Paradoxes, Prices, and Preferences – Essays on Decision Making under Risk and Economic Outcomes

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Abstract

This doctoral thesis contains three theoretical essays on the predictive power of leading descriptive decision theories and one empirical essay on the impact of stock market investors' probability distortion on future economic growth. Chapter 1 provides an extensive summary and motivation of all essays.

The first essay (Chapter 2, co-authored with Maik Dierkes) shows that Cumulative Prospect Theory cannot explain both the St. Petersburg paradox and the common ratio version of the Allais paradox simultaneously if probability weighting and value functions are continuous. This result holds independently of parametrizations of the value and probability weighting function. Using both paradoxes as litmus tests, Cumulative Prospect Theory with the majority of popular weighting functions loses its superior predictive power over Expected Utility Theory. However, neo-additive weighting functions (which are discontinuous) do solve the Allais - St. Petersburg conflict.

The second essay in Chapter 3 (co-authored with Maik Dierkes) shows that Salience Theory explains both a low willingness to pay, for example \$7.86 (\$12.33), for playing the St. Petersburg lottery truncated at around \$1 million (\$1 trillion) and reasonable preference reversal probabilities around 0.33 in Allais' common ratio paradox. Typical calibrations of other prominent theories (for example, Cumulative Prospect Theory or

Expected Utility Theory) cannot solve both paradoxes simultaneously. With unbounded payoffs, however, Salience Theory's ranking-based probability distortion prevents such a solution - regardless of parametrizations. Furthermore, the probability distortion in Salience Theory can be significantly stronger than in Cumulative Prospect Theory, fully overriding the value function's risk attitude.

The third essay in Chapter 4 (co-authored with Maik Dierkes) proves that subproportionality as a property of the probability weighting function alone does not automatically imply the common ratio effect in the framework of Cumulative Prospect Theory. Specifically, the issue occurs in the case of equal-mean lotteries because both risk-averse and risk-seeking behavior have to be predicted there. As a solution, we propose three simple properties of the probability weighting function which are sufficient to accommodate the empirical evidence of the common ratio effect for equal-mean lotteries for any S-shaped value function. These are (1) subproportionality, (2) indistinguishability of small probabilities, and (3) an intersection point with the diagonal lower than 0.5. While subproportionality and a fixed point lower than 0.5 are common assumptions in the literature, the property indistinguishability of small probabilities is introduced for the first time. The ratio of decision weights for infinitesimally small probabilities characterizes indistinguishability and is also an informative measure for the curvature of the probability weighting function at zero. The intuition behind indistinguishability is that, even though the ratio of probabilities stays constant at a moderate level, individuals tend to neglect this relative difference when probabilities get smaller.

Finally, the fourth essay in Chapter 5 (co-authored with Maik Dierkes and Stephan Germer) links stock market investors' probability distortion to future economic growth. The empirical challenge is to quantify the

optimality of today's decision making to test for its impact on future economic growth. Fortunately, risk preferences can be estimated from stock markets. Using monthly aggregate stock prices from 1926 to 2015, we estimate risk preferences via an asset pricing model with Cumulative Prospect Theory agents and distill a recently proposed probability distortion index. This index negatively predicts GDP growth in-sample and out-of-sample. Predictability is stronger and more reliable over longer horizons. Our results suggest that distorted asset prices may lead to significant welfare losses.

Keywords: Cumulative Prospect Theory, Salience Theory, Allais - St. Petersburg Conflict, Common Ratio Effect, Probability Distortion, Economic Growth

Zusammenfassung

Diese Doktorarbeit enthält drei theoretische Abhandlungen über die Vorhersagekraft führender deskriptiver Entscheidungstheorien und eine empirische Abhandlung über die Auswirkung der Wahrscheinlichkeitsverzerrung von Aktienmarktinvestoren auf das zukünftige Wirtschaftswachstum. Kapitel 1 bietet eine ausführliche Zusammenfassung und Motivation aller Aufsätze.

Der erste Aufsatz (Kapitel 2, gemeinsam mit Maik Dierkes verfasst) zeigt, dass die kumulative Prospect-Theorie das St. Petersburg-Paradoxon und die Common-Ratio-Version des Allais-Paradoxons nicht gleichzeitig erklären kann, wenn Wahrscheinlichkeitsgewichtungs- und Wertfunktion stetig sind. Dieses Ergebnis gilt unabhängig von den Parametrisierungen der Wert- und Wahrscheinlichkeitsgewichtungsfunktion. Wenn beide Paradoxe als Lackmustest verwendet werden, verliert die kumulative Prospect-Theorie mit den meisten gängigen Gewichtungsfunktionen ihre überlegene Vorhersagekraft gegenüber der Erwartungsnutzentheorie. Neoadditive Gewichtungsfunktionen (die unstetig an den Stellen 0 und 1 sind) lösen jedoch den Konflikt zwischen dem Allais-Paradoxon und St. Petersburg-Paradoxon.

Der zweite Aufsatz in Kapitel 3 (gemeinsam mit Maik Dierkes verfasst) zeigt, dass die Salience-Theorie sowohl eine geringe Zahlungsbereitschaft von beispielsweise 7,86 USD (12,33 USD) für das Spielen der auf rund 1 Million USD (1 Billion USD) gekürzten St. Petersburg-Lotterie als

auch angemessene Präferenzumkehrwahrscheinlichkeiten um Wahrscheinlichkeiten von 0,33 in Allais' Common-Ratio-Paradoxon erklärt. Typische Kalibrierungen anderer bekannter Theorien (z. B. kumulative Prospect-Theorie oder Erwartungsnutzentheorie) können nicht beide Paradoxien gleichzeitig lösen. Bei unbegrenzten Auszahlungen verhindert jedoch die auf der Rangfolge basierende Wahrscheinlichkeitsverzerrung der Salience-Theorie eine solche Lösung - unabhängig von den Parametrisierungen. Darüber hinaus kann die Wahrscheinlichkeitsverzerrung der Salience-Theorie erheblich stärker sein als in der kumulativen Prospect-Theorie, wodurch die Risikoeinstellung der Wertfunktion vollständig außer Kraft gesetzt werden kann.

Der dritte Aufsatz in Kapitel 4 (gemeinsam mit Maik Dierkes verfasst) belegt, dass im Rahmen der kumulativen Prospect-Theorie die Subproportionalität als Eigenschaft der Wahrscheinlichkeitsgewichtungsfunktion allein nicht automatisch den Common-Ratio-Effekt impliziert. Insbesondere tritt das Problem bei Lotterien mit gleichem Erwartungswert auf, da dort sowohl risikoaverses als auch risikofreudiges Verhalten vorhergesagt werden muss. Als Lösung schlagen wir drei einfache Eigenschaften der Wahrscheinlichkeitsgewichtungsfunktion vor, die ausreichen, um den empirischen Nachweis des Common-Ratio-Effekts für Lotterien mit gleichem Mittelwert für jede S-förmige Wertefunktion aufzunehmen. Diese sind (1) Subproportionalität, (2) Ununterscheidbarkeit kleiner Wahrscheinlichkeiten und (3) ein Schnittpunkt mit der Diagonalen unterhalb von 0,5. Während in der Literatur Subproportionalität und ein Fixpunkt unter 0,5 gängige Annahmen sind, wird erstmals die Eigenschaft "Ununterscheidbarkeit kleiner Wahrscheinlichkeiten" eingeführt. Das Verhältnis der Entscheidungsgewichte für unendlich kleine Wahrscheinlichkeiten kennzeichnet die Ununterscheidbarkeit und ist auch ein aussagekräftiges Maß für die Krümmung der Wahrscheinlichkeitsgewichtungsfunktion bei

Null. Die Intuition hinter der Ununterscheidbarkeit ist, dass Individuen den relativen Unterschied zwischen Wahrscheinlichkeiten vernachlässigen, wenn die Wahrscheinlichkeiten kleiner werden, obwohl das Verhältnis der Wahrscheinlichkeiten auf einem moderaten Niveau konstant bleibt.

Abschließend verknüpft der vierte Aufsatz in Kapitel 5 (gemeinsam mit Maik Dierkes und Stephan Germer verfasst) die Wahrscheinlichkeitsverzerrung von Aktienmarktinvestoren mit dem zukünftigen Wirtschaftswachstum. Die empirische Herausforderung besteht hierbei, die Optimalität der heutigen Entscheidungsfindung zu quantifizieren, um ihre Auswirkungen auf das zukünftige Wirtschaftswachstum zu testen. Glücklicherweise können Risikopräferenzen von den Aktienmärkten geschätzt werden. Unter Verwendung der monatlichen aggregierten Aktienkurse von 1926 bis 2015 schätzen wir die Risikopräferenzen über ein Asset-Pricing-Modell mit kumulativen Prospect-Theorie-Agenten und destillieren einen kürzlich vorgeschlagenen Wahrscheinlichkeitsverzerrungsindex. Dieser Index prognostiziert ein negatives GDP-Wachstum innerhalb und außerhalb der Stichprobe. Dabei ist die Vorhersagbarkeit über längere Zeiträume hinweg stärker und zuverlässiger. Unsere Ergebnisse legen nahe, dass verzerrte Vermögenspreise zu erheblichen Wohlfahrtsverlusten führen können.

Schlagwörter: Kumulative Prospect-Theorie, Salience-Theorie, Allais - St. Petersburg-Konflikt, Common-Ratio-Effekt, Wahrscheinlichkeitsverzerrung, Wirtschaftswachstum

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Chapter 1

Introduction

Expected Utility Theory has been the leading theory of choice under risk and the core of rational decision making for nearly three centuries. The idea of Expected Utility Theory goes back to Bernoulli (1738, 1954) who proposed a concave utility transformation of final wealth to solve the St. Petersburg paradox which describes the fact that hardly anyone would be willing to pay an infinite amount of money for a lottery with infinite expected value which promises an amount of $\$2^k$ with probability 2^{-k} for $k \in \mathbb{N}_{>0}$. This fact was used as evidence against Expected Value Theory and was a key motivation to include risk aversion in normative decision theory to restore a minimum level of descriptive power. Roughly two centuries after Daniel Bernoulli's publication, von Neumann & Morgenstern (1944) provided an axiomatic foundation of Expected Utility Theory.[1]

Nowadays, Expected Utility Theory is still widely accepted and applied as a normative model of rational behavior but hardly as a

[1] Besides the significant contributions of Bernoulli (1738, 1954) and von Neumann & Morgenstern (1944), the works of Ramsey (1931) and Savage (1954) were also of great importance for the further development and understanding of Expected Utility Theory and should therefore not remain unmentioned. For a detailed historical review, see Fishburn (1988).

descriptive model. Over the last seven decades, the literature that questions Expected Utility Theory's descriptive power is consistently growing. One of the first studies that revealed persistent and systematic violations of Expected Utility Theory was Allais (1953) with his prominent Allais paradox. He showed that individuals' choice behavior violates Expected Utility Theory's independence axiom. In particular, he observed that individuals seem to process probabilities of risky outcomes in a non-linear way. A simple demonstration of the Allais paradox is the common ratio effect which involves choices between the two-outcome lotteries $L_1(p) = (\$6000, 0.5p; \$0, 1 - 0.5p)$ and $L_2(p) = (\$3000, p; \$0, 1 - p)$ where p is a probability. Empirically, subjects choose the safer lottery L_2 for high probabilities p and the riskier lottery L_1 for low probabilities p (see, e.g., Kahneman & Tversky, 1979). The independence axiom, however, does not allow for this change in preference over L_1 and L_2 for varying $p \in (0, 1]$.

This observation initiated the development of behavioral decision theories such as Kahneman & Tversky's (1979) prominent Prospect Theory. The advanced version, Cumulative Prospect Theory by Tversky & Kahneman (1992), is largely considered to be the most powerful model to describe individual decision making under risk. It captures experimental evidence such as reference dependence, diminishing value sensitivity, loss aversion and probability weighting (see Kahneman & Tversky, 1979; Tversky & Kahneman, 1992).

The first essay of this thesis (Chapter 2, co-authored with Maik Dierkes) tests whether Cumulative Prospect Theory is able to accommodate both Bernoulli's (1738, 1954) St. Petersburg paradox and Allais' (1953) common ratio paradox with one set of parameters. The main result of this paper is that only discontinuous probability weighting functions – such as neoadditive weighting functions (Wakker, 2010) – can solve both paradoxes simultaneously. This result holds independently of parametrizations of the value and probability weighting function. If value and probability weighting functions are continuous, as it is usually assumed in the literature, then Cumulative Prospect Theory cannot explain both the choice behavior in the Allais paradox and the finite willingness to pay to participate in the St. Petersburg lottery at the same time. For example, consider the originally proposed parametrization in Tversky & Kahneman (1992) where the probability weighting function is given by $w(p) = p^\gamma / \left(p^\gamma + (1-p)^\gamma\right)^{1/\gamma}$ with $\gamma > 0$ and the value function over gains is given by $v(x) = x^\alpha$ with $\alpha > 0$. Finite willingness to pay for the St. Petersburg lottery requires the parameter restriction $\alpha < \gamma$ while predicting Allais' common ratio effect requires the opposite inequality $\alpha \geq \gamma$. The particular strength of this paper stems from the fact that we generalize this result to *all* continuous and strictly increasing value functions v and *all* continuous and strictly increasing probability weighting functions w with $w(0) = 0$ and $w(1) = 1$. Put differently, this joint test dismisses large classes of popular probability weighting functions (e.g. Tversky & Kahneman, 1992; Prelec, 1998; Goldstein & Einhorn, 1987; Rieger & Wang, 2006) and considerably reduces the set of potentially promising weighting functions. Hence, future research shall rather embrace discontinuous weighting functions, such as neo-additive weighting functions (Wakker, 2010) and their obvious nonlinear extensions.

We motivate our test procedure with the fact that virtually all theories of decision making under risk are motivated by *either* the St. Petersburg paradox *or* the Allais paradox, but a discrepancy between these two paradoxes has, to the best of our knowledge, never been addressed before. We therefore propose the *joint* consideration of both paradoxes as the new minimum standard to test descriptive decision theories.

The second essay in Chapter 3 (co-authored with Maik Dierkes) tests whether Bordalo et al.'s (2012) Salience Theory is able to resolve the

Allais - St. Petersburg conflict documented in the previous chapter. Salience Theory is a relatively new promising context-dependent descriptive theory of choice under risk which models a Local Thinker who re-weights probabilities in favor of salient payoffs. Bordalo et al. (2012, p. 1243) argue that Salience Theory "provides a novel and unified account of many empirical phenomena, including frequent risk-seeking behavior, invariance failures such as the Allais paradox, and preference reversals." Their analysis, however, does not include the St. Petersburg paradox. Our paper complements the relatively new strand of literature which tests Salience Theory empirically and theoretically.[2] In particular, we are, to the best of our knowledge, the first to investigate the St. Petersburg paradox under Salience Theory.

Our main result is that Salience Theory can resolve the Allais - St. Petersburg conflict but only under the assumptions of finite resources and a value function which generates substantial risk aversion (such as bounded value functions). A simple parametrization of Salience Theory that performs sufficiently well consists of the exponential value function $v(x) = 1 - e^{-x}$, a probability-distortion parameter value $\delta \approx 0.4$, and any salience function for ranking states with the properties ordering and diminishing sensitivity, as proposed by Bordalo et al. (2012). This simple parametrization simultaneously predicts a reasonable willingness to pay of \$7.86 (\$12.33) for the St. Petersburg lottery truncated at the maximum payoff of $\$2^{20} \approx 1$ million ($\$2^{40} \approx 1$ trillion) dollars and an empirically substantiated preference reversal probability $p^* \approx \frac{1}{3}$ for the common ratio lotteries L_1 and L_2. For this specification, the Allais - St. Petersburg conflict emerges only asymptotically, i.e. when considering the original St. Petersburg lottery with infinite expected payoff. In a realistic, resource-constrained environment, this Salience Theory

[2] See, for example, Kontek (2016); Frydman & Mormann (2018); Nielsen et al. (2018); Königsheim et al. (2019)

specification has an edge over Expected Utility Theory and most Cumulative Prospect Theory calibrations. Recall that, for Cumulative Prospect Theory with continuous preference functions, any solution to Allais' common ratio effect predicts – at odds with experimental evidence – huge willingness to pay for the St. Petersburg lottery, easily exceeding the expected payoff of the truncated St. Petersburg lottery.

Furthermore, we show that the probability distortion in Salience Theory can be significantly stronger than in Cumulative Prospect Theory. An interesting implication of the latter finding is that the use of bounded value functions does not necessarily solve the St. Petersburg paradox under Salience Theory. Note that, under Cumulative Prospect Theory or Expected Utility Theory, bounded value functions always solve the St. Petersburg paradox (e.g. Dierkes & Sejdiu, 2019b; Rieger & Wang, 2006).

The third essay in Chapter 4 (co-authored with Maik Dierkes) clarifies frequent misunderstandings about the relationship of subproportionality as a property of the probability weighting function and the common ratio effect. As selected quotes in Table A.1 in Dierkes & Sejdiu (2019a) show, many researchers equate subproportionality to Allais' common ratio effect. Our paper, however, points out that this is not always the case.

Kahneman & Tversky (1979, p. 282) call a probability weighting function w subproportional "if and only if $\log w(p)$ is a convex function of $\log p$". Note that this definition allows probability weighting functions to be inverse S-shaped *or* convex (Fehr-Duda & Epper, 2012). Convex probability weighting functions, however, do not overweight small probabilities and, therefore, are not able to override the risk attitude predetermined by a S-shaped value function. Being able to predict both risk-averse and risk-seeking behavior is, however, imperative when it comes to the prediction of the common ratio effect for choices between equal-mean lotteries.

As a solution, we propose three simple properties of the probability weighting function (including subproportionality) which are sufficient to explain the common ratio effect for equal-mean lotteries for any S-shaped value function (i.e. concave over gains and convex over losses). For given lotteries $L_r(p) = (\$z, \Delta p\ ;\ \$0, 1-\Delta p)$ and $L_s(p) = (\$\Delta z, p\ ;\ \$0, 1-p)$ with payoff $z > 0$, common ratio $\Delta \in (0,1)$, and varying probability $p \in (0,1]$, the three conditions on w are:

(1) Subproportionality: $w(\Delta p)/w(p) > w(\Delta q)/w(q)$, $\forall\ 0 < p < q \leq 1$,

(2) Indistinguishability of small probabilities: $\lim_{p \to 0^+} w(\Delta p)/w(p) = 1$,

(3) Inner fixed point restriction: $w(\Delta) \leq \Delta$.

While properties (1) and (3) are common assumptions in the literature, the second property which we call *indistinguishability of small probabilities* (abbreviated indistinguishability) is introduced for the first time. This property is key to ensure that for any S-shaped value function, a risk-seeking behavior for choices between two simple equal-mean lotteries can always be predicted by just decreasing the probabilities of winning towards zero by equal proportion. The intuition behind this property is that, even though the ratio of probabilities stays constant at a moderate level, individuals tend to neglect this relative difference when probabilities get smaller and focus solely on the outcomes.

The ratio of decision weights for infinitesimally small probabilities ($\lim_{p \to 0^+} w(\Delta p)/w(p)$), which characterizes indistinguishability, is also an informative measure for the probabilistic risk attitude of individuals. In particular, it can be used to classify probability weighting functions according to their processing of small probabilities. Specifically, we show that $\lim_{p \to 0^+} w(\Delta p)/w(p)$ is directly linked to a probabilistic counterpart of the Arrow-Pratt measure of relative risk aversion at probability $p = 0$. While Prelec (1998) considers the *absolute* Arrow-Pratt measure and relates the *relative* version in log-log scales to subproportionality, we prove for the first time that the *relative* Arrow-Pratt measure at $p = 0$ is closely related to indistinguishability of small probabilities.

The last essay in Chapter 5 (co-authored with Maik Dierkes and Stephan Germer) studies the impact of probability distortion on future economic growth. A cornerstone concept of modern economics is that prices for goods are set to match demand and supply. The good's price ensures an efficient allocation such that it is used for projects with superior profitability. But what if prices are wrong? What if demand is determined by irrational preferences and, hence, prices reflect this irrationality? According to the logic above, such irrationality poses a threat to economic welfare. Irrationality would lead to lower GDP growth due to the inefficient allocation of resources. As Lamont & Thaler (2003, pp. 227-228) put it:

> "Do asset markets offer rational signals to the economy about where to invest real resources? If some firms have stock prices that are far from intrinsic value, then those firms will attract too much or too little capital."

In this paper, we infer potentially irrational preferences from stock prices and find that lower future GDP growth is linked to a higher degree of irrationality. We use the term "rational" to refer to preferences which are consistent with Expected Utility Theory and "irrational" in case they conflict. Our interest centers on violations of Expected Utility Theory's independence axiom because this axiom is key to rational behavior. In Cumulative Prospect Theory, the independence axiom is relaxed by the non-linear processing of probabilities. To infer a probability distortion index, we employ the equilibrium asset pricing model of Barberis & Huang (2008) which assumes that prices on financial market are set by investors who behave according to Cumulative Prospect Theory. Our estimation of potential irrationality is possible because stock prices reflect aggregate preferences of an economy.

Our key finding is that stronger probability distortion today reliably predicts lower future GDP growth in-sample and out-of-sample. This negative link is stronger and statistically more reliable over longer prediction horizons. Our results are robust to numerous variations, such as different calibration procedures of the asset pricing model (simple average returns vs. moving average estimators; GARCH vs. EGARCH), different measures for probability distortion (likelihood insensitivity and Prelec's (1998) probability weighting function), and sample splits (1953-1984 and 1985-2015). Our conjecture is that suboptimal decision making is one channel by which today's market prices and future GDP growth are linked. Implicitly, we provide evidence that stock prices can deviate from their rationally warranted fundamental value.

All chapters of this thesis are self-contained. Therefore, in each chapter, variables and acronyms are redefined. Of course, the notation was adapted whenever possible to promote readability.

Chapter 2

The Need for Discontinuous Probability Weighting Functions: How Cumulative Prospect Theory is torn between the Allais Paradox and the St. Petersburg Paradox*

2.1 Introduction

Descriptive theories of decision making under risk are typically required to pass one or more litmus tests. Such tests help to determine whether a

*This chapter is based on the Working Paper "The Need for Discontinuous Probability Weighting Functions: How Cumulative Prospect Theory is torn between the Allais Paradox and the St. Petersburg Paradox" authored by Maik Dierkes and Vulnet Sejdiu, 2019. We are particularly grateful to Johannes Jaspersen and Walther Paravicini for comments and very helpful advice.

theory of decision making is able to predict human behavior to a desired extent. The most prominent tests in decision theory include Bernoulli's (1738, 1954) St. Petersburg paradox and the Allais paradox (Allais, 1953). Both paradoxes have paved the way for new decision theories and helped define the prevailing standard of a certain era. Specifically, the St. Petersburg paradox criticized Expected Value Theory (EVT) and fostered the dominance of Expected Utility Theory (EUT) thereafter. Similarly, the Allais paradox revealed inconsistencies of EUT with actually observed choice behavior and, thus, initiated the development of descriptive decision theories such as Kahneman & Tversky's (1979) prominent Prospect Theory. The advanced version, Cumulative Prospect Theory (CPT) by Tversky & Kahneman (1992) which is based on Rank Dependent Utility Theory (RDU; Quiggin, 1982), is largely considered to be the most powerful model to describe individual decision making under risk and uncertainty.

Virtually all theories of decision making under risk are motivated by *either* the St. Petersburg paradox *or* the Allais paradox, but a potential discrepancy between these two paradoxes has, to the best of our knowledge, never been addressed before. We propose the *joint* consideration of both paradoxes as the new minimum standard to test descriptive decision theories.

In this paper, we show that CPT is torn between both paradoxes in the following way. If value and weighting functions are continuous, CPT cannot explain both the choice behavior in the Allais paradox and the finite willingness to pay to participate in the St. Petersburg lottery at the same time. For example, consider the originally proposed parametrization in Tversky & Kahneman (1992) where the probability weighting function is given by $w(p) = p^\gamma / (p^\gamma + (1-p)^\gamma)^{1/\gamma}$ with $\gamma > 0$ and the value function over gains is given by $v(x) = x^\alpha$ with $\alpha > 0$. Finite willingness to pay for the St. Petersburg lottery requires the parameter restriction $\alpha < \gamma$

while predicting Allais' common ratio effect requires the opposite inequality $\alpha \geq \gamma$. The more interesting novelty of this paper stems from the fact that we generalize this result to *all* continuous and strictly increasing value functions v and *all* continuous and strictly increasing probability weighting functions w with $w(0) = 0$ and $w(1) = 1$. Put differently, this joint test dismisses large classes of popular probability weighting functions and considerably reduces the set of potentially promising weighting functions.

If the probability weighting function is discontinuous, however, a solution to both paradoxes is possible. Neo-additive weighting functions, as formalized by Wakker (2010) via $w(0) = 0$, $w(1) = 1$, and $w(p) = a + bp$ for $p \in (0, 1)$ with $a, b > 0$, $a + b \leq 1$, presumably constitute the simplest class of such weighting functions. Kilka & Weber (2001, p. 1717) use it for approximating continuous weighting functions while Baillon et al. (2018) regard it as a full-fledged alternative to popular continuous weighting functions. Neo-additive weighting functions are popular for decision making under ambiguity (e.g. Abdellaoui et al., 2011; Baillon et al., 2017; Chateauneuf et al., 2007). To account for more complex choice behavior for moderate probabilities (Harless & Camerer, 1994; Wu & Gonzalez, 1996) neo-additive weighting functions might be too restrictive and should be amended by some non-linearities. Discontinuities are, however, indispensable to accommodate both Allais' common ratio effect and the St. Petersburg paradox.

In light of these results, we find it stunning how much more predictive power Kahneman & Tversky's (1979) originally proposed discontinuous probability weighting function has (when applied in a rank-dependent framework, of course) compared to Tversky & Kahneman's (1992) continuous weighting function. In the original paper, Figure 4 depicts a hypothesized, yet discontinuous weighting function which "is relatively shallow in the open interval and changes abruptly near the end-points where

$w(0) = 0$ and $w(1) = 1$" and which "is not well-behaved near the end-points" (Kahneman & Tversky, 1979, p. 282f.).

Our theoretical findings are consistent with and, in fact, strongly support the empirical estimates in Barseghyan et al. (2013). Based on more than 4000 households' insurance deductible choices, they fit a quadratic polynomial on the probability interval $[0, 0.16]$ and estimate the intercept at 0.061, indicating a discontinuity at probability zero. Thus, our theoretical results virtually echo their estimate which they find "is striking in its resemblance to the probability weighting function originally posited by Kahneman & Tversky (1979). In particular, it is consistent with a probability weighting function that exhibits overweighting of small probabilities, exhibits mild insensitivity to changes in probabilities, and trends toward a positive intercept as [the probability] approaches zero [...]. By contrast, the probability weighting functions later suggested by Tversky & Kahneman (1992), Lattimore, Baker, & Witte (1992), and Prelec (1998) – which are commonly used in the literature [...] – will not fit our data well, because they trend toward a zero intercept [...]" (Barseghyan et al., 2013, p. 2515).

