then have that $i \in nND^c(T)$. Therefore, from Lemma 3.6.22 it follows that there is a $T' \in E \setminus \{T\}$ such that $\pi_i T' = T$. Then

$$\operatorname{cont}(\operatorname{col}_{T_1} \operatorname{col}_{T'}^{-1}) = \{i\} \cup \operatorname{cont}(\operatorname{col}_{T_1} \operatorname{col}_{T}^{-1}) \subseteq A^c,$$

where we use $i \in A^c$, $T \in E_A$ and the characterization of E_A from Lemma 3.6.13 for the inclusion. That is, $T' \in E_A$ by the same lemma. If $\widetilde{T} \in E \setminus \{T\}$ with $\pi_i \widetilde{T} = T$ then $s_i \widetilde{T} = T = s_i T'$ and thus $\widetilde{T} = T'$. Hence T' is the only element of $E \setminus \{T\}$ that is mapped to T by π_i. As a consequence,

$$[T]\pi_i u = [T] \sum_{\widetilde{T} \in E_A} (-1)^{\delta(\widetilde{T})} \pi_i \widetilde{T} = (-1)^{\delta(T)} + (-1)^{\delta(T')} = 0$$

where we use that $\delta(T) = \delta(T') + 1$ since T covers T' in E. □

We emphasize that Proposition 3.6.24 implies that the modules U_A for $A \in \mathcal{FD}^{\mathrm{c}}$ are pairwise non-isomorphic and thus distinct.

The next step is to show that the simple submodules U_A for $A \in \mathcal{FD}^{\mathrm{c}}$ are in fact all the simple sumbmodules of $\boldsymbol{S}_{\alpha /\!/ \beta, E}$. We therefore now consider the elements of $\mathcal{U}$ in general.

Let $v \in \boldsymbol{S}_{\alpha /\!/ \beta, E}$. Then we can expand v $\mathbb{K}$-linearly in the basis E of $\boldsymbol{S}_{\alpha /\!/ \beta, E}$. Recall that the *support* of v, $\operatorname{supp}(v)$, is the set of $T \in E$ appearing in this expansion with nonzero coefficient.

Consider $U \in \mathcal{U}$. Then $U \neq 0$ and hence there is an $u \in U \setminus \{0\}$. Recall from Section 2.3 that the simple $H_n(0)$-modules are one dimensional. Thus $U = \mathbb{K}u$. Moreover, if

$u' \in U \setminus \{0\}$ then $u' = au$ with $a \in \mathbb{K}$ and $a \neq 0$. Hence $\operatorname{supp}(u) = \operatorname{supp}(u')$. Therefore, the following is well defined.

Definition 3.6.25. *Let $U \in \mathcal{U}$.*

(1) *The* support *of U is denoted by E_U and given by* $\operatorname{supp}(u)$ *for any $u \in U \setminus \{0\}$.*
(2) *The* descent set *of U is given by $D(U) := \bigcup_{T \in E_U} D(T)$.*
(3) *The* ascent set *of U is given by $D^c(U) := [n-1] \setminus D(T)$.*

Example 3.6.26. (1) Let $A \in \mathcal{FD}^c$. As $U_A = \mathbb{K}u_A$ with $u_A = \sum_{T \in E_A} (-1)^{\delta(T)} T$, it follows that the support E_{U_A} of U_A is the set of A-sortable tableaux E_A.

(2) The modules $U_{D^c(T_1)}$ and $U_{FD^c(T_1)}$ from Example 3.6.21 have support $\{T_1\}$ and $\{T_1, T_2, T_3\}$, respectively. Moreover, one can check in Example 3.6.21 that

$$D^c(U_{D^c(T_1)}) = D^c(T_1) \quad \text{and} \quad D^c(U_{FD^c(T_1)}) = FD^c(T_1).$$

Let $U \in \mathcal{U}$. We proceed with the study of U. In Lemma 3.6.28 we will show that U is isomorphic to $\boldsymbol{F}_{D(U)}$. We will further see in Lemma 3.6.31 that U is already determined by its support E_U. The next lemma is a simple but useful property of the action of the π_i on U.

Lemma 3.6.27. *Let $U \in \mathcal{U}$ and $u \in U$. Then $\pi_i u \in \{0, u\}$ for all $i \in [n-1]$.*

Proof. Let $i \in [n-1]$. Because $\pi_i 0 = 0$, we can assume $u \neq 0$. Then $U = \mathbb{K}u$ as all simple $H_n(0)$-modules are one-dimensional. Thus, there is an $a \in \mathbb{K}$ such that $\pi_i u = au$. Since $\pi_i^2 = \pi_i$, it follows that

$$a^2 u = \pi_i \pi_i u = \pi_i u = au.$$

Hence $a \in \{0, 1\}$, i.e. $\pi_i u \in \{0, u\}$. □

We now show that the descent set $D(U)$ determines the simple submodule $U \in \mathcal{U}$ up to isomorphism.

Lemma 3.6.28. *Let $U \in \mathcal{U}$ and $u \in U \setminus \{0\}$. Then for $i \in [n-1]$*

$$\pi_i u = \begin{cases} 0 & \text{if } i \in D(U) \\ u & \text{if } i \notin D(U). \end{cases}$$

That is, U and $\boldsymbol{F}_{D(U)}$ are isomorphic as $H_n(0)$-modules.

Proof. Let $i \in [n-1]$. Suppose first that $i \in D(U)$. Then there is a $T \in E_U$ such that $i \in D(T)$ and Lemma 3.6.23 implies $\pi_i T' \neq T$ for each $T' \in E$. Therefore, $[T]\pi_i u = 0$ which means $\pi_i u \neq u$. In addition, $\pi_i u \in \{0, u\}$ by Lemma 3.6.27. Hence $\pi_i u = 0$.

Suppose now that $i \in D^c(U)$. Let $u = \sum_{T \in E_U} a_T T$ be the expansion of u into the $\mathbb{K}$-basis E of $\boldsymbol{S}_{\alpha /\!\!/ \beta, E}$. Since by definition $D^c(U) = \bigcap_{T \in E_U} D^c(T)$, we have $i \in D^c(T)$ for all $T \in E_U$. As a consequence,

$$\pi_i u = \sum_{T \in E_U} a_T \pi_i T = \sum_{T \in E_U} a_T T = u$$

as desired. $\square$

Example 3.6.29. Let U_A with $A \in \mathcal{FD}^c = \{D^c(T_1), FD^c(T_1)\}$ be a simple module from Example 3.6.21. There we have seen that $U_{D^c(A)} \cong \boldsymbol{F}_{A^c}$ as $H_n(0)$-modules. Further from Example 3.6.26 we have $D(U_A) = A^c$. Thus $U_A \cong \boldsymbol{F}_{D(U_A)}$.

Our next objective is to show that the support E_U completely determines U. To do this we need the following result.

Lemma 3.6.30. *Let $U \in \mathcal{U}$, $u \in U \setminus \{0\}$ and $T', T'' \in E$ be arbitrary. For $T \in E$ let $a_T \in \mathbb{K}$ be such that $u = \sum_{T \in E} a_T T$. If $a_{T'} \neq 0$ and T'' covers T' in E then $a_{T''} = -a_{T'}$.*

Proof. Let $T' \in E_U$ and $T'' \in E$ be such that T'' covers T' in E. Then there is an $i \in nAD(T')$ such that $\pi_i(T') = T''$ and $\pi_i T'' = T''$. It is easy to see that $\pi_i T \neq T''$ for all $T \in E \setminus \{T', T''\}$. In addition, from Lemma 3.6.28 it follows that $\pi_i u = 0$ because $i \in D(U)$. Therefore,

$$a_{T''} + a_{T'} = [T'']\pi_i u = [T'']0 = 0$$

as desired. $\square$

Let $A \in \mathcal{FD}^c$. By Lemma 3.6.15, E_A is a filter. The simple submodule U_A is generated by an element u_A which is an alternating sum of the elements of E_A. We now show for each simple submodule U of $\boldsymbol{S}_{\alpha /\!/ \beta, E}$ that the support E_U is a filter as well and that U is generated by a similar element. As a consequence, we obtain that U is uniquely determined in $\mathcal{U}$ by E_U.

Lemma 3.6.31. *Let $U \in \mathcal{U}$.*

(*1*) *E_U is a filter in E. In particular, $T_1 \in E_U$.*
(*2*) *We have $U = \mathbb{K}u$ for $u := \sum_{T \in E_U} (-1)^{\delta(T)} T$ where δ is the rank function of E.*
(*3*) *U is the only simple submodule of $\boldsymbol{S}_{\alpha /\!/ \beta, E}$ with support E_U.*

Proof. Let $U \in \mathcal{U}$. Then $U \neq 0$ and hence $E_U \neq \emptyset$. Moreover, Lemma 3.6.30 implies that E_U is a filter. As the sink tableau T_1 is the greatest element of E, it follows that $T_1 \in E_U$. This shows (1).

Let $u := \sum_{T \in E_U} (-1)^{\delta(T)} T \in \boldsymbol{S}_{\alpha /\!/ \beta, E}$ and $v \in U \setminus \{0\}$. Then we can write v as a $\mathbb{K}$-linear combination $v = \sum_{T \in E_U} a_T T$. Fix a $T \in E_U$ and consider a saturated chain from T to T_1 is E. Applying Lemma 3.6.30 to each covering relation in such a chain yields $a_T = (-1)^{\delta(T_1) - \delta(T)} a_{T_1}$. Therefore, $v = (-1)^{\delta(T_1)} a_{T_1} u$ and we have (2). Lastly note that (3) is a direct consequence of (2). $\square$

Recall that we want show that $\mathcal{U} = \{U_A \mid A \in \mathcal{FD}^c\}$ where only the inclusion $\subseteq$ remains by Proposition 3.6.24. From Lemma 3.6.31 we know that $U \in \mathcal{U}$ is already determined by its support E_U. In addition, we have seen in Example 3.6.26 that the support of U_A is E_A for $A \in \mathcal{FD}^c$. Therefore, our task is to find an $A \in \mathcal{FD}^c$ such that $E_U = E_A$ for each $U \in \mathcal{U}$. This will be $A = D^c(U)$.

We do this as follows. Let $U \in \mathcal{U}$. We first show in Lemma 3.6.35 that for $T \in E$ we have $T \in E_U$ if and only if $\operatorname{cont}(\operatorname{col}_{T_1} \operatorname{col}_T^{-1}) \subseteq D(U)$. This result is similar to the characterization of the A-sortable tableaux E_A from Lemma 3.6.13. Second, we show in Lemma 3.6.37 that $D^c(U) \in \mathcal{FD}^c$. The two results then imply that $E_U = E_A$ for $A = D^c(U)$.

We continue with the study of E_U. The next result is analogous to Lemma 3.6.19

Lemma 3.6.32. *Let $U \in \mathcal{U}$ and $T \in E_U$. Then*

$$ND^c(T) \subseteq D^c(U) \subseteq D^c(T)$$

Proof. By definition $D^c(U) = \bigcap_{T \in E_U} D^c(T)$. Therefore we have the right inclusion.

For the left one, let $u \in U \setminus \{0\}$, $T \in E_U$ and $i \in ND^c(T)$. Then $\pi_i T = T$. Moreover $\pi_i T' \neq T$ for all $T' \in E \setminus \{T\}$. Since if there were a $T' \in E \setminus \{T\}$ with $\pi_i T' = T$ then we would obtain T' from T by interchanging i and $i+1$ in T. But $i+1$ is the left neighbor of i in T so that in T' the entries would not decrease from left to right in the row of i and $i+1$. Thus, T' would not be an SCT and this would contradict $T' \in E$.

From $\pi_i T = T$ and $\pi_i T' \neq T$ for $T' \in E \setminus \{T\}$ it follows that $[T]\pi_i u = [T]u \neq 0$. Hence, Lemma 3.6.27 implies that $\pi_i u = u$ and thus Lemma 3.6.28 yields $i \in D^c(U)$. □

Example 3.6.33. Let $U_A = \mathbb{K}(T_1 - T_2 + T_3)$ with $A = \mathcal{FD}^c(T_1) = \{3, 6\}$ be one of the two simple submodules from Example 3.6.21. In Example 3.6.26 we have seen that $E_{U_A} = E_A = \{T_1, T_2, T_3\}$ and $D^c(U_A) = A$. Moreover, from Example 3.6.18 we have that $ND^c(T) \subseteq A \subseteq D^c(T)$ for each $T \in E_A$. Thus, $ND^c(T) \subseteq D^c(U_A) \subseteq D^c(T)$ for all $T \in E_A$ as well.

Let $U \in \mathcal{U}$ and $T \in E_U$. In Lemma 3.6.30 we have seen that if $i \in nAD(T)$ then $\pi_i T \in E_U$. We now show that we have the dual statement for $i \in nND^c(T) \cap D(U)$.

Lemma 3.6.34. *Let $U \in \mathcal{U}$ and $T \in E_U$ such that there is an $i \in nND^c(T) \cap D(U)$. Then there exists a unique $T' \in E_U$ such that $\pi_i T' = T$ and $T' \neq T$.*

Proof. Let $u \in U$ with $u \neq 0$ and $\sum_{T \in E_U} a_T T$ be the $\mathbb{K}$-expansion of u in E. Fix a $T \in E_U$ and assume that there is an $i \in nND^c(T) \cap D(U)$. Because $i \in nND^c(T)$, Lemma 3.6.22 yields that there is a $T' \in E \setminus \{T\}$ such that $\pi_i T' = T$. Then $T' = s_i T$, which means that T' is unique in E. Hence,

$$[T]\pi_i u = a_T + a_{T'}.$$

In addition, Lemma 3.6.27 implies that $\pi_i u = 0$ since $i \in D(U)$. Therefore,

$$a_T + a_{T'} = [T]\pi_i u = [T]0 = 0,$$

i.e. $a_{T'} = -a_T$. Moreover, we have $T \in E_U$ by assumption which means that $a_T \neq 0$. Hence, $a_{T'} \neq 0$ and $T' \in E_U$ as desired. □

Let $U \in \mathcal{U}$. We now come to the characterization of the elements of E_U which is similar to Lemma 3.6.13. This is one of the two major arguments in the proof of $\mathcal{U} \subseteq \{U_A \mid A \in \mathcal{FD}^c\}$.

Lemma 3.6.35. *Let $U \in \mathcal{U}$. For each $T \in E$ the following are equivalent.*

(1) $T \in E_U$.

(2) $\operatorname{cont}(\operatorname{col}_{T_1} \operatorname{col}_T^{-1}) \subseteq D(U)$.

In other words,

$$E_U = \left\{T \in E \mid \operatorname{cont}(\operatorname{col}_{T_1} \operatorname{col}_T^{-1}) \subseteq D(U)\right\}.$$

Proof. Let $T \in E$ and

$$T = T_k \preceq T_{k-1} \preceq \cdots \preceq T_1$$

be a saturated chain from T to the sink tableau T_1 in E. Then for each $j \in [2,k]$ there is a $i_j \in nAD(T_j)$ such that $\pi_{i_j} T_j = T_{j-1}$. That is, $\operatorname{cont}(\operatorname{col}_{T_1} \operatorname{col}_T^{-1}) = \{i_2, \dots, i_k\}$.

In order to show (1) $\implies$ (2) assume that $T \in E_U$. From Lemma 3.6.31 we have that E_U is a filter. Therefore, $T \in E_U$ implies that $T_j \in E_U$ for each $j \in [k]$ and hence $D(T_j) \subseteq D(U)$ for each $j \in [k]$. As $i_j \in D(T_j)$ for all $j \geq 2$, it follows that

$$\operatorname{cont}(\operatorname{col}_{T_1} \operatorname{col}_T^{-1}) = \{i_2, \dots, i_k\} \subseteq D(U)$$

as desired.

For (2) $\implies$ (1) assume that $\operatorname{cont}(\operatorname{col}_{T_1} \operatorname{col}_T^{-1}) \subseteq D(U)$. Then we also have that $\operatorname{cont}(\operatorname{col}_{T_1} \operatorname{col}_{T_j}^{-1}) \subseteq D(U)$ for all $j \in [k]$.

We show $T_j \in E_U$ for each $j \in [k]$ by induction on j. From Lemma 3.6.31 we know that $T_1 \in E_U$. Thus, we can assume that $T_{j-1} \in E_U$ for a $j > 1$. Set $i := i_j$. Then $i \in nAD(T_j)$ and $\pi_i T_j = T_{j-1}$. Moreover, $i \in \operatorname{cont}(\operatorname{col}_{T_1} \operatorname{col}_T^{-1})$ so that $i \in D(U)$.

We show that $i \in nND^c(T_{j-1})$. Because $i \in nAD(T_j)$, i is strictly left of $i+1$ in T_j. In addition, $i+1$ cannot be the right neighbor of i in T_j since entries decrease in rows of SCTx from left to right. As we obtain T_{j-1} from T_j by swapping i and $i+1$, it follows that $i \in nND^c(T_{j-1})$.

Because $T_{j-1} \in E_U$, $i \in nND^c(T_{j-1})$ and $i \in D(U)$, Lemma 3.6.34 implies that $T_j \in E_U$. □

Example 3.6.36. We consider the simple submodule $U_A = \mathbb{K}(T_1 - T_2 + T_3)$ with $A = FD^c(T_1)$ from Example 3.6.21. By Example 3.6.26, $E_{U_A} = E_A$ and $D(U) = A^c$. In Example 3.6.14 we have seen for $T \in E$ that $T \in E_A$ if and only if $\operatorname{cont}(\operatorname{col}_{T_1} \operatorname{col}_T^{-1}) \subseteq A^c$. As $D(U_A) = A^c$, the latter is equivalent to $\operatorname{cont}(\operatorname{col}_{T_1} \operatorname{col}_T^{-1}) \subseteq D(U_A)$.

For $A \in \mathcal{FD}^{\mathrm{c}}$ and $U \in \mathcal{U}$ we have

$$E_A = \left\{T \in E \mid \operatorname{cont}(\operatorname{col}_{T_1} \operatorname{col}_T^{-1}) \subseteq A^c\right\},$$
$$E_U = \left\{T \in E \mid \operatorname{cont}(\operatorname{col}_{T_1} \operatorname{col}_T^{-1}) \subseteq D(U)\right\}.$$

by Lemma 3.6.13 and Lemma 3.6.35, respectively. Thus, in order to show that for $U \in \mathcal{U}$ there is an $A \in \mathcal{FD}^{\mathrm{c}}$ with $E_U = E_A$, it remains to show that $D^c(U) \in \mathcal{FD}^{\mathrm{c}}$. This is the purpose of the next lemma. Example 3.6.38 serves as an illustration for the result and its rather technical proof.

Lemma 3.6.37. *If $U \in \mathcal{U}$ then $D^c(U) \in \mathcal{FD}^{\mathrm{c}}$.*

Proof. Let $U \in \mathcal{U}$ and T_1 be the sink tableau of E. We use the definitions associated to the ascents of T_1 in Notation 3.6.1. By definition, $D^c(U) \in \mathcal{FD}^{\mathrm{c}}$ if and only if

$$FD^c(T_1) \subseteq D^c(U) \subseteq D^c(T_1).$$

By Lemma 3.6.31, we have that $T_1 \in E_U$. Therefore the second inclusion holds by definition of $D^c(U)$. We will now prove the first inclusion.

From Lemma 3.6.32 we have that

$$ND^c(T) \subseteq D^c(U) \tag{3.19}$$

for each $T \in E_U$. Since $T_1 \in E_U$ and $ND^c(T_1) \subseteq FD^c(T_1)$, it thus remains to show that

$$FD^c(T_1) \cap nND^c(T_1) \subseteq D^c(U).$$

We prove this by contradiction and thus assume that there exists a k such that

$$a_k \in FD^c(T_1) \cap nND^c(T_1) \cap D(U).$$

Our strategy is to infer the existence of a $T_* \in E_U$ and an $i \in ND^c(T_*) \cap D(U)$ contradicting (3.19).

In order to obtain T_* we need some notation. Since $a_k \in FD^c(T_1)$, we have $J_{k+1} \wr_{T_1} J_k$. Hence the maximal element of J_k having a left neighbor from J_{k+1} in T_1

$$a := \max\left\{j \in J_k \mid J_{k+1} \wr_{T_1} j\right\}$$

is well defined. Let $b \in J_{k+1}$ be this left neighbor of a. Moreover, we define

$$J'_k := [a, a_k], \quad J'_{k+1} := [a_k + 1, b], \quad J'_{k,k+1} := [a, b] \quad \text{and} \quad \mathring{J}'_{k,k+1} := [a, b-1].$$

Note that $J'_j \subseteq J_j$ for $j = k, k+1$ and $J'_{k,k+1} \subseteq J_{k,k+1}$. The corresponding sets of cells are denoted by $C'_j := T_1^{-1}(J'_j)$ for $j = k, k+1$ and $C'_{k,k+1} := T_1^{-1}(J'_{k,k+1})$. We further set $\square_j := T_1^{-1}(j)$ for $j \in [n]$. Similar to the filters E_A associated to $A \in \mathcal{FD}^c$, we define

$$F := \left\{ T \in E \mid \operatorname{cont}(\operatorname{col}_{T_1} \operatorname{col}_T^{-1}) \subseteq \mathring{J}'_{k,k+1} \right\}.$$

Clearly $T_1 \in F$ which means $F \neq \emptyset$. Therefore, there exists a $T_* \in F$ which is minimal in F. Lastly set $i := T_*(\square_a)$.

Our goal is to show that $T_* \in E_U$ and $i \in ND^c(T_*) \cap D(U)$. This yields the desired contradiction. The remainder of the proof is divided into seven parts.

(1) We show $T_* \in E_U$.

Recall that by definition, a_k and a_{k+1} are the only ascents of T_1 in $J_{k,k+1}$. Therefore,

$$\mathring{J}'_{k,k+1} \setminus \{a_k\} \subseteq J_{k,k+1} \setminus \{a_k, a_{k+1}\} \subseteq D(T_1) \subseteq D(U)$$

where we use $T_1 \in E_U$ for the rightmost inclusion. Moreover, we have $a_k \in D(U)$ by assumption. Hence, $\mathring{J}'_{k,k+1} \subseteq D(U)$ and Lemma 3.6.35 implies that $F \subseteq E_U$. In particular $T_* \in E_U$.

The next three parts are preparations for proving $i \in D(U)$ and $i \in ND^c(T_1)$.

(2) We consider the geometry of $C'_{k,k+1}$. For $x \in J'_k$ and $y \in J'_{k+1}$ we show the following.

(a) $\square_y$ is located strictly left of $\square_x$.

(b) $\square_y \wr \square_x \implies x = a$ and $y = b$.

The chain of inequalities

$$c(\square_y) \leq c(\square_b) < c(\square_a) \leq c(\square_x)$$

implies (a). The outer inequalities are consequences of Lemma 3.6.3. In addition, by choice of a and b we have $\square_b \wr \square_a$ so that $c(\square_b) = c(\square_a) - 1$. Hence, $c(\square_b) < c(\square_a)$ as well.

From the definitions of a and C'_k it follows that $\square_a$ is the only element of C'_k that has an element of C_{k+1} as left neighbor. As the left neighbor of $\square_a$ is $\square_b$, we obtain (b).

(3) We show $T_*(C'_{k,k+1}) = J'_{k,k+1}$.

Recall that for an SCT T of size n and $j \in [0,n]$ we have $\operatorname{sh}(T^{>j}) = T^{-1}([j+1,n])$. From $T_* \in F$ and $\mathring{J}'_{k,k+1} = [a, b-1]$ it follows that $a-1, b \notin \operatorname{cont}(\operatorname{col}_{T_1} \operatorname{col}_T^{-1})$. Therefore, Proposition 3.2.9 implies that

$$\operatorname{sh}(T_*^{>a-1}) = \operatorname{sh}(T_1^{>a-1}) \quad \text{and} \quad \operatorname{sh}(T_*^{>b}) = \operatorname{sh}(T_1^{>b}).$$

Consequently,

$$\begin{aligned} T_*^{-1}(J'_{k,k+1}) = T_*^{-1}([a,b]) &= \operatorname{sh}(T_*^{>a-1}) \setminus \operatorname{sh}(T_*^{>b}) \\ &= \operatorname{sh}(T_1^{>a-1}) \setminus \operatorname{sh}(T_1^{>b}) = T_1^{-1}([a,b]) = C'_{k,k+1}. \end{aligned}$$

That is, $T_*(C'_{k,k+1}) = J'_{k,k+1}$.

(4) We show that $nND^c(T_*) \cap \mathring{J}'_{k,k+1} = \emptyset$.

For the sake of contradiction, assume that there is a $j \in nND^c(T_*) \cap \mathring{J}'_{k,k+1}$. Set $\sigma_T := \mathrm{col}_{T_1} \mathrm{col}_T^{-1}$ for $T \in E$. Then $T_* \in F$ implies $\mathrm{cont}(\sigma_{T_*}) \subseteq \mathring{J}'_{k,k+1}$. Lemma 3.6.22 provides the existence of a $T \in E$ such that $\pi_j T = T_*$. It follows that

$$\mathrm{cont}(\sigma_T) = \{j\} \cup \mathrm{cont}(\sigma_{T_*}) \subseteq \mathring{J}'_{k,k+1}$$

where we use $j \in \mathring{J}'_{k,k+1}$ and $\mathrm{cont}(\sigma_{T_*}) \subseteq \mathring{J}'_{k,k+1}$ for the inclusion. Thus, $T \in F$. But this contradicts the minimality of T_* in F. Therefore, we have $nND^c(T_*) \cap \mathring{J}'_{k,k+1} = \emptyset$ as claimed.

(5) We show that $i \in \mathring{J}'_{k,k+1}$ and $i \in D(U)$.

Define $b_* := T_*(\Box_b)$. From Part (3) we have that $T_*(C'_{k,k+1}) = J'_{k,k+1}$. Thus, $i, b_* \in J'_{k,k+1}$. Moreover, T being an SCT implies $b_* > i$ so that $i \in \mathring{J}'_{k,k+1}$. In Part (1) we have seen that $\mathring{J}'_{k,k+1} \subseteq D(U)$. Hence, $i \in D(U)$ as well.

(6) We show that $i \in ND^c(T_*)$.

Let $b_* = T_*(\Box_b)$ still be the left neighbor of i in T_*. We will show that $b_* = i+1$. Define

$$t := \max\left\{j \in T_*(C'_k) \mid j < b_*\right\}.$$

As $i \in T_*(C'_k)$ and $i < b_*$, the set is not empty and thus t is well defined. Since $T_*(C'_{k,k+1}) = J'_{k,k+1}$, we have that $t+1 \in T_*(C'_{k,k+1})$ and from the maximality of t it follows that $t+1 \in T_*(C'_{k+1})$ (if $t = b_* - 1$ then this also holds since $b_* = T_*(\Box_b) \in T_*(C'_{k+1})$). Because $t \in T_*(C'_k)$ and $t+1 \in T_*(C'_{k+1})$, we obtain from (a) that $t+1$ is strictly left of t in T_* and thus $t \in D^c(T_*)$. Since $nND^c(T_*) \cap \mathring{J}'_{k,k+1} = \emptyset$ by Part (4), it follows that $t \in ND^c(T_*)$. Then (b) implies

$$t+1 = T_*(\Box_b) \quad \text{and} \quad t = T_*(\Box_a).$$

Consequently, $t = i$ and $b_* = i+1$, i.e. $i \in ND^c(T_*)$.

(7) We summarize our results. On the one hand, we have $T_* \in E_U$ from Part (1) so that (3.19) implies

$$ND^c(T_*) \subseteq D^c(U).$$

On the other hand, Parts (5) and (6) provide the existence of an $i \in ND^c(T_*) \cap D(U)$ contradicting the formula above. Recall that we deduced this contradiction from the assumption $FD^c(T_1) \cap nND^c(T_1) \cap D(U) \neq \emptyset$. Therefore, this intersection must be empty which means by the discussion from the beginning of the proof that $D^c(U) \in \mathcal{FD}^c$. □

Example 3.6.38. Consider the tableaux

$$T_* = \begin{array}{|c|c|c|} \hline 2 & & \\ \hline & 5 & 4 \\ \hline & 3 & 1 \\ \hline & & 6 \\ \hline \end{array} \xrightarrow{\pi_3} T_2 = \begin{array}{|c|c|c|} \hline 2 & & \\ \hline & 5 & 3 \\ \hline & 4 & 1 \\ \hline & & 6 \\ \hline \end{array} \xrightarrow{\pi_2} T_1 = \begin{array}{|c|c|c|} \hline 3 & & \\ \hline & 5 & 2 \\ \hline & 4 & 1 \\ \hline & & 6 \\ \hline \end{array}.$$

Let E be their equivalence class and $\boldsymbol{S}_{\alpha /\!\!/ \beta, E}$ be the corresponding module. We illustrate the proof of Lemma 3.6.37 by directly showing that $D^c(U) \in \mathcal{FD}^{\mathrm{c}}$ for each simple submodule U of $\boldsymbol{S}_{\alpha /\!\!/ \beta, E}$. We also use its notation. In addition, it may be instructive to check that Statements (1) – (6) from the proof hold in this example.

Let $U \in \mathcal{U}$. We have $D^c(T_1) = FD^c(T_1) = \{a_1 = 2\}$. Thus $\{2\}$ is the only element of $\mathcal{FD}^{\mathrm{c}}$. Moreover, note that 2 is an non-neighborly ascent of T_1. We show $2 \in D^c(U)$. From this it follows that $D^c(U) \in \mathcal{FD}^{\mathrm{c}}$ as described in the beginning of the proof of Lemma 3.6.37. We proceed accordingly.

Assume for the sake of contradiction that $a_1 = 2 \in D(U)$. We have $J_1 = [1, 2]$, $J_2 = [3, 6]$ and $J_2 \wr_{T_1} J_1$. In the tableaux above, the cells of C_1 and C_2 respectively have the same shade of gray. The maximal element of J_1 with a left neighbor from J_2 is $a = 2$. Its left neighbor is $b = 5$. Then

$$J_1' = \{2\}\,, \quad J_2' = [3, 5], \quad J_{1,2}' = [2, 5], \quad \mathring{J}_{1,2}' = [2, 4]$$

and

$$F = \left\{ T \in E \mid \operatorname{cont}(\operatorname{col}_{T_1} \operatorname{col}_T^{-1}) \subseteq \{2, 3, 4\} \right\}.$$

In the picture the elements of $\{1, 6\} = J_{1,2} \setminus J_{1,2}'$ are printed in gray. Let $i = T_*(\square_a) = 4$ where $\square_a = T_1^{-1}(a)$ is the cell containing $a = 2$ in T_1. We show that $T_* \in E_U$ and $4 \in ND^c(T_*) \cap D(U)$. Note that we obtain T_1 from T_* by shuffling elements of $J_{1,2}'$ around using operators π_j with $j \in \mathring{J}_{1,2}'$. The other elements 1 and 6 are not affected.

From the picture we obtain that $T_1, T_2, T_* \in F$. Besides, we remark that T_* is minimal in F since $nND^c(T_*) = \{1\}$ which is disjoint to $\mathring{J}_{1,2}'$ (cf. Lemma 3.6.22). Since $3, 4 \in D(T_1)$, the definition of $D(U)$ yields $3, 4 \in D(U)$. Furthermore $2 \in D(U)$ by assumption. Hence, Lemma 3.6.35 implies that $F \subseteq E_U$. In particular $T_* \in E_U$.

We can directly check that $4 \in ND^c(T_*)$. Moreover, we have already seen that $4 \in D(U)$. Hence, $4 \in ND^c(T_*) \cap D(U)$. Yet, $T_* \in E_U$ and thus Lemma 3.6.32 demands that $ND^c(T_*) \subseteq D^c(U)$. We therefore have a contradiction which tells us that $2 \in D^c(U)$ as desired.

We are now in the position to determine the socle of $\boldsymbol{S}_{\alpha /\!\!/ \beta, E}$. Recall that for $A \in \mathcal{FD}^{\mathrm{c}}$ we have $u_A = \sum_{T \in E_A} (-1)^{\delta(T)} T$ where δ is the rank function of E and $U_A = \mathbb{K} u_A$.

Theorem 3.6.39. *Let $\alpha /\!\!/ \beta$ be a skew composition of size n and $E \in \mathcal{E}(\alpha /\!\!/ \beta)$.*

(1) For $A \in \mathcal{FD}^{\mathrm{c}}$, U_A is a simple $H_n(0)$-submodule of $\boldsymbol{S}_{\alpha /\!\!/ \beta, E}$ is isomorphic to $\boldsymbol{F}_{A^{\mathrm{c}}}$.

(2) We have $\operatorname{soc}(\boldsymbol{S}_{\alpha /\!\!/ \beta, E}) = \bigoplus_{A \in \mathcal{FD}^{\mathrm{c}}} U_A$.

Proof. Part (1) is a repetition of Proposition 3.6.24 which we have included for convenience. For Part (2) we first show that

$$\mathcal{U} = \{U_A \mid A \in \mathcal{FD}^{\mathrm{c}}\}. \tag{3.20}$$

Let $U \in \mathcal{U}$ and $A := D^c(U)$. From Lemma 3.6.37 we have that $A \in \mathcal{FD}^{\mathrm{c}}$. Then Part (1) yields that U_A is a simple submodule of $\boldsymbol{S}_{\alpha/\!\!/\beta,E}$. As $U_A = \mathbb{K}u_A$, it follows that the support E_{U_A} of U_A is the set of A-sortable tableaux E_A. On the other hand, Lemma 3.6.35 implies that the support of U is given by

$$E_U = \left\{T \in E \mid \operatorname{cont}(\operatorname{col}_{T_1} \operatorname{col}_T^{-1}) \subseteq A^c\right\}.$$

Thus, Lemma 3.6.13 yields that the support E_U is the set of A-sorted tableaux E_A as well. That is, $E_U = E_{U_A}$. But by Lemma 3.6.31 there is only one simple submodule of $\boldsymbol{S}_{\alpha/\!\!/\beta,E}$ with support E_U. Therefore, $U = U_A$. This proves (3.20).

Part (1) ensures that the $H_n(0)$-submodules U_A of $\boldsymbol{S}_{\alpha/\!\!/\beta,E}$ for $A \in \mathcal{FD}^{\mathrm{c}}$ are all pairwise non-isomorphic. Moreover, we have that $\operatorname{soc}(\boldsymbol{S}_{\alpha/\!\!/\beta,E}) = \sum_{U\in\mathcal{U}} U$ by definition. Therefore, (3.20) implies (2). □

Example 3.6.40. Let T_1, T_2 and T_3 be as in Figure 3.4, E be their equivalence class and $\boldsymbol{S}_{\alpha/\!\!/\beta,E}$ the corresponding $H_n(0)$-module. Then $\mathcal{FD}^{\mathrm{c}} = \{D^c(T_1), FD^c(T_1)\}$ where $FD^c(T_1) = \{3, 6\}$ and $D^c(T_1) = \{1, 3, 6\}$. by Example 3.6.7. Therefore, Theorem 3.6.39 implies

$$\operatorname{soc}(\boldsymbol{S}_{\alpha/\!\!/\beta,E}) = U_{D^c(T_1)} \oplus U_{FD^c(T_1)}.$$

In addition, we have seen in Example 3.6.21 that

$$U_{D^c(T_1)} = \mathbb{K}T_1 \quad \text{and} \quad U_{FD^c(T_1)} = \mathbb{K}(T_1 - T_2 + T_3).$$

Thus,

$$\operatorname{soc}(\boldsymbol{S}_{\alpha/\!\!/\beta,E}) = \mathbb{K}T_1 \oplus \mathbb{K}(T_1 - T_2 + T_3).$$

From Theorem 3.6.39 we have that, up to isomorphism, the socle of $\boldsymbol{S}_{\alpha/\!\!/\beta,E}$ only depends on the ascents and the flanking ascents of the sink tableau of E. We thus obtain the following formula for the socle dual to that for the top from Theorem 3.5.42.

Recall that in this section we always assumed that $\alpha/\!\!/\beta$ is a skew composition of size n, $E \in \mathcal{E}(\alpha/\!\!/\beta)$ and T_1 is the sink tableau of E. Under this assumptions we associated the set

$$\mathcal{FD}^{\mathrm{c}} = \{A \subseteq [n-1] \mid FD^c(T_1) \subseteq A \subseteq D^c(T_1)\}.$$

to T_1 where $FD^c(T_1)$ is the set of flanking ascents of T_1 from Definition 3.6.6. This is the main ingredient in the following formula.

Corollary 3.6.41. *Let $\alpha /\!\!/ \beta$ be a skew composition of size n and $E \in \mathcal{E}(\alpha /\!\!/ \beta)$. Then*

$$\operatorname{soc}(\boldsymbol{S}_{\alpha /\!\!/ \beta, E}) \cong \bigoplus_{A \in \mathcal{FD}^{\mathrm{c}}} \boldsymbol{F}_{A^c}$$

as $H_n(0)$-modules.

Example 3.6.42. Let $\boldsymbol{S}_{\alpha /\!\!/ \beta, E}$ be the $H_n(0)$-modules of the equivalence class E from Figure 3.4. By Example 3.6.7 we then have $\mathcal{FD}^{\mathrm{c}} = \{\{3,6\}, \{1,3,6\}\}$. Thus, Corollary 3.6.41 yields that

$$\operatorname{soc}(\boldsymbol{S}_{\alpha /\!\!/ \beta, E}) \cong \boldsymbol{F}_{\{2,4,5,7\}} \oplus \boldsymbol{F}_{\{1,2,4,5,7\}}$$

as $H_8(0)$-modules.

We gather some direct consequences of Corollary 3.6.41. For Part (3) we use that a module with simple socle is always indecomposable.

Corollary 3.6.43. *Let $\alpha /\!\!/ \beta$ be a skew composition of size n and $E \in \mathcal{E}(\alpha /\!\!/ \beta)$ with sink tableau T_1. Then we have the following.*

(*1*) $\dim \operatorname{soc}(\boldsymbol{S}_{\alpha /\!\!/ \beta, E}) = |\mathcal{FD}^{\mathrm{c}}|$.

(*2*) $\operatorname{soc}(\boldsymbol{S}_{\alpha /\!\!/ \beta, E})$ *is simple if and only if* $FD^c(T_1) = D^c(T_1)$.

(*3*) *If* $FD^c(T_1) = D^c(T_1)$ *then* $\boldsymbol{S}_{\alpha /\!\!/ \beta, E}$ *is indecomposable.*

In Corollary 3.5.44 we have seen that a module $\boldsymbol{S}_{\alpha /\!\!/ \beta, E}$ is indecomposable if each descent of the source tableau $T_{0,E}$ is offensive. By Part (3) of Corollary 3.6.41 we now have a similar condition depending on the flanking ascents of the sink tableau $T_{1,E}$. Yet, even combining Corollary 3.5.44 and Corollary 3.6.43 does not result in a necessary condition for $\boldsymbol{S}_{\alpha /\!\!/ \beta, E}$ to be indecomposable. In other words, there are modules $\boldsymbol{S}_{\alpha /\!\!/ \beta, E}$ which are indecomposable despite having nonsimple top and socle. An example is given below.

Example 3.6.44. We consider $\alpha = (2,2,4,4)$ and the module $\boldsymbol{S}_{\alpha, E}$ whose equivalence class E has source and sink tableau

$$T_0 = \begin{array}{|c|c|cc} \hline 2 & 1 & & \\ \hline 4 & 3 & & \\ \hline 9 & 8 & 7 & \multicolumn{1}{c|}{6} \\ \hline 12 & 11 & 10 & \multicolumn{1}{c|}{5} \\ \hline \end{array} \quad \text{and} \quad T_1 = \begin{array}{|c|c|cc} \hline 4 & 3 & & \\ \hline 7 & 5 & & \\ \hline 10 & 8 & 6 & \multicolumn{1}{c|}{2} \\ \hline 12 & 11 & 9 & \multicolumn{1}{c|}{1} \\ \hline \end{array}\,,$$

respectively. The cells of T_0 and T_1 are shaded according to the associated decomposition of the diagram of α in horizontal strips and vertical strips, respectively. Then

$$\begin{aligned} D(T_0) &= \{2,4,5,9\} \\ OD(T_0) &= \{2,5,9\} \end{aligned} \quad \text{and} \quad \begin{aligned} D^c(T_1) &= \{2,3,6,9,11\} \\ FD^c(T_1) &= \{3,6,9,11\}, \end{aligned}$$

that is, $OD(T_0) \subsetneq D(T_0)$ and $FD^c(T_1) \subsetneq D^c(T_1)$. Thus, top and socle of $\boldsymbol{S}_{\alpha,E}$ are not simple by Corollary 3.5.44 and Corollary 3.6.43, respectively. However, the straight module $\boldsymbol{S}_{\alpha,E}$ certainly is indecomposable by Theorem 3.3.11.

Using the decomposition $\boldsymbol{S}_{\alpha/\!\!/\beta} = \bigoplus_{E\in\mathcal{E}(\alpha/\!\!/\beta)} \boldsymbol{S}_{\alpha/\!\!/\beta,E}$ from Proposition 3.1.13 and the compatibility of soc with direct sums, we infer from Corollary 3.6.41 the following formula for $\operatorname{soc}(\boldsymbol{S}_{\alpha/\!\!/\beta})$. Recall that for $E \in \mathcal{E}(\alpha/\!\!/\beta)$, $T_{1,E}$ denotes the sink tableau of E.

Corollary 3.6.45. *Let $\alpha/\!\!/\beta$ be a skew composition of size n. Then*

$$\operatorname{soc}(\boldsymbol{S}_{\alpha/\!\!/\beta}) \cong \bigoplus_{E\in\mathcal{E}(\alpha/\!\!/\beta)} \bigoplus_{A\in\mathcal{FD}^{\mathrm{c}}_E} \boldsymbol{F}_{A^{\mathrm{c}}}$$

as $H_n(0)$-modules where $\mathcal{FD}^{\mathrm{c}}_E := \{A \subseteq [n-1] \mid FD^c(T_{1,E}) \subseteq A \subseteq D^c(T_{1,E})\}$.

We end the section by showing that the formula for the socle of pacific modules from Corollary 3.4.21 is a special case of Corollary 3.6.45 above. Let $\alpha/\!\!/\beta$ be a pacific skew composition of size n. Recall from Lemma 3.4.5 that then all SCTx of shape $\alpha/\!\!/\beta$ form a single equivalence class. Thus, let T_0 be the source and T_1 be the sink tableau of shape $\alpha/\!\!/\beta$. We use the shorthand $n - D = \{n - d \mid d \in D\}$ for $D \subseteq [n-1]$. From Corollary 3.4.21 we have that

$$\operatorname{soc}(\boldsymbol{S}_{\alpha/\!\!/\beta}) \cong \bigoplus_{D\subseteq D(T_0)} \boldsymbol{F}_{n-D} \tag{3.21}$$

as $H_n(0)$-modules. On the other hand, Corollary 3.6.45 yields that

$$\operatorname{soc}(\boldsymbol{S}_{\alpha/\!\!/\beta}) \cong \bigoplus_{A\in\mathcal{FD}^{\mathrm{c}}} \boldsymbol{F}_{A^{\mathrm{c}}}$$

with $\mathcal{FD}^{\mathrm{c}}$ associated to T_1. Hence, in order to infer (3.21) from Corollary 3.6.45, it remains to show that $\mathcal{FD}^{\mathrm{c}} = \{n - D^c \mid D \subseteq D(T_0)\}$ in the pacific case. This is done in Lemma 3.6.47 which we state after giving an example.

Example 3.6.46. Let $\boldsymbol{S}_{\alpha/\!\!/\beta}$ be the pacific module formed by the tableaux from Figure 3.3. The source and the sink tableau of shape $\alpha/\!\!/\beta$ are

$$T_0 = \begin{array}{cccc} & & & \boxed{4} \\ & & \boxed{3} & \\ \boxed{2} & \boxed{1} & & \end{array} \quad \text{and} \quad T_1 = \begin{array}{cccc} & & & \boxed{1} \\ & & \boxed{2} & \\ \boxed{4} & \boxed{3} & & \end{array},$$

respectively. We have $D(T_0) = \{2, 3\}$, $D^c(T_1) = \{1, 2, 3\}$, $FD^c(T_1) = \{3\}$ and

$$\mathcal{FD}^{\mathrm{c}} = \{\{3\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

By applying Corollary 3.6.45 we get

$$\operatorname{soc}(\boldsymbol{S}_{\alpha/\!\!/\beta}) \cong \bigoplus_{A\in\mathcal{FD}^{\mathrm{c}}} \boldsymbol{F}_{A^{\mathrm{c}}} = \boldsymbol{F}_{\emptyset} \oplus \boldsymbol{F}_{\{1\}} \oplus \boldsymbol{F}_{\{2\}} \oplus \boldsymbol{F}_{\{1,2\}}.$$

As in Example 3.4.22, we obtain from (3.21) that

$$\operatorname{soc}(\boldsymbol{S}_{\alpha/\!\!/\beta}) \cong \bigoplus_{D \subseteq D(T_0)} \boldsymbol{F}_{n-D} = \boldsymbol{F}_{\emptyset} \oplus \boldsymbol{F}_{\{1\}} \oplus \boldsymbol{F}_{\{2\}} \oplus \boldsymbol{F}_{\{1,2\}}.$$

Thus, we can directly see that both sums run over the same index set.

We now state the result that allows to derive Corollary 3.4.21 from Corollary 3.6.45.

Lemma 3.6.47. *Let $\alpha/\!\!/\beta$ be a pacific skew composition of size n and T_0 be the source tableau of shape $\alpha/\!\!/\beta$. Then*

$$\mathcal{FD}^{\mathrm{c}} = \{n - D^c \mid D \subseteq D(T_0)\}\,.$$

Proof. Let T_1 be the sink tableau of the pacific shape $\alpha/\!\!/\beta$. On the one hand, T_1 is a sink tableau so that $D(T_1) = AD(T_1)$ by Theorem 3.1.14. On the other hand, T_1 is pacific and therefore $AD(T_1) = \emptyset$. Hence, $D^c(T_1) = [n-1]$ and thus $\mathcal{FD}^{\mathrm{c}} = \{A \subseteq [n-1] \mid \mathcal{FD}^{\mathrm{c}}(T_1) \subseteq A\}$. We show

$$FD^c(T_1) = n - D^c(T_0)$$

since then it follows for all $D \subseteq [n-1]$ that

$$D \subseteq D(T_0) \iff n - D^c(T_0) \subseteq n - D^c \iff FD^c(T_1) \subseteq n - D^c \iff n - D^c \in \mathcal{FD}^{\mathrm{c}}$$

which yields the claim.

Let let $m_0 := |D(T_0)|$ and $m_1 := |D^c(T_1)|$. We also use the definitions associated to T_0 and T_1 in Notations 3.5.1 and 3.6.1, respectively. Because $D^c(T_1) = [n-1]$, we have $a_k = k$ and $J_k = \{k\}$ for all $k \in [m_1+1]$. Thus for each $k \in [m_1]$ we have that $C_{k+1} \wr C_k$ if and only if $k+1 \wr_{T_1} k$. That is, $FD^c(T_1) = ND^c(T_1)$. By Proposition 3.4.12,

$$\operatorname{col}_{T_1} = n\ n-1\ \cdots\ 1.$$

Moreover, Lemma 3.4.8 yields that the horizontal strip B_k is strictly left of B_{k+1} for each $k \in [m_0]$. In addition, $|B_{1,k}| = |I_{1,k}| = d_k$ for all $k \in [m_0+1]$. Therefore, the definition of the column word implies

$$T_1(B_{1,k}) = [n - d_k + 1, n]$$

for all $k \in [m_0+1]$. As a consequence,

$$\begin{aligned} T_1(B_k) &= T_1(B_{1,k}) \setminus T_1(B_{1,k-1}) \\ &= [n - d_k + 1, n] \setminus [n - d_{k-1} + 1, n] \\ &= [n - d_k + 1, n - d_{k-1}]. \end{aligned}$$

for all $k \in [m_0+1]$.

Fix a $k \in [m_0]$. Because

$$T_1(B_{k+1}) = [n - d_{k+1} + 1, n - d_k],$$

the connected horizontal strip B_{k+1} is filled as follows in T_1.

$n - d_k$	$n - d_k - 1$	$\cdots$	$n - d_{k+1} + 1$

Thus, we have $[n - d_{k+1} + 1, n - d_k - 1] \subseteq ND^c(T_1)$. We claim that $n - d_k \notin ND^c(T_1)$. This can be seen as follows. First, note that $T_0(B_k) = I_k < I_{k+1} = T_0(B_{k+1})$. Second, we have seen above that B_k is strictly left of B_{k+1}. As entries decrease from left to right in the rows of the SCT T_0, we therefore have that B_k and B_{k+1} cannot occupy the same row of the diagram $\alpha /\!\!/ \beta$. Because $n - d_k \in T_1(B_{k+1})$ and $n - d_k + 1 \in T_1(B_k)$, it follows that $n - d_k + 1 \wr_{T_1} n - d_k$ is impossible. That is, $n - d_k \notin ND^c(T_1)$.

Since $k \in [m_0]$ was chosen arbitrarily, it finally follows that

$$FD^c(T_1) = ND^c(T_1) = [n-1] \setminus \{n - d_k \mid k \in [m_0]\} = n - D^c(T_0)$$

as desired. $\square$

3.7 Modules of permuted composition tableaux

Tewari and van Willigenburg generalize in [TvW19] standard straight compositions tableaux to standard permuted compositions tableaux. Let $\alpha \vDash n$ and $\sigma \in \mathfrak{S}_{\ell(\alpha)}$. A standard permuted composition tableau (SPCT) of shape α and type σ is defined as an SCT of shape α in Definition 3.1.4 except that the relative order of the entries in the first column when read from top to bottom is now demanded to be that of σ. We write $\mathrm{SPCT}^\sigma(\alpha)$ for the SPCTx (plural form of SPCT) of shape α and type σ. Tewari and van Willigenburg show that the $\mathbb{K}$-span of $\mathrm{SPCT}^\sigma(\alpha)$ can be endowed with a 0-Hecke action which yields an $H_n(0)$-module which we denote with $\boldsymbol{S}^\sigma_\alpha$. The modules $\boldsymbol{S}^\sigma_\alpha$ and $\boldsymbol{S}_\alpha$ share many properties. In particular $\boldsymbol{S}^\sigma_\alpha$ can be decomposed as $\boldsymbol{S}^\sigma_\alpha = \bigoplus_{E \in \mathcal{E}^\sigma(\alpha)} \boldsymbol{S}^\sigma_{\alpha,E}$ where $\mathcal{E}^\sigma(\alpha)$ is the set of equivalence classes of $\mathrm{SPCT}^\sigma(\alpha)$ with respect to the equivalence relation $\sim$.

The purpose of this section is to transfer the main results of this chapter on the modules $\boldsymbol{S}_{\alpha,E}$ to the modules $\boldsymbol{S}^\sigma_{\alpha,E}$. In particular, we will describe how the arguments of the chapter can be adapted for $E \in \mathcal{E}^\sigma(\alpha)$ in order to show that $\boldsymbol{S}^\sigma_{\alpha,E}$ is indecomposable and to obtain formulas for the top and the socle of $\boldsymbol{S}^\sigma_{\alpha,E}$. As for $\boldsymbol{S}_\alpha$, one can then obtain the corresponding results on $\boldsymbol{S}^\sigma_\alpha$ by using the decomposition from above.

The proof of the indecomposability of $\boldsymbol{S}_{\alpha,E}$ has already been published as the article [Kön19] by the author. The contents of [Kön19] correspond to Sections 3.1 to 3.3 of this chapter. Choi, Kim, Nam and Oh show in [CKNO21] how the arguments from [Kön19] can be adapted to obtain the indecomposability of $\boldsymbol{S}^\sigma_{\alpha,E}$. Mainly, they substitute [Kön19, Proposition 3.8] (corresponding to Proposition 3.2.9 of this thesis) by [CKNO21, Lemma A.3]. In this section however, we will generalize Proposition 3.2.9 to SPCTx instead. This is also necessary for the generalization of the formulas for top and socle. We begin with introducing the permuted modules.

0-Hecke modules of standard permuted composition tableaux

Let $w = w_1 \cdots w_n$ be a word with letters in $\mathbb{N}$. The *standardization of w* is the unique element $\sigma \in \mathfrak{S}_n$ such that $\sigma(i) > \sigma(j)$ if and only if $w_i > w_j$.

Definition 3.7.1. *Let $\alpha \vDash n$ and $\sigma \in \mathfrak{S}_{\ell(\alpha)}$. A* standard permuted composition tableau *(SPCT) of shape α and type σ is a bijective filling $T\colon \alpha \to [n]$ satisfying the following conditions:*

(*1*) *The entries are decreasing in each row from left to right.*

(*2*) *The standardization of the word obtained by reading the first column from top to bottom is σ.*

(*3*) (*Triple rule*). *If $(i, k-1), (j, k) \in \alpha$ such that $j > i$ and $T(j, k) < T(i, k-1)$ then $(i, k) \in \alpha$ and $T(j, k) < T(i, k)$.*

For $\alpha \vDash n$ and $\sigma \in \mathfrak{S}_{\ell(\alpha)}$ we denote the set of standard permuted composition tableaux of shape α and type σ with $\mathrm{SPCT}^\sigma(\alpha)$. Note that we have only defined straight SPCTx. This is the reason why the triple rule above is simpler than the one of Definition 3.1.4. We also remark that for $\alpha \vDash n$ and $\mathrm{id} \in \mathfrak{S}_{\ell(\alpha)}$ we have $\mathrm{SCT}(\alpha) = \mathrm{SPCT}^{\mathrm{id}}(\alpha)$. We do not associate SPCTx with chains of a composition poset as we have done with SCTx in Proposition 3.1.6. But all other notation introduced for SCTx in Section 3.1 up to and including Definition 3.1.9 can be used for SPCTx as well.

Example 3.7.2. The SPCT

$$T_0 = \begin{array}{|c|c|c|} \hline 7 & 6 & 5 \\ \hline 1 \\ \cline{1-1} 8 & 4 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}$$

has shape $\alpha = (3, 1, 3, 1)$ and type $\sigma = 3142$. Moreover, $D(T_0) = \{1, 2, 4, 7\}$ and $AD(T_0) = \{1, 7\}$.

We can define $H_n(0)$-modules $\boldsymbol{S}_\alpha^\sigma$ formed by SPCTx as we have defined the modules $\boldsymbol{S}_\alpha$ in Theorem 3.1.11.

Theorem 3.7.3 ([TvW19, Theorem 3.1])**.** *Let $\alpha \vDash n$ and $\sigma \in \mathfrak{S}_{\ell(\alpha)}$. Then $\boldsymbol{S}_\alpha^\sigma := \operatorname{span}_{\mathbb{K}} \mathrm{SPCT}^\sigma(\alpha)$ is an $H_n(0)$-module with respect to the following action. For $T \in \mathrm{SPCT}^\sigma(\alpha)$ and $i \in [n-1]$,*

$$\pi_i T = \begin{cases} T & \text{if } i \notin D(T) \\ 0 & \text{if } i \in AD(T) \\ s_i T & \text{if } i \in nAD(T) \end{cases}$$

where $s_i T$ is the tableau obtained from T by interchanging i and $i+1$.

Example 3.7.4. Let T_0 be the SPCT from Example 3.7.2. Then

$$\pi_i T_0 = \begin{cases} T_0 & \text{for } i = 3, 5, 6 \\ 0 & \text{for } i = 1, 7 \\ s_i T_0 & \text{for } i = 2, 4. \end{cases}$$

Let $\alpha \vDash n$ and $\sigma \in \mathfrak{S}_{\ell(\alpha)}$. All the results on the modules $\boldsymbol{S}_\alpha$ from Section 3.1 succeeding Theorem 3.1.11 generalize to SPCTx. The most important ones are the following.

(1) We can use the equivalence relation $\sim$ on $\mathrm{SPCT}^\sigma(\alpha)$ and define $\mathcal{E}^\sigma(\alpha)$ as the *set of equivalence classes* of $\mathrm{SPCT}^\sigma(\alpha)$ with respect to $\sim$.

(2) For each $E \in \mathcal{E}^\sigma(\alpha)$, $\boldsymbol{S}^\sigma_{\alpha,E} := \operatorname{span}_{\mathbb{K}} E$ is an $H_n(0)$-module. The $H_n(0)$-module $\boldsymbol{S}^\sigma_\alpha$ decomposes as

$$\boldsymbol{S}^\sigma_\alpha = \bigoplus_{E \in \mathcal{E}^\sigma(\alpha)} \boldsymbol{S}^\sigma_{\alpha,E}.$$

(3) Each $E \in \mathcal{E}^\sigma(\alpha)$ can be endowed with the partial order $\preceq$. The resulting poset $E = (E, \preceq)$ has a smallest element $T_{0,E}$ and a greatest element $T_{1,E}$ which are characterized in E by $D^c(T_{0,E}) = ND^c(T_{0,E})$ and $D(T_{1,E}) = AD(T_{1,E})$ and called *source* and *sink tableau* of E, respectively.

(4) To each SPCT T of size n we can associate the column word col_T which can be regarded as an element of $\mathfrak{S}_n$. For $E \in \mathcal{E}^\sigma(\alpha)$ the poset E is isomorphic to the left weak order interval $[\mathrm{col}_{T_{0,E}}, \mathrm{col}_{T_{1,E}}]_L$ via the map $T \mapsto \mathrm{col}_T$.

Most of the proofs for the results of Section 3.1 (including those cited from [TvW15]) can directly be applied on SPCTx. There are two exceptions also mentioned in [TvW19]. First, a basic result on the operation of the π_i on SCTx [TvW15, Lemma 3.7] has to be substituted by [TvW19, Lemma 3.2]. Second, the proof of the uniqueness of the source and the sink tableau has to be altered as described in [TvW19, Remark 3.8].

A 0-Hecke action on subdiagrams

In Section 3.2 we considered a 0-Hecke action on chains of the composition poset $\mathcal{L}_w$ that lead to a characterization of $\mathrm{cont}(\mathrm{col}_{T_2} \mathrm{col}_{T_1}^{-1})$ for two SCTx $T_1 \preceq T_2$ in Proposition 3.2.9. This result was essential for our results on the indecomposability, the top and the socle of the modules $\boldsymbol{S}_{\alpha,E}$ from Theorem 3.3.11, Corollary 3.5.46 and Corollary 3.6.41, respectively. As said before, for the SPCTx we do not have a correspondence to chains of a poset of compositions and we do not intend to give one. Nevertheless, Proposition 3.2.9 was proven by considering $\mathrm{osh}(T^{>m})$, the outer shape of the tableau corresponding to the enties $> m$ of the SCT T. The connection to chains in $\mathcal{L}_w$ was provided by Lemma 3.2.3. Since we do not have such chains for SPCTx, we simply use Lemma 3.2.3 as a definition.

