# Sasakian quiver gauge theories and instantons on cones over round and squashed seven-spheres

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## Abstract

We study quiver gauge theories on the round and squashed seven-spheres, and orbifolds thereof. They arise by imposing $G$-equivariance on the homogeneous space $G/H = \mathrm{SU}(4)/\mathrm{SU}(3)$ endowed with its Sasaki-Einstein structure, and $G/H = \mathrm{Sp}(2)/\mathrm{Sp}(1)$ as a 3-Sasakian manifold. In both cases we describe the equivariance conditions and the resulting quivers. We further study the moduli spaces of instantons on the metric cones over these spaces by using the known description for Hermitian Yang-Mills instantons on Calabi-Yau cones. It is shown that the moduli space of instantons on the hyper-Kähler cone can be described as the intersection of three Hermitian Yang-Mills moduli spaces. We also study moduli spaces of translationally invariant instantons on the metric cone $\mathbb{R}^8/\mathbb{Z}_k$ over $S^7/\mathbb{Z}_k$.

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## 1. Introduction

Equivariant dimensional reduction and quiver gauge theories in the sense of [1–5] have been extended recently from applications on Kähler coset spaces [6–8] to homogeneous Sasaki-Einstein manifolds. This *Sasakian quiver gauge theory* [9] is motivated by the close relation between Kähler and Sasakian geometry, as well as by the prominent role of Sasaki-Einstein manifolds in the AdS/CFT correspondence. The examples covered so far include the sphere $S^5$ (and orbifolds thereof) [10] and the conifold $T^{1,1}$ [11] in five dimensions, and the Aloff-Wallach space $X_{1,1}$ [12] in seven dimensions which also involves aspects of its 3-Sasakian structure. The aim of this paper is to compare Sasakian and 3-Sasakian quiver gauge theories on the round and squashed sphere $S^7$ in seven dimensions, realized respectively as the homogeneous spaces $SU(4)/SU(3)$ and $Sp(2)/Sp(1)$. Developing these two examples then essentially exhausts the list of physically interesting Sasakian quiver gauge theories.

As Sasaki-Einstein manifolds, the round and squashed seven-spheres naturally appear in the context of $AdS_4/CFT_3$ duality in M-theory. The effective $\mathcal{N} = 8$ supergravity theory on $AdS_4 \times S^7$ is broken to an $\mathcal{N} = 1$ theory when the usual round metric of $S^7$ is deformed to that of the squashed seven-sphere [13,14]. After orbifolding the seven-sphere by the cyclic group $\mathbb{Z}_k$, these backgrounds describe the near horizon geometry of coincident M2-branes situated at the conical singularity of an eight-dimensional cone $X^8$ over $S^7/\mathbb{Z}_k$. Their low energy effective worldvolume theories are three-dimensional superconformal Chern-Simons theories at level $k$, with $\mathcal{N} = 6$ supersymmetry and global R-symmetry group $SU(4)$ for the round metric on $S^7$ [15], and with $\mathcal{N} = 1$ supersymmetry and global R-symmetry group $Sp(2) \subset SU(4)$ for the squashed metric on $S^7$ [16]. In these theories the Chern-Simons gauge fields on $\mathbb{R}^3$ couple to scalar and spinor fields, so that our equivariant dimensional reductions over the eleven-dimensional space-times $\mathbb{R}^3 \times X^8$ lead to Sasakian quiver gauge theories whose Higgs branch moduli spaces could shed light on the generic vacuum structure of the possible low energy descriptions.

In the general context of dimensional reduction of gauge theories, a natural condition to impose on connections over internal homogeneous spaces $G/H$ is equivariance with respect to the group $G$; this is known as *equivariant dimensional reduction*. It has a natural close relation with the representation theory of quivers. Quiver gauge theories allow one to organise the physical degrees of freedom that are present in a chosen representation of the group $G$ inside the structure group in terms of directed graphs which represent the quivers. In this way, they take into account more general solutions to the equivariance condition than the scalar solution, used for instance in [17–19]. An equivariant connection is then characterized by a quiver for a chosen representation of $G$, and imposing instanton equations on this connection yields relations for the quiver and gradient flow equations. In this paper we shall follow the same route for the examples we consider: We will first discuss the equivariance conditions in detail and then describe the moduli spaces of instantons, which determine vacuum moduli spaces for the supersymmetric field theories discussed above.

This paper is organized as follows. In Section 2 we briefly review pertinent aspects of quiver gauge theories in the context of equivariant dimensional reduction and their relation to moduli spaces of instantons on homogeneous manifolds. The core of the present work is the description of Sasakian quiver gauge theory on the round seven-sphere in Section 3 and of 3-Sasakian quiver gauge theory on the squashed seven-sphere in Section 4. In both cases we describe the geometry of the coset spaces $G/H$, derive the equivariance conditions and illustrate them with explicit examples for some low-dimensional representations of $G$. Then we consider instantons on the metric cones, building on the general theory for Calabi-Yau cones, and show that the moduli

spaces of instantons on hyper-Kähler cones can be reduced to the intersection of a $\mathbb{C}P^1$-family of Hermitian Yang-Mills moduli spaces. In Section 5 we study moduli spaces of translationally invariant instantons on the metric cone $\mathbb{R}^8/\mathbb{Z}_k$ over $S^7/\mathbb{Z}_k$. Finally, in Section 6 we conclude with an overview of the main achievements of this paper. Two appendices at the end of the paper contain technical details about geometric structures, connections, and explicit representations of the Lie algebras of $G = \mathrm{SU}(4)$ and $G = \mathrm{Sp}(2)$ which are used throughout the main text.

## 2. Quiver gauge theory and equivariant dimensional reduction

In this section we briefly review some technical preliminaries that are needed in this paper. We begin by reviewing the theory of equivariant vector bundles and their relation to quiver representations, and then relate the equivariance condition to the conventional approach used in studies of connections on homogeneous spaces. We also discuss the generalized instanton equation.

### 2.1. Equivariant vector bundles

Let us begin with the basics of quiver gauge theory that we will apply in the remainder of this article. For details on the physical motivation and an outline of the construction, we refer to the reviews [2,5], whereas a rigorous mathematical account can be found for example in [20].

The general setup is that of a gauge theory on a product $M^d \times G/H$ of a $d$-dimensional Riemannian manifold $M^d$ and a homogeneous manifold $G/H$. The natural objects in geometric considerations of gauge theories are principal fibre bundles, but in this paper we will work with (associated) complex vector bundles; the formulation of equivariant dimensional reduction and the corresponding quiver gauge theories in the setting of principal bundles can be found in [21, 22]. Thus let $\pi : \mathcal{E} \to M^d \times G/H$ be a Hermitian vector bundle of rank $r$, and assume that the group $G$ acts trivially on the Riemannian manifold $M^d$. The bundle $\mathcal{E}$ is $G$-equivariant if the $G$-action on the base manifold and on $\mathcal{E}$ commute with the projection map $\pi$, and if it induces isomorphisms among the fibres. $G$-equivariant bundles $\mathcal{E} \to M^d \times G/H$ are in one-to-one correspondence with $H$-equivariant bundles $E \to M^d$, with the correspondence given by induction of vector bundles $\mathcal{E} = G \times_H E$ [20].

Since the subgroup $H$ acts trivially on the base space, the fibres $E_x \cong \mathbb{C}^r$ carry $H$-representations due to equivariance. We assume that this $H$-representation stems from a $G$-representation $\mathcal{D}$ which decomposes under restriction to $H$ as

$$\mathcal{D}\big|_H = \bigoplus_{j=0}^{m} \rho_j \tag{2.1}$$

into representations $\rho_j$ of $H$.[1] Then the structure group of the bundle is broken as

$$\mathrm{U}(r) \longrightarrow \prod_{j=0}^{m} \mathrm{U}(r_j) \qquad \text{with} \quad \sum_{j=0}^{m} r_j = r \ . \tag{2.2}$$

The bundle $E \to M^d$ decomposes in the same way under the action of $H$ as a Whitney sum and admits an isotopical decomposition

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[1] We will denote the representations of the corresponding Lie algebras with the same symbols.

$$E = \bigoplus_{j=0}^{m} E_j \otimes V_j , \tag{2.3}$$

where the vector space $V_j$ carries the representation $\rho_j$ and $H$ acts trivially on the vector bundles $E_j \to M^d$. The induction map $\mathcal{E} = G \times_H E$ yields an isotopical decomposition of the bundle $\mathcal{E}$ as well, where the action of the group $G$ then connects different summands of the decomposition (2.3) by bundle maps.

This induces a representation (in the category of vector bundles) of a quiver $\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1)$ [23]: For each representation $\rho_j$ one associates a vertex $v_j \in \mathcal{Q}_0$, representing a vector bundle $E_j$, and, if the $G$-action connects $\rho_j$ and $\rho_i$, an arrow $\phi_{ij} \in \mathcal{Q}_1$ between vertices $v_j$ and $v_i$ is associated, representing a homomorphism from $E_j$ to $E_i$; such an entity is sometimes also called a quiver bundle [20]. In this way, equivariant bundles over the homogeneous space $G/H$ correspond to linear quiver representations (in the category of vector spaces), and their construction reduces to studying representations of $G$ and their weight diagrams after suitably collapsing along the generators of the subgroup $H$.[2]

### 2.2. Instantons on homogeneous spaces

The condition giving rise to the quiver, which is referred to as the *equivariance condition*, ensures the invariance of gauge connections on coset spaces, and therefore naturally occurs when studying instantons over reductive homogeneous spaces $G/H$ where the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ of the Lie groups $G$ and $H$ decompose according to

$$\mathrm{span}\langle I_\mu \rangle = \mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h} = \mathrm{span}\langle I_a \rangle \oplus \mathrm{span}\langle I_j \rangle . \tag{2.4}$$

The typical approach we will apply here is to start from a generalized instanton and express the connection locally in terms of matrices. On Riemannian manifolds with Killing spinors, a connection $\Gamma$ is called a *generalized instanton* [24,25] if its curvature $\mathcal{F}_\Gamma = d\Gamma + \Gamma \wedge \Gamma$ satisfies [17]

$$\star \mathcal{F}_\Gamma = -\mathcal{F}_\Gamma \wedge \star Q , \tag{2.5}$$

where $Q$ is an invariant 4-form constructed as a bilinear in the Killing spinors. Solutions to this first-order equation also satisfy the second-order Yang-Mills equation. A potentially occurring torsion term vanishes for our cases of interest – Sasaki-Einstein and 3-Sasakian geometries – due to the properties of the Killing spinors in these instances. A special instanton solution is constructed in [17] which is based solely on the geometry induced by the Killing spinors. This instanton is referred to as the *canonical connection*, and its explicit expression is used below.

Given any instanton $\Gamma$ (not necessarily the canonical one) the general form of a gauge connection $\mathcal{A}$ on $M^d \times G/H$ (see e.g. [18,19,26]) then reads

$$\mathcal{A} = A + \Gamma + \sum_{a=1}^{\dim(\mathfrak{m})} X_a \otimes e^a , \tag{2.6}$$

where $A$ is a connection on the vector bundle $E$ over the Riemannian manifold $M^d$ and $\{e^a\}$ is a local frame on $T_e(G/H) \cong \mathfrak{m}$. For our highly symmetric cases, where the quotient of the

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[2]  This collapsing is nothing other than obtaining the $H$-representations $\rho_i$ from the weight diagram.

isometry group $G$ by the structure group $H$ of the principal bundle $G \to G/H$ coincides with the given realization as a homogeneous space $G/H$,[3] the canonical connection is simply given by

$$\Gamma = \sum_{j=1}^{\dim(\mathfrak{h})} I_j \otimes e^j \tag{2.7}$$

in (2.6).[4] To yield an invariant connection over the homogeneous space, the matrices $X_a$ must act on $I_i$ in the same way that the generators of $\mathfrak{g}$ do,

$$\left[ I_i, X_a \right] = f_{ia}^b \, X_b \,, \tag{2.8}$$

which means that no terms containing the mixed 2-forms $e^i \wedge e^a$ may appear in the curvature $\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$. This leads to the quivers discussed before with the bundle endomorphisms $X_a$ represented by the arrows. The Cartan generators contained in the subalgebra $\mathfrak{h}$ determine the shape of the quiver, once one has chosen a representation for them. Collapsing the weight diagram along the ladder operators of $\mathfrak{h}$, one obtains the decomposition (2.1), and the $G$-action then yields the arrows of the quiver. This procedure will be illustrated for the two examples $G = \mathrm{SU}(4)$ with $H = \mathrm{SU}(3)$ and $G = \mathrm{Sp}(2)$ with $H = \mathrm{Sp}(1)$ in the remainder of this paper.

Because (2.6) with (2.8) only ensure the equivariance of the connection, one still has to impose the instanton equation on the curvature. On cones over Sasaki-Einstein manifolds, one can use the Hermitian Yang-Mills equation to impose the instanton condition. This will be exploited later on.

The equivariance condition determines the general block form of the matrices $X_a$, expressing the gauge connection, which induces the quivers. The homomorphisms from $\mathrm{Hom}(E_j, E_i)$ as entries in these matrices, or equivalently as arrows of the quivers, are restricted by relations induced by the instanton equation. If one considers only rank one vector bundles $E_i$ and takes all functions occurring in each $X_a$ to be the same, then one obtains the scalar solution $X_a = \lambda_a(x) \, I_a$ which has been previously used for explicit constructions [17,27].

## 3. Sasakian quiver gauge theory on the round seven-sphere

In this section we consider quiver gauge theory on $S^7 \cong \mathrm{SU}(4)/\mathrm{SU}(3)$, regarded as a Sasaki-Einstein manifold. Since the canonical connection, the structure equations and the instanton equations of any particular odd-dimensional sphere can be easily generalized to all odd-dimensional spheres, the exposition will closely follow that of the five-sphere in [10]. We start by describing the geometry of the homogeneous space and orbifolds thereof, including the canonical connection with respect to the Sasaki-Einstein structure. We then use this to derive the general form for the equivariant connection and provide some explicit examples of the quivers induced by the equivariance condition. We conclude by formulating the Hermitian Yang-Mills equation on the metric cone and describing the moduli space of solutions.

### 3.1. Geometry of $S^7 \cong \mathrm{SU}(4)/\mathrm{SU}(3)$

*Local section*  The geometric description of $S^7 \cong \mathrm{SU}(4)/\mathrm{SU}(3)$ used in this paper is based on its realization as a circle bundle over the Kähler 3-fold $\mathbb{C}P^3$ via a commuting diagram of fibrations

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[3]  This is the guiding principle of the construction in [19].

[4]  Notice that the generators of $H$ as well as the connection $A$ on $M^d$ have a block diagonal form with respect to the isotopical decomposition. Hence they commute when evaluating the curvature $\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$.

$$\begin{array}{ccc}
\mathrm{SU}(4) & \xrightarrow{\;\;\mathrm{SU}(3)\;\;} & S^7 \\[2pt]
 & {}^{S(U(3)\times U(1))}\searrow & \big\downarrow \mathrm{U}(1) \\[2pt]
 & & \mathbb{C}P^3
\end{array}
\tag{3.1}$$

This allows for the introduction of local coordinates on $S^7$ by considering a section of the bundle $\mathrm{SU}(4) \to \mathbb{C}P^3$ in the following way, which is analogous to the procedure employed in [7,10]. In a patch $\mathcal{U}_0 := \big\{ [z^0 : z^1 : z^2 : z^3] \in \mathbb{C}P^3 \mid z^0 \neq 0 \big\} \subset \mathbb{C}P^3$, one defines

$$Y = (y^1, y^2, y^3)^\top := \left( \frac{z^1}{z^0}, \frac{z^2}{z^0}, \frac{z^3}{z^0} \right)^\top
\tag{3.2}$$

and the complex $4 \times 4$ matrix

$$V := \frac{1}{\gamma} \begin{pmatrix} 1 & Y^\dagger \\ -Y & \Lambda \end{pmatrix}
\quad \text{with} \quad \Lambda := \gamma\, \mathbb{1}_3 - \frac{1}{1+\gamma} Y Y^\dagger
\quad \text{and} \quad \gamma := \sqrt{1 + Y^\dagger Y}.
\tag{3.3}$$

By definition, they have the properties

$$\Lambda Y = Y, \quad Y^\dagger \Lambda = Y^\dagger \quad \text{and} \quad \Lambda^2 = \gamma^2\, \mathbb{1}_3 - Y Y^\dagger,
\tag{3.4}$$

so that $V$ indeed is an element of $\mathrm{SU}(4)$, i.e. $V^\dagger V = V V^\dagger = \mathbb{1}_4$. Hence the matrix $V$ is a local section of the bundle $\mathrm{SU}(4) \to \mathbb{C}P^3$. The Maurer-Cartan form $A_0 := V^{-1}\,\mathrm{d}V$ provides left $\mathrm{SU}(4)$-invariant 1-forms on $\mathbb{C}P^3$:

$$A_0 = V^\dagger\,\mathrm{d}V =: \begin{pmatrix} -3a & \beta^\dagger \\ -\beta & B \end{pmatrix}
\quad \text{with} \quad a = -\frac{1}{2\gamma^2} \left( Y^\dagger\,\mathrm{d}Y - \mathrm{d}Y^\dagger\, Y \right),
\tag{3.5}$$

$$\beta = \frac{1}{\gamma^2} \Lambda\,\mathrm{d}Y \quad \text{and} \quad
B = \frac{1}{\gamma^2} \left( Y\,\mathrm{d}Y^\dagger + \Lambda\,\mathrm{d}\Lambda - \frac{1}{2}\,\mathrm{d}(Y^\dagger Y)\,\mathbb{1}_3 \right).$$

The flatness of the connection $A_0$ yields the equations

$$3\,\mathrm{d}a = -\beta^\dagger \wedge \beta = \sum_{\alpha=1}^{3} \beta^\alpha \wedge \bar\beta^{\bar\alpha}, \quad
\mathrm{d}\beta = -3a \wedge \beta - B \wedge \beta \quad \text{and} \quad
\mathrm{d}B = \beta \wedge \beta^\dagger - B \wedge B.
\tag{3.6}$$

A section of the $\mathrm{SU}(3)$-bundle $\mathrm{SU}(4) \to S^7$ can now be obtained by including the additional $\mathrm{U}(1)$ factor in the fibration (3.1) as

$$S^7 \ni \left( y^1, y^2, y^3, \phi \right) \longmapsto \tilde V := V\,\mathrm{diag}\big( \mathrm{e}^{3\mathrm{i}\phi}, \mathrm{e}^{-\mathrm{i}\phi}, \mathrm{e}^{-\mathrm{i}\phi}, \mathrm{e}^{-\mathrm{i}\phi} \big).
\tag{3.7}$$

The corresponding canonical flat connection $\tilde A_0 := \tilde V^\dagger\,\mathrm{d}\tilde V$ reads

$$\tilde A_0 = \begin{pmatrix} -3a + 3\mathrm{i}\,\mathrm{d}\phi & \mathrm{e}^{-4\mathrm{i}\phi}\,\beta^\dagger \\ -\beta\,\mathrm{e}^{4\mathrm{i}\phi} & B - \mathrm{i}\,\mathrm{d}\phi\,\mathbb{1}_3 \end{pmatrix}$$

$$=: \begin{pmatrix}
3\mathrm{i}\,\mu_7\,e^7 & \zeta_1\,\Theta^1 & \zeta_2\,\Theta^2 & \zeta_3\,\Theta^3 \\
-\zeta_1\,\bar\Theta^{\bar 1} & -\mathrm{i}\,\mu_7\,e^7 + 2\mathrm{i}\,\mu_8\,e^8 & \lambda_4\,\Theta^4 & \lambda_5\,\Theta^5 \\
-\zeta_2\,\bar\Theta^{\bar 2} & -\lambda_4\,\bar\Theta^{\bar 4} & -\mathrm{i}\,\mu_7\,e^7 - \mathrm{i}\,\mu_8\,e^8 - \mathrm{i}\,\mu_9\,e^9 & \lambda_6\,\Theta^6 \\
-\zeta_3\,\bar\Theta^{\bar 3} & -\lambda_5\,\bar\Theta^{\bar 5} & \lambda_6\,\bar\Theta^{\bar 6} & -\mathrm{i}\,\mu_7\,e^7 - \mathrm{i}\,\mu_8\,e^8 + \mathrm{i}\,\mu_9\,e^9
\end{pmatrix}
\tag{3.8}$$

which defines left SU(4)-invariant 1-forms and the corresponding generators of SU(4) in the fundamental representation. We also introduce real 1-forms

$$e^{2\alpha-1} - \mathrm{i}\, e^{2\alpha} := \Theta^{\alpha} \qquad \text{for} \quad \alpha = 1, 2, 3 \,, \tag{3.9}$$

and an orthonormal frame metric

$$\mathrm{d}s^2 = \sum_{\mu=1}^{7} e^{\mu} \otimes e^{\mu} \tag{3.10}$$

on $T_e S^7 \cong \mathrm{span}\langle e^1, \dots, e^7 \rangle \cong \mathfrak{m}$, where the generators of the reductive homogeneous space split according to

$$\mathfrak{su}(4) = \mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h} \cong T_e S^7 \oplus \mathfrak{su}(3) \tag{3.11}$$

with respect to a left-invariant metric. The real parameters $\mu_i$ and $\zeta_i$ in the definition of the 1-forms (3.8) will be fixed by the condition that (3.10) defines a Sasaki-Einstein metric.

