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Loop groups in Yang–Mills theory

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ABSTRACT

We consider the Yang–Mills equations with a matrix gauge group G on the de Sitter dS_4, anti-de Sitter AdS_4 and Minkowski $\mathbb{R}^{3,1}$ spaces. On all these spaces one can introduce a doubly warped metric in the form $ds^2 = -du^2 + f^2 dv^2 + h^2 ds^2_{H^2}$, where f and h are the functions of u and $ds^2_{H^2}$ is the metric on the two-dimensional hyperbolic space H^2. We show that in the adiabatic limit, when the metric on H^2 is scaled down, the Yang–Mills equations become the sigma-model equations describing harmonic maps from a two-dimensional manifold (dS_2, AdS_2 or $\mathbb{R}^{1,1}$, respectively) into the based loop group $\Omega G = C^\infty(S^1, G)/G$ of smooth maps from the boundary circle $S^1 = \partial H^2$ of H^2 into the gauge group G. For compact groups G these harmonic map equations are reduced to equations of geodesics on ΩG, solutions of which yield magnetic-type configurations of Yang–Mills fields. The group ΩG naturally acts on their moduli space.

1. Introduction

It is well known that the self-dual Yang–Mills equations in the Euclidean space $\mathbb{R}^{4,0}$ have an infinite-dimensional algebra of "hidden symmetries" (see [1–6] for discovering, reviews and more references). For the Yang–Mills potentials with value in a Lie algebra $\mathfrak{g} = \mathrm{Lie}\, G$, where G is a matrix gauge group, among these symmetries there is the Lie algebra of the loop group $LG = C^\infty(S^1, G)$. Here we shall show that the same group is a part of the moduli space of solutions to the Yang–Mills equations on the Lorentzian manifolds dS_4, AdS_4 and $\mathbb{R}^{3,1}$ of constant positive, negative and zero curvature.

We will use the *adiabatic limit* method which was applied to the first-order self-dual Yang–Mills equations on the product $\Sigma_1 \times \Sigma_2$ of two Riemann surfaces in [7]. It was shown that when the metric on the Riemann surface Σ_2 shrinks to a point, the Yang–Mills instantons converge to holomorphic maps from Σ_1 to the moduli space of flat connections on Σ_2. In [8] this limit was discussed in the framework of topological Yang–Mills theories on $\Sigma_1 \times \Sigma_2$. We will apply the adiabatic method to the second-order Yang–Mills equations on Lorentzian four-manifolds of constant curvature and describe how to construct approximate solutions of the Yang–Mills equations. It will be shown that these configurations become exact solutions in the adiabatic limit and their moduli space is the tangent space $T\Omega G$ of the based loop group ΩG. Thus, ΩG is a "hidden symmetry group" not only of the first-order self-dual Yang–Mills equations but also of the second-order Yang–Mills equations on Lorentzian manifolds $\mathbb{R}^{3,1}$, AdS_4 and dS_4.

2. Metrics

It is known that on the de Sitter dS_4 and anti-de Sitter AdS_4 spaces one can introduce (local) coordinates such that the metrics on these spaces will be a double warped metrics of the form [10]

$$dS_4:\ ds^2_{+1} = -du^2 + \cosh^2 u\, dv^2 + \sinh^2 u\, ds^2_{H^2}\ , \tag{1}$$

$$AdS_4:\ ds^2_{-1} = -du^2 + \sin^2 u\, dv^2 + \cos^2 u\, ds^2_{H^2}\ , \tag{2}$$

where the first two terms are metrics on the spaces dS_2 and AdS_2, respectively. Here,

$$ds^2_{H^2} = d\chi^2 + \sinh^2 \chi\, d\varphi^2\ , \tag{3}$$

is the metric on the two-dimensional hyperbolic space H^2. This space has two-sheets $H^2 = H^2_+ \cup H^2_-$ with the common boundary S^1 at $\chi \to +\infty$ for H^2_+ and $\chi \to -\infty$ for H^2_-.