To provide some intuition for our results, recall that the St. Petersburg paradox describes the fact that hardly anyone would be willing to pay an infinite amount of money for the lottery with infinite expected value which promises an amount of $\$2^k$ with probability 2^{-k} for $k = 1, 2, 3 \dots$. This fact was used as evidence against EVT and was a key motivation to include risk aversion in normative decision theory – such as EUT – to restore a minimum level of descriptive power. It dates back to 1713 and is among the oldest and most prominent litmus tests for models of risky decision making. Based on the ratio test, we then show that, under CPT, a necessary condition for finite willingness to pay is

$$\lim_{p\to 0^+} \frac{w(0.5p) - w(0.25p)}{w(p) - w(0.5p)} \leq \lim_{z\to\infty} \frac{v(0.5z)}{v(z)}. \tag{STP}$$

For continuous (and strictly increasing) probability weighting functions w, this necessary condition is equivalent to the following simpler necessary condition which is useful when analyzing the common ratio effect:

$$\lim_{p\to 0^+} \frac{w(0.5p)}{w(p)} \leq \lim_{z\to\infty} \frac{v(0.5z)}{v(z)} \tag{STP^*}$$

Now consider the Allais paradox. The Allais paradox exists in different versions and uncovers a violation of EUT's independence axiom. We focus on the common ratio version which involves choices between equal mean lotteries such as $L_1(p) = (\$z, 0.5p; \$0, 1 - 0.5p)$ and $L_2(p) = (\$z/2, p; \$0, 1 - p)$ where p is a probability. Kahneman & Tversky (1979) use payoff $z = 6000$ and $p \in \{0.002, 0.9\}$ in Problems 7 and 8. Empirically, subjects choose the safer lottery L_2 for high probabilities p and the riskier lottery L_1 for low probabilities p. EUT's independence axiom, however, does not allow for this change in preference over L_1 and L_2 for varying probabilities p. In our analysis, we make explicit use of Allais' (1953) notion that the common ratio effect emerges in particular for large payoffs z in lotteries L_1 and L_2. He used payoffs in the millions. We then derive the following necessary condition for the common ratio effect:

$$\lim_{p\to 0^+} \frac{w(0.5p)}{w(p)} \geq \lim_{z\to\infty} \frac{v(0.5z)}{v(z)}. \tag{CRE^*}$$

When comparing conditions (CRE^*) and (STP^*), both inequalities turn into a single equality and we also rule out this equality as a potentially remaining case. Put differently, with continuous (and strictly increasing) v and w there does not exist a simultaneous solution to both paradoxes – independent of the exact parametrizations. To the best of our knowledge, we are the first to prove this general result.

Furthermore, in the CPT framework, any simultaneous solution to both paradoxes must drive a wedge between $\lim_{p \to 0^+} \frac{w(0.5p)-w(0.25p)}{w(p)-w(0.5p)}$ and $\lim_{p \to 0^+} \frac{w(0.5p)}{w(p)}$ and thus involves discontinuous probability weighting functions. Neo-additive weighting functions w_{neo} (see Wakker, 2010) are presumably the simplest class of such weighting functions. For those functions it holds $\lim_{p \to 0^+} \frac{w_{neo}(0.5p)-w_{neo}(0.25p)}{w_{neo}(p)-w_{neo}(0.5p)} = 0.5$ and $\lim_{p \to 0^+} \frac{w_{neo}(0.5p)}{w_{neo}(p)} = 1$. Specifically, if we choose, for example, $a = 0.1$, $b = 0.8$, and $v(x) = x^{0.7}$, the CPT decision maker is willing to pay \$5.89 to be entitled to the St. Petersburg lottery and exhibits the typical choice pattern between the common ratio lotteries L_1 and L_2 with preference reversal probability $p^* = 0.42$.

Using large payoffs, $z \to \infty$, in the Allais paradox above might appear extreme at first glance, but is supported by experimental evidence. If CPT's value function is parameterized by the power value function $v(x) = x^{\alpha}$, as is most often the case in empirical calibration studies, then the v-ratio $\frac{v(0.5z)}{v(z)}$ is independent of payoff z and using large payoffs is irrelevant. In other words, the power value function inhibits a solution to both paradoxes if the weighting function is continuous. With other value functions and continuous weighting functions, solutions might theoretically exist for moderate payoffs only - at odds with Camerer (1989), Conlisk (1989), Fan (2002), Huck & Müller (2012), and Agranov & Ortoleva (2017) who report less frequent Allais-type violations of EUT for small payoffs. A sensitivity analysis, however, explicitly rejects other typical value functions (exponential, logarithmic, and HARA) because of their unrealistic predictions. For example, De Giorgi & Hens (2006) motivate an exponential value function, i.e. $v(x) = \beta\left(1 - e^{-\alpha x}\right)$ with absolute risk aversion coefficient $\alpha \approx 0.2$ and $\beta > 0$. This bounded value function always ensures finite willingness to pay in the St. Petersburg paradox. Presumably, optimal conditions for the emergence of the common ratio effect are then a low preference reversal probability p^* and Prelec's (1998) subproportional weighting function with low curvature parameter γ. If we conservatively set $\gamma = 0.53$ from Bleichrodt & Pinto (2000) and fix $p^* = 0.002$ (which Problem 8 in Kahneman & Tversky (1979) indicates as an extreme lower boundary for p^*) then a preference switch from safe lottery L_2 to risky lottery L_1 happens for payoffs $z < \$18.08$ only – undoubtedly an unrealistic prediction. The appendix provides the details.

To the best of our knowledge, we are the first to show that solving the St. Petersburg paradox rules out practically all CPT preferences that explain the common ratio version of the Allais paradox as long as preferences are given by the same continuous value and weighting function across both paradoxes. Some authors analyze the restrictions that finite willingness to pay for the St. Petersburg lottery places on CPT (e.g. Blavatskyy, 2005; Camerer, 2005; Rieger & Wang, 2006; De Giorgi & Hens, 2006; Cox & Sadiraj, 2008; Pfiffelmann, 2011), but the conflict with the common ratio effect, independent of parametrizations, has not been discovered before.

Our results are not fabricated by the infinite expected payoff of the original St. Petersburg lottery and we refer to the original St. Petersburg gamble because of its prominence. An analysis of truncated St. Petersburg lotteries does not change our conclusions. For example, assume the lottery's maximum payoff is truncated at $\$2^{30}$ which equals roughly 1 billion dollars and corresponds to 29 possible rounds of coin flipping. The usual parametrization in Tversky & Kahneman (1992) with $\alpha = 0.88$ and $\gamma = 0.61$ which predicts the common ratio effect implies that the CPT decision maker would pay up to $\$2,899.88$ to play this lottery – about 93 times more than the expected payoff of \$31. Such price predictions are absurd and inconsistent with the empirical evidence (e.g. Hayden & Platt, 2009; Cox et al., 2011). Further, Rieger & Wang's (2006) arguments show that CPT can predict *infinite* willingness to pay (certainty equivalent) even in cases of risks with *finite* expected value.

We further clarify that the slope of continuous probability weighting functions at probability zero is less important for the St. Petersburg paradox than often thought. Rather the trade-off between the limits of the w-ratio $\lim_{p \to 0^+} \frac{w(0.5p)}{w(p)}$ and v-ratio $\lim_{z \to \infty} \frac{v(0.5z)}{v(z)}$ is key. For smooth weighting functions, the w-ratio limit is equivalent to a probabilistic counterpart of the Arrow-Pratt measure of relative risk aversion, $\lim_{p \to 0^+} p \frac{w''(p)}{w'(p)}$ (see Dierkes & Sejdiu, 2019a). In particular, the limit of the w-ratio reflects the risk attitude induced by the probability weighting function better than the slope at probability zero. For example, the inverse S-shaped versions of the Tversky & Kahneman (1992) and Prelec (1998) weighting functions have w-ratio limits of $0.5^\gamma \in (0, 1)$ and 1, respectively, while both have infinite slope (first derivative) and curvature (second derivative) at zero and, thus, appear indistinguishable on these latter metrics. It is noteworthy that a strictly concave value function v over gains and a positive, but finite slope $w'(0)$ cannot explain the common ratio effect (provided w is smooth, of course) because then $\lim_{p \to 0^+} \frac{w(0.5p)}{w(p)} = 0.5$ and $0.5 < \frac{v(0.5z)}{v(z)} < 1$ for fixed $z > 0$, and the necessary condition is violated. Put differently, scaling down probabilities in the common ratio lotteries counterfactually leads to a risk-averse choice if the continuous probability weighting function has a finite slope at zero and the value function is strictly concave.

The probabilistic and classical Arrow-Pratt measures of relative risk aversion allow for an intuitive interpretation in, for example, Tversky & Kahneman's (1992) original parametrization. The sum of probabilistic risk aversion at probability $p \to 0^+$ and the value function's risk aversion has to be strictly positive for finite willingness to pay in the St. Petersburg paradox, while the common ratio effect emerges only for negative overall risk aversion (risk proclivity).

Finally, the discrepancy between both the St. Petersburg and the Allais paradox is not an artifact of the preference reversal phenomenon whereby there can occur inconsistencies between choice and valuation tasks (see, e.g., Lichtenstein & Slovic (1971) for an early reference).[1] In fact, the preference reversal phenomenon makes CPT's difficulties to predict both paradoxes even greater. According to Tversky et al. (1990), there is more overweighting of small probabilities for high payoffs in pricing tasks, such as in the St. Petersburg paradox, than in choice tasks, such as in the Allais paradox. So, let us assume we elicited an individual's preference parameter combination (α, γ) by a clever sequence of Allais type choices. In particular, we would typically get $\alpha > \gamma$. Then, Tversky et al. (1990) suggest to *lower* γ to predict the individual's willingness to pay in the St. Petersburg game. An even lower γ, however, would counterfactually predict an infinite certainty equivalent for the St. Petersburg lottery. Moreover, the preference reversal phenomenon typically occurs when individuals deal with specific types of lotteries – the so-called P-bet and \$-bet. And the characteristics of these bets do not match those in the two paradoxes.

Next to discontinuous probability weighting functions, another potential explanation for both paradoxes within the CPT framework might be varying preferences across both paradoxes. It is well known that CPT preferences can be driven by, for example, affect (Rottenstreich & Hsee, 2001), feelings (Hsee & Rottenstreich, 2004), or perceived self-competence (Kilka & Weber, 2001). Similarly, Harrison & Rutström (2009) deliberately model decision makers with a latent process which switches between evaluation according to EUT or CPT. Whether differences in the Allais paradox and St. Petersburg paradox trigger such changes in preferences is an open question, though.

[1] Schmidt et al.'s (2008) third generation Prospect Theory enhances CPT to allow, for example, for such preference reversals.

The remainder of this paper illustrates our conclusions by formal proofs and numerical examples.

2.2 The Allais - St. Petersburg Conflict

We make the following assumptions throughout our discussion:

Assumption 2.1 (Preference Calculus)

a) *The decision maker's utility for a lottery* $(x_1, p_1; x_2, p_2; \ldots)$ *is given by Cumulative Prospect Theory. That is the decision maker has a value function* v *and a probability weighting function* w*. Assuming without loss of generality that payoffs are rank ordered such that* $0 \leq x_1 \leq x_2 \leq \ldots$*, the CPT value is given by* $v(x_1)[w(p_1 + p_2 + \ldots) - w(p_2 + p_3 + \ldots)] + v(x_2)[w(p_2 + p_3 + \ldots) - w(p_3 + p_4 + \ldots)] + \ldots$*.*

b) *The value function* v *is continuous and strictly monotonically increasing with* $v(0) = 0$*.*

c) *The probability weighting function* w *is continuous and strictly monotonically increasing with* $w(0) = 0$ *and* $w(1) = 1$*.*

d) *The reference point is the current wealth level. In particular, all lottery payoffs considered here are perceived as gains.*[2]

Assumption 2.2 (Mathematical Notation) *Whenever we use limits, e.g.* $\lim_{x \to z} f(x)$*, we implicitly assume these limits exist in a weak sense, i.e. limes superior and limes inferior coincide and* $\lim_{x \to z} f(x) \in [-\infty, \infty]$*.*

[2] Without loss of generality, we follow the typical assumption that the reference points is fixed at zero in both paradoxes. In particular, theories with stochastic reference points (Kőszegi & Rabin, 2006) are not applicable.

2.2.1 The St. Petersburg Paradox under CPT

Bernoulli's (1738, 1954) St. Petersburg lottery L_{STP} promises an amount of $\$2^k$ with probability 2^{-k} for $k \in \mathbb{N}_{>0}$. Although the expected value of L_{STP} is infinite, real decision makers are only willing to pay a low price for lottery L_{STP}.[3] Under CPT, the decision maker assigns the following utility to the St. Petersburg lottery L_{STP}:

$$\begin{aligned} CPT(L_{STP}) &= \sum_{k=1}^{\infty} v\left(2^k\right) \cdot \left[w\left(\sum_{i=k}^{\infty} \frac{1}{2^i} \right) - w\left(\sum_{i=k+1}^{\infty} \frac{1}{2^i} \right) \right] \\ &= \sum_{k=1}^{\infty} v\left(2^k\right) \cdot \left[w\left(2^{1-k}\right) - w\left(2^{-k}\right) \right]. \end{aligned} \tag{2.1}$$

A CPT decision maker's willingness to pay for the lottery L_{STP} is given by the certainty equivalent $v^{-1}(CPT(L_{STP}))$. The following theorem states conditions for a finite certainty equivalent under CPT.

Theorem 2.1 (Emergence of the St. Petersburg paradox)
Let $v : \mathbb{R} \to \mathbb{R}$ be a strictly increasing value function and $w : [0, 1] \to [0, 1]$ be a strictly increasing probability weighting function with $w(0) = 0$ and $w(1) = 1$. Then, it holds for the St. Petersburg lottery L_{STP}:

a) A CPT decision maker reports finite willingness to pay for L_{STP} if v is bounded from above.

b) Assume v is unbounded. Then, a CPT decision maker reports finite willingness to pay for L_{STP} if (sufficient condition)

$$\lim_{p \to 0^+} \frac{w(0.5p) - w(0.25p)}{w(p) - w(0.5p)} < \lim_{z \to \infty} \frac{v(0.5z)}{v(z)}. \tag{2.2}$$

c) A necessary condition for finite willingness to pay for L_{STP} is

$$\lim_{p \to 0^+} \frac{w(0.5p) - w(0.25p)}{w(p) - w(0.5p)} \leq \lim_{z \to \infty} \frac{v(0.5z)}{v(z)}. \tag{2.3}$$

[3] A comprehensive analysis of the St. Petersburg paradox is provided by Samuelson (1977).

Proof of Theorem 2.1: Using a bounded value function $v^b(\cdot)$ is the simplest way to guarantee a finite CPT value. Assuming that v^b is monotonically increasing, strictly concave and bounded, i.e $\lim_{z\to\infty} v^b(z) = c$, it is straightforward to prove statement a) that, independent of the specification of the probability weighting function w, Equation (2.1) is always strictly smaller than c:

$$\sum_{k=1}^{\infty} v^b\left(2^k\right) \cdot \left[w\left(2^{1-k}\right) - w\left(2^{-k}\right)\right] < \sum_{k=1}^{\infty} c \cdot \left[w\left(2^{1-k}\right) - w\left(2^{-k}\right)\right] = c. \tag{2.4}$$

Hence, the maximum willingness to pay (certainty equivalent) for lottery L_{STP} is finite.

For unbounded value functions v, finite willingness to pay is equivalent to convergence of the infinite sum (2.1). The ratio test to assess the convergence of (2.1) in case of unbounded value functions implies finite willingness to pay if

$$\lim_{k\to\infty} \left| \frac{v\left(2^{k+1}\right) \cdot \left[w\left(2^{-k}\right) - w\left(2^{-k-1}\right)\right]}{v\left(2^k\right) \cdot \left[w\left(2^{1-k}\right) - w\left(2^{-k}\right)\right]} \right| < 1. \tag{2.5}$$

If we substitute p for 2^{1-k} (probability) and z for 2^{k+1} (payoff), we can restate the convergence criterion as

$$\lim_{z\to\infty} \frac{v\left(z\right)}{v\left(0.5z\right)} \cdot \lim_{p\to 0^+} \left| \frac{w(0.5p) - w(0.25p)}{w(p) - w(0.5p)} \right| < 1, \tag{2.6}$$

which corresponds to part b). Part c) follows because, according to the ratio test, a necessary condition for convergence is the weak version of the inequalities above.

Note that statement c) holds for bounded as well as unbounded value functions because for bounded and strictly increasing value functions we yield $\lim\limits_{z\to\infty} \frac{v(0.5z)}{v(z)} = 1$. To see this, recall that $v(0) = 0$ so that strict monotonicity and boundedness leads to $\lim\limits_{x\to\infty} v(x) = c$ for some upper bound $c > 0$ and, thus, $\frac{v(0.5z)}{v(z)} \underset{z\to\infty}{\longrightarrow} \frac{c}{c} = 1$. ∎

The next theorem specializes to the case of continuous preference functions and already adumbrates that, for smooth weighting functions, a finite derivative of the probability weighting function at zero, $w'(0) < \infty$, does not guarantee a finite certainty equivalent as long as we allow for various forms for the value function. Rather, the trade-off between the limit of the w-ratio $\lim_{p \to 0^+} \frac{w(0.5p)}{w(p)}$ and the limit of the v-ratio $\lim_{z \to \infty} \frac{v(0.5z)}{v(z)}$ is important.

Theorem 2.2 (Continuous w and the St. Petersburg paradox) *Let $v : \mathbb{R} \to \mathbb{R}$ be a continuous and strictly increasing value function and $w : [0, 1] \to [0, 1]$ be a continuous and strictly increasing probability weighting function with $w(0) = 0$ and $w(1) = 1$. Then, it holds for the St. Petersburg lottery L_{STP}:*

a) A CPT decision maker reports finite willingness to pay for L_{STP} if (sufficient condition)

$$\lim_{p \to 0^+} \frac{w(0.5p)}{w(p)} < \lim_{z \to \infty} \frac{v(0.5z)}{v(z)}. \tag{2.7}$$

b) A necessary condition for finite willingness to pay for L_{STP} is

$$\lim_{p \to 0^+} \frac{w(0.5p)}{w(p)} \leq \lim_{z \to \infty} \frac{v(0.5z)}{v(z)}. \tag{STP^*}$$

c) If in part b) the limits are equal and less than one, that is

$$\lim_{p \to 0^+} \frac{w(0.5p)}{w(p)} = \lim_{z \to \infty} \frac{v(0.5z)}{v(z)} \in (0, 1), \tag{2.8}$$

then the decision maker's willingness to pay is arbitrarily large. Put differently, no reported finite willingness to pay for L_{STP} can be captured by these CPT preferences.

Proof of Theorem 2.2: The case of bounded value functions is clear from Theorem 2.1. So, let us consider unbounded value functions. Lemma 2.1 in

Appendix A.1 proves that for continuous probability weighting functions, it holds:

$$\lim_{p\to 0^+} \frac{w(0.5p) - w(0.25p)}{w(p) - w(0.5p)} = \lim_{p\to 0^+} \frac{w(0.5p)}{w(p)} \ . \tag{2.9}$$

Then, statements a) and b) are clear from Theorem 2.1.

In the situation of statement c), Lemma 2.2 in Appendix A.1 shows that for all $\epsilon > 0$ there exists $p_0 \in (0,1)$ such that

$$w(p) \geq \text{const} \cdot \left(\frac{p}{2}\right)^{\gamma+\epsilon} \tag{2.10}$$

for all $p \in (0, p_0]$. Similarly, Lemma 2.3 in Appendix A.1 ensures that for all $\epsilon > 0$ there exist $x_0 > 0$ such that

$$v(x) \geq \text{const} \cdot x^{\alpha-\epsilon} \tag{2.11}$$

for all $x \geq x_0$.

Now let $\lim\limits_{p\to 0^+} \frac{w(0.5p)}{w(p)} = \lim\limits_{z\to\infty} \frac{v(0.5z)}{v(z)} = 0.5^\gamma \in (0,1)$ for some $\gamma > 0$. Observe that v is unbounded because otherwise, with our convention $v(0) = 0$, we would have had $\lim\limits_{z\to\infty} \frac{v(0.5z)}{v(z)} = 1$. We use Lemma 2.3 and Lemma 2.2 which give lower boundaries for the value function v for larges payoffs z and for the probability weighting function w for small probabilities, respectively. For any $\epsilon_1, \epsilon_2 > 0$ the CPT value can be assessed as

$$\begin{aligned} CPT(L_{STP}) &= \sum_{k=1}^{\infty} v\left(2^k\right) \cdot \left[w\left(2^{1-k}\right) - w\left(2^{-k}\right)\right] && (2.12) \\ &= \sum_{k=1}^{\infty} v\left(2^k\right) w\left(2^{1-k}\right) \cdot \left[1 - \underbrace{\frac{w\left(2^{-k}\right)}{w\left(2^{1-k}\right)}}_{\approx 0.5^\gamma \text{ for sufficiently large } k \text{ and some } \gamma > 0}\right] && (2.13) \\ &\geq \text{const} \cdot \sum_{k=k_0}^{\infty} \left(2^k\right)^{\gamma-\epsilon_1} \left(\frac{2^{1-k}}{2}\right)^{\gamma+\epsilon_2} && (2.14) \\ &= \text{const} \cdot \sum_{k=k_0}^{\infty} \left(2^{-\epsilon_1-\epsilon_2}\right)^k && (2.15) \end{aligned}$$

where k_0 is a sufficiently large index. Equation (2.15) equals infinity if and only if $\epsilon_1 + \epsilon_2 = 0$. Since we can choose $\epsilon_1 > 0$ and $\epsilon_2 > 0$ arbitrarily small, the sum in (2.15), and hence the willingness to pay, grows arbitrarily large.

■

Without probability weighting (i.e. $w(p) = p$), the v-ratio limit $\lim_{z\to\infty} \frac{v(0.5z)}{v(z)}$ has to be strictly greater than 0.5 to ensure a finite subjective CPT value. Note that the value function's concavity alone does not automatically imply a v-ratio limit greater than 0.5. For example, for Bell's (1988) one-switch function $v(x) = \beta x - e^{-\alpha x} + 1$ with $\alpha, \beta > 0$, we prove in Example 2.4 that the v-ratio limit equals 0.5. Applying Theorem 2.2, statement c), re-establishes the well-known fact that this concave value function yields infinite willingness to pay for the St. Petersburg lottery under EUT. This fact foreshadows the insights of Menger (1934) who shows that even within the EUT framework it is possible to construct a *Super St. Petersburg paradox* where many of the strictly concave utility functions are unable to guarantee finite willingness to pay.

Continuity of the probability weighting function in Theorem 2.2 is crucial as the following example shows:

Example 2.1 *Assume a neo-additive probability weighting function w. That is*

$$w(p) = \begin{cases} 0 & \text{for } p = 0 \\ a + b \cdot p & \text{for } p \in (0,1) \\ 1 & \text{for } p = 1 \end{cases}, \tag{2.16}$$

where $a + b \leq 1$ and $a, b > 0$. Note that although $\lim_{p\to 0^+} \frac{w(0.5p)}{w(p)} = \frac{a}{a} = 1$, finite willingness to pay is well possible for $\lim_{z\to\infty} \frac{v(0.5z)}{v(z)} < 1$ (contrary to the case of *continuous weighting functions) because the CPT value is given by*

$$CPT(L_{STP}) = \sum_{k=1}^{\infty} v\left(2^k\right) \cdot \left[w\left(2^{1-k}\right) - w\left(2^{-k}\right)\right] \tag{2.17}$$

$$= \sum_{k=1}^{\infty} v\left(2^k\right) \cdot \left[a + b \cdot \left(2^{1-k}\right) - a - b \cdot \left(2^{-k}\right)\right] \tag{2.18}$$

$$= \sum_{k=1}^{\infty} v\left(2^k\right) \cdot b \cdot 2^{-k}. \tag{2.19}$$

If $\lim_{z\to\infty} \frac{v(0.5z)}{v(z)} = 0.5$ *then Lemma 2.3 in Appendix A.1 applies and* (2.19) *is larger than* $b\sum_{k=k_0}^{\infty} 2^{(1-\epsilon)k-k} = b\sum_{k=k_0}^{\infty}\left(2^{-\epsilon}\right)^k$ *for any* $\epsilon > 0$ *and sufficiently large* k_0*. Since we can choose* $\epsilon > 0$ *arbitrarily small, the CPT value of the St. Petersburg lottery,* $CPT(L_{STP})$*, is unbounded if* $\lim_{z\to\infty} \frac{v(0.5z)}{v(z)} = 0.5$*. Furthermore,* v *cannot be bounded from above. Together with the ratio test applied to* (2.19)*, it follows that a necessary and sufficient condition for finite willingness to pay for* L_{STP} *is*

$$\lim_{z\to\infty} \frac{v\left(0.5z\right)}{v(z)} > \frac{1}{2}. \tag{2.20}$$

In other words, neo-additive probability weighting functions have the same implications for the St. Petersburg paradox as EUT despite the w*-ratio limit being equal to one. An obvious value function that now produces finite willingness to pay is* $v(x) = x^{0.88}$ *because* $\frac{v(0.5z)}{v(z)} = 0.5^{0.88} = 0.543$.

It is clear that the limit $\lim_{p\to 0^+} \frac{w(0.5p)}{w(p)}$ is in the interval $[0,1]$. Intuitively, the limit of this w-ratio is an index of concavity of w at probability $p = 0$. More precisely, for sufficiently smooth weighting functions, Dierkes & Sejdiu (2019a) show that it relates to a probabilistic counterpart of relative risk aversion at $p = 0$ as defined here.

Definition 2.1 *Let* w *be a probability weighting function which is twice continuously differentiable on a subset* $(0, p_0)$*,* $p_0 \in (0,1)$*, and strictly increasing. Then we define a probabilistic counterpart of relative risk* *aversion at infinitesimally small probabilities:*

$$RRA_w^0 = \lim_{p \to 0^+} p \frac{w''(p)}{w'(p)}. \tag{2.21}$$

Note that $RRA_w^0 > 0$ indicates probabilistic risk aversion and $RRA_w^0 < 0$ probabilistic risk proclivity when processing infinitesimally small probabilities. Using our previous definition, Dierkes & Sejdiu (2019a) show that for smooth w and all $\Delta \in (0,1)$ it holds

$$\lim_{p \to 0^+} \frac{w(\Delta p)}{w(p)} = \Delta^{1+RRA_w^0}. \tag{2.22}$$

That is, the limit $\lim_{p \to 0^+} \frac{w(0.5p)}{w(p)}$ is informative about the curvature of w at $p = 0$. In particular, a higher w-ratio limit indicates more concavity of w at $p = 0$.

The discussion of the power value function $v_{Power}(x) = x^\alpha$ for $\alpha \in (0,1)$ is now rather simple and a particularly worthwhile example because it is by far the most frequently used parametrization in CPT. Recall that v_{Power} exhibits constant relative risk aversion equal to $1 - \alpha$. There is now an intuitive interpretation for finite willingness to pay for the St. Petersburg lottery L_{STP}. Corollary 2.1 below shows that willingness to pay for L_{STP} is finite if and only if the decision maker exhibits strictly positive total relative risk aversion. Here, total relative risk aversion is the sum of probabilistic relative risk aversion RRA_w^0 induced by the probability weighting function and relative risk aversion of the value function as defined by the Arrow-Pratt measure. Put differently, to produce a lower certainty equivalent than the expected value (infinity in this case) the decision maker must exhibit strict risk aversion. In the CPT framework, risk aversion is driven by both the value and the probability weighting function. Conversely, with risk neutrality or risk proclivity, gambling for an infinite expected payoff is desirable and decision makers are willing to pay any amount.

Corollary 2.1 *Provided w is twice continuously differentiable on a subset $(0, p_0)$, $p_0 \in (0,1)$, and strictly increasing and the value function is given by $v_{Power}(x) = x^\alpha$ with $\alpha \in (0,1)$ then the CPT decision maker has finite willingness to pay for the St. Petersburg lottery L_{STP} if and only if*

$$RRA_w^0 + RRA_v > 0, \tag{2.23}$$

where $RRA_v = -x\frac{v''(x)}{v'(x)} = 1 - \alpha$ is the constant relative risk aversion of the power value function v.

Proof of Corollary 2.1: Note that $\frac{v(0.5z)}{v(z)} \in (0,1)$ for all $\alpha > 0$. According to Theorem 2.2, parts a) and c), it suffices to show that $\lim_{p \to 0^+} \frac{w(0.5p)}{w(p)} < 0.5^\alpha$ is equivalent to Equation (2.23). Using $\Delta = 0.5$ in Equation (2.22), we get

$$\lim_{p \to 0^+} \frac{w(0.5p)}{w(p)} < 0.5^\alpha \tag{2.24}$$

$$\Leftrightarrow \quad 0.5^{RRA_w^0 + 1} < 0.5^{-RRA_v + 1} \tag{2.25}$$

$$\Leftrightarrow \quad RRA_w^0 > -RRA_v \tag{2.26}$$

which is equivalent to Equation (2.23). ■

To illustrate the applications of our findings, we discuss some typical parametrizations of v and w from the literature.