*Sasaki-Einstein geometry*   Among the available equivalent definitions of a Sasaki-Einstein manifold, we use the one that declares the Riemannian manifold $S^7$ to be Sasaki-Einstein if its eight-dimensional metric cone $(\mathbb{R}^+ \times S^7, g_{\mathrm{cone}})$ is Calabi-Yau, i.e. a Ricci-flat Kähler manifold, with the cone metric[5]

$$\mathrm{d}s^2_{\mathrm{cone}} = r^2\, \mathrm{d}s^2 + \mathrm{d}r \otimes \mathrm{d}r = r^2 \left( \mathrm{d}s^2 + \frac{\mathrm{d}r}{r} \otimes \frac{\mathrm{d}r}{r} \right) = r^2\, \mathrm{d}s^2_{\mathrm{cyl}} \,. \tag{3.12}$$

One introduces a complex structure $J$ by declaring the 1-forms $\Theta^{\alpha}$ for $\alpha = 1, 2, 3$ and the 1-form

$$\Theta^0 := \frac{\mathrm{d}r}{r} - \mathrm{i}\, e^7 =: e^{\tau} - \mathrm{i}\, e^7 \tag{3.13}$$

to be holomorphic, $J\Theta^{\alpha} = \mathrm{i}\,\Theta^{\alpha}$. In terms of these 1-forms, the cone metric reads

$$\mathrm{d}s^2_{\mathrm{cone}} = r^2 \sum_{\alpha=0}^{3} \Theta^{\alpha} \otimes \bar{\Theta}^{\bar{\alpha}} \tag{3.14}$$

and it yields the Kähler form

$$\Omega^{1,1} := -\tfrac{\mathrm{i}}{2}\, r^2 \left( \Theta^{0\bar{0}} + \Theta^{1\bar{1}} + \Theta^{2\bar{2}} + \Theta^{3\bar{3}} \right) =: -\tfrac{\mathrm{i}}{2}\, r^2\, \Theta^{0\bar{0}} + \omega \,, \tag{3.15}$$

where we generally denote $\Theta^{\alpha\bar{\beta}\gamma\cdots} = \Theta^{\alpha} \wedge \bar{\Theta}^{\bar{\beta}} \wedge \Theta^{\gamma} \wedge \cdots$, etc. Using the structure equations (A.2) induced by flatness of the connection (3.8), one shows that the closure of this form requires

$$\zeta := \zeta_1 = \zeta_2 = \zeta_3 \qquad \text{and} \qquad \mu_7 = \tfrac{1}{3}\,\zeta^2 \,. \tag{3.16}$$

Furthermore, in order for the cone to be Calabi-Yau, the holomorphic 4-form

$$\Omega^{4,0} := r^4\, \Theta^1 \wedge \Theta^2 \wedge \Theta^3 \wedge \Theta^0 \tag{3.17}$$

must be closed. This condition yields

$$\zeta^2 = 1 \qquad \text{and} \qquad \mu_7 = \tfrac{1}{3} \,, \tag{3.18}$$

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[5] For explicit calculations we will use the conformally equivalent cylinder metric $\mathrm{d}s^2_{\mathrm{cyl}}$.

and for definiteness we fix the undetermined parameters $\lambda_i = 1$ for $i = 4, 5, 6$, $\mu_8 = \frac{1}{6}$ and $\mu_9 = \frac{1}{2}$. This choice leads to the structure equations

$$d\Theta^1 = -\tfrac{4}{3}\,\mathrm{i}\,e^7 \wedge \Theta^1 + \tfrac{1}{3}\,\mathrm{i}\,e^8 \wedge \Theta^1 + \Theta^{2\bar{4}} + \Theta^{3\bar{5}}\,,$$

$$d\Theta^2 = -\tfrac{4}{3}\,\mathrm{i}\,e^7 \wedge \Theta^2 - \tfrac{1}{6}\,\mathrm{i}\,e^8 \wedge \Theta^2 - \tfrac{1}{2}\,\mathrm{i}\,e^9 \wedge \Theta^2 - \Theta^{14} + \Theta^{3\bar{6}}\,,$$

$$d\Theta^3 = -\tfrac{4}{3}\,\mathrm{i}\,e^7 \wedge \Theta^3 - \tfrac{1}{6}\,\mathrm{i}\,e^8 \wedge \Theta^3 + \tfrac{1}{2}\,\mathrm{i}\,e^9 \wedge \Theta^3 - \Theta^{15} - \Theta^{26}\,,$$

$$de^7 = -\,\mathrm{i}\left(\Theta^{1\bar{1}} + \Theta^{2\bar{2}} + \Theta^{3\bar{3}}\right) = 2\,\omega\,. \tag{3.19}$$

*Canonical connection*   According to the general construction in [17], the Sasaki-Einstein metric provides an instanton solution. From the last structure equation in (3.19) we see that, as expected, the 1-form $\eta := e^7$ is the contact form dual to the Reeb vector field of the circle fibration, and $\omega$ is the Kähler form of the leaf space underlying the Sasaki-Einstein structure. The *canonical connection* of such a structure, in the sense of [17], is determined by the 3-form

$$P := \eta \wedge \omega = \tfrac{1}{2}\,\eta \wedge d\eta = e^7 \wedge \left(e^{12} + e^{34} + e^{56}\right)\,, \tag{3.20}$$

where $e^{\mu\nu\cdots} = e^\mu \wedge e^\nu \wedge \cdots$, and the torsion

$$T^7 = P_{7\mu\nu}\,e^{\mu\nu} \qquad \text{and} \qquad T^a = \tfrac{2}{3}\,P_{a\mu\nu}\,e^{\mu\nu} \qquad \text{for} \quad a = 1, \ldots, 6\,. \tag{3.21}$$

With these torsion components, the structure equations take the form

$$d\Theta^1 = -2\,\Theta^1 \wedge \mathrm{i}\,e^8 + \Theta^{2\bar{4}} + \Theta^{3\bar{5}} + \left(T^1 - \mathrm{i}\,T^2\right)\,,$$

$$d\Theta^2 = \Theta^2 \wedge \mathrm{i}\,e^8 + \Theta^2 \wedge \mathrm{i}\,e^9 - \Theta^{14} + \Theta^{3\bar{6}} + \left(T^3 - \mathrm{i}\,T^4\right)\,,$$

$$d\Theta^3 = \Theta^3 \wedge \mathrm{i}\,e^8 - \Theta^3 \wedge \mathrm{i}\,e^9 - \Theta^{15} - \Theta^{26} + \left(T^5 - \mathrm{i}\,T^6\right)\,,$$

$$de^7 = T^7\,. \tag{3.22}$$

By writing $de^\mu = -\Gamma^\mu_\nu \wedge e^\nu + T^\mu$ one obtains the connection matrix

$$\Gamma = \begin{pmatrix} -2\,\mathrm{i}\,e^8 & \bar{\Theta}^{\bar{4}} & \bar{\Theta}^{\bar{5}} & 0 \\ -\Theta^4 & \mathrm{i}\,e^8 + \mathrm{i}\,e^9 & \bar{\Theta}^{\bar{6}} & 0 \\ -\Theta^5 & -\Theta^6 & \mathrm{i}\,e^8 - \mathrm{i}\,e^9 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\,. \tag{3.23}$$

Consequently, the canonical connection of the Sasaki-Einstein structure on $S^7$ is given by[6]

$$\Gamma = I_8\,e^8 + I_9\,e^9 + I_4^+\,\Theta^4 + I_5^+\,\Theta^5 + I_6^+\,\Theta^6 + I_{\bar{4}}^-\,\bar{\Theta}^{\bar{4}} + I_{\bar{5}}^-\,\bar{\Theta}^{\bar{5}} + I_{\bar{6}}^-\,\bar{\Theta}^{\bar{6}} =: \sum_{\mu=8}^{15} \hat{I}_\mu\,e^\mu\,. \tag{3.24}$$

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[6] This connection coincides with the one obtained by declaring the torsion to be given by $T(X, Y) = -[X, Y]_{\mathrm{m}}$ for vector fields $X, Y \in T S^7$.

The canonical connection involves all the generators of $\mathfrak{h} = \mathfrak{su}(3)$ in this way because the realization of $S^7$ as the homogeneous space $\mathrm{SU}(4)/\mathrm{SU}(3)$ is isomorphic to the quotient of its isometry group $\mathrm{SU}(4)$ by the structure group $\mathrm{SU}(3)$ of the principal bundle $\mathrm{SU}(4) \to S^7$, as used in the general construction of [19]. It may be compared to the more involved situation on the Aloff-Wallach space $X_{1,1}$ [12], where the representation as a coset space is related to the 3-Sasakian structure instead.

The curvature $\mathcal{F}_\Gamma = \mathrm{d}\Gamma + \Gamma \wedge \Gamma$ of the canonical connection is given by

$$\mathcal{F}_\Gamma = I_4^+ \, \Theta^{\bar{1}2} + I_5^+ \, \Theta^{\bar{1}3} + I_6^+ \, \Theta^{\bar{2}3} + I_4^- \, \Theta^{12} + I_5^- \, \Theta^{13} + I_6^- \, \Theta^{23} \tag{3.25}$$
$$+ I_8 \left( 2\,\mathrm{i}\,\Theta^{1\bar{1}} - \mathrm{i}\,\Theta^{2\bar{2}} - \mathrm{i}\,\Theta^{3\bar{3}} \right) + I_9 \left( -\,\mathrm{i}\,\Theta^{2\bar{2}} + \mathrm{i}\,\Theta^{3\bar{3}} \right)$$

and it indeed solves the instanton equation (2.5) with the 4-form $Q$ defined as [17]

$$Q := \tfrac{1}{2}\,\omega \wedge \omega = -\tfrac{1}{4} \left( \Theta^{1\bar{1}2\bar{2}} + \Theta^{1\bar{1}3\bar{3}} + \Theta^{2\bar{2}3\bar{3}} \right) = e^{1234} + e^{1256} + e^{3456} \,. \tag{3.26}$$

By the geometric construction using the Killing spinor equations, such an instanton solution implies the usual torsion-free Yang-Mills equation. In the following, we will study connections and instanton solutions based on this canonical connection.

*Orbifolds*  We conclude by briefly describing the corresponding geometry of the orbifold $S^7/\mathbb{Z}_k$, closely following the treatment of [10]. For compatibility with the bundle structure of the homogeneous space $S^7 = \mathrm{SU}(4)/\mathrm{SU}(3)$, the cyclic group $\mathbb{Z}_k$ is embedded in $G = \mathrm{SU}(4)$ in a way that it commutes with $H = \mathrm{SU}(3)$, i.e. the action of $\mathbb{Z}_k$ is embedded in the U(1)-factor associated to the contact direction generated by $I_7$. Hence we modify the section (3.7) to

$$S^7/\mathbb{Z}_k \ni \left( y^1, y^2, y^3, \phi/k \right) \longmapsto V' := V \operatorname{diag}\!\left( e^{3\,\mathrm{i}\,\phi/k}, e^{-\mathrm{i}\,\phi/k}\,\mathbb{1}_3 \right) . \tag{3.27}$$

As $\mathbb{Z}_k$-action on the coordinates of $\mathbb{C}^4$, we use

$$h_k \cdot z := \operatorname{diag}\!\left( \zeta_k^3, \zeta_k^{-1}\,\mathbb{1}_3 \right) z \qquad \text{with} \quad \zeta_k := e^{2\pi\,\mathrm{i}/k} \,, \tag{3.28}$$

where $h_k$ is a generator of $\mathbb{Z}_k$ and $z \in \mathbb{C}^4$. Recalling the definition (3.2) of the local coordinates $y^\alpha$ based on a quotient of $\mathbb{C}^4$ in a local patch, the action of $\mathbb{Z}_k$ on them is given by

$$y^\alpha \longmapsto \frac{\zeta_k^{-1}\,z^\alpha}{\zeta_k^3\,z^0} = \zeta_k^{-4}\,y^\alpha \,, \qquad \bar{y}^{\bar\alpha} \longmapsto \frac{\zeta_k\,\bar{z}^{\bar\alpha}}{\zeta_k^{-3}\,\bar{z}^0} = \zeta_k^4\,\bar{y}^{\bar\alpha} \qquad \text{and} \qquad e^7 \longmapsto e^7 \tag{3.29}$$

with $\alpha = 1, 2, 3$; for details see [10]. The local section (3.27) provides the very same structure equations as those of $S^7$, with the replacement $\phi \to \phi_k := \phi/k$ (and correspondingly for the dual 1-form $\eta$). In particular, they still define a Sasaki-Einstein manifold, and – with a slight abuse of notation – we will use the same symbols as before also for the orbifold case.

## 3.2. Quivers

The equivariance condition (2.8) enables us to depict the allowed endomorphisms as arrows in a quiver, starting from the weight diagram of chosen SU(4)-representations. In the following we will consider five explicit examples of this construction and elaborate on some generic features of Sasakian quiver gauge theories on odd-dimensional spheres.

*Fundamental representation* **4** The weight diagram of the fundamental representation **4** of SU(4) is the tetrahedron (A.11) consisting of the four vertices $(3, 0, 0)$, $(-1, 2, 0)$, $(-1, -1, -1)$ and $(-1, -1, 1)$. Under restriction to SU(3) it decomposes into the trivial representation and the fundamental representation of SU(3),

$$\mathbf{4}\big|_{\mathrm{SU}(3)} = \underline{(\mathbf{3, 0, 0})_{\mathbf{1}}} \oplus \underline{(\mathbf{-1, -1, 1})_{\mathbf{3}}} , \tag{3.30}$$

where the subscripts indicate the dimension of the representation and the triples label the quantum numbers of the highest weight states. Collapsing the weight diagram along the ladder operators of SU(3) in this way and implementing the equivariance conditions yields the quiver



$$\tag{3.31}$$

Defining $\phi_{(\alpha)} := X_{2\alpha-1} + \mathrm{i}\, X_{2\alpha}$ for $\alpha = 1, 2, 3$, the endomorphism part of the gauge connection $\mathcal{A}$ from (2.6) reads

$$\phi_{(\alpha)} = \begin{pmatrix} 0 & 0 \\ \phi \otimes I_\alpha & \mathbf{0}_3 \end{pmatrix} \qquad \text{with} \quad I_1 := \big(1, 0, 0\big)^\top , \ \ I_2 := \big(0, 1, 0\big)^\top , \ \ I_3 := \big(0, 0, 1\big)^\top \tag{3.32}$$

and

$$X_7 = \begin{pmatrix} \psi_3 & 0 \\ 0 & \psi_{-1} \otimes \mathbb{1}_3 \end{pmatrix} . \tag{3.33}$$

Here the constant matrices $I_\alpha$ stem from the collapsing of the weight diagram along the subalgebra and realize the part living on the representation spaces $V_j$ of the isotopical decomposition (2.3). The homomorphisms $\phi$, $\psi_3$ and $\psi_{-1}$, which are represented by the arrows $\mathcal{Q}_1$ of the quiver, are morphisms between the vector bundles $E_j$ attached to the vertices $\mathcal{Q}_0$.

The quiver (3.31) is precisely the higher-dimensional analogue of the corresponding quiver in the case of the five-sphere $S^5$ [10]. Due to the straightforward generalization of sections of $S^{2n+1} \cong \mathrm{SU}(n+1)/\mathrm{SU}(n)$ for all $n \geq 2$ and the fact that the fundamental representation $\boldsymbol{n+1}$ of SU($n+1$) splits under restriction to SU($n$) into the fundamental representation and the trivial representation,

$$\boldsymbol{n+1}\big|_{\mathrm{SU}(n)} = \boldsymbol{n} \oplus \boldsymbol{1} , \tag{3.34}$$

the quiver (3.31) will govern the solutions of the equivariance conditions on all odd-dimensional spheres $S^{2n+1}$ for $n \geq 2$.

*Representation* **6** Due to the accidental isomorphism[7] Spin(6) $\cong$ SU(4), there is also an irreducible six-dimensional representation **6** of SU(4). Its weight diagram is the octahedron (A.16), and the representation decomposes under restriction into the fundamental and anti-fundamental representation of SU(3),

$$\mathbf{6}\big|_{\mathrm{SU}(3)} = \underline{(\mathbf{2, -1, 1})_{\mathbf{3}}} \oplus \underline{(\mathbf{-2, -2, 0})_{\mathbf{3}}} . \tag{3.35}$$

---

[7] The Dynkin diagrams $A_3$ and $D_3$ coincide. For the representation theory of SU(4) and SL(4, $\mathbb{C}$), respectively, see for instance [28].

This symmetric splitting yields the quiver

$$
\begin{array}{ccc}
\psi_{-2} & & \psi_{2} \\
\curvearrowright & & \curvearrowright \\
(-2,-2,0) & \xleftarrow{\quad \phi \quad} & (2,-1,1)
\end{array}
\tag{3.36}
$$

The endomorphisms are given as

$$
\phi_{(\alpha)} = \begin{pmatrix} \mathbf{0}_3 & \mathbf{0}_3 \\ \phi \otimes I_\alpha & \mathbf{0}_3 \end{pmatrix}
$$

$$
\text{with } I_1 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad I_2 := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad I_3 := \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\tag{3.37}
$$

and

$$
X_7 = \begin{pmatrix} \psi_2 \otimes \mathbb{1}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \psi_{-2} \otimes \mathbb{1}_3 \end{pmatrix} .
\tag{3.38}
$$

*Representation* **10**   The ten-dimensional representation of SU(4) can be realized by the generators (A.18), and its weight diagram is a tetrahedron consisting of three layers, according to the SU(3) decomposition

$$
\mathbf{10}\big|_{\text{SU}(3)} = \underline{(-2,-2,2)_6} \oplus \underline{(2,-1,1)_3} \oplus \underline{(6,0,0)_1} .
\tag{3.39}
$$

The quiver therefore has three vertices

$$
\begin{array}{ccccc}
\psi_{-2} & & \psi_{2} & & \psi_{6} \\
\curvearrowright & & \curvearrowright & & \curvearrowright \\
(-2,-2,2) & \xleftarrow{\quad \phi_2 \quad} & (2,-1,1) & \xleftarrow{\quad \phi_1 \quad} & (6,0,0)
\end{array}
\tag{3.40}
$$

and the Higgs fields read

$$
X_7 = \begin{pmatrix} \psi_6 & 0 & 0 \\ 0 & \psi_2 \otimes \mathbb{1}_3 & 0 \\ 0 & 0 & \psi_{-2} \otimes \mathbb{1}_6 \end{pmatrix} \quad \text{and} \quad \phi_{(\alpha)} = \begin{pmatrix} 0 & 0 & 0 \\ \phi_1 \otimes I_\alpha^1 & \mathbf{0}_3 & 0 \\ 0 & \phi_2 \otimes I_\alpha^2 & \mathbf{0}_6 \end{pmatrix},
\tag{3.41}
$$

with $I_\alpha^1$ as in (3.32) and $I_\alpha^2$ given by

$$
I_1^2 = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad I_2^2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad I_3^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} .
\tag{3.42}
$$

*Adjoint representation* **15**    Implementing the equivariance conditions in the 15-dimensional adjoint representation transforms the weight diagram (A.20) into the quiver



$$\tag{3.43}$$
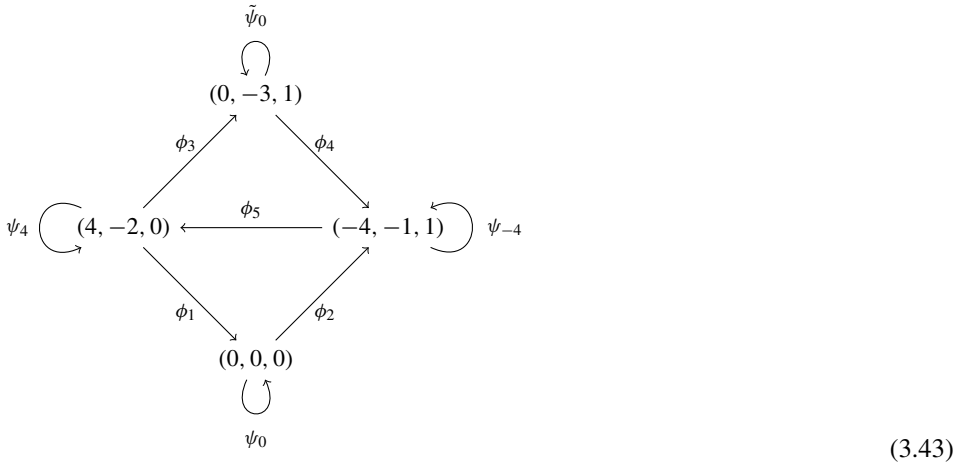
The Higgs fields are given by

$$
X_7 = \begin{pmatrix} \psi_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \psi_4 \otimes \mathbb{1}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \psi_{-4} \otimes \mathbb{1}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \tilde{\psi}_0 \otimes \mathbb{1}_3 \end{pmatrix} ,
$$

$$
\phi_{(\alpha)} = \begin{pmatrix} 0 & \phi_1 \otimes I_\alpha^1 & 0 & 0 \\ 0 & 0 & \phi_5 \otimes I_\alpha^5 & 0 \\ \phi_2 \otimes I_\alpha^2 & 0 & 0 & \phi_4 \otimes I_\alpha^4 \\ 0 & \phi_3 \otimes I_\alpha^3 & 0 & 0 \end{pmatrix}
$$

$$\tag{3.44}$$

where the explicit forms of the matrices $I_\alpha^j$ follow from collapsing the weight diagram along the generators of SU(3).

The quiver associated to the adjoint representation also allows for an immediate generalization to higher-dimensional spheres: The adjoint representation $\boldsymbol{(n{+}1)^2{-}1}$ of SU($n{+}1$) decomposes under restriction to SU($n$) into the adjoint, fundamental, antifundamental, and trivial representation as

$$
\boldsymbol{(n{+}1)^2{-}1}\big|_{\mathrm{SU}(n)} = \boldsymbol{n^2{-}1} \ \oplus \ \boldsymbol{n} \ \oplus \ \boldsymbol{\bar{n}} \ \oplus \ \boldsymbol{1} . \tag{3.45}
$$

Thus, after collapsing along the ladder operators of the subalgebra $\mathfrak{su}(n)$, one obtains a quiver consisting of four vertices.

