Generalised Convex Geometries with Applications to Systems of Subsemilattices

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Referent: Prof. Dr. M. Erné
Korreferent: Prof. RNDr. Ales Pultr
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Abstract

Keywords: Convex geometry, Anti-exchange property, Subsemilattices

In this work we develop a theory for the study of infinite convex geometries, which allows us to analyse systems of convex sets on an infinite set. Here, the term “convex” is used in a very general sense, but still close to what would intuitively be considered “geometric”, as opposed to e.g. the notion of convexity used by M. van de Vel. In this process we look at a large number of properties for closure systems as well as properties in lattice theory, and we probe them for their application to a general description of convexity. This yields a whole net of different notions for convexity of varying strength, which we document with numerous examples that separate systems with these properties.

Some of these descriptions of abstract convexity are analysed more thoroughly, especially those which cover important systems like convex subsets of $\mathbb{R}^n$, convex bodies of $\mathbb{R}^n$ or intervals of subsemilattices. For closure systems with these particular properties, we develop explicit lattice-theoretical descriptions.

This theory makes it possible to study abstract convexity without a restriction to finite sets, or the still very restrictive algebraic closure systems R. Jamison-Waldner proposed, and it covers many classical examples. These are not only classified and ordered in a large diagram, we also get general results concerning important structural properties like (semi-)distributivity or decompositions.

In the next chapters we apply the previously developed theory to systems of (meet-) subsemilattices. In this we mainly focus on infinite meets. These systems are rarely algebraic, but they satisfy some of the strongest properties of the general convexity theory. We therefore get a first idea of the structure of systems of subsemilattices. We obtain further insight into these systems through additional considerations and combinatorial constructions, including a rather surprising characterisation of algebraic intervals of subsemilattices.

The consultation of other theories for a generalisation of convexity gives mixed results. Properties which have their origin in geometric observations in $\mathbb{R}^n$ are rarely satisfied by systems of subsemilattices. Nevertheless we can quite often deduce additional properties of these systems, which give further building blocks for a characterisation of intervals of subsemilattices or more general structures.

We close with some first steps into the theory on convex invariants, like e.g. Radon or Carathéodory numbers and their corresponding notions of dependence, applied to systems of subsemilattices. We obtain some interesting results, headed by the fact that these numbers do not coincide as in the standard convex geometry of $\mathbb{R}^n$ but can differ quite strongly from each other.

Zusammenfassung

Schlagwörter: Konvexe Geometrie, Anti-Austausch-Eigenschaft, Unterhalbverbände

In dieser Arbeit wird eine Theorie zur Behandlung unendlicher konvexer Geometrien entwickelt, wobei hier der Begriff „konvex" sehr weit gefasst aber doch intuitiv „geometrischer" als bei M. van de Vel ist. Im Zuge dieser Entwicklung werden eine große Anzahl von hüllen- und verbandstheoretischen Eigenschaften analysiert und bezüglich ihrer Anwendung für die Beschreibung von Konvexität untersucht. Das Ergebnis ist ein ganzes Netz verschiedener Konvexitätsbegriffe verschiedenster Stärke, was durch zahlreiche Trennbeispiele belegt wird.

Einzelne Begriffe aus diesem Netz werden intensiver betrachtet. Hierbei konzentrieren wir uns auf solche, die auf wichtige Systeme wie konvexe Teilmengen des $\mathbb{R}^n$, konvexe und kompakte Teilmengen des $\mathbb{R}^n$ oder Intervalle von Unterhalbverbänden zutreffen. Dabei werden für Hüllensysteme mit diesen Eigenschaften verbandstheoretische Charakterisierungen entwickelt.

Diese Theorie ermöglicht die Betrachtung abstrakter Konvexität ohne eine Beschränkung auf endliche oder algebraische Hüllensysteme wie bei R. Jamison-Waldner, und umfasst viele klassische Beispiele. Diese werden nicht nur in ein großes Diagramm von Abstufungen von Konvexität eingeordnet, es werden auch allgemeine Ergebnisse über wichtige Struktureigenschaften wie (Semi-)Distributivität oder Zerlegungen erarbeitet.