Example 2.2 *The convergence rate of the w-ratio $\frac{w(0.5p)}{w(p)}$ for tiny probabilities for commonly employed probability weighting functions w are given as follows:*

a) For $w_{TK92}(p) = p^\gamma / \left(p^\gamma + (1-p)^\gamma\right)^{1/\gamma}$, $\gamma \in (0,1)$ proposed by Tversky & Kahneman (1992), we have $\lim_{p \to 0^+} \frac{w_{TK92}(0.5p)}{w_{TK92}(p)} = 0.5^\gamma$.

b) For $w_{log-odds}(p) = \delta p^\gamma / \left(\delta p^\gamma + (1-p)^\gamma\right)$, $\gamma \in (0,1)$, $\delta > 0$ proposed by Goldstein & Einhorn (1987)[4], we have $\lim_{p \to 0^+} \frac{w_{log-odds}(0.5p)}{w_{log-odds}(p)} = 0.5^\gamma$.

[4] It is also used by e.g Tversky & Fox (1995), and Lattimore et al. (1992).

c) *For $w_{Prelec}(p) = e^{-(-\log p)^\gamma}$, $\gamma \in (0,1)$ proposed by Prelec (1998), we have*

$$\lim_{p \to 0^+} \frac{w_{Prelec}(0.5p)}{w_{Prelec}(p)} = \begin{cases} 1 & \text{if } \gamma \in (0,1), \\ 0.5 & \text{if } \gamma = 1, \\ 0 & \text{if } \gamma > 1. \end{cases}$$

d) *Consider polynomial probability weighting functions $w_{Poly}(p) = \sum_{i=1}^{N} a_i \cdot p^i$ with parameters $a_i \in \mathbb{R}$ for $i = 1, \ldots, N$ and $a_N \neq 0$. Let j be the smallest index $i < N$ such that $a_j \neq 0$, i.e. $w_{Poly}(p) = \sum_{i=j}^{N} a_i \cdot p^i$. Then the limit is $\lim\limits_{p \to 0^+} \frac{w_{Poly}(0.5p)}{w_{Poly}(p)} = 0.5^j$.*

The third degree polynomial weighting function $w_{RW06}(p) = \frac{3-3b}{a^2-a+1}\left(p^3 - (a+1)p^2 + ap\right) + p$ with $a, b \in (0,1)$ proposed by Rieger & Wang (2006) is a special case with limit $\lim\limits_{p \to 0^+} \frac{w_{RW06}(0.5p)}{w_{RW06}(p)} = 0.5$ because $a, b \in (0,1)$ imply $a_1 \neq 0$.

Before we get to the crux of our paper, namely the restrictions on finite willingness to pay for L_{STP} inhibit the emergence of the common ratio effect, it is now easy to restate some selected results from the literature on the willingness to pay for the St. Petersburg lottery L_{STP} (see, e.g., Blavatskyy (2005) and Rieger & Wang (2006)):

Example 2.3 *Suppose the CPT decision maker exhibits a power value function $v_{Power}(x) = x^\alpha$ with $\alpha \in (0,1)$. Hence, $\frac{v(0.5z)}{v(z)} = 0.5^\alpha$ is independent of z. From Theorem 2.2 and Example 2.2, we know that:*

(i) *With parametrization of the probability weighting function w as in Tversky & Kahneman (1992) or Goldstein & Einhorn (1987), i.e. $w_{TK92}(p) = p^\gamma / \left(p^\gamma + (1-p)^\gamma\right)^{1/\gamma}$ or $w_{log-odds}(p) = \delta p^\gamma / \left(\delta p^\gamma + (1-p)^\gamma\right)$, respectively, the willingness to pay for the St. Petersburg lottery L_{STP} is finite if and only if $\alpha < \gamma$.*

(ii) *With the Prelec (1998) parametrization $w_{Prelec}(p) = e^{-(-\log p)^{\gamma}}$, finite willingness to pay for the St. Petersburg lottery L_{STP} implies $\gamma \geq 1$. In other words, finite willingness to pay is not possible with the inverse S-shaped probability weighting function w_{Prelec}. This result foreshadows the conflict with the common ratio effect which exactly requires an inverse S-shaped probability weighting function.*

(iii) *The CPT decision maker states finite willingness to pay for L_{STP} for all polynomial probability weighting functions because $0.5^{\alpha} > 0.5^{j}$ for all coefficient indices $j = 1, 2, \ldots$ as in Example 2.2, part d).*

Example 2.4 *Suppose the CPT decision maker exhibits Bell's (1988) one-switch function $v(x) = \beta x - e^{-\alpha x} + 1$ which is unbounded, strictly increasing, and strictly concave for $\alpha > 0$ and $\beta > 0$. Then,*

$$\lim_{z\to\infty} \frac{v(0.5z)}{v(z)} = \lim_{z\to\infty} \frac{\beta 0.5z - e^{-\alpha 0.5z} + 1}{\beta z - e^{-\alpha z} + 1} \overset{l'Hospital}{=} \lim_{z\to\infty} \frac{0.5\beta + 0.5\alpha e^{-\alpha 0.5z}}{\beta + \alpha e^{-\alpha z}} = 0.5$$

and from Theorem 2.2, statement c) and Examples 2.2, we know that:

(i) *For parametrizations of the probability weighting function w as in Tversky & Kahneman (1992), Goldstein & Einhorn (1987), or Prelec (1998) i.e. $w_{TK92}(p) = p^{\gamma}/\left(p^{\gamma} + (1-p)^{\gamma}\right)^{1/\gamma}$, $w_{log-odds}(p) = \delta p^{\gamma}/\left(\delta p^{\gamma} + (1-p)^{\gamma}\right)$, or $w_{Prelec}(p) = e^{-(-\log p)^{\gamma}}$, respectively, an inverse-S shaped weighting function w never leads to finite willingness to pay for the St. Petersburg lottery L_{STP} because $\gamma \in (0, 1)$. If $\gamma > 1$ then willingness to pay is finite.*

(ii) *Consider polynomial probability weighting functions $w_{Poly}(p) = \sum_{i=1}^{N} a_i \cdot p^i$ with parameters $a_i \in \mathbb{R}$ for $i = 1, \ldots, N$ and $a_N \neq 0$. Then, willingness to pay for L_{STP} is finite if and only if $a_1 = 0$. In particular, a probability weighting function's finite slope at probability $p = 0$ is not enough to guarantee finite willingness to pay for the St. Petersburg lottery.*

2.2.2 The common ratio effect under CPT

The Allais paradox is a traditional counterexample against EUT and comes along in different versions. We focus on the common ratio version which involves choices between equal mean lotteries[5]

$$L_1(p) = (\$z, 0.5p; \$0, 1-0.5p) \quad \text{and} \quad L_2(p) = (\$0.5z, p; \$0, 1-p) \tag{2.27}$$

where $z > 0$ denotes a payoff amount and p a probability. Empirically, subjects choose the safer lottery L_2 for high probabilities p and the riskier lottery L_1 for low probabilities p.[6] EUT's independence axiom, however, does not allow for this change in preference over L_1 and L_2 for varying probabilities p.

By introducing probability weighting, CPT is able to explain this choice behavior. In the CPT framework, a risk seeking choice is predicted when

$$CPT(L_1) > CPT(L_2)$$

$$\Leftrightarrow \; v(z) \cdot w(0.5p) + v(0) \cdot [1 - w(0.5p)] > v(0.5z) \cdot w(p) + v(0) \cdot [1 - w(p)] \tag{2.28}$$

$$\Leftrightarrow \qquad \frac{w(0.5p)}{w(p)} - \frac{v(0.5z) - v(0)}{v(z) - v(0)} > 0 \tag{2.29}$$

and a risk averse choice results vice versa. Using Assumption 2 whereby $v(0) = 0$, we define the common ratio effect with the help of Equation (2.29).

Definition 2.2 (Common ratio effect) *Let v be a value function and w be a probability weighting function. We say that, in this CPT framework with equal mean lotteries $L_1(p) = (\$z, 0.5p; \$0, 1-0.5p)$ and $L_2(p) =$*

[5]More generally, the results of this section easily extend to lotteries $L_1(p) = (\$z, \Delta p; \$0, 1-\Delta p)$ and $L_2(p) = (\$\Delta z, p; \$0, 1-p)$ with $\Delta \in (0, 1)$. However, to analyze the conflict with the St. Petersburg paradox, we focus on lotteries with payoff and probability ratio of $\Delta = 0.5$ which is a common choice in experiments.

[6]Lottery $L_1(p)$ is called riskier than the lottery $L_2(p)$ because $L_1(p)$ is a mean-preserving spread of $L_2(p)$.

($\$0.5z, p; \$0, 1-p$) *with payoff* $z > 0$*, the common ratio effect is predicted if and only if there exists exactly one sign change in function*

$$f_{CRE}(p) = \frac{w(0.5p)}{w(p)} - \frac{v(0.5z)}{v(z)}, \tag{2.30}$$

that is, there exists exactly one preference reversal probability p^* *such that* $f_{CRE}(p) > 0$ *for* $p \in (0, p^*)$ *and* $f_{CRE}(p) < 0$ *for* $p \in (p^*, 1]$.

This definition already foreshadows the conflict between the common ratio effect and the necessary conditions for finite willingness to pay for the St. Petersburg Lottery as stated in Theorem 2.2. This definition is also a little stricter than we need for our purposes. We can relax the assumption of a single preference reversal probability as long as there exists a probability p^* such that $f_{CRE}(p) > 0$ for $p \in (0, p^*)$. Put differently, we need to assume that, for sufficiently small probabilities p, the decision maker chooses the riskier lottery which we find is an intuitive criterion. Nevertheless, multiple preference reversal probabilities appear awkward as they would imply, at odds with lab results, rather erratic behavior.

The following proposition is a trivial consequence of the previous definition and explicitly states necessary conditions for the common ratio effect.

Proposition 2.1 (Emergence of the common ratio effect)
Let v *be a value function and* w *be a probability weighting function. Consider the equal mean lotteries* $L_1(p) = (\$z, 0.5p; \$0, 1-0.5p)$ *and* $L_2(p) = (\$0.5z, p; \$0, 1-p)$*. Then it holds:*

a) For fixed payoff z*, one necessary condition*[7] *for the prediction of the*

[7] There are more necessary conditions, of course. This one, however, unveils the conflict with finite willingness to pay for the St. Petersburg lottery L_{STP}. Another obvious necessary condition is $\lim\limits_{p \to 1} \frac{w(0.5p)}{w(p)} = w(0.5) \leq \frac{v(0.5z)}{v(z)}$.

common ratio effect as defined in Definition 2.2 is

$$\lim_{p\to 0^+} \frac{w(0.5p)}{w(p)} \geq \frac{v(0.5z)}{v(z)}. \tag{CRE}$$

b) Allais (1953) suggests that statement a) holds for all payoffs z, in particular large payoffs. This leads to the necessary condition:

$$\lim_{p\to 0^+} \frac{w(0.5p)}{w(p)} \geq \lim_{z\to\infty} \frac{v(0.5z)}{v(z)}. \tag{CRE^*}$$

Clearly, if the v-ratio $\frac{v(0.5z)}{v(z)}$ is independent of the payoff z then the two necessary conditions CRE and CRE^* coincide. This is the case for the power value function $v_{Power}(x) = x^\alpha$, $\alpha \in (0,1)$ which is the, by far, most often employed parametrization of the value function. Further, the necessary conditions CRE^* in Proposition 2.1 and STP^* in Theorem 2.2, respectively, leave at best a corner solution for many CPT calibrations. Put together, both restrictions require for continuous w

$$\lim_{p\to 0^+} \frac{w(0.5p)}{w(p)} = \lim_{z\to\infty} \frac{v(0.5z)}{v(z)}. \tag{2.31}$$

However, statement c) in Theorem 2.2 rules out such cases where

$$\lim_{p\to 0^+} \frac{w(0.5p)}{w(p)} = \lim_{z\to\infty} \frac{v(0.5z)}{v(z)} \in (0,1). \tag{2.32}$$

In particular, using a power value function v_{Power} for which always $\lim_{z\to\infty} \frac{v(0.5z)}{v(z)} \in (0,1)$, will always result in a strict conflict between the restrictions on the St. Petersburg and the Allais paradox if the probability weighting function is continuous.

The remaining case $\lim_{p\to 0^+} \frac{w(0.5p)}{w(p)} = \lim_{z\to\infty} \frac{v(0.5z)}{v(z)} = 1$ is ruled out by the following proposition.

Proposition 2.2 *Let w be a strictly increasing and continuous probability weighting function and v be a strictly increasing value function with $v(0) =$*

0. If $\lim_{z\to\infty} \frac{v(0.5z)}{v(z)} = 1$ *then the common ratio effect does not emerge for large payoffs* z *in lotteries* $L_1(p) = (\$z, 0.5p; \$0, 1-0.5p)$ *and* $L_2(p) = (\$0.5z, p; \$0, 1-p)$. *Put differently, by increasing* z*, the preference reversal probability* p^**, given by the intersection of function* $f_{CRE}(p) = \frac{w(0.5p)}{w(p)} - \frac{v(0.5z)}{v(z)}$ *with the abscissa, can be moved arbitrarily close to zero if it exists at all.*

Proof of Proposition 2.2: Since $0 \leq \frac{w(0.5p)}{w(p)} \leq 1$ for all probabilities p and $\lim_{z\to\infty} \frac{v(0.5z)}{v(z)} = 1$, the function $f_{CRE}(p) = \frac{w(0.5p)}{w(p)} - \frac{v(0.5z)}{v(z)}$ is, for $p \to 0^+$, either negative or arbitrarily close to zero for sufficiently high payoffs z. That is, the intersection probability p^*, also denoted the preference reversal probability, moves for large z arbitrarily close to zero if it exists at all. ■

Bounded value functions always imply finite willingness to pay for the St. Petersburg lottery. However, the following corollary shows that they have difficulties predicting the Allais paradox because bounded value functions always have a v-ratio limit equal to one.

Corollary 2.2 *Let* v *be a bounded and strictly increasing value function with* $v(0) = 0$*. Then the common ratio effect does not emerge for large payoffs* z *in lotteries* $L_1(p) = (\$z, 0.5p; \$0, 1-0.5p)$ *and* $L_2(p) = (\$0.5z, p; \$0, 1-p)$ *because, by increasing* z*, the preference reversal probability* p^**, given by the intersection of function* $f_{CRE}(p) = \frac{w(0.5p)}{w(p)} - \frac{v(0.5z)}{v(z)}$ *with the abscissa, can be moved arbitrarily close to zero if it exists at all.*

Proof of Corollary 2.2: For bounded strictly increasing value functions v with $v(0) = 0$, it holds: $\lim_{z\to\infty} \frac{v(0.5z)}{v(z)} = 1$. Assume $\lim_{x\to\infty} v(x) = b$ for some upper bound $b > 0$. Hence, $\frac{v(0.5z)}{v(z)} \underset{z\to\infty}{\longrightarrow} \frac{b}{b} = 1$. Then, the statement follows from Proposition 2.2. ■

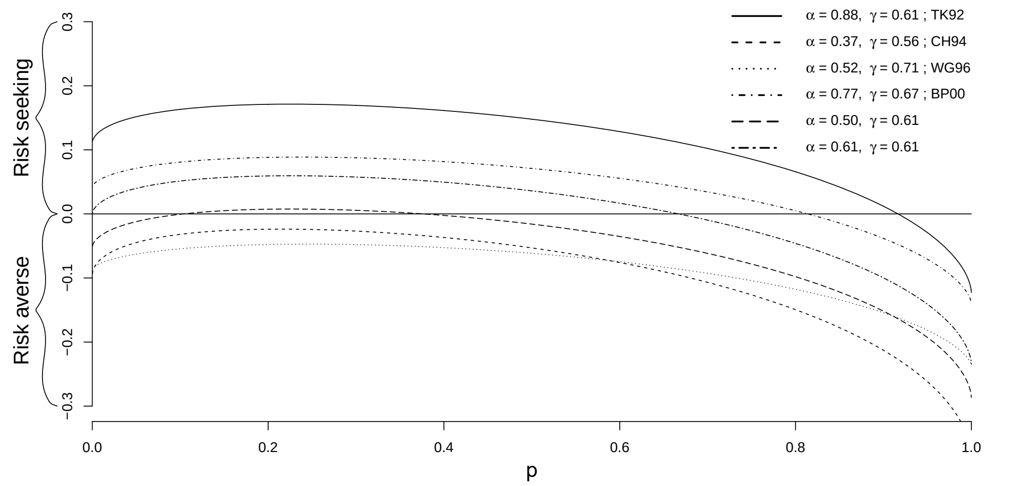
Before we summarize these findings in our main Theorem 2.4 and discuss further the, by now clear, tension between the two presumably most prominent paradoxes in decision theory, we illustrate implications of

Definition 2.2 with Figure 2.1. This illustration is worthwhile for several reasons. First, some calibrations from the literature do not predict the common ratio effect while others do. Second, we show that several preference reversal points are theoretically possible. Third, it nicely hints at the role of Proposition 2.2 although we use the power value function. Fourth, it indicates similarities and differences between the weighting functions w_{TK92}, $w_{log-odds}$, w_{Prelec}, w_{Poly}, and w_{neo}.

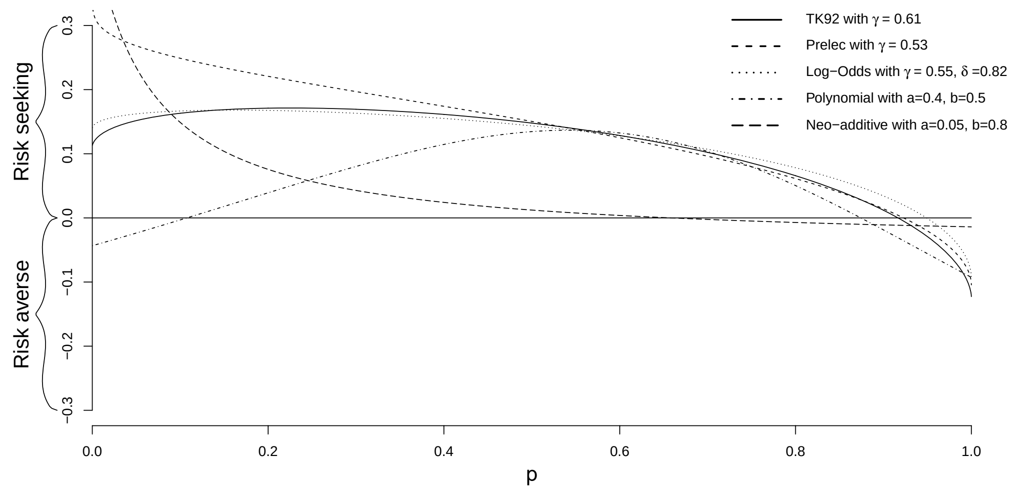
Figure 2.1 depicts function f_{CRE} as defined in Definition 2.2. We focus on Tversky & Kahneman's (1992) CPT parametrization in Panel A and vary the specification of the probability weighting function in Panel B. Since we assume the power value function $v_{Power}(x) = x^{\alpha}$, no assumption about the lottery payoffs z is needed. Specifically, Panel A depicts the following function:

$$f_{TK92}(p) = 0.5^{\gamma} \cdot \left[\frac{p^{\gamma} + (1-p)^{\gamma}}{(0.5p)^{\gamma} + (1-0.5p)^{\gamma}}\right]^{\frac{1}{\gamma}} - 0.5^{\alpha}. \tag{2.33}$$

Positive function values of (2.33) indicate risk seeking behavior and negative values risk averse behavior. The black solid line depicts an individual's risk attitude when assuming Tversky & Kahneman's (1992) suggested median parameters $\alpha = 0.88$ and $\gamma = 0.61$. In line with Definition 2.2, a risk seeking choice is predicted for $p < 0.91$ and a risk averse choice otherwise. Estimates $(\alpha, \gamma) = (0.77, 0.67)$, taken from Bleichrodt & Pinto (2000), make similar predictions. The decision maker behaves risk averse roughly for probabilities $p > 0.81$ and risk seeking otherwise. Interestingly, parameter estimations by Camerer & Ho (1994) and Wu & Gonzalez (1996) who estimate the parameter sets $(\alpha, \gamma) = (0.37, 0.56)$ and $(\alpha, \gamma) = (0.50, 0.71)$, respectively, uniformly display risk aversion for all probabilities p and, hence, do not



Panel A: Common ratio effect for different parameter sets (α, γ) of Tversky & Kahneman's (1992) CPT parametrization.



Panel B: Common ratio effect for various probability weighting functions.

Figure 2.1: Common ratio effect under CPT.

Notes: This figure illustrates the common ratio effect under various parametrizations of CPT by depicting the function $f_{CRE}(p) = w(0.5p)/w(p) - v(0.5z)/v(z)$ as a function of probability p. The decision maker has the choice between the equal mean lotteries $L_1(p) = (\$z, 0.5p; \$0, 1-0.5p)$ and $L_2(p) = (\$0.5z, p; \$0, 1-p)$. For positive function values f_{CRE}, she behaves risk seeking and prefers the riskier lottery L_1 over the safer lottery L_2. Conversely, for negative values f_{CRE}, she behaves risk averse and prefers L_2 over L_1. In Panel A, the individual's preferences are given by the value function $v(x) = x^\alpha$ and the weighting function w_{TK92} for different parameter sets (α, γ) including those estimated in Tversky & Kahneman (1992), Camerer & Ho (1994), Wu & Gonzalez (1996), and Bleichrodt & Pinto (2000) denoted in the legend by TK92, CH94, WG96, and BP00, respectively. In Panel B, the value function is fixed as $v(x) = x^{0.88}$ (Tversky & Kahneman, 1992) and the weighting function takes the forms w_{TK92}, $w_{log-odds}$, w_{Prelec}, w_{RW06}, and w_{neo}. Due to the form of the value function, no assumption about z is needed. Function values of f_{CRE} are depicted for weighting functions with parameter estimates of Tversky & Kahneman (1992) for TK92, estimates of Bleichrodt & Pinto (2000) for Prelec and Log-odds, parameter values motivated by Rieger & Wang (2006) for a cubic weighting function, and the neo-additive weighting function with intercept $a = 0.05$ and slope $b = 0.8$, respectively.

explain the common ratio effect.[8] However, as argued in Section 2.2.1, these parameter combinations are in line with finite willingness to pay for the St. Petersburg lottery because $\alpha < \gamma$.

In addition, we plot two illustrative pairs of preference parameters. Cases where $\alpha = \gamma < 1$ are interesting in the Tversky & Kahneman (1992) parametrization because they robustly predict a single preference reversal point with risk prone behavior for low probabilities p (positive f_{CRE}) and risk averse behavior for larger probabilities p (negative f_{CRE}), but the limiting case in Equation (2.33) yields $\lim_{p\to 0} f_{CRE}(p) = 0$. Panel A illustrates the case $\alpha = \gamma = 0.61$.

Interestingly, the pair $\alpha = 0.5$ and $\gamma = 0.61$ has two preference reversal points, yielding risk prone behavior for intermediate probabilities between 0.1 and 0.38 and risk averse behavior otherwise. Such cases are bad empirical predictors because real decision makers exhibit a single preference reversal point with lower probabilities typically leading to more risk prone choices.

Panel B of Figure 2.1 illustrates that this problem also arises for weighting functions with finite slope at zero such as the cubic weighting function $w_{RW06}(p) = \frac{3-3b}{a^2-a+1}\left(p^3 - (a+1)p^2 + ap\right) + p$ of Rieger & Wang (2006), see also Proposition 2.3 and Example 2.5 below. In the context of our paper, the failure of the polynomial weighting function w_{RW06} to explain the common ratio effect is particularly interesting since it was primarily designed to explain the St. Petersburg paradox (Rieger & Wang, 2006).

Moreover, Panel B fixes the value function as $v(x) = x^{0.88}$ and shows that the image of f_{CRE} is for the approximate range $0.56 \leq p \leq 1$ very similar for all four continuous probability weighting functions

[8] Neilson & Stowe (2002) note that the parameter estimates of Camerer & Ho (1994) and Wu & Gonzalez (1996) are unable to predict gambling on unlikely gains and the choice behavior of Allais' original common consequence example. In contrast to Neilson & Stowe (2002), we do not call for new parametrizations of CPT. Our Allais - St. Petersburg test evaluates CPT in its most general form and does not rely on specific parametrizations of the value and probability weighting function.

w_{TK92}, $w_{log-odds}$, w_{Prelec}, and w_{RW06} with standard parameter values but significantly different for lower probabilities $0 \le p \le 0.56$. While w_{TK92} and $w_{log-odds}$ seem to treat all probabilities very similar, w_{Prelec} and w_{RW06} go into the opposite direction when probabilities get smaller. This discrepancy nicely foreshadows the conflict between the conditions of the Allais and St. Petersburg paradox since both weighting functions are motivated by one paradox, respectively. Note that Prelec (1998) motivates his probability weighting function $w_{Prelec}(p) = e^{-(-\log p)^\gamma}$ with Allais' common ratio effect. Similarly, the discontinuous neo-additive weighting function results in a strictly decreasing function f_{CRE} and predicts the common ratio effect. Figure 2.1 also shows that f_{CRE} does not need be a monotone function in p. Monotonicity is guaranteed, however, if we use a subproportional probability weighting function such as w_{Prelec} (Prelec, 1998) or neo-additive ones.

The next proposition shows that a combination of a probability weighting function with finite slope at zero and a strictly concave value function cannot explain the common ratio effect.

Proposition 2.3 *Let the value function v be strictly concave and let the probability weighting function w be right-differentiable around zero and the slope of w at zero be finite, i.e. $w'(0^+) = c$ with $0 \le c < \infty$. Then, $\lim\limits_{p \to 0^+} \frac{w(0.5p)}{w(p)} \le 0.5$ and the CPT decision maker always prefers the safer lottery $L_2(p) = (\$0.5z, p; \$0, 1-p)$ over $L_1(p) = (\$z, 0.5p; \$0, 1-0.5p)$ for any payoff $z > 0$ when the probabilities p and $0.5p$ tend to zero. In particular, the common ratio effect does not emerge.*

Proof of Proposition 2.3: If w is monotonically increasing with $w(0) = 0$, $w(1) = 1$ and $w'(0^+) = c$, $0 < c < \infty$, then applying the rule of l'Hopital directly shows that

$$\lim_{p \to 0^+} \frac{w(0.5p)}{w(p)} \overset{l'Hopital}{=} \lim_{p \to 0^+} \frac{w'(0.5p) \cdot 0.5}{w'(p)} = \frac{w'(0) \cdot 0.5}{w'(0)} = 0.5 \,. \tag{2.34}$$

For $w'(0) = 0$, the general monotonicity assumption of w ensures that there exists a $p_0 \in (0, 1]$ such that w is convex for $p \in [0, p_0]$, i.e. $w(0.5p) \leq 0.5w(p)$.

Since v is strictly concave it follows that $0.5 < \frac{v(0.5z)}{v(z)} < 1$ for any $z > 0$ and

$$\lim_{p \to 0^+} f_{CRE}(p) = \lim_{p \to 0^+} \underbrace{\frac{w(0.5p)}{w(p)}}_{\leq 0.5} - \underbrace{\frac{v(0.5z)}{v(z)}}_{>0.5} < 0 \tag{2.35}$$

which is equivalent to $L_2 \succ L_1$. This, however, contradicts the empirical evidence of the common ratio effect and, hence, Definition 2.2. ■

In light of Example 2.1, the next proposition shows that neo-additive weighting functions are a prime candidate to solve both paradoxes.