We introduce on the Minkowski space–time a metric similar to (1) and (2). In the Cartesian coordinates x^μ, $\mu = 0, 1, 2, 3$, the metric has the form

$$ds^2_0 = \eta_{\mu\nu} dx^\mu dx^\nu \quad \text{with} \quad (\eta_{\mu\nu}) = \mathrm{diag}(-1, 1, 1, 1)\ . \tag{4}$$

Let us introduce coordinates u, χ and φ by

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$$x^0 = u\cosh\chi\ , \quad x^1 = u\sinh\chi\cos\varphi \quad \text{and} \quad x^2 = u\sinh\chi\sin\varphi\ , \tag{5}$$

and keep x^3 untouched. The coordinates (5) have a range

$$u = \Big((x^0)^2 - (x^1)^2 - (x^2)^2\Big)^{1/2} > 0\ , \quad \chi \in (-\infty, +\infty) \quad \text{and} \quad \varphi \in [0, 2\pi]\ . \tag{6}$$

They cover the interior of the light cone in $\mathbb{R}^{2,1}$ and we denote this subset of $\mathbb{R}^{2,1}$ by $\mathbb{R}^{2,1}_+$. The region $\mathbb{R}^{2,1}_- = \mathbb{R}^{2,1} \setminus \mathbb{R}^{2,1}_+$ can be covered by other choice of pseudospherical coordinates. In these coordinates the metric (4) acquires the form

$$ds_0^2 = -du^2 + (dx^3)^2 + u^2\, ds^2_{H^2}\ . \tag{7}$$

From (7) we recognize a cone over H^2, i.e. $\mathbb{R}^{2,1}_+ = C(H^2)$ and we restrict ourselves to the subset $\mathbb{R}^{3,1}_+ = \mathbb{R} \times \mathbb{R}^{2,1}_+ \subset \mathbb{R}^{3,1}$. For the metrics (1) and (2) we also consider $u > 0$, since for $u = 0$ they degenerate.

After denoting $x^3 = v$, we see that the metrics (1), (2) and (7) have the same form

$$ds^2 = -du^2 + f^2 dv^2 + h^2 ds^2_{H^2}\ , \tag{8}$$

where

$$\begin{aligned} &f = \cosh u \quad \text{and} \quad h = \sinh u \quad \text{for} \quad dS_4 \\ &f = \sin u \quad \text{and} \quad h = \cos u \quad \text{for} \quad AdS_4 \\ &f = 1 \quad \text{and} \quad h = u \quad \text{for} \quad \mathbb{R}^{3,1} \end{aligned} \tag{9}$$

Therefore, we will consider all three spaces together by using the metric (8), specifying f and h if necessary. Recall that we work in local coordinates which cover only part of any of the considered spaces. This is enough for our purposes. For further unification we introduce the coordinates $(y^\mu) = (y^a, y^i) = (u, v, \chi, \varphi)$, where $\mu = (a, i)$ with $a, b, \ldots = 0, 1$ and $i, j, \ldots = 2, 3$. Then metric (8) can be written as

$$ds^2 = g_{\mu\nu} dy^\mu dy^\nu = g_{ab} dy^a dy^b + g_{ij} dy^i dy^j\ , \tag{10}$$

where $g_\Sigma = (g_{ab})$ is the metric on the two-dimensional space Σ which is (a patch of) dS_2, AdS_2 or $\mathbb{R}^{1,1}$ and $(g_{ij}) = h^2(g^{H^2}_{ij})$ where $g_{H^2} = (g^{H^2}_{ij})$ is the metric on H^2.

Finally, as H^2 we will consider only the upper sheet of the two-dimensional hyperbolic space with $\chi \geq 0$ for all three metrics (1), (2), (7) and consider only $y^0 = u > 0$ in (8)–(10). All other regions of our spaces dS_4, AdS^4, and $\mathbb{R}^{3,1}$ can be considered similarly.