In den darauf folgenden Kapiteln wird die vorher entwickelte Theorie benutzt, um Systeme von (Infimum-)Unterhalbverbänden zu analysieren, wobei hier das Hauptaugenmerk auf unendlichen Infima liegt. Diese Systeme sind in der Regel nicht algebraisch, haben aber einige der stärksten Eigenschaften der allgemeinen Theorie. Dadurch ergibt sich bereits ein erstes Bild über die Struktur von Systemen von Unterhalbverbänden. Durch weitergehende Betrachtungen und kombinatorische Konstruktionen gelangen wir zu weiteren Ergebnissen, unter anderem zu einer relativ überraschenden expliziten Charakterisierung der algebraischen Intervalle von Unterhalbverbänden.

Die Betrachtung weiterer Theorien zur Verallgemeinerung von Konvexität liefert gemischte Ergebnisse. Eigenschaften, die hauptsächlich auf geometrische Beobachtungen im $\mathbb{R}^n$ zurückgehen, werden in den seltensten Fällen von Intervallen von Unterhalbverbänden erfüllt. Trotzdem können hieraus häufig neue Erkenntnisse über diese Systeme abgeleitet werden, die wieder einen kleinen Baustein zur Beschreibungen von Intervallen von Unterhalbverbänden oder allgemeineren Strukturen beisteuern.

Die Arbeit endet mit einem Einstieg in die Theorie der konvexen Invarianten, wie z.B. Radon- oder Carathéodory-Zahlen, und der damit verbundenen Abhängigkeitsbegriffe, angewandt auf Intervalle von Unterhalbveränden. Einige Ergebnisse sind sehr überraschend, angeführt von der Tatsache, dass diese Zahlen hier nicht wie in der reellen konvexen Geometrie zusammenfallen, sondern völlig verschieden voneinander sein können.

Introduction

The analysis and discussion of abstract properties of systems of substructures like subsemilattices of a semilattice, convex subsets of a poset or other objects has a long history, see e.g. [8] or [1]. For this work we have looked at quite a number of contributions to this theme, but the original motivation was a relatively young publication. In [26] the authors used for the first time not the complete system of all subsemilattices of a semilattice, but intervals therein. These intervals were then put on a more fundamental basis in [24]. These articles aroused our interest in more general approaches to systems of substructures, in particular in those which could be applied to intervals of subsemilattices.

At first glance, the joint work of P. Edelman and R. Jamison-Waldner to develop an abstract theory of convex geometries in [18], which covers many known classes of convex subsystems, deals with this problem comprehensively, were it not for the small remark in the introduction that all sets considered are finite. This is a very strong restriction and led directly to the question how much of their general convexity theory for finite sets is still true if we allow infinite sets. A slightly more general approach was used by R. Jamison-Waldner in [35], in which he restricts considerations to algebraic closure systems. Most results from the finite theory remain true for these systems, but it is still very restrictive and does not cover intervals of subsemilattices, which are not algebraic in general. Incidentally, we are not the first to ask for an infinite convexity theory. In [3] the authors try to start a discussion of this topic, but they use an approach different to the one we use. We hope that this work helps to start or adds to the requested discussion and instigates further research in this area.

First steps into a theory for infinite convexity that covers intervals of subsemilattices were taken in [22] and are continued here in chapter 2, after we establish some notation and basics in chapter 1. In [18], the main focus is on the anti-exchange property, which in a finite setting is equivalent to a number of other properties. If we allow infinite sets, these equivalences become a whole net of implications between properties which all make sense in the discussion of convexity theory. We accentuate some of these properties or combinations thereof. This is accompanied by a long list of examples which distinguishes the various combinations. This includes systems like convex subsets of n-dimensional euclidean space $\mathbb{E}^n$ or intervals of subsemilattices. By doing so it becomes obvious that we cannot give **the** property to characterise an infinite convexity theory, but a list of possible or suitable characterisations.