Recall that a diagram is a finite set of cells and that a tableau is a filling of a diagram with elements of $\mathbb{N}$.

Definition 3.7.5 (cf. Lemma 2.2.3). *Let $\alpha \vDash n$, $\sigma \in \mathfrak{S}_{\ell(\alpha)}$, $T \in \mathrm{SPCT}^{\sigma}(\alpha)$ and $m \in [0,n]$.*

(*1*) *Define $T^{>m}$ to be the tableau obtained from T by removing the cells containing $1,\dots,m$ and subtracting m from the remaining entries.*

(*2*) *Let $\mathrm{sh}(T^{>m})$ be the diagram formed by the cells occupied by entries of $T^{>m}$.*

As for ordinary compositions, we define the *diagram* of a weak composition $\alpha = (\alpha_1,\dots,\alpha_{\ell(\alpha)}) \vDash_0 n$ as $\{(i,j) \in \mathbb{N}\times\mathbb{N} \mid i \le \ell(\alpha), j \le \alpha_i\}$ and may identify α with its diagram.

Let $\alpha \vDash n$, $m \in [0,n]$ and T be a SPCT of shape α. Because the entries in the rows of T decrease from left to right, the cells of $T^{>m}$ are left aligned. That is, $\mathrm{sh}(T^{>m})$ is the diagram of a weak composition α^m of $n-m$. However, in general $\mathrm{sh}(T^{>m})$ is not a composition and therefore $\mathrm{sh}(T^{>m})$ is not an SPCT as can be seen in the following example.

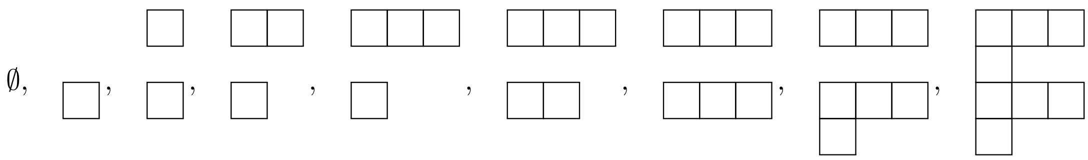
Example 3.7.6. Consider the SPCT

$$T_0 = \begin{array}{|c|c|c|}\hline 7 & 6 & 5 \\ \hline 1 \\ \cline{1-1} 8 & 4 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}$$

from Example 3.7.2. Then

$$T_0^{>3} = \begin{array}{|c|c|c|}\hline 4 & 3 & 2 \\ \hline \multicolumn{3}{c}{} \\ \hline 5 & 1 \\ \cline{1-2} \end{array}$$

is a tableau of shape $(3,0,2)$ and therefore not an SPCT. The sequence of diagrams $\mathrm{sh}(T_0^{>m})$ for $m = 8,7,\dots,0$ associated to T_0 is shown below.



In Definition 3.2.4 we introduced $|\alpha|_j = |\{i \in [l] \mid \alpha_i \ge j\}|$ for $\alpha \vDash n$ and $j \ge 1$ and the preorder $\trianglelefteq$ on the set compositions of size n. These definitions directly generalize to weak compositions. Moreover, $|\alpha|_j$ still is the number of cells in column j of the diagram of the weak composition α.

With these generalized notions, the proof of Proposition 3.2.9 goes trough for SPCTx as well.

Proposition 3.7.7 (cf. Proposition 3.2.9). *Let $\alpha \vDash n$, $\sigma \in \mathfrak{S}_{\ell(\alpha)}$, $i \in [n-1], E \in \mathcal{E}^\sigma(\alpha)$ and $T_1, T_2 \in E$ be such that $T_1 \preceq T_2$. Then*

$$i \in \operatorname{cont}(\operatorname{col}_{T_2} \operatorname{col}_{T_1}^{-1}) \text{ if and only if } \operatorname{sh}(T_2^{>i}) \neq \operatorname{sh}(T_1^{>i}).$$

The decomposition of permuted modules

Let $\alpha \vDash n$ and $\sigma \in \mathfrak{S}_{\ell(\alpha)}$. In Theorem 3.3.11 of Section 3.3 we proved that the module $\boldsymbol{S}_{\alpha,E}$ is indecomposable for all $E \in \mathcal{E}(\alpha)$. Choi, Kim, Nam and Oh show in [CKNO21] how the argumentation can be adapted in order to make it work for $\boldsymbol{S}^\sigma_{\alpha,E}$ with $E \in \mathcal{E}^\sigma(\alpha)$ as well.

Here we present a slightly different approach using the generalized version of Proposition 3.2.9, Proposition 3.7.7. Choi et al. do not generalize Proposition 3.2.9 in their approach.

The argumentation of Section 3.3 leading to Theorem 3.3.11 has to be adapted on two occasions in order to make it work for $\boldsymbol{S}^\sigma_{\alpha,E}$ with $E \in \mathcal{E}^\sigma(\alpha)$ as well. First, the proof of Lemma 3.3.3 exploits the fact that all elements of $\mathrm{SCT}(\alpha)$ have the entry n at position $(\ell(\alpha), 1)$. This however can be generalized to the result that all elements of $\mathrm{SPCT}^\sigma(\alpha)$ have the entry n at position $(\sigma^{-1}(\ell(\alpha)), 1)$ which can then be applied instead. The result is a direct consequence of the ordering conditions of SPCTx. This slight alteration is not mentioned in [CKNO21]. Second, Proposition 3.7.7 has to be used instead of Proposition 3.2.9 in Lemma 3.3.7.

Therefore, we have the following.

Theorem 3.7.8 ([CKNO21, Theorem 4.11]). *Let $\alpha \vDash n$, $\sigma \in \mathfrak{S}_{\ell(\alpha)}$ and $E \in \mathcal{E}^\sigma(\alpha)$. Then $\operatorname{End}_{H_n(0)}(\boldsymbol{S}^\sigma_{\alpha,E}) = \mathbb{K}\operatorname{id}$. In particular, $\boldsymbol{S}^\sigma_{\alpha,E}$ is an indecomposable $H_n(0)$-module.*

Corollary 3.7.9 (cf. Corollary 3.3.12). *Let $\alpha \vDash n$ and $\sigma \in \mathfrak{S}_{\ell(\alpha)}$. Then*

$$\boldsymbol{S}^\sigma_\alpha = \bigoplus_{E \in \mathcal{E}^\sigma(\alpha)} \boldsymbol{S}^\sigma_{\alpha,E}$$

is a decomposition into indecomposable submodules.

Top and socle

In Theorem 3.5.42 of Section 3.5 we gave a combinatorial formula for the top of $\boldsymbol{S}_{\alpha/\!\!/\beta,E}$ for each skew composition $\alpha/\!\!/\beta$ and $E \in \mathcal{E}(\alpha/\!\!/\beta)$. From this we obtained a formula for $\operatorname{top}(\boldsymbol{S}_{\alpha/\!\!/\beta})$ in Corollary 3.5.46. All results of the section hold for the modules $\boldsymbol{S}^\sigma_{\alpha,E}$ as well. There is one minor exception. In the preface of Example 3.5.2 it is noted that the horizontal strip sequence $B_{k,l}$ can be realized as skew composition. In the case of standard permuted composition tableaux, this can be wrong. However, this note is not important for the further argumentation. The only necessary adjustment in the proofs from Section 3.5 is to replace Proposition 3.2.9 by Proposition 3.7.7 again. This has to be done in Lemma 3.5.18. In particular, Theorem 3.5.42 generalizes to SPCTx.

Theorem 3.7.10 (cf. Theorem 3.5.42)**.** *Let $\alpha \vDash n$, $\sigma \in \mathfrak{S}_{\ell(\alpha)}$ and $E \in \mathcal{E}^\sigma(\alpha)$. Then*

$$\operatorname{top}(\boldsymbol{S}^\sigma_{\alpha,E}) \cong \bigoplus_{D \in \mathcal{OD}} \boldsymbol{F}_D$$

as $H_n(0)$-modules.

Example 3.7.11. Let

$$T_0 = \begin{array}{|c|c|c|}\hline 7 & 6 & 5 \\ \hline 1 & \multicolumn{2}{c}{} \\ \hline 8 & 4 & 3 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$$

be the SPCT from Example 3.7.2 and α, σ and E be its shape, type and equivalence class, respectively. Recall $D(T_0) = \{1, 2, 4, 7\}$. Since $D^c(T_0) = ND^c(T_0)$, T_0 is the source tableau of E. The cells are shaded according to the set partition of the diagram of α given by the descents of T_0 (cf. Notation 3.5.1). We have $OD(T_0) = \{1, 4, 7\}$. Therefore, Theorem 3.7.10 yields

$$\operatorname{top}(\boldsymbol{S}^\sigma_{\alpha,E}) \cong \boldsymbol{F}_{\{1,4,7\}} \oplus \boldsymbol{F}_{\{1,2,4,7\}}.$$

as $H_8(0)$-modules.

The formula for the socle of $\boldsymbol{S}_{\alpha /\!\!/ \beta,E}$ from Corollary 3.6.41 also generalizes to $\boldsymbol{S}^\sigma_{\alpha,E}$. In fact, we have again that all results of Section 3.6 hold to the permuted case as well. The necessary alterations are the following. First, in order to obtain Lemma 3.6.13 on has to use Proposition 3.2.9 instead of Proposition 3.7.7. The same is true for Lemma 3.6.37. Second, one has to cite [TvW19, Lemma 3.2] instead of [TvW15, Lemma 3.7] in order to justify Lemma 3.6.22. Therefore we have the following.

Theorem 3.7.12 (cf. Corollary 3.6.41)**.** *Let $\alpha \vDash n$, $\sigma \in \mathfrak{S}_{\ell(\alpha)}$ and $E \in \mathcal{E}^\sigma(\alpha)$. Then*

$$\operatorname{soc}(\boldsymbol{S}^\sigma_{\alpha,E}) \cong \bigoplus_{A \in \mathcal{FD}^c} \boldsymbol{F}_{A^c}$$

as $H_n(0)$-modules.

Example 3.7.13. Consider the SPCT of shape $\alpha = (3, 2, 2)$ and type $\sigma = 213$

$$T_1 = \begin{array}{|c|c|c|}\hline 6 & 5 & 1 \\ \hline 3 & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} 7 & 4 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}\,.$$

It is the sink tableau of its equivalence class E. We have $D^c(T_1) = \{1, 2, 5\}$. The cells above are shaded according to the set partition of the diagram of α given by the ascents of T_1 (cf. Notation 3.6.1). Observe $FD^c(T_1) = \{2, 5\}$. Hence, Theorem 3.7.12 yields

$$\mathrm{soc}(\boldsymbol{S}^{\sigma}_{\alpha,E}) \cong \boldsymbol{F}_{\{3,4,6\}} \oplus \boldsymbol{F}_{\{1,3,4,6\}}$$

as $H_7(0)$-modules.

4 Centers and cocenters of 0-Hecke algebras

Let W be a finite Coxeter group with Coxeter generators S. Fayers asks in [Fay05] for the center $Z(H_W(0))$ of the 0-Hecke algebra $H_W(0)$ of W. Brichard gives a formula for the dimension of the center in type A [Bri08]. In [He15] He describes a basis of $Z(H_W(0))$ in arbitrary type indexed by certain equivalence classes of W. These classes are rather subtle. In fact, [He15] contains no result on the number of these classes which is the dimension of the center. Yang and Li give a lower bound for the dimension of $Z(H_W(0))$ for irreducible W in several types other than A [YL15]. Moreover, they specify the dimension in type $I_2(n)$ for $n \geq 5$. This thesis is mainly concerned with the center of the 0-Hecke algebra $H_n(0)$ of the symmetric group $\mathfrak{S}_n$ and deals with the approaches of Brichard and He.

Let ℓ be the length function of W and define $W_{\min}$ and $W_{\max}$ to be the set of elements of W whose length is minimal and maximal in their conjugacy class, respectively. Geck and Pfeiffer introduce in [GP93] a relation $\to$ on W known as *cyclic shift relation.* It is the reflexive and transitive closure of the relations $\xrightarrow{s}$ for $s \in S$ where we have $w \xrightarrow{s} w'$ if $w' = sws$ and $\ell(w') \leq \ell(w)$.

In the case where W is a Weyl group, Geck and Pfeiffer show that $W_{\min}$ in conjunction with $\to$ has remarkable properties and how these properties can be used in order to define a character table for Hecke algebras of W with invertible parameters [GP93]. Since then their results have been generalized to finite [GHL+96], affine [HN14] and finally to all Coxeter groups [Mar21]. The relation $\to$ can also be used to describe the conjugacy classes of Coxeter groups [GP00, Mar20] in particular for computational purposes [GHL+96, GP00]. Geck, Kim and Pfeiffer introduce a twisted version $\to_\delta$ of the relation belonging to twisted conjugacy classes of W in [GKP00].

By setting $w \approx w'$ if and only if $w \to w'$ and $w' \to w$ one obtains an equivalence relation $\approx$ on W. The $\approx$-equivalence classes of W are known as cyclic shift classes. For an element Σ of the quotient set $W_{\max}/_{\approx}$, He defines the element $\bar{\pi}_{\leq \Sigma} := \sum_x \bar{\pi}_x$ where x runs over all the elements of the order ideal in Bruhat order of W generated by Σ [He15]. Then he shows that the elements $\bar{\pi}_{\leq \Sigma}$ for $\Sigma \in W_{\max}/_{\approx}$ form a basis of $Z(H_W(0))$. Moreover, he defines a basis of the cocenter of $H_W(0)$ indexed by $W_{\min}/_{\approx}$. We review the approach of He together with further preliminary results in Section 4.1.

Motivated by the above connection to $Z(H_n(0))$, the main subject of this chapter is the study of $W_{\max}/_{\approx}$ in the case where W is the symmetric group $\mathfrak{S}_n$. To be precise, we determine its cardinality, obtain sets of representatives for $(\mathfrak{S}_n)_{\max}/_{\approx}$ and develop a combinatorial description for certain elements of $(\mathfrak{S}_n)_{\max}/_{\approx}$.

Cardinalities, parametrizations and sets of representatives are the subject of Section 4.2. Brichard provides the dimension of $Z(H_n(0))$ and thus the cardinality of ${(\mathfrak{S}_n)_{\max}}/{\approx}$ [Bri08]. Her argumentation is based on a calculus on altered braid diagrams on the Möbius strip which we call *crossing diagrams.* Brichard shows that there is a basis of $Z(H_n(0))$ indexed by certain crossing diagrams. By counting the diagrams, she obtains the dimension of $Z(H_n(0))$. In contrast to that of He, the expansion of Brichards basis elements into a basis of $H_n(0)$ is involved and no explicit description is given in [Bri08]. As in [Kim98] we call a composition of n $\alpha \vDash n$ *maximal* and write $\alpha \vDash_e n$ if there is a $k \geq 0$ such that the first k parts of α are even and the remaining parts are odd and weakly decreasing. In Proposition 4.2.10 we give a system of representatives for ${(\mathfrak{S}_n)_{\max}}/{\approx}$ which corresponds to Brichards diagrams and is indexed by the maximal compositions of n.

We obtain another set of representatives for ${W_{\max}}/{\approx}$ from the work of Geck, Kim and Pfeiffer [GKP00]. For each composition $\alpha \vDash n$, Kim defines the *element in stair form* $\sigma_\alpha \in \mathfrak{S}_n$ in [Kim98]. Geck, Kim and Pfeiffer show that $\sigma_\alpha \in (\mathfrak{S}_n)_{\max}$ if and only if α is a maximal composition [GKP00]. We show in that the elements in stair form σ_α for $\alpha \vDash_e n$ form a system of representatives of ${(\mathfrak{S}_n)_{\max}}/{\approx}$ in Proposition 4.2.14.

Besides ${(\mathfrak{S}_n)_{\max}}/{\approx}$ we briefly consider other quotient sets. On ${W}/{\approx}$ the relation $\to$ gives rise to a partial order. Gill considers the corresponding subposets ${\mathcal{O}}/{\approx}$ where $\mathcal{O}$ is a conjugacy class of W and determines the cardinality of ${\mathcal{O}_{\min}}/{\approx}$ in types A, B and D [Gil00]. We infer a parametrization of ${W_{\min}}/{\approx}$ based on compositions, the cardinality of ${W_{\min}}/{\approx}$ and the dimension of the cocenter of $H_W(0)$ in these types. In types B_n and D_{2n} we transfer the results to ${W_{\max}}/{\approx}$. This allows to determine the dimension of $Z(H_W(0))$ in types B_n and D_{2n}. Moreover, we describe a system of representatives of ${(\mathfrak{S}_n)_{\min}}/{\approx}$ given by Coxeter elements. This is the content of Subsections 4.2.3 and 4.2.4.

In Section 4.3 we strife for a description of the elements in Σ for $\Sigma \in {(\mathfrak{S}_n)_{\max}}/{\approx}$. Via the elements in stair form, these equivalence classes can be indexed by maximal compositions. For $\alpha \vDash_e n$ let $\Sigma_\alpha \in {(\mathfrak{S}_n)_{\max}}/{\approx}$ denote the equivalence class of the element in stair form σ_α under $\approx$. Then the elements $\bar{\pi}_{\leq \Sigma_\alpha}$ for $\alpha \vDash_e n$ form a basis of $Z(H_n(0))$. Since $\bar{\pi}_{\leq \Sigma_\alpha}$ is the sum over all $\bar{\pi}_x$ where x is an element of the order ideal generated by Σ_α, a description of the elements of Σ_α is desirable.

The main results of Section 4.3 are combinatorial characterizations of the equivalence classes $\Sigma_{(n)}$ (Theorem 4.3.20) and $\Sigma_{(k,1^{n-k})}$ with k odd (Theorem 4.3.40) and a decomposition rule $\Sigma_{(\alpha_1,\ldots,\alpha_l)} = \Sigma_{(\alpha_1)} \odot \Sigma_{(\alpha_2,\ldots,\alpha_l)}$ if α_1 is even given by an injective operator $\odot$ which we call the *inductive product* (Corollary 4.3.56). This allows us to describe Σ_α for all $\alpha \vDash_e n$ whose odd parts form a hook. Moreover, we will see how these Σ_α can be computed recursively. The results of Section 4.3 will be applied in Chapter 5 whose topic is the action of the elements $\bar{\pi}_{\leq \Sigma_\alpha}$ on the simple $H_n(0)$-modules.

4.1 Centers and cocenters with a twist

Throughout the section let W be a finite Coxeter group with Coxeter generators S and δ be a W-automorphism that fixes S. We also use the shorthand $H := H_W(0)$ for the 0-Hecke algebra of W. The purpose of this section is to introduce the center and the cocenter of H twisted by δ and $\mathbb{K}$-bases of them which are due to He [He15]. These bases are indexed by certain equivalence classes of W under an equivalence relation $\approx_\delta$ depending on δ. At the end of the section we consider a way of parametrizing these index sets which also goes back to [He15]. We are particularly interested in the center of $H_n(0)$ which results from setting $W = \mathfrak{S}_n$ and $\delta = \mathrm{id}$. In Section 4.2 we will discuss further and more explicit parametrizations in types A, B and D.

The following exposition is mainly based on [He15]. We begin with clarifying which choices for δ are possible. Of course, $\delta = \mathrm{id}$ is a valid choice. Another example is given by the conjugation with w_0. Recall that w_0 denotes the longest element of W. For $u, w \in W$ we use the shorthand $w^u = uwu^{-1}$. Define $\nu\colon W \to W$, $w \mapsto w^{w_0}$. Then ν is a group automorphism and by Proposition 2.2.6 it is also an automorphism of the Bruhat order. Consequently, $\ell(\nu(w)) = \ell(w)$ for all $w \in W$ so that $\nu(S) = S$. Hence, ν is another possibility for δ. In general, each graph automorphism of the Coxeter graph of W gives rise to a W-automorphism that fixes S. By the next lemma, the converse direction is also true. The result is not new. For instance, it was already used implicitly in [GKP00, Section 2.10].

Lemma 4.1.1. *Let δ be a group automorphism of W with $\delta(S) = S$.*

(1) δ is an automorphism of the Coxeter graph of W.

(2) δ is an automorphism of the Bruhat order of W.

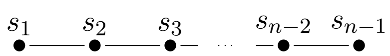
Proof. For $w \in W$ denote the order of w with $\mathrm{ord}(w)$. Let m be the Coxeter matrix and Γ be the Coxeter graph of W. Then $m(s, s') = \mathrm{ord}(ss')$ for all $s, s' \in S$. Since δ is a group automorphism, we have $\mathrm{ord}(\delta(w)) = \mathrm{ord}(w)$ for all $w \in W$. Hence for all $s, s' \in S$

$$m(\delta(s), \delta(s')) = \mathrm{ord}(\delta(s)\delta(s')) = \mathrm{ord}(ss') = m(s, s'). \tag{4.1}$$

Consequently, $\{s, s'\}$ is an edge of Γ (labeled with $m(s, s')$) if and only if $\{\delta(s), \delta(s')\}$ is an edge of Γ (labeled with $m(s, s')$). That is, δ is an automorphism of Γ.

By a comment following [BB05, Proposition 2.3.4], from each graph automorphism φ of Γ we obtain a automorphism in Bruhat order by extending multiplicatively to W. The reason for this is that φ only relabels the generators of W leaving the Coxeter relations intact. Thus, δ is also an automorphism of the Bruhat order of W. ☐

Remark 4.1.2. We determine all automorphism δ of $\mathfrak{S}_n$ with $\delta(S) = S$. The Coxeter graph of $\mathfrak{S}_n$ is shown below.



This graph has at most two automorphisms: the identity and the mapping given by $s_i \mapsto s_{n-i}$. For $n \geq 3$ these maps are distinct. Let w_0 be the longest element of $\mathfrak{S}_n$. Then $w_0(j) = n - j + 1$ for all $j \in [n]$ and therefore $s_i^{w_0} = (n - i + 1, n - i) = s_{n-i}$. Hence the second map is ν. Now Lemma 4.1.1 and the fact that $\nu(S) = S$ imply that $\delta \in \{\mathrm{id}, \nu\}$ if $W = \mathfrak{S}_n$.

Since δ is an automorphism of the Bruhat order by Lemma 4.1.1, it follows that we obtain an algebra automorphism of H by setting $\pi_s \mapsto \pi_{\delta(s)}$ for all $s \in S$ and extending multiplicatively and linearly. This algebra automorphism is also denoted by δ. Note that we have $\delta(\bar{\pi}_s) = \bar{\pi}_{\delta(s)}$ for all $s \in S$ as well.

For $a, b \in H$ define $[a, b]_\delta := ab - b\delta(a)$ the *δ-commutator* of a and b. The *δ-commutator* of H is the $\mathbb{K}$-linear subspace $[H, H]_\delta$ spanned by all δ-commutators of elements of H. We define the *δ-cocenter* of H as the quotient of $\mathbb{K}$-vector spaces $\overline{H}_\delta := {}^{H}\!/_{[H,H]_\delta}$. The *$\delta$-center* of H is given by

$$Z(H)_\delta := \{z \in H \mid az = z\delta(a) \text{ for all } a \in H\}.$$

In the case $\delta = \mathrm{id}$ we may omit the index δ.

Let $\delta' := \nu \circ \delta$. Our next goal is to prove a correspondence between $Z(H)_\delta$ and the dual of $\overline{H}_{(\delta')^{-1}}$ which is quite natural in terms of Frobenius algebras. Afterwards, we continue with He's construction of bases of $\overline{H}_\delta$ and $Z(H)_\delta$. Note that the correspondence is not necessary for the construction of the bases.

We first review some basics of Frobenius algebras and then identify H as an algebra of this kind. Details on Frobenius algebras can be found in textbooks such as [CR62, Lam99]. In [DHT02] Duchamp, Hivert and Thibon use the Frobenius algebra structure of $H_n(0)$ in order to define a comultiplication on $H_n(0)$.

Let A be a finite dimensional $\mathbb{K}$-algebra. We write $A^* := \mathrm{Hom}_{\mathbb{K}}(A, \mathbb{K})$ for its dual space. Then A^* becomes a left A-module by setting $(af)(b) = f(ba)$ for all $f \in A^*$ and $a, b \in A$. We call A *Frobenius algebra* if there is a $\mathbb{K}$-linear map $\chi\colon A \to \mathbb{K}$ such that $\chi(J) \neq 0$ for each left or right ideal $J \neq 0$ of A. If A is a Frobenius algebra then the map $A \to A^*$, $a \mapsto a\chi$ is an isomorphism of A-modules. In other words, χ is an A-basis of A^*.

Let's get back to H. In [Fay05, Proposition 4.1] it is shown that the map $\chi\colon H \to \mathbb{K}$ given by

$$\pi_w \mapsto \begin{cases} 1 & \text{if } w = w_0 \\ 0 & \text{if } w \neq w_0 \end{cases}$$

for $w \in W$ and linear extension makes H a Frobenius algebra. We remark that from Lemma 2.3.2 it follows that $\chi(\bar{\pi}_w) = \chi(\pi_w)$ for all $w \in W$. Proposition 4.2 of [Fay05] yields that

$$\chi(ab) = \chi(\nu(b)a)$$

for all $a, b \in H$. In general, if A together with χ is a Frobenius algebra then there exists a unique algebra automorphism of A satisfying the above equation called the *Nakayama automorphism* of A. In the case of H, we additionally have $\nu^2 = \mathrm{id}$ so that

$$\chi(b\nu(a)) = \chi(\nu^2(a)b) = \chi(ab)$$

as well.

In [Bri08, Claim 1.2] Brichard relates $Z(H)_\delta$ with $\left(\overline{H}_{(\delta')^{-1}}\right)^*$ by a $\mathbb{K}$-isomorphism in the case $W = \mathfrak{S}_n$. Since there are some flaws in the proof we give a proof of an even more general result.

Theorem 4.1.3. *The $\mathbb{K}$-vector spaces $Z(H)_\delta$ and $\left(\overline{H}_{(\delta')^{-1}}\right)^*$ are isomorphic via the map*

$$\Psi\colon Z(H)_\delta \to \left(\overline{H}_{(\delta')^{-1}}\right)^*, \quad z \mapsto \overline{z\chi}$$

where $\pi\colon H \to \overline{H}_{(\delta')^{-1}}$ is the canonical projection and $\overline{z\chi}\colon \overline{H}_{(\delta')^{-1}} \to \mathbb{K}$ is the unique $\mathbb{K}$-linear map satisfying $z\chi = \overline{z\chi} \circ \pi$.

Proof. Let $\delta^\diamond = (\delta')^{-1}$. Then for $z \in Z(H)_\delta$, $\overline{z\chi}$ is given by setting

$$\overline{z\chi}(a + [H,H]_{\delta^\diamond}) = z\chi(a)$$

for all $a \in H$. We have to show that Ψ is well defined, $\mathbb{K}$-linear and bijective. The linearity should be clear. For all $a, b, c \in H$ we have

$$\chi(b[a,c]_\delta) = -c\chi([\delta'(a), b]_{\delta^\diamond}) \tag{4.2}$$

because

$$\begin{aligned} -c\chi\left([\delta'(a), b]_{\delta^\diamond}\right) &= -c\chi(\delta'(a)b - b\delta^\diamond(\delta'(a))) \\ &= -\chi(\delta'(a)bc - bac) \\ &= \chi(bac) - \chi(\delta'(a)bc) \\ &= \chi(bac) - \chi(bc\nu(\delta'(a))) \\ &= \chi(b(ac - c\delta(a)) \\ &= \chi(b[a,c]_\delta) \end{aligned}$$

where we use that $\chi(xy) = \chi(y\nu(x))$ for the forth and $\delta' = \nu \circ \delta$ for the fifth equality. For all $c \in H$ we have

$$\begin{aligned} c \in Z(H)_\delta &\iff [a,c]_\delta = 0 \quad \forall a \in H \\ &\iff \chi(b[a,c]_\delta) = 0 \quad \forall a,b \in H \\ &\iff c\chi([a,b]_{\delta^\diamond}) = 0 \quad \forall a,b \in H \\ &\iff [H,H]_{\delta^\diamond} \subseteq \ker(c\chi) \end{aligned} \tag{4.3}$$

where the second equivalence holds because $\chi(J) \neq 0$ for each left ideal $J \neq 0$ of H

and the third equivalence is a consequence of (4.2) and the fact that $-\delta'$ is an H-automorphism.

Given $z \in Z(H)_\delta$, Equation (4.3) implies that $z\chi$ factors through $[H,H]_{\delta^\diamond}$. Thus, there exists a unique $\overline{z\chi}$ as claimed and Ψ is well defined.

Now we show that Ψ is surjective. Let $\pi^*\colon (\overline{H}_{\delta^\diamond})^* \to H^*$, $f \mapsto f \circ \pi$ be the dual map of the canonical projection. Consider $f \in (\overline{H}_{\delta^\diamond})^*$. Since χ is an H-basis of H^*, there is a $c \in H$ such that $c\chi = \pi^*(f) = f \circ \pi$. Then

$$[H,H]_{\delta^\diamond} = \ker(\pi) \subseteq \ker(c\chi)$$

and from (4.3) it follows that $c \in Z(H)_\delta$. Moreover $\overline{c\chi} \circ \pi = c\chi = f \circ \pi$. Hence, $\overline{c\chi} = f$ by the uniqueness of $\overline{c\chi}$. Consequently, Ψ is surjective.

Lastly, we show that Ψ is injective. Let $z \in Z(H)_\delta$ such that $\overline{z\chi} = 0$. Then $z\chi = 0$ since $z\chi = \overline{z\chi} \circ \pi$. Moreover, χ is an H-basis of H^*. Therefore $z = 0$. $\square$

If $\delta \in \{\mathrm{id}, \nu\}$ then $\delta^{-1} = \delta$ and we obtain the following result from Theorem 4.1.3. For $W = \mathfrak{S}_n$ this is [Bri08, Claim 1.2].

Corollary 4.1.4. *If $\delta \in \{\mathrm{id}, \nu\}$ then*

$$Z(H) \cong \left(\overline{H}_\nu\right)^* \quad \text{and} \quad Z(H)_\nu \cong \left(\overline{H}\right)^*$$

as $\mathbb{K}$-vector spaces.

If $W = \mathfrak{S}_n$ then by Remark 4.1.2 we have $\delta \in \{\mathrm{id}, \nu\}$. Hence, Corollary 4.1.4 covers all possibilities for δ in this case.

We need some more notions from [He15] in order to introduce bases for $\overline{H}_\delta$ and $Z(H)_\delta$. Two elements $w, w' \in W$ are called *δ-conjugate* if there is an $x \in W$ such that $w' = xw\delta(x)^{-1}$. The set of δ-conjugacy classes of W is denoted by $\mathrm{cl}(W)_\delta$.

Example 4.1.5. The ν-conjugacy classes of $\mathfrak{S}_3$ are

$$\{1, (1,2,3), (1,3,2)\}, \ \{(1,2), (2,3)\} \ \text{and} \ \{(1,3)\}.$$

For $\mathcal{O} \in \mathrm{cl}(W)_\delta$ the set of elements of minimal length in $\mathcal{O}$ and the set of elements of maximal length in $\mathcal{O}$ is denoted by $\mathcal{O}_{\min}$ and $\mathcal{O}_{\max}$, respectively. We want to decompose these sets using an equivalence relation.

Let $w, w' \in W$. For $s \in S$ we write $w \xrightarrow{s}_\delta w'$ if $w' = sw\delta(s)$ and $\ell(w') \leq \ell(w)$. We write $w \to_\delta w'$ if there is a sequence $w = w_1, w_2, \ldots, w_{k+1} = w'$ of elements of W such that for each $i \in [k]$ there exists an $s \in S$ such that $w_i \xrightarrow{s}_\delta w_{i+1}$. If $w \to_\delta w'$ and $w' \to_\delta w$ we write $w \approx_\delta w'$.

Clearly, $\approx_\delta$ is an equivalence relation. For $w \in W$ let $[w]_\delta$ denote its equivalence class in W with respect to $\approx_\delta$. If $w \approx_\delta w'$ then $\ell(w) = \ell(w')$. Thus, for all $\mathcal{O} \in \mathrm{cl}(W)_\delta$, $\mathcal{O}_{\min}$ and $\mathcal{O}_{\max}$ decompose in equivalence classes of $\approx_\delta$. Define $W_{\delta,\min} := \bigcup_{\mathcal{O} \in \mathrm{cl}(W)_\delta} \mathcal{O}_{\min}$ and ${W_{\delta,\min}}/{\approx_\delta}$ to be the quotient set of $W_{\delta,\min}$ by $\approx_\delta$. Analogously, define the sets $W_{\delta,\max} := \bigcup_{\mathcal{O} \in \mathrm{cl}(W)_\delta} \mathcal{O}_{\max}$ and ${W_{\delta,\max}}/{\approx_\delta}$. As before, we may omit the index δ if $\delta = \mathrm{id}$.

Example 4.1.6. We consider $\mathfrak{S}_3$.

(1) We have $(1,2) \overset{(1,2)}{\to}_\nu (2,3) \overset{(1,2)}{\to}_\nu (1,2)$ so that $(1,2) \approx_\nu (2,3)$. Thus, from Example 4.1.5 it follows that the elements of ${}^{(\mathfrak{S}_3)_{\nu,\min}}\!/\!_{\approx_\nu}$ are

$$\{1\}\,,\ \{(1,2),(2,3)\} \text{ and } \{(1,3)\}\,.$$

(2) We have $(1,2,3) \overset{(1,2)}{\to} (1,3,2) \overset{(1,2)}{\to} (1,2,3)$ so that $(1,2,3) \approx (1,3,2)$. Hence, the elements of ${}^{(\mathfrak{S}_3)_{\max}}\!/\!_{\approx}$ are

$$\{1\}\,,\ \{(1,2,3),(1,3,2)\} \text{ and } \{(1,3)\}\,.$$

We now come to the bases of $\overline{H}_\delta$ and $Z(H)_\delta$ found by He.

Theorem 4.1.7 ([He15, Theorem 6.5])**.** *Let $w_1,\dots,w_k \in W_{\delta,\min}$ be a system of representatives of ${}^{W_{\delta,\min}}\!/\!_{\approx_\delta}$. The elements $\bar{\pi}_{w_i} + [H,H]_\delta$ for $i = 1,\dots,k$ form a basis of $\overline{H}_\delta$.*

We remark that by [He15, Proposition 3.1] for all $\Sigma \in {}^{W_{\delta,\min}}\!/\!_{\approx_\delta}$ the element $\bar{\pi}_w + [H,H]_\delta$ of $\overline{H}_\delta$ does not depend on the choice of the representative $w \in \Sigma$.

For $\Sigma \in {}^{W_{\delta,\max}}\!/\!_{\approx_\delta}$ set

$$W_{\leq\Sigma} := \{x \in W \mid x \leq w \text{ for some } w \in \Sigma\}$$

and

$$\bar{\pi}_{\leq\Sigma} := \sum_{x \in W_{\leq\Sigma}} \bar{\pi}_x.$$

Theorem 4.1.8 ([He15, Theorem 5.4])**.** *The elements $\bar{\pi}_{\leq\Sigma}$ for $\Sigma \in {}^{W_{\delta,\max}}\!/\!_{\approx_\delta}$ form a basis of $Z(H)_\delta$.*

Example 4.1.9. We consider $H_3(0)$.

(1) From Example 4.1.6 and Theorem 4.1.7 it follows that

$$\{\bar{\pi}_w + [H_3(0),H_3(0)]_\nu \mid w = 1,(1,2),(1,3)\}$$

is a basis of $\overline{H_3(0)}_\nu$.

(2) We use Theorem 4.1.8 in order to determine a basis of $Z(H_3(0))$. In Example 4.1.6 the index set ${}^{(\mathfrak{S}_3)_{\max}}\!/\!_{\approx}$ is given. In addition,

$$(1,2,3) = s_1s_2,\ (1,3,2) = s_2s_1 \text{ and } (1,3) = w_0.$$

Thus, Theorem 4.1.8 yields that the elements

$$1,\ 1+\bar{\pi}_1+\bar{\pi}_2+\bar{\pi}_1\bar{\pi}_2+\bar{\pi}_2\bar{\pi}_1 \text{ and } \sum_{w\in\mathfrak{S}_3}\bar{\pi}_w$$

form a basis of $Z(H_3(0))$.

Remark 4.1.10. Regarding the roles played by ${W_{\delta,\min}}/{\approx_\delta}$ and ${W_{\delta,\max}}/{\approx_\delta}$ in Theorems 4.1.7 and 4.1.8 it is natural to ask for the description of a system of representatives or at least the cardinalities of these sets. This is the subject of Section 4.2. There the question is answered for $W=\mathfrak{S}_n$. Also the cardinalities of the quotient sets are given in type B and in some cases in type D. In [He15] He does not discuss these matters.

We have seen in Proposition 2.2.6 that $w\mapsto ww_0$ is an antiautomorphism of the Bruhat order of W. This map gives rise to a bijection from ${W_{\delta,\min}}/{\approx_\delta}$ to ${W_{\delta',\max}}/{\approx_{\delta'}}$ as follows. This bijection will often be used in Section 4.2.

Lemma 4.1.11. *Let $w,w'\in W$ and $\Sigma\subseteq W$.*

(1) $w\to_\delta w'$ if and only if $w'w_0\to_{\delta'} ww_0$.

(2) $w\in W_{\delta,\min}$ if and only if $ww_0\in W_{\delta',\max}$.

(3) $\Sigma\in {W_{\delta,\min}}/{\approx_\delta}$ if and only if $\Sigma w_0\in {W_{\delta',\max}}/{\approx_{\delta'}}$.

Proof. The proofs of (1) and (2) are slight generalizations of the argumentation at the beginning of [GKP00, Section 2.9]. For Part (3) let $\Sigma\in {W_{\delta,\min}}/{\approx_\delta}$ and $w\in\Sigma$. Then $w\in W_{\delta,\min}$ and by Part (2), $ww_0\in W_{\delta',\max}$. Hence, there is a $\mathrm{T}\in {W_{\delta',\max}}/{\approx_{\delta'}}$ such that $ww_0\in\mathrm{T}$. From Part (1) and the definition of $\approx_\delta$ we infer that for all $w'\in W$

$$w'\approx_\delta w \iff w'w_0\approx_{\delta'} ww_0.$$

In addition, the map from W to W given by right multiplication with w_0 is bijective. Therefore it follows that $\Sigma w_0=\mathrm{T}$. Thus, $\Sigma w_0\in {W_{\delta',\max}}/{\approx_{\delta'}}$. Analogously, we obtain $\mathrm{T}w_0\in {W_{\delta,\min}}/{\approx_\delta}$ if we start with an arbitrary $\mathrm{T}\in {W_{\delta',\max}}/{\approx_{\delta'}}$. $\square$

Combining Theorem 4.1.7, Theorem 4.1.8 and Lemma 4.1.11 we get a result similar to Theorem 4.1.3. Note that this result depends on the $\mathbb{K}$-bases given in the theorems whereas the proof of Theorem 4.1.3 is $\mathbb{K}$-basis-free.

Corollary 4.1.12. *The $\mathbb{K}$-vector spaces $Z(H)_\delta$ and $\overline{H}_{\delta'}$ are isomorphic.*

Elliptic conjugacy classes

We now come to a parametrization of the sets ${W_{\delta,\min}}/{\approx_\delta}$ and ${W_{\delta,\max}}/{\approx_\delta}$ which is due to He. This parametrization is valid for all choices of W and δ. We first state the parametrization in Proposition 4.1.14 and then infer some results that are of use in Section 4.3 and Chapter 5. In Section 4.2 we consider more explicit parametrizations in the cases where W is the symmetric group and make some remarks about the situation in types B and D.

A δ-conjugacy class $\mathcal{O} \in \mathrm{cl}(W)_\delta$ is called *elliptic* if $\mathcal{O} \cap W_I = \emptyset$ for all $I \subsetneq S$ such that $\delta(I) = I$. Elliptic conjugacy classes are also called *cuspidal* in the literature (e.g. in [GP00]).

Define

$$\Gamma_\delta := \{(I, C) \mid I \subseteq S,\ I = \delta(I) \text{ and } C \in \mathrm{cl}(W_I)_\delta \text{ is elliptic}\}.$$

Note Γ_δ has a recursive structure. If all elliptic δ-conjugacy classes of all parabolic subgroups W_I of W with $\delta(I) = I$ are known then the problem of determining Γ_δ reduces to the elliptic δ-conjugacy classes of W itself.

Example 4.1.13. We determine the set Γ_ν associated to $\mathfrak{S}_3$. The simple reflections of $\mathfrak{S}_3$ are $S = \{s_1, s_2\}$. Since $\nu(s_1) = s_2$, the subsets of S that are stable under ν are $\emptyset$ and S. The sole ν-equivalence class of $(\mathfrak{S}_3)_\emptyset$ is $\{1\}$. Trivially, this class is elliptic. The ν-equivalence classes of $(\mathfrak{S}_3)_S = \mathfrak{S}_3$ are given in Example 4.1.5. Observe that $\{1, (1,2,3), (1,3,2)\}$ is the only element of $\mathrm{cl}(\mathfrak{S}_3)_\nu$ that is not elliptic. Hence,

$$\Gamma_\nu = \{(\emptyset, \{1\}), (S, \{(1,2), (2,3)\}), (S, \{(1,3)\})\}.$$

Proposition 4.1.14 ([He15, Corollaries 4.2 and 4.3])**.** *The maps*

$$\begin{aligned} \Gamma_\delta &\to W_{\delta,\min}/{\approx_\delta} \\ (I, C) &\mapsto C_{\min} \end{aligned} \quad \textit{and} \quad \begin{aligned} \Gamma_{\delta'} &\to W_{\delta,\max}/{\approx_\delta} \\ (I, C) &\mapsto C_{\min} w_0 \end{aligned}$$

are bijections.

One may check that by applying Proposition 4.1.14 on Example 4.1.13, we obtain the sets $(\mathfrak{S}_3)_{\nu,\min}/{\approx_\nu}$ and $(\mathfrak{S}_3)_{\max}/{\approx}$ from Example 4.1.6. We continue with consequences of Proposition 4.1.14 which we prepare for later use.

Lemma 4.1.15.

(*1*) *For all* $\Sigma \in W_{\delta,\min}/{\approx_\delta}$ *we have* $\delta(\Sigma) = \Sigma$.

(*2*) *For all* $\Sigma \in W_{\delta,\max}/{\approx_\delta}$ *we have* $\delta'(\Sigma) = \Sigma$.

Proof. (1) Let $\Sigma \in W_{\delta,\min}/{\approx_\delta}$ and $w \in \Sigma$. By Proposition 4.1.14 there exists a tuple $(I, C) \in \Gamma_\delta$ such that $C \in \mathrm{cl}(W_I)_\delta$ and $\Sigma = C_{\min}$. Hence $w \in W_I$ and therefore $w^{-1} \in W_I$. It follows that

$$\delta(w) = w^{-1} w \delta(w^{-1})^{-1} \in C.$$

Moreover, $\ell(\delta(w)) = \ell(w)$ because δ is a Bruhat order automorphism. Therefore, $\delta(w) \in C_{\min} = \Sigma$. Hence, $\delta(\Sigma) = \Sigma$.

(2) Let $\Sigma \in {W_{\delta,\max}}/{\approx_\delta}$. From Lemma 4.1.11 it follows that $\Sigma w_0 \in {W_{\delta',\min}}/{\approx_{\delta'}}$. Hence,

$$\delta'(\Sigma)w_0 = \delta'(\Sigma w_0) = \Sigma w_0$$

where we use that δ' is a group homomorphism with $\delta'(w_0) = w_0$ for the first and Part (1) for the second equality. Now multiply from the right with w_0. $\square$

The following result will be used repeatedly in Section 4.3 and Chapter 5 for $W = \mathfrak{S}_n$. We obtain it by setting $\delta = \mathrm{id}$ in the second part of Lemma 4.1.15.

Corollary 4.1.16. *For all $\Sigma \in {W_{\max}}/{\approx}$ we have $\nu(\Sigma) = \Sigma$.*

A similar result on parabolic subgroups will also be handy in Chapter 5.

Lemma 4.1.17. *For all $I \subseteq S$ we have $\nu(W_I) = W_{\nu(I)}$.*

Proof. Let $I \subseteq S$ and $w \in W$. Recall that ν is an Bruhat order automorphism. In particular, $\nu(I) \subseteq S$. Let $u_1 \cdots u_k$ with $u_j \in S$ be a reduced word for w. Then $\nu(u_1)\cdots\nu(u_k)$ is a reduced word for $\nu(w)$. Thus,

$$\begin{aligned} w \in W_I &\iff u_j \in I \text{ for all } 1 \leq j \leq k \\ &\iff \nu(u_j) \in \nu(I) \text{ for all } 1 \leq j \leq k \iff \nu(w) \in W_{\nu(I)}. \end{aligned}$$ $\square$

4.2 Parametrizations in classical types

Let W be a finite Coxeter group with Coxeter generators S, $H := H_W(0)$ its 0-Hecke algebra and δ be an W-automorphism with $\delta(S) = S$. Theorems 4.1.7 and 4.1.8 introduced bases of the δ-cocenter $\overline{H}_\delta$ and the δ-center $Z(H)_\delta$ that are indexed by ${W_{\delta,\min}}/{\approx_\delta}$ and ${W_{\delta,\max}}/{\approx_\delta}$, respectively. In this section we consider parametrizations of these quotient sets in types A, B and D. We focus on ${(\mathfrak{S}_n)_{\max}}/{\approx}$ since this set indexes the basis of the center of the 0-Hecke algebra of $\mathfrak{S}_n$.

We often use the following correspondence. Recall that $\delta' = \nu \circ \delta$ where ν is the W-automorphism given by conjugating with w_0. From Lemma 4.1.11 it follows that $\Sigma \mapsto \Sigma w_0$ is a bijection from ${W_{\delta,\min}}/{\approx_\delta}$ to ${W_{\delta',\max}}/{\approx_{\delta'}}$. Hence, a parametrization of the one set entails a parametrization of the other.

The section is structured as follows. In Subsection 4.2.1 and Subsection 4.2.2 we consider parametrizations of ${(\mathfrak{S}_n)_{\max}}/{\approx}$. In both cases we obtain a parametrization by certain compositions of n and a set of representatives for ${(\mathfrak{S}_n)_{\max}}/{\approx}$. Subsection 4.2.1 is based on the calculus on *crossing diagrams* done by Brichard [Bri08]. It allows to determine the dimension of $Z(H_n(0))$. In Subsection 4.2.2 we consider certain permutations called *elements in stair form* which were introduced by Kim [Kim98].

In Subsection 4.2.3 we introduce a parametrization of ${(\mathfrak{S}_n)_{\min}}/{\approx}$ given by Coxeter elements which is based on the results of Gill from [Gil00]. Recall from Remark 4.1.2

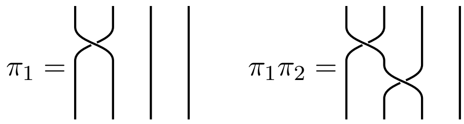


Figure 4.1: Two crossing diagrams on four strands

that we have $\delta \in \{\mathrm{id}, \nu\}$ if $W = \mathfrak{S}_n$. Hence the findings of Subsections 4.2.1 to 4.2.3 cover all possibilities for ${(\mathfrak{S}_n)_{\delta,\min}}/{\approx_\delta}$ and ${(\mathfrak{S}_n)_{\delta,\max}}/{\approx_\delta}$.

Finally, in Subsection 4.2.4 we briefly discuss results concerning ${W_{\min}}/{\approx}$ and ${W_{\max}}/{\approx}$ in types B and D which are also consequences of [Gil00]. We obtain dimension formulas for the cocenter of H in type B_n and D_n and the center of H in type B_n and D_{2n}.

Subsection 4.2.2 is the only part of this section which is significant for the subsequent argumentation in Section 4.3 and Chapter 5. It can be read independently from the other subsections.

4.2.1 Crossing diagrams

In this and the following two subsections we consider the symmetric group $\mathfrak{S}_n$ with its set of simple reflections S and 0-Hecke algebra $H_n(0)$. Our goal is to obtain a parametrization of ${(\mathfrak{S}_n)_{\max}}/{\approx}$ and we use a calculus on topological diagrams due to Brichard [Bri08] to achieve it. We call these diagrams *crossing diagrams*. Each π_w for $w \in \mathfrak{S}_n$ can be represented by a crossing diagram. Brichard uses them in order to obtain a basis of $\overline{H_n(0)}_\nu$ and determine the dimensions of $\overline{H_n(0)}_\nu$ and $Z(H_n(0))$.

Crossing diagrams are similar to the braid diagrams associated with the Artin braid groups. For a textbook treatment of the braid groups and braid diagrams we refer to [KT08]. In the present subsection we first review the findings related to $H_n(0)$ of [Bri08] and then use them to obtain new parametrizations of ${(\mathfrak{S}_n)_{\nu,\min}}/{\approx_\nu}$ and ${(\mathfrak{S}_n)_{\max}}/{\approx}$.

The following definition of crossing diagrams is based on the definition of braid diagrams from [KT08, Section 1.2.2]. Let J be the real interval $[0, 1]$. A *topological interval* is a topological space homeomorphic to J.

Definition 4.2.1. *A* crossing diagram *on n strands $D \subseteq \mathbb{R} \times J$ is the union of n topological intervals called* strands *of D such that the following holds.*

(*1*) *The projection $\mathbb{R} \times J \to J$ maps each strand homeomorphically onto J.*

(*2*) *Every point of $\{1, 2, \dots, n\} \times \{0, 1\}$ is the endpoint of a unique strand of D.*

(*3*) *Every point of $\mathbb{R} \times J$ belongs to at most two strands of D. At each intersection point of two strands, the strands meet transversely.*

The intersection points are also called *crossings*. The difference to the definition of usual braid diagrams in [KT08, Section 1.2.2] is that at a crossing we do not care which

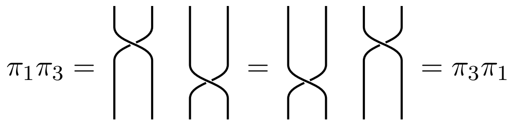


Figure 4.2: Isotopic diagrams

strand is overgoing and which is undergoing. Two diagrams on four strands are shown in Figure 4.1.

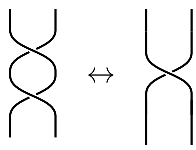
Two crossing diagrams D and D' are called *isotopic* if there is a continuous map $F\colon D \times J \to \mathbb{R} \times J$ such that $F(D \times \{s\})$ is a crossing diagram for each $s \in J$ (see Figure 4.2). It follows that the number of crossings is invariant under isotopy. Note that Brichard uses another notion of isotopy in [Bri08]. Equality of crossing diagrams is considered up to isotopy.

For two diagrams D_1 and D_2 the product D_1D_2 is the diagram obtained by writing D_2 under D_1 and resizing the result to $\mathbb{R} \times J$. For $i = 1, \dots, n-1$ we identify the generator π_i of $H_n(0)$ with the diagram on n strands where exactly strand i and $i+1$ cross. The diagrams on four strands π_1 and $\pi_1\pi_2$ are shown in Figure 4.1. The diagram on n strands without crossings is denoted by 1.

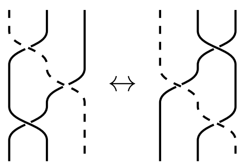
Let D be a crossing diagram on n strands. Then we can use an isotopy to slightly move the crossings of D so that the second coordinates of all crossings are distinct. Then there are $i_1, \dots, i_k \in [n-1]$ such that we can expand D as a product of diagrams $\pi_{i_1} \cdots \pi_{i_k}$. Note that isotopies that preserve the relative order of the second coordinates of the crossings do not change this expansion. Conversely, if a isotopy does changes this order then $\pi_i\pi_j$ is substituted by $\pi_j\pi_i$ for some $i, j \in [n-1]$ such that $|i-j| \geq 2$ within the expansion. Therefore, the expansion of D is unique up to this braid relation.

In order to obtain the other defining relations of $H_n(0)$, we introduce the following two manipulations of sub diagrams of D, which we call *moves*.

(1) Replace two consecutive crossings of the same strands with one crossing or vice versa.



(2) Move a strand completely over or under a crossing.



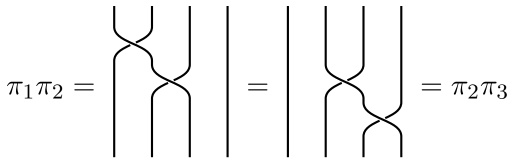


Figure 4.3: Two diagrams on the Möbius strip which are the same. We pushed the first crossing of the left diagram upward in order to move it around the Möbius strip.

Moves 1 and 2 correspond to the relations $\pi_i^2 = \pi_i$ and $\pi_i\pi_{i+1}\pi_i = \pi_{i+1}\pi_i\pi_{i+1}$ of $H_n(0)$, respectively. The second move is known as the *third Reidemeister move.*

We say that two diagrams are *equivalent* if we can transform one diagram into the other by a series of moves (and of course isotopies). That is, the equivalence classes of the crossing diagrams corresponding to π_i satisfy the same relations as the π_i themselves.

Therefore, mapping each $\pi_i \in H_n(0)$ to the equivalence class of its crossing diagram yields a surjective algebra homomorphism from $H_n(0)$ to the $\mathbb{K}$-algebra generated by the equivalence classes of crossing diagrams. This map is also injective: Assume that $\pi_{i_1}\cdots\pi_{i_k}$ and $\pi_{j_1}\cdots\pi_{j_k}$ correspond to equivalent diagrams D and D', respectively. Then we can transform D into D' by a series of isotopies and moves. But since $\pi_{i_1}\cdots\pi_{i_k}$ and $\pi_{j_1}\cdots\pi_{j_k}$ are the expansions of D and D' we can transform $\pi_{i_1}\cdots\pi_{i_k}$ into $\pi_{j_1}\cdots\pi_{j_k}$ by applying the defining relations of $H_n(0)$ corresponding to the isotopies and moves. Hence $\pi_{i_1}\cdots\pi_{i_k} = \pi_{j_1}\cdots\pi_{j_k}$ as elements in $H_n(0)$. Therefore, $H_n(0)$ and the the $\mathbb{K}$-algebra generated by the equivalence classes of crossing diagrams are isomorphic $\mathbb{K}$-algebras.

For each $w \in \mathfrak{S}_n$ we can represent π_w by the crossing diagram $\pi_{i_1}\cdots\pi_{i_k}$ where $s_{i_1}\cdots s_{i_k}$ is a reduced word of w. Since two reduced words of w can be transformed into each other by a series of braid moves, the diagram π_w is unique up to the application of the third Reidemeister move.

From using the isomorphism from above and the fact that the π_w for $w \in \mathfrak{S}_n$ form a basis of $H_n(0)$, it follows that the diagrams π_w for $w \in \mathfrak{S}_n$ form a system of representatives of the equivalence classes of crossing diagrams.

A diagram is called *reduced* if its number of crossings is minimal in its equivalence class. Let $s_{i_1}\cdots s_{i_k}$ be a word in $\mathfrak{S}_n$ and $D := \pi_{i_1}\cdots\pi_{i_k}$ the corresponding diagram. Then $s_{i_1}\cdots s_{i_k}$ is a reduced word if and only if D is reduced. The reason for this is that both statements are equivalent to

$$k = \min\left\{l \mid s_{j_1},\ldots,s_{j_l} \in \mathfrak{S}_n \text{ such that } \pi_{j_1}\cdots\pi_{j_l} = \pi_{i_1}\cdots\pi_{i_k} \text{ in } H_n(0)\right\}.$$

It follows that for all $w \in \mathfrak{S}_n$ the reduced diagrams of the equivalence class of π_w are those whose expansions are given by reduced words of w.

So far, we related $H_n(0)$ to crossing diagrams on n strands in the plane. In order to obtain diagrams that correspond to $\overline{H_n(0)}_\nu$, we consider diagrams on the Möbius

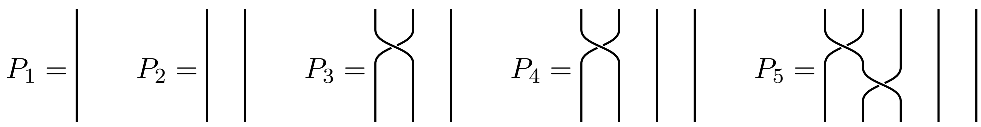


Figure 4.4: Some prime diagrams

strip. That is, we still draw diagrams in the plane but we identify the top position $(i, 1)$ with the bottom position $(n - i + 1, 0)$ for $i = 1, \ldots, n$. Let $\pi_i \pi_w$ with $w \in \mathfrak{S}_n$ be a diagram with topmost crossing between strand i and strand $i + 1$. Then we can use an isotopy to push this crossing upward and move it around the Möbius strip so that we obtain a crossing between strand $n - i + 1$ and $n - i$ below π_w (see Figure 4.3). Hence, $\pi_i \pi_w = \pi_w \pi_{n-i}$, i.e. $[\pi_i, \pi_w]_\nu = 0$.

From this it follows that the equivalence classes of the diagrams on the Möbius strip corresponding to π_w for $w \in \mathfrak{S}_n$ satisfy the same relations as the elements $\pi_w + [H_n(0), H_n(0)]_\nu$ of $\overline{H_n(0)}_\nu$. Thus, the $\mathbb{K}$-vector space formed by these equivalence classes is isomorphic to $\overline{H_n(0)}_\nu$. As a consequence, maximal sets of pairwise nonequivalent reduced diagrams on the Möbius strip correspond to bases of $\overline{H_n(0)}_\nu$.

From now on we understand all diagrams as diagrams on the Möbius strip. Since we have identified top and bottom positions the strands of of an n-strand diagram now form circles around the Möbius strip. Let c be the number of these circles. Then $c \leq n$. A single circle is called *component* of the diagram. The *thickness* of a component is the number of top positions contained in the component. In other words, the thickness is the number of times the circle goes around the Möbius strip. A diagram is called *prime* if it has only one component.

Brichard shows in [Bri08, Section 3.1] for each n that $P_n := \pi_1 \pi_2 \cdots \pi_{\lfloor \frac{n-1}{2} \rfloor}$ is the only reduced prime diagram of thickness n. Some prime diagrams are shown in Figure 4.4. We now add a prime component to a diagram. The following is a reformulation of results from [Bri08, Section 3.2].

