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SU(2|2) supersymmetric mechanics

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Abstract: We introduce a new kind of non-relativistic $\mathcal{N}=8$ supersymmetric mechanics, associated with worldline realizations of the supergroup SU(2|2) treated as a deformation of flat $\mathcal{N}=8$, $d=1$ supersymmetry. Various worldline SU(2|2) superspaces are constructed as coset manifolds of this supergroup, and the corresponding superfield techniques are developed. For the off-shell SU(2|2) multiplets $(\mathbf{3},\mathbf{8},\mathbf{5})$, $(\mathbf{4},\mathbf{8},\mathbf{4})$ and $(\mathbf{5},\mathbf{8},\mathbf{3})$, we construct and analyze the most general superfield and component actions. Common features are mass oscillator-type terms proportional to the deformation parameter and a trigonometric realization of the superconformal group OSp(4*|4) in the conformal cases. For the simplest $(\mathbf{5},\mathbf{8},\mathbf{3})$ model the quantization is performed.

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Contents

1 Introduction

In recent years, interest has grown in theories invariant under some "curved" analogs of rigid Poincaré supersymmetry in diverse dimensions [1–3]. The main motivation was to check general gauge/gravity correspondences in concrete field-theoretical examples, classically as well as quantum mechanically. One construction of such theories is by the localization method [4], which proceeds from the relevant supergravity theories in component formulation. Alternatively, one can start from the supergroup of the corresponding "curved" supersymmetry, list its various coset superspaces and develop appropriate superfield techniques. These permit the derivation of invariant actions as superspace Berezin integrals, with Lagrangians being functions of superfields and their covariant derivatives. This second approach was used in [5–7] and goes back to [8] where superfield techniques for $\mathrm{OSp}(1|4)$ supersymmetry in four dimension were fully developed for the first time.

Supersymmetric mechanics [9, 10] represents the extreme d=1 case of Poincaré-supersymmetric field theory. In the underlying d=1 "Poincaré superalgebra" the supercharges square to the Hamiltonian (and perhaps some constant or operator-valued central charges). Mechanical analogs of higher-dimensional curved rigidly supersymmetric theories can be based on semi-simple supergroups which yield flat d=1 supersymmetries through some contraction. In other words, mechanical models on such supergroups can be treated as deformations of standard supersymmetric mechanics. The main difference between the two types of supersymmetric mechanics models lies in the closure of the supercharges: in the deformed case it contains not only the Hamiltonian but also generators of some non-trivial internal symmetry. As a consequence, the corresponding Hilbert spaces and spectra essentially differ from each other. In particular, in the deformed case an energy level may carry unequal numbers of bosonic and fermionic states.

The simplest examples of such deformed supersymmetric mechanics substitute flat $\mathcal{N}=4, d$=1 supersymmetry with the supergroup $\mathrm{SU}(2|1)$. At the component level, they were constructed in [11–15]. The $\mathrm{SU}(2|1)$ symmetry in the model built on the multiplet $(\mathbf{1}, \mathbf{4}, \mathbf{3})$,[1] has been considered first in [11] and christened there "weak supersymmetry". Models based on the chiral $(\mathbf{2}, \mathbf{4}, \mathbf{2})$ multiplet were considered in [12–15], including a computation of the superconformal index.

The superfield approach to $\mathrm{SU}(2|1)$ mechanics was worked out in [16–19]. The superfield techniques not only reproduced the models of [11–15] (and their interacting extensions) but also revealed some new models, in particular those associated with the $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ multiplet. The latter admits a harmonic $\mathrm{SU}(2|1)$ superspace description generalizing flat $\mathcal{N}=4$, d=1 harmonic superspace [20]. Many notions and models of $\mathcal{N}=4$ mechanics still await a deformation to the $\mathrm{SU}(2|1)$ case, for instance $\mathcal{N}=4$ supersymmetric Calogero-Moser systems [21, 22], the gauging procedure in $\mathcal{N}=4$ mechanics [23, 24], or $\mathcal{N}=4$ models coupled to background abelian or non-abelian gauge fields [25–27].

[1] Multiplets of the standard and deformed $\mathcal{N}=4$, d=1 supersymmetry are denoted as $(\mathbf{k}, \mathbf{4}, \mathbf{4}-\mathbf{k})$ with $\mathbf{k}$=$0, 1, 2, 3, 4$. These numbers correspond to the numbers of bosonic physical fields, fermionic physical fields and bosonic auxiliary fields, respectively. $\mathcal{N}=8$, d=1 multiplets are denoted in the same way as $(\mathbf{k}, \mathbf{8}, \mathbf{8}-\mathbf{k})$, where $\mathbf{k}$=$0, 1, \ldots, 8$.

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The present paper makes one step in a new direction, initiating a study of deformed supersymmetric mechanics models associated with worldline realizations of the supergroup SU(2|2). This supergroup contains eight supercharges and so can be viewed as a deformation of flat $\mathcal{N}=8$, $d=1$ supersymmetry. $\mathcal{N}=8$ mechanics has appeared in many contexts (see, e.g. [28–41]). Our aim here is to construct and discuss SU(2|2) analogs for some of these models, employing the appropriate worldline superfield approach generalizing the SU(2|1) one. Our consideration relies essentially on [29, 31, 33, 34], where superfield methods were efficiently applied for flat $\mathcal{N}=8$, $d=1$ supersymmetry. In the contraction limit, when SU(2|2) goes over into its flat counterpart, the models of this paper specialize to those considered in [29, 31, 33, 34].

In [42], Berenstein, Maldacena and Nastase proposed an M-theory matrix model with 16 supercharges, which spurred investigations of massive super Yang-Mills mechanics (see, e.g. [43–48]). Since their matrix model has SU(2|4) supersymmetry [44, 45], SU(2|1) and SU(2|2) supersymmetric mechanics are expected to be relevant for massive matrix models with 4 and 8 supercharges, respectively.

The plan of the paper is as follows. In section 2 we describe coset superspaces of SU(2|2), to be used in the following sections for defining superfields carrying various irreducible SU(2|2) multiplets. Besides the standard real SU(2|2) superspace we introduce the chiral superspace, the harmonic superspace and the biharmonic superspace. We define the necessary elements of the corresponding superfield technique: covariant derivatives, transformation laws, and invariant integration measures. In sections 3, 4 and 5 we present the models associated with the off-shell SU(2|2) multiplets $(\mathbf{3},\mathbf{8},\mathbf{5})$, $(\mathbf{4},\mathbf{8},\mathbf{4})$ and $(\mathbf{5},\mathbf{8},\mathbf{3})$. We give both the superfield and component-field actions for all cases. Some of these actions reveal enhanced superconformal-type symmetries, some do not. Common features of most actions are an oscillator-type mass term for the fields and a trigonometric realization of the superconformal symmetries. As an example of a quantum model, in subsection 5.3 we discuss SU(2|2) quantum mechanics based on a free $(\mathbf{5},\mathbf{8},\mathbf{3})$ multiplet. In the concluding section 6 we mention links with other models and outline some directions for further study. We also adduce arguments why certain flat $\mathcal{N}=8, d=1$ multiplets (in particular the "root" multiplet $(\mathbf{8},\mathbf{8},\mathbf{0})$) seem not to admit a deformation to SU(2|2) mechanics. We transferred into three appendices some technical points, including the calculation of various harmonic integrals, the embedding of the superalgebra $su(2|2)$ into the $\mathcal{N}=8, d=1$ superconformal algebra $osp(4^*|4)$, the realization of the latter on the SU(2|2) multiplets considered, as well as a short account of the off-shell SU(2|1) multiplet $(\mathbf{3},\mathbf{4},\mathbf{1})$. The latter is an important constituent of our SU(2|2) multiplets but was not properly treated in previous papers on SU(2|1) mechanics.

2 Deformed $\mathcal{N}=8$, $d=1$ superspaces

In this section, we formulate a deformed real $\mathcal{N}=8$, $d=1$ superspace where worldline realizations of the supergroup SU(2|2) can be defined. Then we construct the corresponding chiral, analytic harmonic and analytic biharmonic SU(2|2) superspaces. In the following sections, these types of superspaces will be used for defining different types of superfields and for setting up SU(2|2) invariant actions of the latter, generalizing those constructed in [29, 31, 33, 34] in the presence of flat $\mathcal{N}=8$, $d=1$ supersymmetry.

2.1 Superalgebra

Our starting point is the superalgebra $su(2|2)$ with three central charges:

$$\begin{aligned}
&\left\{Q^{ia},S^{jb}\right\}=2im\left(\varepsilon^{ab}L^{ij}-\varepsilon^{ij}R^{ab}\right)+2\varepsilon^{ab}\varepsilon^{ij}C, && \\
&\left\{Q^{ia},Q^{jb}\right\}=2\,\varepsilon^{ij}\varepsilon^{ab}\left(H+C_1\right), && \left\{S^{ia},S^{jb}\right\}=2\,\varepsilon^{ij}\varepsilon^{ab}\left(H-C_1\right),\\
&\left[L^{ij},L^{kl}\right]=\varepsilon^{il}L^{kj}+\varepsilon^{jk}L^{il}, && \left[R^{ab},R^{cd}\right]=\varepsilon^{ad}R^{bc}+\varepsilon^{bc}R^{ad},\\
&\left[L^{ij},Q^{ka}\right]=\frac{1}{2}\left(\varepsilon^{ik}Q^{ja}+\varepsilon^{jk}Q^{ia}\right), && \left[R^{ab},Q^{kc}\right]=\frac{1}{2}\left(\varepsilon^{ac}Q^{kb}+\varepsilon^{bc}Q^{ka}\right),\\
&\left[L^{ij},S^{ka}\right]=\frac{1}{2}\left(\varepsilon^{ik}S^{ja}+\varepsilon^{jk}S^{ia}\right), && \left[R^{ab},S^{kc}\right]=\frac{1}{2}\left(\varepsilon^{ac}S^{kb}+\varepsilon^{bc}S^{ka}\right).
\end{aligned} \tag{2.1}$$

All other (anti)commutators are vanishing.

The superalgebra $su(2|2)$ contains in general three central charges C, C_1 and H. The generators $L^{ij}=L^{ji}$, $R^{ab}=R^{ba}$ form two mutually commuting $su(2)$ algebras, $su(2)_{\rm L}$ and $su(2)_{\rm R}$. The conjugation rules are as follows:[2]

$$\begin{aligned}
&(Q_{ia})^\dagger=Q^{ia},\quad (S_{ia})^\dagger=S^{ia},\quad (L_{ij})^\dagger=-L^{ij},\quad (R_{ab})^\dagger=-R^{ab},\\
&H^\dagger=H,\qquad (C)^\dagger=C,\qquad (C_1)^\dagger=C_1\,.
\end{aligned} \tag{2.2}$$

The mass dimension parameter m plays the same role as in the SU(2|1) case: by contraction $m\to 0$ the relations (2.1) are reduced to those of the flat $\mathcal{N}=8, d=1$ "Poincaré" superalgebra extended by central charges C, C_1 and possessing a restricted R-symmetry group $\mathrm{SO}(4)\sim\mathrm{SU}(2)_{\rm L}\times\mathrm{SU}(2)_{\rm R}$.[3] Correspondingly, (2.1) can be considered as a deformation of the flat $\mathcal{N}=8, d=1$ supersymmetry, with m as a deformation parameter.

To understand the origin of the central charge operators in (2.1), let us note that these relations in fact coincide with those defining a deformation of the flat $\mathcal{N}=(4,4)$, $d=2$ Poincaré superalgebra. Indeed, in the $m=0$ limit (2.1) can be identified with a sum of two independent $\mathcal{N}=4\,,d=2$ algebras in the left and right sectors of $d=2$ Minkowski space-time in the light-cone parametrization, with $H+C_1$ and $H-C_1$ being the mutually commuting translation operators along two light-cone directions. Moreover, one can realize the $d=2$ Lorentz group SO(1,1) as an additional automorphism group of (2.1) acting as real rescalings of the mutually (anti)commuting sets $(Q^{ia},H+C_1)$ and $(S^{ia},H-C_1)$ (with the weights $(1/2,1)$ and $(-1/2,-1)$, respectively). In such an interpretation, the generator C in (2.1) is SO(1,1) singlet and so it is the central charge from the $d=2$ perspective as well, while C_1 generates the translation along the spatial $d=2$ direction. The natural and

[2]The doublet indices are raised and lowered in the standard way by the ε symbols, e.g.,

$$Q_{ia}=\varepsilon_{ij}\varepsilon_{ab}Q^{jb},\qquad \varepsilon_{12}=-\,\varepsilon^{12}=1\,.$$

[3]If $C=C_1=0$, the R-symmetry group enhances to SO(8).

simplest reduction from d=2 to d=1 proceeds by eliminating altogether the dependence on the spatial coordinate, i.e. just by putting to zero the generator C_1. In what follows we will deal with such a limited $su(2|2)$ superalgebra, corresponding to the choice $C_1 = 0$ in (2.1). In principle, it is easy to construct the $\mathrm{SU}(2|2), d$=1 superfield formalism with $C_1 \neq 0$,[4] but in all examples considered below there is no need to activate this central charge. It is not the case for the "genuine" central charge C which defines an actual symmetry, e.g., in the models based on the multiplet $(\mathbf{4}, \mathbf{8}, \mathbf{4})$ (section 4).

One can rewrite the superalgebra (2.1) (hereafter with $C_1 = 0$) in a different form by defining the complex supercharges

$$\Pi^{ia} := \frac{1}{\sqrt{2}}\left(Q^{ia} - iS^{ia}\right), \qquad \bar{\Pi}^{ia} := \left(\Pi_{ia}\right)^{\dagger} = \frac{1}{\sqrt{2}}\left(Q^{ia} + iS^{ia}\right). \tag{2.3}$$

In the complex basis, the (anti)commutators of (2.1) become

$$\begin{aligned}
\left\{\Pi^{ia}, \bar{\Pi}^{jb}\right\} &= -\,2m\left(\varepsilon^{ab}L^{ij} - \varepsilon^{ij}R^{ab}\right) + 2\,\varepsilon^{ab}\varepsilon^{ij}H, && \\
\left\{\Pi^{ia}, \Pi^{jb}\right\} &= -\,2i\,\varepsilon^{ij}\varepsilon^{ab}C, & \left\{\bar{\Pi}^{ia}, \bar{\Pi}^{jb}\right\} &= 2i\,\varepsilon^{ij}\varepsilon^{ab}C, \\
\left[L^{ij}, L^{kl}\right] &= \varepsilon^{il}L^{kj} + \varepsilon^{jk}L^{il}, & \left[R^{ab}, R^{cd}\right] &= \varepsilon^{ad}R^{bc} + \varepsilon^{bc}R^{ad}, \\
\left[L^{ij}, \Pi^{ka}\right] &= \frac{1}{2}\left(\varepsilon^{ik}\Pi^{ja} + \varepsilon^{jk}\Pi^{ia}\right), & \left[R^{ab}, \Pi^{kc}\right] &= \frac{1}{2}\left(\varepsilon^{ac}\Pi^{kb} + \varepsilon^{bc}\Pi^{ka}\right), \\
\left[L^{ij}, \bar{\Pi}^{ka}\right] &= \frac{1}{2}\left(\varepsilon^{ik}\bar{\Pi}^{ja} + \varepsilon^{jk}\bar{\Pi}^{ia}\right), & \left[R^{ab}, \bar{\Pi}^{kc}\right] &= \frac{1}{2}\left(\varepsilon^{ac}\bar{\Pi}^{kb} + \varepsilon^{bc}\bar{\Pi}^{ka}\right).
\end{aligned} \tag{2.4}$$

The supergroup $\mathrm{SU}(2|2)$ contains a few $\mathrm{SU}(2|1)$ subgroups. One of them has the bosonic subgroup $\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{U}(1)_{\mathrm{R}}$ with $\mathrm{U}(1)_{\mathrm{R}} \subset \mathrm{SU}(2)_{\mathrm{R}}$, while another has the bosonic subgroup $\mathrm{SU}(2)_{\mathrm{R}} \times \mathrm{U}(1)_{\mathrm{L}}$ with $\mathrm{U}(1)_{\mathrm{L}} \subset \mathrm{SU}(2)_{\mathrm{L}}$. These supergroups are equivalent up to switching $\mathrm{SU}(2)_{\mathrm{R}} \leftrightarrow \mathrm{SU}(2)_{\mathrm{L}}$. In what follows, we will mainly deal with the first choice, where $\mathrm{SU}(2|1)$ generators [16] are singled out as[5]

$$\Pi^{i1} =: Q^{i}, \qquad \bar{\Pi}_{j1} =: \bar{Q}_{j}, \qquad L^{i}_{j} =: I^{i}_{j}, \qquad R^{12} =: F, \tag{2.5}$$

$$\{Q^{i}, \bar{Q}_{j}\} = 2\delta^{i}_{j}\left(H - mF\right) + 2mI^{i}_{j}, \quad \{Q^{i}, Q^{j}\} = 0, \qquad [F, Q^{i}] = \frac{1}{2}\,Q^{i}, \; [F, \bar{Q}_{i}] = -\frac{1}{2}\,\bar{Q}_{i}. \tag{2.6}$$

The second basic $su(2|1)$ subalgebra is formed by the generators Π^{1a}, $\bar{\Pi}_{1b}$, R^{ab}, L^{12}, H. Actually, the generators (2.5) form the *centrally extended* superalgebra $\hat{su}(2|1)$ with the central charge H, and the same is true for the second $\mathrm{SU}(2|1)$. The central charge H in (2.5) is a difference of external and internal $\mathrm{U}(1)$ generators in the extended superalgebra $su(2|1) \oplus u(1)_{\mathrm{ext}}$ [17]. If the generator $H - mR^{12}$ is chosen as the full internal $\mathrm{U}(1)$ generator of $su(2|1)$ (such a choice is admissible since H commutes with everything), then R^{12} decouples and becomes a generator of the external $\mathrm{U}(1)_{\mathrm{ext}}$ R-symmetry, such that $u(1)_{\mathrm{ext}} \subset su(2)_{\mathrm{R}}$.

[4] Non-zero C_1 could be generated within the Scherk-Schwarz type d=2 $\to$ d=1 reduction (see, e.g., [49]).

[5] The other pair of supercharges Π^{i2}, $\bar{\Pi}_{j2}$ also form an $su(2|1)$ superalgebra, with the same set of bosonic generators.

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2.2 Basic SU(2|2) supercoset, Cartan forms and covariant derivatives

We will be first interested in the realization of SU(2|2) supersymmetry in a real $\mathcal{N}=8$, d=1 superspace identified with the following supercoset of the supergroup with the superalgebra (2.1):

$$\frac{P\mathrm{SU}(2|2)\times\mathbb{R}^2}{\mathrm{SU}(2)_{\mathrm{L}}\times\mathrm{SU}(2)_{\mathrm{R}}\times\mathbb{R}^1}=\frac{\left\{Q^{ia},S^{jb},L^{ij},R^{ab},C,H\right\}}{\left\{L^{ij},R^{ab},C\right\}}\,. \tag{2.7}$$

Here, the supergroup $P\mathrm{SU}(2|2)$ is a corresponding supergroup SU(2|2) without central charges. Further, we will use the notation SU(2|2) as a supergroup with central extensions. An element of this supercoset is defined as

$$g=e^{\theta_{ia}Q^{ia}}e^{\hat{\theta}_{ia}S^{ia}}e^{itH},\qquad g^{\dagger}=g^{-1}, \tag{2.8}$$

and the supercoset parameters are treated as a set of superspace coordinates

$$\zeta=\left\{t,\theta^{ia},\hat{\theta}^{jb}\right\},\qquad\overline{(\theta_{ia})}=\theta^{ia},\quad\overline{(\hat{\theta}_{ia})}=\hat{\theta}^{ia}. \tag{2.9}$$

The central charge generator H is associated with a translation generator along $\mathbb{R}^2/\mathbb{R}^1\sim\mathbb{R}^1$ parametrized by the time coordinate t.

Before presenting the realization of SU(2|2) on these coordinates induced by the left shifts of the element (2.8), it will be convenient to calculate the left-covariant Cartan 1-forms defined by

$$g^{-1}dg=i\,\Delta t\,H+\Delta\theta_{ia}\,Q^{ia}+\Delta\hat{\theta}_{ia}\,S^{ia}+\Omega_{ij}\,L^{ij}+\Omega_{ab}\,R^{ab}+\Omega_{(C)}\,C. \tag{2.10}$$

The explicit expressions for these forms are

$$\begin{aligned}
&\Delta t=dt-i\theta^{ia}d\theta_{ia}-i\hat{\theta}^{ia}d\hat{\theta}_{ia}+\frac{4m}{3}\,\hat{\theta}^{jd}\hat{\theta}^{i}_{d}\hat{\theta}^{a}_{(i}d\theta_{j)a}\,,\\
&\Delta\theta^{ia}=d\theta^{ia},\qquad\qquad\Delta\hat{\theta}^{ia}=d\hat{\theta}^{ia}+2im\,\hat{\theta}^{ja}\hat{\theta}^{ib}d\theta_{jb}\,,\\
&\Omega_{ij}=2im\,\hat{\theta}^{a}_{(i}d\theta_{j)a}\,,\qquad\Omega_{ab}=2im\,\hat{\theta}_{k(a}d\theta^{k}_{b)}\,,\qquad\Omega_{(C)}=2\,\hat{\theta}_{ia}d\theta^{ia}.
\end{aligned} \tag{2.11}$$

The covariant derivatives of some superfield $\Phi^{A}\big(t,\theta^{ia},\hat{\theta}^{jb}\big)$ can be found from the general expression for its covariant differential

$$\begin{aligned}
\Delta\Phi^{A}&=d\Phi^{A}+\Big[\Omega_{ij}\,\tilde{L}^{ij}+\Omega_{ab}\,\tilde{R}^{ab}+\Omega_{(C)}\,\tilde{C}\Big]^{A}_{\;\;B}\,\Phi^{B}\\
&=\Big[\Delta t\,D_{(t)}+\Delta\theta_{ia}D^{ia}+\Delta\hat{\theta}_{ia}\nabla^{ia}\Big]\,\Phi^{A}.
\end{aligned} \tag{2.12}$$

Here, $\tilde{L}^{ij}$ and $\tilde{R}^{ab}$ are "matrix" parts of the full SU(2) generators (realized as well on Grassmann coordinates), which act on the external indices of covariant derivatives as

$$\tilde{L}^{ij}D^{ka}=-\,\frac{1}{2}\left(\varepsilon^{ik}D^{ja}+\varepsilon^{jk}D^{ia}\right),\quad\tilde{R}^{ab}D^{kc}=-\,\frac{1}{2}\left(\varepsilon^{ac}D^{kb}+\varepsilon^{bc}D^{ka}\right),\quad\text{etc}\,. \tag{2.13}$$

In the same way, they act on the external $\mathrm{SU}(2)_{\mathrm{L}}\times\mathrm{SU}(2)_{\mathrm{R}}$ indices of superfields. The rule of complex conjugation for these matrix parts is as follows

$$\big(\tilde{L}_{ij}\big)^{\dagger}=\tilde{L}^{ij},\quad\big(\tilde{R}_{ab}\big)^{\dagger}=\tilde{R}^{ab},\quad\big(\tilde{C}\,\big)^{\dagger}=-\,\tilde{C}. \tag{2.14}$$