In Chapter 2 we mainly consider closure systems with additional properties as models for convex geometries, but we also look at these systems as complete lattices and try to characterise the occuring systems lattice-theoretically. In addition to this, we deviate from the properties considered in [18] and related papers and discuss additional ones in the context of abstract convexity, some of which are novel or new variations of established properties.

Chapter 3 deals exclusively with intervals of subsemilattices, but in a much larger generality than has been done before. We cover a large spectrum of aspects, starting with a basic definition and the question to what extent these intervals can be considered as general convex geometries in the terms of Chapter 2. We continue with properties specific to intervals of subsemilattices, and we discuss structural properties like algebraicity, form and existence of irreducible elements and the related question on existence of decompositions. The oberservations and results of this part give us a much better insight into the class of intervals of subsemilattices. We apply some of these insights to construction problems connected to our intervals, i.e. the questions of how to algorithmically construct (finite) intervals of subsemilattice from a given semilattice, or the much more difficult and as of yet not completely solved problem of how to reconstruct a semilattice from a given closure system or complete lattice.

The approach to a general convexity theory as it is given here, based mainly on [18] and related papers, is only one of a number of different approaches to abstract convexity. To reflect this fact, we look at a small number of alternative approaches in Chapter 4, with special interest to the question inasfar intervals of subsemilattices are admissible objects in the theories that arise. It does not come as a big surprise that we get mixed results by doing so. “Convex Geometry” is a very general term and, as can be seen from books like [32], it covers a multitude of mathematical disciplines. Even in the relatively few alternative approaches we discuss here, most of which are even not too different in their origins to the one we chose as central basis, we can see many different behaviours, and intervals of subsemilattices rarely fit into these theories.

We close this work with a few steps into a discussion of classical convex invariants like Radon or Carathéodory numbers for intervals of subsemilattices. We make a number of observations and get some first results. Two of the most important observation are probably that these classical numbers are all well-defined for intervals of subsemilattices, a fact which is not trivial, and that these numbers are far from equal, the related theories differ quite strongly. All in all, some more thorough analysis of these aspects might be required for a better understanding of intervals of subsemilattices.

We can say that for the study of general intervals of subsemilattices, the general convexity theory which we discuss in Chapter 2 is a solid basis, but for a more detailed and elaborated analysis we have to use additional angles and approaches. Some are given here, others might be discovered or applied in the future.

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Chapter 1

Order and Lattice Theory

1.1 Relations

1.1.1 Definition

The set

$$X \times Y := \{(x, y) \mid x \in X,\, y \in Y\}$$

is the **cartesian product** of X and Y.

Every subset $R \subseteq X \times Y$ is called a **relation**. It is called a **relation on X** if $X = Y$. We write xRy for $(x, y) \in R$.

The set $1_X = \Delta_X = \mathrm{id}_X = \{(x, x) \mid x \in X\}$ is the **identity on X**, or the **diagonal in** $\mathbf{X} \times \mathbf{X}$.

A set X with a relation R, written as (X, R), is called a **relative**.

All relations and relatives we discuss in this work have some well-known and established properties. We give a short list of the most important ones.

1.1.2 Definition

A relation R on X is called

reflexive, if xRx,

transitive, if xRy and $yRz \Rightarrow xRz$,

symmetric, if $xRy \Rightarrow yRx$,

antisymmetric, if xRy and $yRx \Rightarrow x = y$,

total, if xRy or yRx,

irreflexive, if $(x, x) \notin R$,

for all $x, y, z \in X$.