*Representation* **6** $\oplus$ **4**    Our last example stems from a *reducible* representation of $G$. Since the arrows of the quiver depend only on the equivariance relations with respect to the Cartan generators of the subalgebra $\mathfrak{h}$, the Higgs fields might have more entries than the corresponding ladder operators; see for instance the morphism between fundamental and antifundamental representations in the previous example of the adjoint representation. For reducible representations, one can expect this effect to be even more prominent.

Let us consider as example the direct sum of the SU(4)-respresentations **6** and **4**. The quiver contains the two individual quivers (3.31) and (3.36), but there can also be morphisms between them for the case of $S^7$ (without orbifolding):



$$(3.46)$$

Anticipating Section 3.4, note that the homomorphisms $\phi_3$, $\phi_4$, $\phi_5$ and $\chi$, which are additional degrees of freedom to those anticipated from the shape of the ladder operators, have to vanish if one imposes also equivariance with respect to the finite subgroup in the orbifold case. Then the quiver decomposes into the direct sum of the relevant quivers (3.31) and (3.36), as is to be expected from the existence of invariant subspaces in a reducible $G$-representation.

### 3.3. Yang-Mills-Higgs theories

Having seen how to obtain an SU(4)-equivariant gauge connection from graphical techniques on the weight diagrams of the Lie algebra $\mathfrak{su}(4)$, we now derive the dimensional reduction of the Yang-Mills action

$$S_{\mathrm{YM}} := -\frac{1}{4} \int_{M^d \times S^7} \mathrm{tr}(\mathcal{F} \wedge \star \mathcal{F}) \tag{3.47}$$

for the connection (2.6) over $M^d \times S^7$ to a Yang-Mills-Higgs theory on the manifold $M^d$. Using the Sasaki-Einstein metric (3.10), one obtains for the Lagrangian[8]

$$\begin{aligned}
\mathcal{L}_{\mathrm{YM}} &= -\tfrac{1}{4}\sqrt{\hat{g}}\,\mathrm{tr}\!\left(\mathcal{F}_{\hat{\mu}\hat{v}}\,\mathcal{F}^{\hat{\mu}\hat{v}}\right) \\
&= -\tfrac{1}{4}\sqrt{\hat{g}}\,\mathrm{tr}\!\left(\mathcal{F}_{\mu v}\,\mathcal{F}^{\mu v} + 8\,g^{\mu v}\,\mathcal{F}_{\alpha\mu}\,\mathcal{F}_{v\bar{\alpha}} + 2\,g^{\mu v}\,\mathcal{F}_{\mu 7}\,\mathcal{F}_{v 7} \right. \\
&\qquad\qquad \left. + 8\,\mathcal{F}_{\alpha\beta}\,\mathcal{F}_{\bar{\alpha}\bar{\beta}} + 8\,\mathcal{F}_{\alpha\bar{\beta}}\,\mathcal{F}_{\bar{\alpha}\beta} + 8\,\mathcal{F}_{\alpha 7}\,\mathcal{F}_{\bar{\alpha} 7}\right).
\end{aligned} \tag{3.48}$$

Inserting the non-vanishing components of the curvature and writing $|X|^2 := X\,X^\dagger$, one ends up with the result

$$S_{\mathrm{YM}} = \mathrm{vol}(S^7) \int_{M^d} \mathrm{d}^d y\,\sqrt{g}\,\mathrm{tr}\!\left(\frac{1}{4}\,F_{\mu v}\left(F^{\mu v}\right)^\dagger + 2\sum_{\mu=1}^{d}\sum_{\alpha=1}^{3}\left|D_\mu\phi_{(\alpha)}\right|^2 + \frac{1}{2}\sum_{\mu=1}^{d}\left|D_\mu X_7\right|^2\right) \tag{3.49}$$

---

[8] Here we denote by $\hat{\mu}, \hat{v}, \dots$ generic directions along $M \times G/H$, with $\mu, v, \dots$ directions along $M$ and $\alpha, \beta, \dots$ directions along the coset.

$$+ 2 \sum_{\alpha=1}^{3} \left| [X_7, \phi_{(\alpha)}] + \frac{3}{4} i \, \phi_{(\alpha)} \right|^2 + 4 \left| [\phi_{(1)}, \phi_{(2)}] \right|^2 + 4 \left| [\phi_{(1)}, \phi_{(3)}] \right|^2$$

$$+ 4 \left| [\phi_{(2)}, \phi_{(3)}] \right|^2 + 2 \left| [\phi_{(1)}, \phi_{(1)}^\dagger] - i X_7 + 2 i I_8 \right|^2$$

$$+ 2 \left| [\phi_{(2)}, \phi_{(2)}^\dagger] - i X_7 - i I_8 - i I_9 \right|^2 + 4 \left| [\phi_{(2)}, \phi_{(1)}^\dagger] + I_{\bar{4}}^- \right|^2$$

$$+ 4 \left| [\phi_{(3)}, \phi_{(1)}^\dagger] + I_{\bar{5}}^- \right|^2 + 4 \left| [\phi_{(3)}, \phi_{(2)}^\dagger] + I_{\bar{6}}^- \right|^2$$

$$+ 2 \left| [\phi_{(3)}, \phi_{(3)}^\dagger] - i X_7 - i I_8 + i I_9 \right|^2 \Big),$$

where $F := \mathrm{d}A + A \wedge A$ is the curvature of the gauge connection $A$ on $M^d$, while $D_\mu \phi_{(\alpha)}$ and $D_\mu X_7$ denote the covariant derivatives of the Higgs fields,

$$D\phi_{(\alpha)} := \mathrm{d}\phi_{(\alpha)} + [A, \phi_{(\alpha)}] \qquad \text{for} \quad \alpha = 1, 2, 3 \qquad \text{and} \qquad DX_7 := \mathrm{d}X_7 + [A, X_7].$$

$$(3.50)$$

Since we assumed, due to equivariance, that the endomorphisms $X_a$ are independent of the coordinates of $S^7$, the integration over the coset space here simply produced its volume in the metric (3.10).

### 3.4. Orbifold quivers and reduction to $\mathbb{C}P^3$

Due to the fibration structure (3.1) of $S^7$ as a U(1)-bundle over the complex projective space $\mathbb{C}P^3$, it is natural to consider the reduction of the Sasakian quiver gauge theory on $S^7$ to that of the underlying Kähler coset structure on $\mathbb{C}P^3$. In this reduction, M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$ becomes IIA string theory on $AdS_4 \times \mathbb{C}P^3$ [29], which in the 't Hooft limit is dual to $\mathcal{N} = 6$ superconformal Chern-Simons theories with matter fields [15] to which our constructions apply. Similar reductions to the underlying Kähler leaf spaces have been carried out for the Sasaki-Einstein manifolds considered in [9–11].

For this, one has to further factor by the generator $I_7$, which corresponds to the Reeb vector field of the Sasakian structure, by setting $X_7 = I_7$ so that one obtains the further equivariance conditions

$$[\hat{I}_7, \phi_{(\alpha)}] = -4 \, \phi_{(\alpha)} \qquad \text{for} \quad \alpha = 1, 2, 3 \,. \tag{3.51}$$

This forces the Higgs fields to have the same form as the ladder operators of $G$, i.e. the remaining three complex Higgs fields must act in the weight diagrams of SU(4) as

$$\phi_{(1)} : (\nu_7, \nu_8, \nu_9) \longmapsto (\nu_7 - 4, \nu_8 + 2, \nu_9) \,,$$

$$\phi_{(2)} : (\nu_7, \nu_8, \nu_9) \longmapsto (\nu_7 - 4, \nu_8 - 1, \nu_9 - 1) \,, \tag{3.52}$$

$$\phi_{(3)} : (\nu_7, \nu_8, \nu_9) \longmapsto (\nu_7 - 4, \nu_8 - 1, \nu_9 + 1) \,.$$

Since we removed the contact direction as degree of freedom by setting $X_7 = I_7$, the loops in the quivers disappear, as expected. Apart from this, in our examples from Section 3.2 only the quiver (3.43) associated to the adjoint representation is altered: The morphism $\phi_5$ is ruled out by the additional conditions (3.52) with respect to $I_7$.

The quiver gauge theory on the orbifold $S^7/\mathbb{Z}_k$ shares some features with that on $\mathbb{C}P^3$, because the action of the finite subgroup $\mathbb{Z}_k$ on the fibres is embedded in the U(1) subgroup

generated by $I_7$. Combining this $\mathbb{Z}_k$-action on the fibres with the action (3.29), one has to impose the conditions [9,10]

$$\gamma(h_k)\,\phi_{(\alpha)}\,\gamma(h_k)^{-1} = \zeta_k^4\,\phi_{(\alpha)} \qquad \text{and} \qquad \gamma(h_k)\,X_7\,\gamma(h_k)^{-1} = X_7\,, \tag{3.53}$$

where $\gamma$ is the embedding of $\mathbb{Z}_k$ in the group U(1) generated by the chosen representation of $I_7$. In order to satisfy this condition for *all* integers $k$, the Higgs fields have to act on the quantum number $\nu_7$ in the same way that the ladder operators of SU(4) do, so that – combining this condition with SU(4)-equivariance – their form must be the same as that of the ladder operators of $G$.

In contrast to the additional equivariance condition on $\mathbb{C}P^3$, however, the endomorphism $X_7$ is still a degree of freedom. Furthermore, when considering only a particular given fixed integer $k$, the condition (3.53) might still be satisfied for fields with more entries than the ladder operators have, because the equation on the powers of roots of unity holds *modulo k*. Therefore other contributions may also match the condition for a special value of $k$. For details of $\mathbb{Z}_k$-equivariance, and comparisons with orbifolds of $S^3$ and $S^5$, see [9] and [10] respectively.

### 3.5. Hermitian Yang-Mills instantons on the Kähler cone

We will now extend the form of the gauge connection to the metric cone $C(S^7)$ and make use of its Kähler structure to formulate instanton equations. Starting from the canonical connection $\Gamma$,[9] we now include also the radial direction

$$\mathcal{A} = \Gamma + \sum_{a=1}^{7} X_a\,e^a + X_\tau\,e^\tau =: \Gamma + \sum_{\alpha=0}^{3}\left(Y_\alpha\,\Theta^\alpha + \bar{Y}_{\bar{\alpha}}\,\bar{\Theta}^{\bar{\alpha}}\right) \tag{3.54}$$

with $Y_0 := \frac{1}{2}(X_\tau + \mathrm{i}\,X_7)$, and we assume that the Higgs fields depend only on the radial coordinate, $Y_\alpha = Y_\alpha(r)$ for $\alpha = 0, 1, 2, 3$. Writing the *Hermitian Yang-Mills equations*[10] [32]

$$\mathcal{F}^{2,0} = 0 = \mathcal{F}^{0,2} \qquad \text{and} \qquad \Omega^{1,1}\,\lrcorner\,\mathcal{F} = 0 \tag{3.55}$$

for the components of the curvature yields the algebraic conditions

$$\left[\bar{Y}_{\bar{1}}, \bar{Y}_{\bar{2}}\right] = \left[\bar{Y}_{\bar{1}}, \bar{Y}_{\bar{3}}\right] = \left[\bar{Y}_{\bar{2}}, \bar{Y}_{\bar{3}}\right] = 0 \tag{3.56}$$

and the flow equations

$$r\,\dot{\bar{Y}}_{\bar{\alpha}} = -\tfrac{4}{3}\,\bar{Y}_{\bar{\alpha}} + 2\left[\bar{Y}_{\bar{\alpha}}, Y_0\right] \qquad \text{for} \quad \alpha = 1, 2, 3\,, \tag{3.57a}$$

$$r\left(\dot{Y}_0 - \dot{\bar{Y}}_{\bar{0}}\right) = -6\left(Y_0 - \bar{Y}_{\bar{0}}\right) + 2\left[Y_0, \bar{Y}_{\bar{0}}\right] + 2\left[Y_1, \bar{Y}_{\bar{1}}\right] + 2\left[Y_2, \bar{Y}_{\bar{2}}\right] + 2\left[Y_3, \bar{Y}_{\bar{3}}\right], \tag{3.57b}$$

where a dot indicates the $r$-derivative. From (3.56) one sees that the Hermitian Yang-Mills equations force the complex Higgs fields $\phi_{(\alpha)}$ of the Sasakian quiver gauge theory to commute with each other, i.e. they impose relations on the quivers.

---

[9] The instanton $\Gamma$ also lifts to an instanton on the metric cone and the cylinder.

[10] They are also known as the *Donaldson-Uhlenbeck-Yau equations* and are related to the stability of holomorphic vector bundles [30,31].

*Examples*   The utility of the quiver gauge theory is in its depiction of the matter fields as arrows in a quiver. Having chosen a representation of $G$, the quiver is fixed, and the decomposition encoded in the quiver yields the form of the system of equations one has to solve. In the following, we collect the Hermitian Yang-Mills equations for some of the examples from Section 3.2.

For the fundamental representation (3.31), we only have the arrow $\phi$, and the two loops $\psi_1$ and $\psi_3$. Plugging the Higgs fields (3.32) and (3.33) into the algebraic conditions (3.56) shows that they are automatically satisfied without any restriction on the fields. From (3.57a) we get

$$r\,\dot{\phi} = -\tfrac{3}{4}\,\phi - \mathrm{i}\,\phi\,\psi_3 + \mathrm{i}\,\psi_1\,\phi \;. \tag{3.58}$$

From (3.57b), one obtains two equations for the endomorphisms

$$r\,\dot{\psi}_3 = -6\,\psi_3 + 6\,\mathrm{i}\,\phi^{\dagger}\,\phi \;, \tag{3.59a}$$

$$r\,\dot{\psi}_{-1} = -6\,\psi_{-1} - 2\,\mathrm{i}\,\phi\,\phi^{\dagger} \;. \tag{3.59b}$$

These equations are the analogues of those for the fundamental representation in the quiver gauge theory on $S^5$ [10], as is to be expected from the identical quivers.

Although the quiver (3.36) of the representation **$\underline{6}$** looks formally the same as (3.31), there is a crucial subtlety arising from the different dimensions of the SU(3) representations in the decomposition (3.35) compared to the decomposition (3.30) of the fundamental representation. While the algebraic conditions do not provide further constraints, and the flow equations (3.57a) again yield

$$r\,\dot{\phi} = -\tfrac{3}{4}\,\phi - \mathrm{i}\,\phi\,\psi_2 + \mathrm{i}\,\psi_{-2}\,\phi \;, \tag{3.60}$$

the flow equations for the loop contributions are slightly changed. We obtain

$$r\,\dot{\psi}_2 = -6\,\psi_2 + 4\,\mathrm{i}\,\phi^{\dagger}\,\phi \;, \tag{3.61a}$$

$$r\,\dot{\psi}_{-2} = -6\,\psi_{-2} - 4\,\mathrm{i}\,\phi\,\phi^{\dagger} \;, \tag{3.61b}$$

so that both equations have coefficients with the same modulus, in contrast to those in the example of the fundamental representation.

*Constant endomorphisms*   Before we proceed with the general description of the moduli space of the Hermitian Yang-Mills equations under the constraints imposed by equivariance, we consider the special case of constant endomorphisms $Y_{\tilde{\alpha}}$. In this situation, the radial coordinate $r$ enters the setup simply as a label of the foliations comprising the underlying Sasaki-Einstein structure along the cone direction. Gauging $X_{\tau} = 0$, one obtains from the flow equations the algebraic conditions

$$\left[X_7, \bar{Y}_{\tilde{\alpha}}\right] = -\tfrac{4}{3}\,\mathrm{i}\,\bar{Y}_{\tilde{\alpha}} \;, \tag{3.62}$$

which implies the vanishing of many contributions to the action (3.49), as is to be expected of an instanton solution. As we have seen in Section 3.4, this condition can be satisfied, for instance, by the quiver gauge theory on the projective space $\mathbb{C}P^3$.

*Moduli space of Hermitian Yang-Mills equations*   For the analysis of the flow equations under the given constraints, one can apply the general results of [33] concerning Hermitian Yang-Mills instantons on metric cones over generic $2n+1$-dimensional Sasaki-Einstein manifolds $M^{2n+1}$; we briefly review the main aspects here, referring to [33] and references therein for details. Rescaling the matrices

$$\bar{Y}_{\bar{\alpha}} =: r^{-4/3} W_\alpha \qquad \text{for} \quad \alpha = 1, 2, 3 \qquad \text{and} \qquad \bar{Y}_{\bar{0}} =: r^{-6} Z \tag{3.63}$$

and changing the argument to $s := -\frac{1}{6} r^{-6}$ in (3.57) leads to the equations

$$\frac{\mathrm{d} W_\alpha}{\mathrm{d} s} = 2 \left[ W_\alpha, Z \right], \tag{3.64a}$$

$$\frac{\mathrm{d}}{\mathrm{d} s} \left( Z + Z^\dagger \right) + 2 \left[ Z, Z^\dagger \right] + 2 \left( -6s \right)^{-14/9} \sum_{\alpha=1}^{3} \left[ W_\alpha, W_\alpha^\dagger \right] = 0, \tag{3.64b}$$

in correspondence with the general results of [33] for $n = 3$. The first equations (3.64a) are referred to as the *complex equations* and the second equation (3.64b) as the *real equation*. The description of the moduli space $\mathcal{M}$ of these Nahm-type equations is based on the invariance of the complex equations under the gauge transformations

$$W_\alpha \longmapsto W_\alpha^g := g \, W_\alpha \, g^{-1} \qquad \text{and} \qquad Z \longmapsto Z^g := g \, Z \, g^{-1} - \frac{1}{2} \left( \frac{\mathrm{d} g}{\mathrm{d} s} \right) g^{-1} \tag{3.65}$$

for $g \in \mathcal{G}$, where $\mathcal{G}$ is the subgroup of gauge transformations $g : (-\infty, 0) \to \mathrm{GL}(r, \mathbb{C})$ that also preserve the equivariance conditions; this allows for application of techniques used in the study of Nahm equations [34,35].

On the one hand, one may start from a local gauge in which $Z^g$ vanishes, i.e. $Z = \frac{1}{2} g^{-1} \frac{\mathrm{d} g}{\mathrm{d} s}$ is pure gauge. From the complex equations it then follows that the complex matrices $W_\alpha$ are constant, i.e. one has the local solution

$$Z = \frac{1}{2} g^{-1} \frac{\mathrm{d} g}{\mathrm{d} s} \qquad \text{and} \qquad W_\alpha = g^{-1} T_\alpha \, g \qquad \text{with} \quad \left[ T_\alpha, T_\beta \right] = 0 \tag{3.66}$$

for *constant* matrices $T_\alpha$ obeying the equivariance constraints (2.8). An obvious choice for these matrices is as elements of a Cartan subalgebra of $\mathfrak{gl}(r, \mathbb{C})$. To explicitly solve the instanton equations, one also has to include the real equation and take into account the domain on which the gauge transformation is applicable. The real equation follows as equation of motion for a suitably chosen Lagrangian [33,34]

$$\mathcal{L} = \mathrm{tr}\left( \left| Z^g + Z^{g \, \dagger} \right|^2 + 2 \left( -6s \right)^{-14/9} \sum_{\alpha=1}^{3} |W_\alpha^g|^2 \right). \tag{3.67}$$

The real equation is therefore solved as a variational problem. For uniqueness of the solution and to apply this approach over the entire range $-\infty < s < 0$, one restricts to *framed* instantons and imposes boundary conditions for $s \to -\infty$, i.e. at the conical singularity $r = 0$. One therefore finds $g_0 \in \mathcal{G}$ such that

$$\lim_{s \to -\infty} W_\alpha(s) = g_0^{-1} T_\alpha \, g_0. \tag{3.68}$$

Consequently, the moduli space $\mathcal{M}$ can be described in terms of coadjoint orbits of $\mathrm{GL}(r, \mathbb{C})$ with suitable boundary conditions:

$$\mathcal{M} = \mathcal{O}_{T_1} \times \mathcal{O}_{T_2} \times \mathcal{O}_{T_3}, \tag{3.69}$$

where the orbits $\mathcal{O}_{T_\alpha}$ are generally not regular with closures given by nilpotent cones.

On the other hand, the moduli space also admits a description as a Kähler quotient, making (3.69) into a Kähler manifold. Denoting the space of solutions $\mathcal{A}$ to the complex equations

$\mathcal{F}^{2,0} = 0 = \mathcal{F}^{0,2}$ and the equivariance conditions as $\mathbb{A}^{1,1}$, the real equation can be interpreted in terms of the moment map $\mu : \mathbb{A}^{1,1} \to \mathrm{Lie}(\mathcal{G})$ with $\mu(\mathcal{A}) = \Omega^{1,1} \lrcorner \mathcal{F}$, and the moduli space is given as the Kähler quotient

$$\mathcal{M} = \mu^{-1}(0) /\!/ \mathcal{G} . \tag{3.70}$$

*Sasakian quiver gauge theory on odd-dimensional spheres*   By comparing the various examples of quivers associated to $S^7$, as well as the general realization of odd-dimensional spheres as Sasaki-Einstein spaces $S^{2n+1} \cong \mathrm{SU}(n+1)/\mathrm{SU}(n)$, we suggest that the Sasakian quiver gauge theory is universal on spheres in *all* odd dimensions. We will not attempt to give a rigorous proof of this fact here, but the general construction of the local section, the coset space and the representations of $\mathrm{SU}(n+1)$ strongly support this claim. Of course, exceptional isomorphisms for low-dimensional cases, such as the representation **6**, do not allow for a general treatment. This statement is further supported by the existence of general expressions for Hermitian Yang-Mills instantons on metric cones over *generic* Sasaki-Einstein manifolds [19,33], which we used for the description of the moduli space $\mathcal{M}$ above.