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Explicitly, the covariant derivatives are given by the following expressions

$$
\begin{aligned}
D^{ia} &= \frac{\partial}{\partial\theta_{ia}} - i\left(\theta^{ia} - \frac{2im}{3}\,\hat\theta^{ib}\hat\theta^{ja}\hat\theta_{jb}\right)\partial_t - 2\,\hat\theta^{ia}\tilde C - 2im\,\hat\theta^{ib}\hat\theta^{ja}\frac{\partial}{\partial\hat\theta^{jb}} \\
&\quad - 2im\left[\hat\theta^a_j\,\tilde L^{ij} - \hat\theta^i_b\,\tilde R^{ab}\right], \\
\nabla^{ia} &= \frac{\partial}{\partial\hat\theta_{ia}} - i\hat\theta^{ia}\partial_t\,, \\
D_{(t)} &= \partial_t = \frac{\partial}{\partial t}\,.
\end{aligned}
\tag{2.15}
$$

They satisfy the anticommutation relations

$$
\begin{aligned}
&\left\{D^{ia},\nabla^{jb}\right\} = -\,2im\left(\varepsilon^{ab}\tilde L^{ij} - \varepsilon^{ij}\tilde R^{ab}\right) - 2\,\varepsilon^{ab}\varepsilon^{ij}\tilde C, \\
&\left\{D^{ia},D^{jb}\right\} = -\,2i\,\varepsilon^{ij}\varepsilon^{ab}\partial_t\,, \qquad \left\{\nabla^{ia},\nabla^{jb}\right\} = -\,2i\,\varepsilon^{ij}\varepsilon^{ab}\partial_t\,.
\end{aligned}
\tag{2.16}
$$

By rewriting the covariant derivative as

$$
\mathcal{D}^{ia} = \frac{1}{\sqrt2}\left(D^{ia} - i\nabla^{ia}\right), \qquad \bar{\mathcal{D}}^{ia} = \frac{1}{\sqrt2}\left(D^{ia} + i\nabla^{ia}\right),
\tag{2.17}
$$

we obtain that

$$
\left\{\mathcal{D}^{ia},\bar{\mathcal{D}}^{jb}\right\} = 2m\left(\varepsilon^{ab}\tilde L^{ij} - \varepsilon^{ij}\tilde R^{ab}\right) - 2i\,\varepsilon^{ab}\varepsilon^{ij}\partial_t\,,
\tag{2.18}
$$

$$
\left\{\mathcal{D}^{ia},\mathcal{D}^{jb}\right\} = 2i\,\varepsilon^{ij}\varepsilon^{ab}\tilde C, \qquad \left\{\bar{\mathcal{D}}^{ia},\bar{\mathcal{D}}^{jb}\right\} = -\,2i\,\varepsilon^{ij}\varepsilon^{ab}\tilde C.
\tag{2.19}
$$

2.3 Transformation properties

The transformation properties of the $\mathcal{N}{=}8$ superspace coordinates under the left shifts with the parameters ϵ^{ia} and $\hat\epsilon^{ia}$, as well as the induced stability subgroup infinitesimal transformations, can be found from the general formula

$$
g^{-1}\left(\epsilon_{ia}Q^{ia} + \hat\epsilon_{ia}S^{ia}\right)g = g^{-1}\delta g + \omega_{ij}\,L^{ij} + \omega_{ab}\,R^{ab} + \omega_{(C)}\,C.
\tag{2.20}
$$

The explicit calculations yield the following transformations:

$$
\begin{aligned}
&\delta\theta^{ia} = \epsilon^{ia} + 2im\,\theta^{ib}\theta^{ja}\hat\epsilon_{jb}\,, \qquad \delta\hat\theta^{ia} = \hat\epsilon^{ia} - 2im\left[\hat\theta^{j(b}\theta^{a)}_j\,\hat\epsilon^i_b + \hat\theta^{(j}_b\,\theta^{i)b}\hat\epsilon^a_j\right], \\
&\delta t = -\,i\hat\theta^{ia}\hat\epsilon_{ia} - i\theta^{ia}\epsilon_{ia} + \frac{2m}{3}\,\theta^{ib}\theta^{ja}\theta_{jb}\hat\epsilon_{ia}\,.
\end{aligned}
\tag{2.21}
$$

The induced elements in (2.20) are

$$
\omega_{ij} = -\,2im\,\theta^a_{(i}\,\hat\epsilon_{j)a}\,, \qquad \omega_{ab} = -\,2im\,\theta_{i(a}\,\hat\epsilon^i_{b)}\,, \qquad \omega_{(C)} = 2\,\theta^{ia}\hat\epsilon_{ia}\,.
\tag{2.22}
$$

It is straightforward to check that the coset-space Cartan forms undergo $\mathrm{SU}(2)_{\mathrm{L}}\times\mathrm{SU}(2)_{\mathrm{R}}$ induced transformations under the coordinate transformations (2.21):

$$
\delta\left(\Delta\theta^{ia}\right) = \omega^i_j\,\Delta\theta^{ja} + \omega^a_b\,\Delta\theta^{ib}, \qquad \delta\left(\Delta\hat\theta^{ia}\right) = \omega^i_j\,\Delta\hat\theta^{ja} + \omega^a_b\,\Delta\hat\theta^{ib}.
\tag{2.23}
$$

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Superfields are assumed to transform according to the general law

$$\delta\Phi^A = \left(\omega_{ij}\,\tilde{L}^{ij} + \omega_{ab}\,\tilde{R}^{ab} + \omega_{(C)}\,\tilde{C}\right)^A_{\ B}\Phi^B\,, \tag{2.24}$$

where an external index A of the superfield Φ^A specifies the $\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{SU}(2)_{\mathrm{R}}$ matrix representation by which this superfield is transformed (and that of $\tilde{C}$).

The SU(2|2) invariant $\mathcal{N}=8$, d=1 superspace integration measure is given by

$$d\zeta = dt\,d^4\theta\,d^4\hat{\theta}\,, \qquad \delta\,(d\zeta) = 0\,. \tag{2.25}$$

2.4 Chiral SU(2|2) superspace

We introduce the complex coordinates

$$\zeta_{\mathrm{c}} = \left\{t, \vartheta^{ia}, \bar{\vartheta}^{jb}\right\}, \qquad \overline{(\vartheta_{ia})} = \bar{\vartheta}^{ia}, \tag{2.26}$$

which are related to those defined in (2.9) as

$$\vartheta^{ia} = \frac{1}{\sqrt{2}}\left(\theta^{ia} + i\hat{\theta}^{ia} - \frac{2im}{3}\,\hat{\theta}^{ib}\hat{\theta}^{ja}\hat{\theta}_{jb}\right), \quad \bar{\vartheta}^{ia} = \frac{1}{\sqrt{2}}\left(\theta^{ia} - i\hat{\theta}^{ia} - \frac{2im}{3}\,\hat{\theta}^{ib}\hat{\theta}^{ja}\hat{\theta}_{jb}\right). \tag{2.27}$$

It will be also convenient to pass to the new infinitesimal parameters

$$\eta^{ia} = \frac{1}{\sqrt{2}}\left(\epsilon^{ia} + i\hat{\epsilon}^{ia}\right), \qquad \bar{\eta}^{ia} = \frac{1}{\sqrt{2}}\left(\epsilon^{ia} - i\hat{\epsilon}^{ia}\right), \tag{2.28}$$

in terms of which the transformation properties of the superspace coordinates in (2.26) are as follows

$$\begin{aligned}
\delta\vartheta^{ia} &= \eta^{ia} + 2m\,\vartheta^{ib}\vartheta^{ja}\left(\eta_{jb} - \bar{\eta}_{jb}\right), \qquad \delta\bar{\vartheta}^{ia} = \bar{\eta}^{ia} + 2m\,\bar{\vartheta}^{ib}\bar{\vartheta}^{ja}\left(\eta_{jb} - \bar{\eta}_{jb}\right),\\
\delta t &= -\,i\vartheta^{ia}\bar{\eta}_{ia} - i\bar{\vartheta}^{ia}\eta_{ia} + \frac{2m}{3}\,\theta^{ib}\theta^{ja}\theta_{jb}\hat{\epsilon}_{ia} + \frac{2m}{3}\,\hat{\theta}^{ib}\hat{\theta}^{ja}\hat{\theta}_{jb}\epsilon_{ia}\,.
\end{aligned} \tag{2.29}$$

The measure of integration over (2.26) can be checked to be invariant under these transformations:

$$d\zeta_c = dt\,d^4\vartheta\,d^4\bar{\vartheta}, \qquad \delta\,(d\zeta_c) = 0\,. \tag{2.30}$$

Specializing to the η^{i2}-transformations in (2.29) yields the odd transformations corresponding to the SU(2|1) subgroup (2.5).

The covariant derivatives (2.15) take the following form in the basis (2.26):

$$\begin{aligned}
\mathcal{D}^{ia} &= \partial^{ia} - i\bar{\vartheta}^{ia}\partial_t + i\left(\vartheta^{ia} - \bar{\vartheta}^{ia}\right)\tilde{C} - m\left(\vartheta^{ib} - \bar{\vartheta}^{ib}\right)\left(\vartheta^{ja} - \bar{\vartheta}^{ja}\right)\partial_{jb}\\
&\quad - m\left[\left(\vartheta^a_j - \bar{\vartheta}^a_j\right)\tilde{L}^{ij} - \left(\vartheta^i_b - \bar{\vartheta}^i_b\right)\tilde{R}^{ab}\right],\\
\bar{\mathcal{D}}^{ia} &= \bar{\partial}^{ia} - i\vartheta^{ia}\partial_t + i\left(\vartheta^{ia} - \bar{\vartheta}^{ia}\right)\tilde{C} + m\left(\vartheta^{ib} - \bar{\vartheta}^{ib}\right)\left(\vartheta^{ja} - \bar{\vartheta}^{ja}\right)\bar{\partial}_{jb}\\
&\quad - m\left[\left(\vartheta^a_j - \bar{\vartheta}^a_j\right)\tilde{L}^{ij} - \left(\vartheta^i_b - \bar{\vartheta}^i_b\right)\tilde{R}^{ab}\right].
\end{aligned} \tag{2.31}$$

Hereafter, we use the notation:

$$\partial^{ia} = \partial/\partial\vartheta_{ia}\,, \qquad \bar{\partial}^{ia} = \partial/\partial\bar{\vartheta}_{ia}\,. \tag{2.32}$$

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Now it is easy to show the existence of a left chiral subspace parametrized by the coordinates

$$\zeta_L = \{t_L, \vartheta^{ia}\}, \tag{2.33}$$

where

$$t_L = t - i\vartheta^{ia}\bar{\vartheta}_{ia} - im\left[\frac{1}{3}\,\bar{\vartheta}^{ib}\bar{\vartheta}^{ja}\bar{\vartheta}_{ia} - \frac{1}{2}\,\bar{\vartheta}^{ib}\vartheta^{ja}\bar{\vartheta}_{ia} - \frac{1}{2}\,\vartheta^{ib}\bar{\vartheta}^{ja}\bar{\vartheta}_{ia} + \vartheta^{ib}\vartheta^{ja}\bar{\vartheta}_{ia}\right]\vartheta_{jb}\,. \tag{2.34}$$

Indeed, the set (2.33) is closed under the SU(2|2) transformations

$$\delta\vartheta^{ia} = \eta^{ia} + 2m\,\vartheta^{ib}\vartheta^{ja}\left(\eta_{jb} - \bar{\eta}_{jb}\right), \qquad \delta t_L = -\,2i\vartheta^{ia}\bar{\eta}_{ia} + \frac{4im}{3}\,\vartheta^{ib}\vartheta^{ja}\vartheta_{jb}\bar{\eta}_{ia}\,. \tag{2.35}$$

Actually, the set (2.33) can be identified with the following complex coset superspace of SU(2|2):

$$\frac{\{\Pi^{ia}, \bar{\Pi}^{jb}, L^{ij}, R^{ab}, C, H\}}{\{\bar{\Pi}^{jb}, L^{ij}, R^{ab}, C\}}\,. \tag{2.36}$$

The invariant measure of integration over (2.33), $d\zeta_L$, is defined by

$$d\zeta_L = dt_L\, d^4\vartheta, \qquad \delta\left(d\zeta_L\right) = 0\,. \tag{2.37}$$

In the coordinates (2.33), the covariant derivatives (2.31) are written as

$$\begin{aligned}
\mathcal{D}^{ia} &= \partial^{ia} - 2i\left[\bar{\vartheta}^{ia} + m\left(\vartheta^{ic}\vartheta^{ka}\bar{\vartheta}_{kc} - \frac{1}{3}\,\bar{\vartheta}^{ic}\bar{\vartheta}^{ka}\bar{\vartheta}_{kc}\right)\right]\partial_{t_L} - m\left(\vartheta^{ib} - \bar{\vartheta}^{ib}\right)\left(\vartheta^{ja} - \bar{\vartheta}^{ja}\right)\partial_{jb}\\
&\quad + i\left(\vartheta^{ia} - \bar{\vartheta}^{ia}\right)\tilde{C} - m\left[\left(\vartheta^{a}_{j} - \bar{\vartheta}^{a}_{j}\right)\tilde{L}^{ij} - \left(\vartheta^{i}_{b} - \bar{\vartheta}^{i}_{b}\right)\tilde{R}^{ab}\right],\\
\bar{\mathcal{D}}^{ia} &= \bar{\partial}^{ia} + i\left(\vartheta^{ia} - \bar{\vartheta}^{ia}\right)\tilde{C} + m\left(\vartheta^{ib} - \bar{\vartheta}^{ib}\right)\left(\vartheta^{ja} - \bar{\vartheta}^{ja}\right)\bar{\partial}_{jb}\\
&\quad - m\left[\left(\vartheta^{a}_{j} - \bar{\vartheta}^{a}_{j}\right)\tilde{L}^{ij} - \left(\vartheta^{i}_{b} - \bar{\vartheta}^{i}_{b}\right)\tilde{R}^{ab}\right].
\end{aligned} \tag{2.38}$$

From the structure of the covariant derivative $\bar{\mathcal{D}}^{ia}$ we observe that the general covariantly chiral SU(2|2) superfield Φ^A,

$$\bar{\mathcal{D}}^{ia}\Phi^A = 0\,, \tag{2.39}$$

can be made explicitly chiral after the appropriate $\vartheta, \bar{\vartheta}$-dependent $\mathrm{SU(2)_L}$ and $\mathrm{SU(2)_R}$ rotation of Φ^A with respect to the external indices. For instance, if Φ^A has the $\mathrm{SU(2)_L \times SU(2)_R}$ matrix assignment $(\mathbf{1/2}, \mathbf{1/2})$, this additional redefinition is given by

$$\Phi^{ia}\left(t, \vartheta_{ia}, \bar{\vartheta}^{jb}\right) = \left[e^{-\frac{m}{2}\left(\vartheta_{jb} - \bar{\vartheta}_{jb}\right)\left(\vartheta^{b}_{k} - \bar{\vartheta}^{b}_{k}\right)\tilde{L}^{jk}}\right]^{i}_{l}\left[e^{\frac{m}{2}\left(\vartheta_{jb} - \bar{\vartheta}_{jb}\right)\left(\vartheta^{j}_{c} - \bar{\vartheta}^{j}_{c}\right)\tilde{R}^{bc}}\right]^{a}_{d}\Phi^{ld}_{L}\left(t_L, \vartheta_{ia}\right), \tag{2.40}$$

$$\left(\tilde{L}^{jk}\right)^{i}_{\ l} = \frac{1}{2}\left(\varepsilon^{ij}\delta^{k}_{l} + \varepsilon^{ik}\delta^{j}_{l}\right), \qquad \left(\tilde{R}^{bc}\right)^{a}_{\ d} = \frac{1}{2}\left(\varepsilon^{ab}\delta^{c}_{d} + \varepsilon^{ac}\delta^{b}_{d}\right). \tag{2.41}$$

On the other hand, it is not possible to eliminate $\tilde{C}$ from $\bar{\mathcal{D}}^{ia}$ in a similar way. In fact, $\tilde{C}$ should always be vanishing on chiral superfields, as follows from the anticommutation relations (2.19), which are just the integrability conditions for the chirality constraint (2.39) and its anti-chirality counterpart:

$$\bar{\mathcal{D}}^{ia}\Phi^A = 0 \quad \Rightarrow \quad \tilde{C}\Phi^A = 0\,. \tag{2.42}$$

2.5 Harmonic superspace

We perform a harmonization of the $\mathrm{SU}(2)_{\mathrm{L}}$ indices and define the analytic subspace

$$\zeta_{(A)} = \left\{ t_{(A)}, \theta^{+}_{a}, \hat{\theta}^{+}_{a}, w^{\pm}_{i} \right\}, \tag{2.43}$$

where

$$\begin{aligned} &\theta^{+a} = \theta^{ia} w^{+}_{i}, \qquad \hat{\theta}^{+a} = \hat{\theta}^{ia} w^{+}_{i} + im\, \hat{\theta}^{i}_{b} \hat{\theta}^{jb} \theta^{ka} w^{+}_{i} w^{+}_{j} w^{-}_{k}, \\ &t_{(A)} = t - i\theta^{ia} \theta^{k}_{a} w^{+}_{i} w^{-}_{k} - i\hat{\theta}^{ia} \hat{\theta}^{k}_{a} w^{+}_{i} w^{-}_{k}. \end{aligned} \tag{2.44}$$

It is closed under the following SU(2|2) transformations

$$\begin{aligned} &\delta\theta^{+}_{a} = \epsilon^{i}_{a} w^{+}_{i} - im\, \theta^{+}_{b} \theta^{+b} \hat{\epsilon}^{i}_{a} w^{-}_{i}, \qquad \delta\hat{\theta}^{+}_{a} = \hat{\epsilon}^{i}_{a} w^{+}_{i} - im \left(2\, \hat{\theta}^{+}_{b} \theta^{+b} \hat{\epsilon}^{i}_{a} - \hat{\theta}^{+}_{b} \hat{\theta}^{+b} \epsilon^{i}_{a} \right) w^{-}_{i}, \\ &\delta w^{+}_{i} = 2im\, \theta^{+}_{a} \hat{\epsilon}^{ja} w^{+}_{j} w^{-}_{i}, \qquad \delta w^{-}_{i} = 0, \quad \delta t_{(A)} = 2i \left(\theta^{+}_{a} \epsilon^{ia} + \hat{\theta}^{+}_{a} \hat{\epsilon}^{ia} \right) w^{-}_{i}. \end{aligned} \tag{2.45}$$

Note that the transformation properties of the harmonic variables $w^{\pm}_{i}$, as well as the precise relation between the "central basis" coordinates $(t, \theta^{ia}, \hat{\theta}^{ia})$ and the "analytic basis" coordinates $(t_{(A)}, \theta^{+}_{a}, \hat{\theta}^{+}_{a})$, are uniquely fixed just by requiring (2.43) to be closed under the SU(2|2) transformations.

For further calculations, it will be convenient to pass to another set of harmonic variables in the harmonic superspace (2.43),

$$u^{+i} = w^{+i} - im\, \theta^{+}_{a} \hat{\theta}^{+a} w^{-i}, \quad u^{-i} = w^{-i}. \tag{2.46}$$

With this choice, the realization of the fermionic SU(2|2) transformations in the analytic subspace is as follows,

$$\begin{aligned} &\delta u^{+i} = \Lambda^{++} u^{-i}, \qquad \delta u^{-i} = 0, \\ &\delta\theta^{+}_{a} = \epsilon^{+}_{a} + im \left(\hat{\theta}^{+}_{b} \theta^{+b} \epsilon^{-}_{a} - \theta^{+}_{b} \theta^{+b} \hat{\epsilon}^{-}_{a} \right), \\ &\delta\hat{\theta}^{+}_{a} = \hat{\epsilon}^{+}_{a} + im \left(\hat{\theta}^{+}_{b} \hat{\theta}^{+b} \epsilon^{-}_{a} - \hat{\theta}^{+}_{b} \theta^{+b} \hat{\epsilon}^{-}_{a} \right), \\ &\delta t_{(A)} = 2i \left(\theta^{+}_{a} \epsilon^{-a} + \hat{\theta}^{+}_{a} \hat{\epsilon}^{-a} \right), \end{aligned} \tag{2.47}$$

where

$$\begin{aligned} &\Lambda^{++} = im \left(\theta^{+}_{a} \hat{\epsilon}^{+a} - \hat{\theta}^{+}_{a} \epsilon^{+a} \right) - m^{2} \left(\theta^{+}_{a} \hat{\epsilon}^{-a} + \hat{\theta}^{+}_{a} \epsilon^{-a} \right) \hat{\theta}^{+}_{b} \theta^{+b}, \\ &\epsilon^{\pm a} = \epsilon^{ia} u^{\pm}_{i}, \qquad \hat{\epsilon}^{\pm a} = \hat{\epsilon}^{ia} u^{\pm}_{i}. \end{aligned} \tag{2.48}$$

The SU(2|2) covariant harmonic derivative $\mathcal{D}^{++}$ preserving analyticity is uniquely defined by requiring it to transform as

$$\delta\mathcal{D}^{++} = -\,\Lambda^{++} D^{0}, \quad D^{0} = u^{+i} \frac{\partial}{\partial u^{+i}} - u^{-i} \frac{\partial}{\partial u^{-i}} + \theta^{+a} \frac{\partial}{\partial \theta^{+a}} + \hat{\theta}^{+a} \frac{\partial}{\partial \hat{\theta}^{+a}}. \tag{2.49}$$

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It reads

$$\mathcal{D}^{++} = \partial^{++} + i\left(\theta^{+}_{a}\theta^{+a} + \hat{\theta}^{+}_{a}\hat{\theta}^{+a}\right)\partial_{A} + im\,\hat{\theta}^{+}_{b}\theta^{+b}\left(\theta^{+a}\partial_{+a} - \hat{\theta}^{+a}\hat{\partial}_{+a}\right) + \frac{m^{2}}{2}\left(\theta^{+}\right)^{4}\partial^{--}, \tag{2.50}$$

where

$$\partial^{\pm\pm} = u^{\pm i}\frac{\partial}{\partial u^{\mp i}}\,,\;\partial_{A} = \frac{\partial}{\partial t_{(A)}}\,,\;\partial_{+a} = \frac{\partial}{\partial \theta^{+a}}\,,\;\hat{\partial}_{+a} = \frac{\partial}{\partial \hat{\theta}^{+a}}\,,\quad \left(\theta^{+}\right)^{4} := \left(\theta^{+}_{b}\theta^{+b}\right)\left(\hat{\theta}^{+}_{a}\hat{\theta}^{+a}\right).$$

This harmonic derivative reveals some unusual properties to be used below:

$$\mathcal{D}^{++}u^{-i} = u^{+i},\qquad \mathcal{D}^{++}u^{+i} = \frac{m^{2}}{2}\left(\theta^{+}\right)^{4}u^{-i},\qquad \mathcal{D}^{++}\Lambda^{++} = \frac{m^{2}}{2}\,\delta\left(\theta^{+}\right)^{4}. \tag{2.51}$$

The analytic subspace integration measure

$$d\zeta^{(-4)}_{(A)} := dt_{(A)}\,du\,d^{2}\theta^{+}\,d^{2}\hat{\theta}^{+} \tag{2.52}$$

transforms as

$$\delta\left(d\zeta^{(-4)}_{(A)}\right) = 2\,d\zeta^{(-4)}_{(A)}\,\Lambda\,, \tag{2.53}$$

with

$$\Lambda := im\left(\hat{\theta}^{+}_{a}\epsilon^{-a} - \theta^{+}_{a}\hat{\epsilon}^{-a}\right). \tag{2.54}$$