3. Yang–Mills equations

So, we consider Yang–Mills theory on a Lorentzian 4-manifold M with local coordinates y^μ and the metric given by (8)–(10). We start with the potential $\mathcal{A} = \mathcal{A}_\mu dy^\mu$ with values in the Lie algebra $\mathfrak{g} = \mathrm{Lie}\, G$ having scalar product defined by the trace Tr. Here G is an arbitrary matrix gauge group. The field strength $\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ is the $\mathfrak{g}$-valued two-form:

$$\mathcal{F} = \tfrac{1}{2}\mathcal{F}_{\mu\nu} dy^\mu \wedge dy^\nu \quad \text{with} \quad \mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu]\ . \tag{11}$$

The Yang–Mills equations on M with the metric given by (8)–(10) are

$$D_\mu \mathcal{F}^{\mu\nu} := \frac{1}{\sqrt{|\det g|}}\, \partial_\mu(\sqrt{|\det g|}\, \mathcal{F}^{\mu\nu}) + [\mathcal{A}_\mu, \mathcal{F}^{\mu\nu}] = 0\ , \tag{12}$$

where $g = (g_{\mu\nu})$ and indices are raised by $g^{\mu\nu}$. We have the obvious splitting

$$\mathcal{A} = \mathcal{A}_\mu dy^\mu = \mathcal{A}_a dy^a + \mathcal{A}_i dy^i\ , \tag{13}$$

$$\mathcal{F} = \tfrac{1}{2}\mathcal{F}_{ab} dy^a \wedge dy^b + \mathcal{F}_{ai} dy^a \wedge dy^i + \tfrac{1}{2}\mathcal{F}_{ij} dy^i \wedge dy^j\ . \tag{14}$$

4. Adiabatic limit

By using the adiabatic approach [7–9,11], which is based on the ideas of [12], we deform the metric (8) and introduce

$$ds^2_\varepsilon = -du^2 + f^2 dv^2 + \varepsilon^2 h^2 ds^2_{H^2}\ , \tag{15}$$

where ε is a real parameter. Then $|\det g_\varepsilon| = \varepsilon^4 |\det(g_{ab})| \det(g_{ij})$ and

$$\mathcal{F}^{ab}_\varepsilon = g^{ac}_\varepsilon g^{bd}_\varepsilon \mathcal{F}_{cd} = \mathcal{F}^{ab}, \quad \mathcal{F}^{ai}_\varepsilon = \varepsilon^{-2}\mathcal{F}^{ai} \quad \text{and} \quad \mathcal{F}^{ij}_\varepsilon = \varepsilon^{-4}\mathcal{F}^{ij}\ , \tag{16}$$

where indices of $\mathcal{F}^{\mu\nu}$ are raised by the metric $g^{\mu\nu}$ from (10).

To avoid the divergent term $\varepsilon^{-2}\mathrm{Tr}(\mathcal{F}_{ij}\mathcal{F}^{ij})$ in the Lagrangian, we impose the flatness condition

$$\mathcal{F}_{ij} = 0 \tag{17}$$

on the Yang–Mills curvature along H^2 in M. For the deformed metric (15) the Yang–Mills equations have the form

$$\varepsilon^2 D_a \mathcal{F}^{ab} + D_i \mathcal{F}^{ib} = 0\ , \tag{18}$$

$$D_a \mathcal{F}^{aj} + \varepsilon^{-2} D_i \mathcal{F}^{ij} = 0\ , \tag{19}$$

and in the limit $\varepsilon \to 0$ (after choosing $\mathcal{F}_{ij} = 0$) we have

$$D_i \mathcal{F}^{ib} = 0\ , \tag{20}$$

$$D_a \mathcal{F}^{aj} = 0\ . \tag{21}$$

5. Flat connections

Flat connection $\mathcal{A}_{H^2} = \mathcal{A}_i dy^i$ on H^2 (upper sheet) has a simple form

$$\mathcal{A}_{H^2} = g^{-1}\hat{d}g \quad \text{with} \quad \hat{d} = dy^i \frac{\partial}{\partial y^i}\ , \tag{22}$$

where $g = g(y^a, y^i)$ is a smooth map from H^2 (for any given y^a) into the gauge group G. We consider smooth matrix-valued functions g with smooth boundary value on $S^1 = \partial H^2$ and impose additional condition $g = \mathrm{Id}$ at $1 \in S^1$ (framing of flat connection on H^2 [9]). We denote by $C^\infty_0(H^2, G)$ the space of all such g in (22). On H^2, as on a manifold with the boundary, the group of gauge transformations is defined as [9]

$$\mathcal{G}_{H^2} = \Big\{ g : H^2 \to G \mid g_{|\partial H^2} = \mathrm{Id} \Big\}\ . \tag{23}$$