## 4. 3-Sasakian quiver gauge theory on the squashed seven-sphere

In this section we construct quiver gauge theories on the squashed seven-sphere, making use of its 3-Sasakian structure (see e.g. [36]). A general treatment of 3-Sasakian seven-manifolds and their geometry is given, for instance, in [37], while a description of the representations of $\mathrm{Sp}(2)$ can be again found in [28]. For the geometry and supergravity applications of the squashed seven-sphere, see also [38].

*4.1. Geometry of $S^7 \cong \mathrm{Sp}(2)/\mathrm{Sp}(1)$*

*Local section*   Similarly to Section 3.1, we start our description by providing local coordinates and a basis of 1-forms, again by using certain particular fibrations. As the squashed seven-sphere is a fibration of $\mathrm{SU}(2) \cong \mathrm{Sp}(1)$ over a quaternionic Kähler manifold [39], we can construct a local section by considering the fibration

$$\mathrm{Sp}(2) \longrightarrow S^4 \cong \mathrm{Sp}(2) \big/ \mathrm{Sp}(1) \times \mathrm{Sp}(1) , \tag{4.1}$$

and following the prescription in [32]. A local section of this bundle can be realized by

$$\mathrm{Sp}(2) \ni Q := f^{-1/2} \begin{pmatrix} \mathbb{1}_2 & -x \\ x^\dagger & \mathbb{1}_2 \end{pmatrix} \qquad \text{with} \quad x = x^\mu \tau_\mu , \quad (\tau_\mu) = (-\mathrm{i}\,\sigma_i, \mathbb{1}_2) \tag{4.2}$$

and $f := 1 + x^\dagger x = 1 + \delta_{\mu\nu} x^\mu x^\nu$; here $\sigma_i$ for $i = 1, 2, 3$ are the standard Pauli spin matrices. The canonical flat connection

$$A_0 = Q^{-1}\,\mathrm{d}Q =: \begin{pmatrix} A^- & -\phi \\ \phi^\dagger & A^+ \end{pmatrix} \qquad \text{with} \quad \phi = \frac{1}{f}\,\mathrm{d}x =: \begin{pmatrix} \chi^2 & \chi^1 \\ \bar{\chi}^{\bar{1}} & -\bar{\chi}^{\bar{2}} \end{pmatrix} \tag{4.3}$$

provides complex 1-forms $\chi^1$ and $\chi^2$. An element of $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$ can be written in local coordinates as

$$\mathrm{Sp}(1) \ni g \cdot h := (1 + z\,\bar{z})^{-1/2} \begin{pmatrix} 1 & -\bar{z} \\ z & 1 \end{pmatrix} \cdot \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\varphi} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\varphi} \end{pmatrix} , \tag{4.4}$$

so that a section of $\mathrm{Sp}(2) \to \mathrm{Sp}(2)/\mathrm{Sp}(1)$ can be obtained by

$$\mathrm{Sp}(2) \ni \tilde{Q} := Q \begin{pmatrix} g\,h & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix} . \qquad (4.5)$$

Consider first the flat connection on the twistor space $\mathrm{Sp}(2)/\mathrm{Sp}(1) \times \mathrm{U}(1)$ given by the Maurer-Cartan form of $\hat{Q} := Q\,g$:

$$\hat{A}_0 = \hat{Q}^{-1}\,\mathrm{d}\hat{Q} = g^{-1}\,A_0\,g + g^{-1}\,\mathrm{d}g =: \begin{pmatrix} g^{-1}\,A^-\,g + g^{-1}\,\mathrm{d}g & -g^{-1}\,\phi \\ \phi^{\dagger}\,g & A^+ \end{pmatrix} . \qquad (4.6)$$

Then a section of the bundle $\mathrm{Sp}(2) \to \mathrm{Sp}(2)/\mathrm{Sp}(1)$ is given by $\tilde{Q} := \hat{Q}\,h$, giving

$$\begin{aligned}
\tilde{A}_0 &= h^{-1}\,\hat{A}_0\,h + h^{-1}\,\mathrm{d}h \\[4pt]
&= \begin{pmatrix} h^{-1}\,\hat{A}^-\,h + h^{-1}\,\mathrm{d}h & -h^{-1}\,\hat{\phi} \\ \hat{\phi}^{\dagger}\,h & A^+ \end{pmatrix} \\[4pt]
&=: \begin{pmatrix} i\,e^7 & \Theta^3 & \Theta^2 & \Theta^1 \\ -\bar{\Theta}^{\bar{3}} & -i\,e^7 & \bar{\Theta}^{\bar{1}} & -\bar{\Theta}^{\bar{2}} \\ -\bar{\Theta}^{\bar{2}} & -\Theta^1 & -i\,e^8 & -\bar{\Theta}^{\bar{4}} \\ -\bar{\Theta}^{\bar{1}} & \Theta^2 & \Theta^4 & i\,e^8 \end{pmatrix} ,
\end{aligned} \qquad (4.7)$$

where $\hat{A}^- := g^{-1}\,A^-\,g + g^{-1}\,\mathrm{d}g$ and $\hat{\phi} := g^{-1}\,\phi$. This provides a basis of left-invariant 1-forms for $\mathrm{Sp}(2)$ with structure equations (B.2) and structure constants (B.3). We have defined the 1-forms in (4.7) in such a way that they define a 3-Sasakian structure with metric

$$\mathrm{d}s^2 = \sum_{\mu=1}^{7} e^{\mu} \otimes e^{\mu} . \qquad (4.8)$$

To show this, one uses the structure equations

$$\mathrm{d}e^1 = e^{27} - e^{28} - e^{35} - e^{46} + e^{39} + e^{4\,10}, \quad \mathrm{d}e^2 = -e^{17} + e^{18} + e^{45} - e^{36} + e^{49} - e^{3\,10},$$

$$\mathrm{d}e^3 = e^{47} + e^{48} + e^{15} + e^{26} - e^{19} + e^{2\,10}, \quad \mathrm{d}e^4 = -e^{37} - e^{38} - e^{25} + e^{16} - e^{29} - e^{1\,10},$$

$$\mathrm{d}e^5 = 2e^{67} - 2e^{13} + 2e^{24}, \quad \mathrm{d}e^6 = -2e^{57} - 2e^{14} - 2e^{23}, \quad \mathrm{d}e^7 = 2e^{56} + 2e^{12} + 2e^{34} \qquad (4.9)$$

together with

$$\mathrm{d}e^8 = -2e^{12} + 2e^{34} + 2e^{9\,10}, \quad \mathrm{d}e^9 = 2e^{13} + 2e^{24} - 2e^{8\,10}, \quad \mathrm{d}e^{10} = 2e^{14} - 2e^{23} + 2e^{89} \qquad (4.10)$$

for the real 1-forms $e^{\mu}$ defined as

$$e^1 - i\,e^2 := \Theta^1 , \quad e^3 - i\,e^4 := \Theta^2 , \quad e^5 - i\,e^6 := \Theta^3 \quad \text{and} \quad e^9 - i\,e^{10} := \Theta^4 . \qquad (4.11)$$

From these structure equations, we see that the triple $(\eta^5, \eta^6, \eta^7) := (e^5, e^6, e^7)$ satisfies the defining relations of a 3-Sasakian structure (see e.g. [17])

$$\mathrm{d}\eta^{\alpha} = \epsilon_{\alpha\beta\gamma}\,\eta^{\beta} \wedge \eta^{\gamma} + 2\,\omega^{\alpha} , \qquad \mathrm{d}\omega^{\alpha} = 2\,\epsilon_{\alpha\beta\gamma}\,\eta^{\beta} \wedge \omega^{\gamma} \qquad \text{with} \quad \alpha, \beta, \gamma = 5, 6, 7 , \qquad (4.12)$$

where we identify the 2-forms

$$\omega^5 := -e^{13} + e^{24} , \qquad \omega^6 := -e^{14} - e^{23} \qquad \text{and} \qquad \omega^7 := e^{12} + e^{34} . \tag{4.13}$$

Alternatively, one may also check the closure of some defining forms, as we did in Section 3.1 for the Sasaki-Einstein structure; see also Appendix B.1.

*Orbifolds*   As for the coset SU(4)/SU(3) in Section 3.1, one can introduce an action of the cyclic group $\mathbb{Z}_k$ on the squashed seven-sphere. For this, we embed the action in the U(1) factor $h$ of (4.4) as

$$h_k := \begin{pmatrix} \mathrm{e}^{\,\mathrm{i}\varphi/k} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\varphi/k} \end{pmatrix} . \tag{4.14}$$

The action $\pi$ on 1-forms can be deduced from the action of $g$ in the final section (4.7), giving

$$\pi(h_k)\Theta^\alpha = \zeta_k \,\Theta^\alpha , \ \alpha = 1, 2 , \qquad \pi(h_k)\Theta^3 = \zeta_k^2 \,\Theta^3 \qquad \text{and} \qquad \pi(h_k)e^7 = e^7 . \tag{4.15}$$

### 4.2. Instanton equations

We shall now describe the instanton equations on both Sp(2)/Sp(1) and its metric cone. We will also describe the canonical connection that will appear in the general form of the gauge connection.

*Canonical connection and instanton equation on Sp(2)/Sp(1)*   According to the general results of [17], the torsion of the canonical connection of a 3-Sasakian manifold is given by

$$T^\alpha = 3\, P_{\alpha\mu\nu}\, e^{\mu\nu} , \ \alpha = 5, 6, 7 \qquad \text{and} \qquad T^a = \tfrac{3}{2}\, P_{a\mu\nu}\, e^{\mu\nu} , \ a = 1, 2, 3, 4 \tag{4.16}$$

with the 3-form $P := \tfrac{1}{3}\, \eta^{567} + \tfrac{1}{3} \sum_{\alpha=5}^{7} \eta^\alpha \wedge \omega^\alpha$. In our case we obtain

$$P = \tfrac{1}{3} \left( e^{567} - e^{135} + e^{245} - e^{146} - e^{236} + e^{127} + e^{347} \right) , \tag{4.17}$$

so that the structure equations $de^\mu = -\Gamma^\mu_\nu \wedge e^\nu + T^\mu$ can be written as

$$\mathrm{d} \begin{pmatrix} e^1 \\ e^2 \\ e^3 \\ e^4 \end{pmatrix} = \begin{pmatrix} 0 & e^8 & -e^9 & -e^{10} \\ -e^8 & 0 & e^{10} & -e^9 \\ e^9 & -e^{10} & 0 & -e^8 \\ e^{10} & e^9 & e^8 & 0 \end{pmatrix} \wedge \begin{pmatrix} e^1 \\ e^2 \\ e^3 \\ e^4 \end{pmatrix} + \begin{pmatrix} T^1 \\ T^2 \\ T^3 \\ T^4 \end{pmatrix} \qquad \text{and} \qquad \mathrm{d}e^\alpha = T^\alpha . \tag{4.18}$$

Using the adjoint representation of the generators, one sees that the canonical connection is given as

$$\Gamma = I_8 \otimes e^8 + I_9 \otimes e^9 + I_{10} \otimes e^{10} . \tag{4.19}$$

In particular, as mentioned in Section 2.2, the canonical connection on Sp(2)/Sp(1) has a particularly simple form because the representation of the homogeneous space coincides with the quotient of its isometry and structure groups.[11] Its curvature reads

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[11]  The torsion again coincides with that obtained by setting $T(X, Y) = -[X, Y]_\mathfrak{m}$ for vectors $X, Y \in \mathfrak{m}$.

$$\mathcal{F}_\Gamma = \mathrm{d}\Gamma + \Gamma \wedge \Gamma = 2\, I_8 \otimes \left( -e^{12} + e^{34} \right) + 2\, I_9 \otimes \left( e^{13} + e^{24} \right) + 2\, I_{10} \otimes \left( e^{14} - e^{23} \right)$$
(4.20)

and it solves the instanton equation (2.5) for the 4-form [17] $Q = \frac{1}{6} \sum_{\alpha=5}^{7} \omega^\alpha \wedge \omega^\alpha = e^{1234}$. Written in components of the field strength $\mathcal{F}_\Gamma = \frac{1}{2} \mathcal{F}_{\mu\nu}\, e^{\mu\nu}$, this instanton equation reads

$$\mathcal{F}_{12} = -\mathcal{F}_{34}\,, \qquad \mathcal{F}_{13} = \mathcal{F}_{24}\,, \qquad \mathcal{F}_{14} = -\mathcal{F}_{23}\,, \tag{4.21a}$$

$$\mathcal{F}_{a\alpha} = 0 = \mathcal{F}_{\alpha\beta} \qquad \text{for} \quad a = 1,2,3,4\,, \ \alpha,\beta = 5,6,7\,. \tag{4.21b}$$

*Instantons on the metric cone* The metric cone over Sp(2)/Sp(1) carries, by definition, a hyper-Kähler structure, and the instanton equation (2.5) is written with the 4-form $Q$ defined as [17]

$$Q = \frac{1}{6}\left( \sum_{\alpha=5}^{7} \omega^\alpha \wedge \omega^\alpha + \epsilon_{\alpha\beta\gamma}\, \omega^\alpha \wedge \eta^{\beta\gamma} + 2\, \mathrm{d}\tau \wedge \sum_{\alpha=5}^{7} \eta^\alpha \wedge \omega^\alpha + 6\, \mathrm{d}\tau \wedge \eta^{567} \right). \tag{4.22}$$

Here $\mathrm{d}\tau = \frac{\mathrm{d}r}{r}$ is the 1-form associated to the cone/cylinder direction. In components this instanton equation yields the algebraic conditions

$$\mathcal{F}_{12} = -\mathcal{F}_{34}\,, \qquad \mathcal{F}_{13} = \mathcal{F}_{24}\,, \qquad \mathcal{F}_{14} = -\mathcal{F}_{23} \tag{4.23}$$

and the flow equations

$$\mathcal{F}_{1\tau} = -\mathcal{F}_{35} = -\mathcal{F}_{46} = \mathcal{F}_{27}\,, \qquad \mathcal{F}_{3\tau} = \mathcal{F}_{15} = \mathcal{F}_{26} = \mathcal{F}_{47}\,,$$

$$\mathcal{F}_{2\tau} = \mathcal{F}_{45} = -\mathcal{F}_{36} = -\mathcal{F}_{17}\,, \qquad \mathcal{F}_{4\tau} = -\mathcal{F}_{25} = \mathcal{F}_{16} = -\mathcal{F}_{37} \tag{4.24}$$

together with the triplet of equations

$$\mathcal{F}_{5\tau} = \mathcal{F}_{67}\,, \qquad \mathcal{F}_{6\tau} = -\mathcal{F}_{57}\,, \qquad \mathcal{F}_{7\tau} = \mathcal{F}_{56}\,. \tag{4.25}$$

The first set of conditions (4.23) are again those of the 3-Sasakian manifold in (4.21a), while the flow equations (4.24) and (4.25) demonstrate the SU(2) symmetry of the structure. The canonical connection of the 3-Sasakian manifold is also an instanton on the metric cone (cf. (4.20)), and therefore it can be used as starting point in the general form for the gauge connection on the metric cone.

Since 3-Sasakian manifolds form a special class of Sasaki-Einstein manifolds, one may also expect a corresponding embedding of the instantons, and indeed a gauge connection satisfying the conditions (4.23) and (4.24) is also a solution to the Hermitian Yang-Mills equations. Thus, when studying 3-Sasakian quiver gauge theories, one also obtains implicitly results related to Sasakian quiver gauge theories.[12] In fact, when we describe the moduli space of instantons on the hyper-Kähler cone in Section 4.5, we show that the description is based on intersections of moduli spaces of instantons on cones over Sasaki-Einstein manifolds.

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[12] An example is given in [12], where the connection used as starting point for Sasakian quiver gauge theory is the canonical connection of the 3-Sasakian geometry; it was used because it is better adapted to the structure of the relevant homogeneous space.

### 4.3. Quivers

The general form for the gauge connection on Sp(2)/Sp(1) is given in (2.6), i.e. we express it as

$$\mathcal{A} = A + \Gamma + \sum_{a=1}^{7} X_a \otimes e^a = A + \sum_{j=8}^{10} I_j \otimes e^j + \sum_{a=1}^{7} X_a \otimes e^a \tag{4.26}$$

where again $A$ denotes a connection on the vector bundle $E$ over $M^d$. On the metric cone, one can use exactly the same approach, where the endomorphisms then may depend on the radial coordinate, $X_a = X_a(r)$. As before, equivariance requires the vanishing of the mixed terms (or, equivalently, the condition (2.8)), which here implies that

$$\begin{aligned}
&[\hat{I}_8, \phi_{(1)}] = \phi_{(1)}, &&[\hat{I}_8, \phi_{(2)}] = -\phi_{(2)}, &&[\hat{I}_8, X_\alpha] = 0,\\
&[I_4^+, \phi_{(1)}] = 0, &&[I_4^+, \phi_{(2)}] = \phi_{(1)}, &&[I_4^+, X_\alpha] = 0,\\
&[I_4^-, \phi_{(1)}] = -\phi_{(2)}, &&[I_4^-, \phi_{(2)}] = 0, &&[I_4^-, X_\alpha] = 0
\end{aligned} \tag{4.27}$$

for $\alpha = 5, 6, 7$. Here we have again defined the complex matrices $\phi_{(1)} = \frac{1}{2}(X_1 - iX_2)$ and $\phi_{(2)} = \frac{1}{2}(X_3 - iX_4)$. Since we use the generator dual to $e^7$ as Cartan generator, the quivers might seem to distinguish between $X_7$ and the role of the other two matrices $X_5, X_6$, which we sometimes combine to $\phi_{(3)} = \frac{1}{2}(X_5 - iX_6)$, but from the geometry and the way in which they will appear in the action functional, their contribution is symmetric. Based on the weight diagrams in Appendix B.3, we will consider some examples of quivers in this setting.

On the orbifold, one additionally has to impose equivariance with respect to the finite group $\mathbb{Z}_k$ embedded in the U(1) subgroup generated by $I_7$. Using the action (4.14) and (4.15) for the analogue of (3.53), one again obtains the condition that the Higgs fields must act in the weight diagram in the same way that the ladder operators do, preserving their action on the charge associated to $I_7$, if one wants to solve the equivariance conditions with respect to $\mathbb{Z}_k$ for *all* integers $k$. As mentioned before, for a fixed value of $k$, one might also solve the condition with more general Higgs fields because the powers of the roots of unity $\zeta_k$ enter only modulo $k$. In this sense, equivariance with respect to $\mathbb{Z}_k$ embedded in $I_7$ is a weaker condition than actually imposing $I_7$-equivariance.