It is easy to check that

$$\Lambda^{++} = -\,\mathcal{D}^{++}\Lambda\,. \tag{2.55}$$

2.6 Biharmonic superspace

One can extend the superspace (2.9) by biharmonic coordinates $w^{(\pm1)}_{i}$ and $v^{(\pm1)}_{a}$ associated with the subgroups $\mathrm{SU}(2)_{\mathrm{L}}$ and $\mathrm{SU}(2)_{\mathrm{R}}$, respectively. No such an option exists in the $\mathrm{SU}(2|1)$ case because of presence of only one SU(2) subgroup in $\mathrm{SU}(2|1)$. Like in the previous case, the transformation laws of the double set of harmonics and the precise proper change of the superspace coordinates can be found from the requirement of the existence of the invariant analytic subspace $\zeta_{(B)}$ in the full biharmonic superspace

$$\zeta_{(B)} = \left\{t_{(B)}, \theta^{(1,0)}_{a}, \hat{\theta}^{(0,1)}_{i}, w^{(\pm1)}_{i}, v^{(\pm1)}_{a}\right\}. \tag{2.56}$$

The relation between the coordinates (2.56) and the original coordinates (2.9) is given by the following substitutions,

$$\begin{aligned}
&\theta^{(1,0)}_{a} = \theta^{i}_{a}w^{(1)}_{i},\qquad \hat{\theta}^{(0,1)}_{i} = \left[\hat{\theta}^{a}_{i} - 2im\,\hat{\theta}^{b}_{i}\hat{\theta}^{ja}\theta_{jb} + 4m^{2}\hat{\theta}^{b}_{i}\hat{\theta}^{ja}\hat{\theta}^{kc}\theta_{kb}\theta_{jb}\right]v^{(1)}_{a},\\
&t_{(B)} = t - i\theta^{ia}\theta^{j}_{a}w^{(1)}_{i}w^{(-1)}_{j} - i\hat{\theta}^{ia}\hat{\theta}^{b}_{i}v^{(1)}_{a}v^{(-1)}_{b} + 2m\,\hat{\theta}^{ia}\hat{\theta}^{b}_{i}\hat{\theta}^{kc}\theta^{d}_{k}v^{(-1)}_{a}v^{(-1)}_{b}v^{(1)}_{c}v^{(1)}_{d}.
\end{aligned} \tag{2.57}$$

Now it is straightforward to explicitly find the relevant coordinate $\mathrm{SU}(2|2)$ transformations leaving closed the analytic coordinate set (2.56)

$$\begin{aligned}
&\delta\theta^{(1,0)}_{a} = \epsilon^{(1,0)}_{a} - im\,\theta^{(1,0)}_{b}\theta^{(1,0)\,b}\hat{\epsilon}^{(-1,0)}_{a}, && \delta\hat{\theta}^{(0,1)}_{i} = \hat{\epsilon}^{(0,1)}_{i} - im\,\hat{\theta}^{(0,1)}_{j}\hat{\theta}^{(0,1)\,j}\epsilon^{(0,-1)}_{i},\\
&\delta w^{(1)}_{i} = \Lambda^{(2,0)}\,w^{(-1)}_{i}, && \delta w^{(-1)}_{i} = 0\,,\\
&\delta v^{(1)}_{a} = \Lambda^{(0,2)}\,v^{(-1)}_{a}, && \delta v^{(-1)}_{a} = 0\,,\\
&\delta t_{(B)} = 2i\theta^{(1,0)}_{a}\epsilon^{(-1,0)\,a} + 2i\hat{\theta}^{(0,1)}_{i}\hat{\epsilon}^{(0,-1)\,i}.
\end{aligned} \tag{2.58}$$

Here

$$\epsilon^{(\pm1,0)\,a} = \epsilon^{ia} w_i^{(1)}, \quad \hat\epsilon^{(\pm1,0)\,a} = \hat\epsilon^{ia} w_i^{(1)}, \quad \epsilon^{(0,\pm1)\,i} = \epsilon^{ia} v_a^{(\pm1)}, \quad \hat\epsilon^{(0,\pm1)\,i} = \hat\epsilon^{ia} v_a^{(\pm1)}, \tag{2.59}$$

$$\Lambda^{(2,0)} = 2im\,\theta_a^{(1,0)} \hat\epsilon^{(1,0)\,a}, \qquad \Lambda^{(0,2)} = 2im\,\hat\theta_i^{(0,1)} \epsilon^{(0,1)\,i}. \tag{2.60}$$

The analytic subspace has an invariant integration measure,

$$d\zeta_{(B)}^{(-2,-2)} := dt_{(B)}\, dw\, dv\, d^2\theta^{(1,0)}\, d^2\hat\theta^{(0,1)}, \qquad \delta\left(d\zeta_{(B)}^{(-2,-2)}\right) = 0\,. \tag{2.61}$$

Now we can define the covariant harmonic derivatives

$$\begin{aligned} D^{(2,0)} &= \partial^{(2,0)} + i\theta_a^{(1,0)}\theta^{(1,0)\,a}\,\partial_{(B)}\,, \qquad D_w^0 = \partial_w^0 + \theta^{(1,0)\,a}\frac{\partial}{\partial\theta^{(1,0)\,a}}\,,\\ D^{(0,2)} &= \partial^{(0,2)} + i\hat\theta_i^{(0,1)}\hat\theta^{(0,1)\,i}\,\partial_{(B)}\,, \qquad D_v^0 = \partial_v^0 + \hat\theta^{(0,1)\,i}\frac{\partial}{\partial\hat\theta^{(0,1)\,i}}\,, \end{aligned} \tag{2.62}$$

where

$$\begin{aligned} \partial^{(\pm2,0)} &= w_i^{(\pm1)}\frac{\partial}{\partial w_i^{(\mp1)}}\,, \qquad \partial_w^0 = w_i^{(1)}\frac{\partial}{\partial w_i^{(1)}} - w_i^{(-1)}\frac{\partial}{\partial w_i^{(-1)}}\,, \qquad \partial_{(B)} = \partial/\partial t_{(B)}\,,\\ \partial^{(0,\pm2)} &= v_a^{(\pm1)}\frac{\partial}{\partial v_a^{(\mp1)}}\,, \qquad \partial_v^0 = v_a^{(1)}\frac{\partial}{\partial v_a^{(1)}} - v_a^{(-1)}\frac{\partial}{\partial v_a^{(-1)}}\,. \end{aligned} \tag{2.63}$$

They possess the standard transformation laws

$$\delta D^{(2,0)} = -\,\Lambda^{(2,0)}\,D_w^0\,, \quad \delta D_w^0 = 0\,, \qquad \delta D^{(0,2)} = -\,\Lambda^{(0,2)}\,D_v^0\,, \quad \delta D_v^0 = 0\,. \tag{2.64}$$

One can check that

$$\Lambda^{(2,0)} = D^{(2,0)}\Lambda^{(0,0)}, \qquad \Lambda^{(0,2)} = D^{(0,2)}\Lambda^{(0,0)}, \tag{2.65}$$

$$\Lambda^{(0,0)} = 2im\left[\epsilon_i^{(0,-1)}\hat\theta^{(0,1)\,i} + \hat\epsilon_a^{(-1,0)}\theta^{(1,0)\,a}\right]. \tag{2.66}$$

3 The multiplet $(\mathbf{3}, \mathbf{8}, \mathbf{5})$

3.1 Kinematics

On one hand, the multiplet $(\mathbf{3}, \mathbf{8}, \mathbf{5})$ can be described by a superfield V^{ij} satisfying the constraints

$$D_a^{(i}V^{jk)} = 0\,, \qquad \nabla_a^{(i}V^{jk)} = 0\,, \qquad \tilde C\,V^{ij} = 0\,. \tag{3.1}$$

According to (2.24), the "passive" transformation law of V^{ij} is

$$\delta V^{ij} = \omega_k^i V^{kj} + \omega_k^j V^{ik}. \tag{3.2}$$

On the other hand, one can define the harmonic superfield

$$V^{++}\left(\zeta_{(A)}\right) = V^{ij}w_i^+w_j^+\left[1 + 2im\left(\hat\theta^{ka}w_k^+ + im\,\hat\theta_b^k\hat\theta^{mb}\theta^{na}w_k^+w_m^+w_n^-\right)\theta_a^l w_l^-\right], \tag{3.3}$$

which lives on the analytic harmonic superspace (2.43). The Grassmann analyticity conditions for V^{++} amount just to the constraints (3.1) for V^{ij} in the original central basis.

After passing to the new harmonic variables (2.46), the transformation of V^{++} can be written through the parameter Λ defined in (2.54) as

$$\delta V^{++} = -\,2\Lambda V^{++}. \tag{3.4}$$

The analytic superfield V^{++} satisfies the harmonic condition

$$\mathcal{D}^{++}V^{++} = 0\,, \tag{3.5}$$

which can be proved using (3.3) expressed in terms of the harmonic $u_i^{\pm}$, as well as the explicit expression for $\mathcal{D}^{++}$, eq. (2.50). In fact at this step one can forget about the relation (3.3) and deal with the real analytic harmonic superfield V^{++} ($\widetilde{V^{++}} = V^{++}$) subjected to (3.5). The harmonic constraint implies the following component structure of V^{++}:

$$\begin{aligned}
V^{++} = {}& v^{ij}u_i^+u_j^+ + \theta^{+a}\xi_a^i u_i^+ + \hat\theta^{+a}\hat\xi_a^i u_i^+ + \left(\theta_a^+\theta^{+a} - \hat\theta_a^+\hat\theta^{+a}\right)A_0 - i\dot v^{ij}u_i^+u_j^-\left(\theta_a^+\theta^{+a} + \hat\theta_a^+\hat\theta^{+a}\right)\\
&- 2\theta_a^+\hat\theta^{+a}C_0 + \theta^{+(a}\hat\theta^{+b)}C_{ab} - i\theta^{+a}\hat\theta_b^+\hat\theta^{+b}\left(\dot\xi_a^i + \frac{m}{2}\,\hat\xi_a^i\right)u_i^- - i\hat\theta^{+a}\theta_b^+\theta^{+b}\left(\dot{\hat\xi}{}_a^i - \frac{m}{2}\,\xi_a^i\right)u_i^-\\
&- \left(\theta^+\right)^4\left(\ddot v^{ij} + \frac{m^2}{2}\,v^{ij}\right)u_i^-u_j^- - \mu\left(\theta_a^+\theta^{+a} + \hat\theta_a^+\hat\theta^{+a}\right),
\end{aligned} \tag{3.6}$$

where the fields satisfy the reality conditions:

$$\begin{aligned}
&\overline{(v_{ij})} = v^{ij}, && v^{ij} = v^{ji}, && \overline{(\xi_{ia})} = -\,\xi^{ia}, && \overline{(\hat\xi_{ia})} = -\,\hat\xi^{ia},\\
&\overline{(C_{ab})} = -\,C_{ab}\,, && C_{ab} = C_{ba}\,, && \overline{(C_0)} = -\,C_0\,, && \overline{(A_0)} = -\,A_0\,.
\end{aligned} \tag{3.7}$$

Thus, we are left with three physical bosonic fields $v^{ij}(t)$, eight fermionic fields $\xi_a^i(t)$, $\hat\xi_a^i(t)$ and five bosonic auxiliary fields $A_0(t)$, $C_0(t)$, $C_{ab}(t)$, i.e., just with the $(\mathbf{3},\mathbf{8},\mathbf{5})$ content. A new constant μ, $[\mu] = 1$, came out in the course of solving (3.5). It survives in the flat limit $m = 0$.

We also present here the transformation properties of the component fields,

$$\begin{aligned}
\delta v^{ij} &= \epsilon_a^{(i}\xi^{j)a} + \hat\epsilon_a^{(i}\hat\xi^{j)a}\,,\\
\delta\xi_a^i &= 2i\left(\epsilon_{ja}\dot v^{ij} - m\,\hat\epsilon_{ja}v^{ij}\right) + 2\,\epsilon_a^i\left(A_0 - \mu\right) - 2\,\hat\epsilon_a^i C_0 - \hat\epsilon^{ib}C_{ab}\,,\\
\delta\hat\xi_a^i &= 2i\left(\hat\epsilon_{ja}\dot v^{ij} + m\,\epsilon_{ja}v^{ij}\right) - 2\,\hat\epsilon_a^i\left(A_0 + \mu\right) - 2\,\epsilon_a^i C_0 + \epsilon^{ib}C_{ba}\,,\\
\delta A_0 &= \frac{i}{2}\left(\hat\epsilon_{ia}\dot{\hat\xi}{}^{ia} - \epsilon_{ia}\dot\xi^{ia}\right),\\
\delta C_0 &= \frac{i}{2}\left(\epsilon_{ia}\dot{\hat\xi}{}^{ia} + \hat\epsilon_{ia}\dot\xi^{ia}\right),\\
\delta C_{ab} &= 2i\left[\hat\epsilon_{i(b}\dot\xi_{a)}^i - \epsilon_{i(a}\dot{\hat\xi}{}_{b)}^i\right] + 2im\left[\epsilon_{i(a}\xi_{b)}^i + \hat\epsilon_{i(a}\hat\xi_{b)}^i\right].
\end{aligned} \tag{3.8}$$

Note that the SU(2|2) covariant constraint (3.5) and the transformation law (3.4) can be generalized to an arbitrary analytic superfield q^{+n} of the harmonic U(1) charge n:

$$\mathcal{D}^{++}q^{(+n)} = 0\,, \quad \delta q^{(+n)} = -\,n\,\Lambda\,q^{(+n)}. \tag{3.9}$$

This is similar to the analogous phenomenon observed in the flat $\mathcal{N}=4, d=1$ harmonic superspace [20]. For even n one can impose the reality condition on $q^{(+n)}$. The difference from the $\mathcal{N}=4, d=1$ case is that for $n=1$ the constraint in (3.9) implies the equations of motion for the physical fields and is similar in this respect to the harmonic equation of motion for the analytic hypermultiplet superfield in $\mathcal{N}=2, 4D$ case [50, 51]. For $n=2$, this constraint remains purely kinematic and defines the $d=1$ analog of the $\mathcal{N}=2, d=4$ tensor multiplet, with the constant μ appearing as a solution of the $d=1$ reduction of the well-known "notoph" condition $\partial^\mu A_\mu = 0$ in $4D$. All these features are retained in the flat limit $m=0$.

3.2 Invariant action

Since for $n=2$ the $m=0$ version of (3.9) defines an off-shell $(\mathbf{3}, \mathbf{8}, \mathbf{5})$ multiplet, it is expected that the analytic real superfield $V^{++} := q^{(+2)}$ describes the off-shell multiplet $(\mathbf{3}, \mathbf{8}, \mathbf{5})$ of SU$(2|2)$ supersymmetry as a deformation of the corresponding flat off-shell supermultiplet.

Confronting the transformation law of V^{++} (3.4) with the transformation of the analytic measure (2.53), one concludes that it is impossible to construct any invariant Lagrangian out of V^{++} (even the free one), with harmonic U(1) charge +4 (needed to cancel the negative charge -4 of the measure). To evade this difficulty, we will proceed by analogy with the construction of the superconformal actions in [20, 52].

The procedure is as follows. We introduce an auxiliary constant triplet c^{ij}. Its harmonic projections $c^{+-} = c^{ij}u^+_i u^-_j$ and $c^{\pm\pm} = c^{ij}u^\pm_i u^\pm_j$ satisfy the relation

$$c^{++}c^{--} - \left(c^{+-}\right)^2 = \frac{1}{2}\, c^{ij}c_{ij} \tag{3.10}$$

that follows from the completeness relation for the harmonics. Without loss of generality, we choose $c^{ij}c_{ij} = 1$. Next, we define the "shifted" superfield $\hat{V}^{++}$ as

$$V^{++} = \hat{V}^{++} + \tilde{c}^{++}, \qquad \tilde{c}^{++} = c^{++} - \frac{m^2}{2}\left(\theta^+\right)^4 c^{--}. \tag{3.11}$$

The triplet $\tilde{c}^{++}$ satisfies the condition

$$\mathcal{D}^{++}\tilde{c}^{++} = 0\,, \tag{3.12}$$

and so

$$\mathcal{D}^{++}\hat{V}^{++} = 0\,. \tag{3.13}$$

The appearance of an additional term in $\tilde{c}^{++}$ is related to the properties (2.51) of $\mathcal{D}^{++}$. Note also the useful relation

$$\mathcal{D}^{++}c^{+-} = c^{++} + \frac{m^2}{2}\left(\theta^+\right)^4 c^{--} = \tilde{c}^{++} + m^2\left(\theta^+\right)^4 c^{--}. \tag{3.14}$$

The component structure of the shifted analytic superfield $\hat{V}^{++}$ related to V^{++} by (3.11) is obtained from (3.6) just via the substitution $v^{ij} \to \hat{v}^{ij}$, where $\hat{v}^{ij} = v^{ij} - c^{ij}$.

The newly defined quantities are transformed as

$$
\begin{aligned}
\delta\tilde{c}^{++} &= 4\Lambda^{++}c^{+-} - \mathcal{D}^{++}\left(\Lambda^{++}c^{--}\right),\\
\delta\hat{V}^{++} &= -\,2\Lambda\left(\hat{V}^{++}+\tilde{c}^{++}\right) - 4\Lambda^{++}c^{+-} + \mathcal{D}^{++}\left(\Lambda^{++}c^{--}\right),
\end{aligned}
\tag{3.15}
$$

where $\Lambda^{++} = -\mathcal{D}^{++}\Lambda$ (recall eq. (2.55)). Using these relations, one can construct invariant actions (see appendix A.1) with the superfield Lagrangian

$$
L^{(+4)} = \frac{2\,(\hat{V}^{++})^2}{\left(1+\sqrt{1+2\,c^{--}\hat{V}^{++}}\right)^2} - m^2\,(\theta^+)^4\left(\frac{c^{--}\hat{V}^{++}}{\sqrt{1+2\,c^{--}\hat{V}^{++}}} + \frac{c^{--}\hat{V}^{++}}{1+\sqrt{1+2\,c^{--}\hat{V}^{++}}}\right).
\tag{3.16}
$$

To find the component form of the action

$$
S_{(\mathbf{3},\mathbf{8},\mathbf{5})} = \int d\zeta^{(-4)}_{(A)}\,L^{(+4)} = \int dt\,\mathcal{L}_{(\mathbf{3},\mathbf{8},\mathbf{5})}\,,
\tag{3.17}
$$

we use the normalization

$$
\int d\zeta^{(-4)}_{(A)}(\theta^+)^4 = \int dt\,du\,,\qquad d\zeta^{(-4)}_{(A)} = \frac{1}{16}\,dt\,du\left(D^-_a D^{-a}\right)\left(\nabla^-_a\nabla^{-a}\right).
\tag{3.18}
$$

The main technical problem is to do the relevant harmonic integrals. This can be accomplished using the formulas listed in appendix A.2. The component Lagrangian finally reads

$$
\begin{aligned}
\mathcal{L}_{(\mathbf{3},\mathbf{8},\mathbf{5})} = \frac{1}{2|v|}\Bigg[& \dot{v}_{ij}\dot{v}^{ij} + \frac{i}{2}\left(\dot{\xi}^{ia}\xi_{ia} + \dot{\hat{\xi}}^{ia}\hat{\xi}_{ia}\right) - \frac{i}{2|v|^2}\left(\xi^{(i}_a\xi^{j)a} + \hat{\xi}^{(i}_a\hat{\xi}^{j)a}\right)v_{ik}\dot{v}^k_j\\
&- \frac{v^{ij}}{2|v|^2}\left(\xi^a_i\hat{\xi}^b_j\,C_{ab} - 2\,\xi_{ia}\hat{\xi}^a_j\,C_0 + \xi_{ia}\xi^a_j\,(A_0+\mu) - \hat{\xi}_{ia}\hat{\xi}^a_j\,(A_0-\mu)\right) - \frac{1}{4}\,C^{ab}C_{ab}\\
&- 2\,(A_0+\mu)\,(A_0-\mu) - 2\,(C_0)^2 + \frac{3v_{(ij}v_{kl)}}{8|v|^4}\,\xi^a_i\xi_{ja}\,\hat{\xi}^b_k\hat{\xi}_{lb} + \frac{i}{2}\,m\,\hat{\xi}^{ia}\xi_{ia} - m^2 v_{ij}v^{ij}\Bigg]\\
&- \frac{i\mu\,\dot{v}^{ij}\left(c^k_i v_{jk} + c^k_j v_{ik}\right)}{|v|\left(|v| + c_{ij}v^{ij}\right)}\,.
\end{aligned}
\tag{3.19}
$$

Here, $|v| := \sqrt{v_{ij}v^{ij}}$. The expression within the square brackets can basically be obtained by a dimensional reduction $d{=}4 \to d{=}1$ from the $d{=}4$ Lagrangian of [52]. The new terms are those $\sim \mu$ (they survive in the $m = 0$ limit), the fermionic "mass" mixed term $\sim m$ and the bosonic potential term $\sim m^2$. The last term $\sim \mu$ is a special WZ term for v^{ik} known as a Lorentz-force type coupling to Dirac magnetic monopole [20, 29].

3.3 Duality transformations

In [52], duality transformations of the tensor multiplet was shown to lead to the free hypermultiplet action. Here, we define in the same way duality transformations for the $d{=}1$ multiplet $(\mathbf{3}, \mathbf{8}, \mathbf{5})$.