Hence the solution space of the equation $\mathcal{F}_{H^2} = 0$ is the infinite-dimensional group $\mathcal{N} = C^\infty_0(H^2, G)$ and the moduli space is the based loop group [13]

$$\mathcal{M} = \Omega G = C^\infty_0(H^2, G)/\mathcal{G}_{H^2}\ . \tag{24}$$

Recall that g and $\mathcal{A}_{H^2}$ depend on coordinates y^a.

6. Moduli space

On the group manifold (24) we introduce local coordinates ϕ^α with $\alpha = 1, 2, \dots$ and assume that $\mathcal{A}_\mu$'s depend on u and v only via the moduli parameters $\phi^\alpha = \phi^\alpha(u, v)$. Then moduli of flat connections on H^2 define a map

$$\phi : \Sigma \to \mathcal{M} \quad \text{with} \quad \phi(u, v) = \{\phi^\alpha(u, v)\}\,, \tag{25}$$

where by Σ we denote (a patch of) dS_2, AdS_2 or $\mathbb{R}^{1,1}$ depending on choice of M in (8)–(9). These maps are constrained by the equations (20) and (21). Since $\mathcal{A}_{H^2}$ is a flat connection for any $y^a \in \Sigma$, the derivatives $\partial_a \mathcal{A}_i$ have to satisfy the linearized around $\mathcal{A}_{H^2}$ flatness condition, i.e. $\partial_a \mathcal{A}_i$ belong to the tangent space $T_{\mathcal{A}}\mathcal{N}$ of the space $\mathcal{N} = C_0^\infty(H^2, G)$ of flat connections on H^2. Using the projection π on the moduli space, $\pi : \mathcal{N} \to \mathcal{M}$, one can decompose $\partial_a \mathcal{A}_i$ into the two parts

$$T_{\mathcal{A}}\mathcal{N} = \pi^* T_{\mathcal{A}}\mathcal{M} \oplus T_{\mathcal{A}}\mathcal{G} \quad \Leftrightarrow \quad \partial_a \mathcal{A}_i = (\partial_a \phi^\beta)\xi_{\beta i} + D_i \epsilon_a\,, \tag{26}$$

where $\mathcal{G}$ is the gauge group (restricted to H^2 by fixing $y^a \in \Sigma$, $\mathcal{G}|_{H^2} = \mathcal{G}_{H^2}$), $\{\xi_\alpha = \xi_{\alpha i} dy^i\}$ is a local basis of tangent vectors on $T_{\mathcal{A}}\mathcal{M}$ (they form the Lie algebra $\Omega\mathfrak{g}$) and ϵ_a are $\mathfrak{g}$-valued gauge parameters ($D_i \epsilon_a$ are tangent vectors from $T_{\mathcal{A}}\mathcal{G}$) which are determined by the gauge-fixing equations

$$g^{ij} D_i \xi_{\alpha j} = 0 \quad \Leftrightarrow \quad g^{ij} D_i \partial_a \mathcal{A}_j = g^{ij} D_i D_j \epsilon_a\,. \tag{27}$$

In fact, since ϕ^α depend on $y^a \in \Sigma$, we have $\mathcal{N} = \mathcal{N}(y^a)$, $\mathcal{G} = \mathcal{G}_{H^2}(y^a)$ and $\mathcal{M} = \mathcal{N}/\mathcal{G} = \mathcal{M}(y^a)$.