Proposition 2.4 *Let w be the neo-additive probability weighting function with intercept a and slope b with $a, b > 0$ and $a + b \leq 1$. Then the common ratio effect emerges between Lotteries L_1 and L_2 with payoffs determined by z if and only if*

$$\frac{a + 0.5b}{a + b} < \frac{v(0.5z)}{v(z)} < 1. \tag{2.36}$$

Proof of Proposition 2.4: Observe that the function $f_{CRE}(p) = \frac{w_{neo}(0.5p)}{w_{neo}(p)} - \frac{v(0.5z)}{v(z)}$ is strictly decreasing in p:

$$\frac{\partial f_{CRE}}{\partial p} = -\frac{0.5 \cdot a \cdot b}{(a + b \cdot p)^2} < 0 \quad \forall\, p \in (0, 1) \text{ if } a, b > 0. \tag{2.37}$$

Since $\lim_{p \to 0^+} f_{CRE}(p) = 1 - \frac{v(0.5z)}{v(z)}$ and $\lim_{p \to 1} f_{CRE}(p) = \frac{a+0.5b}{a+b} - \frac{v(0.5z)}{v(z)}$, the common ratio effect then emerges if and only if the conditions in the proposition are fulfilled. ■

The class of neo-additive weighting functions is the simplest class of weighting functions that allows a solution to both paradoxes. In Subsection 2.2.3, we will show how to extend the class of weighting functions so as to accommodate more complex choice behavior for intermediate probabilities (Harless & Camerer, 1994; Wu & Gonzalez, 1996). We conclude this subsection with two examples.

Example 2.5 *Let the value function v be strictly concave. With polynomial probability weighting functions $w_{Poly}(p) = \sum_{i=1}^{N} a_i \cdot p^i$ with parameters $a_i \in \mathbb{R}$ for $i = 1, \ldots, N$ and $a_N \neq 0$, the choice behavior in the common ratio effect according to Definition 2.2 cannot be predicted. To see this, note that it holds $0 \leq w'_{Poly}(0) < \infty$. Then, Proposition 2.3 applies.*

Example 2.6 *Consider $w_{Prelec}(p) = e^{-(-\log p)^{\gamma}}$ proposed by Prelec (1998). It is subproportional and $\lim_{p \to 0^+} \frac{w_{Prelec}(0.5p)}{w_{Prelec}(p)} = 1$ for $\gamma \in (0, 1)$, see Example 2.2. Assume the value function is given by $v_{Log}(x) = \log(1 + x)$. Then, $\lim_{p \to 0^+} \frac{w_{Prelec}(0.5p)}{w_{Prelec}(p)} = \lim_{z \to \infty} \frac{v_{Log}(0.5z)}{v_{Log}(z)} = 1$ is a corner case for the necessary conditions (CRE^*) and (STP^*). However, the empirically observed common ratio effect cannot be predicted because of Proposition 2.2. Intuitively, by using ever larger payoffs z in the common ratio lotteries L_1 and L_2, as is supported by experimental evidence in Allais (1953), we can move the preference reversal probability p^* arbitrarily close to zero.*

Interestingly, Example 2.8 in Appendix A.2 shows that the combination of v_{Log} and w_{Prelec} produces finite willingness to pay for the St. Petersburg lottery. Theoretically then, smaller payoffs z might offer a solution to both paradoxes with v_{Log} and w_{Prelec}. However, a sensitivity analysis in Appendix A.2 unveils for various combinations of v and continuous w that only unreasonably small z would do the trick.

2.2.3 Summary: Allais - St. Petersburg conflict in CPT

We summarize our findings in the following theorems which distinguish between discontinuous and continuous probability weighting functions.

Specifically, in the latter case of continuous weighting functions, no solution exists, in particular for large payoffs. In the former case of discontinuous weighting functions, the class of neo-additive weighting functions opens the door for a broader class of weighting functions which can additionally accommodate more complex choice behavior as found, for example, by Harless & Camerer (1994) than neo-additive functions which are linear for probabilities in $(0, 1)$.

In the following, we shall include explicit statements about the power value function because of its predominant use in the literature although it constitutes a trivial corollary of more general results. Cases where for fixed payoff z and $z/2$ in lotteries L_1 and L_2, respectively, it holds $\frac{v(0.5z)}{v(z)} < \lim_{p \to 0^+} \frac{w(0.5p)}{w(p)} = \lim_{z \to \infty} \frac{v(0.5z)}{v(z)} = 1$ are not covered here. They might theoretically allow for finite willingness to pay for the St. Petersburg lottery L_{STP} and the emergence of the common ratio effect for small or maybe even moderately large payoffs z. A sensitivity analysis in Appendix A.2, however, rules out any realistic cases for specific parametrizations.

Theorem 2.3 (Simultaneous solution to both paradoxes)
Let w_{neo} be defined by

$$w(p) = \begin{cases} 0 & \textit{for } p = 0 \\ a + b \cdot p & \textit{for } p \in (0,1) \\ 1 & \textit{for } p = 1 \end{cases}, \tag{2.38}$$

with $a, b > 0$ and $a + b \leq 1$. Let further v be a continuous and strictly increasing value function. The common ratio lotteries are given by $L_1(p) = (\$z, 0.5p; \$0, 1 - 0.5p)$ and $L_2(p) = (\$0.5z, p; \$0, 1 - p)$ with $z > 0$. Then it holds:

a) Let the probability weighting function be given by w_{neo}. Assume the decision maker strictly prefers the risky lottery L_1 over L_2 for *probabilities near zero and the safe lottery* L_2 *over* L_1 *for probabilities near one for all sufficiently high payoffs* z. *This is equivalent to* $\frac{a+0.5b}{a+b} < \frac{v(0.5z)}{v(z)} < 1$ *for all payoffs* $z \geq z_0$ *for some* $z_0 > 0$. *Furthermore, in those cases, the decision maker states finite willingness to pay for playing the St. Petersburg lottery* L_{STP}.

In other words, $\frac{a+0.5b}{a+b} < \frac{v(0.5z)}{v(z)} < 1$ *for all payoffs* $z \geq z_0$ *for some* $z_0 > 0$ *is equivalent to the simultaneous solution of both the St. Petersburg paradox and the common ratio version of the Allais paradox for all sufficiently large payoffs in the common ratio lotteries.*

b) Assume the probability weighting function is given by $w(p) = w_{neo} \circ w_{cont.}(p) = w_{neo}(w_{cont.}(p))$, *where* $w_{cont.}$ *is a continuous and strictly increasing probability weighting function. If the decision maker states finite willingness to pay for playing the St. Petersburg lottery* L_{STP} *and strictly prefers the risky lottery* L_1 *over* L_2 *for probabilities near zero and the safe lottery* L_2 *over* L_1 *for probabilities near one for all sufficiently high payoffs* $z \geq z_0$, $z_0 > 0$, *then for all* $z \geq z_0$ *it holds (necessary conditions)*

$$\frac{a + bw_{cont.}(0.5)}{a+b} < \frac{v(0.5z)}{v(z)}, \tag{2.39}$$

$$1 > \frac{v(0.5z)}{v(z)}, \tag{2.40}$$

$$\lim_{p \to 0^+} \frac{w_{cont.}(0.5p)}{w_{cont.}(p)} \leq \lim_{z \to \infty} \frac{v(0.5z)}{v(z)}. \tag{2.41}$$

c) Assume that, in the situation of part b), all inequalities hold strictly and the function $f_{CRE}(p) = \frac{w(0.5p)}{w(p)} - \frac{v(0.5z)}{v(z)}$ *has, for every fixed* z, *a single intersection point* $p^* \in (0,1)$ *with the abscissa, that is, there is a single preference reversal probability* p^* *between the common ratio lotteries* L_1 *and* L_2. *Then, both Allais' common ratio effect for all fixed* *payoffs z and a finite willingness to pay for the St. Petersburg lottery emerge.*

Proof of Theorem 2.4: We start by proving a). The common ratio effect for all payoffs $z \geq z_0$ for some $z_0 > 0$ is, by Proposition 2.4, equivalent to $1 > v(0.5z)/v(z) > \frac{a+0.5b}{a+b}$ for all $z \geq z_0$. In those cases, it holds that $1 \geq \lim\limits_{z\to\infty} \frac{v(z)}{v(0.5z)} \geq \frac{a+0.5b}{a+b} > 0.5$ because $\frac{a+0.5b}{a+b} > 0.5$ for $a > 0$. From Example 2.1, then, $\lim\limits_{z\to\infty} \frac{v(z)}{v(0.5z)} > 0.5$ is equivalent to the finite willingness to pay for playing L_{STP}.

For statement b), similar arguments as for Proposition 2.4 show that the common ratio effect implies $\frac{a+bw_{cont.}(0.5)}{a+b} < v(0.5z)/v(z) < 1$ which proves Equations (2.39) and (2.40). Theorem 2.1 gives a necessary condition for convergence of the CPT value for the St. Petersburg lottery as follows:

$$\lim_{z\to\infty} \frac{v\,(z)}{v\,(0.5z)} \cdot \lim_{p\to 0^+} \left| \frac{w(0.5p) - w(0.25p)}{w(p) - w(0.5p)} \right| \leq 1. \tag{2.42}$$

After substituting $w(p) = w_{neo}(w_{cont.}(p))$ we get

$$\lim_{z\to\infty} \frac{v\,(z)}{v\,(0.5z)} \cdot \lim_{p\to 0^+} \left| \frac{w_{cont.}(0.5p) - w_{cont.}(0.25p)}{w_{cont.}(p) - w_{cont.}(0.5p)} \right| \leq 1 \tag{2.43}$$

which, given the continuity of $w_{cont.}$ and Lemma 2.1 in Appendix A.1, leads to

$$\lim_{p\to 0^+} \frac{w_{cont.}(0.5p)}{w_{cont.}(p)} \leq \lim_{z\to\infty} \frac{v(0.5z)}{v(z)}. \tag{2.44}$$

This last equation is the same as Equation (2.41).

Statement c) is clear because the first two Inequalities (2.39) and (2.40) guarantee the common ratio effect with a single preference reversal point and the previous arguments with the ratio test show that the strict version of the last Inequality (2.41) is sufficient for the solution of the St. Petersburg paradox (see also Theorem 2.1). ■

Note that although Prelec's (1998) probability weighting function itself is a prime candidate to predict the common ratio effect it is likely less successful to also predict the St. Petersburg paradox when combined with the neo-additive probability weighting function because of condition (2.41).

We provide two simple examples for a solution to both paradoxes.

Example 2.7 *a) Let the value function be given by $v(x) = x^\alpha$ with $\alpha = 0.7$ and the neo-additive probability weighting function be given by $a = 0.1$ and $b = 0.8$. Then, the certainty equivalent or, equivalently, the willingness to pay for the St. Petersburg lottery equals*

$$CE(L_{STP}) := v^{-1}\left(\sum_{k=1}^{\infty} v\left(2^k\right) \cdot \left[w\left(2^{1-k}\right) - w\left(2^{-k}\right)\right]\right) \tag{2.45}$$

$$= \left[\sum_{k=1}^{\infty} 2^{\alpha k} \cdot b \cdot 2^{-k}\right]^{\frac{1}{\alpha}} \tag{2.46}$$

$$= \left[b \cdot \sum_{k=1}^{\infty} \left(0.5^{1-\alpha}\right)^k\right]^{\frac{1}{\alpha}} \tag{2.47}$$

$$\overset{\alpha \leq 1}{=} \left[\frac{b}{2^{1-\alpha} - 1}\right]^{\frac{1}{\alpha}} = 5.892. \tag{2.48}$$

It it also clear from Theorem 2.3, part a), that the common ratio effect is predicted because $\frac{a+0.5b}{a+b} = \frac{5}{9} < \frac{v(0.5z)}{v(z)} = 0.5^\alpha = 0.616 < 1$. The preference reversal probability is given by

$$\frac{a + b \cdot 0.5p^*}{a + b \cdot p^*} \overset{!}{=} 0.5^\alpha \tag{2.49}$$

$$\Leftrightarrow \qquad p^* = \frac{a\left(1 - 0.5^\alpha\right)}{b\left(0.5^\alpha - 0.5\right)} = 0.416 \tag{2.50}$$

and is well in line with ranges proposed by Kahneman & Tversky (1979) and Starmer & Sugden (1989).

b) Let the value function be given by $v(x) = x^\alpha$, $\alpha > 0$, and the probability weighting function be given by $w(p) = w_{neo} \circ w_{cont.}(p) = w_{neo}(w_{cont.}(p))$ where the neo-additive weighting function w_{neo} is given by intercept $a > 0$ and slope $b > 0$ and the continuous weighting function $w_{cont.}(p) = p^\gamma$, $\gamma > 0$.

Then, Inequality (2.40) *is fulfilled because* $\frac{v(0.5z)}{v(z)} = 0.5^{\alpha} < 1 \; \forall \alpha > 0$*; Inequality* (2.41) *holds strictly if* $\alpha < \gamma$*. Function* $f_{CRE}(p) = \frac{w(0.5p)}{w(p)} - \frac{v(0.5z)}{v(z)}$ *is strictly decreasing in* p *since its derivative is negative for all* $p \in (0,1)$*:*

$$\frac{\partial f_{CRE}}{\partial p} = -\frac{ab\gamma \left(1-0.5^{\gamma}\right) p^{\gamma-1}}{\left(a+bp^{\gamma}\right)^2} < 0 \tag{2.51}$$

and Inequality (2.39) *then ensures a single intersection with the abscissa.*

Specifically, for $\alpha = 0.88$*,* $a = 0.1$*,* $b = 0.8$*, and* $\gamma = 2$*, Equation* (2.39) *is fulfilled because* $\frac{0.1+0.8\cdot 0.5^2}{0.1+0.8} = \frac{1}{3} < 0.5^{0.88} = 0.543$*. The preference reversal probability equals*

$$p^* = \left[\frac{a\left(1-0.5^{\alpha}\right)}{b\left(0.5^{\alpha}-0.5^{\gamma}\right)}\right]^{\frac{1}{\gamma}} = 0.441. \tag{2.52}$$

The certainty equivalent of the St. Petersburg lottery is

$$CE(L_{STP}) = \left[\sum_{k=1}^{\infty} 2^{\alpha k} \cdot b \cdot 2^{-\gamma k} \cdot \left(2^{\gamma}-1\right)\right]^{\frac{1}{\alpha}} \tag{2.53}$$

$$= \left[b(2^{\gamma}-1) \cdot \sum_{k=1}^{\infty} \left(0.5^{\gamma-\alpha}\right)^k\right]^{\frac{1}{\alpha}} \tag{2.54}$$

$$= \left[\frac{b(2^{\gamma}-1)}{2^{\gamma-\alpha}-1}\right]^{1/\alpha} = 2.255. \tag{2.55}$$

However, when preferences are given by continuous functions, no simultaneous solution to both paradoxes exists:

Theorem 2.4 (Continuity and the conflict between both paradoxes) *Assume decision makers behave according to CPT with continuous and strictly increasing value function* v *and continuous and strictly increasing probability weighting function* w*. The common ratio lotteries are given by* $L_1(p) = (\$z, 0.5p; \$0, 1-0.5p)$ *and* $L_2(p) = (\$0.5z, p; \$0, 1-p)$ *with* $z > 0$*.*

a) Assume the common ratio effect shows up for all payoffs z*, in particular for large payoffs* z *as argued by Allais (1953). Then, there* *does not exist a simultaneous solution to both the St. Petersburg paradox and the common ratio effect.*

b) *Assume* $v(x) = x^\alpha$ *with* $\alpha > 0$. *Then, there does not exist a simultaneous solution to both the St. Petersburg paradox and the common ratio effect.*

Proof of Theorem 2.4: We start by proving statement a). Any solution to both paradoxes requires that $\lim_{p \to 0^+} \frac{w(0.5p)}{w(p)} = \lim_{z \to \infty} \frac{v(0.5z)}{v(z)}$ because of Theorem 2.2, part b), and Proposition 2.1. Now, two cases can occur:

1) If $\lim_{p \to 0^+} \frac{w(0.5p)}{w(p)} = \lim_{z \to \infty} \frac{v(0.5z)}{v(z)} \in (0,1)$ then the decision maker always states infinite willingness to pay for the St. Petersburg lottery L_{STP} (see Theorem 2.2, statement c)). This is at odds with the empirical observation whereby decision makers report finite willingness to pay.

2) If $\lim_{p \to 0^+} \frac{w(0.5p)}{w(p)} = \lim_{z \to \infty} \frac{v(0.5z)}{v(z)} = 1$ then, by increasing the payoff z in lotteries L_1 and L_2, we can move the preference reversal probability p^* arbitrarily close to zero which rules out a solution to the common ratio effect. This is proven in Proposition 2.2.

Note that the case $\lim_{p \to 0^+} \frac{w(0.5p)}{w(p)} = \lim_{z \to \infty} \frac{v(0.5z)}{v(z)} = 0$ can be ruled out. To see this, observe that a risk averse choice in the common ratio effect for large probabilities ($p \to 1$) implies the necessary condition $\lim_{p \to 1} \frac{w(0.5p)}{w(p)} = w(0.5) \leq \lim_{z \to \infty} \frac{v(0.5z)}{v(z)}$. Since w is strictly increasing and $w(0) = 0$, this necessary condition can never be fulfilled when $\lim_{z \to \infty} \frac{v(0.5z)}{v(z)} = 0$.

Statement b) is a trivial corollary of case 1) because the v-ratio $\frac{v(0.5z)}{v(z)}$ equals $0.5^\alpha \in (0,1)$ independent of payoff z. ■

Admittedly, allowing payoffs z and $0.5z$ in lotteries L_1 and L_2, respectively, to grow infinitely large in Condition (CRE^*) is extreme. In special cases, such as the power value function, this is not necessary because then the v-ratio $\frac{v(0.5z)}{v(z)}$ is independent of z. Here, the conflict between both paradoxes appears directly. For other parametrizations of the value function, calibration exercises in Appendix A.2 unveil that already moderately large payoffs z run counter real world decision makers' behavior.

Note also that the conflict between finite willingness to pay in the St. Petersburg paradox and choice behavior as in the common ratio effect is not a simple artifact of the preference reversal phenomenon, reported in e.g. Lichtenstein & Slovic (1971). The preference reversal phenomenon describes the puzzling fact that while subjects choose lottery A over lottery B they simultaneously state higher certainty equivalents for lottery B than for lottery A. Accounting for preference reversal effects, however, makes the conflict between both paradoxes even more severe. To see this, note that many studies of the preference reversal phenomenon involve mean-preserving spread lotteries, similar to our L_1 and L_2. Empirically, when subjects choose between L_1 and L_2, they have a tendency to exhibit a more risk averse choice behavior going for L_2 than their willingness to pay suggests. Suppose, for given payoff z, choices indicate indifference between L_1 and L_2 for some probability p. Then, switching from a choice to a pricing task and asking for certainty equivalents would typically indicate a preference for the riskier lottery L_1 according the preference reversal effect. Tversky et al. (1990) attribute this to more extreme overweighting of small probabilities in pricing tasks. In the Tversky & Kahneman (1992) framework, for example, this would be equivalent to an even lower probability weighting parameter $\gamma_{\text{Certainty Equivalent}}$ than revealed by previous choices, that is $\gamma_{\text{Certainty Equivalent}} < \gamma_{\text{Choice}}$. But quite to the contrary, the St. Petersburg paradox, which also involves a pricing task by stating a certainty equivalent, requires a *higher* curvature parameter γ for finite willingness to pay.

Finally, truncating the St. Petersburg lottery does not help much for typical parametrizations. Appendix A.3 analyzes various specifications with truncation levels up to a payment of $\$2^{40}$, that is, roughly one trillion dollar. The essence of our previous conclusions remains unchanged.

2.3 Conclusion

It is striking that so many textbooks on decision theory start by outlining the St. Petersburg paradox and the Allais paradox when motivating EUT and CPT, respectively. However, a joint consideration of both paradoxes has, to the best of our knowledge, never been done before. Since CPT is widely accepted as the gold standard of descriptive theories of decision making under risk and uncertainty, we study a potential discrepancy between these two paradoxes within the framework of CPT. Our results can be extended to other theories of decision making under risk with similar additively separable utility across states.

The main result of our paper is that CPT with continuous preference functions is not able to simultaneously explain the two most prominent paradoxes in decision making under risk – the St. Petersburg paradox and the Allais paradox. All attempts to solve to the Allais - St. Petersburg conflict by changing the parametrizations of the CPT preference calculus within the class of continuous functions are in vain. Rather, future research shall embrace discontinuous weighting functions, such as neo-additive weighting functions (Wakker, 2010) and their obvious nonlinear extensions. In particular, exact calibrations are in order because neo-additive weighting functions can have vastly different predictions than their nonlinear extensions (consider, for example, pricing implications in capital markets, as in Barberis & Huang, 2008). Using field data, Barseghyan et al. (2013) made a first attempt in this direction, indicating a jump at probability zero and explicitly rejecting continuous weighting functions. While they do not offer much advice on probabilities larger than 25%, Harless & Camerer (1994) and Wu & Gonzalez (1996) suggest nonlinearities for moderate probabilities.

A Appendix

A.1 Lemmata

Lemma 2.1 *Suppose both limits below exist. Then, for a continuous and strictly increasing probability weighting function w, it holds*

$$\lim_{p\to 0^+} \frac{w(0.5p) - w(0.25p)}{w(p) - w(0.5p)} = \lim_{p\to 0^+} \frac{w(0.5p)}{w(p)} \ . \tag{A.1}$$

Proof of Lemma 2.1: Strict monotonicity of w ensures that $\frac{w(0.5p)}{w(p)} \in [0,1]$. Therefore, $\lim\limits_{p\to 0^+} \frac{w(0.5p)}{w(p)} \in [0,1]$. The same is true for $\frac{w(0.5p)-w(0.25p)}{w(p)-w(0.5p)}$ as the following arguments show.

Monotonicity of w implies that $\frac{w(0.5p)-w(0.25p)}{w(p)-w(0.5p)} \geq 0$. Assume, for proof by contradiction, that

$$\lim_{p\to 0^+} \frac{w(0.5p) - w(0.25p)}{w(p) - w(0.5p)} = \lambda > 1. \tag{A.2}$$

Then, we can find $p_0 \in (0,1)$ such that

$$\frac{w(0.5p) - w(0.25p)}{w(p) - w(0.5p)} > 1 \tag{A.3}$$

for all sufficiently small probabilities $p \in (0, p_0]$ and for those p we have

$$w(0.5p) - w(0.25p) > w(p) - w(0.5p). \tag{A.4}$$

Note that the last inequality also applies to probabilities $\hat{p} = 0.5p$ which leads to

$$w(0.5^2p) - w(0.5^3p) > w(0.5p) - w(0.5^2p) > w(p) - w(0.5p) \tag{A.5}$$

and by iteration

$$w(0.5^np) - w(0.5^{n+1}p) > w(p) - w(0.5p) \ \text{ for all } n = 1, 2, \ldots. \tag{A.6}$$

Then $w(p)$ can be written as a telescoping series for all probabilities $p \in (0, p_0)$ and we yield the following inequality:

$$w(0.5p) = \sum_{n=1}^{\infty} \left(w\left(0.5^n p\right) - w(0.5^{n+1} p) \right) \geq \sum_{n=1}^{\infty} \left(w(p)\right) - w(0.5p)) = \infty \tag{A.7}$$

which is a contradiction. Hence, $\frac{w(0.5p)-w(0.25p)}{w(p)-w(0.5p)} \in [0, 1]$.

We now prove Equation (A.1) by distinguishing cases. As a first case, suppose that $\lim\limits_{p \to 0^+} \frac{w(0.5p)}{w(p)} < 1$. Then

$$\begin{aligned} \lim_{p \to 0^+} \frac{w(0.5p) - w(0.25p)}{w(p) - w(0.5p)} &= \lim_{p \to 0^+} \frac{1 - w(0.25p)/w(0.5p)}{1 - w(0.5p)/w(p)} \cdot \lim_{p \to 0^+} \frac{w(0.5p)}{w(p)} \\ &= \lim_{p \to 0^+} \frac{w(0.5p)}{w(p)} \end{aligned}$$

and both limits are equal.

In particular, if $\lim\limits_{p \to 0^+} \frac{w(0.5p)-w(0.25p)}{w(p)-w(0.5p)} = 1$ then it cannot be that $\lim\limits_{p \to 0^+} \frac{w(0.5p)}{w(p)} < 1$. Hence, Equation (A.1) holds if $\lim\limits_{p \to 0^+} \frac{w(0.5p)-w(0.25p)}{w(p)-w(0.5p)} = 1$.

As the last case, suppose $\lim\limits_{p \to 0^+} \frac{w(0.5p)}{w(p)} = 1$. Assume, for proof by contradiction, that $\lim\limits_{p \to 0^+} \frac{w(0.5p)-w(0.25p)}{w(p)-w(0.5p)} < 1$. By similar arguments as above, we can find $\lambda \in (0, 1)$ such that

$$\frac{w(0.5p) - w(0.25p)}{w(p) - w(0.5p)} \leq \lambda < 1 \tag{A.8}$$

for all sufficiently small probabilities p less than some $p_0 \in (0, 1)$. By iteration and using a telescoping series, we get

$$\begin{aligned} w(0.5p) &= \sum_{n=1}^{\infty} \left(w(0.5^n p) - w(0.5^{n+1} p) \right) &\text{(A.9)} \\ &\leq \sum_{n=1}^{\infty} \lambda^n \left(w(p)\right) - w(0.5p)) = \frac{\lambda}{1-\lambda} \left(w(p)\right) - w(0.5p)) \,. &\text{(A.10)} \end{aligned}$$

This last inequality is equivalent to

$$\frac{w(0.5p)}{w(p)} \leq \lambda \tag{A.11}$$

and, given that $\lambda < 1$, this is a contradiction. Hence, in any case, Equation (A.1) holds provided both limits exist. ■

Lemma 2.2 *Let* $\lim\limits_{p\to 0^+} \frac{w(0.5p)}{w(p)} = 0.5^\gamma \in (0,1)$. *Then, for all* $\epsilon > 0$ *there exists* $p_0 \in (0,1)$ *such that*

$$w(p) \geq const \cdot \left(\frac{p}{2}\right)^{\gamma+\epsilon} \tag{A.12}$$

for all $p \in (0, p_0]$.

Proof of Lemma 2.2: Assume $\lim\limits_{p\to 0^+} \frac{w(0.5p)}{w(p)} = 0.5^\gamma \in (0,1)$. Then for any $\epsilon > 0$ it holds $0.5^\gamma \geq 0.5^{\gamma+\epsilon}$. Hence, it exists a $p_0 \in (0,1)$ such that for all $p \in (0, p_0]$

$$w\left(0.5p\right) \geq 0.5^{\gamma+\epsilon} w(p) \ . \tag{A.13}$$

Equation (A.13) holds in particular for p_0 and iterating n times yields

$$w\left(0.5^n p_0\right) \geq 0.5^{n(\gamma+\epsilon)} w(p_0) = \left(0.5^n p_0\right)^{\gamma+\epsilon} \frac{w(p_0)}{p_0^{\gamma+\epsilon}} \geq \left(0.5^n p_0\right)^{\gamma+\epsilon} \frac{w(p_0)}{p_0^{\gamma}} \ . \tag{A.14}$$

For any $p \in (0, p_0]$ we choose $n \in \{1, 2, \ldots\}$ such that

$$0.5^n p_0 < p \leq 0.5^{n-1} p_0. \tag{A.15}$$

Then, it holds

$$\begin{aligned} w(p) >& w\left(0.5^n p_0\right) && \text{(A.16)} \\ \geq& \left(0.5^n p_0\right)^{\gamma+\epsilon} \frac{w(p_0)}{p_0^{\gamma}} && \text{(A.17)} \\ =& 0.5^{\gamma+\epsilon} \left(0.5^{n-1} p_0\right)^{\gamma+\epsilon} \frac{w(p_0)}{p_0^{\gamma}} && \text{(A.18)} \\ \geq& \left(\frac{p}{2}\right)^{\gamma+\epsilon} \frac{w(p_0)}{p_0^{\gamma}} && \text{(A.19)} \\ =& \text{const} \cdot \left(\frac{p}{2}\right)^{\gamma+\epsilon} && \text{(A.20)} \end{aligned}$$

■

Lemma 2.3 *Let* $\lim\limits_{z\to \infty} \frac{v(0.5z)}{v(z)} = 0.5^\alpha \in (0,1)$. *Then, for all* $\epsilon > 0$ *there exist* $x_0 > 0$ *such that*

$$v(x) \geq const \cdot x^{\alpha-\epsilon} \tag{A.21}$$

for all $x \geq x_0$.