Definition 4.2.2. *Let $m \in \mathbb{N}$, $k := \lfloor \frac{m-1}{2} \rfloor$ and $D := \pi_{i_1} \cdots \pi_{i_l}$ be a diagram with n strands. Then we define the* composite diagram *of P_m and D to be the crossing diagram with $m + n$ strands*

$$P_m \circ D := \pi_1 \pi_2 \cdots \pi_k \cdot \pi_{i_1+k+1} \pi_{i_2+k+1} \cdots \pi_{i_l+k+1} \cdot \eta$$

where

$$\eta := \begin{cases} 1 & \text{if } m \text{ is even} \\ \pi_{k+1} \pi_{k+2} \cdots \pi_{k+n} & \text{if } m \text{ is odd.} \end{cases}$$

See Figure 4.5 for examples. Geometrically, we obtain $P_m \circ D$ by splitting P_m vertically (almost if m is odd) in the middle and attach the two parts right and left to D. If $m \geq 2$

$$D_{(1,2)} = P_1 \circ P_2 = 1 \cdot 1 \cdot \pi_1 \pi_2 =$$

$$D_{(3,1)} = P_3 \circ P_1 = \pi_1 \cdot 1 \cdot \pi_2 =$$

$$D_{(4,3,1)} = P_4 \circ D_{(3,1)} = \pi_1 \cdot \pi_3 \pi_4 \cdot 1 =$$

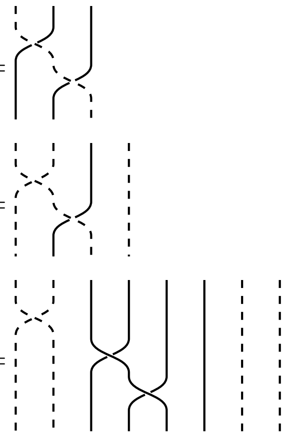


Figure 4.5: The crossing diagrams of some compositions. In each composite diagram $P_m \circ D$ the dashed strands belong to P_m.

then P_m occupies the outer top and bottom positions and D the inner positions in $P_m \circ D$. If m is odd, then P_m has the top positions

$$([k+1] \cup [n+m-k+1, n+m]) \times \{1\}$$

and the bottom positions

$$([k] \cup [n+m-k, n+m]) \times \{0\} .$$

Therefore, the strand of P_m ending in $(k+1, 0)$ corresponds to the strand of $P_m \circ D$ ending in $(m+n-k, 0)$. That is, the latter strand has to cross each strand of D in $P_m \circ D$ which results in the factor η. If m is even then this does not occur since P_m can symmetrically be split in the middle and has no crossing $\pi_{\frac{m}{2}}$.

Definition 4.2.3. *Let $\alpha = (\alpha_1, \dots, \alpha_l) \vDash n$. Inductively, we define*

$$P_{\alpha_1} \circ P_{\alpha_2} \circ \cdots \circ P_{\alpha_l} := P_{\alpha_1} \circ (P_{\alpha_2} \circ \cdots \circ P_{\alpha_l}) .$$

This crossing diagram on the Möbius strip is called the crossing diagram *of α and denoted by D_α.*

Some crossing diagrams of compositions are shown in Figure 4.5. Note that the diagram D_α is not necessarily reduced. For example, $D_{(1,2)} = \pi_1 \pi_2$ is equivalent to π_1 and therefore not reduced (see Figure 4.6). We want to characterize the compositions α for which D_α is reduced.

Definition 4.2.4 ([Kim98])**.** *Let $\alpha = (\alpha_1, \dots, \alpha_l) \vDash n$. We call α* maximal *and write $\alpha \vDash_e n$ if there exists a k with $0 \le k \le l$ such that α_i is even for $i \le k$, α_i is odd for $i > k$ and $\alpha_{k+1} \ge \alpha_{k+2} \ge \cdots \ge \alpha_l$.*

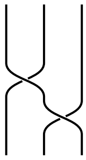
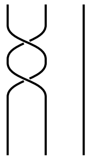
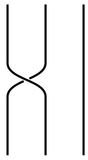
$D_{(1,2)} = \pi_1\pi_2 =$  $=$  $\leftrightarrow$  $= \pi_1$

Figure 4.6: These diagrams on the Möbius strip are equivalent. We obtain the second from the first diagram by pushing the lower crossing around the Möbius strip. Thus, the diagram $D_{(1,2)}$ is not reduced.

We write $\alpha \vDash_e n$ if α is a maximal composition of n to indicate, that only the order of the *even* parts matters. We emphasize that if α is a composition with only odd parts that are weakly decreasing then it is maximal. In other words, each partition λ with only odd parts is a maximal composition. Regarding the compositions from Figure 4.5, we get that $(3,1)$ and $(4,3,1)$ are maximal whereas $(1,2)$ is not.

Section 3.2 of [Bri08] deals with the composition of prime diagrams. Brichard does not use the notions of maximal compositions or crossing diagrams of compositions. However, her results can be rephrased as follows.

Lemma 4.2.5 ([Bri08, Section 3.2])**.**

(*1*) *Let $\alpha \vDash n$. The diagram D_α is reduced if and only if α is maximal.*

(*2*) *Let $\alpha, \beta \vDash_e n$ with D_α equivalent to D_β. Then $\alpha = \beta$.*

(*3*) *Each diagram on n strands is equivalent to D_α for some $\alpha \vDash_e n$.*

Let $\alpha \vDash_e n$ and $D_\alpha = \pi_{i_1} \cdots \pi_{i_k}$. Define $d_\alpha := s_{i_1} \cdots s_{i_k} \in \mathfrak{S}_n$. Since D_α is reduced by Lemma 4.2.5, $s_{i_1} \cdots s_{i_k}$ is a reduced word for d_α and therefore $\pi_{d_\alpha} = D_\alpha$ as diagrams. Note that by using Definitions 4.2.2 and 4.2.3, we can recursively compute d_α.

Lemma 4.2.5 implies that $\{D_\alpha \mid \alpha \vDash_e n\}$ is a system of representatives of the equivalence classes of diagrams with n strands on the Möbius strip. Therefore, we have the following.

Theorem 4.2.6 ([Bri08, Section 5.1])**.** *The elements $\pi_{d_\alpha} + [H_n(0), H_n(0)]_\nu$ for $\alpha \vDash_e n$ form a basis of $\overline{H_n(0)}_\nu$.*

Example 4.2.7. Let $n = 3$. The maximal compositions of 3 are (3), $(2,1)$ and $(1,1,1)$. The corresponding diagrams are

$$\begin{aligned} D_{(3)} &= P_3 = \pi_1, \\ D_{(2,1)} &= P_2 \circ P_1 = 1, \\ D_{(1^3)} &= P_1 \circ (P_1 \circ P_1) = P_1 \circ \pi_1 = \pi_2\pi_1\pi_2. \end{aligned}$$

Hence, $\{\pi_w + [H_3(0), H_3(0)]_\nu \mid w = 1, (1,2), (1,3)\}$ is a basis of $\overline{H_3(0)}_\nu$.

Since the basis of $\overline{H_n(0)}_\nu$ of Theorem 4.2.6 is indexed by maximal compositions, it is easy to determine the dimension of $\overline{H_n(0)}_\nu$. Recall that by Corollary 4.1.4 this is also the dimension of $Z(H_n(0))$.

Corollary 4.2.8 ([Bri08, Section 5.1]). *The dimensions of $Z(H_n(0))$ and $\overline{H_n(0)}_\nu$ both equal*

$$\sum_{\lambda \vdash n} \frac{n_\lambda!}{m_\lambda}$$

where for $\lambda = (1^{k_1}, 2^{k_2}, \dots) \vdash n$, $m_\lambda := \prod_{i \geq 1} k_{2i}!$ and $n_\lambda := \sum_{i \geq 1} k_{2i}$ is the number of even parts of λ.

Proof. Each summand is the number of maximal compositions that rearrange the partition $\lambda \vdash n$. Hence, the sum is the number of maximal compositions of n. By Theorem 4.2.6 this is the dimension of $\overline{H_n(0)}_\nu$. □

From Theorem 4.1.7 we already know a basis of $\overline{H_n(0)}_\nu$. It yields that

$$\{\bar{\pi}_{w_i} + [H_n(0), H_n(0)]_\nu \mid i = 1, \dots, k\}$$

is a basis of $\overline{H_n(0)}_\nu$ where $w_1, \dots, w_k$ is a system of representatives of ${}^{(\mathfrak{S}_n)_{\nu,\min}}\!/\!_{\approx_\nu}$. Alternatively, by Proposition 4.1.14 one can use a system of representatives of

$$\{C_{\min} \mid (I, C) \in \Gamma_\nu\}$$

with the Γ_ν corresponding to $\mathfrak{S}_n$.

In Remark 4.1.10 we raised the question for the cardinality and the description of a system of representatives of ${}^{(\mathfrak{S}_n)_{\nu,\min}}\!/\!_{\approx_\nu}$. Our next aim is to show that with $\{d_\alpha \mid \alpha \vDash_e n\}$ we have found such a system. Its cardinality is given by Corollary 4.2.8. This leads to our desired parametrization and extends the findings of Brichard [Bri08] and He [He15].

Lemma 4.2.9. *Let $\alpha \vDash_e n$. Then $d_\alpha \in (\mathfrak{S}_n)_{\nu,\min}$.*

Proof. Let $\mathcal{O} \in \mathrm{cl}(\mathfrak{S}_n)_\nu$ such that $d_\alpha \in \mathcal{O}$. We have to show that $d_\alpha \in \mathcal{O}_{\min}$, i.e. that $\ell(d_\alpha)$ is minimal in $\mathcal{O}$. Assume that $\ell(d_\alpha)$ is not minimal. Then by [He15, Theorem 2.2] there are $w \in \mathcal{O}$ and $s_i \in S$ such that $d_\alpha \xrightarrow{s_i}_\nu w$ and $\ell(w) < \ell(d_\alpha)$. Thus, $d_\alpha = s_i w s_{n-i}$ and $\ell(w) = \ell(d_\alpha) - 2$. Let $s_{j_1} \cdots s_{j_r}$ be a reduced word for w. Then in $\overline{H_n(0)}_\nu$ we have

$$\pi_{d_\alpha} = \pi_i \pi_{j_1} \cdots \pi_{j_r} \pi_{n-i} = \pi_i \pi_i \pi_{j_1} \cdots \pi_{j_r}.$$

That is, the diagram of D_α contains two consecutive crossings of the same strands. Thus we can apply Move 1 and obtain a diagram equivalent to D_α with one crossing less then D_α. But this is a contradiction since D_α is reduced by Lemma 4.2.5. □

We now come to the parametrization of ${}^{(\mathfrak{S}_n)_{\max}}\!/\!_{\approx}$.

Proposition 4.2.10. *The maps*

$$\begin{array}{ccc} \{\alpha \vDash_e n\} \to {(\mathfrak{S}_n)_{\nu,\min}}/{\approx_\nu} & \text{and} & \{\alpha \vDash_e n\} \to {(\mathfrak{S}_n)_{\max}}/{\approx} \\ \alpha \mapsto [d_\alpha]_\nu & & \alpha \mapsto [d_\alpha w_0] \end{array}$$

are bijections.

Proof. Let $\alpha \vDash_e n$, f be the first and g be the second map. From Lemma 4.2.9 we have that $d_\alpha \in (\mathfrak{S}_n)_{\nu,\min}$. Hence, $[d_\alpha]_\nu \in {(\mathfrak{S}_n)_{\nu,\min}}/{\approx_\nu}$ as claimed. Since $\{\alpha \vDash_e n\}$ and ${(\mathfrak{S}_n)_{\nu,\min}}/{\approx_\nu}$ both parametrize bases of $\overline{H_n(0)}_\nu$ by Theorem 4.2.6 and Theorem 4.1.7, respectively, the two sets have the same cardinality. Therefore, in order to show that f is a bijection, it suffices to prove its injectivity.

Let $\beta \vDash_e n$ be such that $[d_\alpha]_\nu = [d_\beta]_\nu$. Then $d_\alpha \approx_\nu d_\beta$ and from [He15, Proposition 3.1] it follows that $\pi_{d_\alpha} + [H_n(0), H_n(0)]_\nu = \pi_{d_\beta} + [H_n(0), H_n(0)]_\nu$. Now Theorem 4.2.6 implies $\alpha = \beta$. Hence, f is a bijection.

Now consider g. Lemma 4.1.11 provides the bijection

$$h\colon {(\mathfrak{S}_n)_{\nu,\min}}/{\approx_\nu} \to {(\mathfrak{S}_n)_{\max}}/{\approx}, \quad \Sigma \mapsto \Sigma w_0.$$

Then $g = h \circ f$ and hence g is a bijection too. □

By Proposition 4.2.10 we have that the d_α and $d_\alpha w_0$ for $\alpha \vDash_e n$ form a system of representatives for ${(\mathfrak{S}_n)_{\nu,\min}}/{\approx_\nu}$ and ${(\mathfrak{S}_n)_{\max}}/{\approx}$, respectively. Moreover, with Proposition 4.1.14 it follows that the d_α form such a system for $\{C_{\min} \mid (I, C) \in \Gamma_\nu\}$.

Example 4.2.11. Consider $n = 3$. From Example 4.2.7 we obtain that

$\alpha \vDash_e 3$	(3)	$(2,1)$	(1^3)
d_α	$(1,2)$	1	$(1,3)$
$d_\alpha w_0$	$(1,3,2)$	$(1,3)$	1

where $w_0 = (1, 3)$. One may check with Example 4.1.6 that the d_α and the $d_\alpha w_0$ form systems of representatives of ${(\mathfrak{S}_3)_{\nu,\min}}/{\approx_\nu}$ and ${(\mathfrak{S}_3)_{\max}}/{\approx}$, respectively.

From Proposition 4.2.10 and Theorem 4.1.8 we deduce the following.

Corollary 4.2.12. *The elements $\bar{\pi}_{\leq[d_\alpha w_0]}$ for $\alpha \vDash_e n$ form a basis of $Z(H_n(0))$.*

Let $\alpha \vDash_e n$. Recall that $\bar{\pi}_{\leq[d_\alpha w_0]} = \sum_{x \in (\mathfrak{S}_n)_{\leq[d_\alpha w_0]}} \bar{\pi}_x$ where

$$(\mathfrak{S}_n)_{\leq[d_\alpha w_0]} = \{x \in \mathfrak{S}_n \mid x \leq w \text{ for some } w \in [d_\alpha w_0]\}.$$

That is, we have an explicit description of the expansion of elements of the basis of $Z(H_n(0))$ into a basis of $H_n(0)$. In Section 5.1 of [Bri08] Brichard describes how one can use the inverse of the isomorphism from Theorem 4.1.3 in order to obtain a basis of $Z(H_n(0))$ from the basis $\{\pi_{d_\alpha} + [H_n(0), H_n(0)]_\nu \mid \alpha \vDash_e n\}$ of $\overline{H_n(0)}_\nu$. In comparison with the description above, her procedure is less explicit. Indeed, [Bri08] contains no formula for the expansion of the basis elements of the center in terms of a basis of $H_n(0)$.

4.2.2 Elements in stair form

Recall from Definition 4.2.4 that a composition α is called maximal if there is a $k \geq 0$ such that the first k parts of α are even and the remaining parts are odd and weakly decreasing. We mentioned in the introduction of the chapter that Kim defined the *elements in stair form* $\sigma_\alpha \in \mathfrak{S}_n$ for $\alpha \vDash n$ [Kim98]. Geck, Kim and Pfeiffer showed that these elements have maximal length in their respective conjugacy class if and only if α is a maximal composition in [GKP00]. In this subsection we show that the σ_α for $\alpha \vDash_e n$ form a system of representatives of ${(\mathfrak{S}_n)_{\max}}/{\approx}$ and by that give another parametrization of ${(\mathfrak{S}_n)_{\max}}/{\approx}$. This is the foundation of Section 4.3 and Chapter 5.

Definition 4.2.13. *Let $\alpha = (\alpha_1, \dots, \alpha_l) \vDash n$. Define the list $(x_1, x_2, \dots, x_n)$ by setting $x_{2i-1} := i$ and $x_{2i} := n - i + 1$. The* element in stair form $\sigma_\alpha \in \mathfrak{S}_n$ corresponding to α *is given by*

$$\sigma_\alpha := \sigma_{\alpha_1}\sigma_{\alpha_2}\cdots\sigma_{\alpha_l}$$

where σ_{α_i} is the α_i-cycle

$$\sigma_{\alpha_i} := \left(x_{\alpha_1+\dots+\alpha_{i-1}+1}, x_{\alpha_1+\dots+\alpha_{i-1}+2}, \dots, x_{\alpha_1+\dots+\alpha_{i-1}+\alpha_i}\right).$$

For instance, $\sigma_{(4,2)} = (1,6,2,5)(3,4)$. We obtain σ_α for $\alpha = (\alpha_1, \dots, \alpha_l) \vDash n$ as follows. Let $d_i := \sum_{j=1}^{i} \alpha_i$ for $i = 1, \dots, l$ and consider the list $(x_1, x_2, \dots, x_n)$ given as above. Then split the list between x_{d_i} and x_{d_i+1} for $i = 1, \dots, l-1$. The resulting sublists are the cycles of σ_α. In particular, if α and β are compositions with $\sigma_\alpha = \sigma_\beta$ then $\alpha = \beta$.

This following parametrization of ${(\mathfrak{S}_n)_{\max}}/{\approx}$ is the main result of this subsection.

Proposition 4.2.14. *The map*

$$\begin{aligned} \{\alpha \vDash_e n\} &\to {(\mathfrak{S}_n)_{\max}}/{\approx} \\ \alpha &\mapsto [\sigma_\alpha] \end{aligned}$$

is a bijection.

Before we begin with the proof of Proposition 4.2.14 we discuss some immediate consequences. First of all, using Lemma 4.1.11 as in Proposition 4.2.10 we obtain the corresponding parametrization of ${(\mathfrak{S}_n)_{\nu,\min}}/{\approx_\nu}$.

Corollary 4.2.15. *The map*

$$\begin{aligned} \{\alpha \vDash_e n\} &\to {(\mathfrak{S}_n)_{\nu,\min}}/{\approx_\nu} \\ \alpha &\mapsto [\sigma_\alpha w_0]_\nu \end{aligned}$$

is a bijection.

Example 4.2.16. Consider $n = 3$. Then we have

$\alpha \vDash_e 3$	(3)	$(2,1)$	(1^3)
σ_α	$(1,3,2)$	$(1,3)$	1
$\sigma_\alpha w_0$	$(1,2)$	1	$(1,3)$

One may check against Example 4.1.6 that the σ_α and the $\sigma_\alpha w_0$ form a system of representatives of ${}^{(\mathfrak{S}_3)_{\max}}\!/\!_{\approx}$ and ${}^{(\mathfrak{S}_3)_{\nu,\min}}\!/\!_{\approx_\nu}$, respectively.

By comparing with Example 4.2.11, we obtain that $\sigma_\alpha = d_\alpha w_0$ for all $\alpha \vDash_e 3$. Hence, one might be tempted to conjecture that this is always the case. However, $\alpha = (3,3)$ provides the smallest counter example:

$$\sigma_{(3,3)} = (1,6,2)(3,4,5) \quad \text{whereas} \quad d_{(3,3)}w_0 = (1,6,2)(3,5,4).$$

Let $\alpha \vDash_e n$. In comparison with the permutation d_α corresponding to the crossing diagram D_α, the element in stair form σ_α is defined in cycle notation, whereas d_α is given by a reduced word defined by a recursion. In Section 4.3 and Chapter 5 we will only work with elements in stair form exploiting the fact that we know their cycle notation. Therefore, we introduce the following notation.

Definition 4.2.17. *For $\alpha \vDash_e n$ define $\Sigma_\alpha \in {}^{(\mathfrak{S}_n)_{\max}}\!/\!_{\approx}$ to be the equivalence class of the element in stair form σ_α with respect to $\approx$.*

From Proposition 4.2.14 and Theorem 4.1.8 we obtain the following.

Corollary 4.2.18. *The elements $\bar{\pi}_{\leq \Sigma_\alpha}$ for $\alpha \vDash_e n$ form a basis of $Z(H_n(0))$.*

Now we come to the proof of Proposition 4.2.14. The first result in this direction goes back to [Kim98]. See [GKP00, Theorem 3.3] for a proof.

Lemma 4.2.19. *Let $\alpha \vDash n$. Then $\sigma_\alpha \in (\mathfrak{S}_n)_{\max}$ if and only if α is a maximal composition.*

Because of the Lemma 4.2.19, it remains to show the following in order to prove Proposition 4.2.14.

(a) For each $\Sigma \in {}^{(\mathfrak{S}_n)_{\max}}\!/\!_{\approx}$ there is an $\alpha \vDash_e n$ such that $\sigma_\alpha \in \Sigma$.

(b) If $\alpha, \beta \vDash_e n$ and $\sigma_\alpha \approx \sigma_\beta$ then $\alpha = \beta$.

From Proposition 4.2.10 we know that $|\{\alpha \vDash_e n\}| = \left|{}^{(\mathfrak{S}_n)_{\max}}\!/\!_{\approx}\right|$ and therefore it suffices to prove either (a) or (b). However, we show both statements here as both proofs involve intermediate results that will be useful in later sections. By doing so, we also get an alternative proof of the dimension formula given in Corollary 4.2.8.

In order to prove Statement (a) we need the following result.

Lemma 4.2.20. *Let W be a finite Coxeter group and $w, w' \in W$ be such that $w \to w'$ and $\ell(w) = \ell(w')$. Then $w \approx w'$.*

Proof. Let S be the set of Coxeter generators of W. It suffices to consider the case where $w \xrightarrow{s} w'$ for some $s \in S$ because by definition $\to$ is the transitive closure of all the relations $\xrightarrow{t}$ with $t \in S$. Then $w' = sws$. Thus, $w = sw's$ and since $\ell(w) = \ell(w')$, we have $w' \xrightarrow{s} w$. Hence $w \approx w'$. $\square$

Proof of Statement (a). Let $\Sigma \in {}^{(\mathfrak{S}_n)_{\max}}\!/\!_{\approx}$ and $\sigma \in \Sigma$. In [Kim98, Section 3] it is shown that there is a $\beta \vDash n$ such that $\sigma_\beta \to \sigma$. Moreover, Statement (a$''$) of Section 3.1 in [GKP00] provides the existence of an $\alpha \vDash_e n$ such that $\sigma_\alpha \to \sigma_\beta$. Therefore, $\sigma_\alpha \to \sigma$. Hence, σ_α and σ are conjugate and $\ell(\sigma_\alpha) \geq \ell(\sigma)$. But the length of σ is maximal in its conjugacy class. Hence, $\ell(\sigma_\alpha) = \ell(\sigma)$ and Lemma 4.2.20 yields $\sigma_\alpha \approx \sigma$. $\square$

We begin working towards Statement (b). As before, we will trace the relation $\approx$ back to the elementary steps $\xrightarrow{s_i}$ with $i \in [n-1]$. Consider $\sigma \in \mathfrak{S}_n$ and $\tau = s_i \sigma s_i$. Then we have $\tau \xrightarrow{s_i} \sigma$ or $\sigma \xrightarrow{s_i} \tau$ depending on $\ell(s_i\sigma s_i) - \ell(\sigma)$. Moreover $\sigma \approx \tau$ if and only if the difference vanishes. Thus our first goal is to determine $\ell(s_i\sigma s_i) - \ell(\sigma)$ depending on σ and s_i.

Recall that for all $\sigma \in \mathfrak{S}_n$ and $i \in [n-1]$

$$\begin{aligned} s_i \in D_R(\sigma) &\iff \sigma(i) > \sigma(i+1), \\ s_i \in D_L(\sigma) &\iff \sigma^{-1}(i) > \sigma^{-1}(i+1). \end{aligned}$$

Lemma 4.2.21. *Let $\sigma \in \mathfrak{S}_n$ and $i, j \in [n-1]$. Then $\{\sigma(i), \sigma(i+1)\} \neq \{j, j+1\}$ if and only if $(s_j \in D_L(\sigma) \iff s_j \in D_L(\sigma s_i))$.*

Proof. We consider all permutations in one-line notation. Note that for all $\sigma \in \mathfrak{S}_n$ we have that $j \in D_L(\sigma)$ if and only if $j+1$ is left of j in σ.

Now fix a $\sigma \in \mathfrak{S}_n$. Observe that we obtain σs_i from σ by swapping $\sigma(i)$ and $\sigma(i+1)$. Since these are two consecutive characters in the the one-line notation of σ, the relative positioning of j and $j+1$ is affected by this interchange if and only if $\{\sigma(i), \sigma(i+1)\} = \{j, j+1\}$. Now use the note on left descents from the beginning to deduce the claim. $\square$

Lemma 4.2.22. *Let $\sigma \in \mathfrak{S}_n$ and $i \in [n-1]$.*

(1) If $\{\sigma(i), \sigma(i+1)\} \neq \{i, i+1\}$ then

$$\ell(s_i\sigma s_i) = \begin{cases} \ell(\sigma) - 2 & \text{if } \sigma(i) > \sigma(i+1) \text{ and } \sigma^{-1}(i) > \sigma^{-1}(i+1), \\ \ell(\sigma) + 2 & \text{if } \sigma(i) < \sigma(i+1) \text{ and } \sigma^{-1}(i) < \sigma^{-1}(i+1), \\ \ell(\sigma) & \text{else.} \end{cases}$$

(2) If $\{\sigma(i), \sigma(i+1)\} = \{i, i+1\}$ then either i and $i+1$ are fixpoints or form a 2-cycle in σ. In particular, $s_i\sigma s_i = \sigma$.

Proof. Part (2) should be clear. For Part (1) assume that $\{\sigma(i), \sigma(i+1)\} \neq \{i, i+1\}$. We have

$$\ell(s_i\sigma s_i) - \ell(\sigma) = \ell(s_i\sigma s_i) - \ell(\sigma s_i) + \ell(\sigma s_i) - \ell(\sigma)$$

where each of the two differences on the right hand side is -1 or 1 depending the truth value of the statements $s_i \in D_L(\sigma s_i)$ and $s_i \in D_R(\sigma)$, respectively. From Lemma 4.2.21 we have that $s_i \in D_L(\sigma s_i)$ if and only if $s_i \in D_L(\sigma)$. That is, the first difference depends on whether $s_i \in D_L(\sigma)$ or not. Thus, the description of $D_L(\sigma)$ and $D_R(\sigma)$ preceding Lemma 4.2.21 implies the claim. $\square$

We now show that for $\alpha \vDash_e n$ all elements of Σ_α have the same orbits of even length on $[n]$.

Lemma 4.2.23. *Let $\alpha \vDash_e n$ and $\sigma \in \mathfrak{S}_n$ such that $\sigma_\alpha \approx \sigma$. Then we have the following.*

(*1*) *The orbits of even length of σ and σ_α on $[n]$ coincide.*

(*2*) *Let $\mathcal{O}$ be an σ-orbit on $[n]$ of even length. Then the orbits of σ^2 and σ_α^2 on $\mathcal{O}$ coincide.*

Proof. Since $\sigma_\alpha \approx \sigma$, we have $\sigma_\alpha \to \sigma$ and $\ell(\sigma_\alpha) = \ell(\sigma)$. Using induction on the minimal number of elementary steps $w \xrightarrow{s} w'$ (with some $w, w' \in \mathfrak{S}_n$ and $s \in S$) necessary to relate σ_α to σ, we may assume that there is a $\tau \in \mathfrak{S}_n$ and an $s_i \in S$ such that $\sigma_\alpha \to \tau \xrightarrow{s_i} \sigma$ and τ satisfies properties (1) and (2) (σ_α certainly does). Then $\ell(\sigma_\alpha) \geq \ell(\tau) \geq \ell(\sigma)$ so that in fact $\ell(\sigma_\alpha) = \ell(\tau) = \ell(\sigma)$ and $\sigma_\alpha \approx \tau \approx \sigma$.

It remains to show that $\xrightarrow{s_i}$ transfers properties (1) and (2) from τ to σ. Because $\sigma = s_i \tau s_i$, we obtain σ from τ by interchanging i and $i+1$ in the cycle notation of τ. If i and $i+1$ both appear in orbits of uneven length of τ then properties (1) and (2) are not affected by this interchange. Thus, we are left with two cases.

Case 1. Assume that i and $i+1$ appear in different orbits of τ, say $\mathcal{O}_1$ and $\mathcal{O}_2$ such that at least one of them, say $\mathcal{O}_1$, has even length. We show that this case does not occur. To do this, let m_1 and m_2 be the minimal elements of $\mathcal{O}_1$ and $\mathcal{O}_2$, respectively. If $\mathcal{O}_2$ also has even length, we assume $m_1 < m_2$.

For $w \in \mathfrak{S}_n$ and $j \in [n]$ let $\langle w \rangle$ denote the subgroup of $\mathfrak{S}_n$ generated by w and $\langle w \rangle j$ be the orbit of j under the natural action of $\langle w \rangle$ on $[n]$. Since τ satisfies property (2) and $\mathcal{O}_1$ has even length, there is a $p_1 \geq m_1$ such that

$$\begin{aligned} \mathcal{O}_1^< &:= \langle \tau^2 \rangle m_1 = \langle \sigma_\alpha^2 \rangle m_1 = \{m_1, m_1+1, \dots, p_1\}, \\ \mathcal{O}_1^> &:= \langle \tau^2 \rangle \tau(m_1) = \langle \sigma_\alpha^2 \rangle \sigma_\alpha(m_1) = \{n-m_1+1, n-m_1, \dots, n-p_1+1\}. \end{aligned} \tag{4.4}$$

Claim. *Let $a \in \mathcal{O}_1^<, b \in \mathcal{O}_2$ and $c \in \mathcal{O}_1^>$. Then $a < b < c$.*

To prove the claim consider the positions of elements of $[n]$ in the cycle notation $\sigma_\alpha = \sigma_{\alpha_1} \cdots \sigma_{\alpha_l}$ given by the definition. The elements on odd positions $1, 2, 3, \dots$ form an strictly increasing sequence. The elements on even positions $n, n-1, \dots$ form an strictly decreasing sequence but they are always greater than the entries on odd positions.

We want to show that the elements of $\mathcal{O}_2$ all appear right of the cycle consisting of the elements of $\mathcal{O}_1$. If $\mathcal{O}_2$ has even length this is clear. If $\mathcal{O}_2$ has odd length, we can use that by property (1), the unions of odd orbits of τ and σ_α coincide and that in σ_α the elements of odd orbits are all located right of the elements of the even orbits.

Let $a \in \mathcal{O}_1^<$. Then a is on an odd position and thus it is smaller than any entry right of it. On the other hand, $c \in \mathcal{O}_1^>$ implies that c is on an even position and thus is greater then any entry right of it. Finally, in the last paragraph we have shown that each $b \in \mathcal{O}_2$ is located right of a and c. This establishes the claim.

Now, we have to deal with two cases.

If $i \in \mathcal{O}_1$ and $i+1 \in \mathcal{O}_2$ then the claim implies $i \in \mathcal{O}_1^<$. Then $\tau^{-1}(i), \tau(i) \in \mathcal{O}_1^>$. Since $\tau^{-1}(i+1), \tau(i+1) \in \mathcal{O}_2$, our claim yields $\tau^{-1}(i) > \tau^{-1}(i+1)$ and $\tau(i) > \tau(i+1)$. In addition, since $\mathcal{O}_1$ has even length and $i+1 \notin \mathcal{O}_1$, $\tau(i) \neq i, i+1$. Thus, we obtain from Lemma 4.2.22 that $\ell(\sigma) < \ell(\tau)$, a contradiction to $\ell(\tau) = \ell(\sigma)$.

If $i+1 \in \mathcal{O}_1$ and $i \in \mathcal{O}_2$ then the claim implies $i+1 \in \mathcal{O}_1^>$ and similarly as before we obtain $\tau^{-1}(i) > \tau^{-1}(i+1)$ and $\tau(i) > \tau(i+1)$ and thus the same contradiction using Lemma 4.2.22. That is, we have shown that i and $i+1$ cannot appear in two different orbits if one of the latter has even length.

Case 2. Assume that i and $i+1$ appear in the same orbit with even length $\mathcal{O}_1$ of τ. Then (1) also holds for σ.

To show (2), assume $i+1 \in \langle \tau^2 \rangle i$ first. Then both elements appear in the same cycle of τ^2. As we obtain σ^2 from τ^2 by swapping i and $i+1$ in cycle notation, (2) also holds for σ.

Lastly, we show that $i+1 \in \langle \tau^2 \rangle i$ is always true. For the sake of contradiction, assume $i+1 \notin \langle \tau^2 \rangle i$.

Suppose in addition that $|\mathcal{O}_1| = 2$. Then $\{\tau(i), \tau(i+1)\} = \{i, i+1\}$ and from Lemma 4.2.22 we obtain $\sigma = s_i \tau s_i = \tau$. This contradicts the minimality of the sequence of arrow relations from σ_α to σ.

Now suppose $|\mathcal{O}_1| > 2$. Then $\{\tau(i), \tau(i+1)\} \neq \{i, i+1\}$. Since $i+1 \notin \langle \tau^2 \rangle i$, it follows from (4.4) that $i = \max \mathcal{O}_1^<$ and $i+1 = \min \mathcal{O}_1^>$. Consequently, $\tau^{-1}(i), \tau(i) \in \mathcal{O}_1^>$ and $\tau^{-1}(i+1), \tau(i+1) \in \mathcal{O}_1^<$. But this means that

$$\tau^{-1}(i) > \tau^{-1}(i+1) \text{ and } \tau(i) > \tau(i+1).$$

Because $\{\tau(i), \tau(i+1)\} \neq \{i, i+1\}$, we can now apply Lemma 4.2.22 and obtain that $\ell(\sigma) < \ell(\tau)$. Again, we end up with a contradiction. $\square$

Let $\sigma \in \mathfrak{S}_n$. Then the set of orbits of σ on $[n]$ is a set partition of $[n]$. We denote this partition by $P(\sigma)$. The set of even orbits of σ is given by

$$P_e(\sigma) := \{\mathcal{O} \in P(\sigma) \mid |\mathcal{O}| \text{ is even}\}$$

If $P(\sigma) = P(\sigma')$ for $\sigma, \sigma' \in \mathfrak{S}_n$ then σ and σ' have the same type, i.e. they are conjugate.

Lemma 4.2.24. *Let $\alpha, \beta \vDash_e n$ such that σ_α and σ_β are conjugate. If $P_e(\sigma_\alpha) = P_e(\sigma_\beta)$ then $\alpha = \beta$.*

Proof. Let $\alpha = (\alpha_1, \ldots, \alpha_l), \beta = (\beta_1, \ldots, \beta_{l'}) \vDash_e n$ and $(x_1, x_2, \ldots, x_n)$ be the sequence with $x_{2i-1} = i$ and $x_{2i} = n - i + 1$. Since α is maximal, there is a $k \in [0, l]$ such that

α_i is even for $i \leq k$ and odd for $i > k$. Assume that σ_α and σ_β are conjugate and $P_e(\sigma_\alpha) = P_e(\sigma_\beta)$.

Because σ_α and σ_β are conjugate, α and β have the same multiset of parts. In particular, $l = l'$. Since α and β are maximal, the odd parts of α and β form an weakly decreasing sequence at the end of α and β, respectively. As both compositions have the same length and multiset of parts, it follows that $\alpha_i = \beta_i$ for $i = k+1, \ldots, l$.

We show that $\alpha_i = \beta_i$ for $i = 1, \ldots, k$ with induction. Assume that $i \in [k]$ and $\alpha_j = \beta_j$ for all $1 \leq j < i$. Define $d := \sum_{j=1}^{i-1} \alpha_i$. Then by assumption $d = \sum_{j=1}^{i-1} \beta_i$. Moreover, let $\mathcal{O}_{\alpha_i}$ and $\mathcal{O}_{\beta_i}$ be the orbits of x_{d+1} under σ_α and σ_β, respectively. From the definition of elements in stair form it follows that

$$\mathcal{O}_{\alpha_i} = \{x_{d+1}, x_{d+2}, \ldots, x_{d+\alpha_i}\},$$
$$\mathcal{O}_{\beta_i} = \{x_{d+1}, x_{d+2}, \ldots, x_{d+\beta_i}\}.$$

In particular $|\mathcal{O}_{\alpha_i}| = \alpha_i$ and $|\mathcal{O}_{\beta_i}| = \beta_i$. Since $i \leq k$, α_i and β_i are even. Consequently, $\mathcal{O}_{\alpha_i}$ and $\mathcal{O}_{\beta_i}$ both have even length. Moreover, they have the element x_{d+1} in common. Hence, $P_e(\sigma_\alpha) = P_e(\sigma_\beta)$ implies $\mathcal{O}_{\alpha_i} = \mathcal{O}_{\beta_i}$. Thus, $\alpha_i = |\mathcal{O}_{\alpha_i}| = |\mathcal{O}_{\beta_i}| = \beta_i$. □

We are now in the position to prove Statement (b). This finishes the proof of Proposition 4.2.14.

Proof of Statement (b). Let $\alpha, \beta \vDash_e n$ such that $\sigma_\alpha \approx \sigma_\beta$. Then σ_α and σ_β are conjugate. Moreover, Lemma 4.2.23 implies $P_e(\sigma_\alpha) = P_e(\sigma_\beta)$. Hence $\alpha = \beta$ by Lemma 4.2.24. □

We use some of the intermediary results that lead to Proposition 4.2.14 in order to prepare a result for later use in Subsection 4.3.3.

Proposition 4.2.25. *Let $\alpha \vDash_e n$ and $\sigma \in \mathfrak{S}_n$. Then $\sigma \in \Sigma_\alpha$ if and only if*

(1) σ and σ_α are conjugate in $\mathfrak{S}_n$,

(2) $\ell(\sigma) = \ell(\sigma_\alpha)$,

(3) $P_e(\sigma) = P_e(\sigma_\alpha)$.

Proof. First, assume $\sigma \in \Sigma_\alpha$. Because $\sigma_\alpha \in \Sigma_\alpha$ and $\Sigma_\alpha \in {}^{(\mathfrak{S}_n)_{\max}}\!/\!\approx$, σ satisfies (1) and (2). By Lemma 4.2.23, (3) holds as well.

Second, assume that σ satisfies (1) – (3). By (1), σ is in the same conjugacy class as σ_α. From (2) it follows, that σ is maximal in its conjugacy class. Then Proposition 4.2.14 provides the existence of a $\beta \vDash_e n$ such that $\sigma \in \Sigma_\beta$. Using the already proven implication from left to right, we obtain that σ and σ_β are conjugate and $P_e(\sigma) = P_e(\sigma_\beta)$. But as σ satisfies (1) and (3), it follows that σ_β and σ_α are conjugate and $P_e(\sigma_\beta) = P_e(\sigma_\alpha)$. Thus, Lemma 4.2.24 yields $\beta = \alpha$ as desired. □

We end this subsection with a remark on conjugacy classes.

Remark 4.2.26. The conjugacy classes of $\mathfrak{S}_n$ are parametrized by the partitions of n via the cycle type. Let $\lambda \vdash n$ and $\mathcal{O}$ be the conjugacy class whose elements have cycle type λ. From Definition 4.2.13 it follows that for $\alpha \vDash_e n$ the element in stair form σ_α is contained in $\mathcal{O}$ if and only if $\widetilde{\alpha} = \lambda$. Hence, Proposition 4.2.14 implies that $\{\sigma_\alpha \mid \alpha \vDash_e n, \widetilde{\alpha} = \lambda\}$ is a system of representatives for ${}^{\mathcal{O}_{\max}}\!/_{\approx}$. In particular, we have that

$$\left| {}^{\mathcal{O}_{\max}}\!/_{\approx} \right| = 1 \text{ if and only if the even parts of } \lambda \text{ are all equal.}$$

4.2.3 Coxeter elements

We introduce a set of representatives of ${}^{(\mathfrak{S}_n)_{\min}}\!/_{\approx}$ which is due to Gill [Gil00]. From this we obtain parametrizations of the bases of the cocenter $\overline{H_n(0)}$ and the twisted center $Z(H_n(0))_\nu$ of $H_n(0)$ from Section 4.1 by the compositions of n.

Definition 4.2.27. *Let W be a finite Coxeter group with generators S. A* Coxeter element *of W is a product of all $s \in S$ in arbitrary order.*

For each $\alpha \vDash n$ we now fix a Coxeter element c_α of the parabolic subgroup $\mathfrak{S}_\alpha$ of $\mathfrak{S}_n$. Gill showed in [Gil00] that the elements c_α are transversal for ${}^{(\mathfrak{S}_n)_{\min}}\!/_{\approx}$. This leads to the following parametrization of ${}^{(\mathfrak{S}_n)_{\min}}\!/_{\approx}$. Recall that for $w \in \mathfrak{S}_n$ its equivalence class in $\mathfrak{S}_n$ with respect to $\approx_\delta$ is denoted by $[w]_\delta$.

Proposition 4.2.28 ([Gil00, Theorem 5])**.** *The map $\{\alpha \vDash n\} \to {}^{(\mathfrak{S}_n)_{\min}}\!/_{\approx}, \alpha \mapsto [c_\alpha]$ is a bijection.*

With Lemma 4.1.11 we obtain the corresponding parametrization of ${}^{(\mathfrak{S}_n)_{\nu,\max}}\!/_{\approx_\nu}$.

Corollary 4.2.29. *The map $\{\alpha \vDash n\} \to {}^{(\mathfrak{S}_n)_{\nu,\max}}\!/_{\approx_\nu}, \alpha \mapsto [c_\alpha w_0]_\nu$ is a bijection where w_0 is the longest element of $\mathfrak{S}_n$.*

Example 4.2.30. For $n = 3$ we obtain the following representatives for ${}^{(\mathfrak{S}_n)_{\min}}\!/_{\approx}$ and ${}^{(\mathfrak{S}_n)_{\nu,\max}}\!/_{\approx_\nu}$ which we represent via a reduced word and in cycle notation.

$\alpha \vDash 3$	$(1,1,1)$	$(1,2)$	$(2,1)$	(3)
c_α	1	s_2	s_1	s_1s_2
	1	$(2,3)$	$(1,2)$	$(1,2,3)$
$c_\alpha w_0$	$s_1s_2s_1$	s_1s_2	s_2s_1	s_2
	$(1,3)$	$(1,2,3)$	$(1,3,2)$	$(2,3)$

Remark 4.2.31. Let $\alpha = (\alpha_1, \dots, \alpha_l) \vDash n$. In [Gil00, Lemma 4] Gill showed that the $\approx$-equivalence class $[c_\alpha]$ is exactly the set of Coxeter elements of $\mathfrak{S}_\alpha$ and that the cardinality of this set is $\prod_i 2^{\alpha_i - 2}$ where the product runs over all $i \in [l]$ such that $\alpha_i \geq 2$. The reason for the latter is that the Coxeter elements of $\mathfrak{S}_\alpha$ are in one to one correspondence with the orientations of the Coxeter graph of $\mathfrak{S}_\alpha$ (see [Shi97, Theorem 1.5]).

We obtain the following bases of $\overline{H_n(0)}$ and $Z(H_n(0))_\nu$ parametrized by the compositions of n.

Figure 4.7: The Coxeter graphs of type B_n and D_n each with n vertices.

Theorem 4.2.32. *For each $\alpha \vDash n$ let c_α be a Coxeter element of the parabolic subgroup $\mathfrak{S}_\alpha$ of $\mathfrak{S}_n$.*

(*1*) *The elements $\bar{\pi}_{c_\alpha} + [H_n(0), H_n(0)]$ for $\alpha \vDash n$ form a basis of $\overline{H_n(0)}$.*

(*2*) *The elements $\bar{\pi}_{\leq [c_\alpha w_0]_\nu}$ for $\alpha \vDash n$ form a basis of $Z(H_n(0))_\nu$.*

In particular, $\overline{H_n(0)}$ and $Z(H_n(0))_\nu$ both have dimension 2^{n-1}.

Proof. For Part (1) combine Theorem 4.2.6 and proposition 4.2.28. For Part (2) do the same with Theorem 4.1.8 and Corollary 4.2.29. The number of compositions of n is 2^{n-1} since via $\alpha \mapsto \operatorname{Set}(\alpha)$ we have a bijection between the compositions of n and the subsets of $[n-1]$. This yields the dimensions. □

4.2.4 Remarks on types B and D

In [Gil00] Gill determines the the cardinality of ${}^{\mathcal{O}_{\min}}\!/\!_{\approx}$ for $\mathcal{O} \in \operatorname{cl}(W)$ in types A, B and D. We translated his results in type A to Proposition 4.2.28. We now briefly discuss the types B and D and infer dimension formulas for the cocenter and (in types B_n and D_{2n}) for the center of the related 0-Hecke algebras. The Coxeter graphs of types B_n and D_n are shown in Figure 4.7. For background information on these Coxeter groups we refer to [BB05, GP00].

Let $n \geq 2$ and $\mathfrak{B}_n$ be an irreducible Coxeter group of type B_n with Coxeter generators S (that is $|S| = n$). From [Gil00, Theorem 10] it follows that ${}^{(\mathfrak{B}_n)_{\min}}\!/\!_{\approx}$ is parametrized by the pairs (α, β) such that α and β are compositions, β is (weakly) increasing and $|\alpha| + |\beta| = n$.

Let w_0 be the longest element of $\mathfrak{B}_n$. By [Fay05, Proposition 2.4] w_0 is central and thus ν is the identity on $\mathfrak{B}_n$. Hence, Lemma 4.1.11 implies that $\Sigma \mapsto \Sigma w_0$ is a bijection from ${}^{(\mathfrak{B}_n)_{\min}}\!/\!_{\approx}$ to ${}^{(\mathfrak{B}_n)_{\max}}\!/\!_{\approx}$. Using Theorem 4.1.8 and Theorem 4.2.6, we now obtain that

$$\dim Z(H_{\mathfrak{B}_n}(0)) = \dim \overline{H_{\mathfrak{B}_n}(0)} = \sum_{m=0}^{n} c(m)p(n-m)$$

where $c(m)$ and $p(m)$ are the numbers of compositions of m and partitions of m, respectively. Of course, $c(0) = 1$ and $c(m) = 2^{m-1}$ for $m \geq 1$.

Regarding Figure 4.7 it is clear that id is the only graph automorphism of the Coxeter graph of $\mathfrak{B}_n$ if $n \geq 3$. Hence, it follows from Lemma 4.1.1 that the only $\mathfrak{B}_n$-automorphism δ with $\delta(S) = S$ is the identity. That is, we treated ${}^{(\mathfrak{B}_n)_{\delta,\min}}\!/\!_{\approx_\delta}$ and ${}^{(\mathfrak{B}_n)_{\delta,\max}}\!/\!_{\approx_\delta}$ for $n \geq 3$ and all possibilities of δ.

Let $n \geq 4$ and $\mathfrak{D}_n$ be an irreducible Coxeter group of type D_n (with n Coxeter generators). Then one can infer from [Gil00, Theorem 10] that ${}^{(\mathfrak{D}_n)_{\min}}\!/\!_{\approx}$ is parametrized by the pairs of compositions (α, β) such that β is increasing and has even length and $|\alpha| + |\beta| = n$ together with the pairs $(\alpha, -)$ such that $\alpha \vDash n$ and $\alpha_1 > 1$. However, note that in [Gil00, Theorem 10] there is an error in type D (see Remark 4.2.33 below).

Using Theorem 4.2.6, it follows that

$$\dim \overline{H_{\mathfrak{D}_n}(0)} = \sum_{m=0}^{n} c(m) p_e(n-m) + 2^{n-2}$$

where $p_e(n-m)$ is the number of partitions of even length of m. The number of $\alpha \vDash n$ with $\alpha_1 > 1$ is the number of compositions of $n-1$ and therefore 2^{n-2}. If n is even, then by [Fay05, Proposition 2.4] the longest element of $\mathfrak{D}_n$ is central and it follows as in type B that also $\dim Z(H_{\mathfrak{D}_n}(0))$ is given by the above formula.

Remark 4.2.33. In [Gil00, Theorem 10] there is a flaw in type D. It occurs in the case where $\mathcal{O}$ is a conjugacy class of $\mathfrak{D}_n$ which is labeled by the pair $(\emptyset, \lambda)$ (in the notation of [Gil00]) where $\lambda \vdash n$ has an odd part. In this case it can be deduced from the proof that the cardinality of ${}^{\mathcal{O}_{\min}}\!/\!_{\approx}$ is the number of $\alpha \vDash n$ with $\widetilde{\alpha} = \lambda$ plus the number of $\alpha \vDash n$ with $\widetilde{\alpha} = \lambda$ and $\alpha_1 > 1$.

Let $l := \ell(\lambda)$ and consider $\lambda = (1^{l_1}, 2^{l_2}, \ldots, n^{l_n})$ in exponential notation, i.e. l_i is the number of parts of λ that are equal to i. Then

$$|\{\alpha \vDash n \mid \widetilde{\alpha} = \lambda, \alpha_1 > 1\}| = \left(1 - \frac{l_1}{l}\right) \binom{l}{l_1, l_2, \ldots, l_n}$$

where the second factor is a multinomial coefficient. However, in [Gil00, Theorem 10] it is claimed that this cardinality is $l_2 + \cdots + l_n$ which is wrong in general (for instance, consider $\lambda = (2, 2, 1, 1)$).

4.3 Equivalence classes of $(\mathfrak{S}_n)_{\max}$ under $\approx$

Recall that by Definition 4.2.4 we call $\alpha \vDash n$ maximal and write $\alpha \vDash_e n$ if there is a $k \geq 0$ such that the first k parts of α are even and the remaining parts are odd and weakly decreasing. For $\alpha \vDash_e n$ we defined $\Sigma_\alpha \in {}^{(\mathfrak{S}_n)_{\max}}\!/\!_{\approx}$ to be the equivalence class of the element in stair form σ_α under $\approx$. From Proposition 4.2.14 we have that the elements of ${}^{(\mathfrak{S}_n)_{\max}}\!/\!_{\approx}$ are precisely the Σ_α with $\alpha \vDash_e n$. In Corollary 4.2.18 we concluded that the elements $\bar{\pi}_{\leq \Sigma_\alpha}$ for $\alpha \vDash_e n$ form a basis of $Z(H_n(0))$. The subject of this section is the description of the sets Σ_α and bijections between them.

In Subsection 4.3.1 we consider the case where α has only one part. The first result is the characterization of the elements of $\Sigma_{(n)}$ by properties of their cycle notation. From this we obtain bijections relating $\Sigma_{(n-1)}$ with $\Sigma_{(n)}$ for $n \geq 4$ and a closed formula for the cardinality of $\Sigma_{(n)}$.

Table 4.1: The elements of $\Sigma_{(n)}$ for small n. The respective topmost element is the element in stair form $\sigma_{(n)}$.

α	(1)	(2)	(3)	(4)	(5)	(6)
	(1)	$(1,2)$	$(1,3,2)$	$(1,4,2,3)$	$(1,5,2,4,3)$	$(1,6,2,5,3,4)$
			$(1,2,3)$	$(1,3,2,4)$	$(1,5,2,3,4)$	$(1,6,2,4,3,5)$
Σ_α					$(1,5,3,2,4)$	$(1,6,3,4,2,5)$
					$(1,4,2,3,5)$	$(1,5,2,4,3,6)$
					$(1,4,3,2,5)$	$(1,5,3,4,2,6)$
					$(1,3,4,2,5)$	$(1,4,3,5,2,6)$

In Subsection 4.3.2 we generalize the characterization of $\Sigma_{(n)}$ to odd hooks, where a hook $\alpha := (k, 1^{n-k})$ is called *odd* if k is odd and *even* otherwise. Moreover, we define a bijection $\Sigma_{(k)} \times [m+1, n-m] \to \Sigma_{(k,1^{n-k})}$ where $m := \frac{k-1}{2}$. From this we obtain the cardinality of $\Sigma_{(k,1^{n-k})}$.

In Subsection 4.3.3 we consider the inductive product $\odot$ that allows the decomposition $\Sigma_{(\alpha_1,\ldots,\alpha_l)} = \Sigma_{(\alpha_1)} \odot \Sigma_{(\alpha_2,\ldots,\alpha_l)}$ if α_1 is even. Using the results of the previous subsections, we infer a description of Σ_α for all $\alpha \vDash_e n$ whose odd parts form a hook

In Subsection 4.3.4 we use the inductive product in order to obtain necessary conditions and sufficient conditions for $\sigma \in \mathfrak{S}_n$ to be an element of Σ_α for arbitrary $\alpha \vDash_e n$. A maximal compositions with at most one odd part or all odd parts equal to 1 is called *mild*. We show that the conditions from above are both necessary and sufficient for $\sigma \in \Sigma_\alpha$ if and only if α is mild. Even hooks are mild and therefore treated in this subsection.

In Chapter 5 we use results of Subsections 4.3.1 to 4.3.3 in order to study the operation of $\bar{\pi}_{\leq \Sigma_\alpha}$ on the simple modules of $H_n(0)$ for certain α.

4.3.1 Equivalence classes of n-cycles

In this subsection we seek a combinatorial description of the elements of $\Sigma_{(n)}$. Examples are given in Table 4.1. The description is given by two properties: *being oscillating* and *having connected intervals*. We begin with the property of being oscillating.

Definition 4.3.1. *We call the n-cycle $\sigma \in \mathfrak{S}_n$* oscillating *if there exists a positive integer $m \in \left\{\frac{n-1}{2}, \frac{n}{2}, \frac{n+1}{2}\right\}$ such that $\sigma([m]) = [n-m+1, n]$.*

In Corollary 4.3.7 we will obtain a more descriptive characterization of oscillating n-cycles. It turns out that the n-cycle σ of $\mathfrak{S}_n$ (represented in cycle notation) is oscillating if n is even and the entries of σ alternate between the sets $[1, \frac{n}{2}]$ and $[\frac{n}{2}+1, n]$ or n is odd and after deleting the entry $\frac{n+1}{2}$ from σ the remaining entries alternate between the sets $[\frac{n-1}{2}]$ and $[\frac{n+3}{2}, n]$.

Example 4.3.2. (1) Recall that for $n \in \mathbb{N}$ the element in stair form $\sigma_{(n)}$ is an n-cycle of $\mathfrak{S}_n$. For

$$\sigma_{(5)} = (1,5,2,4,3), \quad \sigma_{(5)}^{-1} = (1,3,4,2,5) \quad \text{and} \quad \sigma_{(6)} = (1,6,2,5,3,4)$$

we have

$$\sigma_{(5)}([2]) = [4,5], \quad \sigma_{(5)}^{-1}([3]) = [3,5] \quad \text{and} \quad \sigma_{(6)}([3]) = [4,6].$$

Hence, they are oscillating and the integer m used in Definition 4.3.1 is given by

$$m = 2 = \frac{5-1}{2}, \quad m = 3 = \frac{5+1}{2} \quad \text{and} \quad m = 3 = \frac{6}{2},$$

respectively. Note that the entries in the cycles alternate as described after Definition 4.3.1.

(2) All the elements shown in Table 4.1 are oscillating.

We explicitly write down the three cases for m in Definition 4.3.1.

Remark 4.3.3. Let σ be an oscillating n-cycle $\sigma \in \mathfrak{S}_n$ with parameter m from Definition 4.3.1. Then we have

(1) n is even and $\sigma([\frac{n}{2}]) = [\frac{n}{2}+1, n]$ if $m = \frac{n}{2}$,
(2) n is odd and $\sigma([\frac{n-1}{2}]) = [\frac{n+3}{2}, n]$ if $m = \frac{n-1}{2}$,
(3) n is odd and $\sigma([\frac{n+1}{2}]) = [\frac{n+1}{2}, n]$ if $m = \frac{n+1}{2}$.

Our next aim is to give a characterization of the term *oscillating* in Lemma 4.3.6. By considering complements in $[n]$ we obtain the following.

Lemma 4.3.4. *Let $\sigma \in \mathfrak{S}_n$ be an n-cycle and $m \in [n]$. Then $\sigma([m]) = [n-m+1, n]$ if and only if $\sigma([m+1, n]) = [n-m]$.*

Lemma 4.3.4 implies that an n-cycle $\sigma \in \mathfrak{S}_n$ is oscillating with parameter m if and only if $\sigma([m+1,n]) = [n-m]$.

Lemma 4.3.5. *Let $\sigma \in \mathfrak{S}_n$ be an n-cycle. Then σ is oscillating if and only if σ^{-1} is oscillating.*

Proof. Let $M := \mathbb{N} \cap \left\{\frac{n-1}{2}, \frac{n}{2}, \frac{n+1}{2}\right\}$. If $n = 1$ then $\sigma = \mathrm{id} = \sigma^{-1}$ (which is oscillating). Thus assume $n \geq 2$. It suffices to show the implication from left to right. Suppose that σ is oscillating. Then there is an $m \in M$ such that $\sigma([m]) = [n-m+1, n]$. Consequently, $\sigma([m+1, n]) = [n-m]$ by Lemma 4.3.4 and hence

$$\sigma^{-1}([n-m]) = [m+1, n].$$

Moreover, $m+1 = n-(n-m)+1$ and we have $n-m \in M$ since $m \in M$ and $n \geq 2$. Therefore, σ^{-1} is oscillating. □

In the following we rephrase Definition 4.3.1 from a more local point of view.

Lemma 4.3.6. *Let $\sigma \in \mathfrak{S}_n$ be an n-cycle. We consider the four implications for all $i \in [n]$*

(i) $i < \frac{n+1}{2} \implies \sigma(i) \geq \frac{n+1}{2}$,
(ii) $i < \frac{n+1}{2} \implies \sigma^{-1}(i) \geq \frac{n+1}{2}$,
(iii) $i > \frac{n+1}{2} \implies \sigma(i) \leq \frac{n+1}{2}$,
(iv) $i > \frac{n+1}{2} \implies \sigma^{-1}(i) \leq \frac{n+1}{2}$,

and if n is odd the statement

(A) *either* $\sigma^{-1}(\frac{n+1}{2}) > \frac{n+1}{2}$ *or* $\sigma(\frac{n+1}{2}) > \frac{n+1}{2}$.

Then the following are equivalent.

(1) *σ is oscillating.*
(2) *One of (i) – (iv) is true and if n is odd and $n \geq 3$ then also (A) is true.*
(3) *Each one of (i) – (iv) is true and if n is odd and $n \geq 3$ then also (A) is true.*

Proof. First suppose that n is odd. If $n = 1$ then $\sigma = \mathrm{id}$ is oscillating and the implications (i) – (iv) are trivially satisfied.