Recall that $\mathcal{A}_i$ are fixed by (22) and $\mathcal{A}_a$ are yet free. For the mixed components of the field strength we have

$$\mathcal{F}_{ai} = \partial_a \mathcal{A}_i - D_i \mathcal{A}_a = (\partial_a \phi^\beta)\xi_{\beta i} - D_i(\mathcal{A}_a - \epsilon_a)\,. \tag{28}$$

It is natural to choose $\mathcal{A}_a = \epsilon_a$ [12,14] and obtain

$$\mathcal{F}_{ai} = (\partial_a \phi^\beta)\xi_{\beta i} = \pi_* \partial_a \mathcal{A}_i \in T_{\mathcal{A}}\mathcal{M}\,. \tag{29}$$

On the other hand, since $\mathcal{A}_i(\phi^\alpha, y^j)$ depends on y^a only via ϕ^α, we have

$$\partial_a \mathcal{A}_i = \frac{\partial \mathcal{A}_i}{\partial \phi^\beta}\frac{\partial \phi^\beta}{\partial y^a} \quad \overset{(27)}{\Longrightarrow} \quad \epsilon_a = \mathcal{A}_a = (\partial_a \phi^\beta)\,\epsilon_\beta \tag{30}$$

with the gauge parameters ϵ_α defined by (27) via the equations

$$g^{ij} D_i D_j \epsilon_\alpha = g^{ij} D_i \frac{\partial \mathcal{A}_j}{\partial \phi^\alpha}\,. \tag{31}$$

Thus, if we know $\phi^\alpha(u, v)$ then we can construct

$$(\mathcal{A}_\mu) = (\mathcal{A}_a, \mathcal{A}_i) = \Big((\partial_a \phi^\beta)\,\epsilon_\beta\,,\ g^{-1}(\phi^\beta, y^j)\partial_i g(\phi^\beta, y^k)\Big) \tag{32}$$

which should solve the equations (20) and (21).

7. Harmonic maps

Substituting (29) into (20), one can see that these equations are resolved due to (27), (30) and (31). On the other hand, substitution of (29) and (30) into (21) gives us the equations

$$\frac{1}{\sqrt{|\det g_\Sigma|}}\,\partial_a\Big(\sqrt{|\det g_\Sigma|}\; g^{ab}\partial_b \phi^\beta\Big)\, g^{ij}_{H^2}\xi_{\beta j} + g^{ab} g^{ij}_{H^2}(D_a \xi_{\beta j})\partial_b \phi^\beta = 0\,, \tag{33}$$

where $g_\Sigma = (g_{ab})$ is the metric on Σ and $g_{H^2} = (g^{H^2}_{ij})$ is the metric on H^2. Before proceeding further we introduce a metric $\mathbb{G} = (G_{\alpha\beta})$ on the moduli space $\mathcal{M} = \Omega G$ of flat connections on H^2 as

$$G_{\alpha\beta} \equiv \langle \xi_\alpha, \xi_\beta \rangle = -\int_{H^2} d\,\mathrm{vol}\, g^{ij}_{H^2}\,\mathrm{Tr}(\xi_{\alpha i}\xi_{\beta j})\,, \tag{34}$$

where the integral is taken over $H^2 = H^2_+$. Multiplying (33) by $\xi_{\alpha i}$ and integrating over H^2 (projection on the moduli space), we obtain

$$\begin{aligned}
&\frac{1}{\sqrt{|\det g_\Sigma|}}\,\partial_a\Big(\sqrt{|\det g_\Sigma|}\; g^{ab}\partial_b \phi^\beta\Big)\langle \xi_\alpha, \xi_\beta\rangle + g^{ab}\langle \xi_\alpha, D_a \xi_\beta\rangle \partial_b \phi^\beta \\
&= \frac{1}{\sqrt{|\det g_\Sigma|}}\,\partial_a\Big(\sqrt{|\det g_\Sigma|}\; g^{ab}\partial_b \phi^\beta\Big) G_{\alpha\beta} + \langle \xi_\alpha, \nabla_\gamma \xi_\beta\rangle g^{ab}\partial_a \phi^\gamma \partial_b \phi^\beta \\
&= G_{\alpha\sigma}\Big\{\frac{1}{\sqrt{|\det g_\Sigma|}}\,\partial_a\Big(\sqrt{|\det g_\Sigma|}\; g^{ab}\partial_b \phi^\sigma\Big) + \Gamma^\sigma_{\beta\gamma} g^{ab}\partial_a \phi^\beta \partial_b \phi^\gamma\Big\} = 0\,,
\end{aligned} \tag{35}$$