Proof of Lemma 2.3: Assume $\lim_{z\to\infty} \frac{v(0.5z)}{v(z)} = 0.5^{\alpha} \in (0,1)$. Then for any $\epsilon > 0$ it holds $0.5^{\alpha} \leq 0.5^{\alpha-\epsilon}$. Hence, it exists a $x_0 > 0.5$ such that for all $x \geq x_0$

$$v(0.5x) \leq 0.5^{\alpha-\epsilon} v(x). \tag{A.22}$$

Equation (A.22) holds in particular for x_0 and iterating n times yields

$$v(2^n x_0) \geq 2^{n(\alpha-\epsilon)} v(x_0). \tag{A.23}$$

For any $x \geq x_0$, we choose $n \in \{0, 1, 2, \ldots\}$ such that

$$2^n x_0 \leq x < 2^{n+1} x_0. \tag{A.24}$$

Then, it holds

$$\begin{aligned} v(x) \geq & v\left(2^n x_0\right) && \text{(A.25)} \\ \geq & 2^{n(\alpha-\epsilon)} v(x_0) && \text{(A.26)} \\ = & \left(2^{n+1} x_0\right)^{\alpha-\epsilon} \frac{v(x_0)}{(2x_0)^{\alpha-\epsilon}} && \text{(A.27)} \\ \geq & x^{\alpha-\epsilon} \frac{v(x_0)}{(2x_0)^{\alpha}} && \text{(A.28)} \\ = & \text{const} \cdot x^{\alpha-\epsilon} && \text{(A.29)} \end{aligned}$$

■

A.2 Sensitivity analysis for the common ratio effect with small payoffs

From our analyses in the main part of the paper it is clear that potentially interesting cases require small or only moderately large payoffs z in lotteries L_1 and L_2 and value functions with $\lim_{z\to\infty} \frac{v(0.5z)}{v(z)} = 1$. This restriction excludes Tversky & Kahneman's (1992) suggested power value function $v(x) = x^{\alpha}$ and Bell's (1988) one-switch function. Linear and quadratic utility functions are clearly unable to explain both paradoxes. Table A.1 lists the remaining

Table A.1: Functional forms for the value function v in gains.

Type	Function $v(x)$	Parameter restriction	Bounded utility	$\frac{v(0.5z)}{v(z)}$	$\lim_{z\to\infty}\frac{v(0.5z)}{v(z)}$
Exponential	$1-e^{-\alpha x}$	$\alpha > 0$	yes	$\frac{1-e^{-\alpha 0.5z}}{1-e^{-\alpha z}}$	1
Logarithmic	$\log(1+\alpha x)$	$\alpha > 0$	no	$\frac{\log(1+\alpha 0.5z)}{\log(1+\alpha z)}$	1
HARA	$\frac{1-\alpha}{\alpha}\left[\left(\frac{x}{1-\alpha}+\beta\right)^{\alpha}-\beta^{\alpha}\right]$	$\alpha < 0$, $\beta > 0$	yes	$\frac{[0.5z/(1-\alpha)+\beta]^{\alpha}-\beta^{\alpha}}{[z/(1-\alpha)+\beta]^{\alpha}-\beta^{\alpha}}$	1

typical forms of promising value functions. For these ones, we perform a sensitivity analysis to gauge the set of payoff amounts z for which the common ratio effect can be predicted.

Exponential value function

A promising alternative parametrization is the use of the exponential value function $v(x) = 1 - e^{-\alpha x}$ with $\alpha > 0$. This value function is bounded and, thus, automatically ensures finite willingness to pay for the St. Petersburg lottery (Theorem 2.2, a)). It holds that $\lim_{z\to\infty}\frac{v(0.5z)}{v(z)} = 1$. To simultaneously predict the common ratio effect, a probability weighting function with $\lim_{p\to 0^+}\frac{w(0.5p)}{w(p)} = 1$ is presumably most promising. A prime candidate would be the commonly used weighting function $w_{Prelec}(p) = e^{-(-\log p)^{\gamma}}$ with $\gamma \in (0,1)$ as proposed by Prelec (1998).

Recall that, by Definition 2.2 of the common ratio effect, the decision maker behaves risk seeking for all probabilities $p \in (0, p^*)$ and risk averse for all $p \in (p^*, 1]$. Technically, we solve the following equation for any pair

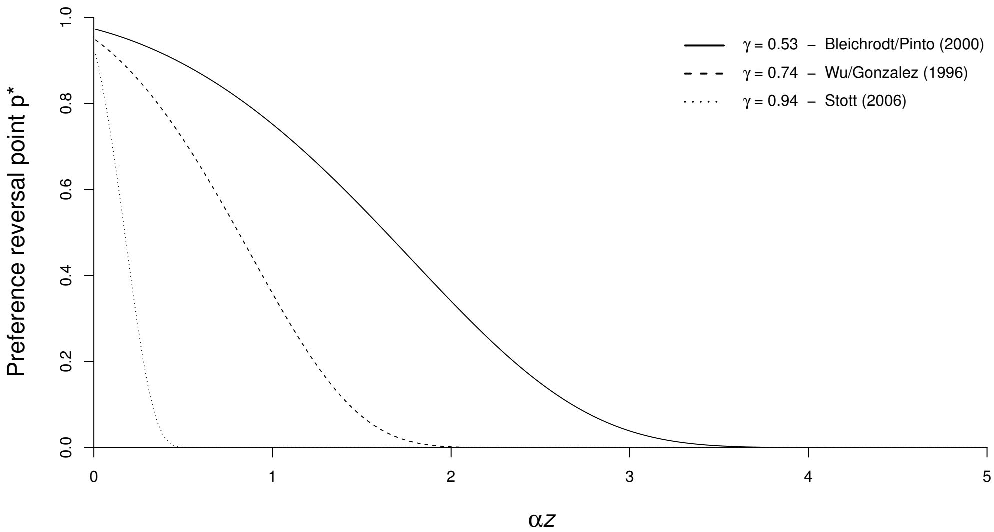


Figure 2.2: Preference reversal points for the exponential value and Prelec weighting function combination.

Notes: This figure depicts preference reversal points p^* under CPT by solving $w(0.5p^*)/w(p^*) - v(0.5z)/v(z) = 0$ for the pairs $(p^*, \alpha z)$. The decision maker has the choice between the equal mean lotteries $L_1(p) = (\$z, 0.5p; \$0, 1 - 0.5p)$ and $L_2(p) = (\$0.5z, p; \$0, 1 - p)$. The individual's preferences are given by the bounded value function $v(x) = 1 - e^{-\alpha x}$, $\alpha > 0$ and Prelec's (1998) probability weighting function $w(p) = e^{-(-\log p)^\gamma}$, $\gamma \in (0, 1)$. The three lines indicate individual's preference reversal points for $\gamma = \{0.53, 0.74, 0.94\}$. The use of these three parameter estimates is motivated by Bleichrodt & Pinto (2000), Wu & Gonzalez (1996), and Stott (2006), respectively.

$(p^*, \alpha z)$:

$$CPT(L_1) \overset{!}{=} CPT(L_2)$$

$$\Leftrightarrow \quad \frac{w(0.5p^*)}{w(p^*)} = \frac{v(0.5z)}{v(z)} \tag{A.30}$$

$$\Leftrightarrow \quad \frac{e^{-(-\log(0.5p^*))^\gamma}}{e^{-(-\log p^*)^\gamma}} = \frac{1 - e^{-0.5\alpha z}}{1 - e^{-\alpha z}} \tag{A.31}$$

Equation (A.31) determines the decision maker's indifference point between the lotteries L_1 and L_2 in (2.27) for given constant absolute risk aversion coefficient α and payoff z.

Figure 2.2 depicts the preference reversal point p^* as a function of αz for the three weighting function parameter estimates $\gamma = \{0.53, 0.74, 0.94\}$

reported in Bleichrodt & Pinto (2000), Wu & Gonzalez (1996), and Stott (2006), respectively. For all three specifications, p^* quickly converges to zero when αz increases. Problems 7 and 8 in Kahneman & Tversky (1979) conservatively suggest that, empirically, $0.002 < p^* < 0.9$. Thus, even the lowest estimate $\gamma = 0.53$ of Bleichrodt & Pinto (2000) requires $\alpha z < 3.617$ to ensure the lower boundary of the empirically observed reversal point $p^* > 0.002$. The typical payoff $z = \$6000$ leads to $\alpha < 3.617/6000 \approx 0.000603$ which is an unreasonably low constant absolute risk aversion. To the contrary, De Giorgi & Hens (2006) suggest for the exponential value function the parameter value $\alpha \approx 0.2$. The typical $z = \$6000$ implies $\alpha z = 1,200$ which results in a v-ratio that is just infinitesimally smaller than one. Hence, only when γ tends to zero – quite at odds with all calibration studies – the w-ratio equals one and the common ratio effect emerges theoretically. An alternative interpretation is that CPT counterfactually predicts a preference for risky lottery $L_1(p = 0.002)$ only for $z < \$18.08$. This is at odds with experiments that show *less* frequent Allais type behavior for low payoffs (Camerer, 1989; Conlisk, 1989; Fan, 2002; Huck & Müller, 2012; Agranov & Ortoleva, 2017). We conclude that the combination of the exponential value function and Prelec's (1998) subproportional weighting function is no viable solution to both paradoxes.

Recall that for probability weighting functions in, e.g., Tversky & Kahneman (1992) and Goldstein & Einhorn (1987), it holds $\lim\limits_{p \to 0^+} \frac{w(0.5p)}{w(p)} = 0.5^\gamma$. Because $\lim_{\gamma \to 0} 0.5^\gamma = 1$, a sufficiently low γ and sufficiently low payoff z, can theoretically explain both paradoxes in similar cases with low αz combinations. Table A.2 provides a sensitivity analysis. It depicts for exponential, logarithmic and HARA value functions and various (α, z) combinations the maximum curvature parameter γ of the probability weighting function that still predicts the common ratio effect. Evidently, with the exponential value functions, payoffs of the order of $z = 6.000$ are

Table A.2: Numerical upper boundaries for the curvature parameter γ of the Tversky & Kahneman (1992) or Goldstein & Einhorn (1987) weighting function for different values of α and z such that the common ratio effect is predicted.

			Payoff z				
Type v	Upper boundary for γ		\$100	\$1,000	\$6000	\$20,000	\$100,000
Exponential	$\frac{\log\left(\frac{1-e^{-\alpha 0.5z}}{1-e^{-\alpha z}}\right)}{\log(0.5)}$	$\alpha = 0.001$	0.96	0.68	0.07	0	0
		$\alpha = 0.01$	0.68	0.01	0	0	0
		$\alpha = 0.1$	0.01	0	0	0	0
Logarithmic	$\frac{\log\left(\frac{\log(1+\alpha 0.5z)}{\log(1+\alpha z)}\right)}{\log(0.5)}$	$\alpha = 0.5$	0.27	0.17	0.13	0.11	0.10
		$\alpha = 1$	0.23	0.15	0.12	0.10	0.09
		$\alpha = 2$	0.2	0.14	0.11	0.10	0.08
HARA ($\beta = 1$)	$\frac{\log\left(\frac{[0.5z/(1-\alpha)+\beta]^{\alpha}-\beta^{\alpha}}{[z/(1-\alpha)+\beta]^{\alpha}-\beta^{\alpha}}\right)}{\log(0.5)}$	$\alpha = -0.25$	0.14	0.06	0.04	0.03	0.02
		$\alpha = -0.5$	0.08	0.02	0.01	0.01	0
		$\alpha = -1$	0.03	0	0	0	0

sufficient to wipe out the common ratio effect unless unrealistic preference parameters are assumed. For example, with $z = 6.000$ and an extremely low $\alpha = 0.001$, the curvature parameter γ cannot exceed 0.07. Larger α values are even more problematic.

Logarithmic value function

Next, we consider a logarithmic value function $v(x) = \log(1 + \alpha x)$ with $\alpha > 0$. This value function exhibits sufficiently high risk aversion such that the CPT decision maker reports finite willingness to pay for the St. Petersburg lottery for typical probability weighting functions w_{TK92}, $w_{log-odds}$, or w_{Prelec}. While the former two cases are a simple application of Theorem 2.2, finite willingness to pay in the latter case (w_{Prelec}) deserves special attention in Example 2.8 below because $\lim\limits_{p\to 0^+} \frac{w(0.5p)}{w(p)} = \lim\limits_{z\to\infty} \frac{v(0.5z)}{v(z)} = 1$.

Example 2.8 *Assume the value function is given by* $v(x) = \log(1 + \alpha x)$ *with* $\alpha > 0$ *and the probability weighting function is given by* $w_{Prelec}(p) = e^{-(-\log p)^\gamma}$ *proposed by Prelec (1998). Then* $\lim\limits_{p \to 0^+} \frac{w(0.5p)}{w(p)} = \lim\limits_{z \to \infty} \frac{v(0.5z)}{v(z)} = 1$ *and the CPT decision maker states finite willingness to pay for* L_{STP}.

Proof of Example 2.8: Observe that

$$w_{Prelec}\left(2^{1-k}\right) = e^{-((k-1)\log(2))^\gamma} < e^{-(k-1)^\gamma \log(2)} = 2^{-(k-1)^\gamma}. \tag{A.32}$$

For any given $\alpha > 0$, it exists a $\bar{k} \in \mathbb{N}_{\geq 1}$ such that for all $k \in \mathbb{N}_{\geq \bar{k}}$

$$\log\left(1 + \alpha 2^k\right) < \log\left(\alpha 2^{k+1}\right) = \log(2\alpha) + k\log(2). \tag{A.33}$$

Then, the CPT value in Equation (2.1) is finite. As a first step we have

$$\begin{aligned}
CPT(L_{STP}) &= \sum_{k=1}^{\infty} \log\left(1 + \alpha 2^k\right) \cdot \left[w_{Prelec}\left(2^{1-k}\right) - w_{Prelec}\left(2^{-k}\right)\right] && \text{(A.34)}\\
&< \sum_{k=1}^{\infty} \log\left(1 + \alpha 2^k\right) \cdot w_{Prelec}\left(2^{1-k}\right) && \text{(A.35)}\\
&= \underbrace{\sum_{k=1}^{\bar{k}-1} \log\left(1 + \alpha 2^k\right) \cdot w_{Prelec}\left(2^{1-k}\right)}_{=c<\infty} + \sum_{k=\bar{k}}^{\infty} \log\left(1 + \alpha 2^k\right) \cdot w_{Prelec}\left(2^{1-k}\right) && \text{(A.36)}\\
&< c + \sum_{k=\bar{k}}^{\infty} [\log(2\alpha) + k\log(2)] \cdot 2^{-(k-1)^\gamma} && \text{(A.37)}\\
&= c + \sum_{k=\bar{k}-1}^{\infty} [\log(2\alpha) + (k+1)\log(2)] \cdot 2^{-k^\gamma} && \text{(A.38)}\\
&= c + \log(4\alpha) \sum_{k=\bar{k}-1}^{\infty} 2^{-k^\gamma} + \log(2) \sum_{k=\bar{k}-1}^{\infty} k 2^{-k^\gamma}. && \text{(A.39)}
\end{aligned}$$

The first term in (A.39) is a constant and the series $\sum_{k=\bar{k}-1}^{\infty} 2^{-k^\gamma}$ is strictly smaller than the series $\sum_{k=\bar{k}-1}^{\infty} k2^{-k^\gamma}$. To assess convergence, it suffices to prove convergence of $\sum_{k=0}^{\infty} k2^{-k^\gamma}$. The integral test, using the substitution

$x = k^\gamma \cdot \log(2)$ with $\frac{dk}{dx} = \frac{1}{\gamma}\left(\frac{1}{\log(2)}\right)^{\frac{1}{\gamma}} x^{\frac{1}{\gamma}-1}$, shows convergence because

$$\int_0^\infty k2^{-k^\gamma} dk = \int_0^\infty ke^{-k^\gamma \cdot \log(2)} dk \tag{A.40}$$

$$= \frac{1}{\gamma}\log(2)^{-\frac{2}{\gamma}} \int_0^\infty x^{\frac{2}{\gamma}-1} e^{-x} dx \tag{A.41}$$

$$= \frac{1}{\gamma}\log(2)^{-\frac{2}{\gamma}} \cdot \Gamma\left(\frac{2}{\gamma}\right) < \infty, \tag{A.42}$$

where Γ is the well known Gamma function. ■

Further, observe that w_{Prelec} is subproportional and $\gamma \in (0,1)$ is a necessary condition for the common ratio effect. The following function, corresponding to function (2.30) depicted in Figure 2.1,

$$f_{CRE*}^{w_{Prelec},v}(p) = \frac{w(0.5p)}{w(p)} - \lim_{z\to\infty} \frac{v(0.5z)}{v(z)} = \frac{w(0.5p)}{w(p)} - 1, \tag{A.43}$$

is, thus, strictly monotonically decreasing in p and $\lim_{p\to 0^+} f_{CRE*}^{w_{Prelec},v}(p) = 0$. In other words, there is no preference reversal point p^* and, thus, no common ratio effect.

A sensitivity analysis is worthwhile because there might be a realistically large set of moderate payoffs z for which condition (CRE) is satisfied. Solving Equation (A.30) with w_{Prelec}, v_{Log}, and $z = 6000$ gives preference reversal points of virtually zero. The value is roughly $p^* \approx 7.32 \times 10^{-11}$ for $\gamma = 0.53$ reported in Bleichrodt & Pinto (2000). For larger γ, as reported in e.g. Wu & Gonzalez (1996) and Stott (2006), the preference reversal point is even closer to zero.

Figure 2.3 depicts the preference reversal probability p^* dependent on the payoff z in lotteries L_1 and L_2. It unveils that p^* quickly converges to zero for moderately large payoffs. Convergence is faster for higher curvature parameters γ. For $z = 400$, the Bleichrodt & Pinto (2000) estimate $\gamma = 0.53$ is not distinguishable from zero. In fact, this moderate case implies $p^* \approx 4.5 \times 10^{-5}$.

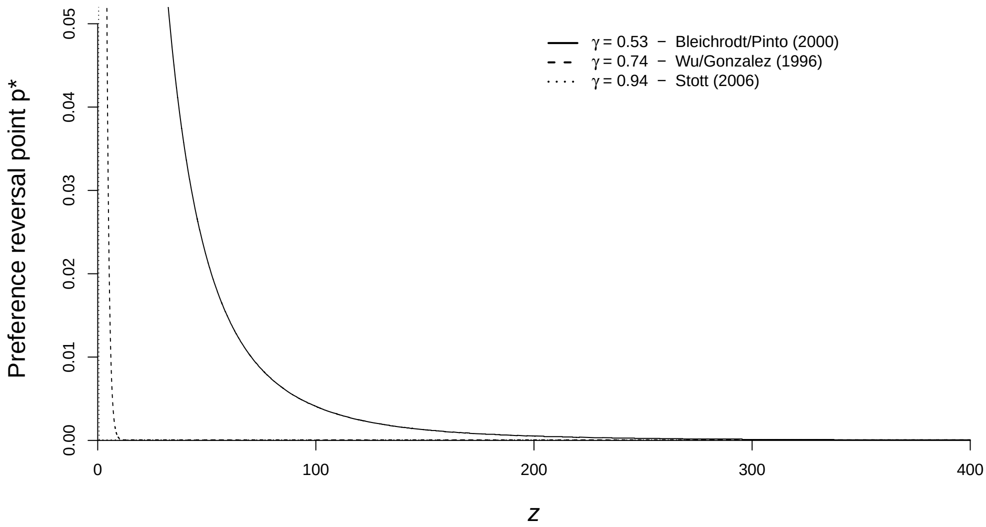


Figure 2.3: Preference reversal points for the logarithmic value and Prelec weighting function combination.

Notes: This figure depicts preference reversal points p^* under CPT by solving $w(0.5p^*)/w(p^*) - v(0.5z)/v(z) = 0$ for the pairs (p^*, z). The decision maker chooses between the equal mean lotteries $L_1(p) = (\$z, 0.5p; \$0, 1 - 0.5p)$ and $L_2(p) = (\$0.5z, p; \$0, 1 - p)$. The individual's preferences are given by the bounded value function $v(x) = \log(1+x)$ and Prelec's (1998) probability weighting function $w(p) = e^{-(-\log p)^{\gamma}}$, $\gamma \in (0, 1)$. The three lines indicate individual's preference reversal points for $\gamma = \{0.53, 0.74, 0.94\}$. The use of these three parameter estimates is motivated by Bleichrodt & Pinto (2000), Wu & Gonzalez (1996), and Stott (2006), respectively.

Table A.2 kills any hope for w_{TK92} and $w_{log-odds}$. For example, using the typical amount $z = 6000$, condition (CRE) implies $\gamma < 0.12$ which is unrealistically low. Further, Camerer & Ho (1994), Rieger & Wang (2006), and Ingersoll (2008) show that for $\gamma \leq 0.28$, w_{TK92} is not monotonically increasing.

HARA value function

Our last candidate value function is the HARA value function $v_{HARA}(x) = \frac{1-\alpha}{\alpha}\left(\left(\frac{x}{1-\alpha} + \beta\right)^{\alpha} - \beta^{\alpha}\right)$ with $\alpha < 0$ and $\beta > 0$. As the function is bounded from above it automatically ensures finite willingness to pay in the

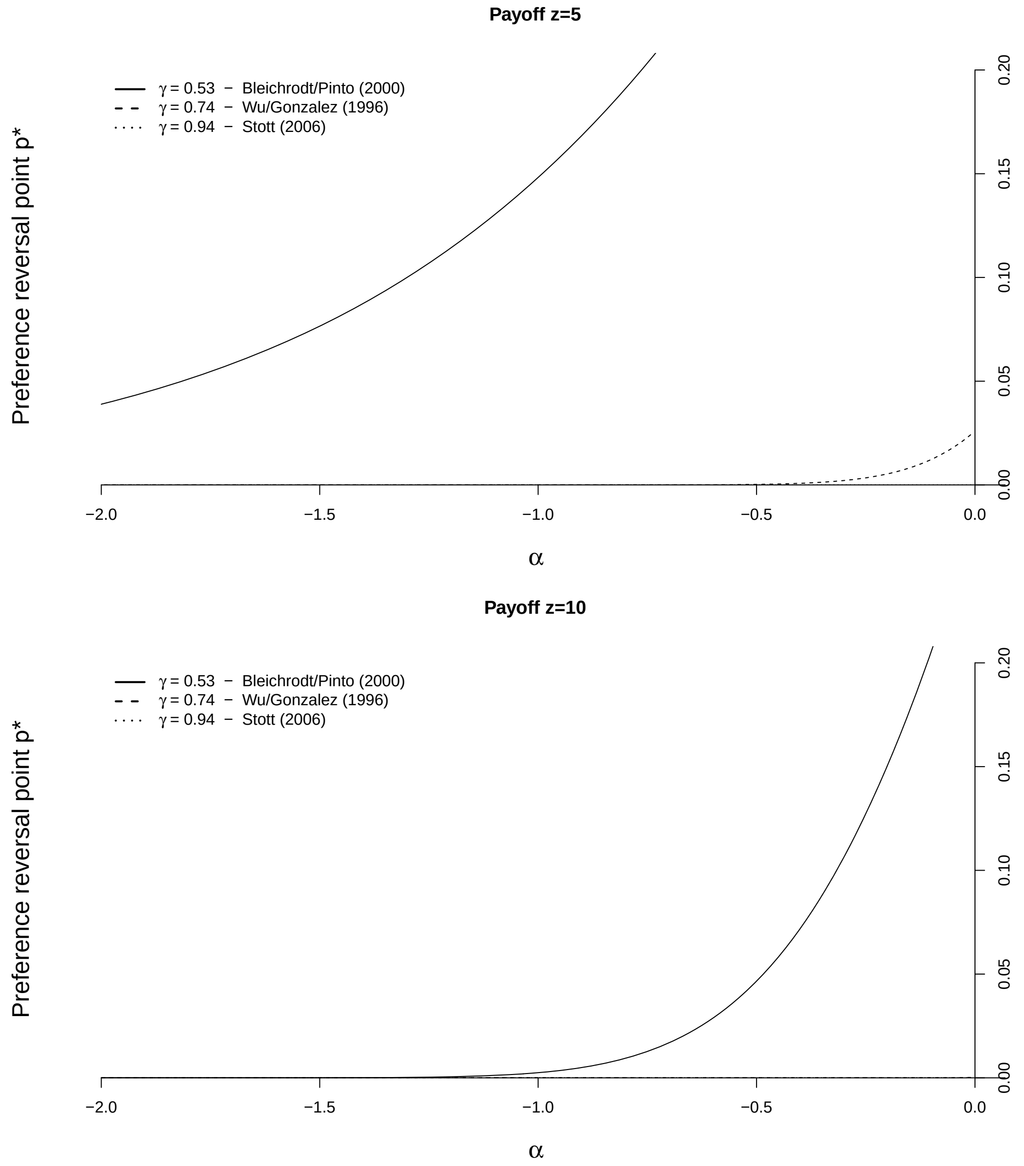


Figure 2.4: Preference reversal points for the HARA value and Prelec weighting function combination.

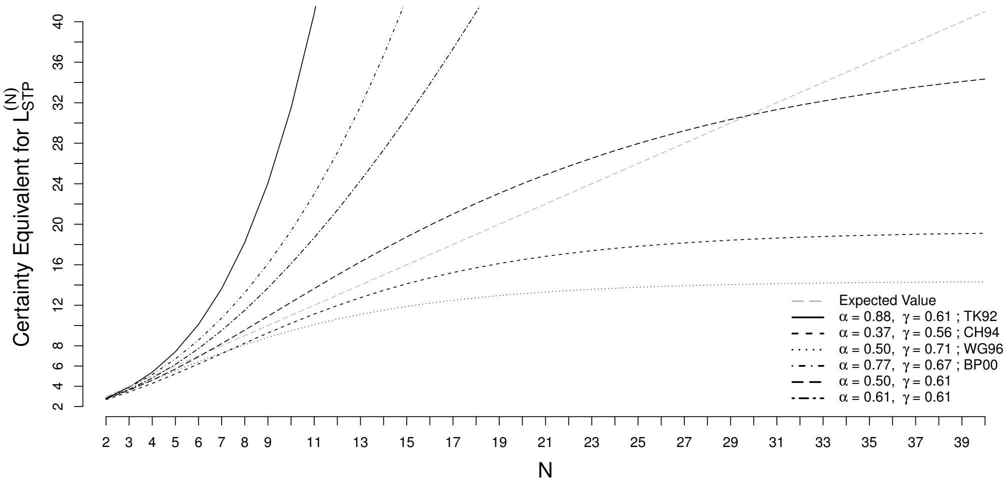
Notes: This figure depicts preference reversal points p^* under CPT on the ordinate and risk aversion parameter α on the abscissa by solving $w(0.5p^*)/w(p^*) - v(0.5z)/v(z) = 0$ for the pairs (p^*, α). The individual's preferences are given by the bounded value function $v(x) = \frac{1-\alpha}{\alpha}\left(\left(\frac{x}{1-\alpha} + \beta\right)^{\alpha} - \beta^{\alpha}\right)$ with $\alpha < 0$ and normalized $\beta = 1$ and Prelec's (1998) probability weighting function $w(p) = e^{-(-\log p)^{\gamma}}$, $\gamma \in (0, 1)$. The three lines indicate individual's preference reversal points for $\gamma = \{0.53, 0.74, 0.94\}$. The use of these three parameter estimates is motivated by Bleichrodt & Pinto (2000), Wu & Gonzalez (1996), and Stott (2006), respectively. The decision maker chooses between the equal mean lotteries $L_1(p) = (\$z, 0.5p; \$0, 1 - 0.5p)$ and $L_2(p) = (\$0.5z, p; \$0, 1 - p)$. The upper panel depicts the risk aversion parameter α if we fix $z = 5$ and the lower panel shows α for $z = 10$.