Assume $n \geq 3$. We show for each of the implications (x) that (A) and (x) is true if and only if σ is oscillating. As n is odd and $n \geq 3$, Statement (A) can be expanded as

$$\begin{array}{ll} \text{either} & \sigma^{-1}(\frac{n+1}{2}) > \frac{n+1}{2} \text{ and } \sigma(\frac{n+1}{2}) < \frac{n+1}{2} \\ \text{or} & \sigma^{-1}(\frac{n+1}{2}) < \frac{n+1}{2} \text{ and } \sigma(\frac{n+1}{2}) > \frac{n+1}{2} \ . \end{array}$$

Moreover, (i) can be rephrased as $\sigma([\frac{n-1}{2}]) \subseteq [\frac{n+1}{2}, n]$. Hence, we have (A) and (i) if and only if

$$\begin{array}{lll} \text{either} & \sigma([\frac{n-1}{2}]) = [\frac{n+3}{2}, n] & (\text{if } \sigma^{-1}(\frac{n+1}{2}) > \frac{n+1}{2} \text{ and } \sigma(\frac{n+1}{2}) < \frac{n+1}{2}) \\ \text{or} & \sigma([\frac{n+1}{2}]) = [\frac{n+1}{2}, n] & (\text{if } \sigma^{-1}(\frac{n+1}{2}) < \frac{n+1}{2} \text{ and } \sigma(\frac{n+1}{2}) > \frac{n+1}{2}). \end{array}$$

In other words, $\sigma([m]) = [n - m + 1, n]$ for either $m = \frac{n-1}{2}$ or $m = \frac{n+1}{2}$, i.e. σ is oscillating.

Similarly, we have (A) and (iii) if and only if

$$\text{either} \quad \sigma([\tfrac{n+1}{2}, n]) = [\tfrac{n+1}{2}] \quad \text{or} \quad \sigma([\tfrac{n+3}{2}, n]) = [\tfrac{n-1}{2}].$$

That is, $\sigma([m + 1, n]) = [n - m]$ for either $m = \frac{n-1}{2}$ or $m = \frac{n+1}{2}$. This is equivalent to σ being oscillating by Lemma 4.3.4.

So far we have shown that

$$\text{(A) and (i)} \iff \sigma \text{ is oscillating} \iff \text{(A) and (iii)}. \tag{4.5}$$

By Lemma 4.3.5 we therefore also have

$$\text{(A) and (ii)} \iff \sigma \text{ is oscillating} \iff \text{(A) and (iv)}. \tag{4.6}$$

This finishes the proof for odd n.

Suppose now that n is even. Note that $\frac{n+1}{2} \notin [n]$ as it is not an integer. It is not hard to see that the equivalences from (4.5) and therefore those from (4.6) hold if we drop Statement (A). □

We continue with two consequences of Lemma 4.3.6. First, we infer the description of oscillating n-cycles mentioned at the beginning of the subsection.

Corollary 4.3.7. *Let $\sigma \in \mathfrak{S}_n$ be an n-cycle. We consider σ in cycle notation. Then σ is oscillating if and only if one of the following is true.*

(1) n is even and the entries of σ alternate between the sets $\left[\frac{n}{2}\right]$ and $\left[\frac{n}{2}+1, n\right]$.

(2) n is odd and after deleting the entry $\frac{n+1}{2}$ from σ, the remaining entries alternate between the sets $\left[\frac{n-1}{2}\right]$ and $\left[\frac{n+3}{2}, n\right]$.

Proof. With (A), (i) and (iii) we refer to the statements of Lemma 4.3.6.

Suppose that n is even. By Lemma 4.3.6, σ is oscillating if and only if the implications (i) and (iii) are satisfied which is the case if and only if the entries of σ alternate between $[\frac{n}{2}]$ and $[\frac{n}{2}+1, n]$.

Suppose that n is odd. If $n \geq 3$ then property (A) states that one of the neighbors $\sigma^{-1}(\frac{n+1}{2})$ and $\sigma(\frac{n+1}{2})$ of $\frac{n+1}{2}$ in σ is an element of $[\frac{n-1}{2}]$ and the other one is an element of $[\frac{n+3}{2}, n]$. Therefore, σ satisfies (A), (i) and (iii) if and only if after deleting $\frac{n+1}{2}$ from the cycle notation of σ, the remaining entries alternate between the sets $[\frac{n-1}{2}]$ and $[\frac{n+3}{2}, n]$. Thus, Lemma 4.3.6 yields that the latter property is satisfied if and only if σ is oscillating. □

By considering σ in cycle notation beginning with 1, we can rephrase Corollary 4.3.7 in a more formal way.

Corollary 4.3.8. *Let $\sigma \in \mathfrak{S}_n$ be an n-cycle. If n is odd, let $0 \leq l \leq n-1$ be such that $\sigma^l(1) = \frac{n+1}{2}$. If n is even, set $l := \infty$. Then σ is oscillating if and only if for all $0 \leq k \leq n-1$ we have*

$$\sigma^k(1) < \frac{n+1}{2} \quad \text{if } k < l \text{ and } k \text{ is even or } k > l \text{ and } k \text{ is odd,}$$
$$\sigma^k(1) > \frac{n+1}{2} \quad \text{if } k < l \text{ and } k \text{ is odd or } k > l \text{ and } k \text{ is even.}$$

We now consider the second property in the characterization of $\Sigma_{(n)}$: the property of *having connected intervals*. Roughly speaking, an n-cycle of $\mathfrak{S}_n$ has connected intervals if in its cycle notation for each $1 \leq k \leq \frac{n}{2}$ the elements of the interval $[k, n-k+1]$ are grouped together.

Definition 4.3.9. *(1) Let $\sigma \in \mathfrak{S}_n$ and $M \subseteq [n]$. We call M* connected *in σ if there is an $m \in M$ such that*

$$M = \left\{m, \sigma(m), \sigma^2(m), \ldots, \sigma^{|M|-1}(m)\right\}.$$

(2) *Let $\sigma \in \mathfrak{S}_n$ be an n-cycle. We say that σ has* connected intervals *if the interval $[k, n-k+1]$ is connected in σ for all integers k with $1 \leq k \leq \frac{n}{2}$.*

Example 4.3.10. All elements shown in Table 4.1 have connected intervals. In particular, the element in stair form $\sigma_{(6)} = (1, 6, 2, 5, 3, 4)$ has connected intervals. In contrast, in $(1, 5, 2, 6, 3, 4)$ the set $[2, 5]$ is not connected.

The main result of this subsection is that an n-cycle $\sigma \in \mathfrak{S}_n$ is an element of $\Sigma_{(n)}$ if and only if σ is oscillating and has connected intervals. We now begin working towards this result.

Lemma 4.3.11. *The element in stair form $\sigma_{(n)} \in \mathfrak{S}_n$ is oscillating and has connected intervals.*

Proof. By Definition 4.2.13,

$$\sigma_{(n)} = \begin{cases} (1, n, 2, n-1, \ldots, \frac{n}{2}, n - \frac{n}{2} + 1) & \text{if } n \text{ is even} \\ (1, n, 2, n-1, \ldots, \frac{n-1}{2}, n - \frac{n-1}{2} + 1, \frac{n+1}{2}) & \text{if } n \text{ is odd.} \end{cases}$$

Thus, $\sigma_{(n)}([\frac{n}{2}]) = [\frac{n}{2} + 1, n]$ if n is even and $\sigma_{(n)}([\frac{n-1}{2}]) = [\frac{n+3}{2}, n]$ if n is odd. That is, $\sigma_{(n)}$ is oscillating.

For all $k \in \mathbb{N}$ with $1 \leq k \leq \frac{n}{2}$ the rightmost $|[k, n-k+1]|$ elements in the cycle of $\sigma_{(n)}$ from above form $[k, n-k+1]$. Thus, $\sigma_{(n)}$ has connected intervals. $\square$

Let $\sigma \in \mathfrak{S}_n$. Sometimes it will be convenient to consider σ^{w_0} instead of σ. We will now show that conjugation with the longest element w_0 of $\mathfrak{S}_n$ preserves the properties of being oscillating and having connected intervals.

Lemma 4.3.12. *Let $\sigma \in \mathfrak{S}_n$ be an n-cycle.*

(1) *If σ is oscillating then σ^{w_0} is oscillating.*

(2) *If σ has connected intervals then σ^{w_0} has connected intervals.*

Proof. If $n = 1$ the result is trivial. Thus suppose $n \geq 2$.

(1) Set $M := \mathbb{N} \cap \left\{\frac{n-1}{2}, \frac{n}{2}, \frac{n+1}{2}\right\}$ and assume that σ is oscillating. Then there is an $m \in M$ such that $\sigma([m]) = [n-m+1, n]$ and from Lemma 4.3.4 it follows that $\sigma([m+1, n]) = [n-m]$. Using $w_0(i) = n-i+1$ for $i \in [n]$, we obtain

$$\begin{aligned} \sigma^{w_0}([n-m]) &= w_0 \sigma w_0([n-m]) \\ &= w_0 \sigma([m+1, n]) \\ &= w_0([n-m]) \\ &= [n-(n-m)+1, n]. \end{aligned}$$

As $n - m \in M$, it follows that σ^{w_0} is oscillating.

(2) Let $I := [k, n-k+1]$ be given by an integer k with $1 \leq k \leq \frac{n}{2}$. Then $w_0(I) = I$. Hence, if I is connected in σ then it is also connected in σ^{w_0}. $\square$

In the following result we study the interplay between the conjugation with w_0 and the relation $\approx$. The generalization to all finite Coxeter groups is straight forward.

Lemma 4.3.13. *Let $w, w' \in \mathfrak{S}_n$ and ν be the automorphism of $\mathfrak{S}_n$ given by $x \mapsto x^{w_0}$.*

(1) If $w \xrightarrow{s_i} w'$ then $\nu(w) \xrightarrow{s_{n-i}} \nu(w')$.

(2) If $w \approx w'$ then $\nu(w) \approx \nu(w')$.

Proof. Assume $w \xrightarrow{s_i} w'$. Then $w' = s_i w s_i$ and $\ell(w') \leq \ell(w)$. Since $\nu(s_i) = s_{n-i}$, we have $\nu(w') = s_{n-i}\nu(w)s_{n-i}$. Moreover, $\ell(\nu(w')) \leq \ell(\nu(w))$ because $\ell(x) = \ell(\nu(x))$ for all $x \in \mathfrak{S}_n$. Thus, $\nu(w) \xrightarrow{s_{n-i}} \nu(w')$. Now, use the definition of $\approx$ to obtain (2) from (1). $\square$

Consider $n = 5$, the oscillating n-cycle $\sigma = (\mathbf{1}, 4, 2, 3, \mathbf{5})$ and its connected interval $I = \{2, 3, 4\}$. In the cycle notation of σ, this interval is enclosed by the two elements $a = 1$ and $b = 5$. Note that $\frac{n+1}{2} = 3$, $a < 3$ and $b > 3$. This illustrates a property of oscillating n-cycles addressed by the next lemma.

Lemma 4.3.14. *Assume that $\sigma \in \mathfrak{S}_n$ is an oscillating n-cycle with a connected interval $I := [i, n-i+1]$ such that $i \in \mathbb{N}$ and $2 \leq i \leq \frac{n+1}{2}$. Let $r := |I|$ and $m \in I$ be such that $I = \left\{\sigma^k(m) \mid k = 0, \dots, r-1\right\}$. Moreover, set $a := \sigma^{-1}(m)$ and $b := \sigma^r(m)$. Then $a, b \neq \frac{n+1}{2}$ and*

$$a < \frac{n+1}{2} \iff b > \frac{n+1}{2}.$$

Proof. Let $p \in [n-1]$ be such that $\sigma^p(1) = a$. Then $\sigma^{p+r+1}(1) = b$. Since $i > 1$, $1 \notin I$ and thus $p + r + 1 \leq n - 1$. We have $r = n - 2i + 2$. Hence, r has the same parity as n.

We want to apply Corollary 4.3.8. If n is odd, let $l \in [0, n-1]$ be such that $\sigma^l(1) = \frac{n+1}{2}$. Then $\frac{n+1}{2} \in I$ so that $p < l < p + r + 1$. In particular, $a, b \neq \frac{n+1}{2}$. Clearly, if n is even then $a, b \neq \frac{n+1}{2}$.

Therefore,

$$\begin{aligned} a = \sigma^p(1) < \frac{n+1}{2} &\iff p \text{ is even} \\ &\iff \begin{cases} p + r + 1 \text{ is odd} & \text{if } n \text{ even} \\ p + r + 1 \text{ is even} & \text{if } n \text{ odd} \end{cases} \\ &\iff b = \sigma^{p+r+1}(1) > \frac{n+1}{2}. \end{aligned}$$

where we use Corollary 4.3.8 (and $p < l < p + r + 1$ if n is odd) for the first and third equivalence. $\square$

Since the $\to$ relation is the transitive closure of the $\xrightarrow{s_i}$ relations, we are interested in the circumstances under which the conjugation with s_i preserves the property of being oscillating with connected intervals.

Lemma 4.3.15. *Let $\sigma \in \mathfrak{S}_n$ be an oscillating n-cycle with connected intervals, $i \in [n-1]$ with $i \leq \frac{n+1}{2}$ and $\sigma' := s_i \sigma s_i$. Then σ' is oscillating and has connected intervals if and only if*

(1) *if $i = \frac{n}{2}$ then $n = 2$,*
(2) *if $i = \frac{n-1}{2}$ or $i = \frac{n+1}{2}$ then $\sigma(i) = i+1$ or $\sigma^{-1}(i) = i+1$,*
(3) *if $i < \frac{n-1}{2}$ then*

$$\sigma(i) \in I \text{ and } \sigma(i+1) \notin I \text{ or } \sigma^{-1}(i) \in I \text{ and } \sigma^{-1}(i+1) \notin I$$

where $I := [i+1, n-i]$.

Proof. We will use Lemma 4.3.6 without further reference. Note that $\sigma' = s_i \sigma s_i$ means that we obtain σ' from σ by interchanging i and $i+1$ in cycle notation. We show the equivalence case by case, depending on i.

Case 1. Suppose $i = \frac{n}{2}$. In this case n is even. If $n = 2$ then $(1,2)$ is the only 2-cycle in $\mathfrak{S}_n$. Thus, $\sigma = \sigma' = (1,2)$. This element is oscillating and has connected intervals.

Assume now that $n > 2$. Since σ is oscillating,

$$\sigma(i) > \frac{n}{2} \text{ and } \sigma^{-1}(i) > \frac{n}{2}.$$

Moreover as $n > 2$, at most one of $\sigma(i)$ and $\sigma^{-1}(i)$ equals $i+1$. Since we obtain σ' from σ by swapping i and $i+1$ in cycle notation we infer

$$\sigma'(i+1) > \frac{n}{2} \text{ or } \sigma'^{-1}(i+1) > \frac{n}{2}.$$

As $i+1 > \frac{n}{2}$, this means that σ' is not oscillating

Case 2. Suppose $i = \frac{n-1}{2}$ or $i = \frac{n+1}{2}$. In this case n is odd and $n \geq 3$. Moreover, $i, i+1 \in [k, n-k+1]$ for $k = 1, \ldots, \frac{n-1}{2}$. Hence, each of the intervals remains connected if we interchange i and $i+1$. Therefore, σ' has connected intervals. It remains to determine in which cases σ' oscillates. We do this for $i = \frac{n-1}{2}$. The proof for $i = \frac{n+1}{2}$ is similar.

For $i = \frac{n-1}{2}$ we have $i+1 = \frac{n+1}{2}$. Since σ is oscillating,

$$\sigma(i) \geq \frac{n+1}{2} \text{ and } \sigma^{-1}(i) \geq \frac{n+1}{2}.$$

Because $n \geq 3$, there is at most one equality among these two inequalities. Assume that there is no equality at all. Then

$$\sigma'\left(\frac{n+1}{2}\right) > \frac{n+1}{2} \text{ and } \sigma'^{-1}\left(\frac{n+1}{2}\right) > \frac{n+1}{2}$$

since $\sigma' = s_i \sigma s_i$. Hence, σ' is not oscillating.

Conversely, assume that $\sigma(i) = i+1$ or $\sigma^{-1}(i) = i+1$. In other words, there exists an $\varepsilon \in \{-1, 1\}$ such that $\sigma^{\varepsilon}(i) = i+1$. Since $i+1 = \frac{n+1}{2}$ and σ is oscillating, we then have $a := \sigma^{-\varepsilon}(i) > \frac{n+1}{2}$. Moreover, $\sigma^{-\varepsilon}(i+1) = i < \frac{n+1}{2}$. Thus σ being oscillating implies that $b := \sigma^\varepsilon(i+1) > \frac{n+1}{2}$. By definition of a and b,

$$\sigma^\varepsilon = (a, i, i+1, b, \dots).$$

As a consequence,

$$\sigma'^\varepsilon = (a, i+1, i, b, \dots)$$

and σ^ε and σ'^ε coincide on the part represented by the dots because $\sigma' = s_i \sigma s_i$. From $a > \frac{n+1}{2}$, $i+1 = \frac{n+1}{2}$, $i < \frac{n+1}{2}$ and $b > \frac{n+1}{2}$ it now follows that σ' is oscillating.

Case 3. Suppose $i < \frac{n-1}{2}$. Note that then $n \geq 4$. Define $I := [i+1, n-i]$ as in the theorem and set $r := |I|$. Since $i+1 < \frac{n+1}{2}$, we have $r > 1$. We show the implication from left to right first. Assume that σ' is oscillating and has connected intervals. Note that

$$\tau^\varepsilon(j) \neq i, i+1 \text{ for all } \tau \in \{\sigma, \sigma'\}, \varepsilon \in \{-1, 1\} \text{ and } j \in \{i, i+1\}$$

since σ and σ' are oscillating and $i, i+1 < \frac{n+1}{2}$. Because I is connected in σ', $i+1 \in I$ and $r > 1$, we have that

$$\exists \varepsilon \in \{-1, 1\} \text{ such that } \sigma'^\varepsilon(i+1) \in I.$$

Therefore,

$$\exists \varepsilon \in \{-1, 1\} \text{ such that } \sigma^\varepsilon(i) \in I$$

as $\sigma' = s_i \sigma s_i$ and $\sigma'^\varepsilon(i+1) \neq i, i+1$. In fact, the statement

$$\exists \varepsilon \in \{-1, 1\} \text{ such that } \sigma^\varepsilon(i) \in I \text{ and } \sigma^{-\varepsilon}(i) \notin I \tag{4.7}$$

is true since otherwise we would have

$$\sigma = (n+i-1, \dots, \sigma^{-1}(i), i, \sigma(i), \dots)$$

with $\sigma^{-1}(i), \sigma(i) \in I$ and $i, n+i-1 \notin I$ in which case I would not be connected in σ.

By interchanging the roles played by σ and σ' in the argumentation leading to (4.7), we get that

$$\exists \varepsilon \in \{-1, 1\} \text{ such that } \sigma'^\varepsilon(i) \in I \text{ and } \sigma'^{-\varepsilon}(i) \notin I.$$

From this we obtain that

$$\exists \varepsilon \in \{-1, 1\} \text{ such that } \sigma^\varepsilon(i+1) \in I \text{ and } \sigma^{-\varepsilon}(i+1) \notin I \tag{4.8}$$

by swapping i and $i+1$ in cycle notation and using that $\sigma'(i), \sigma'^{-1}(i) \neq i, i+1$.

Now, let $\varepsilon \in \{-1,1\}$ be such that $\sigma^{\varepsilon}(i) \in I$ and $\sigma^{-\varepsilon}(i) \notin I$. Then

$$I = \left\{\sigma^{\varepsilon k}(i) \mid k = 1, \ldots, r\right\} \tag{4.9}$$

since I is connected in σ and $i \notin I$. From (4.8) it follows that $i+1$ appears at the border of I in the cycle notation of σ. Hence, (4.9) implies that

$$\sigma^{\varepsilon}(i) = i+1 \text{ or } \sigma^{\varepsilon r}(i) = i+1.$$

As $\sigma^{\varepsilon}(i) \neq i+1$, it follows that $i+1 = \sigma^{\varepsilon r}(i)$. Thus, (4.9) yields that $\sigma^{-\varepsilon}(i+1) \in I$ and $\sigma^{\varepsilon}(i+1) \notin I$. Therefore, we have $\sigma^{\varepsilon}(i) \in I$ and $\sigma^{\varepsilon}(i+1) \notin I$ for an $\varepsilon \in \{-1,1\}$ as desired.

Lastly, we prove the direction from right to left of the equivalence. We are still in the case $i < \frac{n-1}{2}$. Thus, assume that there is an $\varepsilon \in \{-1,1\}$ such that $\sigma^{\varepsilon}(i) \in I$ and $\sigma^{\varepsilon}(i+1) \notin I$. Since σ is oscillating and we interchange two elements $i, i+1 < \frac{n+1}{2}$ in σ in order to obtain σ' from σ, σ' is also oscillating.

It remains to show that σ' has connected intervals. Since $i \notin I$, $\sigma^{\varepsilon}(i) \in I$ and I is connected in σ, we have (4.9). Moreover, from $i+1 \in I$, $\sigma^{\varepsilon}(i+1) \notin I$ and I being connected in σ, it follows that $\sigma^{\varepsilon r}(i) = i+1$. Thus,

$$I = \left\{\sigma'^{\varepsilon k}(i+1) \mid k = 0, \ldots, r-1\right\}$$

because $\sigma' = s_i \sigma s_i$. That is, I is connected in σ'. Let $J := [k, n-k+1]$ for $k \in \mathbb{N}$ with $1 \leq k \leq \frac{n}{2}$ and $k \neq i+1$ be an interval different from I. Then either $i, i+1 \in J$ or $i, i+1 \notin J$. As J is connected in σ and $\sigma' = s_i \sigma s_i$, it follows that J is connected in σ'. Therefore, σ' has connected intervals. $\square$

Example 4.3.16. Consider $\sigma = \sigma_{(6)} = (1,6,2,5,3,4)$ and $\sigma_i := s_i \sigma s_i$ for $i = 1,2$. Then σ is oscillating with connected intervals.

Since $\sigma^{-1}(1) \in [2,5]$ and $\sigma^{-1}(2) \notin [2,5]$, Lemma 4.3.15 yields that σ_1 is oscillating with connected intervals. In contrast, σ_2 is not oscillating with connected intervals because of $\sigma(2), \sigma^{-1}(2) \notin [3,4]$ and Lemma 4.3.15. This can also be checked directly. We have

$$\sigma_1 = (1,5,3,4,2,6) \quad \text{and} \quad \sigma_2 = (1,6,3,5,2,4).$$

For instance, $[3,4]$ is not connected in σ_2.

In the next result we show that the relation $\approx$ is compatible with the concept of oscillating n-cycles with connected intervals.

Lemma 4.3.17. *Let $\sigma \in \mathfrak{S}_n$ be an oscillating n-cycle with connected intervals, $i \in [n-1]$ and $\sigma' := s_i \sigma s_i$. If $\sigma \approx \sigma'$ then σ' is oscillating and has connected intervals.*

Proof. We do a case analysis depending on i.

Case 1. Suppose $i = \frac{n}{2}$. Then n is even. By Lemma 4.3.15, σ' is oscillating with connected intervals if and only if $n = 2$. Thus, we have to show that $\sigma \not\approx \sigma'$ if $n \geq 4$. In this case we have $\sigma(i), \sigma^{-1}(i) > \frac{n}{2}$ and $\sigma(i+1), \sigma^{-1}(i+1) \leq \frac{n}{2}$ because σ is oscillating. But then Lemma 4.2.22 yields $\ell(\sigma') < \ell(\sigma)$ so that $\sigma' \not\approx \sigma$.

Case 2. Suppose $i = \frac{n-1}{2}$ or $i = \frac{n+1}{2}$. We only do the case $i = \frac{n-1}{2}$. The other one is similar. Let $I := [i, n-i+1] = \{i, i+1, i+2\}$. We show the contraposition and assume that σ' is not oscillating or that it does not have connected intervals. Then from Lemma 4.3.15 it follows that $\sigma(i) \neq i+1$ and $\sigma^{-1}(i) \neq i+1$. Furthermore, there is an $m \in I$ such that

$$I = \left\{\sigma^{-1}(m), m, \sigma(m)\right\}$$

since I is connected in σ. Thus, $m = i+2$. Assume $\sigma^{-1}(i+2) = i$ and $\sigma(i+2) = i$ (the proof of the other case with $\sigma(i+2) = i$ is analogous). Then $\sigma^{-1}(i) > i+2$ as σ is oscillating and $\sigma^{-1}(i) \neq i+1, i+2$. Moreover, Lemma 4.3.14 applied to I in σ and $\sigma^{-1}(i) > \frac{n+1}{2}$ yields $\sigma(i+1) < \frac{n+1}{2} = i+1$. Therefore,

$$\sigma(i) = i+2 > \sigma(i+1) \quad \text{and} \quad \sigma^{-1}(i) > i+2 = \sigma^{-1}(i+1)$$

so that $\ell(\sigma') < \ell(\sigma)$ by Lemma 4.2.22 and hence $\sigma' \not\approx \sigma$.

Case 3. Suppose $i < \frac{n-1}{2}$. Then for all $j \in \{i, i+1\}$ we have $\sigma(j), \sigma^{-1}(j) \geq \frac{n+1}{2}$ since $j < \frac{n+1}{2}$ and σ is oscillating. We assume $\sigma \approx \sigma'$ and show that σ' is oscillating and has connected intervals. Define $I_k := [k, n-k+1]$ for all $k \leq \frac{n+1}{2}$ and $I := I_{i+1} = [i+1, n-i]$. Thanks to Lemma 4.3.15 it suffices to show

$$\sigma(i) \in I \text{ and } \sigma(i+1) \notin I \text{ or } \sigma^{-1}(i) \in I \text{ and } \sigma^{-1}(i+1) \notin I.$$

Since $\sigma \approx \sigma'$, $\ell(\sigma) = \ell(\sigma')$. Hence, Lemma 4.2.22 implies that either $\sigma(i) < \sigma(i+1)$ or $\sigma^{-1}(i) < \sigma^{-1}(i+1)$. We assume $\sigma(i) < \sigma(i+1)$ and $\sigma^{-1}(i) > \sigma^{-1}(i+1)$. The other case is similar.

First we show $\sigma(i) \in I$. Assume $\sigma(i) \notin I$ instead. Then $\sigma(i) \geq \frac{n+1}{2}$ implies $\sigma(i) > \max I$. Now we use that $\sigma(i) < \sigma(i+1)$ to obtain $\sigma(i+1) \notin I$. From this it follows that

$$I = \left\{\sigma^{-k}(i+1) \mid k = 0, \ldots, r-1\right\}$$

where $r := |I|$ since I is connected in σ and $i+1 \in I$. Now we consider the interval $I_i = [i, n-i+1]$ in σ. Because σ is oscillating, $\sigma(i+1) > \frac{n+1}{2}$. An application of Lemma 4.3.14 to I in σ yields $\sigma^{-r}(i+1) < \frac{n+1}{2}$. In particular, $\sigma^{-r}(i+1) \neq n-i+1$. But we also have $i \neq \sigma^{-r}(i+1)$ because $\sigma(i) \notin I$. That is $\sigma^{-r}(i+1) \notin I_i$. As a consequence,

$$I_i = \left\{\sigma^{-k}(i+1) \mid k = 0, \ldots, r-1\right\} \cup \left\{\sigma(i+1), \sigma^2(i+1)\right\}$$

since $I \subseteq I_i$ and I_i is connected in σ. Hence

$$\left\{\sigma(i+1), \sigma^2(i+1)\right\} = \{i, n-i+1\}.$$

As $\sigma(i+1) > \frac{n+1}{2}$, it follows that $\sigma(i+1) = n-i+1$ and $\sigma^2(i+1) = i$. Consequently,

$$\sigma(i) > \max I_i = n-i+1 = \sigma(i+1).$$

This is a contradiction to $\sigma(i) < \sigma(i+1)$ and shows that $\sigma(i) \in I$.

It remains to show that $\sigma(i+1) \notin I$. Because $i \notin I$, $\sigma(i) \in I$ and I is connected,

$$I = \left\{ \sigma^k(i) \mid k = 1, \dots, r \right\}.$$

We can apply Lemma 4.3.14 to I in σ and $i < \frac{n+1}{2}$ to obtain $\sigma^{r+1}(i) > \frac{n+1}{2}$. Thus $\sigma^r(i) \leq \frac{n+1}{2}$. In particular, $\sigma^r(i) \neq n-i$.

If $i = \frac{n}{2} - 1$ then $I = \{i+1, n-i\}$ and it follows that $\sigma(i) = n-i$ and $\sigma^2(i) = i+1$. That is, $\sigma(i+1) \notin I$ as desired.

Now suppose $i < \frac{n}{2} - 1$. Then $i+2 \leq \frac{n+1}{2}$ and we consider $I_{i+2} = [i+2, n-i-1]$. Assume for the sake of contradiction that $\sigma(i+1) \in I$. This means that $\sigma^r(i) \neq i+1$. In addition, we have already seen that $\sigma^r(i) \neq n-i$. Therefore, $\sigma^r(i) \in I_{i+2}$. Since I_{i+2} is connected in σ and $I_{i+2} \subseteq I$, we have

$$I_{i+2} = \left\{ \sigma^k(i) \mid k = 3, \dots, r \right\}.$$

and hence $\{\sigma(i), \sigma^2(i)\} = \{i+1, n-i\}$. As $i < \frac{n+1}{2}$, it follows that $\sigma(i) = n-i$ and $\sigma^2(i) = i+1$. But then

$$\sigma(i) = n-i > n-i-1 = \max I_{i+2} \geq \sigma(i+1)$$

which again contradicts the assumption $\sigma(i) < \sigma(i+1)$ and thus shows that $\sigma(i+1) \notin I$.

Case 4. Suppose $i > \frac{n+1}{2}$. Assume $\sigma \approx \sigma'$ and let $\nu\colon \mathfrak{S}_n \to \mathfrak{S}_n, x \mapsto x^{w_0}$, $\tau := \nu(\sigma)$ and $\tau' := \nu(\sigma')$. Since σ is oscillating and has connected intervals, Lemma 4.3.12 implies that τ is oscillating and has connected intervals. In addition, from Lemma 4.3.13 we have $\tau \approx \tau'$. Because $\tau' = s_{n-i} \tau s_{n-i}$ with $n-i < \frac{n+1}{2}$, we now obtain from the already proven cases that τ' is oscillating and has connected intervals. Hence, $\sigma' = \nu(\tau')$ and Lemma 4.3.12 yield that σ' is oscillating with connected intervals. $\square$

In order to show that each oscillating n-cycle with connected intervals is $\approx$-equivalent to $\sigma_{(n)}$, we use an algorithm that takes an oscillating n-cycle $\sigma \in \mathfrak{S}_n$ with connected intervals as input and successively conjugates σ with simple reflections until we obtain $\sigma_{(n)}$. This algorithm has the property that all permutations appearing as interim results are oscillating with connected intervals and $\approx$-equivalent to σ. Eventually, it follows that $\sigma \approx \sigma_{(n)}$.

The mechanism of the algorithm is due to Kim [Kim98]. She used it in order to show that for each $\alpha \vDash_e n$ the element in stair form σ_α has maximal length in its conjugacy class. The next lemma corresponds to one step of the algorithm.

Lemma 4.3.18. *Let $\alpha = (n)$ and $\sigma \in \mathfrak{S}_n$ be an oscillating n-cycle with connected intervals which is different from the element in stair form σ_α. Then there exists a* *minimal integer p such that $1 \leq p \leq n-1$ and $\sigma^p(1) \neq \sigma_\alpha^p(1)$. Set $a := \sigma^p(1)$, $b := \sigma_\alpha^p(1)$ and*

$$\sigma' := \begin{cases} s_{a-1}\sigma s_{a-1} & \text{if } a > b \\ s_a \sigma s_a & \text{if } a < b. \end{cases}$$

Then $\sigma' \approx \sigma$ and σ' is oscillating and has connected intervals.

Proof. Set $I_k := [k, n-k+1]$ for all $k \in \mathbb{N}$ with $k \leq \frac{n+1}{2}$. Because $\sigma \neq \sigma_\alpha$ and both permutations are n-cycles, we have $p \leq n-2$. Recall that by Definition 4.2.13,

$$\sigma_\alpha = \begin{cases} (1, n, 2, n-1, \ldots, \frac{n}{2}, \frac{n}{2}+1) & \text{if } n \text{ is even} \\ (1, n, 2, n-1, \ldots, \frac{n-1}{2}, \frac{n+3}{2}, \frac{n+1}{2}) & \text{if } n \text{ is odd.} \end{cases}$$

If n is odd then $\frac{n+1}{2} = \sigma^{n-1}(1)$ and hence $p \leq n-2$ implies $b \neq \frac{n+1}{2}$. If n is even then $b \neq \frac{n+1}{2}$ anyway.

We assume $b < \frac{n+1}{2}$. The proof in the case $b > \frac{n+1}{2}$ is similar and therefore omitted. By the choice of p, we have $b \neq 1$ so that $1 < b < \frac{n+1}{2}$. The definition of σ_α implies

$$\begin{aligned} \left\{\sigma_\alpha^k(1) \mid k = 0, \ldots, p-1\right\} &= [n] \setminus I_b, \\ \left\{\sigma_\alpha^k(1) \mid k = p, \ldots, n-1\right\} &= I_b. \end{aligned} \tag{4.10}$$

Again by the choice of p, the same equalities hold for σ. Hence, $b < a$ as $a \in I_b$ and $b = \min I_b$. Therefore, we consider $\sigma' = s_{a-1}\sigma s_{a-1}$ and show that $\sigma \approx \sigma'$. Then Lemma 4.3.17 implies that σ' also is oscillating and has connected intervals.

It follows from the definition of σ_α and $b < \frac{n+1}{2}$ that

$$\sigma^{-1}(a) = \sigma_\alpha^{-1}(b) = n - b + 2 > \frac{n+1}{2}. \tag{4.11}$$

As σ is oscillating, we obtain that $a \leq \frac{n+1}{2}$ from Lemma 4.3.6. Since (4.10) holds for σ and $p > 0$,

$$\sigma^{-1}(a) \notin I_b \supseteq I_{a-1} \supseteq I_a.$$

Let $r := |I_a|$. Because I_a is connected in σ, $a \in I_a$ and $\sigma^{-1}(a) \notin I_a$, we have

$$\left\{\sigma^k(a) \mid k = 0, \ldots, r-1\right\} = I_a.$$

Now we can use that $I_{a-1} = I_a \cup \{a-1, n-a+2\}$ is connected in σ and that $\sigma^{-1}(a) \notin I_{a-1}$ to obtain

$$\left\{\sigma^k(a) \mid k = 0, \ldots, r+1\right\} = I_{a-1}$$

The descriptions of I_a and I_{a-1} imply that

$$\left\{\sigma^r(a), \sigma^{r+1}(a)\right\} = \{a-1, n-a+2\}\,.$$

Lemma 4.3.14 applied to I_a in σ and $\sigma^{-1}(a) > \frac{n+1}{2}$ now imply that $\sigma^r(a) < \frac{n+1}{2}$. Thus, $\sigma^r(a) = a-1$ and $\sigma^{r+1}(a) = n-a+2$. That is,

$$\sigma(a-1) = n-a+2 \tag{4.12}$$

Moreover, $\sigma^{-1}(a-1) \in I_a$ implies

$$\sigma^{-1}(a-1) \leq n-a+1. \tag{4.13}$$

We now show

$$\sigma(a) \leq n-a+1. \tag{4.14}$$

and deal with two cases. If $a = \frac{n+1}{2}$ then $n-a+1 = a$. Furthermore, we then have $r = 1$ and therefore $\sigma(a) = a-1 < n-a+1$. If $a < \frac{n+1}{2}$ then $r > 1$ so that $\sigma(a) \in I_a$ and thus $\sigma(a) \leq n-a+1$ as desired.

From (4.11) and (4.13) it follows that

$$\sigma^{-1}(a-1) \leq n-a+1 < n-b+2 = \sigma^{-1}(a).$$

Moreover, (4.12) and (4.14) imply

$$\sigma(a-1) = n-a+2 > n-a+1 \geq \sigma(a).$$

Since $\sigma' = s_{a-1}\sigma s_{a-1}$, Lemma 4.2.22 now yields $\ell(\sigma') = \ell(\sigma)$. Hence, $\sigma' \approx \sigma$ by Lemma 4.2.20. □

Example 4.3.19. Let $n = 5$ and $\alpha = (n)$. The n-cycle $\sigma = (1,3,4,2,5) \in \mathfrak{S}_n$ is oscillating and has connected intervals. We can successively use Lemma 4.3.18 in order to obtain the sequence

$$\begin{aligned}
\sigma = \sigma^{(0)} &= (1,3,4,2,5),\\
\sigma^{(1)} &= (1,4,3,2,5) = s_3\sigma^{(0)}s_3,\\
\sigma^{(2)} &= (1,5,3,2,4) = s_4\sigma^{(1)}s_4,\\
\sigma^{(3)} &= (1,5,2,3,4) = s_2\sigma^{(2)}s_2,\\
\sigma_\alpha = \sigma^{(4)} &= (1,5,2,4,3) = s_3\sigma^{(3)}s_3.
\end{aligned}$$

Moreover, Lemma 4.3.18 ensures that each $\sigma^{(j)}$ is oscillating with connected intervals and all $\sigma^{(j)}$ are $\approx$-equivalent. Therefore, $\sigma \in \Sigma_\alpha$ by Proposition 4.2.14.

We now come to the characterization of $\Sigma_{(n)}$.

$$
\begin{tikzcd}
(1,3,4,2,5) \arrow[dr, shift left, "\mathrm{del}_3"] & (1,4,3,2,5) \arrow[d, shift left, "\mathrm{del}_3"] & (1,4,2,3,5) \arrow[dl, shift left, "\mathrm{del}_3"] \\
 & (1,3,2,4) \arrow[ul, shift left, "\mathrm{ins}_{3,1}"] \arrow[u, shift left, "\mathrm{ins}_{3,2}"] \arrow[ur, shift left, "\mathrm{ins}_{3,3}"] \arrow[d, shift left, "\mathrm{del}_3"] & \\
 & (1,2,3) \arrow[u, shift left, "\mathrm{ins}_{3,1}"] &
\end{tikzcd}
$$

Figure 4.8: Examples for the operators del_k and $\mathrm{ins}_{k,p}$ appearing in Theorem 4.3.21 and its proof. The lower part of the picture serves as an example for the operators used in the case when n is even. The upper part is an example for those used in the case when n is odd. Note that for the integer m from the theorem we have $m = \frac{n}{2} + 1 = 3$ if $n = 4$ and $m = \frac{n+1}{2} = 3$ if $n = 5$.

Theorem 4.3.20. *Let $\sigma \in \mathfrak{S}_n$ be an n-cycle. Then $\sigma \in \Sigma_{(n)}$ if and only if σ is oscillating and has connected intervals.*

Proof. Let $\sigma \in \mathfrak{S}_n$ be an n-cycle. Recall that $\sigma \in \Sigma_{(n)}$ if and only if $\sigma \approx \sigma_{(n)}$ by Proposition 4.2.14. Assume that $\sigma \in \Sigma_{(n)}$. Then $\sigma \approx \sigma_{(n)}$ which by definition of $\approx$ implies that there are sequences $\sigma_\alpha = \sigma^{(0)}, \sigma^{(1)}, \dots, \sigma^{(m)} = \sigma \in \mathfrak{S}_n$ and $i_1, \dots, i_m \in [n-1]$ such that $\sigma^{(j-1)} \approx \sigma^{(j)}$ and $\sigma^{(j)} = s_{i_j}\sigma^{(j-1)}s_{i_j}$ for $j \in [m]$. From Lemma 4.3.11 we have that $\sigma_{(n)}$ is oscillating and has connected intervals. Moreover, Lemma 4.3.17 yields that $\sigma^{(j)}$ is oscillating with connected intervals if $\sigma^{(j-1)}$ is oscillating with connected intervals. Hence, σ is oscillating and has connected intervals by induction.

Conversely, assume that σ is oscillating and has connected intervals. Then we can use Lemma 4.3.18 iteratively to obtain a sequence of $\approx$-equivalent n-cycles starting with σ and eventually ending with σ_α. Thus $\sigma \approx \sigma_\alpha$. $\square$

The goal of the remainder of this subsection is to find bijections that relate $\Sigma_{(n-1)}$ to $\Sigma_{(n)}$. From this we will obtain a recursive description of $\Sigma_{(n)}$ and a formula for the cardinality of $\Sigma_{(n)}$. To achieve our goal, we define two operators ins and del.

Assume that the n-cycle $\sigma \in \mathfrak{S}_n$ is given in cycle notation starting with 1. Then for $k \in [2, n+1]$ $\mathrm{ins}_{k,p}(\sigma) \in \mathfrak{S}_{n+1}$ is the $(n+1)$-cycle obtained from σ by adding 1 to each element greater or equal to k in σ and then inserting k behind the pth element in the resulting cycle. Likewise, for $k \in [2, n]$, $\mathrm{del}_k(\sigma) \in \mathfrak{S}_{n-1}$ is the $(n-1)$-cycle obtained by first deleting k from σ and then decreasing each element greater than k by 1. See Figure 4.8 for examples.

We now define ins and del more formally. Let $\sigma \in \mathfrak{S}_n$ be an n-cycle and $k \in \mathbb{N}$. Set

$$
\varepsilon_r := \begin{cases} 0 & \text{if } \sigma^r(1) < k \\ 1 & \text{if } \sigma^r(1) \geq k \end{cases}
$$

for $r = 0, \dots, n-1$. In the following we will assume $k > 1$. The operators could also be defined for $k = 1$ but this is not necessary for our purposes and would only make the exposition less transparent.

For $k \in [2, n+1]$ and $p \in [n]$, define $\mathrm{ins}_{k,p}(\sigma)$ to be the $(n+1)$-cycle of $\mathfrak{S}_{n+1}$ given by

$$\mathrm{ins}_{k,p}(\sigma)^r(1) := \begin{cases} \sigma^r(1) + \varepsilon_r & \text{if } r < p \\ k & \text{if } r = p \\ \sigma^{r-1}(1) + \varepsilon_{r-1} & \text{if } r > p \end{cases}$$

for $r = 0, \dots, n$. For $k \in [2, n]$, define $\mathrm{del}_k(\sigma)$ to be the $(n-1)$-cycle of $\mathfrak{S}_{n-1}$ given by

$$\mathrm{del}_k(\sigma)^r(1) := \begin{cases} \sigma^r(1) - \varepsilon_r & \text{if } r < p \\ \sigma^{r+1}(1) - \varepsilon_{r+1} & \text{if } r \geq p \end{cases}$$

for $r = 0, \dots, n-2$ where p is the element of $[0, n-1]$ with $\sigma^p(1) = k$.

The next results relates $\Sigma_{(n)}$ with $\Sigma_{(n-1)}$ via a bijection for $n \geq 4$.

Theorem 4.3.21. *Suppose $n \geq 4$. If n is even then set $m := \frac{n}{2} + 1$ and*

$$\psi\colon \Sigma_{(n-1)} \to \Sigma_{(n)}, \quad \sigma \mapsto \mathrm{ins}_{m,p}(\sigma)$$

where p is the element of $[n-1]$ with $\sigma^{p-1}(1) = \min\left\{\sigma^{-1}(\frac{n}{2}), \frac{n}{2}\right\}$. If n is odd then set $m := \frac{n+1}{2}$ and

$$\psi\colon \Sigma_{(n-1)} \times \{0,1,2\} \to \Sigma_{(n)}, \quad (\sigma, q) \mapsto \mathrm{ins}_{m,p+q}(\sigma)$$

where p is the element of $[n-3]$ with $\sigma^{p-1}(1) \notin \{m-1, m\}$ and $\sigma^p(1) \in \{m-1, m\}$. Then ψ is a bijection.

Corollary 4.3.22. *Suppose $n \geq 4$. Then*

$$\left|\Sigma_{(n)}\right| = \begin{cases} \left|\Sigma_{(n-1)}\right| & \text{if } n \text{ is even} \\ 3\left|\Sigma_{(n-1)}\right| & \text{if } n \text{ is odd.} \end{cases}$$

Proof of Theorem 4.3.21. Theorem 4.3.20 states that for all $n \in \mathbb{N}$, $\Sigma_{(n)}$ is the set of oscillating n-cycles of $\mathfrak{S}_n$ with connected intervals. In this proof we repeatedly use this result without further notice.

Let $n \geq 4$. We consider all permutations in the cycle notation where 1 is the leftmost entry in its cycle. In particular, deleting an entry from a permutation or inserting an entry into a permutation means that we do this in the chosen cycle notation. We distinguish two cases depending on the parity of n.

Case 1. Assume that n is even. Then $m = \frac{n}{2} + 1$. For $\tau \in \Sigma_{(n-1)}$ let p be given as in the definition of ψ. Then $\min\left\{\tau^{-1}(\frac{n}{2}), \frac{n}{2}\right\}$ is the pth element in the cycle notation of τ. Hence, we obtain $\psi(\tau)$ by increasing each element in τ greater or equal to m by one and then inserting m behind the element at position p.

Set $\varphi\colon \Sigma_{(n)} \to \Sigma_{(n-1)}, \sigma \mapsto \mathrm{del}_m(\sigma)$. That is, for $\sigma \in \Sigma_{(n)}$ we obtain $\varphi(\sigma)$ by first deleting m from σ and then decreasing each entry greater than m by 1.

We show that φ and ψ are well defined and inverse to each other.

(1) We prove that φ is well defined. Let $\sigma \in \Sigma_{(n)}$ and $\tau := \varphi(\sigma)$. We have to show that $\tau \in \Sigma_{(n-1)}$. That is, we have to prove that τ is oscillating and has connected intervals.

To show the latter, let $1 \leq i \leq \frac{n-1}{2} < \frac{n}{2}$. As $[i, n-i+1]$ is connected in σ there is a $0 \leq q \leq n-1$ such that

$$\left\{\sigma^{q+1}(1), \ldots, \sigma^{q+r}(1)\right\} = [i, n-i+1]$$

where $r := |[i, n-i+1]|$. Moreover, $m \in [i, n-i+1]$. Thus, $\tau = \mathrm{del}_m(\sigma)$ implies

$$\left\{\tau^{q+1}(1), \ldots, \tau^{q+r-1}(1)\right\} = [i, n-i].$$

Hence, $[i, (n-1)-i+1]$ is connected in τ. It follows that τ has connected intervals.

We now show that τ is oscillating. Note that $n-1$ is odd and $\frac{(n-1)+1}{2} = \frac{n}{2}$. By Lemma 4.3.6, it suffices to show that $\tau(i) \geq \frac{n}{2}$ for all $i \in [\frac{n}{2}-1]$ and that either $\tau^{-1}\left(\frac{n}{2}\right) > \frac{n}{2}$ or $\tau\left(\frac{n}{2}\right) > \frac{n}{2}$.

Let $i \in [\frac{n}{2}-1]$. Since $i < \frac{n}{2}$ and σ is oscillating, we infer $\sigma(i) > \frac{n}{2}$ from Lemma 4.3.6. If $\sigma(i) \neq m$ then $\tau(i) = \sigma(i) - 1 \geq \frac{n}{2}$. If $\sigma(i) = m$ then $\sigma^2(i) = \frac{n}{2}$ since $m = \frac{n}{2}+1$, $\left\{\frac{n}{2}, \frac{n}{2}+1\right\}$ is connected in σ and $i \notin \left\{\frac{n}{2}, \frac{n}{2}+1\right\}$. Thus, $\tau(i) = \frac{n}{2}$.

We now show that either $\tau^{-1}\left(\frac{n}{2}\right) > \frac{n}{2}$ or $\tau\left(\frac{n}{2}\right) > \frac{n}{2}$. Since $\left\{\frac{n}{2}, \frac{n}{2}+1\right\}$ is connected in σ there is a $0 \leq q \leq n-1$ such that

$$\left\{\sigma^{q}(1), \sigma^{q+1}(1)\right\} = \left\{\frac{n}{2}, \frac{n}{2}+1\right\}.$$

Hence, $\tau = \mathrm{del}_{\frac{n}{2}+1}(\sigma)$ implies $\tau^q(1) = \frac{n}{2}$. Because $n \geq 4$, we can apply Lemma 4.3.14 to $\left\{\frac{n}{2}, \frac{n}{2}+1\right\}$ in σ and obtain that there are $a < \frac{n}{2}$ and $b > \frac{n}{2}+1$ such that

$$\left\{\sigma^{q-1}(1), \sigma^{q}(1), \sigma^{q+1}(1), \sigma^{q+2}(1)\right\} = \left\{a, b, \frac{n}{2}, \frac{n}{2}+1\right\}.$$

Therefore, $\tau^q(1) = \frac{n}{2}$ and $\tau = \mathrm{del}_{\frac{n}{2}+1}(\sigma)$ yield $\left\{\tau^{-1}\left(\frac{n}{2}\right), \tau\left(\frac{n}{2}\right)\right\} = \{a, b-1\}$. That is, either $\tau^{-1}\left(\frac{n}{2}\right) > \frac{n}{2}$ or $\tau\left(\frac{n}{2}\right) > \frac{n}{2}$. Thus, τ is oscillating.

(2) We check that ψ is well defined. Let $\tau \in \Sigma_{(n-1)}$ and $\sigma := \psi(\tau)$. We have to show $\sigma \in \Sigma_{(n)}$.

The definition of ψ implies that $\frac{n}{2}+1$ is a neighbor of $\frac{n}{2}$ in σ. In addition, $[i, n-i]$ is connected in τ for $i \in [\frac{n}{2}-1]$. Therefore, $[i, n-i+1]$ is connected in σ for $i \in [\frac{n}{2}]$. That is, σ has connected intervals.

We now show that σ is oscillating. By Lemma 4.3.6, it suffices to show that $\sigma(i) > \frac{n}{2}$ for all $i \in [\frac{n}{2}]$. For $i < \frac{n}{2}$ this can be done as before. Thus, we only consider $i = \frac{n}{2}$. As τ is oscillating, Lemma 4.3.6 implies that one of the neighbors of $\frac{n}{2}$ is smaller than $\frac{n}{2}$ and the other one is greater than $\frac{n}{2}$. Let a be the smaller and b be the bigger neighbor of

$\frac{n}{2}$. In the definition of ψ, p is chosen such that $\frac{n}{2}+1$ is inserted in τ between a and $\frac{n}{2}$. Thus, $\frac{n}{2}$ has neighbors $\frac{n}{2}+1$ and $b+1$ in σ. Consequently, $\sigma\left(\frac{n}{2}\right) > \frac{n}{2}$.

(3) We now show that $\psi \circ \varphi = \mathrm{id}$. Let $\sigma \in \Sigma_{(n)}$. Since $\left\{\frac{n}{2}, \frac{n}{2}+1\right\}$ is connected in σ, these two elements are neighbors in σ. As σ is oscillating, there is an $a < \frac{n}{2}$ such that $\frac{n}{2}+1$ has neighbors a and $\frac{n}{2}$. We obtain $\varphi(\sigma)$ from σ by deleting $\frac{n}{2}+1$ so that a and $\frac{n}{2}$ are neighbors in $\varphi(\sigma)$. On the other hand, we obtain $\psi(\varphi(\sigma))$ from $\varphi(\sigma)$ by inserting $\frac{n}{2}+1$ between a and $\frac{n}{2}$. Thus $\psi(\varphi(\sigma)) = \sigma$.

(4) Finally, we show that $\varphi \circ \psi = \mathrm{id}$. Let $\tau \in \Sigma_{(n-1)}$. Then we obtain $\psi(\tau)$ from τ by inserting $\frac{n}{2}+1$ at some position and get $\varphi(\psi(\tau))$ from $\psi(\tau)$ by deleting it again. Hence, $\varphi(\psi(\tau)) = \tau$.

Case 2. Assume that n is odd. Then $m = \frac{n+1}{2}$. For $\tau \in \Sigma_{(n-1)}$ the set $\{m-1, m\}$ is connected. Thus, there is a unique integer p with $1 \le p \le n-3$ such that $\tau^{p-1}(1) \notin \{m-1, m\}$ and $\tau^{p}(1) \in \{m-1, m\}$. That is, the integer p from the definition of ψ in the theorem is well defined. Note that p is the position of the left neighbor of the set $\{m-1, m\}$ in τ.

Conversely, for $\sigma \in \Sigma_{(n)}$, $I := \{m-1, m, m+1\}$ is connected in σ. Hence, there is a unique $0 \le p \le n-1$ such that $I = \left\{\sigma^{p+k}(1) \mid k = 0,1,2\right\}$ and a unique $q \in \{0,1,2\}$ such that $\sigma^{p+q}(1) = m$. We define the map $\varphi\colon \Sigma_{(n)} \to \Sigma_{(n-1)} \times \{0,1,2\}$ by setting $\varphi(\sigma) := (\mathrm{del}_m(\sigma), q)$. Again, we show that φ and ψ are well defined and inverse to each other.

(1) First we show that the two maps are inverse to each other. Let $\sigma \in \Sigma_{(n)}$ and $\varphi(\sigma) = (\tau, q)$. Then we have

$$q = \begin{cases} 0 & \text{if } m \text{ is the left neighbor of } \{m-1, m+1\} \text{ in } \sigma, \\ 1 & \text{if } m \text{ is located between } m-1 \text{ and } m+1 \text{ in } \sigma, \\ 2 & \text{if } m \text{ is the right neighbor of } \{m-1, m+1\} \text{ in } \sigma. \end{cases}$$

Conversely, let $\tau \in \Sigma_{(n-1)}$, $q \in \{0,1,2\}$ and $\sigma = \psi(\tau, q)$ then

$$m \text{ is } \begin{cases} \text{the left neighbor of } \{m-1, m+1\} \text{ in } \sigma & \text{if } q = 0, \\ \text{located between } m-1 \text{ and } m+1 \text{ in } \sigma & \text{if } q = 1, \\ \text{the right neighbor of } \{m-1, m+1\} \text{ in } \sigma & \text{if } q = 2. \end{cases} \tag{4.15}$$

From this it follows that φ and ψ are inverse to each other.

(2) In order to prove that φ is well defined one has to show that $\mathrm{del}_m(\sigma) \in \Sigma_{(n-1)}$. This can be done similarly as in Case 1.

(3) To see that ψ is well defined, let $\tau \in \Sigma_{(n-1)}$, $q \in \{0,1,2\}$ and $\sigma := \psi(\tau, q)$. We first show that σ has connected intervals. Recall that $m = \frac{n+1}{2}$. Let $i \le \frac{n-1}{2} = m-1$. Then $[i, n-i]$ is connected in τ since τ has connected intervals. By the definition of ψ, we obtain the entries $[i, n-i+1]$ in σ by adding 1 to each entry $\ge m$ of $[i, n-i]$ in τ and then inserting m such that by (4.15) at least one of the neighbors of m is $m-1$ or $m+1$. Since $m-1, m, m+1 \in [i, n-i+1]$ it follows that $[i, n-i+1]$ is connected in σ. Therefore, σ has connected intervals.

In order to show that σ is oscillating, let τ' be the $(n-1)$-cycle of $\mathfrak{S}_n$ obtained by adding 1 to each entry of τ which is greater or equal than m. Since τ is oscillating, the entries in τ' alternate between the sets $[m-1]$ and $[m+1,n]$. Furthermore, we obtain σ from τ' by inserting m somewhere in τ'. Thus, Corollary 4.3.7 implies that σ is oscillating. $\square$

From Table 4.1 we know $\Sigma_{(n)}$ for $n = 1, 2, 3$. That is, Theorem 4.3.21 allows us to compute $\Sigma_{(n)}$ recursively for each $n \in \mathbb{N}$. This is illustrated in the following.

Example 4.3.23. We want to compute $\Sigma_{(n)}$ for $n = 4, 5$. To do this we use the bijections ψ and the related notation introduced in Theorem 4.3.21.

(1) Consider $n = 4$. We have

$$\Sigma_{(4)} = \left\{\psi(\sigma) \mid \sigma \in \Sigma_{(3)}\right\}$$

by Theorem 4.3.21. From Table 4.1 we obtain $\Sigma_{(3)} = \{(1,3,2),(1,2,3)\}$.

For $\sigma = (1,3,2)$ we have $p = 3$ since

$$\sigma^{3-1}(1) = 2 = \min\{2,3\} = \min\left\{\sigma^{-1}\left(\frac{4}{2}\right), \frac{4}{2}\right\}.$$

Thus,

$$\psi(\sigma) = \mathrm{ins}_{3,3}((1,3,2)) = (1,3+1,2,3) = (1,4,2,3).$$

For $\sigma = (1,2,3)$ we have $p = 1$ and

$$\psi(\sigma) = \mathrm{ins}_{3,1}((1,2,3)) = (1,3,2,3+1) = (1,3,2,4).$$

Therefore, $\Sigma_{(4)} = \{(1,4,2,3),(1,3,2,4)\}$.

(2) Consider $n = 5$. Theorem 4.3.21 yields

$$\Sigma_{(5)} = \left\{\psi(\sigma,q) \mid \sigma \in \Sigma_{(4)}, q \in \{0,1,2\}\right\}. \tag{4.16}$$

Let $m = \frac{5+1}{2} = 3$ and $I = \{m-1, m\} = \{2,3\}$.

For $\sigma = (1,4,2,3)$ we have $p = 2$ since $\sigma^{2-1}(1) = 4 \notin I$ and $\sigma^2(1) = 2 \in I$. Thus, for instance we have

$$\psi(\sigma,1) = \mathrm{ins}_{3,3}((1,4,2,3)) = (1,4+1,2,3,3+1) = (1,5,2,3,4).$$

For $\sigma = (1,3,2,4)$ we have $p = 1$. Computing $\psi(\sigma,q)$ for all $\sigma \in \Sigma_{(4)}$ and $q \in \{0,1,2\}$, we obtain the following table. By (4.16), it lists all elements of $\Sigma_{(5)}$.