where

$$\Gamma^\sigma_{\beta\gamma} = \tfrac{1}{2} G^{\sigma\lambda}\big(\partial_\gamma G_{\beta\lambda} + \partial_\beta G_{\gamma\lambda} - \partial_\lambda G_{\beta\gamma}\big) \quad \text{with} \quad \partial_\gamma := \frac{\partial}{\partial \phi^\gamma}\,, \tag{36}$$

are the Christoffel symbols and ∇_γ are the corresponding covariant derivatives on the moduli space ΩG of flat connections on H^2.

The equations

$$\frac{1}{\sqrt{|\det g_\Sigma|}}\,\partial_a\Big(\sqrt{|\det g_\Sigma|}\; g^{ab}\partial_b \phi^\alpha\Big) + \Gamma^\alpha_{\beta\gamma} g^{ab}\partial_a \phi^\beta \partial_b \phi^\gamma = 0 \tag{37}$$

are the Euler–Lagrange equations for the effective action

$$S_{\text{eff}} = \int_\Sigma dy^1 dy^2 \sqrt{|\det g_\Sigma|}\; g^{ab} G_{\alpha\beta}\partial_a \phi^\alpha \partial_b \phi^\beta \tag{38}$$

of the Yang–Mills theory on M which appears from the term $\mathrm{Tr}(\mathcal{F}_{aj}\mathcal{F}^{aj})$ in the initial Yang–Mills Lagrangian in the adiabatic limit $\varepsilon \to 0$. The equations (37) are the standard sigma-model equations defining harmonic maps from Σ ($= dS^2$, AdS^2, or $\mathbb{R}^{1,1}$) into the based loop group ΩG parameterized (locally) by coordinates ϕ^α. Note that solutions $\phi^\alpha(u, v)$ exist and depend on both coordinates u and v only if the gauge group G is noncompact. For compact groups G solutions of (37) exist only if ϕ^α do not depend on u. Then equations (37) reduce to the geodesic equations on the loop group ΩG and gives static configurations of Yang–Mills fields. This result can be considered as supplementing the result of [15], where only electric components of adiabatic Yang–Mills fields were nonvanishing. Note that any geodesic on ΩG is parametrized by the initial point $\phi_0 \in \Omega G$ and by the velocity $\dot\phi_0 \in T_0\Omega G$. Therefore, the moduli space of solutions (32) (with $\partial_v \phi = 0$ and $\partial_u \phi =: \dot\phi$) can be identified with the tangent bundle $T\Omega G$ of ΩG. The based loop group ΩG naturally acts on $T\Omega G$ which can be identified with the semi-direct product $\Omega G \ltimes \mathfrak{g}$ of ΩG and its Lie algebra $\mathfrak{g} = \mathrm{Lie}\, G$