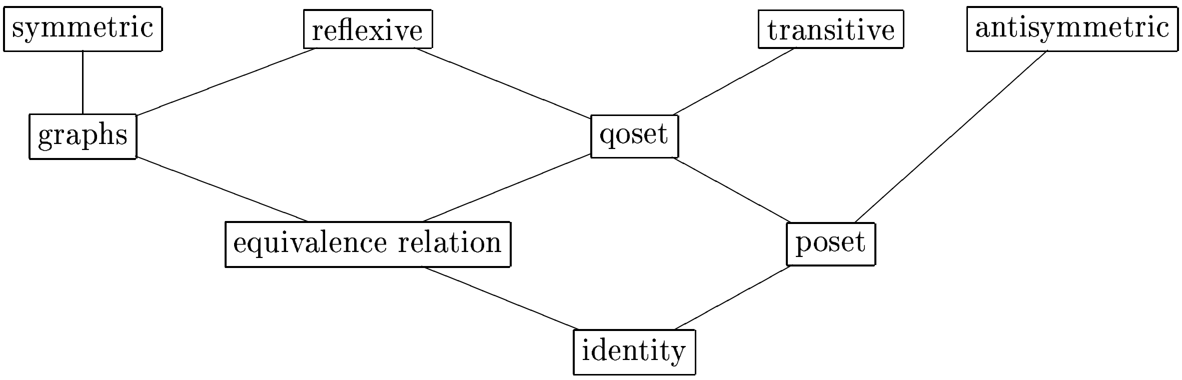
These are the building blocks for some very interesting classes of objects, which we obtain by combining the above-mentioned properties.

1.1.3 Definition

A reflexive and transitive relation is called a **quasi-order**,
a reflexive, transitive and antisymmetric relation is called a **(partial) order**,
a reflexive, transitive and symmetric relation is called an **equivalence relation**.
A total order is also called a **chain**.
An ordered set, in which every nonempty subset has a smallest element (i.e. for every $Y \subseteq X$ there exists a $y \in Y$ such that yRz for all $z \in Y$) is called a **well-ordered set**.

A set with a quasi-order is called a **qoset**, a set with a partial order is called a **poset**.

1.1.4 Diagram

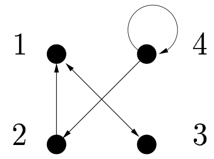


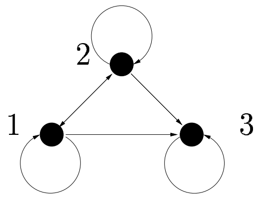
1.1.5 Examples

We give a first small list of relatives with different combinations of properties.

i) The relation $R = \{(1,3), (3,1), (2,1), (4,2), (4,4)\}$ has none of the properties introduced in 1.1.2.

ii) The relation $R = \{(1,1), (2,2), (3,3), (1,2), (2,1), (1,3), (2,3)\}$ is a quasi-order, but neither symmetric nor anti-symmetric.





iii) The relative $(\mathbb{N}, =)$ of the natural numbers with identity is a set with an equivalence relation.

iv) The relative $(\mathbb{Z}, \leq)$ with the relation "lower or equal" is a totally ordered set.

v) The relative $(\mathbb{N}, \leq)$ with the same relation is a well-ordered set.

vi) The relative $(\mathbb{N}, |\,)$ of the natural numbers ordered by divisibility is a partially ordered set.

vii) The relative $(\mathbb{Z}, |\,)$ of the integers ordered by divisibility is only a quasi-ordered set.

1.1.6 Definition

The **cover relation** $\check{R}$ of a relation R is defined by

$$x\check{R}y :\iff xRy,\ x \neq y \text{ and } \forall z \in X : xRzRy \Rightarrow z \in \{x, y\}.$$

If this relation holds, x is called **lower neighbour** or **lower cover** of y,
y is called **upper neighbour** or **upper cover** of x.

We will use the symbol $\prec$ for the cover relation, if there is only one relation involved.

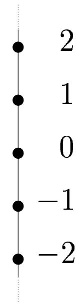
1.1.7 Example

In $(\mathbb{N}, \leq)$, we have $1 \prec 2$, but in $(\mathbb{R}, \leq)$, $1 \not\prec 2$ as $1 \leq \frac{3}{2} \leq 2$.