$\psi(\sigma,q)$	0	1	2
$(1,4,2,3)$	$(1,5,3,2,4)$	$(1,5,2,3,4)$	$(1,5,2,4,3)$
$(1,3,2,4)$	$(1,3,4,2,5)$	$(1,4,3,2,5)$	$(1,4,2,3,5)$

Corollary 4.3.24. *Let $n \in \mathbb{N}$. Then*

$$\left|\Sigma_{(n)}\right| = \begin{cases} 1 & \text{if } n \leq 2 \\ 2 \cdot 3^{\left\lfloor \frac{n-3}{2} \right\rfloor} & \text{if } n \geq 3. \end{cases}$$

Proof. Let $x_n := |\Sigma_{(n)}|$ for $n \geq 1$, $y_1 := y_2 := 1$ and $y_n := 2 \cdot 3^{\left\lfloor \frac{n-3}{2} \right\rfloor}$ for $n \geq 3$. We show that both sequences have the same initial values and recurrence relations. First note that

$$(x_1, x_2, x_3) = (1, 1, 2) = (y_1, y_2, y_3).$$

where we obtain the x_i from Table 4.1. Now let $n \geq 4$. By Corollary 4.3.22 we have to show that $y_n = y_{n-1}$ if n is even and $y_n = 3y_{n-1}$ if n is odd. If n is even, we have

$$\left\lfloor \frac{n-3}{2} \right\rfloor = \left\lfloor \frac{n-4}{2} + \frac{1}{2} \right\rfloor = \frac{n-4}{2} = \left\lfloor \frac{n-1-3}{2} \right\rfloor$$

and thus $y_n = y_{n-1}$. If n is odd, we have

$$\left\lfloor \frac{n-3}{2} \right\rfloor = \frac{n-3}{2} = \frac{n-5}{2} + 1 = \left\lfloor \frac{n-5}{2} + \frac{1}{2} \right\rfloor + 1 = \left\lfloor \frac{n-4}{2} \right\rfloor + 1$$

and hence $y_n = 3y_{n-1}$. $\square$

4.3.2 Equivalence classes of odd hook type

Let $\alpha = (k, 1^{n-k}) \vDash n$ be a hook. Then α is a maximal composition. Recall that a hook α is called *odd* if k is odd and called *even* otherwise. The main result of this subsection is a combinatorial characterization of Σ_α provided that α is an odd hook in Theorem 4.3.40.

Subsection 4.3.4 deals with a characterization of Σ_α for a certain family of maximal compositions called *mild* (see Definition 4.3.68). Since even hooks belong to this family, the characterization of Σ_α in the case where α is an even hook is postponed until Theorem 4.3.72 of Subsection 4.3.4.

We want to generalize the concept of being oscillating and having connected intervals from n-cycles to arbitrary permutations. In order to do this, we standardize cycles in the following way. Let $\sigma := (c_1, \dots, c_k) \in \mathfrak{S}_n$ be a k-cycle. Replace the smallest element among $c_1, \dots, c_k$ by 1, the second smallest by 2 and so on. The result is a k-cycle with entries $1, 2, \dots, k$ which can be regarded as an element $\mathfrak{S}_k$. This permutation is called the *cycle standardization* $\operatorname{cst}(\sigma)$ of σ.

Example 4.3.25. Consider $\sigma = (3, 11, 4, 10, 5) \in \mathfrak{S}_{11}$. Then $\operatorname{cst}(\sigma) = (1, 5, 2, 4, 3) \in \mathfrak{S}_5$ which is oscillating with connected intervals.

We formally define the cycle standardization as follows.

Definition 4.3.26. *(1) Given $\sigma \in \mathfrak{S}_n$ and $i \in [n]$, there is a cycle $(c_1, \ldots, c_k)$ of σ containing i. Then we define*

$$\rho_\sigma(i) := |\{j \in [k] \mid c_j \leq i\}|\,.$$

(2) Let $\sigma = (c_1, \ldots, c_k) \in \mathfrak{S}_n$ be a k-cycle. The cycle standardization *of σ is the k-cycle of $\mathfrak{S}_k$ given by*

$$\mathrm{cst}(\sigma) := (\rho_\sigma(c_1), \rho_\sigma(c_2), \ldots, \rho_\sigma(c_k)).$$

Note that the permutation $\mathrm{cst}(\sigma)$ is independent from the choice of the cycle notation $\sigma = (c_1, c_2, \ldots, c_k)$ in Definition 4.3.26.

Remark 4.3.27. Let $\sigma = (c_1, c_2, \ldots, c_k) \in \mathfrak{S}_n$ be a k-cycle.

(1) The anti-rank of $i \in [n]$ among the elements in its cycle in σ is $\rho_\sigma(i)$.

(2) For all $i, j \in [k]$ we have $c_i < c_j$ if and only if $\rho_\sigma(c_i) < \rho_\sigma(c_j)$.

(3) Let i be an element appearing in the cycle $(c_1, c_2, \ldots, c_k)$. Then we have

$$\mathrm{cst}(\sigma)(\rho_\sigma(i)) = \rho_\sigma(\sigma(i)).$$

We now generalize the notions of being oscillating and having connected intervals to arbitrary permutations via the cycle decomposition and the cycle standardization. Recall that trivial cycles are those of length 1.

Definition 4.3.28. *Let $\sigma \in \mathfrak{S}_n$ and write σ as a product $\sigma = \sigma_1 \cdots \sigma_l$ of disjoint cycles including the trivial ones.*

(1) We say that σ is oscillating *if $\mathrm{cst}(\sigma_i)$ is oscillating for each cycle σ_i.*

(2) We say that σ has connected intervals *if $\mathrm{cst}(\sigma_i)$ has connected intervals for each cycle σ_i*

Let $(c) \in \mathfrak{S}_n$ be a trivial cycle. Then $\mathrm{cst}((c)) = (1) \in \mathfrak{S}_1$ which is oscillating and has connected intervals. Therefore, in order to show that a permutation σ is oscillating (has connected intervals) it suffices to consider the nontrivial cycles.

Example 4.3.29. Let $\alpha = (4, 5, 3, 1) \vDash_e 13$ and

$$\sigma_\alpha = (1, 13, 2, 12)(3, 11, 4, 10, 5)(9, 6, 8)(7).$$

The cycle standardizations of the nontrivial cycles of σ_α are

$$(1, 4, 2, 3), (1, 5, 2, 4, 3) \text{ and } (1, 2, 3).$$

Each of these three permutations is oscillating and has connected intervals (cf. Table 4.1). Thus, σ_α is oscillating and has connected intervals.

Assume that $\sigma \in \mathfrak{S}_n$ is an n-cycle. Then σ has only one cycle σ in cycle notation and $\mathrm{cst}(\sigma) = \sigma$. Thus, for n-cycles our new notion of being oscillating (having connected intervals) from Definition 4.3.28 is equivalent to the old concept from Definition 4.3.1 (Definition 4.3.9).

We now prove some general results on oscillating permutations with connected intervals. As in the last subsection, we are interested in the effect of swapping entries i and $i+1$ in cycle notation (that is, conjugating with s_i). This will in particular be useful to prove our results on odd hooks. We consider the case where i and $i+1$ appear in the same cycle first.

Lemma 4.3.30. *Let $\sigma \in \mathfrak{S}_n$ and write σ as a product $\sigma = \sigma_1 \cdots \sigma_l$ of disjoint cycles. Assume that there is an $i \in [n-1]$ and a $k \in [l]$ such that i and $i+1$ both appear in the cycle σ_k. Set $i' := \rho_\sigma(i)$ and $\tau := \operatorname{cst}(\sigma_k)$. Then we have*

(1) $\operatorname{cst}(s_i \sigma_k s_i) = s_{i'} \tau s_{i'}$,

(2) $s_i \sigma s_i \approx \sigma$ *if and only if* $s_{i'} \tau s_{i'} \approx \tau$.

Proof. By the definition of ρ_σ, we have that $\rho_\sigma(j) = \rho_{\sigma_k}(j)$ for all entries j in the cycle σ_k.

(1) We obtain $s_i \sigma_k s_i$ from σ_k by interchanging i and $i+1$ in cycle notation. Since i and $i+1$ appear in σ_k, we have $\rho_{\sigma_k}(i+1) = i'+1$. Thus, we obtain $\operatorname{cst}(s_i \sigma_k s_i)$ from $\tau = \operatorname{cst}(\sigma_k)$ by interchanging i' and $i'+1$ in cycle notation. That is, $\operatorname{cst}(s_i \sigma s_i) = s_{i'} \tau s_{i'}$.

(2) We have $s_i \sigma s_i \approx \sigma$ if and only if $\ell(s_i \sigma s_i) = \ell(\sigma)$. By Lemma 4.2.22, this is the case if and only if either $\sigma(i) < \sigma(i+1)$ or $\sigma^{-1}(i) < \sigma^{-1}(i+1)$. From the definition of the cycle standardization we obtain that $\tau(\rho_\sigma(j)) = \rho_\sigma(\sigma(j))$ for each entry j in σ_k (cf. Remark 4.3.27). Moreover, by the definition of ρ_σ and the fact that i and $i+1$ appear in the same cycle of σ,

$$\sigma(i) < \sigma(i+1) \iff \rho_\sigma(\sigma(i)) < \rho_\sigma(\sigma(i+1)).$$

Hence,

$$\sigma(i) < \sigma(i+1) \iff \tau(i') < \tau(i'+1).$$

Similarly, one shows that this equivalence is also true for σ^{-1} and τ^{-1}. Therefore, we have $s_i \sigma s_i \approx \sigma$ if and only if either $\tau(i') < \tau(i'+1)$ or $\tau^{-1}(i') < \tau^{-1}(i'+1)$. As for σ, the latter is equivalent to $s_{i'} \tau s_{i'} \approx \tau$. ☐

We now infer from Lemma 4.3.30 that swaps of i and $i+1$ within a cycle that preserve $\approx$ also preserve the properties of being oscillating with connected intervals.

Corollary 4.3.31. *Let $\sigma \in \mathfrak{S}_n$ be oscillating with connected intervals, $i \in [n-1]$ such that i and $i+1$ appear in the same cycle of σ and $\sigma' := s_i \sigma s_i$. If $\sigma \approx \sigma'$ then σ' is oscillating with connected intervals.*

Proof. We write σ as a product $\sigma = \sigma_1 \cdots \sigma_l$ of disjoint cycles and choose k such that i and $i+1$ appear in the cycle σ_k. Moreover, we set $\tau := \operatorname{cst}(\sigma_k)$, $\tau' := \operatorname{cst}(s_i \sigma_k s_i)$ and m to be the length of the cycle σ_k.

As i and $i+1$ only appear in σ_k, $\sigma' = \sigma_1 \cdots \sigma_{k-1}(s_i \sigma_k s_i)\sigma_{k+1} \cdots \sigma_l$ is the decomposition of σ' in disjoint cycles. Since σ is oscillating with connected intervals, $\mathrm{cst}(\sigma_j)$ is oscillating with connected intervals for all $j \in [l]$. Therefore, it remains to show that τ' has these properties. Since $\sigma \approx \sigma'$, Lemma 4.3.30 yields that $\tau \approx \tau'$. In addition, τ is an oscillating m-cycle with connected intervals and thus $\tau \in \Sigma_{(m)}$ by Theorem 4.3.20. Hence, also $\tau' \in \Sigma_{(m)}$, i.e. τ' is oscillating with connected intervals. □

The next result is concerned with the interchange of i and $i+1$ between two cycles.

Lemma 4.3.32. *Let $\sigma \in \mathfrak{S}_n$ be oscillating with connected intervals, $i \in [n-1]$ such that i and $i+1$ appear in different cycles of σ and $\sigma' := s_i \sigma s_i$. Then σ' is oscillating and has connected intervals.*

Proof. We obtain σ' from σ by interchanging i and $i+1$ between two cycles in cycle notation. It is easy to see that this does not affect the cycle standardization of the cycles in question. In addition, all other cycles of σ' appear as cycles of σ. Since σ is oscillating with connected intervals, it follows that the standardization of each cycle of σ' is oscillating with connected intervals. That is, σ' is oscillating with connected intervals. □

We now come to the hooks.

Example 4.3.33. Let $\alpha = (3,1,1) \vDash_e 5$. The elements of Σ_α are

$$(1,5,2), (1,2,5), (1,5,3), (1,3,5), (1,5,4), (1,4,5).$$

Note that 1 and 5 always appear in the cycle of length 3.

Recall that we use *type* as a short form for *cycle type.*

Definition 4.3.34. *Let $\alpha = (k, 1^{n-k}) \vDash_e n$ be a hook, $\sigma \in \mathfrak{S}_n$ of type α, $m := \frac{k-1}{2}$ if k is odd and $m := \frac{k}{2}$ if k is even. We say that σ satisfies the* hook properties *if*

(*1*) *σ is oscillating,*

(*2*) *σ has connected intervals,*

(*3*) *if $k > 1$ then i and $n-i+1$ appear in the cycle of length k of σ for all $i \in [m]$.*

The permutations from Example 4.3.33 satisfy the hook properties. The main result of this subsection is to show that for an odd hook α, the elements of Σ_α are characterized by the hook properties. In Theorem 4.3.72 of Subsection 4.3.4 we will see that the same is true for even hooks.

Example 4.3.35. (1) Let $\sigma \in \mathfrak{S}_n$ be of type (1^n). Then $\sigma = \mathrm{id}$ and σ satisfies the hook properties. Moreover, $\Sigma_{(1^n)} = \{\sigma\}$.

(2) Let $\sigma \in \mathfrak{S}_n$ be of type (n). That is, σ is an n-cycle. Then the third hook property is satisfied by σ since all elements of $[n]$ appear in the only cycle of σ. Thus, σ has the hook properties if and only if σ is oscillating with connected intervals. By Theorem 4.3.20, this is equivalent to $\sigma \in \Sigma_{(n)}$.

(3) Let $\alpha = (3,1,1) \vDash n$. We want to determine all permutations in $\mathfrak{S}_n$ of type α that satisfy the hook properties. Let $\sigma \in \mathfrak{S}_n$ be of type α, σ_1 be the cycle of length 3 of σ and $\mathcal{O}_1$ be the set of elements in σ_1.

Since σ_1 is the only nontrivial cycle of σ, σ is oscillating and has connected intervals if and only if $\tau := \operatorname{cst}(\sigma_1)$ has these properties. The type of τ is (3). By Theorem 4.3.20, the oscillating permutations of type (3) with connected intervals form $\Sigma_{(3)}$. From Table 4.1 we read $\Sigma_{(3)} = \{(1,3,2),(1,2,3)\}$.

Let

$$M = \{\{1,5\} \cup \{j\} \mid j \in [2,4]\} = \{\{1,2,5\},\{1,3,5\},\{1,4,5\}\}$$

The third hook property is satisfied by σ if and only if $\mathcal{O}_1 \in M$.

Therefore, σ fulfills the hook properties if and only if there is a $\tau \in \Sigma_{(3)}$ and an $\mathcal{O}_1 \in M$ such that we obtain σ_1 by writing $\mathcal{O}_1$ in a cycle such that the relative order of entries matches that one in τ. For instance, from $\tau = (1,3,2)$ and $\mathcal{O}_1 = \{1,4,5\}$ we obtain $\sigma = (1,5,4)$. Going through all possibilities for τ and $\mathcal{O}_1$ we obtain the desired set of permutations. These are the ones shown in Example 4.3.33.

For the proof of the characterization of Σ_α when α is an odd hook, we follow the same strategy as in in the case of compositions with one part from Subsection 4.3.1: For any odd hook α we show that σ_α satisfies the hook properties, $\approx$ is compatible with the hook properties and there is an algorithm that computes a sequence of $\approx$-equivalent permutations starting with σ and ending up with σ_α for each permutation σ of type α satisfying the hook properties.

Lemma 4.3.36. *Let $\alpha \vDash_e n$ be an odd hook. Then the element in stair form $\sigma_\alpha \in \mathfrak{S}_n$ satisfies the hook properties.*

Proof. Let $\alpha = (\alpha_1, \ldots, \alpha_l) = (k, 1^{n-k}) \vDash_e n$ be an odd hook. If $k = 1$ then σ_α is the identity which satisfies the hook properties. Assume $k > 1$ and set $m := \frac{k-1}{2}$. By definition, the cycle of length k of σ_α is given by

$$\sigma_{\alpha_1} = (1, n, 2, n-1, \ldots, m, n-m+1, m+1).$$

Hence, σ_α satisfies the third hook property. In order to show that σ_α is oscillating and has connected intervals, it suffices to consider σ_{α_1} because the other cycles of σ_α are trivial. From the description of σ_{α_1} we obtain its cycle standardization

$$\operatorname{cst}(\sigma_{\alpha_1}) = (1, k, 2, k-1, \ldots, m, k-m+1, m+1).$$

That is, $\operatorname{cst}(\sigma_{\alpha_1})$ is the element in stair form $\sigma_{(k)}$ which is oscillating and has connected intervals by Lemma 4.3.11. $\square$

Let $\alpha \vDash_e n$ be an odd hook and $\sigma \in \mathfrak{S}_n$ be of type α satisfying the hook properties. In order to show $\sigma_\alpha \approx \sigma$ we will successively interchange elements i and $i+1$ in the cycle notation of σ. The next lemma considers the case where at least one of i and $i+1$ is a fixpoint of σ.

Lemma 4.3.37. *Let $\alpha = (k, 1^{n-k}) \vDash_e n$ be an odd hook, $m := \frac{k-1}{2}$ and $\sigma \in \mathfrak{S}_n$ of type α satisfying the hook properties. Furthermore, assume that there are $i, i+1 \in [m+1, n-m]$ such that i or $i+1$ is a fixpoint of σ. Then $s_i \sigma s_i \approx \sigma$ and $s_i \sigma s_i$ satisfies the hook properties.*

Proof. If both i and $i+1$ are fixpoints of σ then $s_i \sigma s_i = \sigma$ and there is nothing to show. Therefore, we assume that either i or $i+1$ is not a fixpoint and call this element j. By choice of i and $i+1$, $m < j < n-m+1$. Since σ satisfies the hook properties, the cycle of length k of σ consists of the elements $1, \dots, m, j, n-m+1, \dots, n$.

First we show that $s_i \sigma s_i$ satisfies the hook properties. As σ is oscillating with connected intervals and i and $i+1$ appear in different cycles of σ, Lemma 4.3.32 yields that $s_i \sigma s_i$ is oscillating with connected intervals too. As we obtain $s_i \sigma s_i$ by interchanging i and $i+1$ in cycle notation of σ and

$$i, i+1 \notin \{1, \dots, m, n-m+1, \dots, n\},$$

$s_i \sigma s_i$ satisfies the third hook property.

In order to show $s_i \sigma s_i \approx \sigma$, we assume that $i+1$ is a fixpoint of σ and i is not. The other case is proven analogously. Let $\tau := \operatorname{cst}(\sigma)$ and $i' := \rho_\sigma(i)$. Then $i' = m+1 = \frac{k+1}{2}$ by the description of the cycle of length k from above. Since σ is oscillating, τ is oscillating. Thus, Lemma 4.3.6 implies that there is an $\varepsilon \in \{-1, 1\}$ such that

$$\tau^{\varepsilon}(i') > m+1 \text{ and } \tau^{-\varepsilon}(i') < m+1.$$

Now we use that $\tau^{\delta}(i') = \rho_\sigma(\sigma^{\delta}(i))$ for $\delta = -1, 1$ and obtain that

$$\sigma^{\varepsilon}(i) \geq n-m+1 \text{ and } \sigma^{-\varepsilon}(i) \leq m.$$

As $\sigma(i+1) = i+1 \in [m+2, n-m]$, it follows that

$$\sigma^{\varepsilon}(i) > \sigma^{\varepsilon}(i+1) \text{ and } \sigma^{-\varepsilon}(i) < \sigma^{-\varepsilon}(i+1).$$

Hence, Lemma 4.2.22 implies $\ell(s_i \sigma s_i) = \ell(\sigma)$. Therefore, $s_i \sigma s_i \approx \sigma$. □

The following lemma shows that $\approx$ preserves the hook properties. It is an analogue to Lemma 4.3.17.

Lemma 4.3.38. *Given an odd hook $\alpha = (k, 1^{n-k}) \vDash_e n$, $\sigma \in \mathfrak{S}_n$ of type α satisfying the hook properties and $\sigma' := s_i \sigma s_i$ with $\sigma \approx \sigma'$, we have that also σ' satisfies the hook properties.*

Proof. We show that σ' has the hook properties. If $k = 1$ then $\sigma = \sigma' = \mathrm{id}$ so that σ' satisfies the hook properties. Hence, assume $k > 1$. Set $m := \frac{k-1}{2}$, $\tau := \operatorname{cst}(\sigma)$ and $\tau' := \operatorname{cst}(\sigma')$. We deal with three cases.

First, assume that neither i nor $i+1$ is a fixpoint of σ. Then i and $i+1$ both appear in the cycle of length k of σ. Since σ satisfies the hook properties, it is oscillating and has connected intervals. Therefore, Corollary 4.3.31 yields that also σ' has these properties.

The elements $1, \dots, m, n-m+1, \dots m$ all appear in the cycle of length k of σ because σ satisfies the hook properties. Since we interchange two entries in this cycle to obtain σ' from σ, all the elements also appear in the cycle of length k of σ'.

Second, assume that $i+1$ is a fixpoint of σ but i is not. Since $\sigma \approx \sigma'$, we have $\ell(\sigma) = \ell(\sigma')$ and by Lemma 4.2.22

$$\begin{aligned} &\text{either } \sigma(i) > i+1 \text{ and } \sigma^{-1}(i) < i+1 \\ &\text{or } \sigma(i) < i+1 \text{ and } \sigma^{-1}(i) > i+1 \end{aligned} \tag{4.17}$$

where we used $\sigma(i+1) = i+1$. The elements of the cycle of length k of σ are $1, \dots, m, j, n-m+1, \dots, n$ where $j \in [m+1, n-m]$. We now show that $i, i+1 \in [m+1, n-m]$.

As $i+1$ is a fixpoint, we have $i+1 \leq n-m$ and it remains to show that $i \geq m+1$. Assume that $i \leq m$ instead and set $i' := \rho_\sigma(i)$. Then $i' < \frac{k+1}{2}$. Since $\tau \in \mathfrak{S}_k$ is an oscillating k-cycle, Lemma 4.3.6 yields that $\tau^{-1}(i'), \tau(i') \geq \frac{k+1}{2}$. Because $\rho_\sigma(j) = \frac{k+1}{2}$, it follows that $\sigma^{-1}(i), \sigma(i) \geq j$. Moreover, $i+1$ being a fixpoint and $i \leq m$ imply that $i+1 < j$. Hence, $\sigma^{-1}(i), \sigma(i) > i+1$ which contradicts (4.17).

Since $i, i+1 \in [m+1, n-m]$ and $i+1$ is a fixpoint of σ, we can apply Lemma 4.3.37 which implies that σ' satisfies the hook properties.

In the same vein, one proves the remaining case where i is a fixpoint but $i+1$ is not. $\square$

We now extend Lemma 4.3.18 to the case of odd hooks. That is, we consider one step of the algorithm mentioned earlier.

Lemma 4.3.39. *Let $\alpha = (k, 1^{n-k}) \vDash_e n$ be an odd hook and $\sigma \in \mathfrak{S}_n$ such that σ is of type α, σ satisfies the hook properties and $\sigma \neq \sigma_\alpha$. Then there exists a minimal integer p such that $1 \leq p \leq k-1$ and $\sigma^p(1) \neq \sigma_\alpha^p(1)$. Set $a := \sigma^p(1)$, $b := \sigma_\alpha^p(1)$ and*

$$\sigma' := \begin{cases} s_{a-1}\sigma s_{a-1} & \text{if } a > b \\ s_a \sigma s_a & \text{if } a < b. \end{cases}$$

Then $\sigma' \approx \sigma$ and σ' satisfies the hook properties.

Proof. Set $m := \frac{k-1}{2}$. If $\alpha = (1^n)$ then the only permutation of type α is the identity and there is nothing to show. If $\alpha = (n)$ then this is Lemma 4.3.18. Therefore, assume $1 < k < n$. Since σ satisfies the hook properties, 1 appears in the cycle of length k of σ. By definition, σ_α has the form

$$\sigma_\alpha = \begin{cases} (1, n, 2, n-1, \dots, m+1)(n-m)(m+2) \cdots (\frac{n+3}{2})(\frac{n+1}{2}) & \text{if } n \text{ is odd} \\ (1, n, 2, n-1, \dots, m+1)(n-m)(m+2) \cdots (\frac{n}{2})(\frac{n}{2}+1) & \text{if } n \text{ is even.} \end{cases}$$

In particular, $[m+2, n-m]$ is the set of fixpoints of σ_α and 1 also appears in the cycle of length k of σ_α. Thus, from $\sigma \neq \sigma_\alpha$ it follows that there exists p as claimed. In particular, we can define a, b and σ' as in the theorem.

If n is odd, $k < n$ implies that $\frac{n+1}{2}$ is a fixpoint of σ_α and hence $b \neq \frac{n+1}{2}$. If n is even, we have $b \neq \frac{n+1}{2}$ anyway. Let $\tau := \operatorname{cst}(\sigma)$ and note that $\operatorname{cst}(\sigma_\alpha)$ is just the element in stair form $\sigma_{(k)}$. Moreover set $a' := \rho_\sigma(a)$.

Assume $b < \frac{n+1}{2}$. The proof for $b > \frac{n+1}{2}$ is similar and hence omitted. If $b < \frac{n+1}{2}$ then $b \leq m+1$ by the description of σ_α from above. The choice of p and $1 < b \leq m+1$ imply

$$\sigma^{-1}(a) = \sigma_\alpha^{-1}(b) = n - b + 2 > m + 1$$

and

$$\{1, 2, \ldots, b-1\} \subseteq \{\sigma_\alpha^r(1) \mid r = 0, \ldots, p-1\} = \{\sigma^r(1) \mid r = 0, \ldots, p-1\}.$$

The last equality and $a \neq b$ imply $b < a$. Thus, we consider $\sigma' := s_{a-1}\sigma s_{a-1}$. From the hook properties, we obtain that the elements in the cycle of length k of σ are $1, \ldots, m, j, n-m+1, \ldots n$ where $j \in [m+1, n-m]$. Thus, $\sigma^{-1}(a) > m+1$ implies $\tau^{-1}(a') > m+1$. But since σ is oscillating, τ is oscillating and therefore Lemma 4.3.6 implies $a' \leq m+1$. From the description of the elements in the k-cycle of σ, it now follows that $a \leq n-m$.

To sum up, we have $b < a \leq n-m$ and $\sigma' = s_{a-1}\sigma s_{a-1}$. Now we have two cases depending on $a-1$. If $a-1$ is a fixpoint of σ then because of $a \leq n-m$, we can apply Lemma 4.3.37 and obtain that $\sigma' \approx \sigma$ and σ' satisfies the hook properties.

If $a-1$ is not a fixpoint of σ then $\rho_\sigma(a-1) = a'-1$. Moreover, interchanging $a-1$ and a in σ does not affect the third part of the hook property. Therefore, we obtain from Lemma 4.3.30 that $\sigma' \approx \sigma$ and σ' satisfies the hook properties if $\tau' := s_{a'-1}\tau s_{a'-1} \approx \tau$ and τ' is oscillating with connected intervals. By Lemma 4.3.18, τ' has these properties if $\tau^r(1) = \sigma_{(k)}^r(1)$ for $0 \leq r \leq p-1$, $\tau^p(1) > \sigma_{(k)}^p(1)$ and $\tau^p(1) = a'$. This is what remains be shown.

As $\sigma^r(1) = \sigma_\alpha^r(1)$ for $0 \leq r \leq p-1$, we have the following equality of tuples

$$\begin{aligned}
(\tau^0(1), \tau^1(1), \ldots, \tau^{p-1}(1)) &= (\rho_\sigma(1), \rho_\sigma(n), \rho_\sigma(2), \rho_\sigma(n-1), \ldots, \rho_\sigma(n-b+2)) \\
&= (1, k, 2, k-1, \ldots, k-b+2) \\
&= (\sigma_{(k)}^0(1), \sigma_{(k)}^1(1), \ldots, \sigma_{(k)}^{p-1}(1)).
\end{aligned}$$

Since the cycle of length k of σ contains exactly one element of $[m+1, n-m]$, $a-1$ and a appear in this cycle and $a \leq n-m$, we have that $a \leq m+1$. Moreover, $1, \ldots, m$ appear in the cycle of length k of σ and σ_α. Since $b < a \leq m+1$, this implies

$$\sigma_{(k)}^p(1) = \rho_{\sigma_\alpha}(b) = b \text{ and } \tau^p(1) = \rho_\sigma(a) = a.$$

In particular, $a' = \tau^p(1)$. Moreover, we have $b < a$ so that $\sigma_{(k)}^p(1) < \tau^p(1)$ as desired. □

We now come to the main result of this subsection.

Theorem 4.3.40. *Let $\alpha \vDash_e n$ be an odd hook and $\sigma \in \mathfrak{S}_n$ of type α. Then $\sigma \in \Sigma_\alpha$ if and only if σ satisfies the hook properties.*

Proof. Let $\alpha = (k, 1^{n-k}) \vDash_e n$ be an odd hook and σ_α be the element in stair form. The proof is analogous to the one of Theorem 4.3.20. First, σ_α satisfies the hook properties by Lemma 4.3.36. Let $\sigma \in \mathfrak{S}_n$.

For the direction from left to right assume that $\sigma \in \Sigma_\alpha$. Then $\sigma \approx \sigma_\alpha$. From the definition of $\approx$ and Lemma 4.3.38 it follows that $\approx$ transfers the hook properties from σ_α to σ.

For the converse direction, assume that σ satisfies the hook properties. By using Lemma 4.3.39 iteratively, we obtain a sequence of $\approx$-equivalent permutations starting with σ and ending in σ_α. Hence $\sigma \in \Sigma_\alpha$. □

We continue with a rule for the construction of $\Sigma_{(k,1^{n-k})}$ from $\Sigma_{(k)}$ in the case where k is odd and $k \geq 3$. The rule can be sketched as follows. Given a $\tau \in \Sigma_{(k)}$ we can choose a subset of $[n]$ of size k in accordance with the third hook property. Arranging the elements of this subset in a cycle of length k such that its cycle standardization is τ (and letting the other elements of $[n]$ be fixpoints) then results in an element of $\Sigma_{(k,1^{n-k})}$. See Part (3) of Example 4.3.35 for an illustration.

Corollary 4.3.41. *Let $\alpha = (k, 1^{n-k}) \vDash_e n$ be an odd hook with $k \geq 3$. Set $m := \frac{k-1}{2}$. For $\tau \in \Sigma_{(k)}$ and $j \in [m+1, n-m]$ define $\varphi(\tau, j)$ to be the element $\sigma \in \mathfrak{S}_n$ of type α such that $\operatorname{cst}(\sigma) = \tau$ and the entries in the cycle of length k of σ are $1, \dots, m, j, n-m+1, \dots, n$. Then*

$$\varphi\colon \Sigma_{(k)} \times [m+1, n-m] \to \Sigma_\alpha, \quad (\tau, j) \mapsto \varphi(\tau, j)$$

is a bijection.

Proof. Given a $\tau \in \Sigma_{(k)}$ and a $j \in [m+1, n-m]$ there is only one way (up to cyclic shift) to write the elements $1, 2, \dots, m, j, n-m+1, \dots, n$ in a cycle of length k such that the standardization of the corresponding k-cycle in $\mathfrak{S}_n$ is τ. This k-cycle is $\varphi(\tau, j)$. By construction, $\varphi(\tau, j)$ satisfies the hook properties. Hence, Theorem 4.3.40 yields $\varphi(\tau, j) \in \Sigma_\alpha$. That is, φ is well defined.

Let $\sigma \in \Sigma_\alpha$. Then by Theorem 4.3.40, σ satisfies the hook properties. The third hook property yields that there is a unique $j \in [m+1, n-m]$ such that the elements in the cycle of length k of σ are $1, 2, \dots, m, j, n-m+1, \dots, n$. From the first two hook properties it follows that $\tau := \operatorname{cst}(\sigma)$ is oscillating and has connected intervals. Thus, $\tau \in \Sigma_{(k)}$ by Theorem 4.3.20. By definition of φ, the cycles of length k of $\varphi(\tau, j)$ and σ contain the same elements. Moreover, they have the same cycle standardization τ. Consequently, $\varphi(\tau, j) = \sigma$. That is, φ is surjective. Since τ and j uniquely depend on σ, φ is also injective. □

In the last result of the subsection we determine the cardinality of Σ_α for each odd hook α.

Corollary 4.3.42. *If $\alpha = (k, 1^{n-k}) \vDash_e n$ is an odd hook then*

$$|\Sigma_\alpha| = \begin{cases} 1 & \text{if } k = 1 \\ 2(n-k+1)3^{\frac{k-3}{2}} & \text{if } k \geq 3. \end{cases}$$

Proof. Let $\sigma \in \Sigma_\alpha$. If $k = 1$ then $\Sigma_\alpha = \{1\}$. Now suppose that $k \geq 3$ and set $m := \frac{k-1}{2}$. The cardinality of $[m+1, n-m]$ is $n-k+1$. Hence, Corollary 4.3.41 yields that $|\Sigma_\alpha| = (n-k+1)|\Sigma_{(k)}|$. In addition, we have $|\Sigma_{(k)}| = 2 \cdot 3^{\frac{k-3}{2}}$ from Corollary 4.3.24. □

4.3.3 The inductive product

In this subsection we define the inductive product $\odot$ and use it to obtain in Corollary 4.3.56 a recursion the rule for $\Sigma_{(\alpha_1,\ldots,\alpha_l)}$ in the case where α_1 is even. This leads to a description of Σ_α for all maximal compositions α whose odd parts form a hook (see Remark 4.3.59). Results of this subsection will be applied in Subsection 4.3.4 and Section 5.4.

Recall that we write $\gamma \vDash_0 n$ if γ is a weak composition of n, that is, a finite sequence of nonnegative integers that sum up to n.

Definition 4.3.43. *Let $(n_1, n_2) \vDash_0 n$. The* inductive product *on $\mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2}$ is the binary operator*

$$\begin{aligned} \odot \colon \mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2} &\to \mathfrak{S}_n \\ (\sigma_1, \sigma_2) &\mapsto \sigma_1 \odot \sigma_2 \end{aligned}$$

where $\sigma_1 \odot \sigma_2$ is the element of $\mathfrak{S}_n$ whose cycles are the cycles of σ_1 and σ_2 altered as follows:

(1) in the cycles of σ_1, add n_2 to each entry $> k$,

(2) in the cycles of σ_2, add k to each entry

where $k := \lceil \frac{n_1}{2} \rceil$.

For two sets $X_1 \subseteq \mathfrak{S}_{n_1}$ and $X_2 \subseteq \mathfrak{S}_{n_2}$ we define

$$X_1 \odot X_2 := \{\sigma_1 \odot \sigma_2 \mid \sigma_1 \in \mathfrak{S}_{n_1}, \sigma_2 \in \mathfrak{S}_{n_2}\}.$$

It will follow from Lemma 4.3.47 below that the inductive product is well-defined.

Example 4.3.44. (1) Let $\emptyset \in \mathfrak{S}_0$ be the empty function and $\sigma \in \mathfrak{S}_n$. Then

$$\emptyset \odot \sigma = \sigma \odot \emptyset = \sigma.$$

(2) Consider $n_1 = 6$, $n_2 = 4$, $n = 10$ and the elements in stair form $\sigma_{(6)} \in \mathfrak{S}_{n_1}$ and

$\sigma_{(3,1)} \in \mathfrak{S}_{n_2}$. Then $k = 3$ and

$$\begin{aligned}\sigma_{(6)} \odot \sigma_{(3,1)} &= (1,6,2,5,3,4) \odot (1,4,2)(3)\\ &= (1,6+4,2,5+4,3,4+4)(1+3,4+3,2+3)(3+3)\\ &= (1,10,2,9,3,8)(4,7,5)(6).\end{aligned}$$

(3) Consider $n_1 = 5$, $n_2 = 4$ and the elements in stair form $\sigma_{(5)} = (1,5,2,4,3) \in \mathfrak{S}_{n_1}$ and $\sigma_{(3,1)} = (1,4,2)(3) \in \mathfrak{S}_{n_2}$. Then $\sigma_{(3,1)}^{w_0} = (1,3,4)(2)$ where $w_0 = (1,4)(2,3)$ is the longest element of $\mathfrak{S}_4$. We have $k = 3$ and

$$\begin{aligned}\sigma_{(5)} \odot \sigma_{(3,1)}^{w_0} &= (1,5+4,2,4+4,3)(1+3,3+3,4+3)(2+3)\\ &= (1,9,2,8,3)(7,4,6)(5).\end{aligned}$$

Note that in Parts (2) and (3) we obtain the elements in stair form $\sigma_{(6,3,1)}$ and $\sigma_{(5,3,1)}$, respectively.

In order to work with the inductive product, it is convenient to describe it more formally. To this end we introduce the following notation which we will use throughout the subsection.

Notation 4.3.45. Let $n \geq 0$, $(n_1, n_2) \vDash_0 n$, $k := \lceil \frac{n_1}{2} \rceil$,

$$N_1 := [k] \cup [k + n_2 + 1, n] \quad \text{and} \quad N_2 := [k+1, k+n_2].$$

We have that $|N_1| = n_1, |N_2| = n_2$, N_1 and N_2 are disjoint and $N_1 \cup N_2 = [n]$. Note that $[0] = [1,0] = \emptyset$. Define the bijections $\varphi_1 \colon [n_1] \to N_1$ and $\varphi_2 \colon [n_2] \to N_2$ by

$$\varphi_1(i) := \begin{cases} i & \text{if } i \leq k \\ i + n_2 & \text{if } i > k \end{cases} \quad \text{and} \quad \varphi_2(i) := i + k.$$

The bijections φ_1 and φ_2 formalize the alteration of the cycles of σ_1 and σ_2 in Definition 4.3.43, respectively. Their inverses are given by

$$\varphi_1^{-1}(i) := \begin{cases} i & \text{if } i \leq k \\ i - n_2 & \text{if } i > k \end{cases} \quad \text{and} \quad \varphi_2^{-1}(i) := i - k.$$

For $i = 1,2$ and $\sigma_i \in \mathfrak{S}_{n_i}$, write $\sigma_i^{\varphi_i} := \varphi_i \circ \sigma_i \circ \varphi_i^{-1}$. Then $\sigma_i^{\varphi_i} \in \mathfrak{S}(N_i)$ and $\sigma_i^{\varphi_i}$ can naturally be identified with the element of $\mathfrak{S}_n$ that acts on N_i as $\sigma_i^{\varphi_i}$ and fixes all elements of $[n] \setminus N_i$.

We will see in Lemma 4.3.47 that we obtain $\sigma_i^{\varphi_i}$ by applying φ_i on each entry in of σ_i in cycle notation.

Example 4.3.46. Let $n_1 = 6$ and $n_2 = 4$ and consider the elements in stair form

$$\sigma_1 := \sigma_{(6)} = (1,6,2,5,3,4) \in \mathfrak{S}_6 \quad \text{and} \quad \sigma_2 := \sigma_{(3,1)} = (1,4,2)(3) \in \mathfrak{S}_4.$$

Then $k = 3$ and

$$\begin{aligned}
\sigma_1^{\varphi_1} &= (1, 6+4, 2, 5+4, 3, 4+4) = (1, 10, 2, 9, 3, 8), \\
\sigma_2^{\varphi_2} &= (1+3, 4+3, 2+3)(3+3) = (4, 7, 5)(6).
\end{aligned}$$

Thus, from Example 4.3.44 it follows that $\sigma_1 \odot \sigma_2 = \sigma_1^{\varphi_1}\sigma_2^{\varphi_2}$. The next lemma states that this is true in general.

We now come to the more formal description of the inductive product.

Lemma 4.3.47. *Let $\sigma_r \in \mathfrak{S}_{n_r}$ with decomposition in disjoint cycles $\sigma_r = \sigma_{r,1}\sigma_{r,2}\cdots\sigma_{r,p_r}$ for $r = 1, 2$.*

(*1*) *We have*

$$\sigma_1 \odot \sigma_2 = \sigma_1^{\varphi_1}\sigma_2^{\varphi_2}.$$

(*2*) *Let $r \in \{1, 2\}$ and $\sigma_{r,j} = (c_1, \dots, c_t)$ be a cycle of σ_r. Then*

$$\sigma_{r,j}^{\varphi_r} = (\varphi_r(c_1), \dots, \varphi_r(c_t)).$$

(*3*) *The decomposition of $\sigma_1 \odot \sigma_2$ in disjoint cycles is given by*

$$\sigma_1 \odot \sigma_2 = \sigma_{1,1}^{\varphi_1}\cdots\sigma_{1,p_1}^{\varphi_1} \cdot \sigma_{2,1}^{\varphi_2}\cdots\sigma_{2,p_2}^{\varphi_2}.$$

Proof. Set $\sigma := \sigma_1 \odot \sigma_2$ and $\sigma' := \sigma_1^{\varphi_1}\sigma_2^{\varphi_2}$. It will turn out that $\sigma = \sigma'$.

We first show Part (2). Let $r \in \{1, 2\}$, ξ be a cycle of σ_r and $i \in [n_r]$. Then

$$\xi^{\varphi_r}(\varphi_r(i)) = (\varphi_r \circ \xi \circ \varphi_r^{-1} \circ \varphi_r)(i) = \varphi_r(\xi(i)).$$

Hence, if $\xi = (c_1, \dots, c_t) \in \mathfrak{S}_{n_r}$ then $\xi^{\varphi_r} = (\varphi_r(c_1), \dots, \varphi_r(c_t)) \in \mathfrak{S}(N_r)$.

We continue with showing Part (3) for σ'. For $r = 1, 2$ we have

$$\begin{aligned}
\sigma_r^{\varphi_r} &= \varphi_r \circ \sigma_r \circ \varphi_r^{-1} \\
&= \varphi_r \circ \sigma_{r,1}\cdots\sigma_{r,p_r} \circ \varphi_r^{-1} \\
&= (\varphi_r \circ \sigma_{r,1} \circ \varphi_r^{-1})\cdots(\varphi_r \circ \sigma_{r,p_r} \circ \varphi_r^{-1}) \\
&= \sigma_{r,1}^{\varphi_r}\cdots\sigma_{r,p_r}^{\varphi_r}.
\end{aligned}$$

Thus,

$$\sigma' = \sigma_{1,1}^{\varphi_1}\cdots\sigma_{1,p_1}^{\varphi_1}\sigma_{2,1}^{\varphi_2}\cdots\sigma_{1,p_2}^{\varphi_2}. \tag{4.18}$$

The cycles in this decomposition are given by Part (1). As φ_1 and φ_2 are bijections with disjoint images, the cycles are disjoint.

Lastly, we show $\sigma = \sigma'$. From (4.18), Part (2) and the definition of φ_1 and φ_2 it follows that we obtain the cycles of σ' by altering the cycles of σ_1 and σ_2 as described in Definition 4.3.43. Hence, $\sigma = \sigma'$. □

Corollary 4.3.48. *Let* $\sigma_1 \in \mathfrak{S}_{n_1}, \sigma_2 \in \mathfrak{S}_{n_2}$ *and* $\sigma := \sigma_1 \odot \sigma_2$. *Then*

$$P(\sigma) = \varphi_1(P(\sigma_1)) \cup \varphi_2(P(\sigma_2)).$$

We continue with basic properties of the inductive product.

Lemma 4.3.49. *Let* $\sigma_1 \in \mathfrak{S}_{n_1}, \sigma_2 \in \mathfrak{S}_{n_2}$ *and* $\sigma := \sigma_1 \odot \sigma_2$. *Then for all* $i \in [n]$

$$\sigma(i) = \begin{cases} \sigma_1^{\varphi_1}(i) & \text{if } i \in N_1 \\ \sigma_2^{\varphi_2}(i) & \text{if } i \in N_2. \end{cases}$$

Proof. By Lemma 4.3.47, $\sigma = \sigma_1^{\varphi_1}\sigma_2^{\varphi_2}$. If $n_1 = 0$ or $n_2 = 0$ the claim is trivially true. Thus, suppose $n_1, n_2 \geq 1$ and let $i \in [n]$. Consider $\sigma_1^{\varphi_1}$ and $\sigma_2^{\varphi_2}$ as elements of $\mathfrak{S}_n$. Since $\{N_1, N_2\}$ is a partition of $[n]$ there is exactly one $r \in \{1, 2\}$ such that $i \in N_r$. We have that $\sigma_r^{\varphi_r}(N_r) = N_r$ and that $\sigma_{2-r+1}^{\varphi_{2-r+1}}$ fixes each element of N_r. Hence,

$$\sigma(i) = \sigma_1^{\varphi_1}\sigma_2^{\varphi_2}(i) = \sigma_r^{\varphi_r}(i).$$ □

We now determine the image of the inductive product and show that it is injective.

Lemma 4.3.50. *Let* $(n_1, n_2) \vDash_0 n$.

(*1*) *The image of* $\mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2}$ *under* $\odot$ *is given by*

$$\mathfrak{S}_{n_1} \odot \mathfrak{S}_{n_2} = \{\sigma \in \mathfrak{S}_n \mid \sigma(N_i) = N_i \text{ for } i = 1, 2\}.$$

(*2*) *The inductive product on* $\mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2}$ *is injective.*

Proof. (1) Set $Y := \{\sigma \in \mathfrak{S}_n \mid \sigma(N_i) = N_i \text{ for } i = 1, 2\}$.

We first show $\mathfrak{S}_{n_1} \odot \mathfrak{S}_{n_2} \subseteq Y$. Let $\sigma \in \mathfrak{S}_{n_1} \odot \mathfrak{S}_{n_2}$. Then there are $\sigma_i \in \mathfrak{S}_{n_i}$ for $i = 1, 2$ such that $\sigma = \sigma_1 \odot \sigma_2$. By Lemma 4.3.49 we have $\sigma(N_i) = \sigma^{\varphi_i}(N_i) = N_i$ for $i = 1, 2$. Hence, $\sigma \in Y$.

Now we show $Y \subseteq \mathfrak{S}_{n_1} \odot \mathfrak{S}_{n_2}$. Let $\sigma \in Y$. For $i = 1, 2$ set $\tilde{\sigma}_i = \sigma|_{N_i}$ (the restriction to N_i). Consider $i \in \{1, 2\}$. Since $\sigma \in Y$, $\tilde{\sigma}_i(N_i) = N_i$ and thus $\tilde{\sigma}_i \in \mathfrak{S}(N_i)$. Therefore, $\sigma_i := \varphi_i^{-1} \circ \tilde{\sigma}_i \circ \varphi_i$ is an element of $\mathfrak{S}_{n_i}$. Moreover, $\sigma_i^{\varphi_i}$ considered as an element of $\mathfrak{S}_n$ leaves each element of N_{2-i+1} fixed. Hence, we have

$$(\sigma_1 \odot \sigma_2)|_{N_i} = \sigma_1^{\varphi_1}\sigma_2^{\varphi_2}|_{N_i} = \sigma_i^{\varphi_i}|_{N_i} = \tilde{\sigma}_i|_{N_i} = \sigma|_{N_i}.$$

Consequently, $\sigma = \sigma_1 \odot \sigma_2$.

(2) Since $|N_i| = n_i$ for $i = 1, 2$, the cardinality of Y is $n_1!n_2!$. This is also the cardinality of $\mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2}$. As the image of $\mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2}$ under $\odot$ is Y, it follows that $\odot$ is injective. □

Recall that for $\alpha \vDash_e n$, each element of Σ_α has the property that its length is maximal in its conjugacy class. We want to use this property to prove our main result.

Consider $\sigma = \sigma_1 \odot \sigma_2$ such that σ_1 has type (n_1). We seek a formula for $\ell(\sigma)$ depending on σ_1 and σ_2. We are particularly interested in the case where the n_1-cycle σ_1 is oscillating.

Given $\sigma \in \mathfrak{S}_n$ let $Inv(\sigma) := \{(i,j) \mid 1 \leq i < j \leq n, \sigma(i) > \sigma(j)\}$ be the *set of inversions* of σ. Then $\ell(\sigma) = |Inv(\sigma)|$ by [BB05, Proposition 1.5.2].

Lemma 4.3.51. *Let $\sigma_1 \in \mathfrak{S}_{n_1}$ be an n_1-cycle, $\sigma_2 \in \mathfrak{S}_{n_2}$, $\sigma := \sigma_1 \odot \sigma_2$,*

$$
\begin{aligned}
P &:= \{i \in [k] \mid \sigma_1(i) > k\}\,,\\
Q &:= \{i \in [k+1, n_1] \mid \sigma_1(i) \leq k\}\,,
\end{aligned}
$$

$p := |P|$ and $q := |Q|$. Then we have

$$\ell(\sigma) = \ell(\sigma_1) + \ell(\sigma_2) + (p+q)n_2.$$

Moreover,

(1) $p, q \leq \lfloor \frac{n_1}{2} \rfloor$,
(2) if σ_1 is oscillating, then $p = q = \lfloor \frac{n_1}{2} \rfloor$.

Proof. Let $i, j \in [n]$ and $m := \lfloor \frac{n_1}{2} \rfloor$. We distinguish three types of pairs (i,j) and count the number of inversions of σ type by type.

Type 1. There is an $r \in \{1,2\}$ such that $i, j \in N_r$. In this case let $t \in \{i,j\}$ and set $t' := \varphi_r^{-1}(t)$. Then $t' \in [n_r]$. From Lemma 4.3.49 we obtain

$$\sigma(t) = \varphi_r(\sigma_r(t')).$$

In addition, we have

$$\varphi_r(\sigma_r(i')) > \varphi_r(\sigma_r(j')) \iff \sigma_r(i') > \sigma_r(j')$$

since φ_r is a stricly increasing function. As φ_r^{-1} is stricly increasing as well, we also have that

$$i < j \iff i' < j'.$$

Hence,

$$
\begin{aligned}
(i,j) \in Inv(\sigma) &\iff i < j \text{ and } \sigma(i) > \sigma(j)\\
&\iff i' < j' \text{ and } \varphi_r(\sigma_r(i')) > \varphi_r(\sigma_r(j'))\\
&\iff i' < j' \text{ and } \sigma_r(i') > \sigma_r(j')\\
&\iff (i', j') \in Inv(\sigma_r).
\end{aligned}
$$

Thus, the number of inversions of Type 1 is

$$|Inv(\sigma_1)| + |Inv(\sigma_2)| = \ell(\sigma_1) + \ell(\sigma_2).$$

Type 2. We have $i \in N_1$, $j \in N_2$ and $i < j$. Assume that (i, j) is of this type and recall that $N_1 = [k] \cup [k + n_2 + 1, n]$ and $N_2 = [k + 1, k + n_2]$ where $k = \lceil \frac{n}{2} \rceil$. Since $i < j$, we have $i \leq k$ which in particular means that $\varphi_1^{-1}(i) = i$. As $\sigma(j) \in N_2$, $k + 1 \leq \sigma(j) \leq k + n_2$. Moreover, $\sigma(i) = \sigma_1^{\varphi_1}(i)$ by Lemma 4.3.49. Consequently,

$$\sigma(i) = \sigma_1^{\varphi_1}(i) = \varphi_1(\sigma_1(i)) = \begin{cases} \sigma_1(i) < \sigma(j) & \text{if } \sigma_1(i) \leq k \\ \sigma_1(i) + n_2 > \sigma(j) & \text{if } \sigma_1(i) > k. \end{cases}$$

Therefore,

$$(i, j) \in Inv(\sigma) \iff \sigma_1(i) > k.$$

Hence, the number of inversions of Type 2 is the cardinality of the set $P \times N_2$. Thus, we have pn_2 inversions of Type 2.

Type 3. We have $i \in N_2$, $j \in N_1$ and $i < j$. Let (i, j) be of Type 3. Then from $i < j$ we obtain $j \geq k + n_2 + 1$. In particular this type can only occur if $n_1 > 1$ because otherwise $n = 1 + n_2 < j$.

Since $i \in N_2$, also $\sigma(i) \in N_2$. That is, $k + 1 \leq \sigma(i) \leq k + n_2$. Moreover, from $i < j$ and $i \in N_2$ it follows that $j \geq k + n_2 + 1$. Thus,

$$j' := \varphi_1^{-1}(j) = j - n_2$$

and $j' \in [k + 1, n_1]$. Hence,

$$\sigma(j) = \sigma_1^{\varphi_1}(j) = \varphi_1(\sigma_1(j')) = \begin{cases} \sigma_1(j') < \sigma(i) & \text{if } \sigma_1(j') \leq k \\ \sigma_1(j') + n_2 > \sigma(i) & \text{if } \sigma_1(j') > k. \end{cases}$$

That is,

$$(i, j) \in Inv(\sigma) \iff \sigma_1(j') \leq k \iff j' \in Q \iff j \in \varphi_1(Q)$$

where we use that $j' \in [k + 1, n_1]$ for the second equivalence. Consequently, the set of inversion of Type 3 is the set $N_2 \times \varphi_1(Q)$. Since φ_1 is a bijection, it follows that there are exactly qn_2 inversions of this type.

Summing up the number of inversions of each type, we obtain the formula for the length of σ.

We now prove (1) and (2).

(1) By definition, $\sigma_1(P) \subseteq [k+1, n_1]$ and $Q \subseteq [k+1, n_1]$. The cardinality of $[k+1, n_1]$ is $\lfloor \frac{n_1}{2} \rfloor$. Therefore, $p, q \leq \lfloor \frac{n_1}{2} \rfloor$.

(2) Assume that σ_1 is oscillating. Suppose first that n is even. Then $k = \frac{n_1}{2}$. Because σ_1 is oscillating, we obtain that

$$\sigma_1([k]) = [k + 1, n_1] \quad \text{and} \quad \sigma_1([k + 1, n_1]) = [k]$$

from Definition 4.3.1 and Lemma 4.3.4. Hence, $p = q = k = \lfloor \frac{n_1}{2} \rfloor$.

Suppose now that n is odd. Then $k = \frac{n_1+1}{2}$. Since σ_1 is oscillating, Definition 4.3.1 and Lemma 4.3.4 yield that there is an $m \in \{k-1, k\}$ such that

$$\sigma_1([m]) = [n_1 - m + 1, n_1] \quad \text{and} \quad \sigma_1([m+1, n_1]) = [n_1 - m].$$

It is not hard to see that this implies $p = q = k - 1 = \lfloor \frac{n_1}{2} \rfloor$. $\square$

We have seen in Example 4.3.44 that the elements in stair form $\sigma_{(5,3)}$ and $\sigma_{(6,3)}$ can be decomposed as

$$\sigma_{(5,3)} = \sigma_{(5)} \odot \sigma_{(3)}^{w_0} \quad \text{and} \quad \sigma_{(6,3)} = \sigma_{(6)} \odot \sigma_{(3)}$$

where w_0 is the longest element of $\mathfrak{S}_3$. We want to show that these are special cases of a general rule for decomposing the element in stair form σ_α. Before we state the rule, we compare the sequences used to define the element in stair form in Definition 4.2.13 for compositions of n, n_1 and n_2.

Lemma 4.3.52. *For $m \in \mathbb{N}_0$ let $x^{(m)}$ be the sequence $(x_1^{(m)}, \ldots, x_m^{(m)})$ given by $x_{2i-1}^{(m)} = i$ and $x_{2i}^{(m)} = m - i + 1$. Set $x := x^{(n)}$, $y := x^{(n_1)}$ and $z := x^{(n_2)}$.*

(*1*) *We have $\varphi_1(y_i) = x_i$ for all $i \in [n_1]$.*

(*2*) *If n_1 is even then $\varphi_2(z_i) = x_{i+n_1}$ for all $i \in [n_2]$.*

(*3*) *If n_1 is odd then $\varphi_2(w_0(z_i)) = x_{i+n_1}$ for all $i \in [n_2]$ where w_0 is the longest element of $\mathfrak{S}_{n_2}$.*

Proof. Recall that $k = \lceil \frac{n_1}{2} \rceil$ and $(n_1, n_2) \vDash_0 n$ by Notation 4.3.45. Let $i \in \mathbb{N}$. We mainly do straight forward calculations.

(1) Assume $2i - 1 \in [n_1]$. Then $i \le k$ and thus $\varphi_1(i) = i$. Consequently,

$$\varphi_1(y_{2i-1}) = \varphi_1(i) = i = x_{2i-1}.$$

Now, assume $2i \in [n_1]$. Then

$$\begin{aligned} n_1 - i + 1 = \lceil n_1 - i + 1 \rceil &\ge \left\lceil n_1 - \frac{n_1}{2} + 1 \right\rceil \\ &= \left\lceil \frac{n_1}{2} + 1 \right\rceil = \left\lceil \frac{n_1}{2} \right\rceil + 1 = k + 1, \end{aligned}$$

i.e. $\varphi_1(n_1 - i + 1) = n_1 + n_2 - i + 1$. Therefore,

$$\varphi_1(y_{2i}) = \varphi_1(n_1 - i + 1) = n_1 + n_2 - i + 1 = n - i + 1 = x_{2i}.$$

(2) Assume that n_1 is even. Then $n_1 = 2k$. If $2i - 1 \in [n_2]$ then we have

$$2(k + i) - 1 = n_1 + 2i - 1 \le n_1 + n_2 = n.$$

Thus,

$$\varphi_2(z_{2i-1}) = \varphi_2(i) = k + i = x_{2(k+i)-1} = x_{2i-1+n_1}.$$

Suppose $2i \in [n_2]$. Then $2(k+i) = n_1 + 2i \leq n$ and

$$\begin{aligned}\varphi_2(z_{2i}) = k + n_2 - i + 1 &= (n - 2k - n_2) + k + n_2 - i + 1 \\ &= n - k - i + 1 \\ &= x_{2(k+i)} = x_{2i+n_1}.\end{aligned}$$

(3) Assume that n_1 is odd. In this case $n_1 = 2k - 1$. Let w_0 be the longest element of $\mathfrak{S}_{n_2}$. We have $w_0(j) = n_2 - j + 1$ for all $j \in [n_2]$. If $2i - 1 \in [n_2]$ then $2i - 1 + n_1 \in [n]$ and

$$\begin{aligned}\varphi_2(w_0(z_{2i-1})) &= \varphi_2(w_0(i)) \\ &= \varphi_2(n_2 - i + 1) \\ &= n_2 + k - i + 1 \\ &= (n - 2k + 1 - n_2) + n_2 + k - i + 1 \\ &= n - (k + i - 1) + 1 \\ &= x_{2(i+k-1)} \\ &= x_{2i-1+2k-1} = x_{2i-1+n_1}.\end{aligned}$$

If $2i \in [n_2]$ then $2i + n_1 \in [n]$ and

$$\varphi_2(w_0(z_{2i})) = \varphi_2(w_0(n_2 - i + 1)) = \varphi_2(i) = i + k = x_{2(i+k)-1} = x_{2i+n_1}. \qquad \square$$

Example 4.3.53. Consider $n = 9$, $n_1 = 6$ and $n_2 = 3$. Then $k = 3$. Using the notation from Lemma 4.3.52 we obtain

$$\begin{aligned}x &= (1, 9, 2, 8, 3, 7, 4, 6, 5), \\ y &= (1, 6, 2, 5, 3, 4), \\ z &= (1, 3, 2).\end{aligned}$$

Then $x = (\varphi_1(y_1), \ldots, \varphi_1(y_6), \varphi_2(z_1), \varphi_2(z_2), \varphi_2(z_3))$ as predicted by Lemma 4.3.52. Moreover, x, y and z are the sequences used to define the elements in stair form $\sigma_{(6,3)}$, $\sigma_{(6)}$ and $\sigma_{(3)}$, respectively. Therefore,

$$\sigma_{(6,3)} = (\varphi_1(y_1), \ldots, \varphi_1(y_6))(\varphi_2(z_1), \varphi_2(z_2), \varphi_2(z_3)) = \sigma_{(6)}^{\varphi_1}\sigma_{(3)}^{\varphi_2} = \sigma_{(6)} \odot \sigma_{(3)}.$$

This also illustrates the idea of the proof of the next lemma.