We can rewrite the action (3.17) as

$$S_{(\mathbf{3},\mathbf{8},\mathbf{5})} = \frac{1}{2}\int d\zeta^{(-4)}_{(A)}\left(f^{++}\right)^2, \tag{3.20}$$

where f^{++} is an analytic superfield related to V^{++} and $\hat{V}^{++}$ by

$$\begin{aligned}
f^{++} &= \frac{2\hat{V}^{++}}{1+\sqrt{1+2\,c^{--}\hat{V}^{++}}} - m^2c^{--}\left(\theta^{+}\right)^4\left(1+\frac{1}{2\sqrt{1+2\,c^{--}\hat{V}^{++}}}\right) \quad \Rightarrow \\
V^{++} &= f^{++}\left(1+\frac{1}{2}\,c^{--}f^{++}\right) + c^{++} + m^2c^{--}\left(\theta^{+}\right)^4\left(1+c^{--}f^{++}\right), \\
\hat{V}^{++} &= f^{++}\left(1+\frac{1}{2}\,c^{--}f^{++}\right) + m^2c^{--}\left(\theta^{+}\right)^4\left(\frac{3}{2}+c^{--}f^{++}\right).
\end{aligned} \tag{3.21}$$

In view of this one-to-one correspondence, the harmonic constraint (3.13) implies a nonlinear constraint on the superfield f^{++}. The transformations of f^{++} can be found from (3.21)

$$\delta f^{++} = \frac{2}{1+c^{--}f^{++}}\left[-\Lambda f^{++}\left(1+\frac{1}{2}\,c^{--}f^{++}\right) - \Lambda c^{++} - \Lambda^{++}c^{+-}\right] - 2c^{--}\mathcal{D}^{++}\Lambda^{++}. \tag{3.22}$$

Next, we add to the action (3.20) an additional term with the Lagrange multiplier ω,

$$S^{\text{dual}} = \int d\zeta^{(-4)}_{(A)}\left[\frac{1}{2}\left(f^{++}\right)^2 + \omega\,\mathcal{D}^{++}V^{++}\left(f^{++},u^{\pm}_i\right)\right], \tag{3.23}$$

and thereby get rid of the condition (3.13), ending up with two independent analytic superfields, ω and f^{++}. The requirement of invariance of this action implies ω to transform as

$$\delta\omega = -\,\frac{2\left(c^{+-}\Lambda + c^{--}\Lambda^{++}\right)}{1+c^{--}f^{++}}. \tag{3.24}$$

Integrating by parts the last term in (3.23), we obtain

$$S^{\text{dual}} = \int d\zeta^{(-4)}_{(A)}\left[\frac{1}{2}\left(f^{++}\right)^2 - V^{++}\left(f^{++},u^{\pm}_i\right)\mathcal{D}^{++}\omega\right]. \tag{3.25}$$

By analogy to [52], we can cast the Lagrangian (3.25) in the form of the free action

$$S^{\text{dual}} = -\frac{1}{2}\int d\zeta^{(-4)}_{(A)}\, q^{+i}\,\mathcal{D}^{++}q^{+}_i, \tag{3.26}$$

where

$$\begin{aligned}
q^{+i} :=& \left[f^{++}u^{-i} - 2c^{ij}u^{+}_j + m^2c^{--}u^{-i}\left(\theta^{+}\right)^4\right]\cos\left(\omega/\sqrt{2}\right) \\
&- \sqrt{2}\left[c^{ij}f^{++}u^{-}_j + u^{+i} + m^2c^{ij}u^{-}_j c^{--}\left(\theta^{+}\right)^4\right]\sin\left(\omega/\sqrt{2}\right).
\end{aligned} \tag{3.27}$$

From this relation, one can establish that

$$V^{++} = \frac{1}{2}\,c_{ij}\,q^{+i}q^{+j}. \tag{3.28}$$

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Taking into account (3.22) and (3.24), one can find the superfield transformation of the newly introduced analytic superfield q^{+i}

$$\delta q^{+i} = -\Lambda\, q^{+i}\,. \tag{3.29}$$

By making use of the transformation properties (2.49), (2.53) and (3.29), it is easy to check the SU(2|2) invariance of (3.26).

We observe that the external doublet index i of q^{+i} is inert with respect to the whole SU(2|2) , including the SO(4) transformations. So it a sort of Pauli-Gürsey index and it is convenient to replace it by another letter, e.g. as

$$q^{+i} \longrightarrow q^{+A}. \tag{3.30}$$

The action (3.26) respects an additional invariance under an extra $\mathrm{SU}(2)_{\rm PG}$ rotating the doublet index A.

The superfield q^{+A} has the following θ-expansion:

$$\begin{aligned} q^{+A}\left(\zeta_{(A)}\right) = x^{+A} + \theta^{+a}\lambda^{A}_{a} + \hat{\theta}^{+a}\hat{\lambda}^{A}_{a} + \left(\theta^{+}_{a}\theta^{+a} + \hat{\theta}^{+}_{a}\hat{\theta}^{+a}\right)B^{-A} + \left(\theta^{+}_{a}\theta^{+a} - \hat{\theta}^{+}_{a}\hat{\theta}^{+a}\right)C^{-A} \\ + \theta^{+}_{a}\hat{\theta}^{+a}D^{-A} + \hat{\theta}^{+}_{b}\hat{\theta}^{+b}\theta^{+a}\psi^{(-2)A}_{a} + \theta^{+}_{b}\theta^{+b}\hat{\theta}^{+a}\hat{\psi}^{(-2)A}_{a} + \left(\theta^{+}\right)^{4}A^{(-3)A}. \end{aligned} \tag{3.31}$$

Here, all fields are defined on the extended bosonic space $\left(t_{(A)}, u^{\pm i}\right)$, i.e., their harmonic expansions produce infinite towers of fields [51]. Eliminating auxiliary fields by the relevant part of the equation of motion for (3.26),

$$\mathcal{D}^{++}q^{+A} = 0\,, \tag{3.32}$$

we obtain the on-shell superfield q^{+A} containing a finite set of physical fields,

$$q^{+A} = x^{iA}u^{+}_{i} + \theta^{+a}\lambda^{A}_{a} + \hat{\theta}^{+a}\hat{\lambda}^{A}_{a} - i\left(\theta^{+}_{a}\theta^{+a} + \hat{\theta}^{+}_{a}\hat{\theta}^{+a}\right)\dot{x}^{iA}u^{-}_{i}\,, \tag{3.33}$$

$$\overline{\left(x_{iA}\right)} = x^{iA}, \qquad \overline{\left(\lambda_{Aa}\right)} = -\,\lambda^{Aa}, \qquad \overline{\left(\hat{\lambda}_{Aa}\right)} = -\,\hat{\lambda}^{Aa}. \tag{3.34}$$

The constraint (3.32) puts the residual component fields on-shell:

$$\ddot{x}^{iA} + \frac{m^2}{4}\,x^{iA} = 0\,, \qquad \dot{\lambda}^{Aa} + \frac{m}{2}\,\hat{\lambda}^{Aa} = 0\,, \qquad \dot{\hat{\lambda}}^{Aa} - \frac{m}{2}\,\lambda^{Aa} = 0\,. \tag{3.35}$$

They can be re-derived from the on-shell component Lagrangian

$$\mathcal{L}^{\rm dual} = \frac{1}{2}\,\dot{x}^{iA}\dot{x}_{iA} + \frac{i}{4}\left(\dot{\lambda}^{Aa}\lambda_{Aa} + \dot{\hat{\lambda}}^{Aa}\hat{\lambda}_{Aa}\right) + \frac{i}{4}\,m\,\hat{\lambda}^{Aa}\lambda_{Aa} - \frac{m^2}{8}\,x^{iA}x_{iA}\,, \tag{3.36}$$

which is invariant under the transformations

$$\delta x^{iA} = \epsilon^{i}_{a}\lambda^{Aa} + \hat{\epsilon}^{i}_{a}\hat{\lambda}^{Aa}, \quad \delta\lambda^{Aa} = 2i\epsilon^{a}_{i}\dot{x}^{iA} - im\,\hat{\epsilon}^{a}_{i}x^{iA}, \quad \delta\hat{\lambda}^{Aa} = 2i\hat{\epsilon}^{a}_{i}\dot{x}^{iA} + im\,\epsilon^{a}_{i}x^{iA}. \tag{3.37}$$

These have an on-shell SU(2|2) closure. Note that the translational symmetry $x^{iA} \rightarrow x^{iA} + a^{iA}$ and its fermionic counterpart which are present in the undeformed $m = 0$ version of (3.36) are broken by the oscillator terms.

4 The multiplet $(\mathbf{4},\mathbf{8},\mathbf{4})$

4.1 Kinematics

The multiplet $(\mathbf{4},\mathbf{8},\mathbf{4})$ can be described by the superfield q^{ia}, with $\overline{(q_{ia})} = q^{ia}$. The proper constraints are imposed as

$$D^{(kb}q^{i)a} = 0\,, \qquad \nabla^{k(b}q^{ia)} = 0\,. \tag{4.1}$$

The SU(2|2) covariance of these constraints requires that

$$\tilde{C}q^{ia} = -\,im\,q^{ia}. \tag{4.2}$$

According to (2.24), the odd transformations of q^i can be written as

$$\delta q^{ia} = 2im\left(\hat{\epsilon}_{jb}\theta^{ib}q^{ja} + \hat{\epsilon}^{ja}\theta_{jb}q^{ib}\right). \tag{4.3}$$

Now one can define the analytic biharmonic superfield

$$q^{(1,1)}\left(\zeta_{(B)}\right) = q^{ia}w_i^{(1)}v_a^{(1)} + 2im\,\hat{\theta}_k^{(0,1)}\theta_b^k q^{ib}w_i^{(1)}, \tag{4.4}$$

living on the analytic subspace (2.56) and transforming as

$$\delta q^{(1,1)} = \Lambda^{(0,0)}q^{(1,1)}, \tag{4.5}$$

where $\Lambda^{(0,0)}$ was defined in (2.66). While the Grassmann constraints (4.1) are automatically satisfied for $q^{(1,1)}$ in the analytic basis, the restricted harmonic dependence in (4.4) amounts to the harmonic constraints

$$D^{(2,0)}q^{(1,1)} = D^{(0,2)}q^{(1,1)} = 0\,. \tag{4.6}$$

Taking into account the transformation laws of $D^{(2,0)}$ and $D^{(0,2)}$, eqs. (2.64), as well as the definitions (2.65) and (2.66), it is easy to establish the SU(2|2) covariance of (4.6).

The solution of (4.6) is given by the undeformed superfield

$$\begin{aligned} q^{(1,1)} = {}& f^{ia}w_i^{(1)}v_a^{(1)} + \theta^{(1,0)\,a}v_b^{(1)}\chi_a{}^b + \hat{\theta}^{(0,1)\,i}w_j^{(1)}\hat{\chi}_i{}^j + \theta^{(1,0)\,a}\hat{\theta}^{(0,1)\,i}F_{ia} \\ & - i\left(\theta_b^{(1,0)}\theta^{(1,0)\,b}v_a^{(1)}w_i^{(-1)} + \hat{\theta}_j^{(0,1)}\hat{\theta}^{(0,1)\,j}v_a^{(-1)}w_i^{(1)}\right)\dot{f}^{ia} \\ & - i\hat{\theta}_j^{(0,1)}\hat{\theta}^{(0,1)\,j}\theta^{(1,0)\,a}v_b^{(-1)}\dot{\chi}_a{}^b - i\theta_b^{(1,0)}\theta^{(1,0)\,b}\hat{\theta}^{(0,1)\,i}w_j^{(-1)}\dot{\hat{\chi}}_i{}^j \\ & - \hat{\theta}_j^{(0,1)}\hat{\theta}^{(0,1)\,j}\theta_b^{(1,0)}\theta^{(1,0)\,b}w_i^{(-1)}v_a^{(-1)}\ddot{f}^{ia}. \end{aligned} \tag{4.7}$$

With taking into account (4.5) and (2.58), its components transformations are found to read

$$\begin{aligned} &\delta f^{ia} = -\,\epsilon^{ib}\chi_b{}^a - \hat{\epsilon}^{ja}\hat{\chi}_j{}^i\,, && \delta F_{ia} = 2i\left(\epsilon_{ja}\dot{\hat{\chi}}_i{}^j - \hat{\epsilon}_{ib}\dot{\chi}_a{}^b\right) + 2im\left(\hat{\epsilon}_{ja}\hat{\chi}_i{}^j - \epsilon_{ib}\chi_a{}^b\right), \\ &\delta\chi_a{}^b = 2i\left(\epsilon_{ia}\dot{f}^{ib} - m\,\hat{\epsilon}_{ia}f^{ib}\right) - \hat{\epsilon}^{ib}F_{ia}\,, && \delta\hat{\chi}_i{}^j = 2i\left(\hat{\epsilon}_{ia}\dot{f}^{ja} - m\,\epsilon_{ia}f^{ja}\right) + \epsilon^{ja}F_{ia}\,. \end{aligned} \tag{4.8}$$

Since the superfield $q^{(1,1)}$ in itself is not deformed (only its transformation properties prove to be deformed), we can realize on it the supersymmetry SU(2|2) in parallel with the standard flat $\mathcal{N}=8, d=1$ Poincaré supersymmetry, or even with another SU(2|2) involving the flipped-sign deformation parameter $-m$. The closure of all these symmetries including the original SU(2|2) turns out to constitute an extended superalgebra introduced in [33]:

$$\begin{aligned}
&\left\{Q^0_{ia}, \tilde{S}_{jb}\right\} = \varepsilon_{ij}\, J_{ab} - \varepsilon_{ab}\, I_{ij} + \frac{1}{2}\,\varepsilon_{ab}\,\varepsilon_{ij}\, Z, \\
&\left\{S^0_{ia}, \tilde{Q}_{jb}\right\} = \varepsilon_{ab}\, J_{ij} - \varepsilon_{ij}\, I_{ab} + \frac{1}{2}\,\varepsilon_{ab}\,\varepsilon_{ij}\, Z, \\
&\left\{Q^0_{ia}, Q^0_{jb}\right\} = 2\,\varepsilon_{ij}\,\varepsilon_{ab}\, H, \qquad \left\{S^0_{ia}, S^0_{jb}\right\} = 2\,\varepsilon_{ij}\,\varepsilon_{ab}\, H,
\end{aligned} \tag{4.9}$$

$$\begin{aligned}
&\left[I_{ij}, I_{kl}\right] = \varepsilon_{il}\, I_{kj} + \varepsilon_{jk}\, I_{il}\,, \qquad \left[I_{ab}, I_{cd}\right] = \varepsilon_{ad}\, I_{bc} + \varepsilon_{bc}\, I_{ad}\,, \\
&\left[J_{ij}, J_{kl}\right] = \varepsilon_{il}\, J_{kj} + \varepsilon_{jk}\, J_{il}\,, \qquad \left[J_{ab}, J_{cd}\right] = \varepsilon_{ad}\, J_{bc} + \varepsilon_{bc}\, J_{ad}\,,
\end{aligned} \tag{4.10}$$

$$\begin{aligned}
&\left[I_{ab}, S^0_{kc}\right] = \frac{1}{2}\left(\varepsilon_{ac}\, S^0_{kb} + \varepsilon_{bc}\, S^0_{ka}\right), \qquad \left[I_{ij}, Q^0_{ka}\right] = \frac{1}{2}\left(\varepsilon_{ik}\, Q^0_{ja} + \varepsilon_{jk}\, Q^0_{ia}\right), \\
&\left[I_{ab}, \tilde{Q}_{kc}\right] = \frac{1}{2}\left(\varepsilon_{ac}\, \tilde{Q}_{kb} + \varepsilon_{bc}\, \tilde{Q}_{ka}\right), \qquad \left[I_{ij}, \tilde{S}_{ka}\right] = \frac{1}{2}\left(\varepsilon_{ik}\, \tilde{S}_{ja} + \varepsilon_{jk}\, \tilde{S}_{ia}\right), \\
&\left[J_{ab}, Q^0_{kc}\right] = \frac{1}{2}\left(\varepsilon_{ac}\, Q^0_{kb} + \varepsilon_{bc}\, Q^0_{ka}\right), \qquad \left[J_{ij}, S^0_{ka}\right] = \frac{1}{2}\left(\varepsilon_{ik}\, S^0_{ja} + \varepsilon_{jk}\, S^0_{ia}\right), \\
&\left[J_{ab}, \tilde{S}_{kc}\right] = \frac{1}{2}\left(\varepsilon_{ac}\, \tilde{S}_{kb} + \varepsilon_{bc}\, \tilde{S}_{ka}\right), \qquad \left[J_{ij}, \tilde{Q}_{ka}\right] = \frac{1}{2}\left(\varepsilon_{ik}\, \tilde{Q}_{ja} + \varepsilon_{jk}\, \tilde{Q}_{ia}\right).
\end{aligned} \tag{4.11}$$

In fact, the superalgebra (4.9)–(4.11) contains four SU(2) subalgebras with the generators $I_{ab}\,,\ I_{ij}\,,\ J_{ab}\,,\ J_{ij}$. These generators differently act on the indices of the component fields $\left\{f^{ia}, {\chi_b}^a, {\hat{\chi}_j}^{\ i}, F_{ia}\right\}$. The generators $I_{ab}\,,\ I_{ij}$ rotate only the upper-case indices i and a, while $J_{ab}\,,\ J_{ij}$ act only the lower-case ones (though denoted by the same characters). Thus, the two types of SU(2) indices of the component fields can actually be split into four types.

The SU(2|2) generators of (2.1) can be identified with the following linear combinations of the generators of the extended superalgebra (4.9)–(4.11):

$$\begin{aligned}
&Q_{ia} = Q^0_{ia} - 2im\,\tilde{Q}_{ia}\,, \qquad S_{ia} = S^0_{ia} - 2im\,\tilde{S}_{ia}\,, \\
&L_{ij} = -\left(J_{ij} + I_{ij}\right), \qquad R_{ab} = -\left(J_{ab} + I_{ab}\right).
\end{aligned} \tag{4.12}$$

Hence, the superalgebra (2.1) can be viewed as a subalgebra of the extended superalgebra (4.9)–(4.11), with the central charge

$$C = -\,imZ. \tag{4.13}$$

The second SU(2|2) supergroup is generated by the supercharges

$$Q^0_{ia} + 2im\,\tilde{Q}_{ia}\,, \qquad S^0_{ia} + 2im\,\tilde{S}_{ia}\,. \tag{4.14}$$

The integration measure (2.61) is invariant under the transformations of both SU(2|2) supergroups, with the parameters m and $-m$, i.e. it is also invariant under the transformations produced by all generators of (4.9)–(4.11).

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The generators appearing in (4.12) are realized on the biharmonic superspace (2.56) as

$$
\begin{aligned}
Q^0_{ia} &= w^{(1)}_i \frac{\partial}{\partial\theta^{(1,0)\,a}} - 2iw^{(-1)}_i\theta^{(1,0)}_a\partial_{(B)}\,, \quad S^0_{ia} = v^{(1)}_a \frac{\partial}{\partial\hat\theta^{(1,0)\,i}} - 2iv^{(-1)}_a\hat\theta^{(0,1)}_i\partial_{(B)}\,,\\
\tilde Q_{ia} &= \hat\theta^{(0,1)}_i v^{(1)}_a \partial^{(0,-2)} + \frac{1}{2}\,\hat\theta^{(0,1)}_j\hat\theta^{(0,1)\,j}\,v^{(-1)}_a\frac{\partial}{\partial\hat\theta^{(0,1)\,i}} - \hat\theta^{(0,1)}_i v^{(-1)}_a Z,\\
\tilde S_{ia} &= \theta^{(1,0)}_a w^{(1)}_i \partial^{(-2,0)} + \frac{1}{2}\,\theta^{(1,0)}_b\theta^{(1,0)\,b}\,w^{(-1)}_i\frac{\partial}{\partial\theta^{(1,0)\,a}} - \theta^{(1,0)}_a w^{(-1)}_i Z,\\
I_{ij} &= w^{(1)}_i w^{(1)}_j\partial^{(-2,0)} + \frac{1}{2}\left[w^{(1)}_i w^{(-1)}_j + w^{(1)}_j w^{(-1)}_i\right]\left(\theta^{(1,0)\,a}\frac{\partial}{\partial\theta^{(1,0)\,a}} - Z\right)\\
&\quad + iw^{(-1)}_i w^{(-1)}_j\theta^{(1,0)}_a\theta^{(1,0)\,a}\,\partial_{(B)}\,,\\
I_{ab} &= v^{(1)}_a v^{(1)}_b\partial^{(0,-2)} + \frac{1}{2}\left[v^{(1)}_a v^{(-1)}_b + v^{(1)}_b v^{(-1)}_a\right]\left(\hat\theta^{(0,1)\,k}\frac{\partial}{\partial\hat\theta^{(0,1)\,k}} - Z\right)\\
&\quad + iv^{(-1)}_a v^{(-1)}_b\hat\theta^{(0,1)}_i\hat\theta^{(0,1)\,i}\,\partial_{(B)}\,,\\
J_{ij} &= -\frac{1}{2}\left[\hat\theta^{(0,1)}_i\frac{\partial}{\partial\hat\theta^{(0,1)\,j}} + \hat\theta^{(0,1)}_j\frac{\partial}{\partial\hat\theta^{(0,1)\,i}}\right],\; J_{ab} = -\frac{1}{2}\left[\theta^{(1,0)}_a\frac{\partial}{\partial\theta^{(1,0)\,b}} + \theta^{(1,0)}_b\frac{\partial}{\partial\theta^{(1,0)\,a}}\right].
\end{aligned}
\tag{4.15}
$$

While applying these operators to the superfield $q^{(1,1)}$, one is led to put $Zq^{ia} = q^{ia}$, in accord with (4.2) and (4.13). The algebra of the generators (4.15) can be extended by the generator

$$
K = i\,\theta^{(1,0)}_a\theta^{(1,0)a}\,\partial^{(-2,0)} + i\,\theta^{(0,1)}_i\theta^{(0,1)i}\,\partial^{(0,-2)} - t_{(B)}Z\,, \tag{4.16}
$$

which, together with $H = i\partial_{(B)}$ and Z , form the Heisenberg algebra $\mathbf{h}(2)$

$$
[K, H] = iZ\,. \tag{4.17}
$$

The superalgebra (4.9)–(4.11) with the generator K being included can be treated as $\mathcal{N}=8$ extension of the algebra $\mathbf{h}(2)$ [33].

4.2 Invariant actions

Let us define the new "shifted" superfield

$$
\begin{aligned}
&\hat q^{(1,1)} = q^{(1,1)} - c^{(1,1)}, \qquad X = 2\,c^{(-1,-1)}\hat q^{(1,1)},\\
&c^{(1,1)} = c^{ia}w^{(1)}_i v^{(1)}_a, \quad c^{(-1,-1)} = c^{ia}w^{(-1)}_i v^{(-1)}_a, \quad c^{(1,1)}c^{(-1,-1)} - c^{(-1,1)}c^{(1,-1)} = \frac{1}{2},
\end{aligned}
\tag{4.18}
$$

where c^{ia} is a constant satisfying $c^{ia}c_{ia} = 1$. It is enough to consider the ϵ-transformations

$$
\begin{aligned}
\delta\,\hat q^{(1,1)} &= \Lambda^{(0,0)}_\epsilon\hat q^{(1,1)} + \Lambda^{(0,0)}_\epsilon c^{(1,1)} - \Lambda^{(0,2)}_\epsilon c^{(1,-1)},\\
\delta\,X &= \Lambda^{(0,0)}_\epsilon X + 2\,c^{(-1,-1)}c^{(1,1)}\Lambda^{(0,0)}_\epsilon - 2\,c^{(-1,-1)}c^{(1,-1)}\Lambda^{(0,2)}_\epsilon.
\end{aligned}
\tag{4.19}
$$

Such transformations are similar to the "superconformal" transformations [33]. Then it follows that an SU(2|2) invariant action can be constructed in the same way:

$$
S_{(\mathbf{4,8,4})} = \int dt\mathcal{L}_{(\mathbf{4,8,4})} = \int d\zeta^{(-2,-2)}_{(B)}\,\hat q^{(1,1)}\hat q^{(1,1)}\left[\frac{\ln(1+X)}{X^2} - \frac{1}{(1+X)\,X}\right]. \tag{4.20}
$$

Since the superfield $\hat{q}^{(1,1)}$ is not deformed, this action coincides with the one given in [33] and so it is invariant under the full hidden supersymmetry with the algebra (4.9)–(4.11) and the additional transformations with the generator (4.16). The central charge generator Z acts as a dilatation generator in the target space, $\delta_Z q^{(1,1)} = \omega q^{(1,1)}$, where ω is a constant parameter. Note that (4.20) is not invariant under the standard dilatations which affect not only $q^{(1,1)}$, but also the time coordinate $t_{(B)}$.