8. Concluding remarks

In conclusion we recall that in the Euclidean case Atiyah has shown [16] that the moduli space of instantons over $\mathbb{R}^{4,0}$ is bijective to the moduli space of *holomorphic* maps from S^2 to ΩG. There is a conjecture (see e.g. [17]) that the moduli space of solutions of the second-order Yang–Mills equations on $\mathbb{R}^{4,0}$ is bijective to the moduli space of *harmonic* maps from S^2 to ΩG. Our consideration in this paper can be repeated for the Euclidean space. In [18] it was observed that $\mathbb{R}^4 \cup \{\infty\} \setminus S^1 = S^4 \setminus S^1$ is conformally diffeomorphic to the product manifold $S^2 \times H^2$. Considering $M = S^2 \times H^2$ and literally repeating our calculations for this Euclidean manifold we will arrive to the equations (37) with $\Sigma = S^2$. These equations will define harmonic maps from S^2 into the based loop group ΩG. Furthermore, from the implicit function theorem it follows that near every solution ϕ of (37) with $\Sigma = S^2$ (and the corresponding solution $\mathcal{A}^{\varepsilon=0}$ of the Yang–Mills equations) there exists a solution $\mathcal{A}^{\varepsilon>0}$ of the Yang–Mills equations on M for ε sufficiently small (cf. with the instanton case [7,9,18]). In other words, solutions of (37) with $\Sigma = S^2$ approximate solutions of the Yang–Mills equations on $M = S^2 \times H^2$ (and on $\mathbb{R}^{4,0}$ after some maps [9, 18] from $S^2 \times H^2$ to $\mathbb{R}^{4,0}$) and one can conjecture that the moduli spaces for $\mathcal{A}^{\varepsilon=0}$ and $\mathcal{A}^{\varepsilon>0}$ are bijective.

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References

[1] L. Dolan, A new symmetry group of real self-dual Yang–Mills, Phys. Lett. B 113 (1982) 387.
[2] L. Dolan, Kac–Moody algebras and exact solvability in hadronic physics, Phys. Rep. 109 (1984) 1.
[3] L.-L. Chau, M.-L. Ge, Y.-S. Wu, The Kac–Moody algebra in the self-dual Yang–Mills equation, Phys. Rev. D 25 (1982) 1086.
[4] L.-L. Chau, M.-L. Ge, A. Sinha, Y.-S. Wu, Hidden symmetry algebra for the self-dual Yang–Mills equation, Phys. Lett. B 121 (1983) 391.
[5] A.D. Popov, C.R. Preitschopf, Extended conformal symmetries of the self-dual Yang–Mills equations, Phys. Lett. B 374 (1996) 71, arXiv:hep-th/9512130.
[6] A.D. Popov, Self-dual Yang–Mills: symmetries and moduli space, Rev. Math. Phys. 11 (1999) 1091, arXiv:hep-th/9803183.
[7] S. Dostoglou, D.A. Salamon, Self-dual instantons and holomorphic curves, Ann. Math. 139 (1994) 581.
[8] M. Bershadsky, A. Johansen, V. Sadov, C. Vafa, Topological reduction of 4*d* SYM to 2*d* sigma models, Nucl. Phys. B 448 (1995) 166, arXiv:hep-th/9501096.
[9] D.A. Salamon, Notes on flat connections and the loop group, Preprint, University of Warwick, 1998.
[10] D.V. Alekseevsky, V. Cortes, A.S. Galaev, T. Leistner, Cones over pseudo-Riemannian manifolds and their holonomy, J. Reine Angew. Math. 635 (2009) 23.
[11] S.K. Donaldson, R.P. Thomas, Gauge theory in higher dimensions, in: The Geometric Universe, Oxford University Press, Oxford, 1998.
[12] N.S. Manton, A remark on the scattering of BPS monopoles, Phys. Lett. B 110 (1982) 54.
[13] A.N. Pressley, G.B. Segal, Loop Groups, Oxford University Press, Oxford, 1984.
[14] E.J. Weinberg, P. Yi, Magnetic monopole dynamics, supersymmetry, and duality, Phys. Rep. 438 (2007) 65, arXiv:hep-th/0609055.
[15] O. Lechtenfeld, A.D. Popov, Yang–Mills moduli space in the adiabatic limit, arXiv:1505.05448 [hep-th].
[16] M.F. Atiyah, Instantons in two-dimensions and four-dimensions, Commun. Math. Phys. 93 (1984) 437.
[17] A.G. Sergeev, Twistor interpretation of harmonic spheres and Yang–Mills fields, Mathematics 3 (2015) 17.
[18] S. Jarvis, P. Norbury, Degenerating metrics and instantons on the four-sphere, J. Geom. Phys. 27 (1998) 79.