Lemma 4.3.54. *Let $\alpha = (\alpha_1, \ldots, \alpha_l) \vDash_e n$ with $l \geq 1$. Then we have the following.*

(1) If α_1 is even then $\sigma_\alpha = \sigma_{(\alpha_1)} \odot \sigma_{(\alpha_2,\ldots,\alpha_l)}$.

(2) *If α_1 is odd then $\sigma_\alpha = \sigma_{(\alpha_1)} \odot \left(\sigma_{(\alpha_2,\dots,\alpha_l)}\right)^{w_0}$ where w_0 is the longest element of $\mathfrak{S}_{\alpha_2+\cdots+\alpha_l}$.*

Proof. Set $n_1 := \alpha_1$ and $n_2 := \alpha_2 + \cdots + \alpha_l$. As in Lemma 4.3.52, let $x^{(m)}$ be the sequence $(x_1^{(m)}, \dots, x_m^{(m)})$ given by $x_{2i-1}^{(m)} = i$ and $x_{2i}^{(m)} = m - i + 1$ for $m \in \mathbb{N}_0$ and set $x := x^{(n)}$, $y := x^{(n_1)}$ and $z := x^{(n_2)}$. We have that

(1) σ_α has the cycles

$$\sigma_{\alpha_i} = \left(x_{\alpha_1+\cdots+\alpha_{i-1}+1}, x_{\alpha_1+\cdots+\alpha_{i-1}+2}, \dots, x_{\alpha_1+\cdots+\alpha_{i-1}+\alpha_i}\right)$$

for $i = 1, \dots, l$,

(2) $\sigma_{(\alpha_1)} = (y_1, y_2, \dots, y_{n_1})$ and

(3) $\sigma_{(\alpha_2,\dots,\alpha_l)}$ has the cycles

$$\tilde{\sigma}_{\alpha_i} = \left(z_{\alpha_2+\cdots+\alpha_{i-1}+1}, z_{\alpha_2+\cdots+\alpha_{i-1}+2}, \dots, z_{\alpha_2+\cdots+\alpha_{i-1}+\alpha_i}\right)$$

for $i = 2, \dots, l$.

Assume that α_1 is even and set $\sigma := \sigma_{(\alpha_1)} \odot \sigma_{(\alpha_2,\dots,\alpha_l)}$. From Lemma 4.3.47 we obtain that σ has the cycles $(\sigma_{(\alpha_1)})^{\varphi_1}$ and $(\tilde{\sigma}_{(\alpha_i)})^{\varphi_2}$ for $i = 2, \dots, l$. By Lemma 4.3.52, $\varphi_1(y_j) = x_j$ for $j \in [n_1]$ and $\varphi_2(z_j) = x_{\alpha_1+j}$ for $j \in [n_2]$. As a consequence,

$$(\sigma_{(\alpha_1)})^{\varphi_1} = (\varphi_1(y_1), \dots, \varphi_1(y_{\alpha_1})) = (x_1, \dots, x_{\alpha_1}) = \sigma_{\alpha_1}$$

and

$$\begin{aligned}(\tilde{\sigma}_{\alpha_i})^{\varphi_2} &= \left(\varphi_2(z_{\alpha_2+\cdots+\alpha_{i-1}+1}), \dots, \varphi_2(z_{\alpha_2+\cdots+\alpha_{i-1}+\alpha_i})\right) \\ &= \left(x_{\alpha_1+\cdots+\alpha_{i-1}+1}, \dots, x_{\alpha_1+\cdots+\alpha_{i-1}+\alpha_i}\right) \\ &= \sigma_{\alpha_i}\end{aligned}$$

for $i = 2, \dots, l$. Hence, $\sigma = \sigma_\alpha$.

Now let α_1 be odd. Set $\sigma := \sigma_{(\alpha_1)} \odot \left(\sigma_{(\alpha_2,\dots,\alpha_l)}\right)^{w_0}$ where w_0 is the longest element of $\mathfrak{S}_{\alpha_2+\cdots+\alpha_l}$. Then σ has the cycles $(\sigma_{(\alpha_1)})^{\varphi_1}$ and $((\tilde{\sigma}_{(\alpha_i)})^{w_0})^{\varphi_2}$ for $i = 2, \dots, l$. Moreover, from Lemma 4.3.52 we have that $\varphi_2(w_0(z_j)) = x_{\alpha_1+i}$ for $j \in [n_2]$. Thus,

$$\begin{aligned}(\tilde{\sigma}_{(\alpha_i)})^{w_0})^{\varphi_2} &= \left(\varphi_2(w_0(z_{\alpha_2+\cdots+\alpha_{i-1}+1})), \dots, \varphi_2(w_0(z_{\alpha_2+\cdots+\alpha_{i-1}+\alpha_i}))\right) \\ &= \left(x_{\alpha_1+\cdots+\alpha_{i-1}+1}, \dots, x_{\alpha_1+\cdots+\alpha_{i-1}+\alpha_i}\right) \\ &= \sigma_{\alpha_i}\end{aligned}$$

for $i = 2, \dots, l$. As we have already shown that $(\sigma_{(\alpha_1)})^{\varphi_1} = \sigma_{\alpha_1}$, it follows that $\sigma = \sigma_\alpha$. $\square$

The upcoming Theorem 4.3.55 is the main result of this subsection. It enables us to decompose Σ_α if α_1 is even. Before we can state the result, we need to introduce some more notation. For $\alpha \vDash_e n$ we define

$$\Sigma_\alpha^\times := \{\sigma \in \Sigma_\alpha \mid P(\sigma) = P(\sigma_\alpha)\}.$$

In Theorem 4.3.55 the set $(\Sigma_\alpha^\times)^{w_0}$ appears where w_0 the longest element of $\mathfrak{S}_n$. Let $\sigma \in \Sigma_\alpha$. Then by Corollary 4.1.16, $\sigma^{w_0} \in \Sigma_\alpha$. Since $P(\sigma^{w_0}) = w_0(P(\sigma))$, we have

$$\sigma \in (\Sigma_\alpha^\times)^{w_0} \iff P(\sigma^{w_0}) = P(\sigma_\alpha) \iff P(\sigma) = P(\sigma_\alpha^{w_0}). \tag{4.19}$$

Theorem 4.3.55. *Let $\alpha = (\alpha_1, \dots, \alpha_l) \vDash_e n$ with $l \geq 1$.*

(1) *If α_1 is even then $\Sigma_\alpha = \Sigma_{(\alpha_1)} \odot \Sigma_{(\alpha_2,\dots,\alpha_l)}$.*

(2) *If α_1 is odd then $\Sigma_\alpha^\times = \Sigma_{(\alpha_1)}^\times \odot \left(\Sigma_{(\alpha_2,\dots,\alpha_l)}^\times\right)^{w_0}$ where w_0 is the longest element of $\mathfrak{S}_{\alpha_2+\dots+\alpha_l}$.*

Proof. Let $\alpha^{(1)} := (\alpha_1)$, $\alpha^{(2)} := (\alpha_2, \dots, \alpha_l)$, $n_1 := |\alpha^{(1)}|$, $n_2 := |\alpha^{(2)}|$ and w_0 be the longest element of $\mathfrak{S}_{n_2}$. We use the inductive product on $\mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2}$ and the related notation. The proofs of (1) and (2) have a lot in common. Hence, we do them simultaneously as much as possible and separate the cases α_1 even and α_1 odd only when necessary.

If $l = 1$ then $\alpha = \alpha^{(1)}$, $\alpha^{(2)} = \emptyset$ and thus

$$\Sigma_{\alpha^{(1)}} \odot \Sigma_{\alpha^{(2)}} = \Sigma_\alpha \odot \mathfrak{S}_0 = \Sigma_\alpha.$$

Moreover, $\Sigma_{(\alpha_1)}^\times = \Sigma_{(\alpha_1)}$ and $\left(\Sigma_\emptyset^\times\right)^{w_0} = \Sigma_\emptyset$. Thus we have (1) and (2) in this case.

Now suppose $l \geq 2$. Let $\sigma := \sigma_\alpha$, $\sigma_1 := \sigma_{\alpha^{(1)}}$ and $\sigma_2 := \sigma_{\alpha^{(2)}}$ if α_1 is even and $\sigma_2 = \sigma_{\alpha^{(2)}}^{w_0}$ if α_1 is odd. From Lemma 4.3.54 we have $\sigma = \sigma_1 \odot \sigma_2$. By Proposition 4.2.14, $\sigma_{\alpha^{(i)}} \in \Sigma_{\alpha^{(i)}}$ for $i = 1, 2$. In addition, Corollary 4.1.16 then yields that $\sigma_{\alpha^{(2)}}^{w_0} \in \Sigma_{\alpha^{(2)}}$. Consequently, $\sigma_i \in \Sigma_{\alpha^{(i)}}$ for $i = 1, 2$.

We begin with the inclusions "$\subseteq$". Let $\tau \in \Sigma_\alpha$ with $P(\tau) = P(\sigma)$ if α_1 is odd. First we show $\tau \in \mathfrak{S}_{n_1} \odot \mathfrak{S}_{n_2}$. By Lemma 4.3.50, we have to show $\tau(N_i) = N_i$ for $i = 1, 2$. Since $\{N_1, N_2\}$ is a set partition of $[n]$, it suffices to show $\tau(N_1) = N_1$. As $\sigma_1 \in \mathfrak{S}_{n_1}$ is an n_1-cycle, $P(\sigma_1) = \{[n_1]\}$. Moreover, Corollary 4.3.48 yields $P(\sigma) = \varphi_1(P(\sigma_1)) \cup \varphi_2(P(\sigma_2))$. Thus,

$$N_1 = \varphi_1([n_1]) \in \varphi_1(P(\sigma_1)) \subseteq P(\sigma).$$

If α_1 is even then $N_1 \in P_e(\sigma)$. Moreover, Proposition 4.2.25 yields $P_e(\tau) = P_e(\sigma)$. Thus, $N_1 \in P(\tau)$ which means that $\tau(N_1) = N_1$. If α_1 is odd then $P(\tau) = P(\sigma)$ by assumption. Hence, $N_1 \in P(\sigma) = P(\tau)$ and thus $\tau(N_1) = N_1$.

Because $\tau \in \mathfrak{S}_{n_1} \odot \mathfrak{S}_{n_2}$, there are $\tau_1 \in \mathfrak{S}_{n_1}$ and $\tau_2 \in \mathfrak{S}_{n_2}$ such that $\tau = \tau_1 \odot \tau_2$. Let $i \in \{1, 2\}$. We want to show $\tau_i \in \Sigma_{\alpha^{(i)}}$. Recall that $\sigma_i \in \Sigma_{\alpha^{(i)}}$. Thus, from Proposition 4.2.25 it follows that $\tau_i \in \Sigma_{\alpha^{(i)}}$ if and only if

(i) σ_i and τ_i are conjugate in $\mathfrak{S}_{n_i}$,

(ii) $\ell(\sigma_i) = \ell(\tau_i)$ and

(iii) $P_e(\sigma_i) = P_e(\tau_i)$.

Therefore, we show that τ_i satisfies (i) – (iii). Let i be arbitrary again.

(i) For a permutation ξ, let $C(\xi)$ be the multiset of cycle lengths of ξ. Assume $\xi = \xi_1 \odot \xi_2$ for $\xi_i \in \mathfrak{S}_{n_i}$ and $i = 1, 2$. From Lemma 4.3.47 it follows that

$$C(\xi) = C(\xi_1) \cup C(\xi_2). \tag{4.20}$$

Since $\tau = \tau_1 \odot \tau_2$, Corollary 4.3.48 implies $P(\tau) = \varphi_1(P(\tau_1)) \cup \varphi_2(P(\tau_2))$. Therefore, from $N_1 \in P(\tau)$ it follows that $P(\tau_1) = \{[n_1]\}$. That is, τ_1 is an n_1-cycle of $\mathfrak{S}_{n_1}$. By definition, σ_1 is an n_1-cycle of $\mathfrak{S}_{n_1}$ too. Thus, $C(\tau_1) = C(\sigma_1)$. Since $\tau \in \Sigma_\alpha$, τ and σ are conjugate so that $C(\tau) = C(\sigma)$. Because of (4.20) and $C(\tau_1) = C(\sigma_1)$, it follows that also $C(\tau_2) = C(\sigma_2)$. In other words, τ_i and σ_i are conjugate for $i = 1, 2$.

(ii) Let $m := \lfloor \frac{n_1}{2} \rfloor$. By Lemma 4.3.51, there are $p, q \leq m$ such that

$$\ell(\tau) = \ell(\tau_1) + \ell(\tau_2) + (p + q)n_2.$$

Moreover, we have $\ell(\tau_i) \leq \ell(\sigma_i)$ for $i = 1, 2$ because τ_i and σ_i are conjugate and $\sigma_i \in \Sigma_{\alpha^{(i)}}$. On the other hand, σ_1 is oscillating by Theorem 4.3.20 and hence Lemma 4.3.51 yields

$$\ell(\sigma) = \ell(\sigma_1) + \ell(\sigma_2) + 2mn_2.$$

Since $\tau \in \Sigma_\alpha$, we have $\ell(\tau) = \ell(\sigma)$. Therefore, we obtain from the equalities for $\ell(\tau)$ and $\ell(\sigma)$ and the inequalities for $\ell(\tau_1), \ell(\tau_2), p$ and q that $\ell(\tau_1) = \ell(\sigma_1)$ and $\ell(\tau_2) = \ell(\sigma_2)$.

(iii) Corollary 4.3.48 states that

$$P(\xi) = \varphi_1(P(\xi_1)) \cup \varphi(P(\xi_2)) \tag{4.21}$$

for $\xi = \sigma, \tau$. This equality remains valid if we replace P by P_e. From $\tau \in \Sigma_\alpha$ and Proposition 4.2.25 it follows that $P_e(\tau) = P_e(\sigma)$. Hence,

$$\varphi_1(P_e(\tau_1)) \cup \varphi_2(P_e(\tau_2)) = \varphi_1(P_e(\sigma_1)) \cup \varphi_2(P_e(\sigma_2)).$$

Since φ_1 and φ_2 are bijections and the images of φ_1 and φ_2 are disjoint, it follows that $P_e(\tau_i) = P_e(\sigma_i)$ for $i = 1, 2$. This finishes the proof of $\tau \in \Sigma_{\alpha^{(1)}} \odot \Sigma_{\alpha^{(2)}}$.

It remains to show that $\tau_1 \in \Sigma^\times_{\alpha^{(1)}}$ and $\tau_2 \in \left(\Sigma^\times_{\alpha^{(2)}}\right)^{w_0}$ if α_1 is odd. Thus, assume that α_1 is odd. We have already seen that $P(\tau_1) = P(\sigma_1)$. Hence, $\tau_1 \in \Sigma^\times_{\alpha^{(1)}}$. Since α_1 is odd, $P(\tau) = P(\sigma)$ by assumption and therefore we deduce from (4.21) as above that $P(\tau_2) = P(\sigma_2)$. Now we can use that $\sigma_2 = \sigma^{w_0}_{\alpha^{(2)}}$ and obtain $\tau_2 \in \left(\Sigma^\times_{\alpha^{(2)}}\right)^{w_0}$ from (4.19).

We continue with the inclusions "$\supseteq$". Let $\tau_i \in \Sigma_{\alpha^{(i)}}$ for $i = 1, 2$ and $\tau := \tau_1 \odot \tau_2$. If α_1 is odd, assume that in addition $\tau_1 \in \Sigma^\times_{\alpha^{(1)}}$ and $\tau_2 \in \left(\Sigma^\times_{\alpha^{(2)}}\right)^{w_0}$ which by (4.19) is equivalent to $P(\tau_i) = P(\sigma_i)$ for $i = 1, 2$.

We want to show $\tau \in \Sigma_\alpha$ and again use Proposition 4.2.25 to do this. That is, we show the properties (i) – (iii) for τ and σ.

(i) For $i \in \{1, 2\}$ we have $C(\tau_i) = C(\sigma_i)$ since $\tau_i \in \Sigma_{\alpha^{(i)}}$. Hence, from (4.20) it follows that $C(\tau) = C(\sigma)$, i.e. τ and σ are conjugate.

(ii) Since $\tau_1, \sigma_1 \in \Sigma_{\alpha^{(1)}}$, they are oscillating n_1-cycles by Theorem 4.3.20. Therefore, Lemma 4.3.51 yields

$$\ell(\xi) = \ell(\xi_1) + \ell(\xi_2) + 2mn_2$$

for $\xi = \sigma, \tau$ and $m = \lfloor \frac{n_1}{2} \rfloor$. Moreover, as $\sigma_i, \tau_i \in \Sigma_{\alpha^{(i)}}$, $\ell(\tau_i) = \ell(\sigma_i)$ for $i = 1, 2$. Hence, $\ell(\tau) = \ell(\sigma)$.

(iii) Since $\xi = \xi_1 \odot \xi_2$ for $\xi = \sigma, \tau$, Equation (4.21) holds. This equation remains true if we substitute P by P_e. In addition, from Proposition 4.2.25 we obtain that $P_e(\tau_i) = P_e(\sigma_i)$ for $i = 1, 2$. Thus, $P_e(\tau) = P_e(\sigma)$.

Because of (i) – (iii) we can now apply Proposition 4.2.25 and obtain that $\tau \in \Sigma_\alpha$. In the case where α_1 is odd, it remains to show $P(\tau) = P(\sigma)$. But this is merely a consequence of $P(\tau_i) = P(\sigma_i)$ for $i = 1, 2$ and (4.21). □

We now infer from Theorem 4.3.55 that the inductive product provides a bijection from $\Sigma_{(\alpha_1)} \times \Sigma_{(\alpha_2,\ldots,\alpha_l)}$ to Σ_α for all $\alpha \vDash_e n$ with even α_1.

Corollary 4.3.56. *Let $\alpha = (\alpha_1, \ldots, \alpha_l) \vDash_e n$ with $l \geq 1$.*

(1) If α_1 is even then the map $\Sigma_{(\alpha_1)} \times \Sigma_{(\alpha_2,\ldots,\alpha_l)} \to \Sigma_\alpha$, $(\sigma_1, \sigma_2) \mapsto \sigma_1 \odot \sigma_2$ is a bijection.

(2) If α_1 is odd then the map $\Sigma^\times_{(\alpha_1)} \times \left(\Sigma^\times_{(\alpha_2,\ldots,\alpha_l)}\right)^{w_0} \to \Sigma^\times_\alpha$, $(\sigma_1, \sigma_2) \mapsto \sigma_1 \odot \sigma_2$ where w_0 is the longest element of $\mathfrak{S}_{\alpha_2+\cdots+\alpha_l}$ is a bijection.

Proof. By Lemma 4.3.50 the two maps in question are injective. Theorem 4.3.55 shows that they are also surjective. □

Recall that, given a maximal composition $\alpha = (\alpha_1, \ldots, \alpha_l) \vDash_e n$, there exists $0 \leq j \leq l$ such that $\alpha_1, \ldots, \alpha_j$ are even and $\alpha_{j+1}, \ldots, \alpha_l$ are odd. Using Part (1) of Corollary 4.3.56 iteratively, we obtain the following decomposition of the elements of Σ_α.

Corollary 4.3.57. *Let $\alpha = (\alpha_1, \ldots, \alpha_l) \vDash_e n$, $\sigma \in \mathfrak{S}_n$ of type α and $0 \leq j \leq l$ be such that $\alpha' := (\alpha_{j+1}, \ldots, \alpha_l)$ are the odd parts of α. Then $\sigma \in \Sigma_\alpha$ if and only if there are $\sigma_i \in \Sigma_{(\alpha_i)}$ for $i = 1, \ldots, j$ and $\tau \in \Sigma_{\alpha'}$ such that*

$$\sigma = \sigma_1 \odot \sigma_2 \odot \cdots \odot \sigma_j \odot \tau$$

where the product is evaluated from right to left.

Example 4.3.58. Consider $\alpha = (2, 4, 3, 1, 1) \vDash_e 11$. From Table 4.1 and Example 4.3.33 we obtain

$$\begin{aligned}
\Sigma_{(2)} &= \{(1,2)\}\,, \\
\Sigma_{(4)} &= \{(1,4,2,3), (1,3,2,4)\}\,, \\
\Sigma_{(3,1,1)} &= \{(1,5,2), (1,2,5), (1,5,3), (1,3,5), (1,5,4), (1,4,5)\}\,.
\end{aligned}$$

By Corollary 4.3.57, Σ_α consists of all elements $(1,2) \odot (\sigma \odot \tau)$ with $\sigma \in \Sigma_{(4)}$ and $\tau \in \Sigma_{(3,1,1)}$. Thus, $|\Sigma_\alpha| = 12$. For instance,

$$\begin{aligned}(1,2) \odot ((1,3,2,4) \odot (1,3,5)) &= (1,2) \odot (1,8,2,9)(3,5,7) \\ &= (1,11)(2,9,3,10)(4,6,8)\end{aligned}$$

is an element of Σ_α.

Remark 4.3.59. For compositions with one part $\alpha = (n)$, Theorem 4.3.20 provides a combinatorial characterization of $\Sigma_{(n)}$. Therefore, Corollary 4.3.57 reduces the problem of describing Σ_α for each maximal composition α to the case where α has only odd parts. These α are the partitions consisting of odds parts.

If α is an odd hook, then Theorem 4.3.40 yields that the hook properties characterize the elements of Σ_α. That is, we have a description of Σ_α for all α whose odd parts form a hook. Generalizing this result would be interesting but is out of the scope of this thesis. Remark 4.3.76 gathers some observations on partitions with two odd parts.

Let $\alpha \vDash_e n$ and α' be the composition formed by the odd parts of α. We conclude the subsection with a formula that expresses $|\Sigma_\alpha|$ as a product of $|\Sigma_{\alpha'}|$ and a factor that only depends on the even parts of α. In the case where α' is an odd hook, we can determine $|\Sigma_{\alpha'}|$ explicitly and thus obtain a closed formula for $|\Sigma_\alpha|$.

Corollary 4.3.60. *Let $\alpha = (\alpha_1, \dots, \alpha_l) \vDash_e n$, $0 \leq j \leq l$ be such that $(\alpha_1, \dots, \alpha_j)$ are the even and $\alpha' := (\alpha_{j+1}, \dots, \alpha_l)$ are the odd parts of α, $n' := |\alpha'|$, $P := \{i \in [j] \mid \alpha_i \geq 4\}$, $p := |P|$ and $q := -2p + \frac{1}{2}\sum_{i \in P} \alpha_i$. Then*

$$|\Sigma_\alpha| = 2^p 3^q |\Sigma_{\alpha'}|.$$

Moreover, if α' is a hook $(r, 1^{n'-r})$ then

$$|\Sigma_\alpha| = \begin{cases} 2^p 3^q & \text{if } r \leq 1 \\ (n' - r + 1) 2^{p'} 3^{q'} & \text{if } r \geq 3 \end{cases}$$

where $p' := p + 1$ and $q' := q + \frac{r-3}{2}$.

Proof. Since $\alpha_1, \dots, \alpha_j$ are the even parts of α, Corollary 4.3.57 implies that

$$|\Sigma_\alpha| = |\Sigma_{\alpha'}| \prod_{i=1}^{j} |\Sigma_{(\alpha_i)}|. \tag{4.22}$$

For the same reason, Corollary 4.3.24 yields

$$|\Sigma_{(\alpha_i)}| = \begin{cases} 1 & \text{if } n \leq 2 \\ 2 \cdot 3^{\frac{\alpha_i - 4}{2}} & \text{if } n \geq 4. \end{cases}$$

for $i = 1, \dots, j$. Therefore,

$$\prod_{i=1}^{j} |\Sigma_{(\alpha_i)}| = \prod_{i \in P} 2 \cdot 3^{\frac{\alpha_i - 4}{2}} = 2^p 3^{-2p + \frac{1}{2}\sum_{i \in P} \alpha_i} = 2^p 3^q.$$

and with (4.22) we get the first statement.

For the second part, assume that α' is a hook. Then, by the choice of j, α' is an odd hook. It remains to compute $|\Sigma_{\alpha'}|$. If $\alpha' = \emptyset$ or $\alpha' = (1^{n'})$ we have $|\Sigma'_{\alpha}| = 1$. If $\alpha' = (r, 1^{n'-r})$ with $r \geq 3$ then Corollary 4.3.42 provides the formula

$$|\Sigma_{\alpha'}| = 2(n' - r + 1)3^{\frac{r-3}{2}}. \qquad \square$$

Example 4.3.61. Consider $\alpha = (2, 8, 4, 5, 1, 1, 1) \vDash_e 22$. Then $\alpha' = (5, 1, 1, 1) \vDash_e 8$ is a hook, $P = \{2, 3\}$, $p' = 2 + 1$ and $q' = -2 \cdot 2 + \frac{1}{2}(8 + 4) + \frac{5-3}{2} = 3$. Thus, Corollary 4.3.60 yields $|\Sigma_\alpha| = (8 - 5 + 1)2^3 3^3 = 864$.

4.3.4 Mild equivalence classes

In this subsection we use the inductive product to study oscillating permutations with connected intervals. The first goal is to show that for all $\alpha \vDash_e n$ and $\sigma \in \mathfrak{S}_n$ we have $\sigma \in \Sigma_\alpha$ if σ is oscillating with connected intervals and $P(\sigma) = P(\sigma_\alpha)$. This leads to a characterization of Σ_α for a certain type of compositions which we call *mild*. In this subsection we use the notions related to the inductive product introduced in Notation 4.3.45.

In Lemma 4.3.12 we showed that conjugating n-cycles of $\mathfrak{S}_n$ with w_0 preserves the properties of being oscillating and having connected intervals. We now generalize this result.

Lemma 4.3.62. *Let $\sigma \in \mathfrak{S}_n$.*

(*1*) *If σ is oscillating then σ^{w_0} is oscillating.*

(*2*) *If σ has connected intervals then σ^{w_0} has connected intervals.*

Proof. Let τ be a cycle of σ, t be the length of τ, w_0 be the longest element of $\mathfrak{S}_n$ and u_0 be the longest element of $\mathfrak{S}_t$. We consider the cycle τ^{w_0} of σ^{w_0}.

We show $\mathrm{cst}(\tau^{w_0}) = \mathrm{cst}(\tau)^{u_0}$ first. Fix a presentation of τ in cycle notation $\tau = (a_1, \dots, a_t)$. Then

$$\mathrm{cst}(\tau^{w_0}) = (b_1, \dots, b_t) \quad \text{and} \quad \mathrm{cst}(\tau)^{u_0} = (c_1, \dots, c_t)$$

where

$$b_i := \rho_{\tau^{w_0}}(w_0(a_i)) \quad \text{and} \quad c_i := u_0(\rho_\tau(a_i))$$

for $i = 1, \dots, t$. The t-cycles $\mathrm{cst}(\tau^{w_0})$ and $\mathrm{cst}(\tau)^{u_0}$ are elements of $\mathfrak{S}_t$. Hence,

$$\{b_1, \dots, b_t\} = \{c_1, \dots, c_t\} = [t].$$

Moreover, for all $i, j \in [t]$ we have

$$\begin{aligned} b_i < b_j &\iff \rho_{\tau^{w_0}}(w_0(a_i)) < \rho_{\tau^{w_0}}(w_0(a_j)) \\ &\iff w_0(a_i) < w_0(a_j) \\ &\iff a_i > a_j \\ &\iff \rho_\tau(a_i) > \rho_\tau(a_j) \\ &\iff u_0(\rho_\tau(a_i)) < u_0(\rho_\tau(a_j)) \\ &\iff c_i < c_j. \end{aligned}$$

Therefore, $b_i = c_i$ for all $i \in [t]$. That is, $\operatorname{cst}(\tau^{w_0}) = \operatorname{cst}(\tau)^{u_0}$.

We focus on Part (1). For Part (2) simply substitute all occurrences of the phrase *is oscillating* by *has connected intervals*. Assume that σ is oscillating. Then $\operatorname{cst}(\tau)$ is oscillating by definition. As $\operatorname{cst}(\tau)$ is a t-cycle in $\mathfrak{S}_t$ we can can apply Lemma 4.3.12 and obtain that $\operatorname{cst}(\tau)^{u_0}$ is oscillating too. Because $\operatorname{cst}(\tau^{w_0}) = \operatorname{cst}(\tau)^{u_0}$, it follows that τ^{w_0} is oscillating as well. Since each cycle of σ^{w_0} is given by τ^{w_0} for a cycle τ of σ, we are done. $\square$

We now show that $\sigma = \sigma_1 \odot \sigma_2$ is oscillating with connected intervals if and only if σ_1 and σ_2 have these properties.

Lemma 4.3.63. *Let $\sigma_1 \in \mathfrak{S}_{n_1}$, $\sigma_2 \in \mathfrak{S}_{n_2}$ and $\sigma := \sigma_1 \odot \sigma_2$. Then σ is oscillating (has connected intervals) if and only if σ_1 and σ_2 are oscillating (have connected intervals).*

Proof. Let $\sigma_r = \sigma_{r,1}\sigma_{r,2}\cdots\sigma_{r,p_r}$ be a decomposition in disjoint cycles for $r = 1, 2$. Fix an $r \in \{1, 2\}$ and a cycle $(c_1, \dots c_t) = \sigma_{r,j}$ of σ_r. Then by Lemma 4.3.47 we have that

$$\sigma_{r,j}^{\varphi_r} = (\varphi_r(c_1), \dots, \varphi_r(c_t)).$$

As φ_r is strictly increasing, it preserves the relative order of the cycle elements so that

$$\operatorname{cst}(\sigma_{r,j}) = \operatorname{cst}(\sigma_{r,j}^{\varphi_r}).$$

In addition, Lemma 4.3.47 provides the cycle decomposition

$$\sigma = \sigma_{1,1}^{\varphi_1} \cdots \sigma_{1,p_1}^{\varphi_1} \cdot \sigma_{2,1}^{\varphi_2} \cdots \sigma_{2,p_2}^{\varphi_2}.$$

of σ. Hence, σ is oscillating if and only σ_1 and σ_2 are oscillating. For the same reason, σ has connected intervals if and only if σ_1 and σ_2 have connected intervals. $\square$

We have already seen in Lemma 4.3.11 and Lemma 4.3.36 that the element in stair form σ_α is oscillating and has connected intervals if $\alpha \vDash_e n$ has only one part or is an odd hook. The lemma below generalizes this to all maximal compositions.

Lemma 4.3.64. *Let $\alpha \vDash_e n$. Then the element in stair form σ_α is oscillating and has connected intervals.*

Proof. Let $\alpha = (\alpha_1, \dots, \alpha_l) \vDash_e n$. We do an induction on l. If $l = 1$ then $\alpha = (n)$ and Lemma 4.3.11 states that σ_α is oscillating with connected intervals. Now suppose $l > 1$ and let w_0 be the longest element of $\mathfrak{S}_{\alpha_2+\cdots+\alpha_l}$. Then from Lemma 4.3.54 we obtain

$$\sigma_\alpha = \begin{cases} \sigma_{(\alpha_1)} \odot \sigma_{(\alpha_2,\dots,\alpha_l)} & \text{if } \alpha_1 \text{ is even} \\ \sigma_{(\alpha_1)} \odot \left(\sigma_{(\alpha_2,\dots,\alpha_l)}\right)^{w_0} & \text{if } \alpha_1 \text{ is odd.} \end{cases}$$

By induction hypothesis, $\sigma_{(\alpha_1)}$ and $\sigma_{(\alpha_2,\dots,\alpha_l)}$ are oscillating with connected intervals. Using Lemma 4.3.62, it follows that $\sigma^{w_0}_{(\alpha_2,\dots,\alpha_l)}$ is oscillating with connected intervals as well. Therefore we can apply Lemma 4.3.63 and obtain that in both cases σ is oscillating with connected intervals. $\square$

Let $\alpha \vDash_e n$. We now show that each $\sigma \in \Sigma_\alpha$ is necessarily oscillating and has connected intervals. Moreover, we give a sufficient condition for $\sigma \in \mathfrak{S}_n$ to be an element of Σ_α.

Theorem 4.3.65. *Let $\alpha \vDash_e n$ and $\sigma \in \mathfrak{S}_n$.*

(*1*) *If $\sigma \in \Sigma_\alpha$ then σ is oscillating and has connected intervals.*

(*2*) *Let σ_α be the element in stair form. If σ is oscillating with connected intervals and $P(\sigma) = P(\sigma_\alpha)$ then $\sigma \in \Sigma_\alpha$.*

Proof. Let $\tau \in \mathfrak{S}_n$.

(1) By Lemma 4.3.64 the element in stair form σ_α is oscillating and has connected intervals. In addition, if τ is oscillating with connected intervals and $\tau' := s_i \tau s_i \approx \tau$ for some $i \in [n-1]$ then also τ' is oscillating with connected intervals by Corollary 4.3.31 and Lemma 4.3.32. Hence, we can use an induction argument as in the proof of Theorem 4.3.20 in order to show that τ is oscillating and has connected intervals if $\tau \in \Sigma_\alpha$.

(2) Suppose that $\alpha = (\alpha_1, \dots, \alpha_l)$, $\sigma := \sigma_\alpha$, τ is oscillating with connected intervals and $P(\tau) = P(\sigma)$. We do an induction on l. If $l = 1$ then τ is an n-cycle and the claim is implied by Theorem 4.3.20.

Suppose $l > 1$. Let $\alpha^{(1)} := (\alpha_1)$, $\alpha^{(2)} := (\alpha_2, \dots, \alpha_l)$, $n_1 := |\alpha^{(1)}|$, $n_2 := |\alpha^{(2)}|$ and w_0 be the longest element of $\mathfrak{S}_{n_2}$. From Definition 4.2.13 we have the cycle decomposition $\sigma = \sigma_{\alpha_1}\sigma_{\alpha_2} \cdots \sigma_{\alpha_l}$. Set $\sigma^{(1)} := \sigma_{\alpha^{(1)}}$ and $\sigma^{(2)} := \sigma_{\alpha^{(2)}}$ if α_1 is even and $\sigma^{(2)} := \sigma^{w_0}_{\alpha^{(2)}}$ if α_1 is odd. We use the definitions from Notation 4.3.45.

By Lemma 4.3.54, $\sigma = \sigma^{(1)} \odot \sigma^{(2)}$. As $P(\tau) = P(\sigma)$, we can write τ as a product of disjoint cycles $\tau = \tau_1\tau_2 \cdots \tau_l$ such that the cycles τ_i and σ_{α_i} contain the same elements for $i = 1, \dots, l$. Using Corollary 4.3.48, we obtain

$$N_1 = \varphi_1([n_1]) \in \varphi_1(P(\sigma_{(\alpha_1)})) \subseteq P(\sigma) = P(\tau).$$

Thus, $\tau \in \mathfrak{S}_{n_1} \odot \mathfrak{S}_{n_2}$ by Lemma 4.3.50. It follows that $\tau = \tau^{(1)} \odot \tau^{(2)}$ where $\tau^{(1)} := (\tau_1|_{N_1})^{\varphi_1^{-1}}$ and $\tau^{(2)} := (\tau_2\tau_3 \cdots \tau_l|_{N_2})^{\varphi_2^{-1}}$ (cf. the proof of Lemma 4.3.50). Since each of the two n_1-cycles τ_1 and σ_1 consist of the elements of N_1, each of the n_1-cycles $\tau^{(1)}$ and $\sigma^{(1)}$ consist of the elements $\varphi^{-1}(N_1) = [n_1]$. Hence, $P(\tau^{(1)}) = P(\sigma^{(1)})$. Combining $P(\tau^{(1)}) = P(\sigma^{(1)})$, $P(\tau) = P(\sigma)$ and Corollary 4.3.48, we obtain that also $P(\tau^{(2)}) = P(\sigma^{(2)})$.

Since τ is oscillating with connected intervals by assumption, Lemma 4.3.63 implies that $\tau^{(i)}$ is oscillating and has connected intervals for $i = 1, 2$.

Assume that α_1 is even. Then $\sigma^{(2)} = \sigma_{\alpha^{(2)}}$ and hence $P(\tau^{(i)}) = P(\sigma_{\alpha^{(i)}})$ for $i = 1, 2$. In addition, we have seen that $\tau^{(i)}$ is oscillating and has connected intervals for $i = 1, 2$. Therefore, $\tau^{(i)} \in \Sigma_{\alpha^{(i)}}$ for $i = 1, 2$ by induction hypothesis. Since $\tau = \tau^{(1)} \odot \tau^{(2)}$, an application of Theorem 4.3.55 yields $\tau \in \Sigma_\alpha$ as desired.

Now let α_1 be odd. Then $P(\tau^{(1)}) = P(\sigma_{\alpha^{(1)}})$ and $P(\tau^{(2)}) = P(\sigma^{w_0}_{\alpha^{(2)}})$. Applying the induction hypothesis as above, we obtain $\tau^{(1)} \in \Sigma_{\alpha^{(1)}}$. Moreover, $\tau^{(1)} \in \Sigma^\times_{\alpha^{(1)}}$ since $P(\tau^{(1)}) = P(\sigma_{\alpha^{(1)}})$.

From $P(\tau^{(2)}) = P(\sigma^{w_0}_{\alpha^{(2)}})$ it follows that $P((\tau^{(2)})^{w_0}) = P(\sigma_{\alpha^{(2)}})$. Since $\tau^{(2)}$ is oscillating with connected intervals, Lemma 4.3.62 implies that $(\tau^{(2)})^{w_0}$ is oscillating with connected intervals. Consequently, we can apply the induction hypothesis and obtain $(\tau^{(2)})^{w_0} \in \Sigma_{\alpha^{(2)}}$. Thus, Corollary 4.1.16 implies $\tau^{(2)} \in \Sigma_{\alpha^{(2)}}$. Hence, (4.19) and $P(\tau^{(2)}) = P(\sigma^{w_0}_{\alpha^{(2)}})$ yield $\tau^{(2)} \in \left(\Sigma^\times_{\alpha^{(2)}}\right)^{w_0}$. To sum up, we have $\tau^{(1)} \in \Sigma^\times_{\alpha^{(1)}}$ and $\tau^{(2)} \in \left(\Sigma^\times_{\alpha^{(2)}}\right)^{w_0}$. Therefore, we can apply Theorem 4.3.55 and obtain that $\tau \in \Sigma^\times_\alpha$. $\square$

Let $\alpha \vDash_e n$ and $\sigma \in \mathfrak{S}_n$ be conjugate to σ_α. Then in general, σ being oscillating with connected intervals is not sufficient for $\sigma \in \Sigma_\alpha$. This is shown by the following example.

Example 4.3.66. Consider the maximal composition $\alpha = (2, 1)$. The element in stair form is given by $\sigma_\alpha = (1, 3)(2)$. Let $\sigma = (1, 2)(3)$. Then σ is oscillating with connected intervals and has type α. But $\ell(\sigma) = 1 < \ell(\sigma_\alpha) = 3$. Hence, $\sigma \notin \Sigma_\alpha$.

In general, the sufficient condition for $\sigma \in \Sigma_\alpha$ stated in the second part of Theorem 4.3.65 is not a necessary condition: By Example 4.3.33, it is not satisfied by some elements of $\Sigma_{(3,1,1)}$. Another example is given below.

Example 4.3.67. Let $\alpha = (3, 3)$. The corresponding element in stair form is given by $\sigma_\alpha = (1, 6, 2)(5, 3, 4)$. Let $\sigma = s_2 \sigma_\alpha s_2$. On the one hand, $\sigma = (1, 6, 3)(5, 2, 4)$, i.e. $P(\sigma) \neq P(\sigma_\alpha)$. On the other hand, $\sigma_\alpha(2) = 1 < \sigma_\alpha(3) = 4$ and $\sigma_\alpha^{-1}(2) = 6 > \sigma_\alpha^{-1}(3) = 5$. Consequently, $\ell(\sigma) = \ell(\sigma_\alpha)$ by Lemma 4.2.22. Now Lemma 4.2.20 implies $\sigma \approx \sigma_\alpha$ so that $\sigma \in \Sigma_\alpha$ by Proposition 4.2.14.

One may ask for which compositions α Part (2) of Theorem 4.3.65 is an equivalence. In the following we answer this question.

Definition 4.3.68. *We call a maximal composition $\alpha \vDash_e n$* mild *if α has at most one odd part or each odd part of α is* 1*. In this case, we also call Σ_α* mild.

Proposition 4.3.69. *Let $\alpha \vDash_e n$ and σ_α be the element in stair form. Then α is mild if and only if $P(\sigma) = P(\sigma_\alpha)$ for all $\sigma \in \Sigma_\alpha$.*

Proof. First assume that α is mild and let $\sigma \in \Sigma_\alpha$. From Lemma 4.2.23 follows that σ and σ_α have the same orbits of even length on $[n]$. Thus, if α has no odd part then the implication is clear. If α has exactly one odd part then σ and σ_α each have exactly one odd orbit which contains all the elements not contained in the even orbits. Therefore, also the two odd orbits coincide. If all odd parts of α are 1, then each element $i \in [n]$ which is not contained in an even orbit is a fixed by σ and σ_α. Hence, σ and σ_α have the same orbits on $[n]$.

Assume now that α is not mild. An illustration of the following is given by Example 4.3.67. Let $\alpha = (\alpha_1, \dots, \alpha_{\ell(\alpha)})$. Then α has at least two odd parts and at least one of them is strictly greater than 1. Let r be minimal such that α_r is odd. By the definition of maximal compositions, α_i is even for $i < r$, α_i is odd for $i \geq r$ and $\alpha_r \geq \alpha_{r+1} \geq \dots \geq \alpha_{\ell(\alpha)}$. Hence, $\alpha_r > 1$ and α_{r+1} is odd. Let $k := \frac{1}{2}\sum_{i=1}^{r-1} \alpha_i + 1$ and $l := k + \frac{\alpha_r - 1}{2}$. Then $l = \frac{1}{2}(1 + \sum_{i=1}^{r} \alpha_i) \leq \frac{n}{2}$.

We set $\sigma := s_l \sigma_\alpha s_l$ and show that $\sigma \in \Sigma_\alpha$ and that the orbits of σ and σ_α on $[n]$ are not the same. We deal with two cases depending on α_{r+1}.

If $\alpha_{r+1} > 1$, then the cycles of σ_α corresponding to α_r and α_{r+1} look as follows

$$(k, n-k+1, k+1, n-k, \dots, n-l+2, l)(n-l+1, l+1, n-l, \dots). \tag{4.23}$$

Hence,

$$\begin{aligned} \sigma_\alpha(l) = k &< n-l = \sigma_\alpha(l+1) \\ \sigma_\alpha^{-1}(l) = n-l+2 &> n-l+1 = \sigma_\alpha^{-1}(l+1) \end{aligned}$$

where we use $k < l$ and $l \leq \frac{n}{2}$ for the first inequality. Thus, by Lemma 4.2.22 we have $\ell(\sigma) = \ell(\sigma_\alpha)$. Hence, $\sigma \approx \sigma_\alpha$ by Lemma 4.2.20 and Proposition 4.2.14 implies $\sigma \in \Sigma_\alpha$. On the other hand, we obtain σ from σ_α by interchanging two elements between two nontrivial cycles. Hence, the corresponding orbits on $[n]$ also change.

Assume now that $\alpha_{r+1} = 1$. Since α is a maximal composition, it follows that then $\alpha_i = 1$ for all $i > r$. Then the definition of σ_α implies that $l+1$ is a fixpoint of σ_α and that the cycle corresponding to α_r is the same as in (4.23). Therefore,

$$\begin{aligned} \sigma_\alpha(l) = k &< l+1 = \sigma_\alpha(l+1) \\ \sigma_\alpha^{-1}(l) = n-l+2 &> l+1 = \sigma_\alpha^{-1}(l+1) \end{aligned}$$

where we use that $n - l + 2 \geq \frac{n}{2} + 2 > \frac{n}{2} + 1 \geq l + 1$ for the second inequality. As before this means that $\sigma \in \Sigma_\alpha$. On the other hand, l is a fixpoint of σ but not of σ_α. That is, the sets of orbits on $[n]$ of σ and σ_α are different. □

Let $\alpha \vDash_e n$. We now show that Part (2) of Theorem 4.3.65 characterizes the elements of Σ_α if α is mild. Note that by Proposition 4.3.69, the mild compositions are exactly the maximal compositions for which we can characterize Σ_α in this way.

Theorem 4.3.70. *Let $\alpha \vDash_e n$ be mild, σ_α the element in stair form and $\sigma \in \mathfrak{S}_n$. Then $\sigma \in \Sigma_\alpha$ if and only if σ is oscillating with connected intervals and $P(\sigma) = P(\sigma_\alpha)$.*

Proof. The implication from right to left is given by Theorem 4.3.65.

For the other direction assume that $\sigma \in \Sigma_\alpha$. Then σ is oscillating with connected intervals by Theorem 4.3.65. Moreover, Proposition 4.3.69 yields $P(\sigma) = P(\sigma_\alpha)$ since α is mild. □

Corollary 4.3.71. *Let $\alpha = (\alpha_1, \dots, \alpha_l) \vDash_e n$. Then*

$$|\Sigma_\alpha| \geq \prod_{i=1}^{l} |\Sigma_{(\alpha_i)}|$$

and we have equality if and only if α is mild.

Proof. Since α is maximal, there is a j such that α_i is even for all $i \leq j$ and α_i is odd for all $i > j$. Set $\alpha' := (\alpha_{j+1}, \dots, \alpha_l)$.

We can use Equation (4.22) which states that $|\Sigma_\alpha| = |\Sigma_{\alpha'}| \prod_{i=1}^{j} |\Sigma_{(\alpha_i)}|$. In addition, we have $|\Sigma_{\alpha'}| \geq |\Sigma_{\alpha'}^\times|$. Note that $|(\Sigma_\beta^\times)^{w_0}| = |\Sigma_\beta^\times|$ where w_0 is the longest elements of $\mathfrak{S}_m$ for all $m \in \mathbb{N}$ and $\beta \vDash_e m$. Thus we obtain from using Part (2) of Corollary 4.3.56 inductively that $|\Sigma_{\alpha'}^\times| = \prod_{i=j+1}^{l} |\Sigma_{(\alpha_i)}^\times|$. As (α_i) is a mild composition for all i, we obtain that $|\Sigma_{(\alpha_i)}^\times| = |\Sigma_{(\alpha_i)}|$ from Proposition 4.3.69. Therefore, we have

$$|\Sigma_\alpha| = |\Sigma_{\alpha'}| \prod_{i=1}^{j} |\Sigma_{(\alpha_i)}| \geq |\Sigma_{\alpha'}^\times| \prod_{i=1}^{j} |\Sigma_{(\alpha_i)}| = \prod_{i=1}^{l} |\Sigma_{(\alpha_i)}|. \tag{4.24}$$

Moreover,

$$\alpha \text{ is mild} \iff \alpha' \text{ is mild} \iff |\Sigma_{\alpha'}| = |\Sigma_{\alpha'}^\times|$$

where we use Proposition 4.3.69 for the last equivalence. Therefore, we have equality in (4.24) if and only if α is mild. ☐

We continue with the even hooks. Let α be such a hook. Then α is mild, since each odd part of α equals 1. Thus, we can use Theorem 4.3.70 in order to extend Theorem 4.3.40 to even hooks.

Theorem 4.3.72. *Let $\alpha \vDash_e n$ be a hook and $\sigma \in \mathfrak{S}_n$ of type α. Then $\sigma \in \Sigma_\alpha$ if and only if σ satisfies the hook properties.*

Proof. Let $\alpha = (k, 1^{n-k}) \vDash_e n$ and σ_α be the element in stair form. The case where k is odd was done in Theorem 4.3.40. Therefore, let k be even and $\sigma \in \mathfrak{S}_n$ of type α. Then α is mild and Theorem 4.3.70 implies that $\sigma \in \Sigma_\alpha$ if and only if σ is oscillating, has connected intervals and $P(\sigma) = P(\sigma_\alpha)$. On the other hand, recall that σ satisfies the hook properties if and only if σ is oscillating, σ has connected intervals and $1, 2, \dots, m, n-m+1, n-m+2, \dots, n$ appear in the cycle of length k of σ where $m = \frac{k}{2}$. That is, it remains to show that $P(\sigma) = P(\sigma_\alpha)$ is equivalent to the third hook property.

Two k-cycles have the same orbits on $[n]$ if and only if the same elements occur in their respective cycle of length k since all other elements of $[n]$ are fixpoints. By definition, the cycle of length k of σ_α, consists of the elements $1, 2, \dots, m, n-m+1, n-m+2, \dots, n$. ☐

Let $\alpha = (k, 1^{n-k}) \vDash_e n$ be a hook. From Corollary 4.3.41 we know how to construct Σ_α from $\Sigma_{(k)}$ if k is odd. If k is even, we obtain Σ_α in the following way.

Table 4.2: The cardinalities of Σ_α and $P(\Sigma_\alpha)$ for some partitions α with two odd parts.

α	$\lvert\Sigma_\alpha\rvert$	$\lvert P(\Sigma_\alpha)\rvert$
$(3,3)$	22	6
$(5,3)$	80	10
$(7,3)$	240	10
$(9,3)$	720	10
$(11,3)$	2160	10
$(5,5)$	664	36
$(7,5)$	2156	52
$(9,5)$	6468	52
$(11,5)$	19404	52
$(7,7)$	18596	210
$(9,7)$	57700	274
$(11,7)$	173100	274

Corollary 4.3.73. *Let $\alpha = (k, 1^{n-k}) \vDash_e n$ be an even hook and* $\mathrm{id} \in \mathfrak{S}_{n-k}$*. Then the map $\Sigma_{(k)} \to \Sigma_\alpha$, $\sigma \mapsto \sigma \odot \mathrm{id}$ is a bijection.*

Proof. Recall $\Sigma_{(1^{n-k})} = \{\mathrm{id}\}$. Then Corollary 4.3.56 yields that the map from the claim is a bijection. □

Example 4.3.74. Consider $\alpha = (4, 1, 1)$ and $\mathrm{id} \in \mathfrak{S}_2$. From Table 4.1 we read

$$\Sigma_{(4)} = \{(1, 4, 2, 3), (1, 3, 2, 4)\}$$

Hence, Corollary 4.3.73 yields

$$\sigma_\alpha = \left\{\sigma \odot \mathrm{id} \mid \sigma \in \Sigma_{(4)}\right\} = \{(1, 6, 2, 5), (1, 5, 2, 6)\}.$$

The cardinality of Σ_α in the case where α is a hook is given as follows.

Corollary 4.3.75. *Let $\alpha = (k, 1^{n-k}) \vDash_e n$ be a hook. Then*

$$|\Sigma_\alpha| = \begin{cases} 1 & \text{if } k \leq 2 \\ 2 \cdot 3^{\frac{k-4}{2}} & \text{if } k \geq 3 \text{ and } k \text{ is even} \\ 2(n-k+1)3^{\frac{k-3}{2}} & \text{if } k \geq 3 \text{ and } k \text{ is odd.} \end{cases}$$

Proof. Use the second part of Corollary 4.3.60. □

We end this chapter with a remark on the open cases in the description of the elements of ${}^{(\mathfrak{S}_n)_{\max}}\!/\!_{\approx}$.

Remark 4.3.76. In Remark 4.3.59 we reduced the problem of describing Σ_α for all maximal compositions α to the partitions with only odd parts. Therefore, it would be interesting to find a combinatorial description of Σ_α if α is a partition of odd parts which is not a hook. Unfortunately, the situation is a lot more complex. One reason for this is the following. For any subset Σ of $\mathfrak{S}_n$ define

$$P(\Sigma) := \{P(\sigma) \mid \sigma \in \Sigma\}\,.$$

By Proposition 4.3.69, $P(\sigma_\alpha)$ is not the only element of $P(\Sigma_\alpha)$ and there seems to be no simple way to describe $P(\Sigma_\alpha)$. Moreover, the number of $\sigma \in \Sigma_\alpha$ whose orbits yield the same set partition of $[n]$ depends on this very set partition. For example, $\Sigma_{(3,3)}$ consists of the following elements where elements with the same orbit partition occur in the same row.

$$\begin{array}{llll}
(1,6,2)(3,4,5) & (1,2,6)(3,4,5) & (1,6,2)(3,5,4) & (1,2,6)(3,5,4) \\
(1,6,3)(2,4,5) & (1,6,3)(2,5,4) & (1,3,6)(2,4,5) & (1,3,6)(2,5,4) \\
(1,4,5)(2,6,3) & (1,5,4)(2,3,6) & (1,5,4)(2,6,3) & (1,4,5)(2,3,6) \\
(1,6,4)(2,3,5) & (1,4,6)(2,3,5) & (1,6,4)(2,5,3) & (1,4,6)(2,5,3) \\
(1,6,5)(2,3,4) & (1,5,6)(2,3,4) & (1,5,6)(2,4,3) & (1,6,5)(2,4,3) \\
(1,5,3)(2,4,6) & (1,3,5)(2,6,4) & &
\end{array}$$

However, for partitions with two odd parts, the data shown in Table 4.2 suggests that there are the following recurrence relations. Let $k > l \geq 3$ be two odd integers. Then

$$\begin{aligned}
\left|\Sigma_{(k+2,l)}\right| &= 3\left|\Sigma_{(k,l)}\right| \\
\left|P(\Sigma_{(k+2,l)})\right| &= \left|P(\Sigma_{(k,l)})\right|.
\end{aligned}$$

The first relation also holds for odd n-cycles by Corollary 4.3.22.

Regarding the description of $P(\Sigma_\alpha)$, there is the following property similar to the third hook property satisfied by the compositions $\alpha = (k,l)$ with $k > l$ from Table 4.2. Let $\sigma \in \Sigma_\alpha$ and $m := \frac{k-l}{2}$. Then $1,\ldots,m,n-m+1,\ldots,n$ are elements of the orbit of length k of σ on $[n]$.

5 The center acting on simple modules

In this chapter we study the action of the center of $H_n(0)$ on the simple $H_n(0)$-modules. Unless stated otherwise all notation related to Coxeter groups (such as S and w_0) refers to the symmetric group $\mathfrak{S}_n$. By Corollary 4.2.18, a basis of $Z(H_n(0))$ is given by the elements $\bar{\pi}_{\leq \Sigma_\alpha}$ for $\alpha \vDash_e n$. We are interested in determining $\bar{\pi}_{\leq \Sigma_\alpha} v_I$ for $\alpha \vDash_e n$ and $I \subseteq S$ where v_I is the generator of the simple $H_n(0)$-module $\boldsymbol{F}_I$. For instance, consider the maximal composition (1^n). The element in stair form associated to (1^n) is $\sigma_{(1^n)} = 1$. Thus, $\bar{\pi}_{\leq \Sigma_{(1^n)}} = 1$ and consequently $\bar{\pi}_{\leq \Sigma_{(1^n)}} v_I = v_I$ for all $I \subseteq S$.

If $n \geq 3$ then from Theorem 2.3.10 it follows that $H_n(0)$ has exactly three blocks: one isomorphic to $\boldsymbol{F}_S$, one isomorphic to $\boldsymbol{F}_\emptyset$ and a nontrivial block to which all other simple modules $\boldsymbol{F}_I$ with $I \neq \emptyset, S$ belong to. Calculations for $n \leq 9$ lead to the following conjecture.

Conjecture 5.0.1. *Let $n \geq 3$, $\alpha \vDash_e n$ with $\alpha \neq (1^n)$ and $\boldsymbol{F}_I$ be a simple $H_n(0)$-module belonging to the nontrivial block of $H_n(0)$. Then $\bar{\pi}_{\leq \Sigma_\alpha} v_I = 0$.*

The main result of the chapter is the verification of Conjecture 5.0.1 for all maximal compositions whose odd parts form a hook in Corollary 5.4.10. The proof is based on the combinatorial description of Σ_α for this family of maximal compositions developed in Section 4.3 (cf. Remark 4.3.59). The conjecture is complemented by Lemma 5.1.1 which deals with the remaining simple modules $\boldsymbol{F}_\emptyset$ and $\boldsymbol{F}_S$. It states that

$$\bar{\pi}_{\leq \Sigma_\alpha} v_\emptyset = v_\emptyset \quad \text{and} \quad \bar{\pi}_{\leq \Sigma_\alpha} v_S = \sum_{w \in \mathfrak{S}_{\leq \Sigma_\alpha}} (-1)^{\ell(w)} v_S$$

for all $\alpha \vDash_e n$.

The structure of the chapter is as follows. After some preparations in Section 5.1 we consider the elements $\bar{\pi}_{\leq \Sigma_\alpha}$ for three classes of compositions. We start with compositions with one part in Section 5.2, continue with odd hooks in Section 5.3 and finally use the inductive product in order to extend our results to maximal compositions whose odd parts form a hook in Section 5.4.

5.1 The action of central elements associated to maximal compositions

For $I \subseteq S$ we make use of the shorthand $\mathfrak{S}_I$ for the parabolic subgroup $(\mathfrak{S}_n)_I$. Likewise, for $\alpha \vDash_e n$ we may write $\mathfrak{S}_{\leq \Sigma_\alpha}$ instead of $(\mathfrak{S}_n)_{\leq \Sigma_\alpha}$.

For $I \subseteq S$ consider the simple $H_n(0)$-module $\boldsymbol{F}_I$ generated by v_I. For $i \in [n-1]$ we have

$$\bar{\pi}_i v_I = (\pi_{s_i} - 1)v_I = \begin{cases} -v_I & \text{if } s_i \in I \\ 0 & \text{if } s_i \notin I. \end{cases}$$

Let $\alpha \vDash_e n$ and $I \subseteq S$. Recall from Section 4.1 that

$$\mathfrak{S}_{\leq \Sigma_\alpha} = \{w \in \mathfrak{S}_n \mid w \leq \sigma \text{ for some } \sigma \in \Sigma_\alpha\}$$

and $\bar{\pi}_{\leq \Sigma_\alpha} = \sum_{w \in \mathfrak{S}_{\leq \Sigma_\alpha}} \bar{\pi}_w$. Therefore,

$$\bar{\pi}_{\leq \Sigma_\alpha} v_I = \sum_{w \in \mathfrak{S}_{\leq \Sigma_\alpha}} \bar{\pi}_w v_I = \sum_{\substack{w \in \mathfrak{S}_{\leq \Sigma_\alpha}, \\ w \in \mathfrak{S}_I}} (-1)^{\ell(w)} v_I = \sum_{w \in \mathfrak{S}_I \cap \mathfrak{S}_{\leq \Sigma_\alpha}} (-1)^{\ell(w)} v_I. \tag{5.1}$$

That is, $\bar{\pi}_{\leq \Sigma_\alpha}$ acts as $\sum_{w \in \mathfrak{S}_I \cap \mathfrak{S}_{\leq \Sigma_\alpha}} (-1)^{\ell(w)}$ on $\boldsymbol{F}_I$. We directly obtain three border cases.