Despite the transformations (4.8) are mass-deformed, the component Lagrangian of (4.20) contains no terms with the parameter m. In particular, the bosonic core of this Lagrangian is as follows

$$\mathcal{L}^{\rm bos}_{(\mathbf{4},\mathbf{8},\mathbf{4})} = \frac{1}{f^2}\left(\dot{f}_{ia}\dot{f}^{ia} - \frac{1}{4}\,F_{ia}F^{ia}\right), \qquad f^2 = f_{ia}f^{ia}. \tag{4.21}$$

On the other hand, from the SU$(2|1)$ standpoint, the multiplet $(\mathbf{4},\mathbf{8},\mathbf{4})$ is a direct sum of two SU$(2|1)$ multiplets, $(\mathbf{4},\mathbf{8},\mathbf{4}) = (\mathbf{4},\mathbf{4},\mathbf{0}) \oplus (\mathbf{0},\mathbf{4},\mathbf{4})$, and it is known [19] that the Lagrangian of the multiplet $(\mathbf{4},\mathbf{4},\mathbf{0})$ in the general case explicitly involves the deformation parameter m. In particular, its bosonic core is

$$\sim G(f)\left(\dot{f}^{ia}\dot{f}_{ia} - \frac{m^2}{4}f^{ia}f_{ia}\right).$$

The only option for which the mass term becomes a constant and so fully decouples is just $G(f) = 1/f^2$ required by the SU$(2|2)$ invariance.[6] It can be shown that for this special choice the parameter m disappears also from all other terms in the $(\mathbf{4},\mathbf{4},\mathbf{0})$ Lagrangian.

Note that the Lagrangian (4.21) is invariant under the Z 'dilatations", $\delta_Z f^{ia} = \omega f^{ia}$, $\delta F^{ia} = \omega F^{ia}$ and, up to a total derivative, under the transformations generated by the operator K defined in (4.16), $\delta_K f^{ia} = \omega'\, t f^{ia}$, $\delta_K F^{ia} = \omega'\, t F^{ia}$.

5 The multiplet $(\mathbf{5},\mathbf{8},\mathbf{3})$

5.1 Kinematics

In the standard flat $\mathcal{N}{=}8, d{=}1$ supersymmetry, chiral superfields with some extra constraints can be used to describe the supermultiplets $(\mathbf{2},\mathbf{8},\mathbf{6})$ and $(\mathbf{5},\mathbf{8},\mathbf{3})$ (see the table of $\mathcal{N}{=}8$ supermultiplets in [31]). As we have checked, the constraints defining the multiplet $(\mathbf{2},\mathbf{8},\mathbf{6})$ do not admit a generalization to the SU$(2|2)$ supersymmetry, while those defining the supermultiplet $(\mathbf{5},\mathbf{8},\mathbf{3})$ can be SU$(2|2)$ covariantized. Thus, the supermultiplet $(\mathbf{5},\mathbf{8},\mathbf{3})$ proves to be the only SU$(2|2)$ multiplet for the description of which a chiral superfield can be utilized.

We consider the complex superfield Ψ satisfying the standard chiral constraints

$$\bar{\mathcal{D}}^{ia}\Psi = 0\,, \qquad \tilde{L}^{ij}\Psi = \tilde{R}^{ab}\Psi = \tilde{C}\,\Psi. \tag{5.1}$$

[6] Note that the singularity of this metric at $f^{ia} = 0$ can be avoided, e.g., by assuming that f^{ia} starts with a constant c^{ia}. The 4-dimensional manifold with such a metric is known as Hopf manifold (see e.g. [53]) and it provides a simplest non-trivial example of the so called HKT ("hyper-Kähler with torsion") manifolds.

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This superfield lives as unconstrained on the chiral subspace (2.33). It means that the solution of (5.1) is given by the general ϑ-expansion

$$\Psi\left(t_L,\vartheta^{ia}\right)=z+\sqrt{2}\,\vartheta^{ia}\psi_{ia}+\vartheta^{ia}\vartheta^j_a A_{ij}+\vartheta^{ia}\vartheta^b_i B_{ab}+\frac{2\sqrt{2}}{3}\,\vartheta^{ib}\vartheta^{ja}\vartheta_{jb}\,\pi_{ia}+\frac{1}{3}\,\vartheta^{ib}\vartheta^{ja}\vartheta_{jb}\vartheta_{ia}D,$$
$$B_{ab}=B_{ba}\,,\qquad A_{ij}=A_{ji}\,. \tag{5.2}$$

The passive transformation law $\delta\Psi=0$ implies the following component transformations:

$$\begin{aligned}
\delta z&=-\sqrt{2}\,\eta^{ia}\psi_{ia}\,,\qquad \delta\psi_{ia}=-\sqrt{2}\left(\eta^j_a A_{ij}+\eta^b_i B_{ab}-i\bar{\eta}_{ia}\dot{z}\right),\\
\delta A_{ij}&=-\sqrt{2}\,\eta^a_{(j}\left[\pi_{i)a}+m\,\psi_{i)a}\right]-\sqrt{2}\,\bar{\eta}^a_{(j}\left[i\dot{\psi}_{i)a}-m\,\psi_{i)a}\right],\\
\delta B_{ab}&=-\sqrt{2}\,\eta_{i(b}\left[\pi^i_{a)}+m\,\psi^i_{a)}\right]+\sqrt{2}\,\bar{\eta}_{i(b}\left[i\dot{\psi}^i_{a)}+m\,\psi^i_{a)}\right],\\
\delta\pi_{ia}&=\sqrt{2}\left(-i\bar{\eta}^j_a\dot{A}_{ij}+i\bar{\eta}^b_i\dot{B}_{ab}-\eta_{ia}D\right)+\sqrt{2}\,m\left[\left(\eta^j_a-\bar{\eta}^j_a\right)A_{ij}+\left(\eta^b_i-\bar{\eta}^b_i\right)B_{ab}-i\bar{\eta}_{ia}\dot{z}\right],\\
\delta D&=\sqrt{2}\,i\bar{\eta}^{ia}\left(\dot{\pi}_{ia}+m\,\dot{\psi}_{ia}\right).
\end{aligned} \tag{5.3}$$

Indeed, their Lie brackets are easily checked to form SU$(2|2)$ symmetry. The chiral superfield (5.2) contains 16 bosonic and 16 fermionic fields and so is reducible. To single out the multiplet $(\mathbf{5},\mathbf{8},\mathbf{3})$, we impose the extra SU$(2|2)$ covariant constraints

$$\bar{\mathcal{D}}^{ia}\bar{\mathcal{D}}^b_i\,\bar{\Psi}+\mathcal{D}^{ia}\mathcal{D}^b_i\,\Psi=0\,,\quad 2\sqrt{2}\,\bar{\mathcal{D}}^{ia}\mathcal{V}^{jk}=-\,\varepsilon^{i(j}\mathcal{D}^{k)a}\,\Psi\,,\qquad 2\sqrt{2}\,\mathcal{D}^{ia}\mathcal{V}^{jk}=-\,\varepsilon^{i(j}\bar{\mathcal{D}}^{k)a}\,\bar{\Psi}\,,$$
$$\bar{\mathcal{D}}^{(i}_a\mathcal{V}^{jk)}=0\,,\qquad \mathcal{D}^{(i}_a\mathcal{V}^{jk)}=0\,,\quad \tilde{C}\,\mathcal{V}^{ij}=0\,,\qquad \mathcal{V}^{ij}\big|_{\vartheta=\bar{\vartheta}=0}=v^{ij}, \tag{5.4}$$

where $\mathcal{V}^{ij}$ is an additional deformed $\mathcal{N}=8$ superfield. Solving the constraints, we find that

$$A_{ij}=\sqrt{2}\left(-i\dot{v}_{ij}+m\,v_{ij}\right),\qquad \pi_{ia}=-\,i\dot{\bar{\psi}}_{ia}+m\,\bar{\psi}_{ia}-m\,\psi_{ia}\,,\qquad D=\ddot{\bar{z}}+im\dot{\bar{z}}\,,$$
$$\overline{(z)}=\bar{z}\,,\qquad \overline{(\psi_{ia})}=\bar{\psi}^{ia},\qquad \overline{(v_{ij})}=v^{ij},\qquad \overline{(B_{ab})}=B^{ab}=B^{ba}. \tag{5.5}$$

This field content now corresponds to the multiplet $(\mathbf{5},\mathbf{8},\mathbf{3})$, and the deformed transformations (5.3) are rewritten for the involved fields as

$$\begin{aligned}
&\delta z=-\sqrt{2}\,\eta^{ia}\psi_{ia}\,,\qquad \delta\bar{z}=\sqrt{2}\,\bar{\eta}^{ia}\bar{\psi}_{ia}\,,\qquad \delta v_{ij}=-\,\eta^a_{(j}\bar{\psi}_{i)a}+\bar{\eta}^a_{(j}\psi_{i)a}\,,\\
&\delta\psi_{ia}=2i\eta^j_a\dot{v}_{ij}-2m\,\eta^j_a v_{ij}-\sqrt{2}\,\eta^b_i B_{ab}+\sqrt{2}\,i\bar{\eta}_{ia}\dot{z}\,,\\
&\delta\bar{\psi}_{ia}=-\,2i\bar{\eta}^j_a\dot{v}_{ij}-2m\,\bar{\eta}^j_a v_{ij}-\sqrt{2}\,\bar{\eta}^b_i B_{ab}-\sqrt{2}\,i\eta_{ia}\dot{\bar{z}}\,,\\
&\delta B_{ab}=\sqrt{2}\,\eta_{i(b}\left[i\dot{\bar{\psi}}^i_{a)}-m\,\bar{\psi}^i_{a)}\right]+\sqrt{2}\,\bar{\eta}_{i(b}\left[i\dot{\psi}^i_{a)}+m\,\psi^i_{a)}\right].
\end{aligned} \tag{5.6}$$

5.2 Invariant actions

The $\mathcal{N}=8$ invariant deformed action can be written as an integral over chiral subspaces, like in the case of flat $\mathcal{N}=8$ supersymmetry [32]:

$$S_{(\mathbf{5},\mathbf{8},\mathbf{3})}=\frac{1}{4}\int d\zeta_L\,f\left(\Psi\right)+\frac{1}{4}\int d\zeta_R\,\bar{f}\left(\bar{\Psi}\right)=\int dt\,\mathcal{L}_{(\mathbf{5},\mathbf{8},\mathbf{3})}\,. \tag{5.7}$$

The component Lagrangian reads

$$\begin{aligned}\mathcal{L}_{(\mathbf{5},\mathbf{8},\mathbf{3})} = g\Big[&\dot{\bar{z}}\dot{z} + \dot{v}_{ij}\dot{v}^{ij} + \frac{i}{2}\left(\psi_{ia}\dot{\bar{\psi}}^{ia} - \dot{\psi}_{ia}\bar{\psi}^{ia}\right) - m\,\psi_{ia}\bar{\psi}^{ia} - m^2 v_{ij}v^{ij} + \frac{1}{2}B_{ab}B^{ab}\Big] \\ &- \frac{i}{2}mg\left(\dot{\bar{z}}z - \dot{z}\bar{z}\right) + \frac{i}{2}\left(\dot{\bar{z}}g_{\bar{z}} - \dot{z}g_z\right)\psi_{ia}\bar{\psi}^{ia} - \frac{1}{2}\left(g_z\,\psi^i_a\psi_{ib} + g_{\bar{z}}\,\bar{\psi}^i_a\bar{\psi}_{ib}\right)B^{ab} \\ &+ \frac{i}{\sqrt{2}}\left(g_z\,\psi_{ia}\psi^a_j - g_{\bar{z}}\,\bar{\psi}_{ia}\bar{\psi}^a_j\right)\dot{v}^{ij} - \frac{m}{\sqrt{2}}\left(g_z\,\psi_{ia}\psi^a_j + g_{\bar{z}}\,\bar{\psi}_{ia}\bar{\psi}^a_j\right)v^{ij} \\ &- \frac{1}{12}\left(g_{zz}\,\psi^{ib}\psi^{ja}\psi_{jb}\psi_{ia} + g_{\bar{z}\bar{z}}\,\bar{\psi}^{ib}\bar{\psi}^{ja}\bar{\psi}_{jb}\bar{\psi}_{ia}\right). \end{aligned} \tag{5.8}$$

Here, g is a special Kähler metric defined as

$$g\left(z,\bar{z}\right) = f''\left(z\right) + \bar{f}''\left(\bar{z}\right), \qquad g_z = \frac{\partial g\left(z,\bar{z}\right)}{\partial z}, \quad g_{\bar{z}} = \frac{\partial g\left(z,\bar{z}\right)}{\partial \bar{z}}, \quad \text{etc}. \tag{5.9}$$

As compared to the undeformed case, we observe the appearance of the oscillator-type fermionic ($\sim m$) and bosonic ($\sim m^2$) potential terms, as well as the internal bosonic WZ term accompanied by some new Yukawa-type couplings.

The simplest free action $S^{\text{free}}_{(\mathbf{5},\mathbf{8},\mathbf{3})}$ corresponds to the choice $f\left(\Psi\right) = \Psi^2/4$. Its component off-shell Lagrangian reads

$$\mathcal{L}^{\text{free}}_{(\mathbf{5},\mathbf{8},\mathbf{3})} = \dot{\bar{z}}\dot{z} + \dot{v}_{ij}\dot{v}^{ij} + \frac{i}{2}\left(\psi_{ia}\dot{\bar{\psi}}^{ia} - \dot{\psi}_{ia}\bar{\psi}^{ia}\right) - m\,\psi_{ia}\bar{\psi}^{ia} - \frac{i}{2}m\left(\dot{\bar{z}}z - \bar{z}\dot{z}\right) - m^2 v_{ij}v^{ij} + \frac{1}{2}B_{ab}B^{ab}. \tag{5.10}$$

In [47], SU(2|2) supersymmetry was shown to underlie $\mathcal{N}=8$ massive quantum mechanics of type I inspired by some super Yang-Mills theory. One can show that the relevant Lagrangian in the abelian case with U(1) as a gauge symmetry coincides with the on-shell Lagrangian obtained from (5.10). It would be interesting to inquire to which higher-dimensional system the general Lagrangian (5.8) could correspond.

5.3 Description in terms of SU(2|1) superfields

The supergroup SU(2|2) contains as a subgroup the supergroup SU(2|1). Hence, SU(2|2) supersymmetric mechanics can be equivalently viewed as SU(2|1) supersymmetric mechanics [16–19] associated with a few irreducible SU(2|1) multiplets forming a given SU(2|2) multiplet. Here, we deal with the supergroup SU(2|1) defined in (2.5). The multiplet $(\mathbf{5},\mathbf{8},\mathbf{3})$ can be split into SU(2|1) multiplets as $(\mathbf{4},\mathbf{4},\mathbf{0}) \oplus (\mathbf{1},\mathbf{4},\mathbf{3})$ or $(\mathbf{2},\mathbf{4},\mathbf{2}) \oplus (\mathbf{3},\mathbf{4},\mathbf{1})$. We have restricted our consideration to the latter option.

The SU(2|1) superspace coordinates are defined in the basis (2.26) as $\left\{t\,,\vartheta_{i1}\,,\bar{\vartheta}^{i1}\right\} =: \left\{t\,,\theta_i\,,\bar{\theta}^i\right\}$ and are transformed under SU(2|1) according to

$$\delta\theta_i = \epsilon_i + 2m\,\bar{\epsilon}^j\theta_j\theta_i\,, \qquad \delta\bar{\theta}^i = \bar{\epsilon}^i - 2m\,\epsilon_j\bar{\theta}^j\bar{\theta}^i, \qquad \delta t = i\bar{\epsilon}^i\theta_i + i\epsilon_i\bar{\theta}^i\,. \tag{5.11}$$

Here the parameters $\epsilon_i, \bar{\epsilon}^i$ are related to the parameters in (2.29) as

$$\eta_{i1} =: \epsilon_i\,, \qquad \bar{\eta}^{i1} =: \bar{\epsilon}^i, \qquad \eta_{i2} =: \varepsilon_i\,, \qquad \bar{\eta}^{i2} =: \bar{\varepsilon}^i\,, \tag{5.12}$$

the ε-transformations being associated with the hidden supersymmetry which extends SU(2|1) to SU(2|2).

The ϵ-transformations in (5.6) are split into SU(2|1) transformations corresponding to the chiral multiplet $(\mathbf{2},\mathbf{4},\mathbf{2})$ [16] with the U(1) charge $\kappa = 0$ and the multiplet $(\mathbf{3},\mathbf{4},\mathbf{1})$ (see appendix C):

$$\Phi = \left(z, \psi^{i1}, -B_{22}\right) \Rightarrow \left(z, \xi^{i}, B\right), \quad V_{ij} = \left(v_{ij}, -\bar{\psi}^{i1}, -\psi_{i1}, -\sqrt{2}\, iB_{12}\right) \Rightarrow \left(v_{ij}, \chi^{i}, \bar{\chi}_{i}, A\right). \tag{5.13}$$

Generally, the SU(2|2) invariant Lagrangian can be written in terms of these SU(2|1) superfields as

$$\mathcal{L}_{(\mathbf{5},\mathbf{8},\mathbf{3})} = \int d^2\theta\, d^2\bar{\theta}\left(1 + 2m\,\bar{\theta}^k\theta_k\right)\mathcal{F}\left(\Phi, \bar{\Phi}, V_{ij}\right), \tag{5.14}$$

where $\mathcal{F}$ is an arbitrary real scalar function of SU(2|1) superfields satisfying the five-dimensional Laplace equation [28, 30]:

$$\left(\frac{4\,\partial^2}{\partial\Phi\,\partial\bar{\Phi}} + \frac{\partial^2}{\partial V^{ij}\,\partial V_{ij}}\right)\mathcal{F} = 0\,. \tag{5.15}$$

The metric $g := g\left(z, \bar{z}, v_{ij}\right)$ of the target space is expressed as

$$g\left(z, \bar{z}, v_{ij}\right) = -\,\frac{\partial^2\mathcal{F}\left(z, \bar{z}, v_{ij}\right)}{\partial v^{ij}\,\partial v_{ij}} = \frac{4\,\partial^2\mathcal{F}\left(z, \bar{z}, v_{ij}\right)}{\partial z\,\partial\bar{z}}\,. \tag{5.16}$$

One can explicitly check that (5.15) is the only condition which is required for the invariance under the second subgroup $\mathrm{SU}(2)_{\mathrm{R}}$ of SU(2|2) in the terms quadratic and quartic in fermions. Since the closure of SU(2|1) and $\mathrm{SU}(2)_{\mathrm{R}}$ transformations necessarily yields the supersymmetry SU(2|2), the equations (5.15) is none other than the conditions of the SU(2|2) supersymmetry. One can treat the invariant Lagrangian (5.14) as a Lagrangian constructed in terms of harmonic superfields associated with Ψ, $\bar{\Psi}$ and $\mathcal{V}^{ij}$. This way of obtaining (5.14) can presumably be figured out from the harmonic formalism elaborated in [30].

As a solution of (5.15), the Lagrangian (5.8) can be rewritten in terms of SU(2|1) superfields as

$$\mathcal{L}_{(\mathbf{5},\mathbf{8},\mathbf{3})} = \int d^2\theta\, d^2\bar{\theta}\left(1{+}2m\,\bar{\theta}^k\theta_k\right)\left\{\frac{1}{4}\left[\bar{\Phi} f'\left(\Phi\right){+}\Phi\bar{f}'\left(\bar{\Phi}\right)\right] - \frac{1}{6}\,V_{ij}V^{ij}\left[f''\left(\Phi\right){+}\bar{f}''\left(\bar{\Phi}\right)\right]\right\}. \tag{5.17}$$

Here, the function f is related to (5.7) as

$$f\left(\Phi\right) \equiv f\left(\Psi\right)|_{\vartheta_{i2}=\bar{\vartheta}^{i2}=0}\,, \tag{5.18}$$

and the relevant metric (5.16) coincides with that defined in (5.9).

The metric (5.9) corresponds to the most general solution of (5.15) for $\mathcal{F}$ restricted to the 2-dimensional target space as $\mathcal{F} \equiv \mathcal{F}\left(z, \bar{z}\right), g \equiv g\left(z, \bar{z}\right)$. One can consider more general solutions involving some extra dependence on the triplet v^{ij}. For instance, the most general solution with $g \equiv g\left(v_{ij}\right)$ yields the Lagrangian

$$\mathcal{L}^{*}_{(\mathbf{5},\mathbf{8},\mathbf{3})} = \frac{1}{8}\int d^2\theta\, d^2\bar{\theta}\left(1 + 2m\,\bar{\theta}^k\theta_k\right)\frac{\bar{\Phi}\Phi - 2V_{ij}V^{ij}}{\sqrt{V_{ij}V^{ij}}}\,. \tag{5.19}$$

In the component form it reads

$$
\begin{aligned}
\mathcal{L}^{*}_{(\mathbf{5},\mathbf{8},\mathbf{3})} = \frac{1}{2|v|}\Big[& \dot{\bar z}\dot z + \dot v_{ij}\dot v^{ij} + \frac{i}{2}\left(\psi_{ia}\dot{\bar\psi}^{ia} - \dot\psi_{ia}\bar\psi^{ia}\right) + \frac{1}{2}B^{ab}B_{ab} + \frac{i}{|v|^2}\,\psi^{(i}_{a}\bar\psi^{j)a}v_{ik}\dot v^{k}_{j} \\
& + \frac{v^{ij}}{2\sqrt{2}\,|v|^2}\left(2\,\psi^a_i\bar\psi^b_j\,B_{ab} + i\psi_{ia}\psi^a_j\,\dot{\bar z} + i\bar\psi_{ia}\bar\psi^a_j\,\dot z\right) - \frac{3v_{(ij}v_{kl)}}{8|v|^4}\,\psi^a_i\psi_{ja}\,\bar\psi^b_k\bar\psi_{lb} \\
& - \frac{m}{2}\,\psi_{ia}\bar\psi^{ia} - m^2 v_{ij}v^{ij}\Big]\,,
\end{aligned} \tag{5.20}
$$

with

$$
g\left(v_{ij}\right) = \frac{1}{2|v|}\,. \tag{5.21}
$$

One can explicitly check that this Lagrangian is invariant under (5.6).

There can be many other solutions of (5.15) depending on all five fields. An example of such a solution producing a superconformal model is given in appendix B.3.