St. Petersburg paradox. However, Table A.2 unveils even more unrealistic calibrations for the curvature parameter γ for the Tversky & Kahneman (1992) or Goldstein & Einhorn (1987) probability weighting functions. For example, if we use the typical payoff $z = 6000$ and value function parameters $\alpha = -1$ and $\beta = 1$ the largest plausible parameter would be $\gamma = 0.00048$ which is neither supported by empirical calibrations nor supported by – in the Tversky & Kahneman (1992) case – our stochastic dominance assumption of a strictly increasing w (see, e.g., Camerer & Ho (1994), Rieger & Wang (2006), or Ingersoll (2008)).

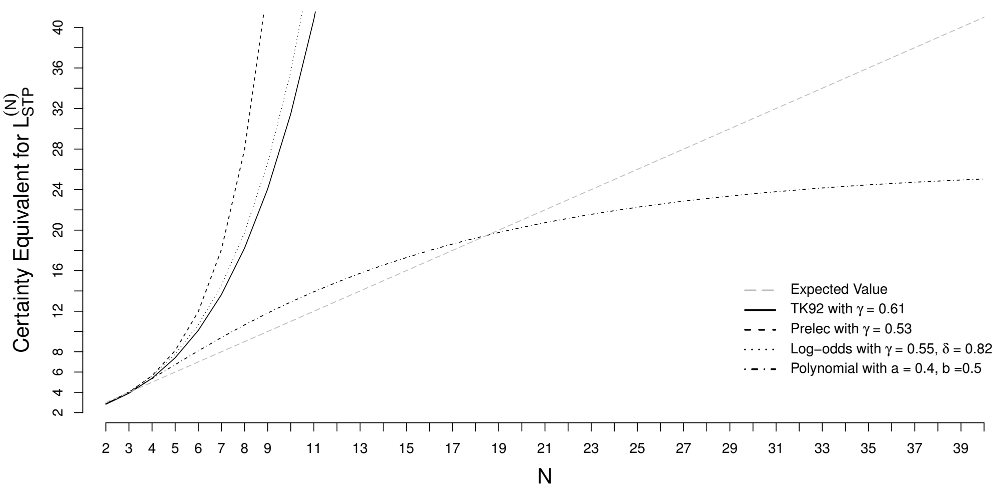
Figure 2.4 provides the results of a similar analysis with Prelec's (1998) probability weighting function w_{Prelec}. It depicts preference reversal points p^* on the ordinate and risk aversion parameter α on the abscissa by solving $\frac{w(0.5p^*)}{w(p^*)} - \frac{v(0.5z)}{v(z)} = 0$ for the pairs (p^*, α). Since the convergence of the preference reversal probability p^* to zero is so fast we depict the special case $z = 5$ and $z = 10$ in the upper and lower panel, respectively. As before, we use $\gamma = \{0.53, 0.74, 0.94\}$. For $\gamma = 0.94$ convergence is too quick to visualize it in one of the graphs. These effects are considerably stronger for larger values z. We conclude that using the HARA value function does not yield practical solutions to both paradoxes.

A.3 Truncated St. Petersburg lotteries

As our testing ground, we propose the St. Petersburg paradox and the Allais paradox because of their outstanding prominence and importance for the development of new theories of decision making under risk throughout the history of risky decision making. Some readers might feel tempted to change this playing field. Especially the infinite expected payoff of the St. Petersburg lottery sometimes spurs criticism. We consider this to be scientific foul play. True, willingness to pay for the original St. Petersburg



Panel A: $CE\left(L_{STP}^{(N)}\right)$ for different parameter sets (α, γ) of Tversky & Kahneman's (1992) parametrization.



Panel B: $CE\left(L_{STP}^{(N)}\right)$ for various probability weighting functions.

Figure 2.5: Certainty equivalents for the truncated St. Petersburg gamble under CPT.

Notes: This figure illustrates the willingness to pay $CE\left(L_{STP}^{(N)}\right) = v^{-1}\left(CPT\left(L_{STP}^{(N)}\right)\right)$ for the truncated St. Petersburg lottery where $N = 2, \ldots, 40$ determines the maximum payoff $\$2^N$. The truncated gamble promises a payoff of $\$2^k$ with probability 0.5^k for $k = 1, \ldots, N-1$ and a payoff of $\$2^N$ with probability 0.5^{N-1}. In Panel A, the individual's preferences are given by the value function $v(x) = x^\alpha$ and the probability weighting function $w_{TK92}(p) = p^\gamma / (p^\gamma + (1-p)^\gamma)^{1/\gamma}$ for different parameter sets (α, γ) including those estimated in Tversky & Kahneman (1992), Camerer & Ho (1994), Wu & Gonzalez (1996), and Bleichrodt & Pinto (2000) denoted in the legend by TK92, CH94, WG96, and BP00, respectively. In Panel B, the value function is consistently $v(x) = x^{0.88}$ (Tversky & Kahneman, 1992) and the weighting function takes the forms w_{TK92}, $w_{log-odds}(p) = \delta p^\gamma / (\delta p^\gamma + (1-p)^\gamma)$, $w_{Prelec}(p) = e^{-(-\log p)^\gamma}$, and $w_{RW06}(p) = \frac{3-3b}{a^2-a+1}\left(p^3 - (a+1)p^2 + ap\right) + p$. Certainty equivalents CE_{STP} are depicted for weighting functions with parameter estimates of Tversky & Kahneman (1992) for w_{TK92}, estimates of Bleichrodt & Pinto (2000) for w_{Prelec} and $w_{log-odds}$, and parameter values motivated by Rieger & Wang (2006) for w_{RW06}, respectively. The gray dashed line represents the expected value of the lottery.

lottery L_{STP} is difficult to elicit with monetary incentives. Nevertheless, stated willingness to pay in hypothetical scenarios is reliable, though noisy and real incentives would not change our conclusions (Holt & Laury, 2002, 2005).[9]

A typical concern is that subjects do not trust promises to payout in the St. Petersburg lottery above a certain threshold (Tversky & Bar-Hillel, 1983). Resorting to truncated versions of the St. Petersburg lottery, however, is not much more than grasping at straws, as we shall see. Let $L_{STP}^{(N)}$ denote the truncated St. Petersburg lottery which yields a payoff \$$2^k$ with probability 0.5^k for $k = 1, \ldots, N-1$ and a payoff of \$$2^N$ with probability 0.5^{N-1}. The expected value of this lottery equals $N+1$ and corresponds to $N-1$ possible rounds of coin flipping.[10]

By all indications, subjects behave risk averse in the original as well as the truncated St. Petersburg lottery $L_{STP}^{(N)}$ (Bernoulli, 1738, 1954; Bottom et al., 1989; Rivero et al., 1990; Baron, 2008; Hayden & Platt, 2009; Neugebauer, 2010; Cox et al., 2011; Seidl, 2013; Erev et al., 2017; Cox et al., 2019). Specifically, Hayden & Platt (2009) show in an experimental study with real monetary payments that individuals' willingness to pay is hardly affected by truncating the lottery. They find that bids on truncated St. Petersburg lotteries are typically smaller than twice the smallest payoff, that is \$4 for $L_{STP}^{(N)}$. Cox et al. (2011) show that the majority of their subjects behave risk averse in the finite St. Petersburg gamble especially for $N \geq 6$. For $N = 9$, 83% of their participants rejected the gamble for a price of \$8.75, thus indicating risk averse behavior. The proportion of subjects rejecting the gamble for a price slightly lower than the expected

[9] Note that infinite willingness to pay can also emerge from finite expected payoff gambles (Rieger & Wang, 2006).

[10] For example, for $N = \{10, 20, 30, 40\}$ the maximum payoffs are \$$2^{10}$, \$$2^{20}$, \$$2^{30}$, and \$$2^{40}$ which roughly equal 1 thousand, 1 million, 1 billion, and 1 trillion dollars, respectively.

value increased monotonically with N. Moreover, Erev et al. (2017) show that risk aversion in the St. Petersburg paradox is also robust to feedback (with real monetary incentives).

In the CPT framework, Figure 2.5 depicts the willingness to pay $CE\left(L_{STP}^{(N)}\right) = v^{-1}\left(CPT\left(L_{STP}^{(N)}\right)\right)$ as a function of N for the truncated St. Petersburg gamble, where $N = 2, \ldots, 40$ determines the maximum payoff $\$2^N$, for the most common parametrizations. Panel A shows the certainty equivalents $CE\left(L_{STP}^{(N)}\right)$ for Tversky & Kahneman's (1992) parametrization of the value and probability weighting function, that is $v_{Power}(x) = x^\alpha$ and $w_{TK92}(p) = p^\gamma / \left(p^\gamma + (1-p)^\gamma\right)^{1/\gamma}$, for different parameter sets (α, γ) as estimated in Tversky & Kahneman (1992), Camerer & Ho (1994), Wu & Gonzalez (1996), and Bleichrodt & Pinto (2000). Parameter combinations with $\alpha \geq \gamma$ predict a certainty equivalent that increases exponentially with N and illustrate once more infinite willingness to pay for $N \to \infty$ (see Theorem 2.2 or Example 2.3). The empirical evidence on truncated St. Petersburg lotteries clearly rejects such parameter combinations.[11] Surprisingly, the hypothetical parameter combination $(\alpha, \gamma) = (0.50, 0.61)$ implies risk proclivity for $N = 7, \ldots, 29$, that is higher certainty equivalents than expected payoff (gray dashed line) although this preference combination yields finite willingness to pay for $N \to \infty$. Predicted and actual willingness to pay can deviate by substantial amounts, though both are finite. One interpretation is that our formerly applied criterion of *finite* willingness to pay is rather conservative if benchmarked against actual willingness to pay.

We derive similar conclusion from Panel B which fixes the value function as $v(x) = x^{0.88}$ and uses probability weighting functions $w_{TK92}(p) =$

[11] The empirical evidence mentioned above overwhelmingly supports risk averse behavior in truncated St. Petersburg lotteries and risk proclivity is merely a thought experiment (Tversky & Bar-Hillel, 1983) or an artificially induced observation where individuals were framed to a risk-seeking choice (Erev et al., 2008).

$p^\gamma / (p^\gamma + (1-p)^\gamma)^{1/\gamma}$ with $\gamma = 0.61$, $w_{log-odds}(p) = \delta p^\gamma / (\delta p^\gamma + (1-p)^\gamma)$ with $\gamma = 0.55$ and $\delta = 0.82$, $w_{Prelec}(p) = e^{-(-\log p)^\gamma}$ with $\gamma = 0.53$, and $w_{RW06}(p) = \frac{3-3b}{a^2-a+1}(p^3 - (a+1)p^2 + ap) + p$ with $a = 0.4$ and $b = 0.5$. The parameter values are motivated by Tversky & Kahneman (1992) for w_{TK92}, Bleichrodt & Pinto (2000) for w_{Prelec} and $w_{log-odds}$, and Rieger & Wang (2006) for the cubic weighting function w_{RW06}.

Just like with w_{TK92}, we see that the standard parametrization of the two weighting functions w_{Prelec} and $w_{log-odds}$ also predict unreasonably high willingness to pay for finite values of N. Interesting is the case of the polynomial weighting function w_{RW06} which implies a willingness to pay of \$26.18 for $N \to \infty$. We yield risk seeking behavior for $N = 3, \ldots, 18$ which correspond to maximum payoffs of $\$8; \$16; \ldots; \$262,144$. This prediction does not match the empirical fact mentioned above.

In summary, truncating the original St. Petersburg lottery does not change the essence of our previous conclusions.

Chapter 3

Salience Theory and the Allais - St. Petersburg Conflict*

3.1 Introduction

Descriptive theories of choice under risk aim to rationalize human choice behavior. But what characterizes a "good" descriptive model? Obviously, the model has to accommodate as many empirically observed choice patterns as possible while being manageable and falsifiable. George Box once remarked that "all models are wrong, but some are useful."(Box, 1979, p. 202) "The practical question is how wrong do they have to be to not be useful."(Box & Draper, 1987, p. 74) In the context of descriptive decision theories, his quotes imply that the first step is to define what the minimum standard for descriptive models is in order to be able to reject a model. Dierkes & Sejdiu (2019b) recently propose the joint consideration of the Allais paradox (Allais, 1953) and Bernoulli's St. Petersburg paradox (Bernoulli, 1738, 1954) as the new minimum standard of descriptive decision

*This chapter is based on the Working Paper "Salience Theory and the Allais - St. Petersburg Conflict" authored by Maik Dierkes and Vulnet Sejdiu, 2019.

theories. They motivate this minimum standard with the outstanding historical importance of both paradoxes and the failure of Tversky & Kahneman's (1992) Cumulative Prospect Theory to accommodate both paradoxes simultaneously.

In this paper, we analyze the predictions of Bordalo et al.'s (2012) Salience Theory in the St. Petersburg paradox and the common ratio version of the Allais' paradox. Salience Theory is a recently developed promising context-depended descriptive theory for choice under risk. Successful applications vary from asset pricing over consumer choice to judicial decisions (see, e.g., Bordalo et al., 2013a,b, 2015). The theory models a Local Thinker whose preferences are determined by a value function v, a salience function σ, and a probability-distortion parameter $\delta \in (0, 1)$. The Local Thinker re-weights probabilities for states dependent on the salience of the respective outcomes in these states. The probability-distortion parameter δ drives the re-weighting of probabilities. $\delta = 1$ corresponds to the case where there is no re-weighting at all and the Local Thinker's risk attitude is solely driven by the value function v. A lower value δ reflects more extreme re-weighting. The exact re-weighting of probabilities also depends on payoff's relative salience across states. Throughout this paper, we follow Bordalo et al. (2012) and focus on the nonparametric ranking-based probability distortion. Specifically, probabilities get distorted by δ^k where k denotes the salience rank given by the general salience function σ. In particular, we only assume that the value function v is strictly monotonically increasing with $v(0) = 0$ and that the salience function σ satisfies the conditions *ordering* and *diminishing sensitivity* (see Bordalo et al., 2012, Definition 1).

Our main result is that Salience Theory can resolve the Allais - St. Petersburg conflict with one set of parameters if we truncate the St. Petersburg lottery – which avoids offering lotteries with infinite expected payoff. Here, Salience Theory has an edge over most Cumulative Prospect Theory calibrations and, of course, Expected Utility Theory. Dierkes & Sejdiu (2019b) prove that, for Cumulative Prospect Theory with continuous preference functions, any solution to Allais' common ratio effect predicts – at odds with experimental evidence – huge willingness to pay for the St. Petersburg lottery, easily exceeding the expected payoff of the truncated St. Petersburg lottery. In contrast, Salience Theory with an exponential value function $v(x) = 1 - e^{-x}$ and a probability-distortion parameter $\delta = 0.4$ simultaneously predicts a reasonable willingness to pay of \$7.86 (\$12.33) for the St. Petersburg lottery truncated at the maximum payoff of $\$2^{20} \approx 1$ million ($\$2^{40} \approx 1$ trillion) dollars and an empirically substantiated preference reversal probability $p^* \approx \frac{1}{3}$ for the common ratio lotteries $L_1(p) = (\$6000, 0.5p; \$0, 1 - 0.5p)$ and $L_2(p) = (\$3000, p; \$0, 1 - p)$ where p is a probability.

With unbounded payoffs, however, Salience Theory cannot predict both paradoxes because that requires the following two conflicting conditions:

$$\lim_{z\to\infty} \frac{v(0.5z)}{v(z)} \geq \frac{1}{2\delta} \;, \qquad \text{(St. Petersburg paradox)}$$

$$\lim_{z\to\infty} \frac{v(0.5z)}{v(z)} < \frac{1}{2\delta} \;. \qquad \text{(Common ratio effect)}$$

Using the limit in the latter condition accommodates the peculiarity that Allais' common ratio effect emerges in particular for large payoffs (Allais, 1953).

Obviously, both conditions directly refuse the often chosen power value function $v(x) = x^\alpha$, $\alpha > 0$, because then the limit of the v-ratio simplifies to $\lim_{z\to\infty} \frac{v(0.5z)}{v(z)} = 0.5^\alpha$. This case also includes the linear value function. Bordalo et al. (2012) choose a linear value function and $\delta = 0.7$ to explain the Allais paradox, consistent with the above condition. However, their specification cannot explain finite willingness to pay for the St. Petersburg lottery.

Furthermore, we prove that the probability distortion in Salience Theory can be significantly stronger than in Cumulative Prospect Theory. Yet another necessary condition for finite willingness to pay in the St. Petersburg paradox is $\delta > 0.5$ – independent of the value function v. Hence, a probability-distortion parameter $\delta \leq 0.5$ can fully override the impact of the value function's curvature. For comparison, under Cumulative Prospect Theory, an individual's overall risk attitude is always determined by both the value and the probability weighting function (Dierkes & Sejdiu, 2019b).

Another interesting implication of this latter finding is that the use of bounded value functions does not necessarily solve the St. Petersburg paradox under Salience Theory. The intuition is that, for $\delta \leq 0.5$, the ever growing payoffs are salient and so strongly overweighted that the willingness to pay for the St. Petersburg lottery diverges for *any* monotonically increasing value function. Note that, under Cumulative Prospect Theory or Expected Utility Theory, bounded value functions solve the St. Petersburg paradox (e.g. Dierkes & Sejdiu, 2019b; Rieger & Wang, 2006). Experimental calibration exercises should, thus, pay special attention to estimating the probability-distortion parameter δ because this is the much more sensitive parameter in Salience Theory (see also Königsheim et al., 2019).

Our paper complements the relatively new strand of literature which tests Salience Theory empirically and theoretically. In particular, we are, to the best of our knowledge, the first to examine the St. Petersburg paradox under Salience Theory. Frydman & Mormann (2018) test Salience Theory by conducting two choice experiments. In their first experiment, they vary the correlation between lottery options and in their second experiment, they use a phantom lottery to manipulate the perception of payoffs. In contrast to Expected Utility Theory and Cumulative Prospect Theory, Salience Theory is able to explain the results caused by the perceptual manipulations. Nielsen et al. (2018) present an alternative experimental test of Salience Theory where subjects have to bet on a risky option under different treatments. The experimental results support Salience Theory's prediction that the salience of preferred consequences increases the attractiveness of the risky option. Königsheim et al. (2019) provide a first calibration of Salience Theory. Their empirical estimates roughly support the parameterization of Bordalo et al. (2012) (linear value function and $\delta \approx 0.7$). Moreover, they find that the estimates of the probability-distortion parameter δ are not significantly affected by the assumption of a concave value function (standard CRRA function). However, the estimate of δ is significantly smaller when a lottery's downside is more salient. Kontek (2016) shows that the certainty equivalent of a lottery can be undefined for some ranges of probabilities and that monotonicity violations can occur.

The rest of this paper is organized as follows. Sections 3.2 and 3.3 evaluate the original St. Petersburg paradox and the common ratio version of the Allais paradox with Salience Theory, respectively. Section 3.4 analyzes the conflict between the conditions of both paradoxes. In particular, we distinguish between an unbounded payoff analysis (Section 3.4.1), where payoffs can grow infinitely large (as in the original example), and a bounded payoff analysis (Section 3.4.2), where the maximum payoff is limited to a certain amount. Finally, we conclude with Section 3.5. An appendix complements the paper with selected formal proofs.

3.2 Bernoulli's St. Petersburg paradox under Salience Theory

Throughout our discussion, we make the following assumptions (see, e.g., Bordalo et al., 2012; Dierkes & Sejdiu, 2019b).

Assumption 3.1 (Preference Calculus)

a) The decision maker (Local Thinker) behaves according to Bordalo et al.'s (2012) Salience Theory where preferences are determined by a value function v, a salience function σ, and a probability-distortion parameter $\delta \in (0,1)$.

b) The Local Thinker's value function v is continuous and strictly monotonically increasing with $v(0) = 0$.

c) The salience function σ satisfies the two conditions ordering *and* diminishing sensitivity *of Definition 1 in Bordalo et al. (2012) on p. 1249.*

Assumption 3.2 (Mathematical Notation) *Our notation $\lim_{x \to z} f(x)$ implies that the limit exists in a weak sense, namely limes superior and limes inferior are equal and $\lim_{x \to z} f(x) \in [-\infty, \infty]$.*

In order to evaluate Bernoulli's (1738, 1954) prominent St. Petersburg lottery with Salience Theory, we consider a truncated St. Petersburg gamble, where \$$2^N$ is the maximum payoff, and compute the limit for $N \to \infty$:[1]

$$L_{STP}^{(N)} = \begin{cases} \left(2^k; 2^{-k}\right) & \text{for} \quad k = 1, \ldots, N-1 \\ \left(2^N; 2^{1-N}\right) & \text{for} \quad k = N \ . \end{cases} \tag{3.1}$$

As suggested by Bordalo et al. (2012) on p. 1271, we infer the willingness to pay for $L_{STP}^{(N)}$ by calculating the certainty equivalent when the lottery is considered in isolation, i.e. against a sure outcome of zero. Then, the state

[1]We do this because the construction of the salience ranking is easier to understand for the truncated St. Petersburg lottery. For fixed N, we work with the natural minimal state space. Splitting states, however, would not alter evaluation or choice under Salience Theory (see Bordalo et al., 2012).

space equals $S := \{(2,0), (4,0), \ldots, (2^N, 0)\}$. For *any* salience function σ satisfying the ordering condition of Definition 1 in Bordalo et al. (2012), the salience ranking for the St. Petersburg lottery is:

$$\sigma(2^N, 0) > \sigma(2^{N-1}, 0) > \ldots > \sigma(2, 0).$$

The original St. Petersburg lottery is given by $L_{STP} = \lim_{N\to\infty} L_{STP}^{(N)}$ and the Local Thinker (LT) assigns the following value to L_{STP}:

$$V^{LT}(L_{STP}) = \lim_{N\to\infty} V^{LT}\left(L_{STP}^{(N)}\right) = \lim_{N\to\infty} \frac{2^{1-N}\cdot\delta\cdot v\left(2^N\right) + \sum_{k=1}^{N-1} 2^{k-N}\cdot\delta^{k+1}\cdot v\left(2^{N-k}\right)}{2^{1-N}\cdot\delta + \sum_{k=1}^{N-1} 2^{k-N}\cdot\delta^{k+1}} \tag{3.2}$$

$$= \lim_{N\to\infty} \frac{\left(\frac{1}{2\delta}\right)^N \cdot v\left(2^N\right) + \sum_{k=0}^{N-1} \left(\frac{1}{2\delta}\right)^{N-k}\cdot v\left(2^{N-k}\right)}{\left(\frac{1}{2\delta}\right)^N + \sum_{k=0}^{N-1} \left(\frac{1}{2\delta}\right)^{N-k}} \tag{3.3}$$

$$= \lim_{N\to\infty} \frac{\left(\frac{1}{2\delta}\right)^N \cdot v\left(2^N\right) + \sum_{k=1}^{N} \left(\frac{1}{2\delta}\right)^{k}\cdot v\left(2^{k}\right)}{\left(\frac{1}{2\delta}\right)^N + \sum_{k=1}^{N} \left(\frac{1}{2\delta}\right)^{k}}. \tag{3.4}$$

The following proposition states necessary conditions for finite willingness to pay for L_{STP}.

Proposition 3.1 (Necessary conditions for STP) *Given the Assumptions 3.1 and 3.2, if Salience Theory predicts a finite willingness to pay for the St. Petersburg lottery L_{STP} then:*

$$\delta > \frac{1}{2}\ , \tag{3.5}$$

and

$$\lim_{z\to\infty} \frac{v(0.5z)}{v(z)} \geq \frac{1}{2\delta}\ . \tag{$ST.STP^*$}$$

Proof of Proposition 3.1: We conduct a proof by contraposition to derive the first necessary condition (3.5). More precisely, we show that $\delta \leq 0.5$

leads to an infinite certainty equivalent for the St. Petersburg lottery L_{STP}. It then follows that $\delta > 0.5$ is a necessary condition for a finite certainty equivalent $v^{-1}\big(V^{LT}(L_{STP})\big)$.

An infinite certainty equivalent is predicted by Salience Theory if and only if $\lim_{N\to\infty} V^{LT}(L_N) = \lim_{x\to\infty} v(x)$. Hence, we have to prove that the parameter restriction $0 < \delta \leq 0.5$ leads to $\lim_{N\to\infty} V^{LT}(L_N) = \lim_{x\to\infty} v(x)$. Since both the numerator and denominator of fraction (3.4) contain variants of the geometric series, we separately consider the two cases $\delta = 0.5$ and $0 < \delta < 0.5$ for ease of exposition.

If $\delta = 0.5$, Equation (3.4) can be rewritten to:

$$V^{LT}(L_{STP}) = \lim_{N\to\infty} \frac{v\left(2^N\right)}{1+N} + \lim_{N\to\infty} \frac{1}{1+N} \sum_{k=1}^{N} v\left(2^k\right) \tag{3.6}$$

$$= \lim_{N\to\infty} \frac{v\left(2^N\right)}{1+N} + \lim_{N\to\infty} v\left(N\right) \tag{3.7}$$

$$= \begin{cases} \infty & \text{if } v \text{ is unbounded, i.e. } \lim_{x\to\infty} v(x) = \infty \\ v(\infty) = c & \text{if } v \text{ is bounded, i.e. } \lim_{x\to\infty} v(x) = c < \infty \end{cases} . \tag{3.8}$$

Going from (3.6) to (3.7) is obvious. The proof, however, is available upon request. Note that Assumption 3.1 only assumes that v is continuous and monotonically increasing. Additionally, we distinguish between bounded and unbounded value functions v. Equation (3.8) shows that the certainty equivalent $v^{-1}\left(V^{LT}(L_{STP})\right)$ is infinite in both cases.