Lemma 5.1.1. *Let $\alpha \vDash_e n$ and $I \subseteq S$.*

(*1*) *If $I = \emptyset$ then $\bar{\pi}_{\leq \Sigma_\alpha} v_I = v_I$.*

(*2*) *If $I = S$ then $\bar{\pi}_{\leq \Sigma_\alpha} v_I = \sum_{w \in \mathfrak{S}_{\leq \Sigma_\alpha}} (-1)^{\ell(w)} v_I$.*

(*3*) *If $\alpha = (1^n)$ then $\bar{\pi}_{\leq \Sigma_\alpha} v_I = v_I$.*

Conjecture 5.0.1 can be rephrased as

$$\bar{\pi}_{\leq \Sigma_\alpha} v_I = 0 \text{ for all } n \geq 3,\ \emptyset \subsetneq I \subsetneq S \text{ and } (1^n) \neq \alpha \vDash_e n.$$

That is, if Conjecture 5.0.1 is true and $\sum_{w \in \mathfrak{S}_{\leq \Sigma_\alpha}} (-1)^{\ell(w)}$ is known for each $\alpha \vDash_e n$ then we have a complete description of the action of $Z(H_n(0))$ on the simple $H_n(0)$-modules. Therefore, determining $\sum_{w \in \mathfrak{S}_{\leq \Sigma_\alpha}} (-1)^{\ell(w)}$ for each $\alpha \vDash_e n$ would be interesting. But this is beyond the scope of this thesis.

The strategy for proving Conjecture 5.0.1 for all $\alpha \vDash_e n$ whose odd parts form a hook is as follows. For the compositions in question we already have a combinatorial description of Σ_α (cf. Remark 4.3.59) from which we can infer properties of $\mathfrak{S}_I \cap \mathfrak{S}_{\leq \Sigma_\alpha}$ that imply $\sum_{w \in \mathfrak{S}_I \cap \mathfrak{S}_{\leq \Sigma_\alpha}} (-1)^{\ell(w)} = 0$. To be precise, we will show that $\mathfrak{S}_I \cap \mathfrak{S}_{\leq \Sigma_\alpha}$ is an interval in Bruhat order. From (5.1) it then follows that $\bar{\pi}_{\leq \Sigma_\alpha} v_I = 0$.

Let $\alpha \vDash_e n$ and $I \subseteq S$ be arbitrary. By Lemma 2.2.10, $\mathfrak{S}_I$ consists of all $w \in \mathfrak{S}_n$ with $w \leq w_0(I)$ and therefore is an order ideal of $\mathfrak{S}_n$ with respect to the Bruhat order. In addition, $\mathfrak{S}_{\leq \Sigma_\alpha}$ is an order ideal in Bruhat order by definition. Consequently,

$$\mathfrak{S}_I \cap \mathfrak{S}_{\leq \Sigma_\alpha} = \{w \in \mathfrak{S}_n \mid w \leq w_0(I) \text{ and } \exists \sigma \in \Sigma_\alpha \colon w \leq \sigma\}.$$

is an order ideal as well.

We are interested in the elements of $\mathfrak{S}_I \cap \mathfrak{S}_{\leq \Sigma_\alpha}$ that are maximal in Bruhat order. Therefore, we first consider $w_0(I)$ and then a characterization of $\leq$ called the *tableau* *criterion.* We will see that for all $y \in \mathfrak{S}_n$ there exists the meet $w_0(I) \wedge y$ in Bruhat order (although for $n \geq 3$, $(\mathfrak{S}_n, \leq)$ is not a lattice). Then it follows that each maximal element of $\mathfrak{S}_I \cap \mathfrak{S}_{\leq \Sigma_\alpha}$ is of the form $w_0(I) \wedge \sigma$ where $\sigma \in \Sigma_\alpha$. Moreover, we will see how one can compute $w_0(I) \wedge \sigma$ from $w_0(I)$ and σ. We begin with a description of $w_0(I)$.

Lemma 5.1.2. *Let $I \subseteq S$, $w := w_0(I)$ and $1 \leq a_1 < a_2 < \cdots < a_m = n$ be indices such that $S \setminus I = \{s_{a_1}, \ldots, s_{a_{m-1}}\}$. Then $w(1) = a_1$ and for $i \in [n-1]$ we have*

$$w(i+1) = \begin{cases} a_{k+1} & \text{if } i = a_k \text{ for some } k \\ w(i) - 1 & \text{otherwise.} \end{cases}$$

Proof. Let $w \in \mathfrak{S}_n$ be defined by the recursion above. We show $w = w_0(I)$. First note that w maps $[a_k]$ to itself for $k = 1, \ldots, m$. Thus, $w \in \mathfrak{S}_{S \setminus \{s_{a_k}\}}$ for $k = 1, \ldots, m$ by Lemma 2.2.4. Consequently,

$$w \in \bigcap_{k=1}^{m} \mathfrak{S}_{S \setminus \{s_{a_k}\}} = \mathfrak{S}_I.$$

Moreover, we obtain from the recursion that $D_R(w) = I$. Hence Proposition 2.2.8 yields $w = w_0(I)$. □

Example 5.1.3. For $n = 11$ and $I = S \setminus \{s_2, s_5, s_9\}$ we have

$$w_0(I) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 2 & 1 & 5 & 4 & 3 & 9 & 8 & 7 & 6 & 11 & 10 \end{pmatrix}.$$

Definition 5.1.4. *Given $x \in \mathfrak{S}_n$ and $k \in [n]$, let $x_{i,k}$ be the i-th element in the increasing rearrangement of $x(1), x(2), \ldots, x(k)$.*

The Tableau Criterion is a well-known characterization of the Bruhat order of the symmetric group [BB05, p. 63]. We use the following version and include a proof based on [BB05].

Theorem 5.1.5 (Tableau Criterion)**.** *For $x, y \in \mathfrak{S}_n$ the following are equivalent.*

(*1*) $x \leq y$.

(*2*) $x_{i,k} \leq y_{i,k}$ *for all $k \in [n-1]$ and $i \in [k]$.*

Proof. Recall from Proposition 2.2.3 that for $I \subseteq S$ we can uniquely factorize each $x \in \mathfrak{S}_n$ as $x = x^I \cdot x_I$ where $x^I \in (\mathfrak{S}_n)^I$ and $x_I \in (\mathfrak{S}_n)_I$. Let $x, y \in \mathfrak{S}_n$. From [BB05, Theorem 2.6.1] it follows that $x \leq y$ if and only if $x^{S \setminus \{s_k\}} \leq y^{S \setminus \{s_k\}}$ for all $k \in [n-1]$. Moreover, by [BB05, Proposition 2.4.8] we have for $k \in [n-1]$ that $x^{S \setminus \{s_k\}} \leq y^{S \setminus \{s_k\}}$ if and only if $x_{i,k} \leq y_{i,k}$ for all $i \in [k]$. This yields the claim. □

The *Bruhat tableau* $B(x)$ of $x \in \mathfrak{S}_n$ is the tableau of shape $(n-1, n-2, \ldots, 1)$ for which the kth row counted from bottom to top is

$$x_{1,k}, x_{2,k}, \ldots, x_{k,k}.$$

Thanks to the tableau criterion, checking for $x, y \in \mathfrak{S}_n$ whether $x \leq y$ can be done by comparing the Bruhat tableaux $B(x)$ and $B(y)$ cellwise.

Example 5.1.6. Let $n = 5$, $I = \{s_1, s_3, s_4\}$, $w = w_0(I) = 21543$ and σ be the element in stair form $\sigma_{(5)} = (1, 5, 2, 4, 3) = 54132$. Then

$$B(w) = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 5 \\ \hline 1 & 2 & 5 \\ \cline{1-3} 1 & 2 \\ \cline{1-2} 2 \\ \cline{1-1} \end{array} \quad \text{and} \quad B(\sigma) = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 5 \\ \hline 1 & 4 & 5 \\ \cline{1-3} 4 & 5 \\ \cline{1-2} 5 \\ \cline{1-1} \end{array} .$$

By comparing the Bruhat tableaux, we see that $w_{i,k} \leq \sigma_{i,k}$ for all i, k. Thus, the tableau criterion yields $w \leq \sigma$.

We continue with some properties of Bruhat tableaux.

Lemma 5.1.7. *Let $x \in \mathfrak{S}_n$.*

(*1*) *For all $k \in [n]$ and $i \in [k]$ we have $i \leq x_{i,k} \leq n - k + i$.*

(*2*) *For all $k \in [n-1]$ we have $\{x_{1,k}, \ldots, x_{k,k}\} \subseteq \{x_{1,k+1}, \ldots, x_{k+1,k+1}\}$.*

(*3*) *For all $k \in [n-1]$ and $i \in [k]$ we have $x_{i,k+1} \leq x_{i,k} \leq x_{i+1,k+1}$.*

Proof. Statements (1) and (2) are direct consequences of the definition of $x_{i,k}$. For (3) consider $x(k+1)$ and recall $x_{i,k+1} < x_{i+1,k+1}$. If $x(k+1) > x_{i,k}$ then $x_{i,k} = x_{i,k+1}$. If $x(k+1) < x_{i,k}$ then $x_{i,k} = x_{i+1,k+1}$. □

Now we come to a sufficient condition for a tableau of shape $(n-1, n-2, \ldots, 1)$ to be a Bruhat tableau of an $x \in \mathfrak{S}_n$. In fact, by the definition of Bruhat tableaux and Lemma 5.1.7 it is also a necessary condition.

Lemma 5.1.8. *Let $b_{i,k} \in [n]$ for $k \in [n-1]$ and $i \in [k]$ be integers such that*

(*1*) *$b_{1,k} < b_{2,k} < \cdots < b_{k,k}$ for all $k \in [n-1]$ and*

(*2*) *$\{b_{1,k}, \ldots, b_{k,k}\} \subseteq \{b_{1,k+1}, \ldots, b_{k+1,k+1}\}$ for all $k \in [n-2]$.*

Then there is a unique $x \in \mathfrak{S}_n$ such that $x_{i,k} = b_{i,k}$ for all $1 \leq i \leq k \leq n-1$.

Proof. From the definition of the $b_{i,k}$, it follows that there exists a unique $x \in \mathfrak{S}_n$ such that

(i) $x(1) = b_{1,1}$,

(ii) $\{x(k)\} = \{b_{1,k}, \ldots, b_{k,k}\} \setminus \{b_{1,k-1}, \ldots, b_{k-1,k-1}\}$ for $k = 2, \ldots, n-1$,

(iii) $\{x(n)\} = [n] \setminus \{x(1), \ldots, x(n-1)\}$.

We show $x_{i,k} = b_{i,k}$ for all $k \in [n-1]$ and $i \in [k]$ by induction on k. For $k = 1$ we have $x_{1,1} = x(1) = b_{1,1}$. Assume now that $1 < k < n$ and that the hypothesis is true for $k-1$. From the choice of $x(k)$ and the induction hypothesis we obtain

$$\begin{aligned} \{x_{1,k}, \ldots, x_{k,k}\} &= \{x(k)\} \cup \{x_{1,k-1}, \ldots, x_{k-1,k-1}\} \\ &= \{x(k)\} \cup \{b_{1,k-1}, \ldots, b_{k-1,k-1}\} = \{b_{1,k}, \ldots, b_{k,k}\} . \end{aligned}$$

Because

$$x_{1,k} < \cdots < x_{k,k} \quad \text{and} \quad b_{1,k} < \cdots < b_{k,k},$$

it follows that $x_{i,k} = b_{i,k}$ for all $i \in [k]$ □

Let $n \geq 3$ and $w, y \in \mathfrak{S}_n$. We have seen in Section 2.2 that w and y may do not have a meet in Bruhat order. However, it turns out that if $w = w_0(I)$ for some $I \subseteq S$ then they have.

Proposition 5.1.9. *Let $I \subseteq S$ and $y \in \mathfrak{S}_n$. Then $w_0(I)$ and y have a meet z in Bruhat order. Moreover, we have $z_{i,k} = \min\{w_0(I)_{i,k}, y_{i,k}\}$ for $1 \leq i \leq k \leq n-1$.*

Example 5.1.10. Let $n = 6$, $I = \{s_1, s_2, s_3, s_4\}$ and $y = (1,6,5)(2,4,3) = 642315$. Then the Bruhat tableaux of $w_0(I)$ and y are given by

$$\begin{array}{|c|c|c|c|c|}\hline 1&2&3&4&5\\\hline 2&3&4&5\\\cline{1-4} 3&4&5\\\cline{1-3} 4&5\\\cline{1-2} 5\\\cline{1-1}\end{array} \quad \text{and} \quad \begin{array}{|c|c|c|c|c|}\hline 1&2&3&4&6\\\hline 2&3&4&6\\\cline{1-4} 2&4&6\\\cline{1-3} 4&6\\\cline{1-2} 6\\\cline{1-1}\end{array},$$

respectively. The tableau containing the cellwise minima of the two tableaux is given by

$$\begin{array}{|c|c|c|c|c|}\hline 1&2&3&4&5\\\hline 2&3&4&5\\\cline{1-4} 2&4&5\\\cline{1-3} 4&5\\\cline{1-2} 5\\\cline{1-1}\end{array}$$

so that the meet of $w_0(I)$ and y is $z = 542316 = (1,5)(2,4,3)$.

Proof. Let $0 = a_0 < a_1 < \cdots < a_m = n$ be integers such that $S \setminus I = \{s_{a_1}, \ldots, s_{a_{m-1}}\}$, $x := w_0(I)$ and $b_{i,k} := \min\{x_{i,k}, y_{i,k}\}$ for $k \in [n-1]$ and $i \in [k]$. Theorem 5.1.5 implies that for each $u \in \mathfrak{S}_n$ we have $u \leq x$ and $u \leq y$ if and only if $u_{i,k} \leq b_{i,k}$ for all $1 \leq i \leq k \leq n-1$. Therefore, if we show that there is a $z \in \mathfrak{S}_n$ such that $z_{i,k} = b_{i,k}$ for all $i \in [k]$ and $k \in [n-1]$, this permutation z is the meet of x and y.

By Lemma 5.1.8, we have to show that $b_{i,k} < b_{i+1,k}$ for all $k \in [n-1]$ and $i \in [k-1]$ and that $\{b_{1,k}, \ldots, b_{k,k}\} \subseteq \{b_{1,k+1}, \ldots, b_{k+1,k+1}\}$ for all $k \in [n-2]$.

The first part is an easy consequence of the definition of the $b_{i,k}$ and the fact that $x_{i,k} < x_{i+1,k}$ and $y_{i,k} < y_{i+1,k}$ for $i \in [k-1]$.

For the second part let $k \in [n-2]$. We deal with two cases

Case 1. There is a $j \in [m]$ such that $k = a_j$ (i.e. $k \notin D_R(x)$). Then x stabilizes $[k]$ since $x \in \mathfrak{S}_I$. Moreover, Lemma 5.1.2 implies $x(k+1) = a_{j+1}$. Hence, $x_{i,k} = x_{i,k+1} = i$ for all $i \in [k]$. But by Lemma 5.1.7 this means that $x_{i,k} \leq y_{i,k}$ and $x_{i,k+1} \leq y_{i,k+1}$ so that $b_{i,k} = x_{i,k} = x_{i,k+1} = b_{i,k+1}$ for all $i \in [k]$.

Case 2. There is a $j \in [m]$ such that $a_{j-1} < k < a_j$ (i.e. $k \in D_R(x)$). Then x stabilizes $[a_{j-1}]$ as $x \in \mathfrak{S}_I$. In addition, we obtain from Lemma 5.1.2 that $x(i) = a_j + a_{j-1} - i + 1$ for $a_{j-1} < i \leq k+1$. Thus, for $i \in [k]$ we have

$$x_{i,k} = \begin{cases} x_{i,k+1} = i & \text{if } i \leq a_{j-1} \\ x_{i,k+1} + 1 = x_{i+1,k+1} & \text{if } i > a_{j-1}. \end{cases} \tag{5.2}$$

Now, fix an $i \in [k]$. If $i \leq a_{j-1}$, we have $b_{i,k} = i = b_{i,k+1}$ as before. If $i > a_j$ then again we have two cases.

Assume first that $x_{i,k} \leq y_{i,k}$ and recall $y_{i,k} \leq y_{i+1,k+1}$ from Lemma 5.1.7. Combining this with (5.2) we obtain $x_{i+1,k+1} = x_{i,k} \leq y_{i+1,k+1}$ so that $b_{i,k} = x_{i,k} = b_{i+1,k+1}$.

Assume now that $x_{i,k} > y_{i,k}$. Note that either $y_{i,k} = y_{i,k+1}$ or $y_{i,k} = y_{i+1,k+1}$. If $y_{i,k} = y_{i,k+1}$ then

$$b_{i,k} = \min\{x_{i,k}, y_{i,k+1}\} = \min\{x_{i,k} - 1, y_{i,k+1}\} \overset{(5.2)}{=} \min\{x_{i,k+1}, y_{i,k+1}\} = b_{i,k+1}.$$

Otherwise $y_{i,k} = y_{i+1,k+1}$ so that

$$b_{i,k} = \min\{x_{i,k}, y_{i+1,k+1}\} \overset{(5.2)}{=} \min\{x_{i+1,k+1}, y_{i+1,k+1}\} = b_{i+1,k+1}. \qquad \square$$

Let $\alpha \vDash_e n$ and $I \subseteq S$. If $w \in \mathfrak{S}_I \cap \mathfrak{S}_{\leq \Sigma_\alpha}$ then there is a $\sigma \in \Sigma_\alpha$ such that $w \leq w_0(I)$ and $w \leq \sigma$. Since $w_0(I) \wedge \sigma$ exists by Proposition 5.1.9, it follows that $w \leq w_0(I) \wedge \sigma$. In particular, if w is maximal in $\mathfrak{S}_I \cap \mathfrak{S}_{\leq \Sigma_\alpha}$ with respect to the Bruhat order then $w = w_0(I) \wedge \sigma$. Hence, there is a subset T of Σ_α such that the maximal elements of $\mathfrak{S}_I \cap \mathfrak{S}_{\leq \Sigma_\alpha}$ are $\{w_0(I) \wedge \tau \mid \tau \in \mathrm{T}\}$.

Recall that for $\nu\colon \mathfrak{S}_n \to \mathfrak{S}_n, w \mapsto w^{w_0}$ we have $\nu(s_i) = s_{n-i}$. As a consequence, $\nu(I) = \{s_{n-i} \mid s_i \in I\}$. Sometimes it will be convenient to consider $\mathfrak{S}_{\nu(I)} \cap \mathfrak{S}_{\leq \Sigma_\alpha}$ instead of $\mathfrak{S}_I \cap \mathfrak{S}_{\leq \Sigma_\alpha}$. The next result shows how the maximal elements of the two sets are related.

Lemma 5.1.11. *Let $\alpha \vDash_e n$, $I \subseteq S$, $\nu\colon \mathfrak{S}_n \to \mathfrak{S}_n$, $w \mapsto w^{w_0}$ and $\sigma \in \Sigma_\alpha$ be such that $w_0(I) \wedge \sigma$ is maximal in $\mathfrak{S}_I \cap \mathfrak{S}_{\leq \Sigma_\alpha}$ with respect to the Bruhat order. Then $\nu(\sigma) \in \Sigma_\alpha$ and $w_0(\nu(I)) \wedge \nu(\sigma)$ is maximal in $\mathfrak{S}_{\nu(I)} \cap \mathfrak{S}_{\leq \Sigma_\alpha}$.*

Proof. Set $U := \mathfrak{S}_I \cap \mathfrak{S}_{\leq \Sigma_\alpha}$. From Lemma 4.1.17 and Corollary 4.1.16 it follows that $\nu(\mathfrak{S}_I) = \mathfrak{S}_{\nu(I)}$ and $\nu(\Sigma_\alpha) = \Sigma_\alpha$. Hence, $\nu(U) = \mathfrak{S}_{\nu(I)} \cap \mathfrak{S}_{\leq \Sigma_\alpha}$. Moreover, we obtain that $\nu(w_0(I)) = w_0(\nu(I))$ and $\nu(\sigma) \in \Sigma_\alpha$.

From the fact that ν is an automorphism in Bruhat order, it follows that

$$\nu(w_0(I) \wedge \sigma) = \nu(w_0(I)) \wedge \nu(\sigma) = w_0(\nu(I)) \wedge \nu(\sigma).$$

Using the automorphism property again yields that $w_0(I) \wedge \sigma$ being maximal in U implies that its image under ν is maximal in $\nu(U)$. As this image is $w_0(\nu(I)) \wedge \nu(\sigma)$, we are done. $\square$

We end the preliminaries with a sufficient condition for $\bar{\pi}_{\Sigma_\alpha} v_I = 0$ which will be used later.

Lemma 5.1.12. *Let $\alpha \vDash_e n$ and $I \subseteq S$. If there is a $u \in \mathfrak{S}_n$ with $u \neq 1$ such that $\mathfrak{S}_I \cap \mathfrak{S}_{\leq \Sigma_\alpha}$ is the interval in Bruhat order $[1, u]$ then $\bar{\pi}_{\leq \Sigma_\alpha} v_I = 0$.*

Proof. Assume that there is a $1 \neq u \in \mathfrak{S}_n$ such that $\mathfrak{S}_I \cap \mathfrak{S}_{\leq \Sigma_\alpha}$ is the Bruhat order interval $[1, u]$. Since $u \neq 1$, the number of elements of even length equals the number of elements of odd length in $[1, u]$ [BB05, Corollary 2.7.11]. Hence, $\mathfrak{S}_I \cap \mathfrak{S}_{\leq \Sigma_\alpha} = [1, u]$ implies $\sum_{w \in \mathfrak{S}_I \cap \mathfrak{S}_{\leq \Sigma_\alpha}} (-1)^{\ell(w)} v_I = 0$. Thus, (5.1) yields $\bar{\pi}_{\leq \Sigma_\alpha} v_I = 0$. □

5.2 Compositions with one part

In this section we prove Conjecture 5.0.1 in the case where α has only one part. That is, we show $\bar{\pi}_{\leq \Sigma_{(n)}} v_I = 0$ for all $\emptyset \subsetneq I \subsetneq S$.

Let $\alpha = (n)$. The proof has three major steps. First, we determine the Bruhat tableau of σ_α. Second, we show that $w_0(I) \leq \sigma_\alpha$ or $w_0(I) \leq \sigma_\alpha^{w_0}$ for all $I \subsetneq S$ using the first result and the tableau criterion. Third, we infer from the second step that $\mathfrak{S}_I \cap \mathfrak{S}_{\leq \Sigma_\alpha} = \mathfrak{S}_I$ for all $I \subsetneq S$.

Let $x \in \mathfrak{S}_n$. By Lemma 5.1.7, $x_{i,j} \leq n - j + i$ for all $j \in [n-1]$ and $i \in [j]$. We call $x_{i,j}$ *maximal* if $x_{i,j} = n - j + i$. In this case we also call the entry in the Bruhat tableau of x corresponding to $x_{i,j}$ maximal. It follows that $y_{i,k} \leq x_{i,k}$ for all $y \in \mathfrak{S}_n$ if $x_{i,k}$ is maximal.

Example 5.2.1. The Bruhat tableau of

$$\sigma = \sigma_{(5)} = (1, 5, 2, 4, 3) = 54132$$

is shown below.

1	3	4	5
1	4	5	
4	5		
5			

Observe that $\sigma_{1,3} = \sigma_{1,4} = 1$ and $\sigma_{i,k} = n - k + i$ otherwise. That is, $\sigma_{i,k}$ is maximal in the second case.

Lemma 5.2.2. *Let $\sigma := \sigma_{(n)}$ be the element in stair form and $m := \left\lceil \frac{n+1}{2} \right\rceil$.*

(*1*) *For $i \in [n]$,*

$$\sigma(i) = \begin{cases} n - i + 1 & \text{if } i < m \\ 1 & \text{if } i = m \\ n - i + 2 & \text{if } i > m. \end{cases}$$

(2) For $k \in [n-1]$ and $i \in [k]$,

$$\sigma_{i,k} = \begin{cases} 1 & \text{if } k \geq m \text{ and } i = 1 \\ n-k+i & \text{otherwise.} \end{cases}$$

Proof. (1) From the definition of the element in stair form, it follows that

$$\sigma = (x_1, x_2, \ldots, x_n)$$

in cycle notation where $x_{2i-1} = i$ and $x_{2i} = n-i+1$. Hence for $j \in [n]$,

$$\sigma(x_j) = \begin{cases} x_{j+1} & \text{if } j < n \\ x_1 & \text{if } j = n. \end{cases}$$

If n is even then $x_n = x_{2\frac{n}{2}} = n - \frac{n}{2} + 1 = \frac{n}{2} + 1 = m$. If n is odd then we have $x_n = x_{2\frac{n+1}{2}-1} = \frac{n+1}{2} = m$. Therefore, $\sigma(m) = 1$.

Let $i \in [n]$. If $i < m$ then $x_{2i-1} = i$ and thus $\sigma(i) = x_{2i} = n-i+1$. If $i > m$ then $x_{2(n-i+1)} = n-(n-i+1)+1 = i$ and hence $\sigma(i) = x_{2(n-i+2)-1} = n-i+2$.

(2) Let $k \in [n-1]$. We have that

$$\sigma([k]) = \sigma([k] \cap [m-1]) \cup \sigma([k] \cap \{m\}) \cup \sigma([k] \cap [m+1, n]).$$

By Part (1),

$$\begin{aligned} \sigma([k] \cap [m-1]) &= \{n-i+1 \mid 1 \leq i \leq m-1 \text{ and } i \leq k\}, \\ \sigma([k] \cap [m+1, n]) &= \{n-i+2 \mid m+1 \leq i \leq k\} \\ &= \{n-i+1 \mid m \leq i \leq k-1\}. \end{aligned}$$

Hence, if $k < m$ then $\sigma([k]) = \{n-k+1, n-k+2, \ldots, n\}$, i.e. $\sigma_{i,k} = n-k+i$ for $i \in [k]$. Moreover, if $k \geq m$ then $\sigma([k]) = \{1\} \cup \{n-k+2, n-k+3, \ldots, n\}$ and thus $\sigma_{1,k} = 1$ and $\sigma_{i,k} = n-k+i$ for $1 < i \leq k$. $\square$

In Example 5.1.6 it is shown that $w_0(I) \leq \sigma_{(5)}$ for $n = 5$ and $I = \{s_1, s_3, s_4\}$ via the tableau criterion. This is a special case of the next result.

Lemma 5.2.3. *Assume $n \geq 2$, $\alpha = (n)$ and $I \subsetneq S$. Let $a \in [n-1]$ be such that $s_a \in S \setminus I$ and*

$$\sigma := \begin{cases} \sigma_{(n)} & \text{if } a \leq \lceil \frac{n}{2} \rceil \\ \sigma_{(n)}^{w_0} & \text{if } a > \lceil \frac{n}{2} \rceil. \end{cases}$$

Then $\sigma \in \Sigma_\alpha$ and $w_0(I) \leq \sigma$.

Proof. As $I \subsetneq S$ and $n \geq 2$, there exists an $a \in [n-1]$ such that $I \subseteq S \setminus \{s_a\}$. Because $w_0(I) \leq w_0(S \setminus \{s_a\})$, we can assume $I = S \setminus \{s_a\}$ without loss of generality. Let σ

be given as in the theorem. By definition we have $\sigma_\alpha \in \Sigma_\alpha$. From Corollary 4.1.16 it follows that also $\sigma_\alpha^{w_0} \in \Sigma_\alpha$. Thus, $\sigma \in \Sigma_\alpha$.

It remains to show $w_0(I) \le \sigma$. First, suppose that $a \le \lceil \frac{n}{2} \rceil$. Let $w := w_0(I)$, $k \in [n-1]$ and $i \in [k]$. By Theorem 5.1.5, we have to show $w_{i,k} \le \sigma_{i,k}$. If $k < \left\lceil \frac{n+1}{2} \right\rceil$ or $i > 1$ then Lemma 5.2.2 implies that $\sigma_{i,k}$ is maximal which means that $w_{i,k} \le \sigma_{i,k}$. Thus, consider the case where $k \ge \left\lceil \frac{n+1}{2} \right\rceil$ and $i = 1$. From Lemma 5.1.2 we obtain $w(a) = 1$. Because $k \ge \left\lceil \frac{n+1}{2} \right\rceil \ge a$, it follows that $w_{1,k} = 1$. Thus, $w_{1,k} \le \sigma_{1,k}$.

Second, let $a > \lceil \frac{n}{2} \rceil$. Consider the automorphism $\nu\colon \mathfrak{S}_n \to \mathfrak{S}_n, w \mapsto w^{w_0}$. Then $\nu(I) = S \setminus \{s_{n-a}\}$ and since $n - a \le \lceil \frac{n}{2} \rceil$, we obtain that $w_0(\nu(I)) \le \sigma_\alpha$ from the already proven case. As ν is order preserving, it follows that $\nu(w_0(\nu(I))) \le \sigma_\alpha^{w_0}$. Moreover, $\nu(w_0(\nu(I))) = w_0(I)$ by Lemma 4.1.17. Hence $w_0(I) \le \sigma$. □

It turns out that $\mathfrak{S}_I \subseteq \mathfrak{S}_{\le \Sigma_{(n)}}$ if $I \subsetneq S$. This is the major step towards the verification of Conjecture 5.0.1 in the case $\alpha = (n)$.

Theorem 5.2.4. *Let $\alpha = (n)$ and $I \subseteq S$ with $I \ne S$ if $n > 1$. Then we have that $\mathfrak{S}_I \cap \mathfrak{S}_{\le \Sigma_\alpha} = \mathfrak{S}_I$.*

Proof. First suppose $n = 1$. Then $\alpha = (1)$ and $I = \emptyset$. Thus, $\mathfrak{S}_I = \{1\} = \mathfrak{S}_{\le \Sigma_\alpha}$.

Now assume $n \ge 2$. Then $I \subsetneq S$. We set $w := w_0(I)$. From Lemma 5.2.3 we have that $\sigma_\alpha, \sigma_\alpha^{w_0} \in \Sigma_\alpha$ and $w \le \sigma_\alpha$ or $w \le \sigma_\alpha^{w_0}$. Consequently, $\mathfrak{S}_I \subseteq \mathfrak{S}_{\le \Sigma_\alpha}$ because by Lemma 2.2.10 $\mathfrak{S}_I$ is the Bruhat order interval $[1, w]$ and $\mathfrak{S}_{\le \Sigma_\alpha}$ is an order ideal with maximal elements Σ_α. □

Corollary 5.2.5. *Conjecture 5.0.1 is true for $\alpha = (n)$.*

Proof. Let $n \ge 3$, $\alpha = (n)$ and $\emptyset \subsetneq I \subsetneq S$. We have to show that $\bar{\pi}_{\le \Sigma_\alpha} v_I = 0$. From Theorem 5.2.4 it follows that $\mathfrak{S}_I \cap \mathfrak{S}_{\le \Sigma_\alpha} = \mathfrak{S}_I$. Furthermore, $\mathfrak{S}_I = [1, w_0(I)]$ by Lemma 2.2.10 and $w_0(I) \ne 1$ since $I \ne \emptyset$. Now Lemma 5.1.12 implies $\bar{\pi}_{\le \Sigma_\alpha} v_I = 0$. □

5.3 Odd hooks

Let $\alpha \vDash_e n$ be an odd hook such that $\alpha \ne (1^n), (n)$. Then $n \ge 4$. In this section we verify Conjecture 5.0.1 for this kind of composition. The case $\alpha = (1^n)$ is a border case and the case $\alpha = (n)$ has been treated in the last section. Indeed, the proof presented in this section fails if $\alpha = (n)$.

In order to motivate the reasoning, we reformulate some results from the $\alpha = (n)$ case. Let $n \ge 2$ and $I \subsetneq S$. Lemma 5.2.3 implies that there is a $\sigma \in \left\{\sigma_{(n)}, \sigma_{(n)}^{w_0}\right\}$ such that $w_0(I) \wedge \sigma = w_0(I)$. As $\mathfrak{S}_I \cap \mathfrak{S}_{\le \Sigma_{(n)}} = \mathfrak{S}_I$ by Theorem 5.2.4, it follows that $\mathfrak{S}_I \cap \mathfrak{S}_{\le \Sigma_{(n)}}$ is the interval in Bruhat order $[1, w_0(I) \wedge \sigma]$.

In this section we define $\tau \in \Sigma_\alpha$ depending on I such that $\mathfrak{S}_I \cap \mathfrak{S}_{\le \Sigma_\alpha} = [1, w_0(I) \wedge \tau]$. However, the construction of τ and the proof that $w_0(I) \wedge \tau$ is the only maximal element of $\mathfrak{S}_I \cap \mathfrak{S}_{\le \Sigma_\alpha}$ requires more work as in the $\alpha = (n)$ case. One reason for the latter is that in general $w_0(I) \wedge \tau \ne w_0(I)$.

We begin with a property of the Bruhat tableaux of all $\sigma \in \Sigma_\alpha$ that will be useful for showing that $w_0(I) \wedge \tau$ is the greatest element of $\mathfrak{S}_I \cap \mathfrak{S}_{\leq \Sigma_\alpha}$. Before stating the general result, we illustrate the property.

Example 5.3.1. Let $\alpha = (3,1,1)$. Then Σ_α consists of the elements

$$(1,5,2),(1,5,3),(1,5,4),(1,2,5),(1,3,5),(1,4,5).$$

The respective Bruhat tableaux are shown below.

$$\begin{array}{|c|c|c|c|}\hline 1&3&4&5\\\hline 1&3&5\\\cline{1-3} 1&5\\\cline{1-2} 5\\\cline{1-1}\end{array}\quad
\begin{array}{|c|c|c|c|}\hline 1&2&4&5\\\hline 1&2&5\\\cline{1-3} 2&5\\\cline{1-2} 5\\\cline{1-1}\end{array}\quad
\begin{array}{|c|c|c|c|}\hline 1&2&3&5\\\hline 2&3&5\\\cline{1-3} 2&5\\\cline{1-2} 5\\\cline{1-1}\end{array}\quad
\begin{array}{|c|c|c|c|}\hline 2&3&4&5\\\hline 2&3&5\\\cline{1-3} 2&5\\\cline{1-2} 2\\\cline{1-1}\end{array}\quad
\begin{array}{|c|c|c|c|}\hline 2&3&4&5\\\hline 2&3&5\\\cline{1-3} 2&3\\\cline{1-2} 3\\\cline{1-1}\end{array}\quad
\begin{array}{|c|c|c|c|}\hline 2&3&4&5\\\hline 2&3&4\\\cline{1-3} 2&4\\\cline{1-2} 4\\\cline{1-1}\end{array}$$

The entries in the gray subtableaux are cellwise bounded from above by the tableau

$$\begin{array}{|c|c|}\hline 2&3\\\hline 2\\\cline{1-1}\end{array}\,.$$

Lemma 5.3.2. *Let $\alpha = (\alpha_1, 1^{n-\alpha_1}) \vDash_e n$ with $1 < \alpha_1 < n$ be an odd hook, $m := \frac{\alpha_1 - 1}{2}$ and $\sigma \in \Sigma_\alpha$. Then $\sigma_{i,k} \leq m + i$ if $m < k < n - m$ and $1 \leq i \leq k - m$.*

Proof. Since $\sigma \in \Sigma_\alpha$, it satisfies the hook properties by Theorem 4.3.40. In particular, there exists a $j \in [m+1, n-m]$ such that $[m+1, n-m] \setminus \{j\}$ is the set of fixpoints of σ. We do an induction on k.

For the base case suppose $k = m+1$. We have to show $\sigma_{1,m+1} \leq m+1$. If $j \neq m+1$ then $\sigma(m+1) = m+1$ so that $m+1 \in \sigma([m+1])$ and consequently

$$\sigma_{1,m+1} = \min \sigma([m+1]) \leq m+1.$$

Assume $j = m+1$. Then $[m+2, n-m]$ is the set of fixpoints of σ. Hence, $\sigma([m+1])$ and $[m+2, n-m]$ are disjoint. In addition, Lemma 5.1.7 yields

$$\sigma_{1,m+1} \leq n - (m+1) + 1 = n - m.$$

Therefore, $\sigma_{1,m+1} \leq m+1$.

Assume now that $k > m+1$ and the claim holds for $k-1$. We distinguish two cases depending on j.

First, suppose $j \neq k$. Then $\sigma(k) = k$. Let $i \in [k-m-1]$. By induction hypothesis $\sigma_{i,k-1} \leq m+i$. As $i \leq k-m-1$, it follows that

$$\sigma_{i,k-1} \leq m + k - m - 1 = k - 1 < k = \sigma(k).$$

Thus, $\sigma_{i,k} = \sigma_{i,k-1}$ and therefore $\sigma_{i,k} \leq m+i$. Now consider $\sigma_{k-m,k}$. We have

$$\sigma_{k-m,k} = \min(\sigma([k]) \setminus \{\sigma_{1,k}, \ldots, \sigma_{k-m-1,k}\}).$$

Since $\sigma(k) = k$, we have $k \in \sigma([k])$. Moreover, from the reasoning above we obtain $\sigma_{i,k} = \sigma_{i,k-1} < k$ for $i \in [k-m-1]$. Therefore, k is an element of the set above. Consequently, $\sigma_{k-m,k} \leq k = m + (k-m)$ as desired.

Second, assume $j = k$. Then all the elements of $[k+1, n-m]$ are fixpoints of σ. Hence, $\sigma([k])$ and $[k+1, n-m]$ are disjoint. Moreover by using Lemma 5.1.7 again, we obtain

$$\sigma_{i,k} \leq n-k+i \leq n-k+k-m \leq n-m.$$

for $i = 1, \ldots, k-m$. Thus, $\sigma_{i,k} \in [k]$ for $i = 1, \ldots, k-m$. The $k-m$ greatest elements of $[k]$ are $m+1, m+2, \ldots, k$. Therefore, $\sigma_{i,k} \leq m+i$ for $i = 1, \ldots, k-m$. $\square$

Let $\alpha \vDash_e n$ be an odd hook with $\alpha \neq (1^n), (n)$ and $I \subseteq S$. In order to obtain $\tau \in \Sigma_\alpha$ such that $w_0(I) \wedge \tau$ is the greatest element of $\mathfrak{S}_I \cap \mathfrak{S}_{\leq \Sigma_\alpha}$, we will use the elements $\sigma^{(j)}$ that are the subject of the next lemma. Again we are interested in the Bruhat tableau of $\sigma^{(j)}$.

Lemma 5.3.3. *Let $\alpha = (\alpha_1, 1^{n-\alpha_1}) \vDash_e n$ be an odd hook with $1 < \alpha_1 < n$, $m := \frac{\alpha_1 - 1}{2}$ and $j \in [m+1, n-m]$. Define the element of $\mathfrak{S}_n$*

$$\sigma^{(j)} := (j, x_{\alpha_1 - 1}, x_{\alpha_1 - 2}, \ldots, x_1)$$

where $x = (x_1, x_2, \ldots, x_n)$ is the sequence with $x_{2i-1} = i$ and $x_{2i} = n - i + 1$.

(*1*) *We have $\sigma^{(j)} \in \Sigma_\alpha$.*

(*2*) *For all $i \in [n]$,*

$$\sigma^{(j)}(i) = \begin{cases} j & \text{if } i = 1 \\ n-i+2 & \text{if } 2 \leq i \leq m \\ i & \text{if } m+1 \leq i \leq n-m \text{ and } i \neq j \\ n-m+1 & \text{if } i = j \\ n-i+1 & \text{if } n-m+1 \leq i \leq n. \end{cases}$$

(*3*) *For all $k \in [n-1]$ and $i \in [k]$,*

$$\sigma^{(j)}_{i,k} = \begin{cases} j & \text{if } k \leq m \text{ and } i = 1 \text{ or } m < k < j \text{ and } i = k-m+1 \\ m+i & \text{if } m < k < n-m \text{ and } 1 \leq i \leq k-m \\ n-k+i & \text{otherwise.} \end{cases}$$

Example 5.3.4. Let α be an odd hook and $m = \frac{\alpha_1 - 1}{2}$. We give examples for the elements $\sigma^{(j)}$ from Lemma 5.3.3 in cycle and one-line notation.

(1) For $\alpha = (3,1,1)$ we have $m = 1$ and

$$\begin{aligned}\sigma^{(2)} &= (1,2,5) = 2\ 5\ 3\ 4\ 1,\\ \sigma^{(3)} &= (1,3,5) = 3\ 2\ 5\ 4\ 1,\\ \sigma^{(4)} &= (1,4,5) = 4\ 2\ 3\ 5\ 1.\end{aligned}$$

The related Bruhat tableaux are shown in Example 5.3.1. They are the three tableaux on the right hand side.

(2) For $\alpha = (7,1,1,1)$ we have $m = 3$ and

$$\begin{aligned}\sigma^{(4)} &= (1,4,8,3,9,2,10) = 4\ 10\ 9\ 8\ 5\ 6\ 7\ 3\ 2\ 1,\\ \sigma^{(5)} &= (1,5,8,3,9,2,10) = 5\ 10\ 9\ 4\ 8\ 6\ 7\ 3\ 2\ 1,\\ \sigma^{(6)} &= (1,6,8,3,9,2,10) = 6\ 10\ 9\ 4\ 5\ 8\ 7\ 3\ 2\ 1,\\ \sigma^{(7)} &= (1,7,8,3,9,2,10) = 7\ 10\ 9\ 4\ 5\ 6\ 8\ 3\ 2\ 1.\end{aligned}$$

Proof of Lemma 5.3.3. We begin with the proof of Part (2). Note that $x_1 = 1$, $x_2 = n$, $x_{\alpha_1-2} = m$ and $x_{\alpha_1-1} = n-m+1$. In particular, $\sigma(1) = j$ and $\sigma(j) = n-m+1$. From the definition of x it follows that

$$\begin{aligned}\{x_r \mid 1 \le r \le \alpha_1 - 1\} &= [m] \cup [n-m+1, n],\\ \{x_r \mid \alpha_1 \le r \le n\} &= [m+1, n-m].\end{aligned} \tag{5.3}$$

Let $i \in [n]$ with $i \neq 1$ and $i \neq j$. If $i \in [m+1, n-m]$ then i is a fixpoint of σ, i.e. $\sigma(i) = i$ as desired. Hence assume $i \in [m] \cup [n-m+1, n]$. Let $r \in [n]$ be such that $x_r = i$. By the definition of σ, $\sigma(i) = x_{r-1}$. If $i \le m$ then $x_{2i-1} = i$ and therefore $\sigma(i) = x_{2(i-1)} = n-i+2$. If $i \ge n-m+1$ then $x_{2(n-i+1)} = i$ and thus $\sigma(i) = x_{2(n-i+1)-1} = n-i+1$. This finishes the proof of Part (2).

We proceed with the proof of Part (1). In order to show that $\sigma \in \Sigma_\alpha$, we use Theorem 4.3.40, i.e. we have to show that σ satisfies the hook properties. By (5.3), the third property is satisfied. It remains to show that σ is oscillating with connected intervals. Let τ be the cycle standartization of the cycle of length α_1 of σ. We have to show that τ is oscillating with connected intervals. We have

$$\tau = (m+1, \alpha_1 - m + 1, m, \ldots, \alpha_1 - 1, 2, \alpha_1, 1).$$

Hence, $\tau([m+1]) = [m+1, \alpha_1]$. Since $m+1 = \frac{\alpha_1+1}{2}$, it follows that τ is oscillating. Moreover, for each $k \in [m]$ and $q := |[k, \alpha_1 - k + 1]|$,

$$\{\tau^r(m+1) \mid r = 0, \ldots, q-1\} = [k, \alpha_1 - k + 1].$$

Thus, τ has connected intervals.

Lastly, we prove Part (3). Let $k \in [n-1]$ and $i \in [k]$. We deal with four cases depending on k. In each case we use Part (2) in order to determine the set $\sigma([k])$.

Case 1. Suppose $k \leq m$. Part (2) implies

$$\sigma([k]) = \{j, n-k+2, n-k+3, \ldots, n\}.$$

Hence, $\sigma_{1,k} = j$ and $\sigma_{i,k} = n-k+i$ if $i > 1$.

Case 2. Suppose $m < k < j$. Then

$$\sigma([k]) = \{m+1, m+2, \ldots, k, j, n-m+2, n-m+3, \ldots, n\}.$$

Thus,

$$\sigma_{i,k} = \begin{cases} m+i & \text{if } i \leq k-m \\ j & \text{if } i = k-m+1 \\ n-k+i & \text{if } i > k-m+1 \end{cases}$$

Case 3. Suppose $j \leq k < n-m$. Then

$$\sigma([k]) = \{m+1, m+2, \ldots, k, n-m+1, n-m+2, \ldots, n\}.$$

Consequently, $\sigma_{i,k} = m+i$ if $i \leq k-m$ and $\sigma_{i,k} = n-k+i$ if $i > k-m$.

Case 4. Suppose $k \geq n-m$. Then

$$\sigma([k]) = \{n-k+1, n-k+2, \ldots, n\}.$$

Thus, $\sigma_{i,k} = n-k+i$. □

Definition 5.3.5. *Let $\alpha = (\alpha_1, 1^{n-\alpha_1}) \vDash_e n$ be an odd hook with $1 < \alpha_1 < n$, $m := \frac{\alpha_1 - 1}{2}$ and $a \in [n-1]$. Define*

$$\tau_{\alpha,a} := \begin{cases} \sigma^{(j)} & \text{if } a \leq \lceil \frac{n}{2} \rceil \\ (\sigma^{(n-j+1)})^{w_0} & \text{if } a > \lceil \frac{n}{2} \rceil \end{cases}$$

with

$$j := \begin{cases} \max\{a, m+1\} & \text{if } a \leq \lceil \frac{n}{2} \rceil \\ \min\{a+1, n-m\} & \text{if } a > \lceil \frac{n}{2} \rceil \end{cases}$$

and $\sigma^{(j)}$ the element from Lemma 5.3.3.

Example 5.3.6. Consider $\alpha = (3,1,1) \vDash_e 5$. Then $m = \frac{\alpha_1 - 1}{2} = 1$. Let $\sigma^{(j)}$ be defined as in Lemma 5.3.3 for $j = 2, 3, 4$. The elements $\tau_{\alpha,a}$ for $a = 1, \ldots, 4$ are

$$\begin{aligned} \tau_{\alpha,1} = \tau_{\alpha,2} = \sigma^{(2)} &= (1,2,5), \\ \tau_{\alpha,3} = \sigma^{(3)} &= (1,3,5), \\ \tau_{\alpha,4} = \left(\sigma^{(2)}\right)^{w_0} &= (1,5,4). \end{aligned}$$

Let $\alpha \vDash_e n$ be an odd hook unequal to (1^n) and (n) and $I \subsetneq S$. Then there is an $a \in [n-1]$ such that $s_a \notin I$. We want to show that $\tau_{\alpha,a}$ is our desired element of Σ_α with $\mathfrak{S}_I \cap \mathfrak{S}_{\leq \Sigma_\alpha} = [1, w_0(I) \wedge \tau_{\alpha,a}]$. We first check that $\tau_{\alpha,a}$ is well defined and an element of Σ_α.

Lemma 5.3.7. *Let $\alpha = (\alpha_1, 1^{n-\alpha_1}) \vDash_e n$ be an odd hook with $1 < \alpha_1 < n$, $m := \frac{\alpha_1 - 1}{2}$ and $a \in [n-1]$.*

(1) *The element $\tau_{\alpha,a}$ is well defined and $\tau_{\alpha,a} \in \Sigma_\alpha$*

(2) *If $a > \lceil \frac{n}{2} \rceil$ then $\tau_{\alpha,a} = \tau_{\alpha,n-a}^{w_0}$.*

Proof. Let j be as in the definition of $\tau_{\alpha,a}$ and $\sigma^{(k)}$ be the element from Lemma 5.3.3 for $k \in [m+1, n-m]$.

(1) Since $\sigma^{(k)}$ is only defined for $k \in [m+1, n-m]$ and $k \in [m+1, n-m]$ if and only if $n-k+1 \in [m+1, n-m]$, we have to show that $j \in [m+1, n-m]$.

Assume first that $a \leq \lceil \frac{n}{2} \rceil$. Then $j = \max\{a, m+1\}$. If $a \leq m+1$ then $j = m+1$ and we are done. If $a > m+1$ then $j = a$ and

$$j = a \leq \left\lceil \frac{n}{2} \right\rceil \leq \frac{n+1}{2} = n - \frac{n-1}{2} \leq n - \frac{\alpha_1 - 1}{2} = n - m,$$

i.e. $j \in [m+1, n-m]$. Therefore, $\tau_{\alpha,a} = \sigma^{(j)}$ is well defined. From Lemma 5.3.3 it follows that $\tau_{\alpha,a} \in \Sigma_\alpha$.

Assume now that $a > \lceil \frac{n}{2} \rceil$. Then $j = \min\{a+1, n-m\}$. If $a+1 \geq n-m$ then $j = n-m$. If $a+1 < n-m$ then $j = a+1$ and

$$m + 1 = \frac{\alpha_1 + 1}{2} \leq \frac{n}{2} \leq \left\lceil \frac{n}{2} \right\rceil < a < j.$$

That is, $j \in [m+1, n-m]$ and $\tau_{\alpha,a}$ is well defined in this case too. Lemma 5.3.3 yields that $\sigma^{(n-j+1)} \in \Sigma_\alpha$. In addition, we have $\Sigma_\alpha = \Sigma_\alpha^{w_0}$ by Corollary 4.1.16. Hence, $\tau_{\alpha,a} = (\sigma^{(n-j+1)})^{w_0} \in \Sigma_\alpha$.

(2) Assume that $a > \lceil \frac{n}{2} \rceil$ and set $a' := n - a$ and $j' := \max\{a', m+1\}$. Then

$$j = \min\{a+1, n-m\}, \quad \tau_{\alpha,a} = (\sigma^{(n-j+1)})^{w_0}, \quad a' \leq \lceil \tfrac{n}{2} \rceil \quad \text{and} \quad \tau_{\alpha,a'} = \sigma^{(j')}.$$

As a consequence, we have $\tau_{\alpha,a} = \tau_{\alpha,a'}^{w_0}$ if and only if $j' = n - j + 1$. We show the latter. Clearly,

$$a + 1 \leq n - m \quad \iff \quad a' = n - a \geq m + 1.$$

Thus, if $j = a+1$ then $a+1 \leq n-m$ so that $a' \geq m+1$ and consequently

$$j' = a' = n - a = n - j + 1$$

as desired. Moreover, if $j = n-m$ then $a+1 \geq n-m$ so that $a' \leq m+1$ and hence

$$j' = m + 1 = n - (n-m) + 1 = n - j + 1. \qquad \square$$

Example 5.3.8. This example illustrates the reasoning in this section. Consider $n = 5$, $\alpha = (3, 1, 1) \vDash_e 5$ and $I = S \setminus \{s_1\}$. Let $m = \frac{\alpha_1 - 1}{2}$, $a = 1$, $j = \max\{a, m+1\}$, $w = w_0(I)$ and $\tau = \tau_{\alpha,a}$. We show that $\mathfrak{S}_I \cap \mathfrak{S}_{\leq \Sigma_\alpha}$ is the interval in Bruhat order $[1, w \wedge \tau]$.

We have $m = 1$, $j = 2$, $w = (2,5)(3,4)$ and $\tau = \sigma^{(2)} = (1,2,5)$ and consider the Bruhat tableaux

$$B(w) = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 5 \\ \hline 1 & 4 & 5 \\ \cline{1-3} 1 & 5 \\ \cline{1-2} 1 \\ \cline{1-1}\end{array} \quad \text{and} \quad B(\tau) = \begin{array}{|c|c|c|c|}\hline 2 & 3 & 4 & 5 \\ \hline 2 & 3 & 5 \\ \cline{1-3} 2 & 5 \\ \cline{1-2} 2 \\ \cline{1-1}\end{array}.$$

Proposition 5.1.9 states that $(w \wedge \sigma)_{i,k} = w_{i,k} \wedge \sigma_{i,k}$ for all $\sigma \in \mathfrak{S}_n$, $k \in [4]$ and $i \in [k]$. Thus,

$$B(w \wedge \tau) = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 5 \\ \hline 1 & \mathbf{3} & 5 \\ \cline{1-3} 1 & 5 \\ \cline{1-2} 1 \\ \cline{1-1}\end{array}.$$

It follows that $w \wedge \tau = (2,5)$. Moreover, we see that for all $k \in [4]$ and $i \in [k]$,

$$(w \wedge \tau)_{i,k} = \begin{cases} \tau_{i,k} = 3 & \text{if } k = 3 \text{ and } i = 2 \\ w_{i,k} & \text{otherwise.} \end{cases}$$

Let $\sigma \in \Sigma_\alpha$. From Example 5.3.1 or applying Lemma 5.3.2 we obtain that

$$\tau_{2,3} = 3 = m + 2 \geq \sigma_{2,3}.$$

Thus,

$$\begin{aligned}(w \wedge \tau)_{i,k} &= \begin{cases} \tau_{i,k} \geq \sigma_{i,k} & \text{if } k = 3 \text{ and } i = 2 \\ w_{i,k} & \text{otherwise} \end{cases} \\ &\geq w_{i,k} \wedge \sigma_{i,k} \\ &= (w \wedge \sigma)_{i,k}\end{aligned}$$

for all $k \in [4]$ and $i \in [k]$. Therefore, the tableau criterion, Theorem 5.1.5, implies that $w \wedge \sigma \leq w \wedge \tau$.

For each $x \in \mathfrak{S}_I \cap \mathfrak{S}_{\leq \Sigma_\alpha}$ there exists a $\sigma \in \Sigma_\alpha$ such that $x \leq w$ and $x \leq \sigma$ and hence $x \leq w \wedge \sigma \leq w \wedge \tau$. Thus, $\mathfrak{S}_I \cap \mathfrak{S}_{\leq \Sigma_\alpha} = [1, w \wedge \tau]$.

Let $I \subsetneq S$ such that there is an $a \leq \lceil \frac{n}{2} \rceil$ with $s_a \notin I$. We want to compare the Bruhat tableaux of $w_0(I)$ and $\tau_{\alpha,a}$. The one of $\tau_{\alpha,a}$ is given by Lemma 5.3.3. We now determine the Bruhat tableau of $w_0(I)$ in the case where I is a maximal subset of S.

Lemma 5.3.9. *Let $n \geq 2$, $a \in [n-1]$, $I := S \setminus \{s_a\}$ and $w := w_0(I)$. Then for all $k \in [n-1]$ and $i \in [k]$,*

$$w_{i,k} = \begin{cases} a-k+i & \text{if } k \leq a \\ i & \text{if } i \leq a \text{ and } a < k \\ n-k+i & \text{if } a < i \text{ and } a < k. \end{cases}$$

Proof. This is not hard to see considering the one line notation of w

$$w = a \; a-1 \; \cdots 1 \; n \; n-1 \; n-2 \; \cdots \; a+1$$

which we obtain from Lemma 5.1.2. □

We now consider $I \subsetneq S$ such there is an $a \leq \lceil \frac{n}{2} \rceil$ with $s_a \notin I$ and compare the Bruhat tableaux of $w_0(I)$ and $\tau_{\alpha,a}$.

Lemma 5.3.10. *Let $\alpha = (\alpha_1, 1^{n-\alpha_1}) \vDash_e n$ be an odd hook with $1 < \alpha_1 < n$, $I \subsetneq S$ be such that there is an $a \in [n-1]$ with $s_a \notin I$ and $a \leq \lceil \frac{n}{2} \rceil$, $k \in [n-1]$ and $i \in [k]$. Set $m := \frac{\alpha_1 - 1}{2}$, $w := w_0(I)$ and $\tau := \tau_{\alpha,a}$. If $w_{i,k} > \tau_{i,k}$ then $m < k < n-m$ and $i \leq k-m$.*

Proof. Let $j := \max\{a, m+1\}$ and $\sigma^{(j)}$ be defined as in Lemma 5.3.3. By the definition of τ, we have that $\tau = \sigma^{(j)}$. Since $w_0(I) \leq w_0(S \setminus \{s_a\})$, we obtain from Theorem 5.1.5 that $w_0(I)_{i,k} \leq w_0(S \setminus \{s_a\})_{i,k}$. Therefore, we can assume without loss of generality that $I = S \setminus \{s_a\}$.

We show the contraposition. That is, we assume that the statement $m < k < n-m$ and $i \leq k-m$ is not true and show that then $w_{i,k} \leq \tau_{i,k}$. Lemma 5.3.3 implies that

$$\tau_{i,k} = \begin{cases} j & \text{if } k \leq m \text{ and } i = 1 \text{ or } m < k < j \text{ and } i = k-m+1 \\ n-k+i & \text{otherwise.} \end{cases}$$

In the second case, $\tau_{i,k}$ is maximal and thus $w_{i,k} \leq \tau_{i,k}$. It remains show that $w_{i,k} \leq \tau_{i,k}$ in the first case.

Suppose $k \leq m$ and $i = 1$. Lemma 5.1.2 yields $w(1) = a$, i.e. $a \in w([k])$. By the choice of j, we have that $j \geq a$. Therefore,

$$\tau_{1,k} = j \geq a \geq \min w([k]) = w_{1,k}.$$

Now, suppose $m < k < j$ and $i = k-m+1$. This case can only occur if $j > m+1$. Then $j = a$. By assumption, $\alpha_1 \geq 3$ and therefore $m \geq 1$. Since $k < j = a$, we obtain from Lemma 5.3.9 that

$$w_{k-m+1,k} = a-k+k-m+1 = a-m+1 \leq a.$$

Thus, $\tau_{k-m+1,k} = j = a \geq w_{k-m+1,k}$. □

Now we come to the main result on $\mathfrak{S}_I \cap \mathfrak{S}_{\leq \Sigma_\alpha}$ in the case where α is an odd hook.