*Instanton equations*  Evaluating the curvature of the gauge connection (4.26) and plugging the components into the instanton equations on the metric cone yields the flow equations of the Higgs fields (B.13), which coincides with the equations one could have obtained from the general formulation in [19]. They can be formulated as

$$r\dot{\phi}_{(1)} = -\phi_{(1)} - [\phi_{(2)}^\dagger, X_5] = -\phi_{(1)} + i[\phi_{(2)}^\dagger, X_6] = -\phi_{(1)} - i[\phi_{(1)}, X_7], \tag{4.28a}$$

$$r\dot{\phi}_{(2)} = -\phi_{(2)} + [\phi_{(1)}^\dagger, X_5] = -\phi_{(2)} - i[\phi_{(1)}^\dagger, X_6] = -\phi_{(2)} - i[\phi_{(2)}, X_7], \tag{4.28b}$$

$$r\dot{X}_\alpha = -2X_\alpha - \frac{1}{2}\epsilon^{\alpha\beta\gamma}[X_\beta, X_\gamma] \quad \text{for} \quad \alpha, \beta, \gamma = 5, 6, 7 \tag{4.28c}$$

together with the algebraic relations

$$[\phi_{(1)}, \phi_{(2)}] = 2\phi_{(3)} \quad \text{and} \quad i[\phi_{(1)}, \phi_{(1)}^\dagger] + i[\phi_{(2)}, \phi_{(2)}^\dagger] = 2X_7. \tag{4.29}$$

*Fundamental representation $\underline{4}$* Using the generators in the fundamental representation of Sp(2) and the weight diagram (B.18), one obtains as decomposition

$$\underline{4}\big|_{\mathrm{Sp}(1)} = \underline{(\mathbf{1,0})_1} \oplus \underline{(\mathbf{-1,0})_1} \oplus \underline{(\mathbf{0,-1})_2} \tag{4.30}$$

and the quiver



$$\tag{4.31}$$

The Higgs fields are given by

$$\phi_{(1)} = \begin{pmatrix} 0 & 0 & \phi_1 & 0 \\ 0 & 0 & \phi_2 & 0 \\ 0 & 0 & 0 & 0 \\ \phi_3 & \phi_4 & 0 & 0 \end{pmatrix}, \quad \phi_{(2)} = \begin{pmatrix} 0 & 0 & 0 & -\phi_1 \\ 0 & 0 & 0 & -\phi_2 \\ \phi_3 & \phi_4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$X_\alpha = \begin{pmatrix} \psi_1^\alpha & \chi^\alpha & 0 & 0 \\ -\chi^{\alpha\,\dagger} & \psi_{-1}^\alpha & 0 & 0 \\ 0 & 0 & \psi_0^\alpha & 0 \\ 0 & 0 & 0 & \psi_0^\alpha \end{pmatrix} \tag{4.32}$$

with $\alpha = 5, 6, 7$. The form of these Higgs fields shows a typical feature: Since not all Cartan generators enter the equivariance condition (4.27), the allowed morphisms are more general than the action of the generators (B.1).[13] Imposing additionally equivariance under the $\mathbb{Z}_k$-action for generic $k$ requires the orbifold quiver to be the collapsed weight diagram



$$\tag{4.33}$$

and the Higgs fields

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[13] This typically occurs for Sasakian quiver gauge theories because, compared to those on Kähler cosets, one has at least one Cartan generator less in the equivariance condition. The usual description is formulated for $H$ being a parabolic subgroup of $G$ [20], which does not apply here.

$$\phi_{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \phi_2 \otimes (1,0) \\ \phi_3 \otimes (0,1)^\top & 0 & \mathbf{0}_2 \end{pmatrix}, \qquad \phi_{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\phi_2 \otimes (0,1) \\ \phi_3 \otimes (1,0)^\top & 0 & \mathbf{0}_2 \end{pmatrix},$$

$$\phi_{(3)} = \begin{pmatrix} 0 & 0 & 0 \\ \phi_6 & 0 & 0 \\ 0 & 0 & \mathbf{0}_2 \end{pmatrix}, \qquad X_7 = \begin{pmatrix} \chi_{-1} & 0 & 0 \\ 0 & \chi_1 & 0 \\ 0 & 0 & \chi_0 \otimes \mathbb{1}_2 \end{pmatrix} \tag{4.34}$$

take the form of the generators with homomorphisms $\phi_i$ and endomorphisms $\chi_j$ as entries. The flow equations for the generic quiver (4.31) yield the complicated system of equations

$$r\,\dot\phi_1 = -\phi_1 + i\,\phi_3^\dagger\,\psi_0^6 - i\,\psi_{-1}^6\,\phi_3^\dagger - i\,\chi^6\,\phi_4^\dagger = -\phi_1 - \phi_3^\dagger\,\psi_0^5 + \psi_{-1}^5\,\phi_3^\dagger + \chi^5\,\phi_4^\dagger$$

$$= -\phi_1 - i\,\phi_1\,\psi_0^7 + i\,\psi_{-1}^7\,\phi_1 + i\,\chi^7\,\phi_2 \,,$$

$$r\,\dot\phi_2 = -\phi_2 + i\,\phi_4^\dagger\,\psi_0^6 + i\,\chi^{6\,\dagger}\,\phi_3^\dagger - i\,\psi_1^6\,\phi_4^\dagger = -\phi_2 - \phi_4^\dagger\,\psi_0^5 - \chi^{5\,\dagger}\,\phi_3^\dagger + \psi_1^5\,\phi_4^\dagger$$

$$= -\phi_2 - i\,\phi_2\,\psi_0^7 - i\,\chi^{7\,\dagger}\,\phi_1 + i\,\psi_1^7\,\phi_2 \,,$$

$$r\,\dot\phi_3 = -\phi_3 - i\,\phi_1^\dagger\,\psi_{-1}^6 + i\,\phi_2^\dagger\,\chi^{6\,\dagger} + i\,\psi_0^6\,\phi_1^\dagger = -\phi_3 + \phi_1^\dagger\,\psi_{-1}^5 - \phi_2^\dagger\,\chi^{5\,\dagger} - \psi_0^5\,\phi_1^\dagger$$

$$= -\phi_3 - i\,\phi_3\,\psi_{-1}^7 + i\,\phi_4\,\chi^{7\,\dagger} + \psi_0^7\,\phi_3 \,,$$

$$r\,\dot\phi_4 = -\phi_4 - i\,\phi_1^\dagger\,\chi^6 - i\,\phi_2^\dagger\,\psi_1^6 + i\,\psi_0^6\,\phi_2^\dagger = -\phi_4 + \phi_1^\dagger\,\chi^5 + \phi_2^\dagger\,\psi_1^5 - \psi_0^5\,\phi_2^\dagger$$

$$= -\phi_4 - i\,\phi_3\,\chi^7 - i\,\phi_4\,\psi_1^7 + i\,\psi_0^7\,\phi_4 \,. \tag{4.35}$$

The flow equations for the entries of the remaining three matrices are given by

$$r\,\dot\psi_{-1}^\alpha = -2\,\psi_{-1}^\alpha - \tfrac{1}{2}\,\epsilon_{\alpha\beta\gamma}\left(\psi_{-1}^{[\beta}\,\psi_{-1}^{\gamma]} - \chi^{[\beta}\,\chi^{\gamma]\,\dagger}\right),$$

$$r\,\dot\chi^\alpha = -2\,\chi^\alpha - \tfrac{1}{2}\,\epsilon_{\alpha\beta\gamma}\left(\psi_{-1}^{[\beta}\,\chi^{\gamma]} + \chi^{[\beta}\,\psi_1^{\gamma]}\right),$$

$$r\,\dot\psi_1^\alpha = -2\,\psi_1^\alpha - \tfrac{1}{2}\,\epsilon_{\alpha\beta\gamma}\left(-\chi^{[\beta\dagger}\,\chi^{\gamma]} + \psi_1^{[\beta}\,\psi_1^{\gamma]}\right),$$

$$r\,\dot\psi_0^\alpha = -2\,\psi_0^\alpha - \tfrac{1}{2}\,\epsilon_{\alpha\beta\gamma}\,\psi_0^{[\beta}\,\psi_0^{\gamma]} \,. \tag{4.36}$$

Furthermore, one has the algebraic conditions, i.e. the quiver relations

$$-i\,\psi_0^7 = -\phi_1\,\phi_1^\dagger - \phi_2\,\phi_2^\dagger + \phi_3\,\phi_3^\dagger + \phi_4\,\phi_4^\dagger \,,$$

$$-i\,\psi_1^7 = \phi_1\,\phi_1^\dagger - \phi_3^\dagger\,\phi_3 \,, \qquad -i\,\psi_{-1}^7 = \phi_2\,\phi_2^\dagger - \phi_4^\dagger\,\phi_4 \,,$$

$$-i\,\chi^7 = \phi_1\,\phi_2^\dagger - \phi_3^\dagger\,\phi_4 \,, \tag{4.37}$$

$$\psi_0 = \phi_3\,\phi_1 + \phi_4\,\phi_2 \,, \qquad \psi_{-1} = \phi_1\,\phi_3 \,, \qquad \psi_1 = \phi_2\,\phi_4 \,,$$

$$\chi = \phi_1\,\phi_4 = -(\phi_2\,\phi_3)^\dagger \,,$$

where we denote $\psi_i := \tfrac{1}{2}\,(\psi_i^5 - i\,\psi_i^6)$ and $\chi := \tfrac{1}{2}\,(\chi^5 - i\,\chi^6)$. Restricting to the case of the simpler orbifold quiver (4.33) reduces the complexity somewhat, but one still has to solve a highly non-trivial system of matrix equations. However, when imposing the scalar form of [17], the system simplifies to (B.15), which has the analytic solutions (B.16).

*Representation* $\underline{5}$   Using the five-dimensional representation (B.20) with the Cartan generators $\hat{I}_7 = \mathrm{diag}(-1, -1, 0, 1, 1)$, $\hat{I}_8 = \mathrm{diag}(1, -1, 0, 1, -1)$ and its weight diagram (B.19), one obtains the decomposition

$$\underline{5}\big|_{\mathrm{Sp}(1)} = \underline{(-1, -1)_2} \oplus \underline{(0, 0)_1} \oplus \underline{(1, -1)_2} \tag{4.38}$$

and the quiver



$$\tag{4.39}$$
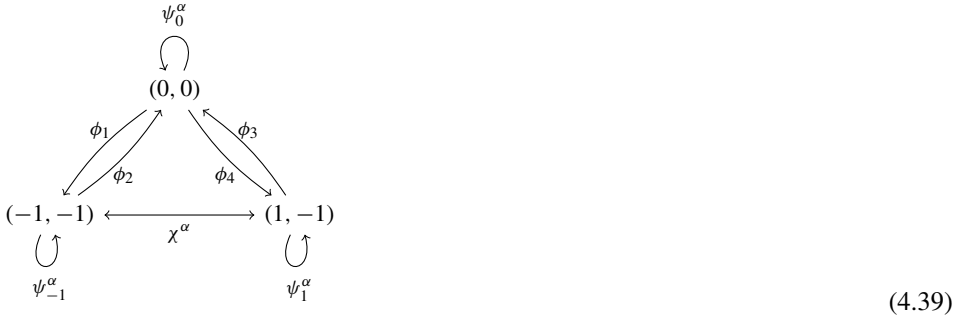
The Higgs fields read

$$\phi_{(1)} = \begin{pmatrix} 0 & 0 & \phi_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \phi_2 & 0 & 0 & \phi_3 \\ 0 & 0 & \phi_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \phi_{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \phi_1 & 0 & 0 \\ -\phi_2 & 0 & 0 & \phi_3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\phi_4 & 0 & 0 \end{pmatrix}, \tag{4.40a}$$

$$X_\alpha = \begin{pmatrix} \psi_{-1}^\alpha & 0 & 0 & \chi^\alpha & 0 \\ 0 & \psi_{-1}^\alpha & 0 & 0 & -\chi^\alpha \\ 0 & 0 & \psi_0^\alpha & 0 & 0 \\ -\chi^{\alpha\,\dagger} & 0 & 0 & \psi_1^\alpha & 0 \\ 0 & \chi^{\alpha\,\dagger} & 0 & 0 & \psi_1^\alpha \end{pmatrix}. \tag{4.40b}$$

As in the previous example, the number of allowed arrows is larger than the number of entries in the generators (B.20), and the quiver is reduced to the weight diagram by further imposing equivariance under $\mathbb{Z}_k \hookrightarrow \langle \exp(\hat{I}_7) \rangle$. Then the orbifold quiver is given by



$$\tag{4.41}$$
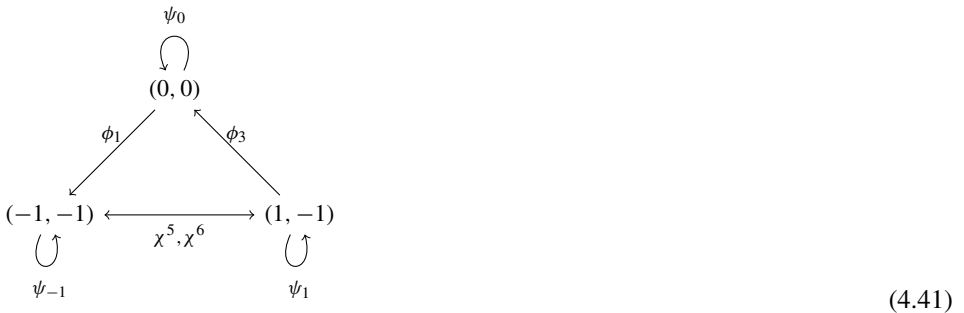
with $\psi_i := \psi_i^7$. The two representations $\underline{4}$ and $\underline{5}$ yield quivers of the same shape, but the decomposition into irreducible representations of Sp(1) differs, so that one obtains a different set of instanton equations. The flow equations read

$$r\,\dot{\phi}_1 = -\phi_1 - \mathrm{i}\,\chi^6\,\phi_3^\dagger = -\phi_1 + \chi^5\,\phi_3^\dagger = -\phi_1 - \mathrm{i}\,\phi_1\,\psi_0 + \mathrm{i}\,\psi_{-1}\,\phi_1\ ,$$

$$r\,\dot{\phi}_3 = -\phi_3 - \mathrm{i}\,\phi_1^\dagger\,\chi^6 = -\phi_3 + \phi_1^\dagger\,\chi^5 = -\phi_3 - \mathrm{i}\,\phi_3\,\psi_1 + \mathrm{i}\,\psi_0\,\phi_3\ ,$$

$$r\,\dot{\chi}^5 = -2\,\chi^5 - \chi^6\,\psi_1 + \psi_{-1}\,\chi^6 = -2\,\chi^5 - \chi^6\,\psi_{-1} + \psi_1\,\chi^6\ ,$$

$$r\,\dot{\chi}^6 = -2\,\chi^6 + \chi^5\,\psi_1 - \psi_{-1}\,\chi^5 = -2\,\chi^6 + \chi^5\,\psi_{-1} - \psi_1\,\chi^5\ , \tag{4.42}$$

$$r\,\dot{\psi}_{-1} = -2\,\psi_{-1} + \chi^5\,\chi^{6\,\dagger} - \chi^6\,\chi^{5\,\dagger}\ ,$$

$$r\,\dot{\psi}_0 = -2\,\psi_0\ ,$$

$$r\,\dot{\psi}_1 = -2\,\psi_1 + \chi^{5\,\dagger}\,\chi^6 - \chi^{6\,\dagger}\,\chi^5\ ,$$

while the algebraic conditions from (4.23) yield the quiver relations

$$\tfrac{1}{2}\left(\chi^5 - \mathrm{i}\,\chi^6\right) = \phi_1\,\phi_3\ , \quad -2\mathrm{i}\,\psi_{-1} = \phi_1\,\phi_1^\dagger\ , \quad 2\mathrm{i}\,\psi_1 = \phi_3^\dagger\,\phi_3\ , \quad \psi_0 = \phi_3\,\phi_3^\dagger - \phi_1^\dagger\,\phi_1\ . \tag{4.43}$$

*Adjoint representation* **10**   The ten-dimensional adjoint representation of Sp(2) decomposes as

$$\mathbf{10}\big|_{\mathrm{Sp}(1)} = \underline{(-2,0)}_{\mathbf{1}} \oplus \underline{(-1,-1)}_{\mathbf{2}} \oplus \underline{(0,-2)}_{\mathbf{3}} \oplus \underline{(0,0)}_{\mathbf{1}} \oplus \underline{(1,-1)}_{\mathbf{2}} \oplus \underline{(2,0)}_{\mathbf{1}}\ . \tag{4.44}$$

For better readability, we specialize to the case when equivariance is imposed as well with respect to the second Cartan generator $\hat{I}_7$. The generic case can be easily recovered by the applying the conditions (4.27) to the weight diagram (B.21), and it will involve a large number of arrows and also several maps between the same vertices, similarly to the more general quivers in the previous examples (4.31) and (4.39). With this restriction, we obtain the orbifold quiver



$$\tag{4.45}$$
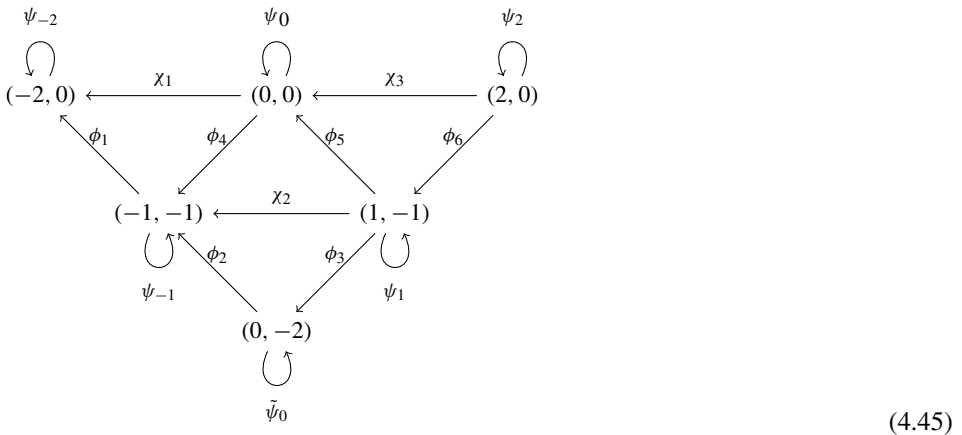
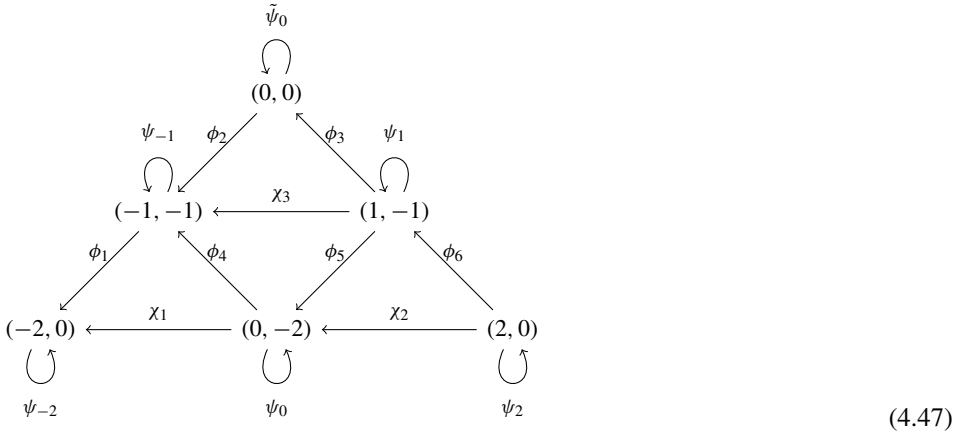where $\psi_i$ and $\tilde{\psi}_0$ denote the endomorphisms contained in $X_7$, while $\chi_i := \tfrac{1}{2}(\chi_i^5 - \mathrm{i}\,\chi_i^6)$ are the entries of $X_5$ and $X_6$. Due to the large number of arrows, we do not write out explicitly the instanton equations for this case, but they can be obtained simply by inserting the Higgs field matrices into the instanton equations (4.28) and (4.29).

*Representation* **14**   As our final example we consider the 14-dimensional representation of Sp(2), which decomposes under restriction to Sp(1) as

$$\mathbf{14}\big|_{\text{Sp}(1)} = \underline{(-2,-2)_3} \oplus \underline{(-1,-1)_2} \oplus \underline{(0,-2)_3} \oplus \underline{(0,0)_1} \oplus \underline{(1,-2)_2} \oplus \underline{(2,-2)_3}\,, \tag{4.46}$$

so that the quiver (after further imposing equivariance under $\mathbb{Z}_k \hookrightarrow \langle \exp(\hat{I}_7) \rangle$) follows from the weight diagram (B.22) and is given by

$$
\begin{array}{c}
\tilde{\psi}_0 \\
\circlearrowright \\
(0,0) \\
\psi_{-1}\quad \phi_2 \qquad \phi_3 \quad \psi_1 \\
\circlearrowright \qquad\qquad \circlearrowright \\
(-1,-1) \xleftarrow{\ \chi_3\ } (1,-1) \\
\phi_1 \qquad \phi_4 \qquad \phi_5 \qquad \phi_6 \\
(-2,0) \xleftarrow{\ \chi_1\ } (0,-2) \xleftarrow{\ \chi_2\ } (2,0) \\
\circlearrowright \qquad\qquad \circlearrowright \qquad\qquad \circlearrowright \\
\psi_{-2} \qquad\qquad \psi_0 \qquad\qquad \psi_2
\end{array}
\tag{4.47}
$$

This quiver has the same general structure as that of the adjoint representation, but the multiplicities of the vertices are different, as was also the case for the fundamental representation in comparison with the representation **5**. These last two examples with a large number of possibly contributing fields clearly demonstrate the advantages of the quiver approach for quickly constructing an equivariant connection.

### 4.4. Yang-Mills-Higgs theories

Plugging in the components of the field strength and using the orthonormality of the basis, one obtains the Yang-Mills action for an equivariant gauge connection on Sp(2)/Sp(1) as

$$
\begin{aligned}
S_{\text{YM}} = \text{vol}\big(\text{Sp}(2)/\text{Sp}(1)\big) \int_{M^d} \mathrm{d}^d y\, \sqrt{g}\, \frac{1}{2}\, \text{tr}\Big( \frac{1}{2} F_{\mu\nu} \big(F^{\mu\nu}\big)^{\dagger} + \sum_{\mu=1}^{d} \sum_{a=1}^{7} \big|D_\mu X_a\big|^2 \\
+ \big|[X_1,X_2]+2X_7-2I_8\big|^2 + \big|[X_1,X_3]-2X_5+2I_9\big|^2 \\
+ \big|[X_1,X_4]-2X_6+2I_{10}\big|^2 + \big|[X_2,X_3]-2X_6-2I_{10}\big|^2 \\
+ \big|[X_2,X_4]+2X_5+2I_9\big|^2 + \big|[X_3,X_4]+2X_7+2I_8\big|^2 \\
+ \big|[X_1,X_5]+X_3\big|^2 + \big|[X_1,X_6]+X_4\big|^2 + \big|[X_1,X_7]-X_2\big|^2 + \big|[X_2,X_5]-X_4\big|^2 \\
+ \big|[X_2,X_6]+X_3\big|^2 + \big|[X_2,X_7]+X_1\big|^2 + \big|[X_3,X_5]-X_1\big|^2 + \big|[X_3,X_6]-X_2\big|^2 \\
+ \big|[X_3,X_7]-X_4\big|^2 + \big|[X_4,X_5]+X_2\big|^2 + \big|[X_4,X_6]-X_1\big|^2 + \big|[X_4,X_7]+X_3\big|^2 \\
+ \big|[X_5,X_6]+2X_7\big|^2 + \big|[X_5,X_7]-2X_6\big|^2 + \big|[X_6,X_7]+2X_5\big|^2 \Big)\,,
\end{aligned}
\tag{4.48}
$$

where the covariant derivatives are defined as before for the action of the Sasakian quiver gauge theory in (3.49). The 3-Sasakian geometry of the squashed seven-sphere is evident in the form of this action. The instanton conditions (4.21b) on $\mathrm{Sp}(2)/\mathrm{Sp}(1)$ imply the vanishing of the terms in the last four lines in (4.48), so that the Yang-Mills action for instantons on $\mathrm{Sp}(2)/\mathrm{Sp}(1)$ is given by

$$
S_{\mathrm{YM}}^{\mathrm{inst}} = \mathrm{vol}\big(\mathrm{Sp}(2)/\mathrm{Sp}(1)\big) \int_{M^d} \mathrm{d}^d y \, \sqrt{g} \, \frac{1}{2} \, \mathrm{tr}\Big( \frac{1}{2} \, F_{\mu\nu} \left( F^{\mu\nu} \right)^\dagger + \sum_{\mu=1}^{d} \sum_{a=1}^{7} \left| D_\mu X_a \right|^2
$$
$$
+ 2 \left| [X_1, X_2] + 2\, X_7 - 2\, I_8 \right|^2 + 2 \left| [X_1, X_3] - 2\, X_5 + 2\, I_9 \right|^2
$$
$$
+ 2 \left| [X_1, X_4] - 2\, X_6 + 2\, I_{10} \right|^2 \Big) . \tag{4.49}
$$

As a by-product one can obtain the action for an equivariant gauge connection on the twistor space $\mathrm{Sp}(2)/\mathrm{Sp}(1) \times \mathrm{U}(1)$, analogously to the reduction from $S^7$ to $\mathbb{C}P^3$ from Section 3.4, and to the quaternionic space $S^4$ underlying the local section $Q$ of the fibration (4.1).