5.4 The free quantum model

As an example, here we present quantization of the simplest free model corresponding to (5.10). Eliminating auxiliary fields, we obtain

$$
\mathcal{L}^{\text{free}}_{(\mathbf{5},\mathbf{8},\mathbf{3})} = \dot{\bar z}\dot z + \dot v_{ij}\dot v^{ij} + \frac{i}{2}\left(\psi_{ia}\dot{\bar\psi}^{ia} - \dot\psi_{ia}\bar\psi^{ia}\right) - m\,\psi_{ia}\bar\psi^{ia} - \frac{i}{2}\,m\left(\dot{\bar z}z - \bar z\dot z\right) - m^2 v_{ij}v^{ij}. \tag{5.22}
$$

After performing Legendre transformations we obtain the canonical Hamiltonian

$$
H = \left(p_{\bar z} + \frac{i}{2}\,mz\right)\left(p_z - \frac{i}{2}\,m\bar z\right) + \frac{p^{ij}p_{ij}}{4} + m^2 v^{ij}v_{ij} + m\,\psi^{ia}\bar\psi_{ia}\,. \tag{5.23}
$$

Other Noether charges are given by

$$
\begin{aligned}
\Pi^{ia} &= \sqrt{2}\left(p_z - \frac{i}{2}\,m\bar z\right)\psi^{ia} + \left(p^{ik} + 2imv^{ik}\right)\bar\psi^a_k, \\
\bar\Pi_{jb} &= \sqrt{2}\left(p_{\bar z} + \frac{i}{2}\,mz\right)\bar\psi_{jb} - \left(p_{jk} - 2imv_{jk}\right)\psi^k_b, \\
L^i_j &= \psi^{ia}\bar\psi_{ja} - \frac{1}{2}\,\delta^i_j\,\psi^{kc}\bar\psi_{kc} + 2iv^{ik}p_{kj} - i\delta^i_j v^{kl}p_{kl}\,, \\
R^a_b &= \psi^{ka}\bar\psi_{kb} - \frac{1}{2}\,\delta^a_b\,\psi^{kc}\bar\psi_{kc}\,.
\end{aligned} \tag{5.24}
$$

The Poisson and Dirac brackets are imposed as

$$
\{z, p_z\} = 1\,,\quad \{\bar z, p_{\bar z}\} = 1\,,\quad \left\{v^{ij}, p_{kl}\right\} = \frac{1}{2}\left(\delta^i_k\delta^j_l + \delta^i_l\delta^j_k\right),\quad \left\{\bar\psi^{ia}, \psi_{jb}\right\} = -\,i\delta^a_b\delta^i_j \tag{5.25}
$$

and they are quantized in the standard way

$$
[z, p_z] = i\,,\quad [\bar z, p_{\bar z}] = i\,,\quad \left[v^{ij}, p_{kl}\right] = \frac{i}{2}\left(\delta^i_k\delta^j_l + \delta^i_l\delta^j_k\right),\quad \left\{\bar\psi^{ia}, \psi_{jb}\right\} = \delta^a_b\delta^i_j. \tag{5.26}
$$

We will use the operators

$$
\left[\nabla_z, \bar\nabla_{\bar z}\right] = m\,,\qquad \left[\nabla^{-ij}, \nabla^{+}_{kl}\right] = \frac{m}{2}\left(\delta^i_k\delta^j_l + \delta^i_l\delta^j_k\right), \tag{5.27}
$$

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where

$$\nabla_z = p_z - \frac{i}{2}\,m\bar{z}, \qquad \bar{\nabla}_{\bar{z}} = p_{\bar{z}} + \frac{i}{2}\,mz, \qquad \nabla^{\pm}_{ij} = \frac{1}{2}\left(p_{ij} \pm 2imv_{ij}\right). \tag{5.28}$$

In terms of the so defined creation and annihilation operators, the quantum version of the generators of (2.4) takes the form

$$\begin{aligned}
&\Pi^{ia} = \sqrt{2}\,\nabla_z\psi^{ia} + 2\nabla^{+ik}\bar{\psi}^a_k, && \bar{\Pi}_{jb} = \sqrt{2}\,\bar{\nabla}_{\bar{z}}\bar{\psi}_{jb} - 2\nabla^-_{jk}\psi^k_b,\\
&L^i_j = \psi^{ia}\bar{\psi}_{ja} - \frac{1}{2}\,\delta^i_j\,\psi^{kc}\bar{\psi}_{kc} + \frac{1}{m}\left(\nabla^{+ik}\nabla^-_{jk} - \nabla^+_{jk}\nabla^{-ik}\right), && R^a_b = \psi^{ka}\bar{\psi}_{kb} - \frac{1}{2}\,\delta^a_b\,\psi^{kc}\bar{\psi}_{kc}\,,\\
&H = \bar{\nabla}_{\bar{z}}\nabla_z + \nabla^{+ij}\nabla^-_{ij} + m\,\psi^{ia}\bar{\psi}_{ia}\,.
\end{aligned} \tag{5.29}$$

As follows from the definition (5.28), the quantum generator L^i_j in fact does not involve the parameter m. So the latter appears only in the supercharges and the Hamiltonian.

To construct the Hilbert space of wave functions, we use the creation operators $\bar{\nabla}_{\bar{z}}$, ∇^{+ij}, ψ^{ia} and the annihilation operators ∇_z, ∇^{-ij}, $\bar{\psi}^{ia}$. Then, the energy spectrum of H is found to be

$$H\Omega^{(\ell)} = m\ell\,\Omega^{(\ell)}, \tag{5.30}$$

where $\Omega^{(\ell)}$ is a wave function at the Landau level ℓ. The ground state corresponds to $\ell = 0$ and the first excited level to $\ell = 1$. The relevant wave functions are given by the expressions:

$$\begin{aligned}
\Omega^{(0)} &= a^{(0)}\left(\bar{z}\right) e^{-\frac{mz\bar{z}}{2}},\\
\Omega^{(1)} &= \left(a^{(1)}\left(\bar{z}\right)\bar{\nabla}_{\bar{z}} + b^{(1)}_{ij}\left(\bar{z}\right)\nabla^{+ij} + c^{(1)}_{ia}\left(\bar{z}\right)\psi^{ia}\right) e^{-\frac{mz\bar{z}}{2}}.
\end{aligned} \tag{5.31}$$

The coefficients $a^{(0)}$, $a^{(1)}$, $b^{(1)}_{ij}$ and $c^{(1)}_{ia}$ are some arbitrary antiholomorphic functions. This infinite degeneracy is caused by action of the additional generators $\nabla_z + im\bar{z}$ and $\bar{\nabla} - imz$ (magnetic translations) which commute with all quantum generators (5.29). All the higher levels $\ell > 1$ have wave functions of more complex structure and we will not consider them here.

A few words about SU(2|2) representations are to the point. The ground state $\Omega^{(0)}$ is annihilated by all quantum generators (5.29), i.e., it is just a singlet. According to [54], the level $\ell = 1$ corresponds to the atypical SU(2|2) representation $\langle 1, 0; 1/2, 0, 0\rangle$, with the overall dimension 8. All the higher ℓ wave functions can be also classified based on the analysis of [54].

6 Conclusions

Using powerful d=1 superfield coset techniques, we have constructed and studied several models of SU(2|2) supersymmetric mechanics based on the off-shell multiplets $(\mathbf{3}, \mathbf{8}, \mathbf{5})$, $(\mathbf{4}, \mathbf{8}, \mathbf{4})$ and $(\mathbf{5}, \mathbf{8}, \mathbf{3})$. This new kind of supersymmetric mechanics is a deformation of flat $\mathcal{N}$=8 supersymmetric mechanics. The corresponding actions were presented, both in terms of superfields and of component fields. The extended symmetries of these actions were revealed, and quantization was explicitly performed in one simple case.

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Off-shell supermultiplets of standard $\mathcal{N}=8$ supersymmetric mechanics [29, 31, 33, 34] other than $(\mathbf{3},\mathbf{8},\mathbf{5})$, $(\mathbf{4},\mathbf{8},\mathbf{4})$ or $(\mathbf{5},\mathbf{8},\mathbf{3})$ do not seem to admit a deformation to SU(2|2) multiplets. A possible explanation is the following. If we take for granted that the SU(2|2) transformations form a subset of $\mathcal{N}=8$ superconformal transformations (like the SU(2|1) ones which are embedded into an appropriate $\mathcal{N}=4$ superconformal group), then such superconformal transformations can correspond only to the superconformal group OSp$(4^*|4)$. Indeed, the superalgebra $osp(4^*|4)$ is the only one which contains $su(2|2)$ as a subalgebra (see appendix B.1). According to [38], OSp$(4^*|4)$ transformations are realized only on the multiplets $(\mathbf{3},\mathbf{8},\mathbf{5})$ and $(\mathbf{5},\mathbf{8},\mathbf{3})$. Hence, the supergroup SU(2|2) also admits an action only on these two multiplets. The multiplet $(\mathbf{4},\mathbf{8},\mathbf{4})$ is exceptional: none of $\mathcal{N}=8$, $d{=}1$ superconformal symmetries can act on it. However, one can realize on it an $\mathcal{N}=8$ extended Heisenberg superalgebra [33]. In section 4, we showed that this extended superalgebra (4.9)–(4.11) contains an $su(2|2)$ superalgebra. Hence, the SU(2|2) transformations of all three multiplets $(\mathbf{3},\mathbf{8},\mathbf{5})$, $(\mathbf{4},\mathbf{8},\mathbf{4})$ and $(\mathbf{5},\mathbf{8},\mathbf{3})$ are embedded into extended supergroups containing SU(2|2) as a subgroup.

As an example for the contrary, the root multiplet $(\mathbf{8},\mathbf{8},\mathbf{0})$ of flat $\mathcal{N}=8, d{=}1$ supersymmetry admits $osp(8|2)$ superconformal transformations [37]. Besides a flat $\mathcal{N}=8, d{=}1$ subalgebra, this superalgebra possesses two "curved" subalgebras with 8 supercharges [55], namely $su(1|4)$ and $osp(4|2)$,[7] but not $su(2|2)$. This is evidence in favor of the non-existence of the multiplet $(\mathbf{8},\mathbf{8},\mathbf{0})$ for SU(2|2) supersymmetry. Still, one might hope to construct the root multiplet $(\mathbf{8},\mathbf{8},\mathbf{0})$ as a sum of two mutually mirror SU(2|1) multiplets $(\mathbf{4},\mathbf{4},\mathbf{0}) \oplus (\mathbf{4},\mathbf{4},\mathbf{0})$ [19]. However, by trial and error, we became convinced that there is no way to extend SU(2|1) supersymmetry to SU(2|2) in such a system, although we are not able to give a direct rigorous proof. Similar arguments also suggest the absence of SU(2|2) analogs of the other "flat" off-shell $\mathcal{N}=8, d{=}1$ multiplets discussed in [31, 34]. For a more systematic search of these "missing" deformed multiplets, one should presumably study general deformations of $\mathcal{N}=8$ supersymmetric mechanics.

In [47], two types of $\mathcal{N}=8$ massive super Yang-Mills quantum mechanics provided matrix descriptions of supermembranes. Type I is based on the supergroup SU(2|2), while type II uses the product supergroup SU(2|1) × SU(2|1). In section 5, we noticed that the type I model of [47] reduced to the simplest U(1) gauge symmetry corresponds to the free Lagrangian (5.10) of the multiplet $(\mathbf{5},\mathbf{8},\mathbf{3})$. Our superfield approach gives the more general SU(2|2) supersymmetric Lagrangian (5.8). It will be interesting to explicitly consider deformations yielding the supergroups SU(2|1) × SU(2|1) [47] or SU(1|4) [46]. Further off-shell deformed $\mathcal{N}=8$ multiplets and the associated mechanics models may be constructed in this way. It is especially interesting to inspect worldline realizations of the supergroup SU(2|4) [44, 45], as they should bear a direct relation to the matrix models of [42] (see also [46]). Such models can be studied directly in an SU(2|2) mechanics language, proceeding from the fact that SU(2|2) is a subgroup of SU(2|4) and representing the multiplets of the latter as direct sums of the appropriate SU(2|2) multiplets. For instance, the SU(2|4) on-shell multiplet $(\mathbf{10},\mathbf{16})$ can hopefully be organized from two copies of the SU(2|2) multiplet $(\mathbf{5},\mathbf{8},\mathbf{3})$.

[7]The superalgebra $osp(4|2)$ is a superconformal $\mathcal{N}=4$ algebra [56].

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A Details of invariant action for the multiplet $(\mathbf{3},\mathbf{8},\mathbf{5})$

A.1 Calculation of the superfield action

The idea is to construct the invariant Lagrangian as a power series in $\hat{V}^{++}$,

$$L^{(+4)} \propto \rho\, \tilde{c}^{++}\hat{V}^{++} + m^2 \left(\theta^+\right)^4 \sum_{n=2}^{\infty} a_n \left(c^{--}\hat{V}^{++}\right)^n + \sum_{n=2}^{\infty} b_n \left(c^{--}\right)^{n-2} \left(\hat{V}^{++}\right)^n. \quad \text{(A.1)}$$

Using the fact that $\hat{V}^{++}$ is transformed inhomogeneously in (3.15), we require that the variations of the adjacent terms in the sum cancel each other modulo a total derivative, which will impose strict relations between the coefficients b_n and, finally, fix the form of the above series. We will properly employ the freedom in normalizing (A.1).

The physical normalization of the kinetic term of the boson triplet fixes $b_2 = 1/2$. We include the transformation of the integration measure (2.53) into the variations of various terms in the Lagrangian (A.1). Then such a generalized variation of the first term in the sum in (A.1) is reduced to (up to a total harmonic derivative)

$$\delta\left[\frac{1}{2}(\hat{V}^{++})^2\right] = \hat{V}^{++}\left[-\Lambda\left(\hat{V}^{++}+2\tilde{c}^{++}\right)-4\Lambda^{++}c^{+-}\right] = -\left(\hat{V}^{++}+6\tilde{c}^{++}\right)\Lambda\hat{V}^{++}. \quad \text{(A.2)}$$

The first piece of this variation is going to be compensated from the variation of the term $\sim \left(\hat{V}^{++}\right)^3$ in (A.1), while the second piece is canceled with the variation of the term $\sim \rho$ in (A.1). Indeed, it is easy to show that, up to a total derivative,

$$\delta\left(\rho\,\tilde{c}^{++}\hat{V}^{++}\right) = 4\Lambda\left(\rho\,\tilde{c}^{++}\hat{V}^{++}\right), \quad \text{(A.3)}$$

and the choice $\rho = 3/2$ ensures the cancelation needed. Next, we consider the cubic term $b_3\, c^{--}\left(\hat{V}^{++}\right)^3$ with the variation

$$\delta\left[b_3\, c^{--}\left(\hat{V}^{++}\right)^3\right] = -\,4\Lambda\left[b_3\, c^{--}\left(\hat{V}^{++}\right)^3\right] - 2b_3\,\Lambda\left(\hat{V}^{++}\right)^2 + \frac{5m^2}{2}\, b_3\left(c^{--}\hat{V}^{++}\right)^2 \delta\left(\theta^+\right)^4. \quad \text{(A.4)}$$

Under the choice $b_3 = -\,b_2 = -\,1/2$, the second piece in (A.4) exactly cancels the second piece in (A.2). The remaining term in (A.4) is canceled by the variation of the term

$$\frac{5m^2}{4}\left(\theta^+\right)^4\left(c^{--}\hat{V}^{++}\right)^2 \quad \Rightarrow \quad a_2 = \frac{5}{4}\,. \quad \text{(A.5)}$$

In this term, the variation of $\hat{V}^{++}$ yields a vanishing contribution due to the presence of the highest-order θ monomial and the fact that both Λ and Λ^{++} involve at least one power of the Grassmann-odd coordinates. So its full variation is exclusively defined by the variations of the explicit θ s. Proceeding further, we find that

$$\begin{aligned} & b_4=\frac{5}{8}\,, \qquad b_5=-\,\frac{7}{8}\,, \quad \text{etc}\,, \\ & a_3=-\,\frac{7}{4}\,, \quad a_4=\frac{45}{16}\,, \quad \text{etc}\,. \end{aligned} \tag{A.6}$$

After some effort, using the general formula

$$\left(c^{--}\right)^n c^{++}=\frac{n}{2n+1}\left(c^{--}\right)^{n-1}+\frac{1}{2\,(n+1)(2n+1)}\left(\mathcal{D}^{++}\right)^2\left(c^{--}\right)^{n+1}, \quad n\geqslant 1, \quad c^{ij}c_{ij}=1\,, \tag{A.7}$$

we find a recurrence relation for the coefficients a_n and b_n :

$$a_n=-\,\frac{(n+1)\,(2n+1)}{2\,(2n-1)}\,b_{n+1}\,, \qquad b_{n+1}=-\,\frac{2n-1}{n+1}\,b_n\,. \tag{A.8}$$

Then, using the property that, up to a total derivative,

$$\tilde{c}^{++}\hat{V}^{++}=-\,m^2\left(\theta^+\right)^4 c^{--}\hat{V}^{++}, \tag{A.9}$$

we represent the total Lagrangian $L^{(+4)}$ as

$$\begin{aligned} L^{(+4)}= & \; 4\sum_{n=2}^{\infty}(-1)^n\,\frac{(2n-3)!}{2^n n!\,(n-2)!}\,(c^{--})^{n-2}\big(\hat{V}^{++}\big)^n \\ & -\,2m^2\left(\theta^+\right)^4\sum_{n=2}^{\infty}(-1)^n\,\frac{(2n-1)\,(2n-4)!}{2^n\,(n-1)!\,(n-2)!}\left(c^{--}\hat{V}^{++}\right)^{n-1}. \end{aligned} \tag{A.10}$$

It remains to learn to which functions these series sum up. Using the Taylor expansions

$$\sum_{n=0}^{\infty}\frac{(2n+1)!}{n!\,(n+2)!}\,x^{n+2}=\frac{1}{4}\left(1-2x-\sqrt{1-4x}\,\right), \tag{A.11}$$

$$\sum_{n=0}^{\infty}\frac{(2n)!}{(n!)^2}\,x^n=\frac{1}{\sqrt{1-4x}}\,, \qquad \sum_{n=0}^{\infty}\frac{(2n)!}{n!\,(n+1)!}\,x^{n+1}=\frac{1}{2}\left(1-\sqrt{1-4x}\,\right)$$

with

$$x:=-\,\frac{\hat{V}^{++}c^{--}}{2}\,,$$

it is straightforward to write (A.10) as

$$L^{(+4)}=\frac{2\left(\hat{V}^{++}\right)^2}{\left(1+\sqrt{1+2\,c^{--}\hat{V}^{++}}\right)^2}-m^2\left(\theta^+\right)^4\left(\frac{c^{--}\hat{V}^{++}}{\sqrt{1+2\,c^{--}\hat{V}^{++}}}+\frac{c^{--}\hat{V}^{++}}{1+\sqrt{1+2\,c^{--}\hat{V}^{++}}}\right). \tag{A.12}$$

Then, the SU(2|2) invariant Lagrangian for the multiplet $(\mathbf{3},\mathbf{8},\mathbf{5})$ is given by

$$\mathcal{L}_{(\mathbf{3},\mathbf{8},\mathbf{5})}=\int d\zeta_A^{(-4)}\,L^{(+4)}. \tag{A.13}$$

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A.2 Harmonic integrals

For the calculation of the component Lagrangian of (A.13), we take as input the known harmonic integrals [20, 52]

$$\int \frac{du}{(1+2\,c^{--}\hat{v}^{++})^{3/2}} = \frac{1}{\sqrt{1+2\,c_{ij}\hat{v}^{ij}+\hat{v}_{ij}\hat{v}^{ij}}}\,, \tag{A.14}$$

$$\int \frac{du\, u^+_{(i}u^-_{j)}}{(1+2\,c^{--}\hat{v}^{++})^{3/2}} = \frac{-\left(c^k_i\hat{v}_{jk}+c^k_j\hat{v}_{ik}\right)}{1{+}2\,c_{ij}\hat{v}^{ij}{+}\hat{v}_{ij}\hat{v}^{ij}{+}(1{+}c_{ij}\hat{v}^{ij})\sqrt{1{+}2\,c_{ij}\hat{v}^{ij}{+}\hat{v}_{ij}\hat{v}^{ij}}}\,. \tag{A.15}$$

After some algebraic manipulations involving integration by parts, the component Lagrangian is reduced to a few terms containing the expressions (all taken at $\theta = 0$)

$$\begin{aligned}
\frac{\partial^2 L^{(+4)}}{\partial\hat{v}^{++2}} &= \frac{1}{(1+2\,c^{--}\hat{v}^{++})^{3/2}}\,,\\
\frac{\partial^3 L^{(+4)}}{\partial\hat{v}^{++3}} &= -\frac{3\,c^{--}}{(1+2\,c^{--}\hat{v}^{++})^{5/2}}\,,\\
\frac{\partial^4 L^{(+4)}}{\partial\hat{v}^{++4}} &= \frac{15\,(c^{--})^2}{(1+2\,c^{--}\hat{v}^{++})^{7/2}}\,.
\end{aligned} \tag{A.16}$$

They appear with the specific combinations of harmonics and the corresponding harmonic integrals are computed as

$$\begin{aligned}
\int du\,\frac{\partial^2 L^{(+4)}}{\partial\hat{v}^{++2}} &= \frac{1}{\sqrt{1+2\,c_{ij}\hat{v}^{ij}+\hat{v}_{ij}\hat{v}^{ij}}}\,,\\
\int du\,u^+_k u^+_l\,\frac{\partial^3 L^{(+4)}}{\partial\hat{v}^{++3}} &= -\frac{c_{kl}+\hat{v}_{kl}}{(1+2\,c_{ij}\hat{v}^{ij}+\hat{v}_{ij}\hat{v}^{ij})^{3/2}}\,,\\
\int du\,u^+_k u^+_l u^+_m u^+_n\,\frac{\partial^4 L^{(+4)}}{\partial\hat{v}^{++4}} &= \frac{3\,(c+\hat{v})_{(kl}\,(c+\hat{v})_{mn)}}{(1+2\,c_{ij}\hat{v}^{ij}+\hat{v}_{ij}\hat{v}^{ij})^{5/2}}\,.
\end{aligned} \tag{A.17}$$

There is also the harmonic integral

$$-\int du\left[\frac{\hat{v}^{++}\hat{v}^{--}}{\sqrt{1{+}2\,c^{--}\hat{v}^{++}}\,(1{+}\sqrt{1{+}2\,c^{--}\hat{v}^{++}})}+\frac{c^{--}\hat{v}^{++}}{1{+}\sqrt{1{+}2\,c^{--}\hat{v}^{++}}}+\frac{c^{--}\hat{v}^{++}}{\sqrt{1{+}2\,c^{--}\hat{v}^{++}}}\right], \tag{A.18}$$

responsible for the potential term $\sim m^2$. It is not immediately obvious how to compute it. It is easier to calculate this integral by considering its series expansion

$$\int du\sum_{n=2}^{\infty}\frac{(-1)^{n-1}\,(2n-4)!}{2^{n-1}\,(n-1)!\,(n-2)!}\left[(2n-3)\,\hat{v}^{--}\hat{v}^{++}+(2n-1)\,c^{--}\hat{v}^{++}\right]\left(c^{--}\hat{v}^{++}\right)^{n-2}. \tag{A.19}$$

Using the identities

$$\begin{aligned}
&\int du\,\left(c^{--}\hat{v}^{++}\right)^n = \frac{2n{-}1}{2n{+}1}\int du\,c_{ij}\hat{v}^{ij}\,\left(c^{--}\hat{v}^{++}\right)^{n-1} - \frac{n{-}1}{2n{+}1}\int du\,\hat{v}^{++}\hat{v}^{--}\,\left(c^{--}\hat{v}^{++}\right)^{n-2},\\
&\int du\,\hat{v}^{--}\hat{v}^{++}\,\left(c^{--}\hat{v}^{++}\right)^{n-2} = \frac{(n-1)\,\hat{v}_{ij}\hat{v}^{ij}}{2n-1}\int du\,\left(c^{--}\hat{v}^{++}\right)^{n-2},\\
&\hat{v}_{ij}\hat{v}^{ij} = 2\left(\hat{v}^{--}\hat{v}^{++}-\hat{v}^{+-}\hat{v}^{+-}\right), \qquad c_{ij}\hat{v}^{ij} = c^{--}\hat{v}^{++}+c^{++}\hat{v}^{--}-2c^{+-}\hat{v}^{+-},
\end{aligned} \tag{A.20}$$

we transform (A.19) to the form

$$\frac{1}{2} + \left(1 + 2\, c_{ij}\hat{v}^{ij} + \hat{v}_{ij}\hat{v}^{ij}\right) \int du \sum_{n=0}^{\infty} \frac{(-1)^{n+1}\,(2n+2)!}{2^{n+2} n!\,(n+1)!}\left(c^{--}\hat{v}^{++}\right)^{n}. \tag{A.21}$$

The final result is given by the integral (A.14) as

$$(\text{A.18}) = -\int \frac{du\,\left(1 + 2\,c_{ij}\hat{v}^{ij} + \hat{v}_{ij}\hat{v}^{ij}\right)}{2\left(1 + 2\,c^{--}\hat{v}^{++}\right)^{3/2}} + \frac{1}{2} = \frac{1}{2}\left(1 - \sqrt{1 + 2\,c_{ij}\hat{v}^{ij} + \hat{v}_{ij}\hat{v}^{ij}}\right). \tag{A.22}$$

B Superconformal symmetry

Superconformal $\mathcal{N}{=}4$ symmetry in SU(2|1) superspace is realized by *trigonometric* transformations [18]. Analogously, superconformal $\mathcal{N}{=}8$ symmetry in SU(2|2) superspace also yields trigonometric superconformal mechanics [57].