Now, we consider the case $0 < \delta < 0.5$. Then, Equation (3.4) can be rewritten as:

$$V^{LT}(L_{STP}) = \lim_{N\to\infty} \frac{1-2\delta}{2-2\delta-(2\delta)^N} \cdot \left[v\left(2^N\right) + \sum_{k=1}^{N} (2\delta)^{N-k} \cdot v\left(2^k\right)\right] \tag{3.9}$$

$$= \frac{1-2\delta}{2-2\delta} \cdot \left[\lim_{N\to\infty} v\left(2^N\right) + \lim_{N\to\infty} \sum_{k=0}^{N-1} (2\delta)^k \cdot v\left(2^{N-k}\right)\right] \tag{3.10}$$

$$= \frac{1-2\delta}{2-2\delta} \cdot \left[1 + \sum_{k=0}^{\infty} (2\delta)^k\right] \cdot \lim_{N\to\infty} v\left(2^N\right) \tag{3.11}$$

$$= \frac{1-2\delta}{2-2\delta} \cdot \left[1 + \frac{1}{1-2\delta}\right] \cdot \lim_{N\to\infty} v\left(2^N\right) \tag{3.12}$$

$$= \lim_{N\to\infty} v\left(2^N\right). \tag{3.13}$$

This shows that $\delta \in (0, 0.5)$ also leads to an infinite certainty equivalent for L_{STP}. Thus, a finite certainty equivalent implies $\delta > 0.5$. Note that the rewriting from (3.10) to (3.11) also holds for bounded value functions. Given that $\lim_{N\to\infty} v\left(2^N\right) = c \in \mathbb{R}^+$, then $\forall \epsilon > 0 \ \exists N_0 : c - v\left(2^{N-k}\right) < \epsilon \ \forall N_0 \leq N-k$ and

$$\lim_{N\to\infty} \left| \frac{c}{1-2\delta} - \sum_{k=0}^{N-1} (2\delta)^k \cdot v\left(2^{N-k}\right) \right|$$

$$\leq \lim_{N\to\infty} \left| \frac{c}{1-2\delta} - \left[\sum_{k=0}^{N-N_0} (2\delta)^k \cdot v\left(2^{N-k}\right) + \sum_{k=N-N_0+1}^{N-1} (2\delta)^k \cdot v\left(2^{N-k}\right)\right] \right| \tag{3.14}$$

$$\leq \lim_{N\to\infty} \left| \frac{c}{1-2\delta} - \left[(c-\epsilon) \cdot \sum_{k=0}^{N-N_0} (2\delta)^k + \sum_{k=N-N_0+1}^{N-1} (2\delta)^k \cdot v\left(2^{N-k}\right)\right] \right| \tag{3.15}$$

$$\leq \lim_{N\to\infty} \left| \frac{c}{1-2\delta} - \left[(c-\epsilon) \cdot \frac{1-(2\delta)^{N-N_0+1}}{1-2\delta} + \sum_{k=N-N_0+1}^{N-1} (2\delta)^k \cdot v\left(2^{N-k}\right)\right] \right| \tag{3.16}$$

$$\leq \lim_{N\to\infty} \left| \frac{c}{1-2\delta} - \left[(c-\epsilon) \cdot \frac{1-(2\delta)^{N-N_0+1}}{1-2\delta} + (N_0-1) \cdot (2\delta)^{N-N_0+1} \cdot v(2)\right] \right| \tag{3.17}$$

$$\leq \left| \frac{\epsilon}{1-2\delta} \right|. \tag{3.18}$$

Hence, $\forall \nu > 0 \ \exists N_0$ such that

$$\lim_{N\to\infty} \left| \frac{c}{1-2\delta} - \sum_{k=0}^{N-1} (2\delta)^k \cdot v\left(2^{N-k}\right) \right| < \nu \quad \forall N \geq N_0. \tag{3.19}$$

In other words

$$\lim_{N\to\infty}\sum_{k=0}^{N-1}(2\delta)^k \cdot v\left(2^{N-k}\right) = \frac{1}{1-2\delta}\cdot \lim_{N\to\infty} v\left(2^N\right). \tag{3.20}$$

Next, we derive the second necessary condition ($ST.STP^*$)by applying the ratio test. From now on, we always assume $\delta > 0.5$. This leads to:

$$V^{LT}(L_{STP}) = (2\delta - 1)\cdot\left[\lim_{N\to\infty}\frac{v\left(2^N\right)}{(2\delta)^N} + \sum_{k=1}^{\infty}\left(\frac{1}{2\delta}\right)^k \cdot v\left(2^k\right)\right]. \tag{3.21}$$

Equation (3.21) converges if and only if the series $\sum\limits_{k=1}^{\infty}\left(\frac{1}{2\delta}\right)^k \cdot v\left(2^k\right)$ converges. The ratio test indicates that if $\sum\limits_{k=1}^{\infty}\left(\frac{1}{2\delta}\right)^k \cdot v\left(2^k\right)$ converges then (necessary condition):

$$\lim_{k\to\infty}\left|\frac{\left(\frac{1}{2\delta}\right)^{k+1}\cdot v\left(2^{k+1}\right)}{\left(\frac{1}{2\delta}\right)^k \cdot v\left(2^k\right)}\right| \leq 1. \tag{3.22}$$

After substituting z for 2^{k+1} and some rearrangements, we get exactly the above stated second condition:

$$\lim_{z\to\infty}\frac{v\left(0.5z\right)}{v\left(z\right)} \geq \frac{1}{2\delta}. \tag{3.23}$$

■

Since we primarily aim to investigate the conflict with the conditions of the Allais paradox, we only state the necessary conditions in Proposition 3.1. However, condition ($ST.STP^*$)can be easily transformed to a sufficient condition by replacing the greater or equal sign with a strict greater sign. The Local-Thinker value (3.4) converges if (sufficient condition)

$$\lim_{z\to\infty}\frac{v\left(0.5z\right)}{v\left(z\right)} > \frac{1}{2\delta}. \tag{3.24}$$

Corollary 3.1 *If $\delta \leq 0.5$, then Salience Theory predicts an infinite certainty equivalent for the St. Petersburg lottery independent of the choice of the value function v.*

Proof of Corollary 3.1: According to Proposition 3.1, $\delta > 0.5$ is a necessary condition for a finite certainty equivalent for L_{STP}. Thus, $\delta \leq 0.5$ is sufficient for an infinite certainty equivalent independent of the choice of the value function. ■

Corollary 1 is a trivial consequence of Proposition 3.1. Nevertheless, we explicitly formalize a statement to highlight the importance of the condition $\delta > 0.5$. This condition reveals the surprising fact that bounded value functions are not sufficient for a finite certainty equivalent for L_{STP} in the framework of Salience Theory. Intuitively, for $\delta \leq 0.5$, the $N_0(< N)$ smallest payoffs are assigned an ever smaller decision weight when the number of coin flips N increases. Note the difference to Cumulative Prospect Theory and Expected Utility Theory. In these latter frameworks, a bounded value function always guarantees a finite willingness to pay for L_{STP} (see, e.g., Rieger & Wang, 2006; Dierkes & Sejdiu, 2019b) because the decision weights for the $N_0(< N)$ smallest payoffs stay constant for varying N.

Interestingly, under Salience Theory, extreme probability distortion ($\delta \leq 0.5$) fully overrides the value function's curvature in the context of the St. Petersburg paradox. For moderate distortion $\delta > 0.5$, though, the value function is a crucial determinant. There is a trade-off between probability distortion (right hand side of (3.24)) and the value function's curvature (left hand side of (3.24)). In particular, the value function is important for $\delta > 0.5$. Consider, for example, Bordalo et al.'s (2012) suggested standard specification of Salience Theory with linear value function $v(x) = x$ and a salience parameter $\delta = 0.7$. This specification of Salience Theory is not able to explain the St. Petersburg paradox because it violates the second St. Petersburg condition ($ST.STP^*$), i.e. $\lim_{z\to\infty} \frac{v(0.5z)}{v(z)} = 0.5 \not> \frac{1}{2\delta}$ for any $\delta \in (0, 1)$. An interpretation of this result is that the risk aversion predetermined by the value function has to be substantial enough to override the risk proclivity generated by the distortion of probabilities induced by the salience mechanism.

Although our focus is primarily on the question of whether Salience Theory is able to resolve Dierkes & Sejdiu's (2019b) Allais - St. Petersburg conflict, we want to briefly discuss the use of alternative value functions and derive some sufficient conditions for finite willingness to pay for the St. Petersburg lottery. Thereby, we are, to the best of our knowledge, the first to examine the St. Petersburg paradox in the framework of Salience Theory.

Table 3.1 lists the most well-known value functions from the literature. The value function *Linear* is a simple linear function, *Power* corresponds to Tversky & Kahneman's (1992) value function, and the functions *Logarithmic*, *Exponential*, and *HARA* (hyperbolic absolute risk aversion) are standard utility functions from the literature. All listed value functions are continuous and monotonically increasing with $v(0) = 0$ and, hence, satisfy Assumption 3.1. Additionally, Table 3.1 differentiates between bounded and unbounded value functions. The last column of Table 3.1 reports the if-and-only-if parameter conditions for a finite certainty equivalent for the St. Petersburg lottery L_{STP} given the respective value function (first column).[2] Table 3.1 shows that the Logarithmic, Exponential, and HARA value functions predict a finite certainty equivalent as long as $\delta > 0.5$ holds. The frequently used power value function $v(x) = x^\alpha$ of Tversky & Kahneman (1992) explains the St. Petersburg paradox if and only if $\delta > 2^{\alpha-1}$. Following Bordalo et al. (2012) and assuming $\delta = 0.7$ leads to the restriction that the risk aversion parameter α has to be strictly lower than $\alpha < \frac{\log(2\delta)}{\log(2)} \approx 0.4854$.

In summary, solutions to the St. Petersburg paradox are similarly restrictive under Salience Theory as they are under Cumulative Prospect Theory. Especially the originally proposed parametrization of Salience

[2]For a detailed derivation of the if-and-only-if conditions, see the appendix.

Table 3.1: If-and-only-if conditions for the explanation of the St. Petersburg paradox for various value functions under Salience Theory.

Type	Function $v(x)$ with $v(0) = 0$	Parameter restriction	Bounded utility	Finite certainty equivalent for L_{STP} iff
Linear	x	–	no	$\delta > 1$
Power	x^{α}	$\alpha > 0$	no	$\delta > 2^{\alpha-1}$
Logarithmic	$\log(1 + \alpha x)$	$\alpha > 0$	no	$\delta > 0.5$
Exponential	$1 - e^{-\alpha x}$	$\alpha > 0$	yes	$\delta > 0.5$
HARA	$\frac{1-\alpha}{\alpha}\left[\left(\frac{x}{1-\alpha} + \beta\right)^{\alpha} - \beta^{\alpha}\right]$	$\alpha < 0$, $\beta > 0$	yes	$\delta > 0.5$

Theory by Bordalo et al. (2012) with linear value function fails to explain the St. Petersburg paradox. And, surprisingly, bounded value functions are not a sufficient condition for the solution of the St. Petersburg paradox in the framework of Salience Theory. The following section shows that the prediction of Allais' common ratio effect further tightens parameter restriction under Salience Theory considerably.

3.3 Allais' common ratio effect under Salience Theory

To investigate under which conditions Salience Theory explains Allais' common ratio effect, we follow Bordalo et al. (2012) and consider a choice between the two equal-mean lotteries

$$L_1(p) = (\$z, 0.5p; 0, 1 - 0.5p) \quad \text{and} \quad L_2(p) = (\$0.5z, p; 0, 1 - p), \tag{3.25}$$

which are determined by the payoff amount $z > 0$ and the probability p.[3] Lab experiments systematically reveal that subjects tend to choose the safer lottery L_2 for high probabilities p and the riskier lottery L_1 for low probabilities p (e.g. Kahneman & Tversky, 1979). A definition of the common ratio effect is given below:

Definition 3.1 (Common ratio effect) *Given the equal-mean lotteries L_1 and L_2 defined in (3.25). The common ratio effect occurs if and only if the preference $L_1 \succ L_2$ holds for any $p \in (0, p^*)$ and $L_1 \prec L_2$ for any $p \in (p^*, 1]$ where $p^* \in (0, 1)$ indicates the preference reversal probability that implies $L_1 \sim L_2$.*

Since the lotteries L_1 and L_2 are uncorrelated, the state space is $S = \{(z, 0.5z); (0, 0.5z); (z, 0); (0, 0)\}$ and the salience ranking is

$$\sigma(z, 0) > \sigma(0, 0.5z) > \sigma(z, 0.5z) > \sigma(0, 0) \quad \forall z > 0.$$

This salience ranking follows from the diminishing sensitivity property of the salience function σ (Bordalo et al., 2012, see p. 1249) and holds for any $z > 0$ without any parametric assumptions about σ.

The Local Thinker evaluates both lotteries as follows:

$$\begin{aligned} V^{LT}(L_1) = {} & 0.5p(1-p)\frac{\delta}{\eta}v(z) + (1-0.5p)p\frac{\delta^2}{\eta}v(0) \\ & + 0.5p^2\frac{\delta^3}{\eta}v(z) + (1-0.5p)(1-p)\frac{\delta^4}{\eta}v(0), \end{aligned} \tag{3.26}$$

$$\begin{aligned} V^{LT}(L_2) = {} & 0.5p(1-p)\frac{\delta}{\eta}v(0) + (1-0.5p)p\frac{\delta^2}{\eta}v(0.5z) \\ & + 0.5p^2\frac{\delta^3}{\eta}v(0.5z) + (1-0.5p)(1-p)\frac{\delta^4}{\eta}v(0), \end{aligned} \tag{3.27}$$

where η is just some normalizing constant such that perceived probabilities sum up to one. It does not affect choices which are determined by the

[3] For numerical examples of the lottery pair (3.25) see, e.g., Kahneman & Tversky (1979) Problem 7 and 8.

maximum $\max\{V^{LT}(L_1), V^{LT}(L_2)\}$. We follow Bordalo et al. (2012) and assume, without loss of generality, $v(0) = 0$. Then, the Local Thinker prefers the riskier lottery $L_1 \succ L_2$ iff

$$V^{LT}(L_1) > V^{LT}(L_2) \tag{3.28}$$

$$\Leftrightarrow \; v(z) \cdot \left[0.5p(1-p)\frac{\delta}{\eta} + 0.5p^2\frac{\delta^3}{\eta}\right] > v(0.5z) \cdot \left[(1-0.5p)p\frac{\delta^2}{\eta} + 0.5p^2\frac{\delta^3}{\eta}\right] \tag{3.29}$$

$$\Leftrightarrow \qquad \frac{1-p\left(1-\delta^2\right)}{2\delta - p(\delta-\delta^2)} > \frac{v(0.5z)}{v(z)} \tag{3.30}$$

and $L_1 \prec L_2$ vice versa. Inequality (3.30) shows that a risk-seeking (risk-averse) choice is predicted only if the ratio of decision weights is strictly greater (smaller) than the ratio of evaluated payoffs $v(0.5z)/v(z)$. Interestingly, it turns out that the ratio of decision weights (left-hand side of Inequality (3.30)) is monotonically decreasing in p:

$$\frac{\partial}{\partial p}\frac{1-p\left(1-\delta^2\right)}{2\delta - p\left(\delta-\delta^2\right)} = -\frac{1-\delta(2\delta-1)}{\delta\left(2-p(1-\delta)\right)^2} < 0 \;\text{ for any } \delta \in (0,1), \tag{3.31}$$

which means that the probability-weighting mechanism of Salience Theory satisfies subproportionality in this setting. This is particularly interesting because subproportionality is a key concept when modeling the common ratio effect under Prospect Theory (Prelec, 1998). It ensures that if a preference change between the lottery pairs (L_1, L_2) for varying $p \in (0, 1]$ exists, it is a unique one. Hence, it excludes unrealistic predictions of multiple preference reversals. Moreover, this property enables us to focus on the corner cases $p \to 0^+$ and $p = 1$ for the prediction of the common ratio effect. The following proposition states the necessary and sufficient conditions for Salience Theory to predict the common ratio effect according to Definition 3.1.

Proposition 3.2 (Necessary and sufficient condition for CRE)
Given Assumption 3.1 and the equal-mean lotteries L_1 and L_2 defined in (3.25)*, then Salience Theory satisfies Definition 3.1 and thus predicts the common ratio effect for varying probability $p \in (0,1]$ and for payoff $z > 0$ if and only if (necessary and sufficient condition):*

$$\frac{\delta}{1+\delta} < \frac{v(0.5z)}{v(z)} < \frac{1}{2\delta} \ . \qquad (ST.CRE)$$

Proof of Proposition 3.2: According to Definition 3.1 and Inequality (3.30), we have to prove that the common ratio condition ($ST.CRE$) implies that the decision-weight ratio (left-hand side of (3.30)) is greater than the v-ratio $v(0.5z)/v(z)$ for any $p \in (0, p^*)$ and smaller for any $p \in (p^*, 1]$. As already shown in (3.31), the decision-weight ratio in Salience Theory is monotonically decreasing in p. Therefore, to satisfy Definition 3.1 and especially to ensure the existence of exactly one preference reversal point $p^* \in (0,1)$, it is enough to focus on the two corner cases $p \to 0^+$ and $p = 1$.

If p tends to zero, then condition (3.30), which implies the preference $L_1 \succ L_2$, simplifies to

$$\lim_{p \to 0^+} \frac{1 - p\,(1-\delta^2)}{2\delta - p(\delta - \delta^2)} = \frac{1}{2\delta} > \frac{v(0.5z)}{v(z)} \ . \qquad (3.32)$$

To accommodate the common ratio effect, Salience Theory also has to predict a preference for the safer lottery $L_2 \succ L_1$ if p tends to one (certainty effect):

$$\lim_{p \to 1} \frac{1 - p\,(1-\delta^2)}{2\delta - p(\delta - \delta^2)} = \frac{\delta}{1+\delta} < \frac{v(0.5z)}{v(z)} \ . \qquad (3.33)$$

Now, considering both boundaries together leads to the above stated condition for the common ratio effect:

$$\frac{\delta}{1+\delta} < \frac{v(0.5z)}{v(z)} < \frac{1}{2\delta} \ . \qquad (3.34)$$

Thus, if and only if (3.34) holds, then Definition 3.1 is satisfied and $L_1 \succ L_2$ is predicted for any $p \in (0, p^*)$ and $L_1 \prec L_2$ for any $p \in (p^*, 1]$ where $p^* \in (0, 1)$ is the preference reversal probability. ■

The next corollary is a trivial consequence of Proposition 3.2 but a very practical one because it highlights that $\delta < 0.5$ satisfies the common ratio condition $(ST.CRE)$ of Proposition 3.2 for any value function v that has a v-ratio $\frac{v(0.5z)}{v(z)} \geq \frac{1}{3}$. Note that $\frac{v(0.5z)}{v(z)} \geq \frac{1}{3}$ includes all concave value functions and even some convex value functions like, e.g., $v(x) = x^{1.5}$.

Corollary 3.2 *Given Assumption 3.1 and the equal-mean lotteries L_1 and L_2 defined in* (3.25) *with payoff $z > 0$, the combination of a value function v with v-ratio $\frac{v(0.5z)}{v(z)} \geq \frac{1}{3}$ for all $z > 0$ and a probability-distortion parameter $\delta < 0.5$ is sufficient for the prediction of the common ratio effect according to Definition 3.1.*

Proof of Corollary 3.2: The parameter restriction $\delta < 0.5$ implies $\delta/(1 + \delta) < 1/3$ and $1/(2\delta) > 1$. The value function's monotonicity assumption in Assumption 3.1 ensures that the v-ratio is bounded from above, i.e. $\frac{v(0.5z)}{v(z)} \leq 1$. Together with the assumption that the v-ratio is greater than or equal to $\frac{1}{3}$, it follows that the v-ratio is bounded between $\frac{1}{3} \leq \frac{v(0.5z)}{v(z)} \leq 1$ for any payoff $z > 0$ (even for $z \to \infty$). Hence,

$$\frac{\delta}{1+\delta} < \frac{1}{3} \leq \frac{v(0.5z)}{v(z)} \leq 1 < \frac{1}{2\delta} \quad \forall z > 0, \tag{3.35}$$

and the common ratio condition $(ST.CRE)$ of Proposition 3.2 is satisfied. ■

Corollary 3.2 shows that Salience Theory with $\delta < 0.5$ explains the common ratio effect for all commonly employed value functions (see Table 3.1 for examples) and also for any level of positive risk aversion. Besides practicability, this result illustrates again the dominant role of δ. If $\delta \in (0, 0.5)$, then the probability-weighting mechanism in Salience

Theory generates such a strong risk proclivity, in the case where the winning probabilities of both lotteries are scaled down ($p \to 0^+$) that any level of risk aversion generated by the value function is fully compensated for – independent of payoff $z > 0$ (even for $z \to \infty$). It turns out that this feature is particularly useful for the prediction of the common ratio effect because it prevents the issue that the preference reversal probability p^* can be moved arbitrarily close to zero just by increasing the payoffs of the lotteries, i.e. by increasing z. Dierkes & Sejdiu (2019b) show that Cumulative Prospect Theory with typical (continuous) weighting functions suffers from this very problem and hence has difficulties to predict the common ratio effect especially when employing bounded value functions.

Moreover, in Salience Theory, the preference reversal probability p^* can be represented as an explicit function of δ and the v-ratio $\frac{v(0.5z)}{v(z)}$:

$$p^* = \frac{1 - 2\delta \cdot \frac{v(0.5z)}{v(z)}}{1 - \delta^2 - (\delta - \delta^2) \cdot \frac{v(0.5z)}{v(z)}} \in [0, 1] \quad \text{for} \quad \frac{\delta}{1+\delta} \leq \frac{v(0.5z)}{v(z)} \leq \frac{1}{2\delta}. \tag{3.36}$$

Figure 3.1 presents a contour plot of (3.36) and nicely summarizes the results of this chapter in one figure. The blue contour lines in Figure 3.1 depict the parameter sets $\left(\frac{v(0.5z)}{v(z)}, \delta\right)$ that yield to a preference reversal probability $p^* \in \{0, 0.1, 0.2, \ldots, 0.9, 1\}$. Note that only the parameter sets that are located strictly in between the two blue solid lines $\frac{v(0.5z)}{v(z)}|_{p^*=0} = \frac{1}{2\delta}$ and $\frac{v(0.5z)}{v(z)}|_{p^*=1} = \frac{\delta}{1+\delta}$ predict a preference reversal probability $p^* \in (0, 1)$ and thus satisfy the common ratio condition ($ST.CRE$) of Proposition 3.2. The parameter sets (I) and (II) including and above the blue solid 0-line and including and below the blue solid 1-line, respectively, do not explain the common ratio effect because they predict either only risk-averse behavior (I) or only risk-seeking behavior (II). For example, in area (I) which corresponds to $\frac{v(0.5)}{v(z)} \geq \frac{1}{2\delta}$, the safer lottery L_2 is always preferred over the riskier lottery L_1 for all $p \in (0, 1]$ which means that no preference change

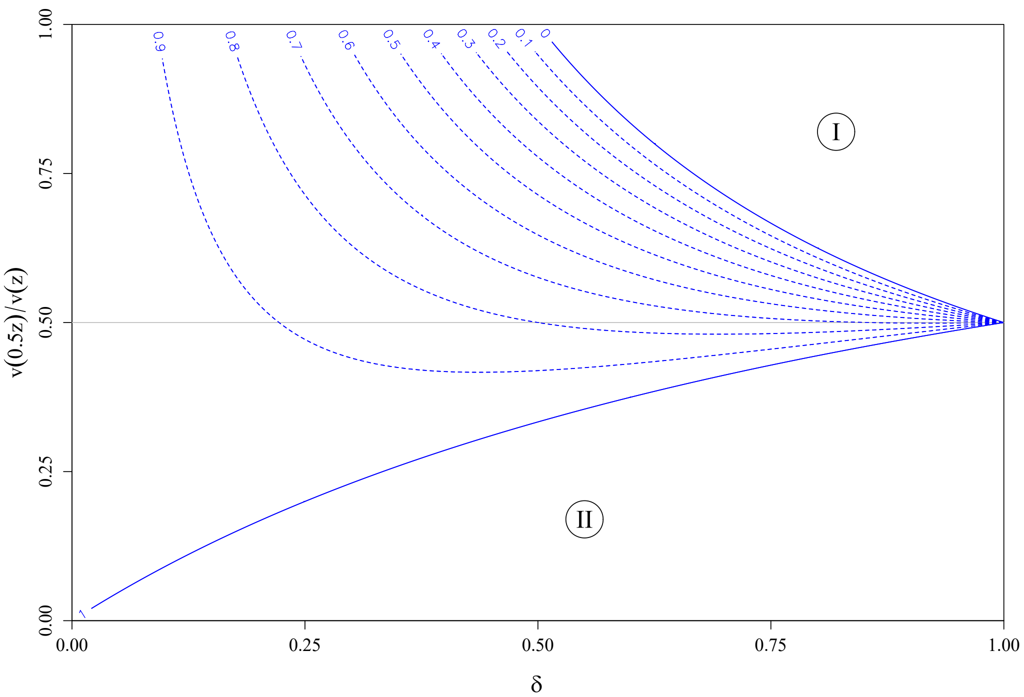


Figure 3.1: Contour plot of the preference reversal probability p^*.
Notes: The figure illustrates the combination of v-ratio $v(0.5z)/v(z)$ and δ that yields to $p^* = \{0, 0.1, 0.2, \ldots, 0.9, 1\}$, respectively. In detail, we depict the v-ratio in dependence of δ and p^* according to $\frac{v(0.5z)}{v(z)} = \frac{1-p^*(1-\delta^2)}{2\delta-p^*(\delta-\delta^2)}$.

occurs. Interestingly, I is at the same time the domain which has the best chances to solve the St. Petersburg paradox. This fact already foreshadows the conflict between the conditions ($ST.STP^*$) and ($ST.CRE^*$) of both paradoxes.

Figure 3.1 also visualizes Corollary 3.2 and the importance of the parameter δ. The gray horizontal line at $\frac{v(0.5z)}{v(z)} = 0.5$ separates concave (upper half) and convex (lower half) value functions. When considering the top-left part of the figure it becomes clear that the combination of a concave value function $\left(\frac{v(0.5z)}{v(z)} \geq 0.5\right)$ and $\delta < 0.5$ is sufficient for the common ratio effect ($ST.CRE$) as implied by Corollary 3.2. Furthermore, the figure illustrates that even when assuming a bounded value function and considering high payoffs, i.e. $\lim_{z\to\infty} v(0.5z)/v(z) = 1$, Salience Theory is still able to predict a reasonable preference reversal probability $0 < p^* < 1$ as long as $\delta < 0.5$. In such a case, function (3.36) simplifies to

$$p^*|_{\frac{v(0.5z)}{v(z)}\approx 1} = \frac{1-2\delta}{1-\delta} \,. \tag{3.37}$$

Interestingly, the model is able to ensure that the preference reversal probability p^* does not fall below some threshold. For example, $\delta = \frac{1}{3}$ ensures that $p^* \in [0.5, 1)$ for any value function with a v-ratio $\frac{v(0.5z)}{v(z)} \in (0.25, 1]$. To see this note that the preference reversal function (3.36) is monotonically decreasing in $\frac{v(0.5z)}{v(z)}$ for any $\delta \in (0, 1)$.[4] Thus, a lower v-ratio, which corresponds to lower risk aversion, leads to a higher preference reversal probability p^* and vice versa. This prediction of Salience Theory is different from that of Cumulative Prospect Theory. Under Cumulative Prospect Theory, the preference reversal probability p^* always converges to zero if the value function's risk aversion increases to infinity. The reason for this is that under Cumulative Prospect Theory, an individual's risk attitude is determined by both the value function and probability weighting function. In contrast, Salience Theory's probability-weighting mechanism can dominate the value function in this setting if $\delta < 0.5$ and hence can easily predict the common ratio effect for *any* concave value function. Recall, however, that $\delta < 0.5$ cannot accommodate the St. Petersburg paradox.

[4]The partial derivative of p^* with respect to the v-ratio $\frac{v(0.5z)}{v(z)}$ is negative for any $\frac{v(0.5z)}{v(z)} \in [0, 1]$ and $\delta \in (0, 1)$:

$$\frac{\partial p^*}{\partial \frac{v(0.5z)}{v(z)}} = -\frac{\delta(1-\delta)\cdot(2\delta+1)}{\left(1-\delta^2-(\delta-\delta^2)\cdot\frac{v(0.5z)}{v(z)}\right)^2} < 0 \;\; \forall\, \delta \in (0, 1). \tag{3.38}$$

3.4 The Allais - St. Petersburg conflict in Salience Theory

After revisiting the St. Petersburg paradox and the common ratio version of the Allais paradox separately, we analyze the conflict between the conditions of both paradoxes. Furthermore, we distinguish between an unbounded payoff analysis where payoffs can grow infinitely large (as in the original example) and a bounded payoff analysis where the maximum payoff is limited to a certain amount. The latter analysis can be considered as a robustness check and aims to investigate whether the potential conflict is driven by the infinite expected payoff of the St. Petersburg lottery and hence the assumption of infinite resources.