### 4.5. $\mathrm{Sp}(2)$-instantons on the hyper-Kähler cone

We shall now describe the moduli space of instantons on the hyper-Kähler cone $C\big(\mathrm{Sp}(2)/\mathrm{Sp}(1)\big) = \mathbb{H}^2$; for an overview of hyper-Kähler geometry, see for instance [40]. The defining property is the existence of an $\mathrm{SU}(2)$-triplet of complex structures $J_\alpha$ satisfying the quaternion relations

$$
J_\alpha J_\beta = -\delta_{\alpha\beta} \mathbb{1} + \epsilon_{\alpha\beta}{}^\gamma J_\gamma , \tag{4.50}
$$

or, equivalently, a triplet of Kähler forms[14] $\Omega_\alpha(X, Y) := g_{\mathrm{cone}}(X, J_\alpha Y)$. By virtue of the quaternion relations (4.50), any $J = s^\alpha J_\alpha$ with $s = (s^\alpha) \in S^2$ yields a complex structure on the tangent bundle of the hyper-Kähler manifold.

We now show that the condition of being an $\mathrm{Sp}(2)$-instanton is equivalent to imposing the condition of being a Hermitian Yang-Mills instanton with respect to any such complex structure $J$. On the metric cone $C(\mathrm{Sp}(2)/\mathrm{Sp}(1))$ the holonomy can be reduced from the generic holonomy group $\mathrm{SO}(8)$ of an oriented eight-dimensional Riemannian manifold to the subgroup $\mathrm{Sp}(2)$. Denoting this splitting of the corresponding Lie algebra as

$$
\mathfrak{so}(8) = \mathfrak{sp}(2) \oplus \mathfrak{sp}(1) \oplus \mathfrak{k} \tag{4.51}
$$

with $\mathfrak{k}$ the orthogonal complement, an $\mathrm{Sp}(2)$-instanton is a connection whose curvature $\mathcal{F} = (\mathcal{F}_{\mu\nu})$, considered as an $\mathfrak{so}(8)$ matrix, is valued in the subalgebra $\mathfrak{sp}(2)$ alone. On hyper-Kähler manifolds there is a $\mathbb{C}P^1$-family of complex structures $J$, and $\mathrm{Sp}(2)$-instantons can be generally described by the condition [41]

$$
\mathcal{F}_J^{0,2} = 0 = \mathcal{F}_J^{2,0} \qquad \text{for} \quad J = s^\alpha J_\alpha \qquad \text{with} \quad s \in S^2 , \tag{4.52}
$$

where the superscripts refer to the $(0, 2)$ and $(2, 0)$ parts of the curvature with respect to the complex structure $J$. Recall that (4.52) is the holomorphicity condition of Hermitian Yang-Mills instantons, i.e. it ensures that in the splitting

---

[14]  With our explicit choices of the defining equations (4.12) and (4.13) of the underlying 3-Sasakian manifold, they can be taken as $\Omega_1 = r^2 \, (e^{12} + e^{34} + e^{56} + e^{\tau 7})$, $\Omega_2 = r^2 \, (e^{31} + e^{24} + e^{67} + e^{\tau 5})$ and $\Omega_3 = r^2 \, (e^{32} + e^{41} + e^{75} + e^{\tau 6})$.

$$\mathfrak{so}(8) = \mathfrak{u}(4) \oplus \mathfrak{p} \tag{4.53}$$

the curvature is valued in the subalgebra $\mathfrak{u}(4)$ alone. In contrast to the case of a single Hermitian Yang-Mills moduli space as in Section 3.5, by imposing the holomorphicity conditions with respect to *all* complex structures, the corresponding stability conditions are automatically fulfilled as the following argument shows. For any $J$ the conditions in (4.52) can be formulated in terms of the projection operator $\frac{1}{2}(\mathbb{1} + \mathrm{i} J)$. In components, we then have

$$\frac{1}{4}\left(\delta_\mu^\lambda + \mathrm{i} J_\mu^\lambda\right)\left(\delta_\nu^\sigma + \mathrm{i} J_\nu^\sigma\right)\mathcal{F}_{\lambda\sigma} = 0\,, \tag{4.54}$$

so that the Sp(2)-instanton equations are equivalent to

$$J_{\alpha\mu}^\sigma \mathcal{F}_{\nu\sigma} = J_{\alpha\nu}^\sigma \mathcal{F}_{\mu\sigma} \qquad \text{and} \qquad J_{\alpha\mu}^\lambda J_{\alpha\nu}^\sigma \mathcal{F}_{\lambda\sigma} = \mathcal{F}_{\mu\nu} \qquad \text{(no sum on } \alpha\text{)}\,. \tag{4.55}$$

The second set of equations in (4.55) in fact follows from the first set, as is demonstrated in Appendix B.2. Imposing these conditions for all $\alpha = 1, 2, 3$ then also implies the conditions

$$\Omega_\alpha^{\mu\nu} \mathcal{F}_{\mu\nu} = 0\,, \tag{4.56}$$

which are the stability conditions of Hermitian Yang-Mills instantons. Therefore, the Sp(2)-instanton equations restrict the curvature to lie in the subalgebra $\mathfrak{su}_\alpha(4) \subset \mathfrak{su}_\alpha(4) \oplus \mathfrak{u}_\alpha(1) \oplus \mathfrak{p}_\alpha = \mathfrak{so}(8)$ for each $\alpha = 1, 2, 3$. Conversely, by the same arguments,[15] such a triplet of Hermitian Yang-Mills equations implies that the connection is an Sp(2)-instanton [41].

Consequently, the moduli space of Sp(2)-instantons on the hyper-Kähler cone is given as the intersection of the three Hermitian Yang-Mills moduli spaces:

$$\mathcal{M} = \mathcal{M}_1 \cap \mathcal{M}_2 \cap \mathcal{M}_3\,. \tag{4.57}$$

For the description of each Hermitian Yang-Mills moduli space

$$\mathcal{M}_\alpha := \left\{ \mathcal{A} \in \mathbb{A}^{1,1} \mid \Omega_\alpha \lrcorner \mathcal{F} = 0 \quad, \quad \mathcal{F}_{J_\alpha}^{2,0} = 0 = \mathcal{F}_{J_\alpha}^{0,2} \right\} \tag{4.58}$$

one can apply the techniques of Section 3.5, making (4.57) into a hyper-Kähler orbit space. Note that, of course, the equivariance conditions differ, and that also the quiver relations imposed by the holomorphicity conditions are slightly different, as the explicit equations (B.13) and (B.14) show. The intersection (4.57) is obviously non-empty because it contains, for instance, the trivial solutions $X_\mu = 0$ and the analytic solution of the scalar form from [17].

## 5. Translationally invariant instantons on the Calabi-Yau cone

The metric cone over the seven-sphere with its round metric is $\mathbb{C}^4$, and this motivates a study of translationally invariant instantons on an orbifold $\mathbb{C}^4/\Gamma$ by a finite group $\Gamma \subset \mathrm{SU}(4)$, as similarly done in [9,10]. Generally, moduli spaces of instantons on $\mathbb{C}^n/\Gamma$ are related to resolutions of orbifold singularities and aspects of the McKay correspondence, see [42] and references therein. For $n = 4$ and $\Gamma = \mathbb{Z}_k$, they determine the vacuum moduli spaces of the Chern-Simons quiver gauge theories discussed in Section 1.

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[15] Alternatively, note that the 4-form $Q$ appearing in the instanton equation (2.5) can be written here as $Q = \frac{1}{3}(Q_1 + Q_2 + Q_3)$, so that a connection satisfying the Hermitian Yang-Mills equations with respect to all three complex structures is also a solution to the Sp(2)-instanton equation.

## 5.1. Equivariant connections

On $\mathbb{C}^4$ we use the coordinates $(z_1, z_2, z_3, z_4)$, equipped with the standard metric and complex structure on $\mathbb{C}^4$, i.e. $Jz_\alpha = i\,z_\alpha$. The differentials of the coordinates provide a translationally invariant basis of 1-forms, and we write a connection as

$$\mathcal{A} = Y_\alpha \otimes dz^\alpha + \bar{Y}_{\bar\alpha} \otimes d\bar{z}^{\bar\alpha} \tag{5.1}$$

with $\bar{Y}_{\bar\alpha} = -Y_\alpha^\dagger$ describing the endomorphism part of the connection (acting on the fibres $\mathbb{C}^r$ of the underlying vector bundle). Translational invariance of this connection, i.e. $d\mathcal{A} = 0$, then implies as conditions

$$dY_\alpha = 0 = d\bar{Y}_{\bar\alpha} \ . \tag{5.2}$$

The form of the endomorphism is determined by equivariance with respect to the finite subgroup $\Gamma = \mathbb{Z}_k$. When introducing the orbifold action on $S^7$ previously, its action on forms followed from the way it acts on the quantities entering the local section. For the round seven-sphere it was induced by the fundamental action of SU(4) on $\mathbb{C}^4$ in (3.28), and the ensuing quotient leading to the local patch in $\mathbb{C}P^3$. Now, however, we start directly from the action of $I_7$ in the fundamental representation, so that one can define

$$\pi(h_k) : z_a \longmapsto \zeta_k^{-1} z_a \ , \ a = 1, 2, 3 \ , \qquad z_4 \longmapsto \zeta_k^3 z_4 \tag{5.3}$$

with $\zeta_k$ a primitive $k$-th root of unity. In [10, Section 6.1] a detailed discussion is given of the different choices of $\mathbb{Z}_k$-actions in the case of $G$-equivariant connections and translationally invariant instantons on the cone over $S^5$. Following this, we do not consider the weights associated to the generator $I_7$, which has been used for our discussion of SU(4)-equivariant instantons on orbifolds of $S^7$, but consider the weights pertaining to the other Cartan generators, as the examples below will clarify. The condition of equivariance then reads [9,10]

$$\gamma(h_k)\, Y_\alpha\, \gamma(h_k)^{-1} = \pi(h_k) Y_\alpha \ , \tag{5.4}$$

where $\gamma(h_k)$ denotes the action of $\Gamma$ on the fibres $\mathbb{C}^r$. With its standard metric and complex structure, the Kähler form on $\mathbb{C}^4$ is given by

$$\Omega = -\frac{i}{2} \sum_{\alpha=1}^{4} dz^\alpha \wedge d\bar{z}^{\bar\alpha} \tag{5.5}$$

and it allows for application of the Hermitian Yang-Mills equations. While the condition $\mathcal{F}_{\alpha\beta} = 0 = \mathcal{F}_{\bar\alpha\bar\beta}$ again yields

$$\left[Y_\alpha, Y_\beta\right] = 0 = \left[\bar{Y}_{\bar\alpha}, \bar{Y}_{\bar\beta}\right] \qquad \text{for} \quad \alpha, \beta = 1, 2, 3, 4 \ , \tag{5.6}$$

one can include a Fayet-Iliopoulos term $\Xi$ in the stability condition $\Omega \lrcorner \mathcal{F} = \Xi$ which yields

$$\sum_{\alpha=1}^{4} \left[Y_\alpha, \bar{Y}_{\bar\alpha}\right] = \Xi \ , \tag{5.7}$$

where $\Xi$ lies in the centre of the Lie algebra $\mathfrak{u}(r)$ of the structure group. Motivated by the study of self-dual connections in four dimensions and their description as hyper-Kähler quotients [43], one might also consider here Sp(2)-instantons, whose equations and moduli space we shall discuss in detail in Section 5.3.

### 5.2. Examples

We shall now study the resulting quiver gauge theories for some representations of SU(4), as used for the description of Section 3.2. Due to the complicated and lengthy equations for hyper-Kähler instantons, we restrict our attention here to Hermitian Yang-Mills instantons with respect to one of the complex structures, as higher-dimensional analogues of the cases considered in [10].

As action $\gamma(h_k)$ on the fibres we assign to each subspace in the decomposition of the chosen SU(4)-representation certain $k$-th roots of unity with weights given by the quantum numbers with respect to the Cartan generator $I_8$. Note that this choice of $\gamma(h_k)$ is neither unique nor necessary, but just a *possible* choice for the action on the fibres (as in [10]). However, in order to actually get an embedding into SU(4), one has to impose a certain condition on the dimensions $r_i$ of the occurring vector spaces in the decomposition.[16]

*Fundamental representation* **4**  For the fundamental representation of SU(4), the generator of $\mathbb{Z}_k$ is chosen based on the decomposition (3.30) and the quantum numbers with respect to $\hat{I}_8$ as

$$\gamma(h_k) = \begin{pmatrix} \zeta_k^{-1}\, \mathbb{1}_3 \otimes \mathbb{1}_{r_{-1}} & 0 \\ 0 & 1 \otimes \mathbb{1}_{r_3} \end{pmatrix} \qquad \text{with} \quad 3r_{-1} \equiv 0 \mod k , \tag{5.8}$$

where $r_{-1}$ denotes the dimension of the vector space attached to the first vertex. The equivariance condition (5.4) then yields the 3-Kronecker quiver

$$(-1,-1,1) \overset{\Phi_\alpha}{\underset{\Longleftarrow}{\Longleftarrow}} (3,0,0) \tag{5.9}$$

and the Higgs fields are of the form

$$Y_\alpha = \begin{pmatrix} 0 & \Phi_\alpha \\ 0 & 0 \end{pmatrix} \qquad \text{for} \quad \alpha = 1, 2, 3 \qquad \text{and} \qquad Y_4 = 0 . \tag{5.10}$$

As expected, this is the higher-dimensional analogue of the quiver for the fundamental representation of SU(3) in [10]. The holomorphicity condition $[Y_\alpha, Y_\beta] = 0$ is trivially satisfied, while the stability condition (5.7) yields

$$\Phi_1\,\Phi_1^\dagger + \Phi_2\,\Phi_2^\dagger + \Phi_3\,\Phi_3^\dagger = -\xi_{-1}\, \mathbb{1}_3 \otimes \mathbb{1}_{r_{-1}} ,$$
$$\Phi_1^\dagger\,\Phi_1 + \Phi_2^\dagger\,\Phi_2 + \Phi_3^\dagger\,\Phi_3 = \xi_3\, 1 \otimes \mathbb{1}_{r_3} , \tag{5.11}$$

where $\xi_i$ are the components of the decomposition of $\Xi$ according to the action $\gamma(h_k)$.

*Representation* **6**  For the six-dimensional representation of SU(4) we consider the embedding of the generator of $\Gamma = \mathbb{Z}_k$ given by

$$\gamma(h_k) = \begin{pmatrix} \zeta_k^{-2}\, \mathbb{1}_3 \otimes \mathbb{1}_{r_{-2}} & 0 \\ 0 & \zeta_k^{-1}\, \mathbb{1}_3 \otimes \mathbb{1}_{r_2}, \end{pmatrix} \qquad \text{with} \quad 6r_{-2} + 3r_2 \equiv 0 \mod k . \tag{5.12}$$

The equivariance condition (5.4) then again yields the 3-Kronecker quiver

---

[16] Since SU(4) has rank three, another obvious choice is to use the weights associated to the Cartan generator $I_9$. For the four examples given here, this yields the same quivers.

$$\Phi_\alpha$$

$$(-2,-2,0) \; \begin{array}{c} \longleftarrow \\ \overline{\phantom{xxxxxxxx}} \\ \longleftarrow \end{array} \; (2,-1,1) \tag{5.13}$$

with the Higgs fields

$$Y_\alpha = \begin{pmatrix} 0 & \Phi_\alpha \\ 0 & 0 \end{pmatrix} \qquad \text{for} \quad \alpha = 1,2,3 \qquad \text{and} \qquad Y_4 = 0 \, . \tag{5.14}$$

The corresponding instanton equations are

$$\Phi_1 \, \Phi_1^\dagger + \Phi_2 \, \Phi_2^\dagger + \Phi_3 \, \Phi_3^\dagger = -\xi_{-2} \, \mathbb{1}_3 \otimes \mathbb{1}_{r_{-2}} \, ,$$
$$\Phi_1^\dagger \, \Phi_1 + \Phi_2^\dagger \, \Phi_2 + \Phi_3^\dagger \, \Phi_3 = \xi_2 \, \mathbb{1}_3 \otimes \mathbb{1}_{r_2} \, , \tag{5.15}$$

which are similar to those of the fundamental representation but, of course, the multiplicities differ, as was already the case for the SU(4)-equivariant connections constructed in Section 3.2.

*Representation* **10**    For the ten-dimensional representation we use the embedding

$$\gamma(h_k) = \begin{pmatrix} \zeta_k^{-2} \, \mathbb{1}_6 \otimes \mathbb{1}_{r_{-2}} & 0 & 0 \\ 0 & \zeta_k^{-1} \, \mathbb{1}_3 \otimes \mathbb{1}_{r_2} & 0 \\ 0 & 0 & 1 \otimes \mathbb{1}_{r_6} \end{pmatrix} \qquad \text{with} \quad 12 r_{-2} + 3 r_2 \equiv 0 \mod k \tag{5.16}$$

to obtain the 3-Beilinson quiver [44]

$$\Phi_\alpha^1 \qquad\qquad\qquad \Phi_\alpha^2$$

$$(-2,-2,2) \; \begin{array}{c} \longleftarrow \\ \overline{\phantom{xxxx}} \\ \longleftarrow \end{array} \; (2,-1,1) \; \begin{array}{c} \longleftarrow \\ \overline{\phantom{xxxx}} \\ \longleftarrow \end{array} \; (6,0,0) \tag{5.17}$$

and the Higgs fields

$$Y_\alpha = \begin{pmatrix} 0 & \Phi_\alpha^1 & 0 \\ 0 & 0 & \Phi_\alpha^2 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{for} \quad \alpha = 1,2,3 \qquad \text{and} \qquad Y_4 = 0 \, . \tag{5.18}$$

The Hermitian Yang-Mills equations require the holomorphicity conditions

$$\Phi_\alpha^1 \, \Phi_\beta^2 - \Phi_\beta^1 \, \Phi_\alpha^2 = 0 \tag{5.19}$$

together with the stability conditions

$$\sum_{\alpha=1}^3 \Phi_\alpha^1 \, \Phi_\alpha^{1\,\dagger} = -\xi_{-2} \, \mathbb{1}_6 \otimes \mathbb{1}_{r_{-2}} \, ,$$

$$\sum_{\alpha=1}^3 \left( \Phi_\alpha^2 \, \Phi_\alpha^{2\,\dagger} - \Phi_\alpha^{1\,\dagger} \, \Phi_\alpha^1 \right) = -\xi_2 \, \mathbb{1}_3 \otimes \mathbb{1}_{r_2} \, ,$$

$$\sum_{\alpha=1}^3 \Phi_\alpha^{2\,\dagger} \, \Phi_\alpha^2 = \xi_6 \, 1 \otimes \mathbb{1}_{r_6} \, . \tag{5.20}$$

Here the arrows related to $Y_\alpha$ for $\alpha = 1,2,3$ occur whenever the difference in the powers of $\zeta_k$ in (5.4) is equal to one, so that the underlying structure of the quivers shares the main features with

those of [10]. While the morphisms induced by $Y_4$ only appear for differences of 2 in the case of the group SU(3) studied in [10], for the higher-dimensional version here it requires differences of 3 in the powers of the root of unity, so that there is no arrow $Y_4$ for the ten-dimensional representation.

*Representation* **20**   As our final example, consider the representation

$$\underline{\mathbf{20}}\big|_{\mathrm{SU}(3)} = \underline{(-3,-3,3)}_{\mathbf{10}} \oplus \underline{(1,-2,2)}_{\mathbf{6}} \oplus \underline{(5,-1,1)}_{\mathbf{3}} \oplus \underline{(9,0,0)}_{\mathbf{1}} \tag{5.21}$$

which leads to the quiver



$$\tag{5.22}$$
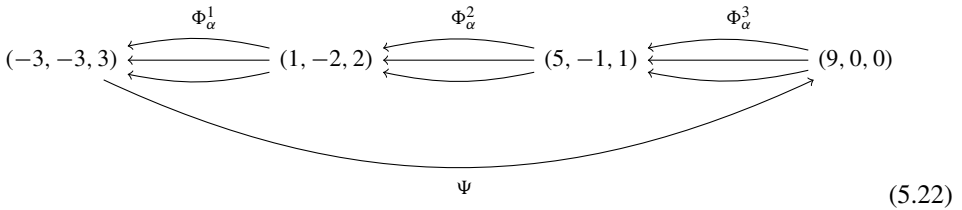
The underlying structure of this quiver is again that of a chain connecting adjacent vertices with three Higgs fields, as its weight diagram is an extension of those for the representations **4** and **10**. As the powers of $\zeta_k$ here are large enough, we encounter also the morphism $\Psi$ induced by $Y_4$.

Since the representations of SU(4) we considered here are constructed via layers of representations of the subgroup SU(3), one may expect the general shape of the quivers with this choice of group action $\gamma(h_k)$ to hold also for generic groups SU($n+1$). Studying other choices for the action on the fibres even for the concrete case $\mathbb{C}^4/\Gamma$ is beyond the scope of this paper, as is the inclusion of other finite subgroups $\Gamma \subset \mathrm{SU}(4)$.