Theorem 5.3.11. *Let $\alpha = (\alpha_1, 1^{n-\alpha_1}) \vDash_e n$ be an odd hook with $1 < \alpha_1 < n$, $I \subsetneq S$ and $a \in [n-1]$ be such that $s_a \notin I$. Then $\mathfrak{S}_I \cap \mathfrak{S}_{\leq \Sigma_\alpha}$ is the interval $[1, w_0(I) \wedge \tau_{\alpha,a}]$ in Bruhat order.*

Proof. Let $\tau := \tau_{\alpha,a}$ and $w := w_0(I)$. Recall from Proposition 5.1.9 that for each $\sigma \in \mathfrak{S}_n$ the meet $w \wedge \sigma$ exists and we have that $(w \wedge \sigma)_{i,k} = w_{i,k} \wedge \sigma_{i,k}$ for all $k \in [n-1]$ and $i \in [k]$. We distinguish two cases depending on a.

Case 1. Suppose $a \leq \lceil \frac{n}{2} \rceil$. Let $x \in \mathfrak{S}_I \cap \mathfrak{S}_{\leq \Sigma_\alpha}$. We want to show that $x \leq w \wedge \tau$. Since $x \in \mathfrak{S}_I \cap \mathfrak{S}_{\leq \Sigma_\alpha}$, there is a $\sigma \in \Sigma_\alpha$ such that $x \leq w$ and $x \leq \sigma$. Thus, $x \leq w \wedge \sigma$ and we can assume that $x = w \wedge \sigma$ without loss of generality.

Let $k \in [n-1]$ and $i \in [k]$. By the tableau criterion, Theorem 5.1.5, we have to show that $(w \wedge \sigma)_{i,k} \leq (w \wedge \tau)_{i,k}$. We deal with two cases. If $w_{i,k} \leq \tau_{i,k}$ then

$$(w \wedge \tau)_{i,k} = w_{i,k} \geq (w \wedge \sigma)_{i,k}.$$

Suppose now that $w_{i,k} > \tau_{i,k}$ and let $m = \frac{\alpha_1 - 1}{2}$. Then Lemma 5.3.10 implies $m < k < n - m$ and $i \leq k - m$. On the one hand, Part (3) of Lemma 5.3.3 yields that $\tau_{i,k} = m + i$. On the other hand, we obtain from Lemma 5.3.2 that $\sigma_{i,k} \leq m + i$. Therefore,

$$(w \wedge \tau)_{i,k} = \tau_{i,k} \geq \sigma_{i,k} \geq (w \wedge \sigma)_{i,k}.$$

Case 2. Suppose that $a > \lceil \frac{n}{2} \rceil$. Set $\tilde{I} = I^{w_0}$, $\tilde{a} = n - a$, $\tilde{w} = w_0(\tilde{I})$ and $\tilde{\tau} = \tau_{\alpha,\tilde{a}}$. Then $\tilde{a} \leq \lceil \frac{n}{2} \rceil$ and by Case 1, $\mathfrak{S}_{\tilde{I}} \cap \mathfrak{S}_{\leq \Sigma_\alpha} = [1, \tilde{w} \wedge \tilde{\tau}]$. Hence, an application of Lemma 5.1.11 yields

$$\mathfrak{S}_I \cap \mathfrak{S}_{\leq \Sigma_\alpha} = [1, \tilde{w}^{w_0} \wedge \tilde{\tau}^{w_0}].$$

Furthermore, $w = \tilde{w}^{w_0}$ and by Lemma 5.3.7, $\tau = \tau_{\alpha,n-a}^{w_0} = \tilde{\tau}^{w_0}$. As a consequence, $\mathfrak{S}_I \cap \mathfrak{S}_{\leq \Sigma_\alpha} = [1, w \wedge \tau]$. $\square$

Lemma 5.3.12. *Let $\alpha \vDash_e n$ with $\alpha \neq (1^n)$. Then $s_i \leq \sigma_\alpha$ for all $i \in [n-1]$.*

Proof. Let $i \in [n-1]$ and $I = S \setminus \{s_i\}$. Then for each $w \in \mathfrak{S}_I$, $w([i]) = [i]$. Consider $\alpha \vDash_e n$ with $\alpha \neq (1^n)$. Then $\alpha_1 \geq 2$. Thus by definition, $\sigma_\alpha(1) = n \notin [i]$ which implies $\sigma_\alpha \notin \mathfrak{S}_I$. As $\mathfrak{S}_I$ is the subgroup of $\mathfrak{S}_n$ generated by $S \setminus \{s_i\}$, we conclude $s_i \leq \sigma_\alpha$. $\square$

Remark. In the proof of Lemma 5.3.12 we showed that $\sigma_\alpha([i]) \neq [i]$ for all $i \in [n-1]$ and all elements in stair form σ_α with $\alpha \vDash_e n$ and $\alpha \neq (1^n)$. Duchamp, Hivert and Thibon call permutations with this property *connected* in [DHT02]. In the paper they show that the connected permutations index a basis of the algebra of free quasisymmetric functions.

Corollary 5.3.13. *Conjecture 5.0.1 is true for all odd hooks $\alpha \vDash_e n$ with $\alpha \neq (n)$.*

Proof. Let $n \geq 3$, α be an odd hook with $\alpha \neq (n)$ and $\emptyset \subsetneq I \subsetneq S$. As Conjecture 5.0.1 only pertains compositions different from (1^n), we also assume that $\alpha \neq (1^n)$. We have to show that $\bar{\pi}_{\leq \Sigma_\alpha} v_I = 0$.

Since $I \neq S$ we can apply Theorem 5.3.11 and obtain that there is a $u \in \mathfrak{S}_n$ such that $\mathfrak{S}_I \cap \mathfrak{S}_{\leq \Sigma_\alpha}$ is the interval in Bruhat order $[1, u]$. Since $\emptyset \neq I$ there is an $i \in [n-1]$ such that $s_i \in \mathfrak{S}_I$. On the other hand, we obtain $s_i \leq \sigma_\alpha$ from Lemma 5.3.12 since $\alpha \neq (1^n)$. Thus, $s_i \in \mathfrak{S}_{\leq \Sigma_\alpha}$. Consequently, $s_i \in \mathfrak{S}_I \cap \mathfrak{S}_{\leq \Sigma_\alpha}$, i.e. $s_i \leq u$. Hence, $u \neq 1$ and Lemma 5.1.12 implies $\bar{\pi}_{\leq \Sigma_\alpha} v_I = 0$. □

5.4 An application of the inductive product

The goal of this section is to show that Conjecture 5.0.1 is true for all maximal compositions whose odd parts form a hook. This is the largest family of compositions for which we validate the conjecture in this thesis. We always assume $n \geq 1$ and use the notions related to the inductive product introduced in Notation 4.3.45.

Let $\emptyset \subsetneq I \subsetneq S$ and $\alpha = (\alpha_1, \ldots, \alpha_l) \vDash_e n$ be such that $\alpha \neq (1^n)$ and the odd parts of α form a hook. We are going show that $(\mathfrak{S}_n)_I \cap (\mathfrak{S}_n)_{\leq \Sigma_\alpha}$ is an interval so that $\bar{\pi}_{\leq \Sigma_\alpha} v_I = 0$. For $\alpha = (n)$ or α an odd hook, we obtained these results in Section 5.2 and Section 5.3. Therefore, we can assume that $\ell(\alpha) \geq 2$ and α_1 is even.

The strategy is the same as before: We construct $\tau \in \Sigma_\alpha$ with the property that $w_0(I) \wedge \tau$ is the greatest element of $(\mathfrak{S}_n)_I \cap (\mathfrak{S}_n)_{\leq \Sigma_\alpha}$. To do this we use the inductive product exploiting the fact that α_1 is even. More precisely, we set $\alpha' := (\alpha_2, \ldots, \alpha_l)$, $n' := |\alpha'|$ and $\tau := \xi \odot \tau'$ for certain $\xi \in \{\sigma_{(\alpha_1)}, \sigma_{(\alpha_1)}^{w_1}\}$ and $\tau' \in \Sigma_{\alpha'}$ depending on I and α where w_1 is the longest element of $\mathfrak{S}_{\alpha_1}$. We will choose τ' such that $w' \wedge \tau'$ is the greatest element of $(\mathfrak{S}_{n'})_{I'} \cap (\mathfrak{S}_{n'})_{\leq \Sigma_{\alpha'}}$ where S' are the simple reflections of $\mathfrak{S}_{n'}$, $I' \subseteq S'$ also depends on I and α, $(\mathfrak{S}_{n'})_{I'}$ is a parabolic subgroup of $\mathfrak{S}_{n'}$ and $w' \in \mathfrak{S}_{n'}$ is the longest element of $(\mathfrak{S}_{n'})_{I'}$. The existence of τ' will be provided by induction.

We start with a lemma which determines the Bruhat tableau of the inductive product of two permutations $\sigma_1 \odot \sigma_2$ where $\sigma_1 \in \mathfrak{S}_{n_1}$ is an oscillating n_1-cycle with n_1 even.

We will use the lemma in situations where $\sigma_1 \in \Sigma_{(n_1)}$ since then σ_1 is oscillating by Theorem 4.3.20. For instance, it can be applied to the element $\tau = \xi \odot \tau'$ mentioned above.

Lemma 5.4.1. *Let $(n_1, n_2) \vDash n$ with n_1 even and $\sigma = \sigma_1 \odot \sigma_2$ where $\sigma_i \in \mathfrak{S}_{n_i}$ for $i = 1, 2$ and σ_1 is oscillating. Then for all $j \in [n-1]$ and $i \in [j]$,*

$$\sigma_{i,j} = \begin{cases} (\sigma_1)_{i,j} + n_2 & \text{if } j < \frac{n_1}{2} \text{ and } i \leq j, \\ (\sigma_2)_{i,j-\frac{n_1}{2}} + \frac{n_1}{2} & \text{if } \frac{n_1}{2} < j < \frac{n_1}{2} + n_2 \text{ and } i \leq j - \frac{n_1}{2}, \\ (\sigma_1)_{i,j-n_2} & \text{if } \frac{n_1}{2} + n_2 < j \text{ and } i \leq j - \frac{n_1}{2} - n_2, \\ n - j + i & \text{otherwise.} \end{cases}$$

Proof. Recall from Definition 4.3.1 that $\sigma_1([\frac{n_1}{2}]) = [\frac{n_1}{2} + 1, n_1]$ as σ_1 is oscillating and n_1 is even. The proof is divided into seven parts. Table 5.1 gives an overview of what we show in which part.

Table 5.1: An overview of the results shown in the seven parts of the proof of Lemma 5.4.1.

(1)	$\sigma_{i,j} = (\sigma_1)_{i,j} + n_2$	if $j \leq \frac{n_1}{2}$ and $i \leq j$,
(2)	$\sigma_{i,j} = n - j + i$	if $j = \frac{n_1}{2}$ and $i \leq j$,
(3)	$\sigma_{i,j} = n - j + i$	if $\frac{n_1}{2} < j$ and $j - \frac{n_1}{2} < i$,
(4)	$\sigma_{i,j} = (\sigma_2)_{i,j-\frac{n_1}{2}} + \frac{n_1}{2}$	if $\frac{n_1}{2} < j \leq \frac{n_1}{2} + n_2$ and $i \leq j - \frac{n_1}{2}$,
(5)	$\sigma_{i,j} = n - j + i$	if $j = \frac{n_1}{2} + n_2$ and $i \leq j$,
(6)	$\sigma_{i,j} = n - j + i$	if $\frac{n_1}{2} + n_2 < j$ and $j - \frac{n_1}{2} - n_2 < i$,
(7)	$\sigma_{i,j} = (\sigma_1)_{i,j-n_2}$	if $\frac{n_1}{2} + n_2 < j$ and $i \leq j - \frac{n_1}{2} - n_2$.

Let $j \in [n-1]$. From Lemma 4.3.49 we obtain that

$$\sigma([j]) = \sigma_1^{\varphi_1}([j] \cap N_1) \cup \sigma_2^{\varphi_2}([j] \cap N_2).$$

(1) Suppose $j \leq \frac{n_1}{2}$. Then $\varphi_1^{-1}([j]) = [j]$ and

$$\sigma([j]) = \sigma_1^{\varphi_1}([j]) = \varphi_1(\sigma_1([j])) = \{\varphi_1((\sigma_1)_{i,j}) \mid 1 \leq i \leq j\}.$$

Since the sequence $(\sigma_1)_{1,j}, (\sigma_1)_{2,j}, \ldots, (\sigma_1)_{j,j}$ is strictly increasing and φ_1 is order preserving, it follows that

$$\sigma_{i,j} = \varphi_1((\sigma_1)_{i,j})$$

for all $i \in [j]$. Using the definition of φ_1 and $\sigma_1([\frac{n_1}{2}]) = [\frac{n_1}{2} + 1, n_1]$, we obtain that $\varphi_1((\sigma_1)_{i,j}) = (\sigma_1)_{i,j} + n_2$ for all $i \in [j]$.

(2) Suppose $j = \frac{n_1}{2}$ and let $i \in [j]$. Then we have $\sigma_1([j]) = [n_1 - j + 1, n_1]$ so that $(\sigma_1)_{i,j} = n_1 - j + i$. Thus,

$$\sigma_{i,j} = (\sigma_1)_{i,j} + n_2 = n_1 + n_2 - j + i = n - j + i$$

where we use Part (1) for the first equality.

(3) Suppose $\frac{n_1}{2} < j$ and let $i \in [j]$ with $j - \frac{n_1}{2} < i$. Set $r := j - \frac{n_1}{2}$. Then $j - r = \frac{n_1}{2}$, $i - r \in \left[\frac{n_1}{2}\right]$ and

$$n - j + i = n - (j - r) + i - r = \sigma_{i-r,j-r} \leq \sigma_{i,j} \leq n - j + i$$

where the second equality is valid by Part (2) and the first and the second inequality are consequences of Lemma 5.1.7 Part (3) and (1), respectively. Thus, $\sigma_{i,j} = n - i + j$. Moreover, it follows that

$$\sigma_{i,j} = \sigma_{i-r,j-r} = \sigma_{i-r,\frac{n_1}{2}}.$$

As a consequence,

$$\sigma\left(\left[\frac{n_1}{2}\right]\right) = \left\{\sigma_{i,\frac{n_1}{2}} \mid 1 \le i \le \frac{n_1}{2}\right\} = \left\{\sigma_{i,j} \mid j - \frac{n_1}{2} < i \le j\right\}.$$

(4) Suppose $\frac{n_1}{2} < j \le \frac{n_1}{2} + n_2$. Then $\varphi_2^{-1}\left([\frac{n_1}{2}+1, j]\right) = [j - \frac{n_1}{2}]$ and

$$\begin{aligned}\sigma([j]) &= \sigma_1^{\varphi_1}\left(\left[\frac{n_1}{2}\right]\right) \cup \sigma_2^{\varphi_2}\left(\left[\frac{n_1}{2}+1, j\right]\right)\\ &= \sigma\left(\left[\frac{n_1}{2}\right]\right) \cup \varphi_2\left(\sigma_2\left(\left[j - \frac{n_1}{2}\right]\right)\right)\\ &= \left\{\sigma_{i,j} \mid j - \frac{n_1}{2} < i \le j\right\} \cup \left\{\varphi_2\left((\sigma_2)_{i,j-\frac{n_1}{2}}\right) \mid 1 \le i \le j - \frac{n_1}{2}\right\}\\ &= \left\{\sigma_{i,j} \mid j - \frac{n_1}{2} < i \le j\right\} \cup \left\{(\sigma_2)_{i,j-\frac{n_1}{2}} + \frac{n_1}{2} \mid 1 \le i \le j - \frac{n_1}{2}\right\}.\end{aligned}$$

Because the sequences $\sigma_{1,j}, \dots, \sigma_{j,j}$ and $(\sigma_2)_{1,j-\frac{n_1}{2}}, \dots, (\sigma_2)_{j-\frac{n_1}{2},j-\frac{n_1}{2}}$ are both strictly increasing, it follows that $\sigma_{i,j} = (\sigma_2)_{i,j-\frac{n_1}{2}} + \frac{n_1}{2}$ for all $i \in [j - \frac{n_1}{2}]$.

(5) Suppose $j = \frac{n_1}{2} + n_2$. Then $\sigma_2([j - \frac{n_1}{2}]) = [n_2]$ and thus for all $i \in [j - \frac{n_1}{2}]$,

$$\sigma_{i,j} = (\sigma_2)_{i,j-\frac{n_1}{2}} + \frac{n_1}{2} = i + \frac{n_1}{2} = n - j + i$$

where we use that $n - j = \frac{n_1}{2}$. From Part (3) we have that $\sigma_{i,j} = n - j + i$ for all $i \in [j - \frac{n_1}{2} + 1, j]$ as well. Therefore, $\sigma_{i,j} = n - j + i$ for all $i \in [j]$.

(6) Suppose $\frac{n_1}{2} + n_2 < j$. We can argue as in Part (3) and obtain that $\sigma_{i,j} = n - j + i$ for all i such that $j - \frac{n_1}{2} - n_2 < i \le j$. Moreover analogous to Part (3), we obtain that

$$\sigma\left(\left[\frac{n_1}{2} + n_2\right]\right) = \left\{\sigma_{i,j} \mid j - \frac{n_1}{2} - n_2 < i \le j\right\}.$$

(7) Suppose $\frac{n_1}{2} + n_2 < j$. Then

$$\sigma([j]) = \sigma\left(\left[\frac{n_1}{2} + n_2\right]\right) \cup \sigma_1^{\varphi_1}\left(\left[\frac{n_1}{2} + n_2 + 1, j\right]\right).$$

Hence, the last equation of Part (6) implies

$$\left\{\sigma_{i,j} \mid 1 \le i \le j - \frac{n_1}{2} - n_2\right\} = \sigma_1^{\varphi_1}\left(\left[\frac{n_1}{2} + n_2 + 1, j\right]\right).$$

Using the definition of φ_1^{-1}, we obtain

$$\varphi_1^{-1}\left(\left[\frac{n_1}{2} + n_2 + 1, j\right]\right) = \left[\frac{n_1}{2} + 1, j - n_2\right].$$

By assumption, $\sigma_1([\frac{n_1}{2}]) = [\frac{n_1}{2}+1, n_1]$ and $\sigma_1([\frac{n_1}{2}+1, n_1]) = [\frac{n_1}{2}]$. Thus, we have that $\sigma_1\left([\frac{n_1}{2}+1, j-n_2]\right) \subseteq [\frac{n_1}{2}]$. As a consequence,

$$\varphi_1\left(\sigma_1\left(\left[\frac{n_1}{2}+1, j-n_2\right]\right)\right) = \sigma_1\left(\left[\frac{n_1}{2}+1, j-n_2\right]\right)$$

and this set contains the $j - \frac{n_1}{2} - n_2$ smallest elements of $\sigma_1([j-n_2])$. To sum up,

$$\begin{aligned}\left\{\sigma_{i,j} \mid 1 \le i \le j - \frac{n_1}{2} - n_2\right\} &= \sigma_1^{\varphi_1}\left(\left[\frac{n_1}{2}+n_2+1, j\right]\right) \\ &= \sigma_1\left(\left[\frac{n_1}{2}+1, j-n_2\right]\right) \\ &= \left\{(\sigma_1)_{i,j-n_2} \mid 1 \le i \le j - \frac{n_1}{2} - n_2\right\}.\end{aligned}$$

That is, $\sigma_{i,j} = (\sigma_1)_{i,j-n_2}$ for all i such that $1 \le i \le j - \frac{n_1}{2} - n_2$. $\square$

Example 5.4.2. Let $n_1 = 6, n_2 = 5$, $n = 11$, $\xi = \sigma_{(6)} = (1,6,2,5,3,4) \in \mathfrak{S}_{n_1}$, $\tau' = (1,2,5) \in \mathfrak{S}_{n_2}$ and $\tau = \xi \odot \tau'$. Then $\tau = (1,11,2,10,3,9)(4,5,8)$. The Bruhat tableaux of ξ, τ' and τ are shown below.

$$B(\xi) = \begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 4 & 5 & 6 \\ \hline 1 & 4 & 5 & 6 & \\ \hline 4 & 5 & 6 & & \\ \hline 5 & 6 & & & \\ \hline 6 & & & & \\ \hline\end{array} \qquad B(\tau') = \begin{array}{|c|c|c|c|}\hline 2 & 3 & 4 & 5 \\ \hline 2 & 3 & 5 & \\ \hline 2 & 5 & & \\ \hline 2 & & & \\ \hline\end{array} \qquad B(\tau) = \begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline 1 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ \hline 1 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & \\ \hline 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & & \\ \hline 5 & 6 & 7 & 8 & 9 & 10 & 11 & & & \\ \hline 5 & 6 & 8 & 9 & 10 & 11 & & & & \\ \hline 5 & 8 & 9 & 10 & 11 & & & & & \\ \hline 5 & 9 & 10 & 11 & & & & & & \\ \hline 9 & 10 & 11 & & & & & & & \\ \hline 10 & 11 & & & & & & & & \\ \hline 11 & & & & & & & & & \\ \hline\end{array}$$

Since $\xi \in \Sigma_{(n_1)}$, it is oscillating by Theorem 4.3.20. Hence, we can apply Lemma 5.4.1 on τ and it follows that the lower, upper and middle white part of $B(\tau)$ are determined by the lower white part of $B(\xi)$, the upper white part of $B(\xi)$ and $B(\tau')$, respectively. Moreover, the gray cells of $B(\xi)$ and $B(\tau)$ contain maximal entries. For the former this is a consequence of $\xi([\frac{n_1}{2}]) = [\frac{n_1}{2}+1, n_1]$ and for the latter this is the case *otherwise* of Lemma 5.4.1.

In the last sections conjugating with w_0 turned out to be a useful tool. We now consider the interplay between the inductive product on $\mathfrak{S}_{n_1}$ and $\mathfrak{S}_{n_2}$ and the conjugation with the longest elements of $\mathfrak{S}_{n_1}$, $\mathfrak{S}_{n_2}$ and $\mathfrak{S}_{n_1+n_2}$.

Lemma 5.4.3. *Let $(n_1, n_2) \vDash n$ with n_1 even. Set $n_0 := n$ and let w_i be the longest element of $\mathfrak{S}_{n_i}$ for $i = 0, 1, 2$.*

(1) For $i = 1, 2$ and φ_i the bijection from Notation 4.3.45 regarded as a function

$\varphi_i \colon [n_i] \to [n]$, *we have*

$$w_0 \circ \varphi_i = \varphi_i \circ w_i.$$

(2) *Given* $\sigma = \sigma_1 \odot \sigma_2 \in \mathfrak{S}_n$ *with* $\sigma_i \in \mathfrak{S}_{n_i}$ *for* $i = 1, 2$, *we have*

$$\sigma^{w_0} = \sigma_1^{w_1} \odot \sigma_2^{w_2}.$$

Proof. (1) Let $j \in [n_1]$. Using the definition of φ_1, we obtain

$$w_0\left(\varphi_1(j)\right) = \begin{cases} w_0(j) = n - j + 1 & \text{if } j \leq \frac{n_1}{2} \\ w_0(j + n_2) = n - n_2 - j + 1 = n_1 - j + 1 & \text{if } j > \frac{n_1}{2}. \end{cases}$$

On the other hand,

$$\varphi_1\left(w_1(j)\right) = \varphi_1(n_1 - j + 1) = \begin{cases} n_1 + n_2 - j + 1 = n - j + 1 & \text{if } n_1 - j + 1 > \frac{n_1}{2} \\ n_1 - j + 1 & \text{if } n_1 - j + 1 \leq \frac{n_1}{2}. \end{cases}$$

Now use

$$j \leq \frac{n_1}{2} \iff n_1 - j + 1 \geq n - \frac{n_1}{2} + 1 \iff n_1 - j + 1 > \frac{n_1}{2}$$

to obtain the claim for φ_1.

Consider φ_2. Let $j \in [n_2]$. Then

$$w_0\left(\varphi_2(j)\right) = w_0(j + \frac{n_1}{2}) = n - \frac{n_1}{2} - j + 1 = \varphi_2(n_2 - j + 1) = \varphi_2\left(w_2(j)\right)$$

as desired.

(2) Let $i \in \{1, 2\}$ and $j \in N_i$. Note that $w_0(N_i) = N_i$ and $w_0^{-1} = w_0$. Hence, $\sigma(w_0^{-1}(j)) = \sigma_i^{\varphi_i}(w_0^{-1}(j))$ by Lemma 4.3.49. Thus,

$$\begin{aligned}
\sigma^{w_0}(j) &= (w_0 \circ \sigma_i^{\varphi_i} \circ w_0^{-1})(j) \\
&= (w_0 \circ \varphi_i \circ \sigma_i \circ \varphi_i^{-1} \circ w_0^{-1})(j) \\
&= ((w_0 \circ \varphi_i) \circ \sigma_i \circ (w_0 \circ \varphi_i)^{-1})(j) \\
&= ((\varphi_i \circ w_i) \circ \sigma_i \circ (\varphi_i \circ w_i)^{-1})(j) \\
&= (\varphi_i \circ w_i \circ \sigma_i \circ w_i^{-1} \circ \varphi_i^{-1})(j) \\
&= ((\sigma_i^{w_i})^{\varphi_i})(j)
\end{aligned}$$

where we use Part (1) for the forth equality. □

Let $\alpha = (\alpha_1, \ldots, \alpha_l) \vDash_e n$ with $l \geq 2$ and α_1 even, $\alpha' := (\alpha_2, \ldots, \alpha_l)$, $n' := |\alpha'|$, w_1 be the longest element of $\mathfrak{S}_{\alpha_1}$, $\tau := \xi \odot \tau'$ with $\xi \in \{\sigma_{(\alpha_1)}, \sigma_{(\alpha_1)}^{w_1}\}$ and $\tau' \in \Sigma_{\alpha'}$, $I \subsetneq S$ and $w := w_0(I)$. We want to compare the Bruhat tableaux of τ and w. From Lemma 5.4.1 we have a nice description of the Bruhat tableau of τ in terms of ξ and τ'. Since w lacks the inductive structure of τ, we have no such description for its Bruhat tableau. Therefore we consider an element $v := \xi \odot w'$ with $w' \in \mathfrak{S}_{n'}$ and the property that $w \leq v$. Then Lemma 5.4.1 implies that the Bruhat tableaux of τ and v are the same, except for the parts determined by τ' and w', respectively. That is, comparing the Bruhat tableaux of τ and v reduces to the comparison of the Bruhat tableaux of τ' and w'. We now introduce the element v.

Lemma 5.4.4. *Let $(n_1, n_2) \vDash n$ with n_1 even, $I \subsetneq S$ and $a \in [n-1]$ with $s_a \notin I$. Set $w := w_0(I)$,*

$$\xi := \begin{cases} \sigma_{(n_1)} & \text{if } a \leq \lceil \frac{n}{2} \rceil \\ \sigma_{(n_1)}^{w_1} & \text{if } a > \lceil \frac{n}{2} \rceil \end{cases}$$

where w_1 is the longest element of $\mathfrak{S}_{n_1}$, S' to be the set of simple reflections of $\mathfrak{S}_{n_2}$,

$$I' := \begin{cases} S' = \emptyset & \text{if } n_2 = 1 \\ S' \setminus \{s_{a'}\} & \text{if } n_2 > 1 \end{cases} \quad \text{where} \quad a' := \begin{cases} \min\{a, n_2 - 1\} & \text{if } a \leq \lceil \frac{n}{2} \rceil \\ \max\{1, a - n_1\} & \text{if } a > \lceil \frac{n}{2} \rceil, \end{cases}$$

$w' \in \mathfrak{S}_{n_2}$ to be the longest element of $(\mathfrak{S}_{n_2})_{I'}$ and $v := \xi \odot w'$. Then $w \leq v$.

Proof. Since $I \subseteq S \setminus \{s_a\}$, it follows that $w \leq w_0(S \setminus \{s_a\})$. Therefore we can assume without loss of generality that $I = S \setminus \{s_a\}$.

(1) Assume $a \leq \lceil \frac{n}{2} \rceil$. Then $\xi = \sigma_{(n)}$. Let $j \in [n-1]$ and $i \in [j]$. We show $w_{i,j} \leq v_{i,j}$. Then we can apply the tableau criterion, Theorem 5.1.5, to obtain $w \leq v$. Since the element in stair form $\sigma_{(n_1)}$ is oscillating by Lemma 4.3.11 we can apply Lemma 5.4.1 on v. We distinguish the four cases that occur in Lemma 5.4.1.

Case 1. Assume $j < \frac{n_1}{2}$. Then Lemma 5.4.1 yields $v_{i,j} = (\sigma_{(n_1)})_{i,j} + n_2$. From Lemma 5.2.2 we have that $(\sigma_{(n_1)})_{p,q} = n_1 - q + p$ unless $p = 1$ and $q > \frac{n_1}{2}$. Thus, $(\sigma_{(n_1)})_{i,j} = n_1 - j + i$, i.e. $v_{i,j} = n - j + i$. That is, $v_{i,j}$ is maximal and hence $v_{i,j} \geq w_{i,j}$.

Case 2. Assume $\frac{n_1}{2} + n_2 < j$ and $i \leq j - \frac{n_1}{2} - n_2$. Then $v_{i,j} = (\sigma_{(n_1)})_{i,j-n_2}$ by Lemma 5.4.1.

First assume $i > 1$. Then it follows from above that

$$(\sigma_{(n_1)})_{i,j-n_2} = n_1 - (j - n_2) + i = n - j + i.$$

Hence, again $v_{i,j}$ is maximal and therefore $v_{i,j} \geq w_{i,j}$.

We now assume $i = 1$. We have

$$a \leq \left\lceil \frac{n}{2} \right\rceil \leq \frac{n_1}{2} + \frac{n_2}{2} + \frac{1}{2} \leq \frac{n_1}{2} + n_2 < j.$$

Consequently, $w_{1,j} = 1$ by Lemma 5.3.9 and thus certainly $w_{1,j} \leq v_{1,j}$.

Case 3. Suppose $\frac{n_1}{2} < j < \frac{n_1}{2} + n_2$ and $i \leq j - \frac{n_1}{2}$. This case can only occur if $n_2 \geq 2$. Hence, $I' = S' \setminus \{s_{a'}\}$. Moreover, in this case $v_{i,j} = w'_{i,j-\frac{n_1}{2}} + \frac{n_1}{2}$ by Lemma 5.4.1.

Assume first that $a \geq n_2 - 1$. Then $a' = n_2 - 1$ and hence $j - \frac{n_1}{2} \leq n_2 - 1 = a'$. Thus,

$$w'_{i,j-\frac{n_1}{2}} = a' - (j - \frac{n_1}{2}) + i = \frac{n_1}{2} + n_2 - 1 - j + i$$

by Lemma 5.3.9. Hence, $v_{i,j} = n - 1 - j + i$. If $j < a$ then Lemma 5.3.9 yields $w_{i,j} = a - j + i$ and since $a \leq n - 1$, it follows that $w_{i,j} \leq v_{i,j}$. We have

$$i \leq j - \frac{n_1}{2} \leq n_2 - 1 \leq a.$$

Therefore, if $j \geq a$ then $w_{i,j} = i$ by Lemma 5.3.9 and hence Lemma 5.1.7 implies $w_{i,j} \leq v_{i,j}$.

Assume now that $a < n_2 - 1$. Then $a' = a$.

(i) Suppose $i \leq a$. If $j \geq a$ then $w_{i,j} = i \leq v_{i,j}$. Thus, assume $j < a$. Then Lemma 5.3.9 yields $w_{i,j} = a - j + i$ and $w'_{i,j-\frac{n_1}{2}} = \frac{n_1}{2} + a - j + i$. Hence $v_{i,j} = n_1 + a - j + i \geq w_{i,j}$.

(ii) Suppose $i > a$. Then also $j - \frac{n_2}{2} > a$. Because $a = a'$, we obtain from Lemma 5.3.9 that $w'_{i,j-\frac{n_1}{2}} = \frac{n_1}{2} + n_2 - j + i$. Thus, $v_{i,j} = n - j + i$. Hence, $v_{i,j} \geq w_{i,j}$ as $v_{i,j}$ is maximal.

Case 4. Assume that i and j do not fall in one of the previous cases. Then we obtain from Lemma 5.4.1 that $v_{i,j} = n - j + i$. Thus, $v_{i,j} \geq w_{i,j}$. This finishes the proof of $w \leq v$ in the case $a \leq \lceil \frac{n}{2} \rceil$.

(2) Assume $a > \lceil \frac{n}{2} \rceil$. Let w_0 be the longest element of $\mathfrak{S}_n$ and w_i be the longest element of $\mathfrak{S}_{n_i}$ for $i = 1, 2$. We use the Bruhat order automorphism ν with $\nu(x) = x^{w_0}$ to trace this case back to Part (1). Set $\tilde{a} := n - a$ and $\tilde{I} := S \setminus \{s_{\tilde{a}}\}$. Define $\tilde{w}$, $\tilde{a}', \tilde{\xi}, \tilde{I}', \tilde{w}'$ and $\tilde{v}$ depending on $\tilde{I}$ and $\tilde{a}$ in the same way as their counterparts without tilde from the theorem are defined depending on I and a.

Since $a > \lceil \frac{n}{2} \rceil$, we have $\tilde{a} = n - a \leq \lceil \frac{n}{2} \rceil$. From Lemma 4.1.17 it follows that $w = \tilde{w}^{w_0}$.

We claim that $w' = (\tilde{w}')^{w_2}$. If $n_2 = 1$ then $w' = \tilde{w}' = 1 \in \mathfrak{S}_{n_2}$ and therefore $w' = (\tilde{w}')^{w_2}$. Now suppose that $n_2 \geq 2$. Then $w' = w_0(S' \setminus \{s'_a\})$ and $\tilde{w}' = w_0(S' \setminus \{s_{\tilde{a}'}\})$ in $\mathfrak{S}_{n_2}$. We show $a' = n_2 - \tilde{a}'$. By definition, $a' = \max\{1, a - n_1\}$ and $\tilde{a}' = \min\{\tilde{a}, n_2 - 1\}$. Moreover,

$$a' = 1 \iff a - n_1 \leq 1 \iff n - \tilde{a} - n_1 \leq 1 \iff n_2 - 1 \leq \tilde{a} \iff \tilde{a}' = n_2 - 1.$$

Hence, $a' = n_2 - \tilde{a}'$ if $a' = 1$. Furthermore it follows from the equivalence that if $a' = a - n_1$ then $\tilde{a} = \tilde{a}'$ and

$$n_2 - a' = n_2 + n_1 - a = n - a = \tilde{a} = \tilde{a}'.$$

Therefore, we have $a' = n_2 - \tilde{a}'$ as desired. This implies $s_{a'} = s_{\tilde{a}'}^{w_2}$ in $\mathfrak{S}_{n_2}$ and thus $w' = (\tilde{w}')^{w_2}$ by Lemma 4.1.17. This finishes the proof of the claim $w' = (w')^{\tilde{w}_2}$.

We have seen in Part (1) that $\sigma_{(n_1)}([\frac{n_1}{2}]) = [\frac{n_1}{2} + 1, n_1]$. Therefore we can apply

Lemma 5.4.3 to $\tilde{v}$ and obtain

$$\tilde{v}^{w_0} = \sigma_{(n_1)}^{w_1} \odot (\tilde{w}')^{w_2} = \sigma_{(n_1)}^{w_1} \odot w' = v.$$

From Part (1) we have $\tilde{w} \leq \tilde{v}$. Since ν is a Bruhat order automorphism, it follows that $w = \tilde{w}^{w_0} \leq \tilde{v}^{w_0} = v$. $\square$

Example 5.4.5. Let $n = 11$, $\alpha = (\alpha_1, \ldots, \alpha_4) = (6, 3, 1, 1) \vDash_e n$, $a = 1$, $I = S \setminus \{s_a\}$ and $w = w_0(I)$. Moreover, set $\alpha' = (3, 1, 1)$, $n_1 = \alpha_1 = 6$ and $n_2 = |\alpha'| = 5$.

(1) We define v as in Lemma 5.4.4. That is, we set $\xi = \sigma_{(6)}$, S' to be the simple reflection of $\mathfrak{S}_5$, $a' = \min\{1, 4\} = 1$, $I' = S' \setminus \{s_1\}$, w' to be the longest element of $(\mathfrak{S}_5)_{I'}$ and $v = \sigma_{(6)} \odot w'$. Then

$$\begin{aligned} w &= (2,11)(3,10)(4,9)(5,8)(6,7), \\ \sigma_{(6)} &= (1,6,2,5,3,4), \\ w' &= (2,5)(3,4), \\ v &= (1,11,2,10,3,9)(5,8)(6,7) \end{aligned}$$

and Lemma 5.4.4 yields $w \leq v$. One can check the latter by comparing the Bruhat tableau of w and v shown below.

$$B(w) = \begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline 1&3&4&5&6&7&8&9&10&11 \\ \hline 1&4&5&6&7&8&9&10&11 \\ \cline{1-9} 1&5&6&7&8&9&10&11 \\ \cline{1-8} 1&6&7&8&9&10&11 \\ \cline{1-7} 1&7&8&9&10&11 \\ \cline{1-6} 1&8&9&10&11 \\ \cline{1-5} 1&9&10&11 \\ \cline{1-4} 1&10&11 \\ \cline{1-3} 1&11 \\ \cline{1-2} 1 \\ \cline{1-1} \end{array} \qquad B(v) = \begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline 1&3&4&5&6&7&8&9&10&11 \\ \hline 1&4&5&6&7&8&9&10&11 \\ \cline{1-9} 4&5&6&7&8&9&10&11 \\ \cline{1-8} 4&6&7&8&9&10&11 \\ \cline{1-7} 4&7&8&9&10&11 \\ \cline{1-6} 4&8&9&10&11 \\ \cline{1-5} 4&9&10&11 \\ \cline{1-4} 9&10&11 \\ \cline{1-3} 10&11 \\ \cline{1-2} 11 \\ \cline{1-1} \end{array}$$

By Lemma 5.4.1 the entries in the gray cells of $B(v)$ are maximal and the other entries only depend on either ξ or w'.

(2) Consider $\tau = \xi \odot \tau'$ with $\tau' = (1, 2, 5) \in \mathfrak{S}_5$. This is the element τ from Example 5.4.2. Then $\xi = \sigma_{(6)} \in \Sigma_{(\alpha_1)}$ and by Example 5.3.1, $\tau' \in \Sigma_{\alpha'}$. Therefore, Theorem 4.3.55 implies $\tau \in \Sigma_\alpha$.

We compare the Bruhat tableaux of w, v and τ. The latter is shown in Example 5.4.2. Since τ and v are elements of $\mathfrak{S}_{11}$ given as inductive products with the same left factor ξ, Lemma 5.4.1 implies that $B(\tau)$ and $B(v)$ coincide outside the white subtableau of shape $(4, 3, 2, 1)$. Therefore, $w \leq v$ implies that $w_{i,j} > \tau_{i,j}$ is possible only for entries within this subtableau, i.e. for $3 < j < 8$ and $i \leq j - 3$. From Lemma 5.4.1 we obtain that

$$v_{i,j} = w'_{i,j'} + 3 \quad \text{and} \quad \tau_{i,j} = \tau'_{i,j'} + 3$$

for these i and j where $j' = j - 3$. Thus, if $w_{i,j} > \tau_{i,j}$ then $v_{i,j} > \tau_{i,j}$ and hence $w'_{i,j'} > \tau'_{i,j'}$. That is, we have established a connection to the smaller Bruhat tableaux of $w', \tau' \in \mathfrak{S}_5$. This is a crucial step in the proof of the following result.

The next lemma gives rise to a recursive definition of the element τ form the introduction of the section. The objects ξ, I', a' and w' occurring in it are defined exactly as in Lemma 5.4.4 for $n_1 = \alpha_1$ and $n_2 = n'$.

Lemma 5.4.6. *Assume that $\alpha = (\alpha_1, \dots, \alpha_l) \vDash_e n$ with α_1 even and $l \geq 2$, $I \subsetneq S$ and $a \in [n-1]$ with $s_a \notin I$ are given. Let $w := w_0(I)$, $\alpha' := (\alpha_2, \dots, \alpha_l)$,*

$$\xi := \begin{cases} \sigma_{(\alpha_1)} & \text{if } a \leq \lceil \frac{n}{2} \rceil \\ \sigma_{(\alpha_1)}^{w_1} & \text{if } a > \lceil \frac{n}{2} \rceil \end{cases}$$

where w_1 is the longest element of $\mathfrak{S}_{\alpha_1}$, $\tau' \in \Sigma_{\alpha'}$ and $\tau := \xi \odot \tau'$. Then $\tau \in \Sigma_\alpha$.

Moreover, let $n' := |\alpha'|$, S' be the set of simple reflections of $\mathfrak{S}_{n'}$,

$$I' := \begin{cases} S' = \emptyset & \text{if } n' = 1 \\ S' \setminus \{s_{a'}\} & \text{if } n' > 1 \end{cases} \quad \text{where} \quad a' := \begin{cases} \min\{a, n'-1\} & \text{if } a \leq \lceil \frac{n}{2} \rceil \\ \max\{1, a - \alpha_1\} & \text{if } a > \lceil \frac{n}{2} \rceil \end{cases}$$

and $w' \in \mathfrak{S}_{n'}$ be the longest element of $(\mathfrak{S}_{n'})_{I'}$. Then $w \wedge \tau$ is the greatest element of $(\mathfrak{S}_n)_I \cap (\mathfrak{S}_n)_{\leq \Sigma_\alpha}$ if $w' \wedge \tau'$ is the greatest element of $(\mathfrak{S}_{n'})_{I'} \cap (\mathfrak{S}_{n'})_{\leq \Sigma_{\alpha'}}$.

Proof. First, we show $\tau \in \Sigma_\alpha$. We know that $\sigma_{(\alpha_1)} \in \Sigma_{(\alpha_1)}$ and by Corollary 4.1.16 also that $\sigma_{(\alpha_1)}^{w_1} \in \Sigma_{(\alpha_1)}$. Thus, $\xi \in \Sigma_{(\alpha_1)}$. Moreover, $\tau' \in \Sigma_{\alpha'}$ by assumption. Since α_1 is even, Theorem 4.3.55 now yields $\tau = \xi \odot \tau' \in \Sigma_\alpha$.

In the following we repeatedly use that for all $m \in \mathbb{N}$ and $u, x \in \mathfrak{S}_m$ such that u is the longest element of a parabolic subgroup of $\mathfrak{S}_m$, the meet $u \wedge x$ exists and we have $(u \wedge x)_{i,j} = u_{i,j} \wedge x_{i,j}$ for all $j \in [m-1]$ and $i \in [j]$ by Proposition 5.1.9.

Assume that $(\mathfrak{S}_{n'})_{I'} \cap (\mathfrak{S}_{n'})_{\leq \Sigma_{\alpha'}}$ has a greatest element and that $w' \wedge \tau'$ is this element. Let $x \in \mathfrak{S}_I \cap \mathfrak{S}_{\leq \Sigma_\alpha}$. Then there is a $\sigma \in \Sigma_\alpha$ such that $x \leq w$ and $x \leq \sigma$. Hence $x \leq w \wedge \sigma$ and without loss of generality we can assume that $x = w \wedge \sigma$. By Theorem 4.3.55, there are $\eta \in \Sigma_{(\alpha_1)}$ and $\sigma' \in \Sigma_{\alpha'}$ such that $\sigma = \eta \odot \sigma'$. An overview of the permutations appearing in this proof and their relations in Bruhat order is given by Figure 5.1.

We have to show that $w \wedge \sigma \leq w \wedge \tau$. By the tableau criterion, Theorem 5.1.5, this is equivalent to $(w \wedge \sigma)_{i,j} \leq (w \wedge \tau)_{i,j}$ for all $j \in [n-1]$ and $i \in [j]$.

Let $j \in [n-1]$ and $i \in [j]$. If $w_{i,j} \leq \tau_{i,j}$ then

$$(w \wedge \tau)_{i,j} = w_{i,j} \geq (w \wedge \sigma)_{i,j}$$

as desired.

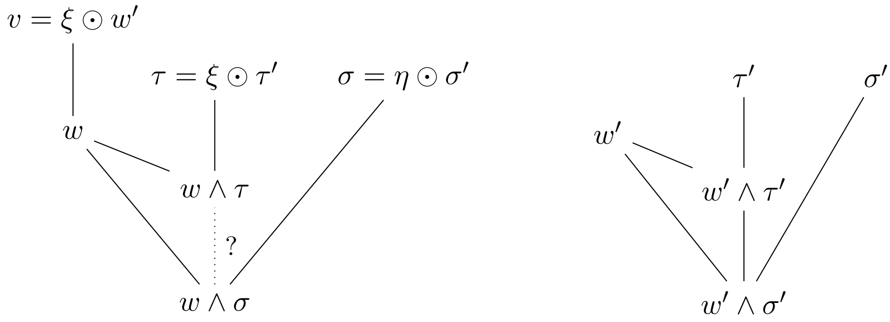


Figure 5.1: The two diagrams show how the elements appearing in the proof of Lemma 5.4.6 are related to each other by the Bruhat order. On the left hand side, we have the elements of $\mathfrak{S}_n$ and on the right hand side the elements of $\mathfrak{S}_{n'}$. If x and y are joined by an edge and x is below y then $x \leq y$.

Assume $w_{i,j} > \tau_{i,j}$. In this case $(w \wedge \tau)_{i,j} = \tau_{i,j}$. Set $n_1 := \alpha_1$, $n_2 := n'$ and $v := \xi \odot w'$. Then v is defined as in Lemma 5.4.4 and the same lemma yields $v \geq w$. Therefore, the tableau criterion implies $v_{i,j} > \tau_{i,j}$.

The elements $\tau = \xi \odot \tau', v = \xi \odot w'$ and $\sigma = \eta \odot \sigma'$ are all contained in $\Sigma_{(n_1)} \odot \mathfrak{S}_{n_2}$. In addition, n_1 is even and each element of $\Sigma_{(n_1)}$ is oscillating by Theorem 4.3.20. Therefore, we can use Lemma 5.4.1 in order to compute the Bruhat tableaux of τ, v and σ from their respective factors in the inductive product.

Since τ and v have the same left factor in the inductive product and $v_{i,j} \neq \tau_{i,j}$, Lemma 5.4.1 implies that $\frac{n_1}{2} < j < \frac{n_1}{2} + n_2$ and $i \leq j - \frac{n_1}{2}$. For this kinds of indices the same lemma yields

$$v_{i,j} = w'_{i,j'} + \frac{n_1}{2}, \quad \tau_{i,j} = \tau'_{i,j'} + \frac{n_1}{2} \quad \text{and} \quad \sigma_{i,j} = \sigma'_{i,j'} + \frac{n_1}{2} \tag{5.4}$$

where $j' := j - \frac{n_1}{2}$. Thus, from $v_{i,j} > \tau_{i,j}$ we obtain that $w'_{i,j'} > \tau'_{i,j'}$. Consequently, $(w' \wedge \tau')_{i,j'} = \tau'_{i,j'}$. Since $w' \wedge \tau'$ is the greatest element of $(\mathfrak{S}_{n'})_{I'} \cap (\mathfrak{S}_{n'})_{\leq \Sigma_{\alpha'}}$, it follows that

$$\tau'_{i,j'} = (w' \wedge \tau')_{i,j'} \geq (w' \wedge \sigma')_{i,j'}.$$

In particular, $(w' \wedge \sigma')_{i,j'} = \sigma'_{i,j'}$ because otherwise we would obtain the contradiction $\tau'_{i,j'} \geq (w' \wedge \sigma')_{i,j'} = w'_{i,j'}$. Therefore, we have $\tau'_{i,j'} \geq \sigma'_{i,j'}$. Using (5.4) again, we get $\tau_{i,j} \geq \sigma_{i,j}$. Consequently,

$$(w \wedge \tau)_{i,j} = \tau_{i,j} \geq \sigma_{i,j} \geq (w \wedge \sigma)_{i,j}. \qquad \square$$

Example 5.4.7. As in Example 5.4.5, let $n = 11$, $\alpha = (6,3,1,1) \vDash_e n$, $I = S \setminus \{s_1\}$, $w = w_0(I)$, $\alpha' = (3,1,1)$, $n' = |\alpha'|$, S' be the simple reflections of $\mathfrak{S}_5$, $a' = 1$ $I' = S' \setminus \{s_{a'}\}$, w' be the longest element of $(\mathfrak{S}_5)_{I'}$, $\xi = \sigma_{(6)}$, $\tau' = (1,2,5) \in \mathfrak{S}_5$ and $\tau = \xi \odot \tau'$.

Using Definition 5.3.5, we have $\tau' = \tau_{\alpha',a'}$. Therefore, Lemma 5.3.7 yields $\tau' \in \Sigma_{\alpha'}$ and from Theorem 5.3.11 it follows that $w' \wedge \tau'$ is the greatest element of $(\mathfrak{S}_{n'})_{I'} \cap (\mathfrak{S}_{n'})_{\leq \Sigma_{\alpha'}}$. That is, we can apply Lemma 5.4.6 and obtain that $\tau \in \Sigma_\alpha$ and that $w \wedge \tau$ is the greatest element of $(\mathfrak{S}_n)_I \cap (\mathfrak{S}_n)_{\leq \Sigma_\alpha}$.

By generalizing the construction from Example 5.4.7, we obtain the main result on $\mathfrak{S}_I \cap \mathfrak{S}_{\leq \Sigma_\alpha}$. It implies that for all $I \subsetneq S$ and $\alpha \vDash_e n$ such that the odd parts of α form a hook, $\mathfrak{S}_I \cap \mathfrak{S}_{\leq \Sigma_\alpha}$ is an interval in Bruhat order.

Theorem 5.4.8. *Let $(\alpha_1, \dots, \alpha_l) \vDash_e n$ be such that the odd entries of α form a hook and $I \subseteq S$ be such that $I \neq S$ if $n > 1$. Then there exists a $\tau \in \Sigma_\alpha$ such that $\mathfrak{S}_I \cap \mathfrak{S}_{\leq \Sigma_\alpha}$ is the interval $[1, w_0(I) \wedge \tau]$ in Bruhat order.*

Proof. Let m be the number of even parts among $\alpha_1, \dots, \alpha_{l-1}$ and $w := w_0(I)$. We do an induction on m.

For the base case assume $m = 0$. We claim that then $\alpha = (n)$, $\alpha = (1^n)$ or $\alpha = (\alpha_1, 1^{n-\alpha_1})$ with $1 < \alpha_1 < n$ and α_1 odd. If $l = 1$ then $\alpha = (n)$. If $l > 1$ then $\alpha_1, \dots, \alpha_{l-1}$ are odd. It follows that α_l is odd as well, since α is a maximal composition. Hence α is an odd hook. That is, either $\alpha = (1^n)$ or $\alpha = (\alpha_1, 1^{n-\alpha_1})$ with $1 < \alpha_1 < n$ and α_1 odd. This finishes the proof of the claim.

Suppose $\alpha = (1^n)$. Then $\Sigma_\alpha = \{1\}$ and therefore $\mathfrak{S}_I \cap \mathfrak{S}_{\leq \Sigma_\alpha} = \{1\} = [1, w \wedge 1]$.

From now on we can assume $n \geq 2$. Then $I \subsetneq S$ and there exists an $a \in [n-1]$ such that $s_a \notin I$.

Suppose $\alpha = (n)$ and set $\tau := \sigma_{(n)}$ if $a \leq \lceil \frac{n}{2} \rceil$ and $\tau := \sigma_{(n)}^{w_0}$ if $a > \lceil \frac{n}{2} \rceil$. Then Lemma 5.2.3 yields that $\tau \in \Sigma_\alpha$ and $w \wedge \tau = w$. Together with Theorem 5.2.4 it follows that $\mathfrak{S}_I \cap \mathfrak{S}_{\leq \Sigma_\alpha} = \mathfrak{S}_I = [1, w \wedge \tau]$.

Suppose $\alpha = (\alpha_1, 1^{n-\alpha_1})$ with $1 < \alpha_1 < n$ and α_1 odd. Set $\tau := \tau_{\alpha,a}$ with $\tau_{\alpha,a}$ as in Definition 5.3.5. Then $\tau \in \Sigma_\alpha$ by Lemma 5.3.7. Moreover, Theorem 5.3.11 yields $\mathfrak{S}_I \cap \mathfrak{S}_{\leq \Sigma_\alpha} = [1, w \wedge \tau]$.

We continue with the induction step. Assume $m \geq 1$. Then $l \geq 2$, $n \geq 2$ and there exists an $a \in [n-1]$ such that $s_a \notin I$. As in Lemma 5.4.6, we set $\alpha' := (\alpha_2, \dots, \alpha_l)$, $n' := |\alpha'|$, S' to be the set of simple reflections of $\mathfrak{S}_{n'}$,

$$I' := \begin{cases} S' = \emptyset & \text{if } n' = 1 \\ S' \setminus \{s_{a'}\} & \text{if } n' > 1 \end{cases} \quad \text{where} \quad a' := \begin{cases} \min\{a, n'-1\} & \text{if } a \leq \lceil \frac{n}{2} \rceil \\ \max\{1, a - \alpha_1\} & \text{if } a > \lceil \frac{n}{2} \rceil \end{cases}$$

and w' to be the longest element of $(\mathfrak{S}_{n'})_{I'}$. The first $\ell(\alpha') - 1 = l - 2$ parts of α' are $\alpha_2, \dots, \alpha_{l-1}$ and hence exactly $m-1$ of these parts are even. Furthermore, $I' \subseteq S'$ with $I' \neq S'$ if $n' > 1$. Thus, we can apply the induction hypotheses and obtain that there is a $\tau' \in \Sigma_{\alpha'}$ such that $(\mathfrak{S}_{n'})_{I'} \cap (\mathfrak{S}_{n'})_{\leq \Sigma_{\alpha'}} = [1, w' \wedge \tau']$. Let

$$\xi := \begin{cases} \sigma_{(\alpha_1)} & \text{if } a \leq \lceil \frac{n}{2} \rceil \\ \sigma_{(\alpha_1)}^{w_1} & \text{if } a > \lceil \frac{n}{2} \rceil \end{cases}$$

where w_1 is the longest element of $\mathfrak{S}_{\alpha_1}$ and $\tau := \xi \odot \tau'$. Then we can apply Lemma 5.4.6 and obtain that $\tau \in \Sigma_\alpha$ and $(\mathfrak{S}_n)_I \cap (\mathfrak{S}_n)_{\leq \Sigma_\alpha} = [1, w \wedge \tau]$. □

From Theorem 5.4.8 it follows that $\mathfrak{S}_I \cap \mathfrak{S}_{\leq \Sigma_\alpha}$ is an interval if $I \neq \emptyset, S$ and the odd parts of $\alpha \vDash_e n$ form a hook. By the following example, this is not true for all maximal compositions.

Example 5.4.9. Let $\alpha = (3,3)$ and $I = \{s_1, s_2, s_3, s_4\}$. Then $w_0(I) = (1,5)(2,4)$. In Remark 4.3.76 the 22 elements of Σ_α are given. In particular, $\sigma_1 = (1,6,5)(2,4,3)$ and $\sigma_2 = (1,5,6)(2,3,4)$ are elements of Σ_α. Set $\tau_i = w_0(I) \wedge \sigma_i$ for $i = 1,2$. Then $\tau_1, \tau_2 \in \mathfrak{S}_I \cap \mathfrak{S}_{\leq \Sigma_\alpha}$. Computing τ_1 and τ_2 with Proposition 5.1.9 yields

$$\tau_1 = (1,5)(2,4,3) \quad \text{and} \quad \tau_2 = (1,5)(2,3,4).$$

We show that both τ_1 and τ_2 are maximal in $\mathfrak{S}_I \cap \mathfrak{S}_{\leq \Sigma_\alpha}$. Then $\mathfrak{S}_I \cap \mathfrak{S}_{\leq \Sigma_\alpha}$ cannot be an interval. One can check that $\ell(\sigma_1) = 10$. Because $\approx$ preserves the length, we have $\ell(\sigma) = 10$ for each $\sigma \in \Sigma_\alpha$. On the other hand, $\ell(w_0(I)) = 10$ but $w_0(I)$ is obviously not an element of Σ_α. Thus, each element of $\mathfrak{S}_I \cap \mathfrak{S}_{\leq \Sigma_\alpha}$ has at most length 9. As $\ell(\tau_i) = 9$ for $i = 1,2$, both elements must be maximal in $\mathfrak{S}_I \cap \mathfrak{S}_{\leq \Sigma_\alpha}$.

Thanks to Theorem 5.4.8, we can now prove the main result of this chapter. The proof is similar to that one of Corollary 5.3.13.

Corollary 5.4.10. *Conjecture 5.0.1 is true for all $\alpha \vDash_e n$ whose odd parts form a hook.*

Proof. Let $n \geq 3$, $\alpha \vDash_e n$ with $\alpha \neq (1^n)$ be such that the odd parts of α form a hook and $\emptyset \subsetneq I \subsetneq S$. We have to show that $\bar{\pi}_{\leq \Sigma_\alpha} v_I = 0$ where v_I is the element generating the simple $H_n(0)$-module $\boldsymbol{F}_I$.

Since $I \neq S$ we can apply Theorem 5.4.8 which provides a $u \in \mathfrak{S}_n$ such that $\mathfrak{S}_I \cap \mathfrak{S}_{\leq \Sigma_\alpha}$ is the interval $[1, u]$ in Bruhat order. Because $\emptyset \neq I$, there is an $i \in [n-1]$ such that $s_i \in \mathfrak{S}_I$. Moreover, as $\alpha \neq (1^n)$, we can use Lemma 5.3.12 and obtain that $s_i \leq \sigma_\alpha$ which implies that $s_i \in \mathfrak{S}_{\leq \Sigma_\alpha}$. Consequently, $s_i \in \mathfrak{S}_I \cap \mathfrak{S}_{\leq \Sigma_\alpha}$ and thus $s_i \leq u$. That is, $u \neq 1$ and Lemma 5.1.12 implies that $\bar{\pi}_{\leq \Sigma_\alpha} v_I = 0$. □

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Acknowledgments

I thank my advisor Christine Bessenrodt for guidance, motivation and patience. I am grateful to the members of the Institute for Algebra, Number Theory and Discrete Mathematics for helpful discussions. Special thanks go to Ruwen Hollenbach, Lucia Morotti and Patrick Wegener for proofreading. Computer explorations were facilitated by the open source mathematical software `sage` and its algebraic combinatorics features developed by the `sage-combinat` community [SMS]. Finally, I like to thank my wife and family for all their support and encouragement.

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Publication list

- S. König. *The decomposition of 0-Hecke modules associated to quasisymmetric Schur functions.* Algebraic Combinatorics, Volume 2 (2019), pp. 735-751.
- S. König. *The decomposition of 0-Hecke modules associated to quasisymmetric Schur functions* (extended abstract). Séminaire Lotharingien de Combinatoire, Volume 80B (2018), Proceedings of the 30th Conference on Formal Power Series and Algebraic Combinatorics (Hanover), Article 32