B.1 Superconformal algebra $osp(4^*|4)$ and its $su(2|2)$ subalgebra

There are four superconformal $\mathcal{N}{=}8$, $d{=}1$ algebras [56]: $osp(8|2)$, $su(1,1|4)$, $F(4)$, $osp(4^*|4)$. According to [55] (table VI), the superalgebra $su(2|2)$ can be embedded only into the superconformal algebra $osp(4^*|4)$. Here, we present this embedding.

The explicit structure of $osp(4^*|4)$ is given by the following nonvanishing (anti)-commutators:

$$\begin{aligned}
&\left\{Q^{ia}, Q^{jb}\right\} = -2\,\varepsilon^{ij}\varepsilon^{ab}P, && \left\{\mathcal{Q}^{i\alpha}, \mathcal{Q}^{j\beta}\right\} = -2\,\varepsilon^{ij}\varepsilon^{\alpha\beta}P,\\
&\left\{S^{ia}, S^{jb}\right\} = -2\,\varepsilon^{ij}\varepsilon^{ab}K, && \left\{\mathcal{S}^{i\alpha}, \mathcal{S}^{j\beta}\right\} = -2\,\varepsilon^{ij}\varepsilon^{\alpha\beta}K,\\
&\left\{Q^{ia}, S^{jb}\right\} = 2\left(\varepsilon^{ab}T^{ij} - \varepsilon^{ij}\varepsilon^{ab}D - 2\,\varepsilon^{ij}T_1^{ab}\right), &&\\
&\left\{\mathcal{Q}^{i\alpha}, \mathcal{S}^{j\beta}\right\} = 2\left(\varepsilon^{\alpha\beta}T^{ij} - \varepsilon^{ij}\varepsilon^{\alpha\beta}D - 2\,\varepsilon^{ij}T_2^{\alpha\beta}\right), &&\\
&\left\{Q^{ia}, \mathcal{S}^{j\alpha}\right\} = -2\,\varepsilon^{ij}U^{a\alpha}, && \left\{\mathcal{Q}^{i\alpha}, S^{ja}\right\} = -2\,\varepsilon^{ij}U^{a\alpha},
\end{aligned} \tag{B.1}$$

$$\begin{aligned}
&[D,P] = -iP, \qquad [D,K] = iK, && [P,K] = 2iD,\\
&\left[T^{ij}, T^{kl}\right] = -i\left(\varepsilon^{ik}T^{jl} + \varepsilon^{jl}T^{ik}\right), && \left[U^{a\alpha}, U^{b\beta}\right] = -2i\left(\varepsilon^{\alpha\beta}T_1^{ab} + \varepsilon^{ab}T_2^{\alpha\beta}\right),\\
&\left[T_1^{ab}, T_1^{cd}\right] = -i\left(\varepsilon^{ac}T_1^{bd} + \varepsilon^{bd}T_1^{ac}\right), && \left[T_2^{\alpha\beta}, T_2^{\gamma\rho}\right] = -i\left(\varepsilon^{\alpha\gamma}T_2^{\beta\rho} + \varepsilon^{\beta\rho}T_2^{\alpha\gamma}\right),\\
&\left[T_1^{ab}, U^{c\alpha}\right] = -\frac{i}{2}\left(\varepsilon^{ac}U^{b\alpha} + \varepsilon^{bc}U^{a\alpha}\right), && \left[T_2^{\alpha\beta}, U^{a\gamma}\right] = -\frac{i}{2}\left(\varepsilon^{\alpha\gamma}U^{a\beta} + \varepsilon^{\beta\gamma}U^{a\alpha}\right),
\end{aligned} \tag{B.2}$$

$$\begin{aligned}
&[P, S^{ia}] = i\,Q^{ia},\ \ [P, \mathcal{S}^{i\alpha}] = i\,\mathcal{Q}^{i\alpha}, && [K, Q^{ia}] = -i\,S^{ia},\ \ [K, \mathcal{Q}^{i\alpha}] = -i\,\mathcal{S}^{i\alpha},\\
&[D, Q^{ia}] = -\frac{i}{2}\,Q^{ia},\ \ [D, \mathcal{Q}^{i\alpha}] = -\frac{i}{2}\,\mathcal{Q}^{i\alpha}, && [D, S^{ia}] = \frac{i}{2}\,S^{ia},\ \ [D, \mathcal{S}^{i\alpha}] = \frac{i}{2}\,\mathcal{S}^{i\alpha},\\
&\left[U^{a\alpha}, Q^{ib}\right] = -i\,\varepsilon^{ab}\mathcal{Q}^{i\alpha}, && \left[U^{a\alpha}, \mathcal{Q}^{i\beta}\right] = -i\,\varepsilon^{\alpha\beta}Q^{ia},\\
&\left[U^{a\alpha}, S^{ib}\right] = -i\,\varepsilon^{ab}\mathcal{S}^{i\alpha}, && \left[U^{a\alpha}, \mathcal{S}^{i\beta}\right] = -i\,\varepsilon^{\alpha\beta}S^{ia},\\
&\left[T_1^{ab}, Q^{ic}\right] = -\frac{i}{2}\left(\varepsilon^{ac}Q^{ib} + \varepsilon^{bc}Q^{ia}\right), && \left[T_1^{ab}, S^{ic}\right] = -\frac{i}{2}\left(\varepsilon^{ac}S^{ib} + \varepsilon^{bc}S^{ia}\right),
\end{aligned}$$

$$
\begin{aligned}
\left[T_2^{\alpha\beta}, \mathcal{Q}^{i\gamma}\right] &= -\frac{i}{2}\left(\varepsilon^{\alpha\gamma}\mathcal{Q}^{i\beta} + \varepsilon^{\beta\gamma}\mathcal{Q}^{i\alpha}\right), &
\left[T_2^{\alpha\beta}, \mathcal{S}^{i\gamma}\right] &= -\frac{i}{2}\left(\varepsilon^{\alpha\gamma}\mathcal{S}^{i\beta} + \varepsilon^{\beta\gamma}\mathcal{S}^{i\alpha}\right), \\
\left[T^{ij}, Q^{ka}\right] &= -\frac{i}{2}\left(\varepsilon^{ik}Q^{ja} + \varepsilon^{jk}Q^{ia}\right), &
\left[T_1^{ij}, S^{ka}\right] &= -\frac{i}{2}\left(\varepsilon^{ik}S^{jb} + \varepsilon^{jk}S^{ia}\right), \\
\left[T^{ij}, \mathcal{Q}^{k\alpha}\right] &= -\frac{i}{2}\left(\varepsilon^{ik}\mathcal{Q}^{j\alpha} + \varepsilon^{jk}\mathcal{Q}^{i\alpha}\right), &
\left[T^{ij}, \mathcal{S}^{k\alpha}\right] &= -\frac{i}{2}\left(\varepsilon^{ik}\mathcal{S}^{j\alpha} + \varepsilon^{jk}\mathcal{S}^{i\alpha}\right).
\end{aligned}
\tag{B.3}
$$

The bosonic subgroup is $\mathrm{SO}(5) \times \mathrm{SU}(2) \times \mathrm{SO}(2,1)$.

There are three $\mathrm{SU}(2)$ groups with generators T^{ij}, T_1^{ab}, $T_2^{\alpha\beta}$ acting on the relevant indices. Let us redefine the $\mathrm{SU}(2)$ indices $a, b, c \ldots$ and $\alpha, \beta, \gamma, \ldots$ as

$$
\mathcal{Q}^{i\alpha} \to \mathcal{Q}^{ia}, \qquad \mathcal{S}^{i\alpha} \to \mathcal{S}^{ia}, \qquad T_2^{\alpha\beta} \to T_2^{ab}, \qquad U^{a\alpha} \to U^{ab}. \tag{B.4}
$$

This redefinition just means that we passed to the equivalent basis where, instead of the $su(2)$ algebra with the generators $T_2^{\alpha\beta}$, we deal with the diagonal $su(2)$ in the direct sum of the former $su(2)$ and the one with the generators T_1^{ab}. Then, we define the m-deformed supercharges

$$
\begin{aligned}
\Pi^{ia}(\pm m) &:= \frac{1}{2}\left[Q^{ia} - i\mathcal{Q}^{ia} \pm m\left(\mathcal{S}^{ia} + iS^{ia}\right)\right], \\
\bar{\Pi}^{ia}(\pm m) &:= \frac{1}{2}\left[Q^{ia} + i\mathcal{Q}^{ia} \pm m\left(\mathcal{S}^{ia} - iS^{ia}\right)\right]
\end{aligned}
\tag{B.5}
$$

and the bosonic generators

$$
\begin{aligned}
&\mathcal{H} = -\frac{1}{2}\left(P + m^2 K\right), \quad U = \frac{1}{2}\,\varepsilon_{cd}\,U^{cd}, \quad U^{(ab)} = \frac{1}{2}\left(U^{ab} + U^{ba}\right), \quad L^{ij} = i\,T^{ij}, \\
&R^{ab} = i\left(T_1^{ab} + T_2^{ab}\right), \quad \tilde{R}^{ab} = i\left(T_1^{ab} - T_2^{ab}\right), \quad \mathcal{T}_{\pm} = \frac{1}{2}\left(P - m^2 K \pm 2imD\right).
\end{aligned}
\tag{B.6}
$$

In terms of these redefined generators the superalgebra $osp(4^*|4)$ takes the form:

$$
\begin{aligned}
\left\{\Pi^{ia}(\pm m), \bar{\Pi}^{jb}(\pm m)\right\} &= \mp 2m\left(\varepsilon^{ab}L^{ij} - \varepsilon^{ij}R^{ab}\right) + 2\,\varepsilon^{ab}\varepsilon^{ij}\left(\mathcal{H} \pm mU\right), \\
\left\{\Pi^{ia}(\pm m), \bar{\Pi}^{jb}(\mp m)\right\} &= -\,\varepsilon^{ij}\varepsilon^{ab}\mathcal{T}_{\pm}, \\
\left\{\Pi^{ia}(m), \Pi^{jb}(-m)\right\} &= 2m\,\varepsilon^{ij}U^{(ab)} + 2m\,\varepsilon^{ij}\tilde{R}^{ab}, \\
\left\{\bar{\Pi}^{ia}(m), \bar{\Pi}^{jb}(-m)\right\} &= 2m\,\varepsilon^{ij}U^{(ab)} - 2m\,\varepsilon^{ij}\tilde{R}^{ab},
\end{aligned}
\tag{B.7}
$$

$$
\begin{aligned}
\left[\mathcal{H}, \mathcal{T}_{\pm}\right] &= \pm\, m\,\mathcal{T}_{\pm}, & \left[\mathcal{T}_{+}, \mathcal{T}_{-}\right] &= -\,2m\mathcal{H}, \\
\left[L^{ij}, L^{kl}\right] &= \varepsilon^{il}L^{kj} + \varepsilon^{jk}L^{il}, & \left[R^{ab}, \tilde{R}^{cd}\right] &= \varepsilon^{ad}\tilde{R}^{bc} + \varepsilon^{bc}\tilde{R}^{ad}, \\
\left[R^{ab}, R^{cd}\right] &= \varepsilon^{ad}R^{bc} + \varepsilon^{bc}R^{ad}, & \left[\tilde{R}^{ab}, \tilde{R}^{cd}\right] &= \varepsilon^{ad}R^{bc} + \varepsilon^{bc}R^{ad}, \\
\left[R^{ab}, U^{(cd)}\right] &= \varepsilon^{ad}U^{(bc)} + \varepsilon^{bc}U^{(ad)}, & \left[\tilde{R}^{ab}, U^{(cd)}\right] &= -\left(\varepsilon^{ac}\varepsilon^{bd} + \varepsilon^{ad}\varepsilon^{bc}\right)U, \\
\left[U^{(ab)}, U^{(cd)}\right] &= -\,\varepsilon^{ad}R^{bc} - \varepsilon^{bc}R^{ad}, & \left[\tilde{R}^{ab}, U\right] &= U^{(ab)}, \quad \left[U^{(ab)}, U\right] = \tilde{R}^{ab},
\end{aligned}
\tag{B.8}
$$

$$
\begin{aligned}
\left[L^{ij}, \Pi^{ka}(\pm m)\right] &= \frac{1}{2}\left(\varepsilon^{ik}\Pi^{ja}(\pm m) + \varepsilon^{jk}\Pi^{ia}(\pm m)\right),\\
\left[R^{ab}, \Pi^{ic}(\pm m)\right] &= \frac{1}{2}\left(\varepsilon^{ac}\Pi^{ib}(\pm m) + \varepsilon^{bc}\Pi^{ia}(\pm m)\right),\\
\left[L^{ij}, \bar\Pi^{ka}(\pm m)\right] &= \frac{1}{2}\left(\varepsilon^{ik}\bar\Pi^{ja}(\pm m) + \varepsilon^{jk}\bar\Pi^{ia}(\pm m)\right),\\
\left[R^{ab}, \bar\Pi^{ic}(\pm m)\right] &= \frac{1}{2}\left(\varepsilon^{ac}\bar\Pi^{ib}(\pm m) + \varepsilon^{bc}\bar\Pi^{ia}(\pm m)\right),\\
\left[\tilde R^{ab}, \Pi^{ic}(\pm m)\right] &= \frac{1}{2}\left(\varepsilon^{ac}\bar\Pi^{ib}(\mp m) + \varepsilon^{bc}\bar\Pi^{ia}(\mp m)\right),\\
\left[\tilde R^{ab}, \bar\Pi^{ic}(\pm m)\right] &= \frac{1}{2}\left(\varepsilon^{ac}\Pi^{ib}(\mp m) + \varepsilon^{bc}\Pi^{ia}(\mp m)\right),
\end{aligned} \tag{B.9}
$$

$$
\begin{aligned}
\left[\mathcal{H}, \Pi^{ia}(\pm m)\right] &= \pm\frac{m}{2}\,\Pi^{ia}(\pm m), & \left[\mathcal{H}, \bar\Pi^{ia}(\pm m)\right] &= \mp\frac{m}{2}\,\bar\Pi^{ia}(\pm m),\\
\left[\mathcal{T}_\pm, \Pi^{ia}(\mp m)\right] &= \pm m\,\Pi^{ia}(\pm m), & \left[\mathcal{T}_\pm, \bar\Pi^{ia}(\mp m)\right] &= \mp m\,\bar\Pi^{ia}(\pm m),\\
\left[U, \Pi^{ia}(\pm m)\right] &= -\frac{1}{2}\,\Pi^{ia}(\pm m), & \left[U, \bar\Pi^{ia}(\pm m)\right] &= \frac{1}{2}\,\bar\Pi^{ia}(\pm m),\\
\left[U^{(ab)}, \Pi^{ic}(\pm m)\right] &= -\frac{1}{2}\left(\varepsilon^{ac}\bar\Pi^{ib}(\mp m) + \varepsilon^{bc}\bar\Pi^{ia}(\mp m)\right),\\
\left[U^{(ab)}, \bar\Pi^{ic}(\pm m)\right] &= \frac{1}{2}\left(\varepsilon^{ac}\Pi^{ib}(\mp m) + \varepsilon^{bc}\Pi^{ia}(\mp m)\right).
\end{aligned} \tag{B.10}
$$

Thus, the deformed supercharges $\Pi^{ia}(m)$, $\bar\Pi^{jb}(m)$ generate the $su(2|2)$ superalgebra (2.4) with central charges $H = \mathcal{H} + mU$ and $C = 0$.[8] Hence, the supercharges $\Pi^{ia}(-m)$, $\bar\Pi^{jb}(-m)$ also form a $su(2|2)$ superalgebra, but with the opposite-sign deformation parameter $-m$. The closure of these two $su(2|2)$ superalgebras yields the whole superconformal algebra $osp(4^*|4)$. In this embedding, the subgroup SO(5) contains the subgroup $\mathrm{SU}(2)_\mathrm{R}$ from SU(2|2), i.e. $\mathrm{SU}(2)_\mathrm{R} \subset \mathrm{SO}(5)$. Switching $\mathrm{SU}(2)_\mathrm{R} \leftrightarrow \mathrm{SU}(2)_\mathrm{L}$ in SU(2|2), one can consider the embedding where $\mathrm{SU}(2)_\mathrm{L} \subset \mathrm{SO}(5)$.

B.2 Superconformal properties of the multiplet (3, 8, 5)

The component fields of (3.6) can be rewritten in a complex notation as

$$
\begin{aligned}
\xi_{ia} &= \frac{1}{\sqrt2}\left(\bar\psi_{ia}\,e^{-\frac{i}{2}mt} - \psi_{ia}\,e^{\frac{i}{2}mt}\right), & \hat\xi_{ia} &= \frac{i}{\sqrt2}\left(\bar\psi_{ia}\,e^{-\frac{i}{2}mt} + \psi_{ia}\,e^{\frac{i}{2}mt}\right),\\
A_0 &= -\frac{i}{2}\left(A\,e^{imt} + \bar A\,e^{-imt}\right), & C_0 &= \frac{1}{2}\left(A\,e^{imt} - \bar A\,e^{-imt}\right).
\end{aligned} \tag{B.11}
$$

Then the Lagrangian (3.19) is rewritten as

$$
\begin{aligned}
\mathcal{L}^{\mathrm{conf.}}_{(\mathbf{3},\mathbf{8},\mathbf{5})} = \frac{1}{2|v|}\Bigg[&\dot v_{ij}\dot v^{ij} + \frac{i}{2}\left(\psi_{ia}\dot{\bar\psi}^{ia} - \dot\psi_{ia}\bar\psi^{ia}\right) + \frac{i}{|v|^2}\,\psi_a^{(i}\bar\psi^{j)a}v_{ik}\dot v_j^k + 2A\bar A - \frac14\,C^{ab}C_{ab}\\
&+ \frac{v^{ij}}{2|v|^2}\left(i\psi_i^a\bar\psi_j^b\,C_{ab} + i\psi_{ia}\psi_j^a\,\bar A + i\bar\psi_{ia}\bar\psi_j^a\,A - 2\mu\,\psi_{ia}\bar\psi_j^a\right) - \frac{3v_{(ij}v_{kl)}}{8|v|^4}\,\psi_i^a\psi_{ja}\,\bar\psi_k^b\bar\psi_{lb}\\
&+ \mu^2 - m^2 v_{ij}v^{ij}\Bigg] - \frac{i\mu\,v^{ij}\left(c_i^k v_{jk} + c_j^k v_{ik}\right)}{|v|\left(|v| + c_{ij}v^{ij}\right)}.
\end{aligned} \tag{B.12}
$$

[8] As discussed, any $\mathcal{N}=8$, $d=1$ superconformal symmetry cannot be realized on the multiplet $(\mathbf{4},\mathbf{8},\mathbf{4})$ which has $C \neq 0$.

The relevant transformations (3.8) leaving this Lagrangian invariant (modulo a total derivative), become

$$
\begin{aligned}
\delta v_{ij} &= -\,\eta^a_{(j}\bar\psi_{i)a}\,e^{-\frac{i}{2}mt} + \bar\eta^a_{(j}\psi_{i)a}\,e^{\frac{i}{2}mt},\\
\delta\psi_{ia} &= \Big(2i\eta^j_a\dot v_{ij} - 2m\,\eta^j_a v_{ij} - i\eta^b_i C_{ab} + 2\mu\,\eta_{ia}\Big)\,e^{-\frac{i}{2}mt} + 2i\bar\eta_{ia}A\,e^{\frac{i}{2}mt},\\
\delta\bar\psi_{ia} &= -\Big(2i\bar\eta^j_a\dot v_{ij} + 2m\,\bar\eta^j_a v_{ij} + i\bar\eta^b_i C_{ab} + 2\mu\,\bar\eta_{ia}\Big)\,e^{\frac{i}{2}mt} - 2i\eta_{ia}\bar A\,e^{-\frac{i}{2}mt},\\
\delta C_{ab} &= 2\eta_{i(b}\left[\dot{\bar\psi}{}^i_{a)} + \frac{i}{2}\,m\,\bar\psi^i_{a)}\right]e^{-\frac{i}{2}mt} + 2\bar\eta_{i(b}\left[\dot\psi^i_{a)} - \frac{i}{2}\,m\,\psi^i_{a)}\right]e^{\frac{i}{2}mt},\\
\delta A &= -\,\eta^{ia}\left(\dot\psi_{ia} + \frac{i}{2}\,m\,\psi_{ia}\right)e^{-\frac{i}{2}mt}, \qquad \delta\bar A = \bar\eta^{ia}\left(\dot{\bar\psi}_{ia} - \frac{i}{2}\,m\,\bar\psi_{ia}\right)e^{\frac{i}{2}mt}.
\end{aligned}
\tag{B.13}
$$

Exploiting the property that the Lagrangian (B.12) depends only on m^2, we can define additional SU(2|2) transformations with $m \to -m$:

$$
\begin{aligned}
\delta' v_{ij} &= -\,\eta'^a_{(j}\bar\psi_{i)a}\,e^{\frac{i}{2}mt} + \bar\eta'^a_{(j}\psi_{i)a}\,e^{-\frac{i}{2}mt},\\
\delta'\psi_{ia} &= \Big(2i\eta'^j_a\dot v_{ij} + 2m\,\eta'^j_a v_{ij} - i\eta'^b_i C_{ab} + 2\mu\,\eta'_{ia}\Big)\,e^{\frac{i}{2}mt} + 2i\bar\eta'_{ia}A\,e^{-\frac{i}{2}mt},\\
\delta'\bar\psi_{ia} &= -\Big(2i\bar\eta'^j_a\dot v_{ij} - 2m\,\bar\eta'^j_a v_{ij} + i\bar\eta'^b_i C_{ab} + 2\mu\,\bar\eta'_{ia}\Big)\,e^{-\frac{i}{2}mt} - 2i\eta'_{ia}\bar A\,e^{\frac{i}{2}mt},\\
\delta' C_{ab} &= 2\eta'_{i(b}\left[\dot{\bar\psi}{}^i_{a)} - \frac{i}{2}\,m\,\bar\psi^i_{a)}\right]e^{\frac{i}{2}mt} + 2\bar\eta'_{i(b}\left[\dot\psi^i_{a)} + \frac{i}{2}\,m\,\psi^i_{a)}\right]e^{-\frac{i}{2}mt},\\
\delta' A &= -\,\eta'^{ia}\left(\dot\psi_{ia} - \frac{i}{2}\,m\,\psi_{ia}\right)e^{\frac{i}{2}mt}, \qquad \delta'\bar A = \bar\eta'^{ia}\left(\dot{\bar\psi}_{ia} + \frac{i}{2}\,m\,\bar\psi_{ia}\right)e^{-\frac{i}{2}mt}.
\end{aligned}
\tag{B.14}
$$

The closure of these two types of SU(2|2) transformations gives a trigonometric realization of the full superconformal symmetry OSp(4^*|4). Thus, the Lagrangian (B.12) is superconformal and is recognized as a deformation of the parabolic Lagrangian given in [31] by the oscillator mass term $\sim m^2$.