### 5.3. Moduli spaces

In Section 5.2 we considered some examples of translationally invariant connections on $\mathbb{R}^8/\mathbb{Z}_k$, where we specialised to the instanton equations with respect to *one* complex structure, i.e. we considered a *single* Hermitian Yang-Mills moduli space over the flat Calabi-Yau orbifold $\mathbb{R}^8/\mathbb{Z}_k$. Analogously to the discussion of the moduli spaces of such instantons on $\mathbb{R}^6/\mathbb{Z}_k$ in [10] and the discussion in Section 3.5, the moduli space can be described as a Kähler quotient involving the pre-image of the Fayet-Iliopoulos term $\Xi$ under a moment map on the space of holomorphic $\Gamma$-equivariant connections. Since we now consider translationally invariant connections, there are only algebraic holomorphicity conditions, in contrast to the SU(4)-instantons discussed in Section 3. This moduli space then resembles a noncommutative crepant resolution, defined by the path algebra of the underlying quiver, of the nilpotent orbits that appeared in (3.69), lending a gauge theory interpretation for some of the constructions of [45].

However, it is also interesting to study Sp(2)-instantons in this context. Then one can include a triplet of Fayet-Iliopoulos terms $\Xi^\alpha$, and the generalized instanton equation reads

$$\star\mathcal{F} + \mathcal{F} \wedge \star Q = \Xi^\alpha \star \Omega_\alpha + \Xi^\alpha\, \Omega_\alpha \wedge \star Q \qquad \text{or} \qquad \star\tilde{\mathcal{F}} + \tilde{\mathcal{F}} \wedge \star Q = 0 \tag{5.23}$$

for a deformed curvature $\tilde{\mathcal{F}} := \mathcal{F} - \Xi^\alpha\,\Omega_\alpha$. This generalizes the four-dimensional anti-self-duality equation with deformations induced by Fayet-Iliopoulos terms $\Xi^\alpha$. In this case, for constant $\Xi^\alpha$ lying in the centre of the Lie algebra $\mathfrak{u}(r)$, the generalized instanton equations with deformations still imply the Yang-Mills equations, but otherwise they induce sources [46]. It was

shown in [43] that the moduli space of the four-dimensional instanton equation can be expressed as a hyper-Kähler quotient $\mathcal{M}_{\Xi} = \left( \mu_1^{-1}(\Xi^1) \cap \mu_2^{-1}(\Xi^2) \cap \mu_3^{-1}(\Xi^3) \right) /\!/\!/ \, \mathcal{G}$, and that it is closely related to the ALE spaces which appear as resolutions of the orbifold $\mathbb{C}^2/\Gamma$.

For vanishing Fayet-Iliopoulos terms in our eight-dimensional setup, the moduli space can again be described as the intersection of three Hermitian Yang-Mills moduli spaces by the general arguments of Section 4.5:

$$\mathcal{M}_0 = \mathcal{M}_1 \cap \mathcal{M}_2 \cap \mathcal{M}_3 \,. \tag{5.24}$$

Since non-vanishing $\Xi^\alpha$ describe deformations of the complex structures [46], one cannot simply impose the holomorphicity conditions on $\mathcal{F}$. Instead, by rewriting the instanton equation (5.23) in terms of the deformed curvature $\tilde{\mathcal{F}}$, one gets formally again the "usual" equation whose moduli space is given as the intersection of three Hermitian Yang-Mills moduli spaces,

$$\mathcal{M}_{\Xi} = \tilde{\mathcal{M}}_1 \cap \tilde{\mathcal{M}}_2 \cap \tilde{\mathcal{M}}_3 \,, \tag{5.25}$$

where each moduli space $\tilde{\mathcal{M}}_\alpha$ is determined by imposing the usual holomorphicity (and stability) condition with respect to the undeformed complex structure $J_\alpha$ on the *deformed* curvature $\tilde{\mathcal{F}} = \mathcal{F} - \Xi^\alpha \, \Omega_\alpha$.

## 6. Summary and conclusions

In this paper we have studied three classes of quiver gauge theories: (1) Sasakian quiver gauge theories on the round seven-sphere SU(4)/SU(3), (2) 3-Sasakian quiver gauge theories on the squashed seven-sphere Sp(2)/Sp(1), and (3) Translationally invariant instantons on the Calabi-Yau cone $\mathbb{R}^8/\Gamma$. In all cases we discussed the equivariance conditions, giving explicit examples of the resulting quivers for some low-dimensional representations of $G$, and described the corresponding moduli spaces.

The Sasakian quiver gauge theory on $S^7$ yields a higher-dimensional analogue of that on $S^5$ [10], but with one completely new class due to the exceptional representation **6** of SU(4). Because of the systematic construction of all odd-dimensional round spheres $S^{2n+1} \cong$ SU($n$+1)/SU($n$), one can expect the regular quivers to be the same for all cases. This fits in the framework of generic expressions for quiver gauge theories [19] and the general description of moduli spaces for Hermitian Yang-Mills instantons on metric cones over Sasaki-Einstein manifolds [33]; indeed, the moduli space of instantons on the Calabi-Yau cone we discussed here is contained in the general description of [33].

Making use of the 3-Sasakian structure on the coset space Sp(2)/Sp(1), we constructed new quiver gauge theories based on representations of Sp(2), again giving some explicit examples of quivers. We discussed the more complicated instanton equations on the metric cone and showed that the moduli space can be described as the intersection of three Hermitian Yang-Mills moduli spaces. In contrast to a single Hermitian Yang-Mills instanton moduli space over a Calabi-Yau cone, in the hyper-Kähler setup the holomorphicity conditions automatically imply the stability conditions of the Hermitian Yang-Mills moduli spaces.

Finally, we discussed some examples of quivers for translationally invariant instantons on $\mathbb{R}^8/\Gamma$ with the finite group $\Gamma = \mathbb{Z}_k$ embedded into SU(4). While the moduli space of Hermitian Yang-Mills instantons can be described as a Kähler quotient for a possibly non-trivial Fayet-Iliopoulos term, as in the case of translationally invariant instantons on $\mathbb{R}^6/\mathbb{Z}_k$ [10], we attributed the moduli space of instantons with respect to the hyper-Kähler structure again to the intersection

of three Hermitian Yang-Mills moduli spaces. Hereby, one has to consider deformed curvatures if non-trivial Fayet-Iliopoulos terms are present.

## Acknowledgements

## Appendix A. Technical details for $S^7 \cong \mathrm{SU}(4)/\mathrm{SU}(3)$

This appendix contains some technical details of calculations involving the round seven-sphere, including the derivation of the parameter dependences in the structure equations, the canonical connection and weight diagrams, and some explicit realizations of SU(4)-representations.

### A.1. Structure equations and Sasaki-Einstein geometry

By the choice of the 1-forms in (3.8), the generators of the Lie algebra of SU(4) in the fundamental representation take the form

$$
I_1^+ := \zeta_1 \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad
I_2^+ := \zeta_2 \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
$$

$$
I_3^+ := \zeta_3 \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad
I_4^+ := \lambda_4 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
$$

$$
I_5^+ := \lambda_5 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad
I_6^+ := \lambda_6 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},
$$

$$
I_7 := \mathrm{i}\,\mu_7 \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad
I_8 := \mathrm{i}\,\mu_8 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},
$$

$$
I_9 := \mathrm{i}\,\mu_9 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{A.1}
$$

with real parameters $\zeta_\alpha$, $\lambda_\beta$ and $\mu_i$. The flatness of the connection (3.8) yields the structure equations

$$d\Theta^1 = 4\,\mu_7\,\Theta^1 \wedge \mathrm{i}\,e^7 - 2\,\mu_8\,\Theta^1 \wedge \mathrm{i}\,e^8 + \zeta_2\,\lambda_4\,\zeta_1^{-1}\,\Theta^{2\bar{4}} + \zeta_3\,\lambda_5\,\zeta_1^{-1}\,\Theta^{3\bar{5}}\,,$$

$$d\Theta^2 = 4\,\mu_7\,\Theta^2 \wedge \mathrm{i}\,e^7 + \mu_8\,\Theta^2 \wedge \mathrm{i}\,e^8 + \mu_9\,\Theta^2 \wedge \mathrm{i}\,e^9 - \zeta_1\,\lambda_4\,\zeta_2^{-1}\,\Theta^{14} + \zeta_3\,\lambda_6\,\zeta_2^{-1}\,\Theta^{3\bar{6}}\,,$$

$$d\Theta^3 = 4\,\mu_7\,\Theta^3 \wedge \mathrm{i}\,e^7 + \mu_8\,\Theta^3 \wedge \mathrm{i}\,e^8 - \mu_9\,\Theta^3 \wedge \mathrm{i}\,e^9 - \zeta_2\,\lambda_6\,\zeta_3^{-1}\,\Theta^{26} - \zeta_1\,\lambda_5\,\zeta_3^{-1}\,\Theta^{15}\,,$$

$$de^7 = -\tfrac{\mathrm{i}}{3\mu_7}\left(\zeta_1^2\,\Theta^{1\bar{1}} + \zeta_2^2\,\Theta^{2\bar{2}} + \zeta_3^2\,\Theta^{3\bar{3}}\right)\,, \tag{A.2}$$

together with the equations for the complex conjugates $\bar{\Theta}^{\bar{\alpha}}$ for $\alpha = 1, 2, 3$. The necessary dependences of the parameters in (3.16) follow from imposing closure of the fundamental form $\Omega^{1,1}$:

$$2\,\mathrm{i}\,d\Omega^{1,1} = r^2\left(\lambda_4\left(\frac{\zeta_2}{\zeta_1} - \frac{\zeta_1}{\zeta_2}\right)\left(\Theta^{\bar{1}2\bar{4}} - \Theta^{1\bar{2}4}\right) + \lambda_5\left(\frac{\zeta_3}{\zeta_1} - \frac{\zeta_1}{\zeta_3}\right)\left(\Theta^{\bar{1}35} - \Theta^{1\bar{3}\bar{5}}\right)\right.$$

$$+ \lambda_6\left(\frac{\zeta_2}{\zeta_3} - \frac{\zeta_3}{\zeta_2}\right)\left(\Theta^{2\bar{3}6} - \Theta^{\bar{2}3\bar{6}}\right) + \left(1 - \frac{\zeta_1^2}{3\mu_7}\right)\Theta^{1\bar{1}} \wedge \left(\Theta^0 + \bar{\Theta}^{\bar{0}}\right) \quad \text{(A.3)}$$

$$\left.+ \left(1 - \frac{\zeta_2^2}{3\mu_7}\right)\Theta^{2\bar{2}} \wedge \left(\Theta^0 + \bar{\Theta}^{\bar{0}}\right) + \left(1 - \frac{\zeta_3^2}{3\mu_7}\right)\Theta^{3\bar{3}} \wedge \left(\Theta^0 + \bar{\Theta}^{\bar{0}}\right)\right)\,.$$

The exterior derivative of the top-degree holomorphic form reads

$$d\Omega^{4,0} = r^4\,(2 - 6\mu_7)\,\Theta^{1230\bar{0}}\,, \tag{A.4}$$

leading to the condition (3.18). With these parameter values one obtains the structure equations (3.19) together with

$$d\Theta^4 = -\tfrac{1}{2}\,\mathrm{i}\,e^8 \wedge \Theta^4 - \tfrac{1}{2}\,\mathrm{i}\,e^9 \wedge \Theta^4 + \Theta^{\bar{1}2} + \Theta^{56}\,,$$

$$d\Theta^5 = -\tfrac{1}{2}\,\mathrm{i}\,e^8 \wedge \Theta^5 + \tfrac{1}{2}\,\mathrm{i}\,e^9 \wedge \Theta^5 + \Theta^{\bar{1}3} - \Theta^{46}\,,$$

$$d\Theta^6 = \mathrm{i}\,e^9 \wedge \Theta^6 + \Theta^{\bar{2}3} + \Theta^{\bar{4}5}\,,$$

$$de^8 = 2\,\mathrm{i}\,\Theta^{1\bar{1}} - \mathrm{i}\,\Theta^{2\bar{2}} - \mathrm{i}\,\Theta^{3\bar{3}} - 3\,\mathrm{i}\,\Theta^{4\bar{4}} - 3\,\mathrm{i}\,\Theta^{5\bar{5}}\,,$$

$$de^9 = -\mathrm{i}\,\Theta^{2\bar{2}} + \mathrm{i}\,\Theta^{3\bar{3}} - \mathrm{i}\,\Theta^{4\bar{4}} + \mathrm{i}\,\Theta^{5\bar{5}} + 2\,\mathrm{i}\,\Theta^{6\bar{6}}\,. \tag{A.5}$$

With respect to the rescaled Cartan generators $\hat{I}_j := -\mathrm{i}\,\mu_j^{-1}\,I_j$, i.e.

$$\hat{I}_7 = \mathrm{diag}\,(3, -1, -1, -1)\,,\quad \hat{I}_8 = \mathrm{diag}\,(0, 2, -1, -1)\quad\text{and}\quad \hat{I}_9 = \mathrm{diag}\,(0, 0, -1, 1)\,, \tag{A.6}$$

the non-vanishing structure constants read

$$C_{84}^4 = C_{85}^5 = 3\,,\quad C_{94}^4 = 1\,,\quad C_{95}^5 = -1\,,\quad C_{96}^6 = -2\,,\quad C_{4\bar{4}}^8 = C_{5\bar{5}}^8 = -\tfrac{1}{2}\,,$$

$$C_{4\bar{4}}^9 = -\tfrac{1}{2}\,,\quad C_{5\bar{5}}^9 = \tfrac{1}{2}\,,\quad C_{6\bar{6}}^9 = 1\,,\quad C_{56}^4 = -1\,,\quad C_{46}^5 = 1\,,\quad C_{45}^6 = -1\,,$$

$$C_{71}^1 = C_{72}^2 = C_{73}^3 = 4\,,\quad C_{81}^1 = -2\,,\quad C_{82}^2 = C_{83}^3 = 1\,,\quad C_{92}^2 = 1\,,\quad C_{93}^3 = -1\,,$$

$$C^7_{1\bar{1}} = C^7_{2\bar{2}} = C^7_{3\bar{3}} = -\tfrac{1}{3}\,, \quad C^8_{1\bar{1}} = \tfrac{1}{3}\,, \quad C^8_{2\bar{2}} = -\tfrac{1}{6}\,, \quad C^8_{3\bar{3}} = -\tfrac{1}{6}\,, \quad C^9_{2\bar{2}} = -\tfrac{1}{2}\,,$$

$$C^9_{3\bar{3}} = \tfrac{1}{2}\,, \quad C^1_{2\bar{4}} = -1\,, \quad C^1_{3\bar{5}} = -1\,, \quad C^2_{14} = 1\,, \quad C^2_{3\bar{6}} = -1\,, \quad C^3_{15} = 1\,,$$

$$C^3_{26} = 1\,, \quad C^4_{\bar{1}2} = -1\,, \quad C^5_{\bar{1}3} = -1\,, \quad C^6_{23} = -1 \tag{A.7}$$

plus the conjugated ones.

Explicit evaluation of the torsion of the canonical connection yields the components

$$T^1 = \tfrac{4}{3}\,e^{27}\,, \qquad T^2 = -\tfrac{4}{3}\,e^{17}\,, \qquad T^3 = \tfrac{4}{3}\,e^{47}\,,$$

$$T^4 = -\tfrac{4}{3}\,e^{37}\,, \qquad T^5 = \tfrac{4}{3}\,e^{67}\,, \qquad T^6 = -\tfrac{4}{3}\,e^{57}\,,$$

$$T^7 = 2\left(e^{12} + e^{34} + e^{56}\right)\,, \tag{A.8}$$

which leads to the results in the main text.

## A.2. *Representations of* SU(4)

We shall now provide the weight diagrams and some of the generators which are used for the explicit examples of quivers. To this end, recall that the ladder operators act, according to the structure constants (A.7), on the quantum numbers as

$$I^-_{\bar{1}} : (\nu_7, \nu_8, \nu_9) \longmapsto (\nu_7 - 4, \nu_8 + 2, \nu_9)\,, \qquad I^-_{\bar{4}} : (\nu_7, \nu_8, \nu_9) \longmapsto (\nu_7, \nu_8 - 3, \nu_9 - 1)\,,$$

$$I^-_{\bar{2}} : (\nu_7, \nu_8, \nu_9) \longmapsto (\nu_7 - 4, \nu_8 - 1, \nu_9 - 1)\,, \qquad I^-_{\bar{5}} : (\nu_7, \nu_8, \nu_9) \longmapsto (\nu_7, \nu_8 - 3, \nu_9 + 1)\,,$$

$$I^-_{\bar{3}} : (\nu_7, \nu_8, \nu_9) \longmapsto (\nu_7 - 4, \nu_8 - 1, \nu_9 + 1)\,, \qquad I^-_{\bar{6}} : (\nu_7, \nu_8, \nu_9) \longmapsto (\nu_7, \nu_8, \nu_9 + 2)\,,$$

$$\tag{A.9}$$

which yields the root system[17]



$$\tag{A.10}$$
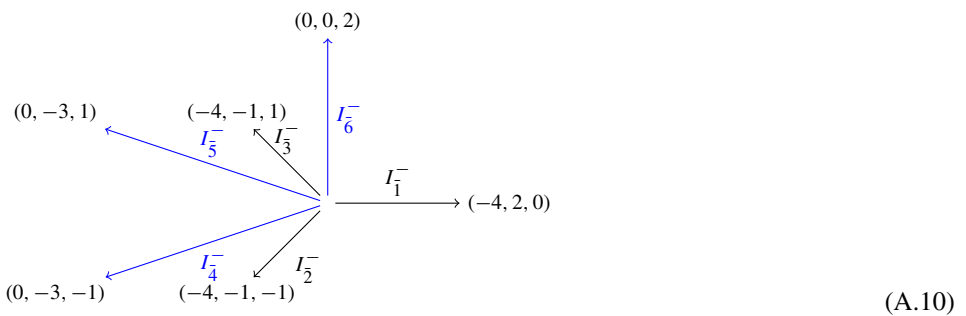
Here we depict the ladder operators of the subalgebra $\mathfrak{h} = \mathfrak{su}(3)$, along which one has to collapse the weight diagram, by blue arrows. (For interpretation of the colours in the figure(s), the reader is referred to the web version of this article.) By using this root system, one can easily construct the representations we list in the following. For details on the representation theory of SU(4) (or SL(4, $\mathbb{C}$)), see [28].

---

[17] For better readability, we restrict to the generators $I^-_{\bar{\alpha}}$ and do not depict the adjoint generators.

*Fundamental representation* $\underline{4}$   The generators are those of the chosen defining representation (A.1), and the weight diagram is the tetrahedron



$$(A.11)$$

*Representation* $\underline{6}$   The generators of the six-dimensional irreducible representation can be chosen as

$$I_{\tilde{\alpha}}^- = \begin{pmatrix} \mathbf{0}_3 & \mathbf{0}_3 \\ \tilde{I}_{\tilde{\alpha}} & \mathbf{0}_3 \end{pmatrix} = -\left(I_\alpha^+\right)^\dagger \quad \text{for } \alpha = 1, 2, 3 \tag{A.12}$$

with

$$\tilde{I}_{\tilde{1}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad \tilde{I}_{\tilde{2}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \tilde{I}_{\tilde{3}} = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{A.13}$$

and

$$I_{\tilde{\beta}}^- = \begin{pmatrix} \tilde{I}_{\tilde{\beta}}^1 & \mathbf{0}_3 \\ \mathbf{0}_3 & \tilde{I}_{\tilde{\beta}}^2 \end{pmatrix} = -\left(I_\beta^+\right)^\dagger \quad \text{for } \beta = 4, 5, 6 \tag{A.14}$$

with

$$\tilde{I}_{\tilde{4}}^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad \tilde{I}_{\tilde{4}}^2 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \tilde{I}_{\tilde{5}}^1 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{A.15a}$$

$$\tilde{I}_{\tilde{5}}^2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \tilde{I}_{\tilde{6}}^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \tilde{I}_{\tilde{6}}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \tag{A.15b}$$

$$\hat{I}_7 = \text{diag}\,(2\mathbb{1}_3, -2\mathbb{1}_3)\,, \quad \hat{I}_8 = \text{diag}\,(2, -1, -1, -2, 1, 1)\,,$$
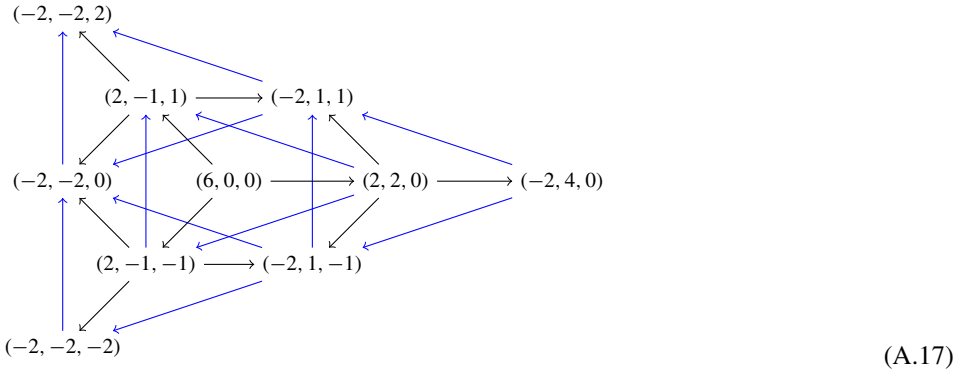$$\hat{I}_9 = \text{diag}\,(0, 1, -1, 0, 1, -1)\,. \tag{A.15c}$$

The weight diagram is the octahedron



$$(A.16)$$
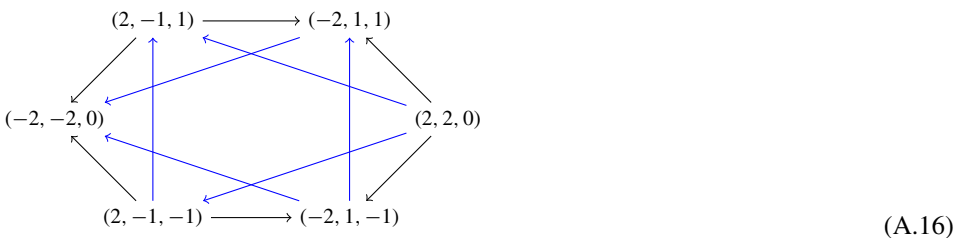
Collapsing this diagram along the action of the ladder operators of SU(3) (blue arrows) then yields the quiver (3.36).