3.4.1 Unbounded payoff analysis

Theorem 3.1 (Main Result) *Assume decision makers behave according to Salience Theory with continuous and strictly increasing value function* v *and* $v(0) = 0$*. The common ratio lotteries are given by* $L_1(p) = (\$z, 0.5p; \$0, 1-0.5p)$ *and* $L_2(p) = (\$0.5z, p; \$0, 1-p)$ *and we assume the common ratio effect shows up for all payoffs* $z > 0$*, in particular for large payoffs as argued by Allais (1953). Then, there does not exist any set of parameters that simultaneously explains both Bernoulli's St. Petersburg paradox and Allais' common ratio effect.*

Proof of Theorem 3.1: Given Allais' (1953) observation that the Allais paradox occurs in particular for large payoffs (he used payoffs in the millions), the following condition can be considered as a special case of the common ratio condition ($ST.CRE$) of Proposition 3.2:

$$\frac{\delta}{1+\delta} < \lim_{z\to\infty} \frac{v(0.5z)}{v(z)} < \frac{1}{2\delta}\ . \qquad (ST.CRE^*)$$

Then, the conflict with the necessary condition of ($ST.STP^*$) for the St. Petersburg paradox

$$\lim_{z\to\infty} \frac{v(0.5z)}{v(z)} \geq \frac{1}{2\delta}$$

is ultimately evident since both conditions are in direct conflict. ■

Theorem 3.1 shows that Salience Theory cannot resolve Dierkes & Sejdiu's (2019b) Allais - St. Petersburg conflict. Put differently, no parameter set is able to explain Bernoulli's original St. Petersburg paradox and Allais' common ratio effect simultaneously. The conflict becomes clear for any value function v when the payoffs in the common ratio lotteries L_1 and L_2 grow infinitely large ($z \to \infty$). Using infinitely large payoffs is supported by empirical evidence. For example, in his original work, Allais (1953) used payoffs in the millions. For such high payoffs $\frac{v(0.5z)}{v(z)} \approx \lim_{z\to\infty} \frac{v(0.5z)}{v(z)}$ for most value functions.

Before we consider a truncated setting where monetary resources are limited and hence payoffs are bounded, we briefly discuss the use of the most popular value function $v(x) = x^{\alpha}$ with $\alpha > 0$ as suggested by Tversky & Kahneman (1992).

Corollary 3.3 *In Salience Theory, if the value function equals the frequently used power function $v(x) = x^{\alpha}$ with $\alpha > 0$, then the parameter sets that explain the common ratio effect and the St. Petersburg paradox, respectively, are strictly disjunct independent of payoff z.*

Proof of Corollary 3.3: Compare again the necessary condition ($ST.STP^*$) for the St. Petersburg paradox and the upper necessary condition ($ST.CRE$)

for the common ratio effect. When applying $v(x) = x^\alpha$, then $\lim_{z\to\infty} \frac{v(0.5z)}{v(z)} = \frac{v(0.5z)}{v(z)} = 0.5^\alpha \; \forall z > 0$ and

$$\lim_{z\to\infty} \frac{v(0.5z)}{v(z)} = 0.5^\alpha \geq \frac{1}{2\delta} \,, \qquad \text{(St. Petersburg paradox)}$$
$$\frac{v(0.5z)}{v(z)} = 0.5^\alpha < \frac{1}{2\delta} \qquad \text{(Common ratio effect)}$$

which proves that the parameter sets are strictly disjunct independent of z.

■

Corollary 3.3 highlights that assuming a power value function, including the linear function as a special case, produces the Allais - St. Petersburg conflict independent of the payoff level z in the common ratio lotteries L_1 and L_2. Hence, it is obvious that Bordalo et al.'s (2012) originally proposed parametrization with linear value function $v(x) = x$ and $\delta = 0.7$ also produces the conflict for any value z.

3.4.2 Bounded payoff analysis

In this section, we assume that monetary resources are finite. Then, payoff z in the common ratio lotteries L_1 and L_2 and the maximum payoff 2^N of the St. Petersburg gamble $L_{STP}^{(N)}$ have to be below a certain threshold. By doing this, we test whether the occurrence of the Allais - St. Petersburg conflict in Salience Theory is only an asymptotic result which is mainly driven by the consideration of infinitely large payoffs.

To keep things simple, we follow Kahneman & Tversky (1979) and fix $z = 6000$ in the common ratio example. Then, the Local Thinker has the choice between the lotteries $L_1(p) = (\$6000, 0.5p; \$0, 1 - 0.5p)$ and $L_2(p) = (\$3000, p; \$0, 1 - p)$ for varying probability $p \in (0, 1]$. Definition 3.1 of the common ratio effect is very general and only requires that a preference reversal point $p^* \in (0, 1)$ exists but not that it lies in a reasonable range which can be substantiated by empirical evidence. For example, in Kahneman & Tversky (1979) Problem 7 and 8, subjects exhibit the preferences $L_1(0.9) \prec L_2(0.9)$ and $L_1(0.002) \succ L_2(0.002)$ which implies that the preference reversal probability p^* is in the range $0.002 < p^* < 0.9$. Obviously, this range of p^* is a conservative estimate, nevertheless, it gives us a first indication in what range an acceptable prediction of the preference reversal probability has to be. Starmer & Sugden (1989, p. 173) provide a more accurate estimate of the p^*-range and argue that "most subjects switch preference in the range $0.6 > p > 0.2$". This more accurate estimate naturally puts tighter restrictions on Salience Theory. Bordalo et al. (2012) already report on p. 1269 that $\delta \in (0.22, 1)$ is a necessary condition for the prediction of $p^* \in (0.002, 0.9)$ when assuming a linear value function. Since we already know that a linear value function cannot explain the St. Petersburg paradox, we are particularly interested in what restrictions for δ follow if we assume more common concave value functions, for example, a bounded value function which has the best chances to solve the St. Petersburg paradox.

Table 3.2 presents parameter sets of δ that predict the preference reversal probability p^* in the ranges $0.002 < p^* < 0.9$ and $0.2 < p^* < 0.6$, respectively, for all value functions v listed in Table 3.1 and a given set of risk-aversion parameters α. Although we already know that the power value function $v(x) = x^\alpha$ with $\alpha > 0$ produces the Allais - St. Petersburg conflict independent of the payoff z, we still include this value function to illustrate the interplay between δ and α in the common ratio example when the preference reversal probability p^* has to lie in a realistic range. For example, if $v(x) = x^{0.4}$ then δ has to be in the interval $(0.124, 0.659)$ to satisfy the minimum requirement $0.002 < p^* < 0.9$. When assuming a linear value function ($\alpha = 1$), then the preference reversal probability p^* is always strictly greater than $\frac{2}{3}$ which explains why no $\delta \in (0, 1)$ is

Table 3.2: Allais' common ratio effect under Salience Theory.

Notes: This table indicates parameter sets of δ that predict the preference reversal probability p^* in a certain range for all respective value functions v listed in Table 3.1 for a given set of risk-aversion parameters α. In detail, we consider a choice between the lotteries $L_1(p) = (\$6000, 0.5p; \$0, 1-0.5p)$ and $L_2(p) = (\$3000, p; \$0, 1-p)$ for varying probability $p \in (0,1]$. The common ratio effect occurs if and only if the preference $L_1 \succ L_2$ holds for any $p \in (0, p^*)$ and $L_1 \prec L_2$ for any $p \in (p^*, 1]$ where $p^* \in (0,1)$ indicates the preference reversal probability that implies $L_1 \sim L_2$. The ranges $0.002 < p^* < 0.9$ and $0.2 < p^* < 0.6$ are motivated by the experimental observations in Kahneman & Tversky (1979) and Starmer & Sugden (1989), respectively.

Function v	v-ratio $\frac{v(0.5z)}{v(z)}$	α-value	δ-condition for a given range of p^*	
			$0.002 < p^* < 0.9$	$0.2 < p^* < 0.6$
Power	0.5^α	$\alpha = 1$	$\delta \in (0.222, 1.000)$	$\emptyset$
		$\alpha = 0.7$	$\delta \in (0.161, 0.812)$	$\delta \in (0.543, 0.762)$
		$\alpha = 0.4$	$\delta \in (0.124, 0.659)$	$\delta \in (0.399, 0.599)$
Logarithmic	$\frac{\log(1+\alpha 3000)}{\log(1+\alpha 6000)}$	$\alpha = 0.5$	$\delta \in (0.100, 0.547)$	$\delta \in (0.317, 0.489)$
		$\alpha = 1$	$\delta \in (0.099, 0.543)$	$\delta \in (0.314, 0.485)$
		$\alpha = 2$	$\delta \in (0.099, 0.539)$	$\delta \in (0.312, 0.482)$
Exponential	$\frac{1-e^{-\alpha 3000}}{1-e^{-\alpha 6000}}$	$\alpha = 0.001$	$\delta \in (0.096, 0.524)$	$\delta \in (0.302, 0.468)$
		$\alpha = 0.01$	$\delta \in (0.091, 0.500)$	$\delta \in (0.286, 0.444)$
		$\alpha = 0.1$	$\delta \in (0.091, 0.500)$	$\delta \in (0.286, 0.444)$
HARA ($\beta = 1$)	$\frac{\left(\frac{0.5z}{1-\alpha}+1\right)^\alpha - 1}{\left(\frac{z}{1-\alpha}+1\right)^\alpha - 1}$	$\alpha = -0.25$	$\delta \in (0.094, 0.513)$	$\delta \in (0.294, 0.457)$
		$\alpha = -0.5$	$\delta \in (0.092, 0.503)$	$\delta \in (0.288, 0.448)$
		$\alpha = -1$	$\delta \in (0.091, 0.500)$	$\delta \in (0.286, 0.445)$

able to predict a $p^* \in (0.2, 0.6)$.[5] For a graphical illustration see contour

[5]Given that the value function's v-ratio is 0.5, then the preference reversal function (3.36) simplifies to

$$p^*\left(\delta, \frac{v(0.5z)}{v(z)} = 0.5\right) = \frac{1-\delta}{1-\delta^2 - 0.5\,(\delta-\delta^2)}. \tag{3.39}$$

The image of $p^*(\delta, 0.5)$ equals the interval $\left(\frac{2}{3}, 1\right)$ for $\delta \in (0,1)$. The lower bound $\frac{2}{3}$ follows when considering the limit $\delta \to 1$:

$$\lim_{\delta\to 1} \frac{1-\delta}{1-\delta^2 - 0.5\,(\delta-\delta^2)} \overset{l'Hopital}{=} \lim_{\delta\to 1} \frac{-1}{-2\delta - 0.5\,(1-2\delta)} = \frac{-1}{-1.5} = \frac{2}{3}\,. \tag{3.40}$$

lines $p^* = \{0.2, 0.6\}$ in Figure 3.1. Both contour lines are monotonically decreasing and consistently greater than 0.5 (horizontal gray line) for any $\delta \in (0,1)$. This shows that $0.2 < p^* < 0.6$ is not possible with a linear value function.

Considering now the δ-conditions for the more interesting value functions Logarithmic, Exponential, and HARA, shows surprisingly narrow ranges for δ. For the conservative range $0.002 < p^* < 0.9$, the upper bounds of the δ-conditions are just slightly higher than 0.5 for all listed values of α. If we require the more accurate prediction $0.2 < p^* < 0.6$, then the upper bounds of δ are strictly lower than 0.5 for all listed specifications. Even for moderate values of z and a wide range of the preference reversal probability p^*, δ is not allowed to be greater than 0.5. The Allais - St. Petersburg conflict is then predetermined because $\delta > 0.5$ is one of the necessary conditions for the solution of the St. Petersburg paradox. Note that choosing an unrealistically low risk-aversion parameter α does not solve the problem, because then the certainty equivalent of the St. Petersburg gamble will grow tremendously. For example, assuming a logarithmic value function $v(x) = \log(1 + \alpha x)$ with $\alpha = 0.5$ and $\delta = 0.54$ satisfies the minimum requirement $0.002 < p^* < 0.9$ but predicts a willingness to pay of \$12,884.57 for St. Petersburg lottery L_{STP} which is an unrealistic prediction of typical human behavior. People are usually unwilling to pay more than \$8 to play the lottery (see, e.g., Hayden & Platt, 2009; Cox et al., 2011). Thus, decreasing the value function's risk aversion does not help. At best, it provides only a theoretical corner solution without much practical benefits. Therefore, the conflict between the conditions of a finite willingness to pay for the St. Petersburg lottery and Allais' common ratio effect is not caused by an unrealistic payoff level z in the common ratio lotteries or any other parameter. In fact, the results show that a realistic prediction of the common ratio effect alone suggests $\delta < 0.5$ for all value functions except the function Power. The power function, however, is immediately ruled out when we want to predict the St. Petersburg paradox in addition to the common ratio effect (see Corollary 3.3).

Next, we test whether the Allais - St. Petersburg conflict is driven by the infinite expected payoff of the St. Petersburg lottery. Technically, we consider a truncated St. Petersburg lottery where the maximum payoff of the lottery is 2^N. Recall that $L_{STP}^{(N)}$ yields a payoff $\$2^k$ with probability 0.5^k for $k = 1, \ldots, N-1$ and a payoff of $\$2^N$ with probability 0.5^{N-1}. The expected value of the lottery is $N+1$ and corresponds to $N-1$ possible rounds of coin flipping. For example, for $N = 40$, the expected payoff equals \$41 and the maximum payoff is $\$2^{40}$ which equals roughly 1 trillion dollars.

In Figure 3.2, we depict the willingness to pay $CE\left(L_{STP}^{(N)}\right) = v^{-1}\left(V^{LT}\left(L_{STP}^{(N)}\right)\right)$ as a function of N for the truncated St. Petersburg gamble, where $N = 2, \ldots, 40$ determines the maximum payoff $\$2^N$, for various parametrizations of Salience Theory. We focus on the value functions Logarithmic $v(x) = \log(1+x)$ and Exponential $v(x) = 1 - e^{-x}$ and two exemplary values of $\delta = \{0.4, 0.7\}$. Both value functions are chosen because they are simple and, more importantly, they resolve the St. Petersburg paradox in Expected Utility Theory and Cumulative Prospect Theory (Bernoulli, 1738, 1954; Menger, 1934; Dierkes & Sejdiu, 2019b). The choice of $\delta = 0.7$ follows the suggestion of Bordalo et al. (2012) and $\delta = 0.4$ is interesting because it violates the St. Petersburg condition $\delta > 0.5$ without being too far away from 0.5.

Figure 3.2 shows that the logarithmic value function $v(x) = \log(1+x)$ predicts acceptable certainty equivalents only for $\delta = 0.7$ (solid line) but not for $\delta = 0.4$ (dashed line). The combination with $\delta = 0.4$ (dashed line) performs poorly because, for all depicted truncation levels N, it predicts certainty equivalents higher than the expected value (risk-seeking behavior). Even worse, the certainty equivalent increases exponentially in N. This

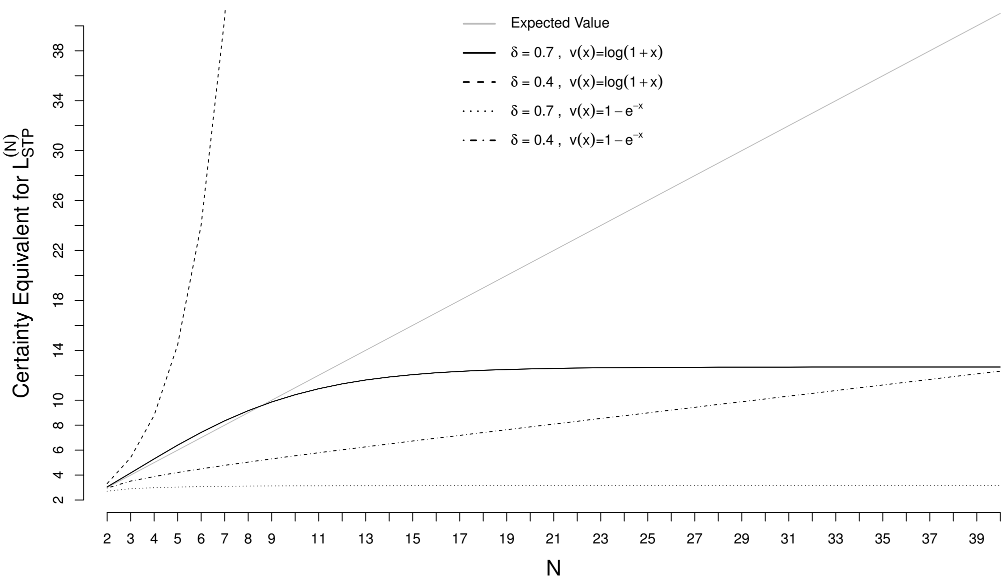


Figure 3.2: Certainty equivalents for the truncated St. Petersburg lottery under Salience Theory.

Notes: This figure depicts the willingness to pay $CE\left(L_{STP}^{(N)}\right) = v^{-1}\left(V^{LT}\left(L_{STP}^{(N)}\right)\right)$ for the truncated St. Petersburg lottery under Salience Theory where $N = 2, \ldots, 40$ determines the maximum payoff \$$2^N$. The truncated gamble promises a payoff of \$$2^k$ with probability 0.5^k for $k = 1, \ldots, N-1$ and a payoff of \$$2^N$ with probability 0.5^{N-1}. The gray line represents the expected value of the lottery. Individual's preferences are given by the value functions Logarithmic ($v(x) = \log(1+x)$) and Exponential ($v(x) = 1 - e^{-x}$) and the salience mechanism of Bordalo et al. (2012) with $\delta = \{0.4, 0.7\}$, respectively.

result is no surprise because $\delta > 0.5$ is one of the necessary conditions for the solution of the St. Petersburg paradox. With $\delta = 0.7$, however, the logarithmic value function predicts certainty equivalents higher than the expected value (risk-seeking behavior) for low truncation levels $N = 2, \ldots, 8$. This is relevant because experimental studies show that subjects also behave risk averse in the truncated St. Petersburg gamble (e.g. Hayden & Platt, 2009).

Consider now the specifications with the exponential value function $v(x) = 1 - e^{-x}$. Figure 3.2 reveals that the exponential value function predicts reasonable certainty equivalents even for $\delta = 0.4$. More precisely, the certainty equivalents are lower than the expected value of the lottery (positive risk premium) throughout all depicted truncation levels, thereby correctly indicating risk-averse behavior for all $2 \leq N \leq 40$. Moreover, this combination implies for $N = 40$ ($\approx$maximum payoff of 1 trillion dollars) a certainty equivalent of only \$12.33 which is a good prediction. Note that the combination of $v(x) = 1 - e^{-x}$ and $\delta = 0.4$ also yields a preference reversal probability of $p^* \approx \frac{1}{3}$ for Kahneman & Tversky's (1979) common ratio lotteries with payoff $z = 6000$. Such a prediction of p^* furthermore satisfies Starmer & Sugden's (1989) tighter suggestion $0.2 < p^* < 0.6$. Finally, the predicted certainty equivalent for the St. Petersburg lottery strictly decreases when the constant absolute risk aversion parameter α increases while the preference reversal probability p^* for the common ratio lotteries remains always higher than $\frac{1}{3}$ as long as $\delta = 0.4$. For example, choosing $v(x) = 1 - e^{-2x}$ predicts a certainty equivalent of \$7.24 for the St. Petersburg lottery with $N = 40$ and preference reversal probability $p^* \approx \frac{1}{3}$ for the common ratio lotteries.

The examples of this section demonstrate that combinations of v and δ exist that are able to predict a reasonable willingness to pay for truncated St. Petersburg lotteries and accommodate the empirical evidence on Allais' common ratio effect simultaneously. This result is by no means trivial and shows that Salience Theory has an edge over other prominent decision theories. For example, Dierkes & Sejdiu (2019b) show that, with typical (continuous) parametrizations of Cumulative Prospect Theory, the Allais - St. Petersburg conflict emerges for unbounded as well as bounded payoff lotteries.

3.5 Conclusion

Bordalo et al.'s (2012) Salience Theory is a new promising context-dependent descriptive theory of choice under risk. Extensive tests of this theory are, however, still work in progress. Our paper contributes to this literature by examining Salience Theory with Dierkes & Sejdiu's (2019b) Allais - St. Petersburg test. We find that Salience Theory can resolve the Allais - St. Petersburg conflict but only under the assumptions of finite resources and a value function which generates substantial risk aversion (e.g. bounded value functions). A simple parametrization of Salience Theory that performs sufficiently well consists of the exponential value function $v(x) = 1 - e^{-x}$, a probability-distortion parameter value $\delta \approx 0.4$, and any salience function for ranking states with the properties ordering and diminishing sensitivity, as proposed by Bordalo et al. (2012). This parametrization simultaneously predicts a reasonable willingness to pay for the truncated St. Petersburg lottery and an empirically substantiated preference reversal probability for Allais' common ratio lotteries with equal-mean lotteries. The Allais - St. Petersburg conflict emerges only asymptotically for this specification, i.e. when considering the original St. Petersburg lottery with infinite expected payoff. Although theoretically imperfect, this specification works sufficiently well in a realistic, resource-constrained environment.

This study also reveals new implications of Salience Theory and highlights differences to related decision theories such as Expected Utility Theory and Cumulative Prospect Theory. One important peculiarity of Salience Theory is that the probability-weighting mechanism can have a significantly stronger impact on the decision maker's choice than the utility concept (i.e. the shape of the value function). In particular, we show that a probability-distortion parameter $\delta \leq 0.5$ can fully override the impact of a value function's curvature in this setting. Under Salience theory, there can be both a trade-off between probability distortion and value function as well as the value function's impact being muted.

B Appendix

In the following, we derive the if-and-only-if conditions for a finite certainty equivalent for the St. Petersburg lottery L_{STP} listed in Table 3.1. Throughout this section, we follow Proposition 3.1 and assume the necessary condition $\delta > 0.5$. Then, Equation (3.4) simplifies to

$$V^{LT}(L_{STP}) = (2\delta - 1)\left[\lim_{N\to\infty} \frac{v\left(2^N\right)}{(2\delta)^N} + \sum_{k=1}^{\infty} \left(\frac{1}{2\delta}\right)^k \cdot v\left(2^k\right)\right]. \tag{B.1}$$

We skip the case of the linear value function $v(x) = x$ and start with the power value function $v(x) = x^\alpha$ since it includes linear utility as a special case ($\alpha = 1$).

1. Power value function $v(x) = x^\alpha$, $\alpha > 0$:

$$V^{LT}(L_{STP}) = (2\delta - 1)\left[\lim_{N\to\infty} \frac{\left(2^N\right)^\alpha}{(2\delta)^N} + \sum_{k=1}^{\infty} \left(\frac{1}{2\delta}\right)^k \cdot \left(2^k\right)^\alpha\right] \tag{B.2}$$

$$= (2\delta - 1)\left[\lim_{N\to\infty} \left(\frac{1}{\delta 2^{1-\alpha}}\right)^N + \sum_{k=1}^{\infty} \left(\frac{1}{\delta 2^{1-\alpha}}\right)^k\right] \tag{B.3}$$

$$= \begin{cases} \infty & \text{if } \delta 2^{1-\alpha} \le 1 \\ \frac{2\delta-1}{\delta 2^{1-\alpha}-1} & \text{if } \delta 2^{1-\alpha} > 1 \end{cases}. \tag{B.4}$$

Series (B.2) converges if and only if $\delta 2^{1-\alpha} > 1$ because $\sum_{k=1}^{\infty} \left(\frac{1}{\delta 2^{1-\alpha}}\right)^k$ is a geometric series. For $\alpha = 1$, which corresponds to a linear value function $v(x) = x$, $\delta > 1$ is then the iff-condition for a finite certainty equivalent for L_{STP}.

2. Logarithmic value function $v(x) = \log(1 + \alpha x)$, $\alpha > 0$:

$$V^{LT}(L_{STP}) = (2\delta - 1)\left[\lim_{N\to\infty} \frac{\log\left(1 + \alpha 2^N\right)}{(2\delta)^N} + \sum_{k=1}^{\infty}\left(\frac{1}{2\delta}\right)^k \cdot \log\left(1 + \alpha 2^k\right)\right] \tag{B.5}$$

$$= (2\delta - 1)\cdot \sum_{k=1}^{\infty}\left(\frac{1}{2\delta}\right)^k \cdot \log\left(1 + \alpha 2^k\right) < \infty \quad \forall \alpha > 0. \tag{B.6}$$

Our general assumption $\delta > 0.5$ and the ratio test imply that (B.5) converges if and only if $\delta > 0.5$ for any $\alpha > 0$:

$$\lim_{k\to\infty}\left|\frac{\left(\frac{1}{2\delta}\right)^{k+1} \cdot \log\left(1 + \alpha 2^{k+1}\right)}{\left(\frac{1}{2\delta}\right)^{k} \cdot \log\left(1 + \alpha 2^{k}\right)}\right| = \frac{1}{2\delta} \cdot \lim_{k\to\infty} \frac{\log\left(1 + \alpha 2^{k+1}\right)}{\log\left(1 + \alpha 2^{k}\right)} = \frac{1}{2\delta} = r < 1. \tag{B.7}$$

3. Exponential value function $v(x) = 1 - e^{-\alpha x}$, $\alpha > 0$:

$$V^{LT}(L_{STP}) = (2\delta - 1)\left[\lim_{N\to\infty} \frac{1 - e^{-\alpha 2^N}}{(2\delta)^N} + \sum_{k=1}^{\infty}\left(\frac{1}{2\delta}\right)^k \cdot \left(1 - e^{-\alpha 2^k}\right)\right] \tag{B.8}$$

$$= (2\delta - 1)\cdot \sum_{k=1}^{\infty}\left(\frac{1}{2\delta}\right)^k \cdot \left(1 - e^{-\alpha 2^k}\right) \tag{B.9}$$

$$< (2\delta - 1)\cdot \sum_{k=1}^{\infty}\left(\frac{1}{2\delta}\right)^k = 1 \quad \forall \alpha > 0. \tag{B.10}$$

The certainty equivalent $v^{-1}\left(V^{LT}(L_{STP})\right)$ is finite if and only if $\delta > 0.5$ for any given $\alpha > 0$ since $V^{LT}(L_{STP}) < v(\infty) = 1$ iff $\delta > 0.5$.

4. HARA value function $v(x) = \frac{1-\alpha}{\alpha}\left[\left(\frac{x}{1-\alpha}+\beta\right)^{\alpha}-\beta^{\alpha}\right]$**,** $\alpha < 0$**,** $\beta > 0$**:**

$$\begin{aligned} V^{LT}(L_{STP}) = (2\delta-1)\cdot &\left[\lim_{N\to\infty}\frac{\frac{1-\alpha}{\alpha}\left[\left(\frac{2^N}{1-\alpha}+\beta\right)^{\alpha}-\beta^{\alpha}\right]}{(2\delta)^N}\right. \\ &\left.+\sum_{k=1}^{\infty}\left(\frac{1}{2\delta}\right)^{k}\cdot\frac{1-\alpha}{\alpha}\left[\left(\frac{2^k}{1-\alpha}+\beta\right)^{\alpha}-\beta^{\alpha}\right]\right] \end{aligned} \tag{B.11}$$

$$= \frac{(\alpha-1)(2\delta-1)}{\alpha}\cdot\sum_{k=1}^{\infty}\left(\frac{1}{2\delta}\right)^{k}\cdot\left[\beta^{\alpha}-\left(\frac{2^k}{1-\alpha}+\beta\right)^{\alpha}\right] \tag{B.12}$$

$$< \frac{(\alpha-1)(2\delta-1)}{\alpha}\cdot\sum_{k=1}^{\infty}\left(\frac{1}{2\delta}\right)^{k}\cdot\beta^{\alpha} \tag{B.13}$$

$$= \frac{\alpha-1}{\alpha}\beta^{\alpha} = v(\infty) \quad \forall\alpha<0,\beta>0. \tag{B.14}$$

The certainty equivalent $v^{-1}\left(V^{LT}(L_{STP})\right)$ is finite if and only if $\delta > 0.5$ for any given $\alpha < 0$ and $\beta > 0$ because $V^{LT}(L_{STP}) < v(\infty) = \frac{\alpha-1}{\alpha}\beta^{\alpha}$ iff $\delta > 0.5$.

Chapter 4

Indistinguishability of Small Probabilities, Subproportionality, and the Common Ratio Effect

This chapter is based on the Article "Indistinguishability of small probabilities, subproportionality, and the common ratio effect" authored by Maik Dierkes and Vulnet Sejdiu, *Journal of Mathematical Psychology, 93*, 2019.

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Chapter 5

Probability Distortion, Asset Prices and Economic Growth

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