B.3 Superconformal properties of the multiplet $(\mathbf{5}, \mathbf{8}, \mathbf{3})$

Redefining component fields in (5.6) as

$$
\begin{aligned}
&z \to z e^{-imt}, \qquad \bar z \to \bar z e^{imt}, \qquad B_{ab} \to B_{ab}\,,\\
&\psi_{ia} \to \psi_{ia}\,e^{-\frac{i}{2}mt}, \qquad \bar\psi_{ia} \to \bar\psi_{ia}\,e^{\frac{i}{2}mt},
\end{aligned}
\tag{B.15}
$$

we cast their SU(2|2) transformations into the form

$$
\begin{aligned}
&\delta z = -\sqrt2\,\eta^{ia}\psi_{ia}\,e^{\frac{i}{2}mt}, \quad \delta\bar z = \sqrt2\,\bar\eta^{ia}\bar\psi_{ia}\,e^{-\frac{i}{2}mt}, \quad \delta v_{ij} = -\,\eta^a_{(j}\bar\psi_{i)a}\,e^{\frac{i}{2}mt} + \bar\eta^a_{(j}\psi_{i)a}\,e^{-\frac{i}{2}mt},\\
&\delta\psi_{ia} = \Big(2i\eta^j_a\dot v_{ij} - 2m\,\eta^j_a v_{ij} - \sqrt2\,\eta^b_i B_{ab}\Big)\,e^{\frac{i}{2}mt} + \sqrt2\,\bar\eta_{ia}\,(i\dot z + mz)\,e^{-\frac{i}{2}mt},\\
&\delta\bar\psi_{ia} = \Big(-2i\bar\eta^j_a\dot v_{ij} - 2m\,\bar\eta^j_a v_{ij} - \sqrt2\,\bar\eta^b_i B_{ab}\Big)\,e^{-\frac{i}{2}mt} - \sqrt2\,\eta_{ia}\,(i\dot{\bar z} - m\bar z)\,e^{\frac{i}{2}mt},\\
&\delta B_{ab} = \sqrt2\,\eta_{i(b}\left[i\dot{\bar\psi}{}^i_{a)} - \frac{3m}{2}\,\bar\psi^i_{a)}\right]e^{\frac{i}{2}mt} + \sqrt2\,\bar\eta_{i(b}\left[i\dot\psi^i_{a)} + \frac{3m}{2}\,\psi^i_{a)}\right]e^{-\frac{i}{2}mt}.
\end{aligned}
\tag{B.16}
$$

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Making the change $m \to -m$ in these transformations, we define additional SU(2|2) transformations. In the same way as in the previous case, the two types of SU(2|2) transformations close on the superconformal symmetry OSp(4*|4) in the trigonometric realization.

Superconformal Lagrangian admits construction in terms of SU(2|1) superfields corresponding to the multiplets $(\mathbf{2},\mathbf{4},\mathbf{2})$ and $(\mathbf{3},\mathbf{4},\mathbf{1})$ as SU(1,1|2) $\subset$ OSp(4*|4) superconformal trigonometric Lagrangian. Superfield Lagrangian satisfying (5.15) is given by

$$\mathcal{L}^{\text{conf.}}_{(\mathbf{5},\mathbf{8},\mathbf{3})} = \int d^2\theta\, d^2\bar{\theta} \left(1 + 2m\,\bar{\theta}^k\theta_k\right) \frac{\log\left(\sqrt{V_{ij}V^{ij}} + \sqrt{V_{ij}V^{ij} + \Phi\bar{\Phi}}\right)}{\sqrt{V_{ij}V^{ij}}}. \tag{B.17}$$

In contrast to section 5.3, the chiral SU(2|1) superfield Φ describing the multiplet $(\mathbf{2},\mathbf{4},\mathbf{2})$ has a central charge[9] $b = -1$ [18]. By analogy with (B.12) and previously constructed trigonometric superconformal Lagrangians [18, 57], the relevant $(\mathbf{5},\mathbf{8},\mathbf{3})$ superconformal Lagrangian is a deformation of the parabolic superconformal Lagrangian [29, 31] by oscillator term. The bosonic truncation of superconformal Lagrangian reads

$$\mathcal{L}^{\text{conf.}}_{(\mathbf{5},\mathbf{8},\mathbf{3})}\,|_{\text{bos}} = \left(v_{ij}v^{ij} + z\bar{z}\right)^{-3/2}\left[\dot{\bar{z}}\dot{z} + \dot{v}_{ij}\dot{v}^{ij} + \frac{1}{2}\,B^{ab}B_{ab} - m^2\left(v_{ij}v^{ij} + z\bar{z}\right)\right]. \tag{B.18}$$

One can see that this Lagrangian is SO(5) × SU(2) invariant, where dynamical bosonic fields form SO(5) vector and auxiliary fields are combined into SU(2) triplet.

C The multiplet $(\mathbf{3},\mathbf{4},\mathbf{1})$

We briefly consider the multiplet $(\mathbf{3},\mathbf{4},\mathbf{1})$ described by the superfield V^{ij} in the framework of the SU(2|1) superspace [16]. It satisfies the SU(2|1) covariant constraints

$$\mathcal{D}^{(k}V^{ij)} = \bar{\mathcal{D}}^{(k}V^{ij)} = 0\,, \qquad \left(V^{ij}\right)^{\dagger} = V_{ij}\,, \qquad V^{ij} \equiv V^{ji}, \tag{C.1}$$

where SU(2|1) covariant derivatives are

$$\begin{aligned}
\mathcal{D}^i &= \left[1{+}m\bar{\theta}^k\theta_k - \frac{3m^2}{4}\left(\bar{\theta}^k\theta_k\right)^2\right]\frac{\partial}{\partial\theta_i} - m\bar{\theta}^i\theta_j\frac{\partial}{\partial\theta_j} - i\bar{\theta}^i\partial_t + m\bar{\theta}^i\tilde{F} - m\bar{\theta}^j\left(1{-}m\bar{\theta}^k\theta_k\right)\tilde{I}^i_j,\\
\bar{\mathcal{D}}_j &= -\left[1{+}m\bar{\theta}^k\theta_k{-}\frac{3m^2}{4}\left(\bar{\theta}^k\theta_k\right)^2\right]\frac{\partial}{\partial\bar{\theta}^j} + m\bar{\theta}^k\theta_j\frac{\partial}{\partial\bar{\theta}^k}{+}i\theta_j\partial_t{-}m\theta_j\tilde{F}{+}m\theta_k\left(1{-}m\bar{\theta}^l\theta_l\right)\tilde{I}^k_j.
\end{aligned} \tag{C.2}$$

The solution is given by

$$\begin{aligned}
V^{ij} &= \left[1 + m\,\bar{\theta}^k\theta_k - m^2\left(\bar{\theta}^k\theta_k\right)^2\right]v^{ij} + \theta^{(i}\chi^{j)} + \bar{\theta}^{(i}\bar{\chi}^{j)} - i\bar{\theta}^{(i}\theta^{j)}A\\
&\quad - i\left[\bar{\theta}^{(i}\theta_k + \bar{\theta}_k\theta^{(i}\right]\dot{v}^{j)k} - i\bar{\theta}^k\theta_k\left[\theta^{(i}\dot{\chi}^{j)} - \bar{\theta}^{(i}\dot{\bar{\chi}}^{j)}\right] + \frac{1}{2}\left(\bar{\theta}^k\theta_k\right)^2\ddot{v}^{ij},\\
\overline{(v^{ij})} &= v_{ij}\,, \quad \overline{(\chi^i)} = \bar{\chi}_i\,, \quad \overline{(B)} = B.
\end{aligned} \tag{C.3}$$

[9]The central charge b is related to a scaling dimension parameter as $b = -\lambda_D$ where λ_d=1 for the multiplet $(\mathbf{5},\mathbf{8},\mathbf{3})$ [38]. Thus, we have that $b = -1$.

The superfield V^{ij} has the following passive transformations

$$\delta V^{ij} = -\,m\Big(1+m\,\bar\theta^l\theta_l\Big)\Big[\Big(\epsilon_k\bar\theta^k+\bar\epsilon^k\theta_k\Big)\,V^{ij}-\big(\epsilon_k\bar\theta^i+\bar\epsilon^i\theta_k\big)\,V^{kj}-\big(\epsilon_k\bar\theta^j+\bar\epsilon^j\theta_k\big)\,V^{ik}\Big], \qquad \text{(C.4)}$$

and its component fields transform as

$$\begin{aligned} &\delta v^{ij} = -\,\epsilon^{(i}\chi^{j)} - \bar\epsilon^{(i}\bar\chi^{j)}, && \delta A = \epsilon_k\dot\chi^k - \bar\epsilon^k\dot{\bar\chi}_k + im\left(\epsilon_k\chi^k + \bar\epsilon^k\bar\chi_k\right),\\ &\delta\bar\chi_j = -\,2i\epsilon^k\dot v_{kj} + i\epsilon_j A + 2m\,\epsilon^k v_{kj}\,, && \delta\chi^i = -\,2i\bar\epsilon_k\dot v^{ik} - i\bar\epsilon^i A - 2m\,\bar\epsilon_k v^{ik}. \end{aligned} \qquad \text{(C.5)}$$

The general σ-model action for the SU(2|1) multiplet $(\mathbf{3},\mathbf{4},\mathbf{1})$ is constructed as

$$S_{(\mathbf{3},\mathbf{4},\mathbf{1})} = \int dt\,\mathcal{L} = -\frac{1}{6}\int dt\,d^2\theta\,d^2\bar\theta\,\Big(1+2m\,\bar\theta^k\theta_k\Big)\,L\left(V^2\right), \qquad V^2 = V_{ij}V^{ij}, \qquad \text{(C.6)}$$

where $dt\,d^2\theta\,d^2\bar\theta\left(1+2m\,\bar\theta^k\theta_k\right)$ is a SU(2|1) invariant measure. The simplest free Lagrangian corresponding to the choice $L = V^2$ reads

$$\mathcal{L}^{\text{free}}_{(\mathbf{3},\mathbf{4},\mathbf{1})} = \dot v_{ij}\dot v^{ij} + \frac{i}{2}\left(\bar\chi_i\dot\chi^i - \dot{\bar\chi}_i\chi^i\right) + m\,\chi^i\bar\chi_i - m^2 v_{ij}v^{ij} + \frac{A^2}{2}\,. \qquad \text{(C.7)}$$

After eliminating the auxiliary field A, this Lagrangian can be viewed as an abelian reduction of the mass-deformed $\mathcal{N}=4$ matrix models of type I [47]. Recently, in [48], the SU(2|1) multiplet $(\mathbf{3},\mathbf{4},\mathbf{1})$ was used at the component level for the description of a new class of $\mathcal{N}=4$ supersymmetric massive quiver matrix models. The component Lagrangian (C.7) corresponds to the simplest case of one node without arrows and with a U(1) gauge group.

References

[1] G. Festuccia and N. Seiberg, *Rigid Supersymmetric Theories in Curved Superspace*, *JHEP* **06** (2011) 114 [arXiv:1105.0689] [INSPIRE].

[2] T.T. Dumitrescu, G. Festuccia and N. Seiberg, *Exploring Curved Superspace*, *JHEP* **08** (2012) 141 [arXiv:1205.1115] [INSPIRE].

[3] C. Closset, T.T. Dumitrescu, G. Festuccia and Z. Komargodski, *Supersymmetric Field Theories on Three-Manifolds*, *JHEP* **05** (2013) 017 [arXiv:1212.3388] [INSPIRE].

[4] V. Pestun, *Localization of gauge theory on a four-sphere and supersymmetric Wilson loops*, *Commun. Math. Phys.* **313** (2012) 71 [arXiv:0712.2824] [INSPIRE].

[5] S.M. Kuzenko, U. Lindström and G. Tartaglino-Mazzucchelli, *Off-shell supergravity-matter couplings in three dimensions*, *JHEP* **03** (2011) 120 [arXiv:1101.4013] [INSPIRE].

[6] I.B. Samsonov and D. Sorokin, *Superfield theories on S^3 and their localization*, *JHEP* **04** (2014) 102 [arXiv:1401.7952] [INSPIRE].

[7] I.B. Samsonov and D. Sorokin, *Gauge and matter superfield theories on S^2*, *JHEP* **09** (2014) 097 [arXiv:1407.6270] [INSPIRE].

JHEP11(2016)031

[8] E.A. Ivanov and A.S. Sorin, *Superfield formulation of* $\mathrm{OSp}(1,4)$ *supersymmetry*, *J. Phys.* **A 13** (1980) 1159 [INSPIRE].

[9] E. Witten, *Dynamical Breaking of Supersymmetry*, *Nucl. Phys.* **B 188** (1981) 513 [INSPIRE].

[10] E. Witten, *Constraints on Supersymmetry Breaking*, *Nucl. Phys.* **B 202** (1982) 253 [INSPIRE].

[11] A.V. Smilga, *Weak supersymmetry*, *Phys. Lett.* **B 585** (2004) 173 [hep-th/0311023] [INSPIRE].

[12] S. Bellucci and A. Nersessian, *(Super)oscillator on* CP^N *and constant magnetic field*, *Phys. Rev.* **D 67** (2003) 065013 [*Erratum ibid.* **D 71** (2005) 089901] [hep-th/0211070] [INSPIRE].

[13] S. Bellucci and A. Nersessian, *Supersymmetric Kähler oscillator in a constant magnetic field*, hep-th/0401232 [INSPIRE].

[14] C. Romelsberger, *Counting chiral primaries in* $N=1$, $D=4$ *superconformal field theories*, *Nucl. Phys.* **B 747** (2006) 329 [hep-th/0510060] [INSPIRE].

[15] C. Romelsberger, *Calculating the Superconformal Index and Seiberg Duality*, arXiv:0707.3702 [INSPIRE].

[16] E. Ivanov and S. Sidorov, *Deformed Supersymmetric Mechanics*, *Class. Quant. Grav.* **31** (2014) 075013 [arXiv:1307.7690] [INSPIRE].

[17] E. Ivanov and S. Sidorov, *Super Kähler oscillator from* $\mathrm{SU}(2|1)$ *superspace*, *J. Phys.* **A 47** (2014) 292002 [arXiv:1312.6821] [INSPIRE].

[18] E. Ivanov, S. Sidorov and F. Toppan, *Superconformal mechanics in* $\mathrm{SU}(2|1)$ *superspace*, *Phys. Rev.* **D 91** (2015) 085032 [arXiv:1501.05622] [INSPIRE].

[19] E. Ivanov and S. Sidorov, $\mathrm{SU}(2|1)$ *mechanics and harmonic superspace*, *Class. Quant. Grav.* **33** (2016) 055001 [arXiv:1507.00987] [INSPIRE].

[20] E. Ivanov and O. Lechtenfeld, $N{=}4$ *supersymmetric mechanics in harmonic superspace*, *JHEP* **09** (2003) 073 [hep-th/0307111] [INSPIRE].

[21] S. Fedoruk, E. Ivanov and O. Lechtenfeld, *Supersymmetric Calogero models by gauging*, *Phys. Rev.* **D 79** (2009) 105015 [arXiv:0812.4276] [INSPIRE].

[22] S. Fedoruk, E. Ivanov and O. Lechtenfeld, *Superconformal Mechanics*, *J. Phys.* **A 45** (2012) 173001 [arXiv:1112.1947] [INSPIRE].

[23] F. Delduc and E. Ivanov, *Gauging* $N{=}4$ *Supersymmetric Mechanics*, *Nucl. Phys.* **B 753** (2006) 211 [hep-th/0605211] [INSPIRE].

[24] F. Delduc and E. Ivanov, *Gauging* $N{=}4$ *supersymmetric mechanics II:* $(1,4,3)$ *models from the* $(4,4,0)$ *ones*, *Nucl. Phys.* **B 770** (2007) 179 [hep-th/0611247] [INSPIRE].

[25] M. Konyushikhin and A.V. Smilga, *Self-duality and supersymmetry*, *Phys. Lett.* **B 689** (2010) 95 [arXiv:0910.5162] [INSPIRE].

[26] E.A. Ivanov, M.A. Konyushikhin and A.V. Smilga, *SQM with Non-Abelian Self-Dual Fields: Harmonic Superspace Description*, *JHEP* **05** (2010) 033 [arXiv:0912.3289] [INSPIRE].

[27] E. Ivanov and M. Konyushikhin, $N{=}4$, $3D$ *Supersymmetric Quantum Mechanics in Non-Abelian Monopole Background*, *Phys. Rev.* **D 82** (2010) 085014 [arXiv:1004.4597] [INSPIRE].

[28] D.-E. Diaconescu and R. Entin, *A nonrenormalization theorem for the D=1, N=8 vector multiplet*, Phys. Rev. **D 56** (1997) 8045 [hep-th/9706059] [INSPIRE].

[29] S. Bellucci, E. Ivanov, S. Krivonos and O. Lechtenfeld, *N=8 superconformal mechanics*, Nucl. Phys. **B 684** (2004) 321 [hep-th/0312322] [INSPIRE].

[30] E.A. Ivanov and A.V. Smilga, *Symplectic σ-models in superspace*, Nucl. Phys. **B 694** (2004) 473 [hep-th/0402041] [INSPIRE].

[31] S. Bellucci, E. Ivanov, S. Krivonos and O. Lechtenfeld, *ABC of N=8, D=1 supermultiplets*, Nucl. Phys. **B 699** (2004) 226 [hep-th/0406015] [INSPIRE].

[32] S. Bellucci, S. Krivonos and A. Nersessian, *N=8 supersymmetric mechanics on special Kähler manifolds*, Phys. Lett. **B 605** (2005) 181 [hep-th/0410029] [INSPIRE].

[33] S. Bellucci, E. Ivanov and A. Sutulin, *N=8 mechanics in* SU(2) × SU(2) *harmonic superspace*, Nucl. Phys. **B 722** (2005) 297 [*Erratum ibid.* **B 747** (2006) 464] [hep-th/0504185] [INSPIRE].

[34] E. Ivanov, O. Lechtenfeld and A. Sutulin, *Hierarchy of N=8 Mechanics Models*, Nucl. Phys. **B 790** (2008) 493 [arXiv:0705.3064] [INSPIRE].

[35] F. Delduc and E. Ivanov, *New Model of N=8 Superconformal Mechanics*, Phys. Lett. **B 654** (2007) 200 [arXiv:0706.2472] [INSPIRE].

[36] Z. Kuznetsova, M. Rojas and F. Toppan, *Classification of irreps and invariants of the N-extended supersymmetric quantum mechanics*, JHEP **03** (2006) 098 [hep-th/0511274] [INSPIRE].

[37] Z. Kuznetsova and F. Toppan, *D-module Representations of N=2, 4, 8 Superconformal Algebras and Their Superconformal Mechanics*, J. Math. Phys. **53** (2012) 043513 [arXiv:1112.0995] [INSPIRE].

[38] S. Khodaee and F. Toppan, *Critical scaling dimension of D-module representations of N = 4,7,8 Superconformal Algebras and constraints on Superconformal Mechanics*, J. Math. Phys. **53** (2012) 103518 [arXiv:1208.3612] [INSPIRE].

[39] M. Gonzales, S. Khodaee, O. Lechtenfeld and F. Toppan, *Target duality in N=8 superconformal mechanics and the coupling of dual pairs*, J. Math. Phys. **54** (2013) 072902 [arXiv:1303.6732] [INSPIRE].

[40] N. Dorey and A. Singleton, *Superconformal quantum mechanics and the discrete light-cone quantisation of $\mathcal{N}$=4 SUSY Yang-Mills*, JHEP **02** (2015) 067 [arXiv:1409.8440] [INSPIRE].

[41] N. Dorey and A. Singleton, *Instantons, Integrability and Discrete Light-Cone Quantisation*, arXiv:1412.5178 [INSPIRE].

[42] D.E. Berenstein, J.M. Maldacena and H.S. Nastase, *Strings in flat space and pp waves from N=4 super Yang-Mills*, JHEP **04** (2002) 013 [hep-th/0202021] [INSPIRE].

[43] K. Dasgupta, M.M. Sheikh-Jabbari and M. Van Raamsdonk, *Matrix perturbation theory for M-theory on a PP wave*, JHEP **05** (2002) 056 [hep-th/0205185] [INSPIRE].

[44] K. Dasgupta, M.M. Sheikh-Jabbari and M. Van Raamsdonk, *Protected multiplets of M-theory on a plane wave*, JHEP **09** (2002) 021 [hep-th/0207050] [INSPIRE].

[45] N. Kim and J.-H. Park, *Superalgebra for M-theory on a pp wave*, Phys. Rev. **D 66** (2002) 106007 [hep-th/0207061] [INSPIRE].

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[46] L. Motl, A. Neitzke and M.M. Sheikh-Jabbari, *Heterotic plane wave matrix models and giant gluons*, *JHEP* **06** (2003) 058 [hep-th/0306051] [INSPIRE].

[47] N. Kim and J.-H. Park, *Massive super Yang-Mills quantum mechanics: Classification and the relation to supermembrane*, *Nucl. Phys.* **B 759** (2006) 249 [hep-th/0607005] [INSPIRE].

[48] C.T. Asplund, F. Denef and E. Dzienkowski, *Massive quiver matrix models for massive charged particles in AdS*, *JHEP* **01** (2016) 055 [arXiv:1510.04398] [INSPIRE].

[49] E.A. Ivanov, S.O. Krivonos and A.I. Pashnev, *Partial supersymmetry breaking in $N=4$ supersymmetric quantum mechanics*, *Class. Quant. Grav.* **8** (1991) 19 [INSPIRE].

[50] A. Galperin, E. Ivanov, S. Kalitsyn, V. Ogievetsky and E. Sokatchev, *Unconstrained $N=2$ Matter, Yang-Mills and Supergravity Theories in Harmonic Superspace*, *Class. Quant. Grav.* **1** (1984) 469 [*Erratum ibid.* **2** (1985) 127] [INSPIRE].

[51] A.S. Galperin, E.A. Ivanov, V.I. Ogievetsky and E.S. Sokatchev, *Harmonic superspace*, Cambridge University Press (2001).

[52] A. Galperin, E. Ivanov and V. Ogievetsky, *Superspace actions and duality transformations for $N=2$ tensor multiplets*, *Phys. Scripta* **T15** (1987) 176.

[53] A.V. Smilga, *Supercharges in the hyper-Kähler with torsion supersymmetric σ-models*, *J. Math. Phys.* **53** (2012) 122105 [arXiv:1209.0539] [INSPIRE].

[54] N. Beisert, *The Analytic Bethe Ansatz for a Chain with Centrally Extended* SU(2|2) *Symmetry*, *J. Stat. Mech.* **01** (2007) P01017 [nlin/0610017] [INSPIRE].

[55] L. Frappat, P. Sorba and A. Sciarrino, *Dictionary on Lie superalgebras*, hep-th/9607161 [INSPIRE].

[56] A. Van Proeyen, *Tools for supersymmetry*, *Ann. U. Craiova Phys.* **9** (1999) 1 [hep-th/9910030] [INSPIRE].

[57] N.L. Holanda and F. Toppan, *Four types of (super)conformal mechanics: D-module reps and invariant actions*, *J. Math. Phys.* **55** (2014) 061703 [arXiv:1402.7298] [INSPIRE].

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