*Representation* **10**   The weight diagram of the ten-dimensional representation reads



$$(\text{A.17})$$

It can be obtained from the representation **6** by adding four *fundamental tetrahedra* to the octahedron of **6**. Each layer with a fixed quantum number $v_7 \in \{6, 2, -2\}$ corresponds to one SU(3)-representation, so that collapsing along the SU(3) generators leads to a quiver with three vertices. The generators in this representation can be chosen so that their only non-vanishing components are

$$
\begin{aligned}
&\left(I_{\bar{1}}^{-}\right)_{21} = \sqrt{2}\,, && \left(I_{\bar{1}}^{-}\right)_{52} = -\sqrt{2}\,, && \left(I_{\bar{1}}^{-}\right)_{63} = -1, && \left(I_{\bar{1}}^{-}\right)_{74} = -1\,, \\
&\left(I_{\bar{2}}^{-}\right)_{31} = \sqrt{2}\,, && \left(I_{\bar{2}}^{-}\right)_{62} = -1\,, && \left(I_{\bar{2}}^{-}\right)_{93} = -\sqrt{2}\,, && \left(I_{\bar{2}}^{-}\right)_{84} = -1\,, \\
&\left(I_{\bar{3}}^{-}\right)_{41} = \sqrt{2}\,, && \left(I_{\bar{3}}^{-}\right)_{72} = -1\,, && \left(I_{\bar{3}}^{-}\right)_{83} = -1\,, && \left(I_{\bar{3}}^{-}\right)_{10\,4} = -\sqrt{2}\,, \\
&\left(I_{\bar{4}}^{-}\right)_{32} = -1\,, && \left(I_{\bar{4}}^{-}\right)_{87} = -1\,, && \left(I_{\bar{4}}^{-}\right)_{65} = -\sqrt{2}\,, && \left(I_{\bar{4}}^{-}\right)_{96} = -\sqrt{2}\,, \\
&\left(I_{\bar{5}}^{-}\right)_{42} = -1\,, && \left(I_{\bar{5}}^{-}\right)_{75} = -\sqrt{2}\,, && \left(I_{\bar{5}}^{-}\right)_{86} = -1\,, && \left(I_{\bar{5}}^{-}\right)_{10\,7} = -\sqrt{2}\,, \\
&\left(I_{\bar{6}}^{-}\right)_{43} = -1\,, && \left(I_{\bar{6}}^{-}\right)_{76} = -1\,, && \left(I_{\bar{6}}^{-}\right)_{10\,8} = -\sqrt{2}\,, && \left(I_{\bar{6}}^{-}\right)_{89} = -\sqrt{2}
\end{aligned}
$$
$$(\text{A.18})$$

and

$$
\begin{aligned}
I_7 &= \operatorname{diag}\left(6, 2, 2, 2, -2, -2, -2, -2, -2, -2\right), \\
I_8 &= \operatorname{diag}\left(0, 2, -1, -1, 4, 1, 1, -2, -2, -2\right), \\
I_9 &= \operatorname{diag}\left(0, 0, -1, 1, 0, -1, 1, 0, -2, 2\right).
\end{aligned}
$$
$$(\text{A.19})$$

*Adjoint representation* **15**   The adjoint representation is described by the weight diagram

$$(A.20)$$
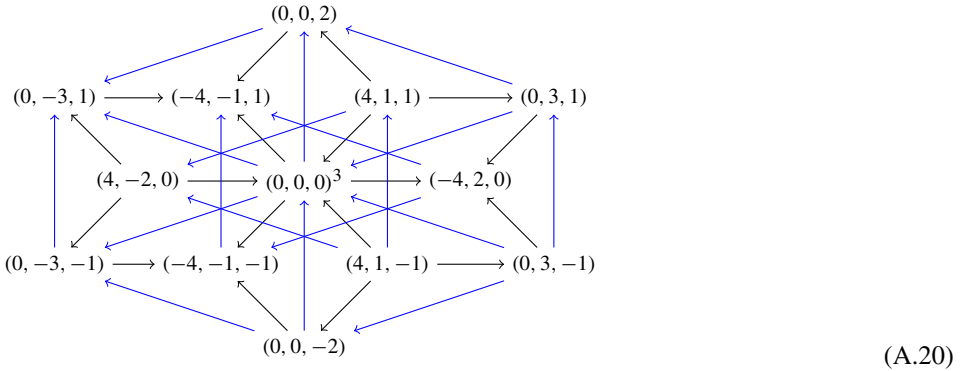
where the centre node at $(0, 0, 0)$ has multiplicity 3. Collapsing this diagram yields four vertices representing one trivial, one fundamental, one anti-fundamental, and one adjoint representation of SU(3). The generators are given by the structure constants (A.7) as $\left(I_{\hat{\lambda}}\right)_{\hat{\mu}}^{\hat{\nu}} = C_{\hat{\lambda}\hat{\mu}}^{\hat{\nu}}$.

## Appendix B. Technical details for $S^7 \cong \mathrm{Sp}(2)/\mathrm{Sp}(1)$

We shall now provide some details on the calculations needed for the squashed seven-sphere $\mathrm{Sp}(2)/\mathrm{Sp}(1)$, in particular the defining properties of a 3-Sasakian manifold, and also representations of $\mathrm{Sp}(2)$ where we again refer to [28].

### B.1. Structure equations and 3-Sasakian geometry

The choice of 1-forms in (4.7) yields the fundamental representation of the generators as

$$
I_{\bar{1}}^- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \qquad
I_{\bar{2}}^- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
$$

$$
I_{\bar{3}}^- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad
I_{\bar{4}}^- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix},
$$

$$(B.1)$$

$$\hat{I}_7 := -\mathrm{i}\, I_7 = \mathrm{diag}\,(1, -1, 0, 0)\ , \qquad \hat{I}_8 := -\mathrm{i}\, I_8 = \mathrm{diag}\,(0, 0, -1, 1)\ ,$$

and the (complex) structure equations read

$$
\begin{aligned}
\mathrm{d}\Theta^1 &= -\mathrm{i}\, e^7 \wedge \Theta^1 + \mathrm{i}\, e^8 \wedge \Theta^1 - \Theta^{\bar{2}3} + \Theta^{2\bar{4}}\ , \\
\mathrm{d}\Theta^2 &= -\mathrm{i}\, e^7 \wedge \Theta^2 - \mathrm{i}\, e^8 \wedge \Theta^2 + \Theta^{\bar{1}3} - \Theta^{14}\ , \\
\mathrm{d}\Theta^3 &= -2\,\mathrm{i}\, e^7 \wedge \Theta^3 - 2\,\Theta^{12}\ , && \mathrm{d}\Theta^4 = -2\,\mathrm{i}\, e^8 \wedge \Theta^4 + 2\,\Theta^{\bar{1}2}\ , \\
\mathrm{d}e^7 &= -\mathrm{i}\left(\Theta^{1\bar{1}} + \Theta^{2\bar{2}} + \Theta^{3\bar{3}}\right)\ , && \mathrm{d}e^8 = \mathrm{i}\left(\Theta^{1\bar{1}} - \Theta^{2\bar{2}} - \Theta^{4\bar{4}}\right)\ .
\end{aligned}
$$

$$(B.2)$$

This yields the non-vanishing structure constants

$$f_{71}^1 = -f_{7\bar1}^{\bar1} = 1\ , \qquad f_{72}^2 = -f_{7\bar2}^{\bar2} = 1, \qquad f_{73}^3 = -f_{7\bar3}^{\bar3} = 2, \qquad f_{81}^1 = -f_{8\bar1}^{\bar1} = -1,$$

$$f_{82}^2 = -f_{8\bar2}^{\bar2} = 1, \qquad f_{84}^4 = -f_{8\bar4}^{\bar4} = 2, \qquad f_{2\bar3}^{\bar1} = f_{23}^1 = 1, \qquad f_{2\bar4}^1 = f_{24}^{\bar1} = -1,$$

$$f_{12}^3 = f_{\bar12}^{\bar3} = 2, \qquad f_{14}^2 = f_{\bar14}^{\bar2} = 1, \qquad f_{13}^2 = f_{\bar13}^{\bar2} = -1, \qquad f_{12}^4 = f_{\bar12}^{\bar4} = -2,$$

$$f_{1\bar1}^7 = f_{2\bar2}^7 = f_{3\bar3}^7 = -1, \qquad f_{1\bar1}^8 = 1, \qquad f_{2\bar2}^8 = -1, \qquad f_{4\bar4}^8 = -1.$$

(B.3)

The generators $\{I_4^+, I_4^-, \hat I_8\}$ span the subalgebra $\mathfrak{sp}(1) \cong \mathfrak{su}(2)$ which is factored when forming the homogeneous space.

The fact that the orthonormal basis $\{e^1, \dots, e^7\}$ describes a 3-Sasakian structure can also be shown by considering the metric cone. By definition, a manifold $M$ is 3-Sasakian if its metric cone $C(M)$ is hyper-Kähler. For this, one can again introduce a fourth holomorphic 1-form,

$$\Theta^0 := \frac{\mathrm{d}r}{r} - \mathrm{i}\, e^7\ , \tag{B.4}$$

as in Section 3.1, and establish the Sasaki-Einstein property because the forms

$$\Omega^{1,1} := -\tfrac{\mathrm{i}}{2}\, r^2 \left( \Theta^{1\bar1} + \Theta^{2\bar2} + \Theta^{3\bar3} + \Theta^{0\bar0} \right) \qquad \text{and} \qquad \Omega^{4,0} := r^4\, \Theta^{1230} \tag{B.5}$$

are closed. For the holonomy to be further reduced from SU(4) to Sp(2), one additionally requires closure of the form [18]

$$\Omega^{2,0} := r^2 \left( \Theta^{12} + \Theta^{30} \right)\ , \tag{B.6}$$

which follows from the structure equations (B.2).

### B.2. Instanton equations on hyper-Kähler and Calabi-Yau cones

We will now provide some technical details of the relation between instanton equations on the metric cones over 3-Sasakian manifolds and those over Sasaki-Einstein manifolds. In components the quaternion relations (4.50) read

$$J_{\alpha\mu}^\sigma J_{\beta\sigma}^\nu = -\delta_{\alpha\beta}\, \delta_\mu^\nu + \epsilon_{\alpha\beta}{}^\gamma\, J_{\gamma\mu}^\nu\ , \tag{B.7}$$

and the associated Kähler forms

$$\Omega_{\alpha\mu\nu} = g_{\mu\sigma} J_{\alpha\nu}^\sigma = -g_{\nu\sigma} J_{\alpha\mu}^\sigma = -g_{\sigma\nu} J_{\alpha\mu}^\sigma\ , \qquad J_{\alpha\mu}^\nu = g^{\nu\sigma}\, \Omega_{\alpha\sigma\mu} \tag{B.8}$$

satisfy

$$J_{\alpha\mu}^\sigma \Omega_{\beta\sigma\nu} = \delta_{\alpha\beta}\, g_{\mu\nu} + \epsilon_{\alpha\beta}{}^\gamma\, \Omega_{\gamma\mu\nu}\ ,$$

$$J_{\alpha\sigma}^\mu \Omega_\beta^{\sigma\nu} = \delta_{\alpha\beta}\, g^{\mu\nu} - \epsilon_{\alpha\beta}{}^\gamma\, \Omega_\gamma^{\mu\nu} \qquad \text{with} \quad \Omega_\alpha^{\mu\nu} := g^{\mu\rho}\, J_{\alpha\rho}^\nu\ . \tag{B.9}$$

Using (B.7), one obtains from the first equation of (4.55) the relation

$$J_{\alpha\mu}^\lambda J_{\beta\nu}^\sigma \mathcal{F}_{\lambda\sigma} = \delta_{\alpha\beta}\, \mathcal{F}_{\mu\nu} + \epsilon_{\alpha\beta}{}^\gamma\, J_{\gamma\mu}^\sigma \mathcal{F}_{\nu\sigma}\ , \tag{B.10}$$

which includes the second equation of (4.55) upon setting $\alpha = \beta$. To show that the stability condition of Hermitian Yang-Mills instantons automatically follows from the holomorphicity conditions, one contracts (4.55) with $\Omega_\beta^{\mu\nu}$ for $\beta \neq \alpha$ to get

$$J^{\lambda}_{\alpha\mu} \, J^{\sigma}_{\alpha\nu} \, \Omega^{\mu\nu}_{\beta} \, \mathcal{F}_{\lambda\sigma} = \Omega^{\mu\nu}_{\beta} \, \mathcal{F}_{\mu\nu} \qquad \text{(no sum on } \alpha \text{)} \, . \tag{B.11}$$

With the help of the properties (B.8) and (B.9), one can show that the left-hand side gives the negative of the right-hand side, so that the stability condition

$$0 = \Omega^{\mu\nu}_{\beta} \, \mathcal{F}_{\mu\nu} \tag{B.12}$$

follows.[18]

For our case of the cone $C(\mathrm{Sp}(2)/\mathrm{Sp}(1))$, the $\mathrm{Sp}(2)$-instanton equation explicitly gives the flow equations

$$\frac{\mathrm{d}X_1}{\mathrm{d}\tau} = -X_1 + \big[X_3, X_5\big] = -X_1 + \big[X_4, X_6\big] = -X_1 - \big[X_2, X_7\big] \, ,$$

$$\frac{\mathrm{d}X_2}{\mathrm{d}\tau} = -X_2 - \big[X_4, X_5\big] = -X_2 + \big[X_3, X_6\big] = -X_2 + \big[X_1, X_7\big] \, ,$$

$$\frac{\mathrm{d}X_3}{\mathrm{d}\tau} = -X_3 - \big[X_1, X_5\big] = -X_3 - \big[X_2, X_6\big] = -X_3 - \big[X_4, X_7\big] \, ,$$

$$\frac{\mathrm{d}X_4}{\mathrm{d}\tau} = -X_4 + \big[X_2, X_5\big] = -X_4 - \big[X_1, X_6\big] = -X_4 + \big[X_3, X_7\big] \, , \tag{B.13}$$

$$\frac{\mathrm{d}X_5}{\mathrm{d}\tau} = -2X_5 - \big[X_6, X_7\big] \, , \qquad \frac{\mathrm{d}X_6}{\mathrm{d}\tau} = -2X_6 + \big[X_5, X_7\big] \, ,$$

$$\frac{\mathrm{d}X_7}{\mathrm{d}\tau} = -2X_7 - \big[X_5, X_6\big] \, ,$$

while the algebraic conditions read

$$4X_5 = \big[X_1, X_3\big] - \big[X_2, X_4\big] \, , \qquad 4X_6 = \big[X_1, X_4\big] + \big[X_2, X_3\big] \, , \tag{B.14}$$

$$4X_7 = -\big[X_1, X_2\big] - \big[X_3, X_4\big] \, .$$

One recognizes immediately the form of the intersection of three Hermitian Yang-Mills instanton equations as discussed in Section 3.5.[19]

*Scalar solution* Setting $X_a = \lambda(r) \, I_a$ for $a = 1, 2, 3, 4$ and $X_\alpha = \psi(r) \, I_\alpha$ for $\alpha = 5, 6, 7$ with functions $\lambda$ and $\psi$, the equivariance conditions are automatically satisfied and the instanton equations reduce to the system

---

[18] As an example, consider the Hermitian Yang-Mills equations associated to $\Omega_7$ where the holomorphicity conditions yield the relations

$$\mathcal{F}_{13} = \mathcal{F}_{24} \, , \quad \mathcal{F}_{15} = \mathcal{F}_{26} \, , \quad \mathcal{F}_{35} = \mathcal{F}_{46} \, , \quad \mathcal{F}_{23} = -\mathcal{F}_{14} \, , \quad \mathcal{F}_{25} = -\mathcal{F}_{16} \, , \quad \mathcal{F}_{45} = -\mathcal{F}_{36} \, ,$$

$$\mathcal{F}_{1\tau} = \mathcal{F}_{27} \, , \quad \mathcal{F}_{2\tau} = -\mathcal{F}_{17} \, , \quad \mathcal{F}_{3\tau} = \mathcal{F}_{47} \, , \quad \mathcal{F}_{4\tau} = -\mathcal{F}_{37} \, , \quad \mathcal{F}_{5\tau} = \mathcal{F}_{67} \, , \quad \mathcal{F}_{6\tau} = -\mathcal{F}_{57} \, .$$

They imply the condition

$$0 = -\mathcal{F}_{13} + \mathcal{F}_{24} + \mathcal{F}_{67} - \mathcal{F}_{5\tau} = \Omega_5 \lrcorner \, \mathcal{F} \, ,$$

which is the stability condition associated to $\Omega_5$.

[19] Note that in these Hermitian Yang-Mills equations one has slightly different algebraic quiver relations and a different scaling factor in the flow equations for $X_5$, $X_6$ and $X_7$.

$$\dot{\lambda} = \lambda\,(\psi - 1)\,,$$
$$\dot{\psi} = 2\,\psi\,(\psi - 1)\,, \tag{B.15}$$
$$\lambda^2 = \psi\,.$$

This is exactly the scalar form for instanton equations on cones over 3-Sasakian manifolds discussed in [17], where they give the analytic solutions

$$\psi(\tau) = \left(1 + e^{2(\tau - \tau_0)}\right)^{-1} \qquad \text{and} \qquad \lambda(\tau) = \pm\,\psi(\tau)^{1/2}\,. \tag{B.16}$$

### B.3. Representations of Sp(2)

In the following we collect the weight diagrams and some explicit choices for the generators used in the main text. Due to the structure constants (B.3), the root system of the Lie algebra of Sp(2) is spanned by



$$\tag{B.17}$$

together with the conjugate operators, where the blue arrow represents the ladder operator $I_{\bar{4}}^-$ of the subalgebra $\mathfrak{sp}(1) \cong \mathfrak{su}(2)$.

*Fundamental representation* **4**    The generators are those in (B.1), and the weight diagram of the fundamental representation **4** is given by



$$\tag{B.18}$$

*Representation* **5**    The five-dimensional representation is characterized by the weight diagram



$$\tag{B.19}$$

and the generators can be chosen as

$$
I_{\bar{1}}^{-} = \begin{pmatrix} 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad
I_{\bar{2}}^{-} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},
$$

$$
I_{\bar{3}}^{-} = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad
I_{\bar{4}}^{-} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},
$$

$$
I_7 = \mathrm{diag}\,(-1,-1,0,1,1)\,, \quad I_8 = \mathrm{diag}\,(1,-1,0,1,-1)\,. \tag{B.20}
$$

*Adjoint representation* **10**   The generators of the adjoint representation are determined by the structure constants (B.3), and one obtains the weight diagram



$$\tag{B.21}$$
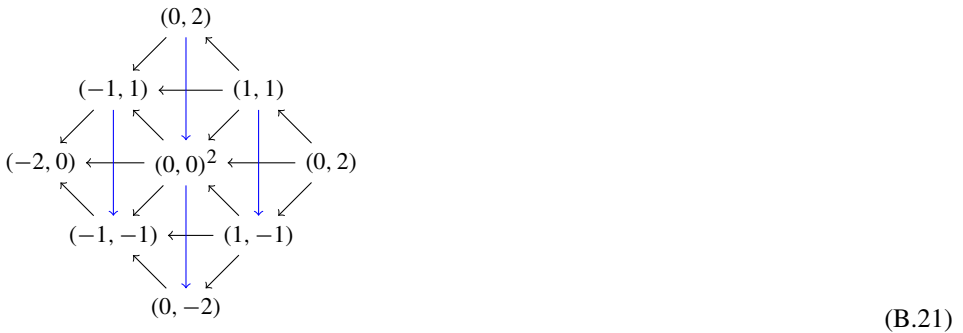
where the centre node at $(0,0)$ has multiplicity 2.

*Representation* **14**   We do not determine the generators explicitly here, but restrict our attention to the weight diagram of the 14-dimensional representation which is given by



$$\tag{B.22}$$
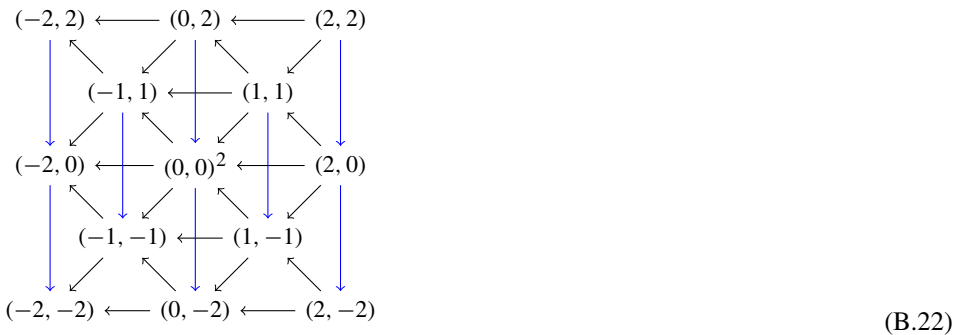
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