In the following, we will usually consider an ordered set X with the relation $\leq$. The restriction to ordered sets plus a few conventions allow us to draw diagrams of ordered sets, sometimes even for infinite sets. Since every element is in relation to itself ($x \leq x$), we do not need to draw loops. If x is a lower cover of y, we draw x below y and connect them by a line. All other relations can be deduced from the covering pairs, using transitivity.

1.1.8 Example

With this set of rules we can give a diagram for $(\mathbb{Z}, \leq)$:



1.1.9 Definition

Consider an ordered set $(X, \leq)$, $Y \subseteq X$ and $x \in X$. We call

$$\uparrow Y := \{z \in X \mid \exists y \in Y : y \leq z\}$$

the **upper set of Y**. Similarly, we define the **upper set of x** by

$$\uparrow x := \{z \in X \mid x \leq z\}.$$

The **lower sets** $\downarrow Y$ and $\downarrow x$ are defined dually.

1.1.10 Definition

Consider an ordered set $(X, \leq)$, and $Y \subseteq X$. We call $x \in X$

maximum or **largest element** of Y	$:\Longleftrightarrow$	$x \in Y$ and $Y \subseteq {\downarrow} x$,
minimum or **least element** of Y	$:\Longleftrightarrow$	$x \in Y$ and $Y \subseteq {\uparrow} x$,
upper bound of Y	$:\Longleftrightarrow$	$Y \subseteq {\downarrow} x$,
lower bound of Y	$:\Longleftrightarrow$	$Y \subseteq {\uparrow} x$,
maximal in Y	$:\Longleftrightarrow$	$(\uparrow x) \cap Y = \{x\}$,
minimal in Y	$:\Longleftrightarrow$	$(\downarrow x) \cap Y = \{x\}$,
supremum or **join** of Y	$:\Longleftrightarrow$	x is least upper bound of Y
	$\Longleftrightarrow$	$Y \subseteq {\downarrow} x$ and $(Y \subseteq {\downarrow} z \Rightarrow x \leq z)$,
infimum or **meet** of Y	$:\Longleftrightarrow$	x is largest lower bound of Y
	$\Longleftrightarrow$	$Y \subseteq {\uparrow} x$ and $(Y \subseteq {\uparrow} z \Rightarrow z \leq x)$.

1.1.11 Notation

If the ordered set X has a maximum, we will denote this by $\top_X$ or $\top$. Similarly, a minimum of X will be denoted by $\bot_X$ or $\bot$.

The supremum of $Y \subseteq X$ will be denoted by $\bigvee Y$, if it exists. Likewise, the infimum of Y will be denoted by $\bigwedge Y$. For two-element sets $\{x, y\}$ we write $x \vee y$ for $\bigvee\{x, y\}$, and $x \wedge y$ for $\bigwedge\{x, y\}$. This is well-defined, since suprema and infima are unique, if they exist.

1.1.12 Definition

Consider a partially ordered set X, and elements $u, v \in X$ with $u \leq v$. The set

$$A = \{x \in X \mid u \leq x \leq v\}$$

is called the **interval** generated by u and v, written as $[u, v]$.

1.1.13 Examples

i) In $[0, 1]$ with the relation $\leq$, we have $\top = 1$ and $\bot = 0$. For an arbitrary $a \in [0, 1]$ we have $\downarrow a = [0, a]$ and $\uparrow a = [a, 1]$.

ii) In the power set $\mathcal{P}M$ of a non-empty set M, $\top = M$ and $\bot = \emptyset$.

iii) In the set $T_0 := (\mathbb{N}_0, |\,)$ of all non-negative integers $\mathbb{N}_0$ ordered by divisibility, $\top = 0$ and $\bot = 1$.

1.2 Ordinals and Cardinals

We will give a short introduction to ordinal and cardinal numbers. The following paragraphs are mainly based on [5].

1.2.1 Definition

An **ordinal** is a set n such that

i) If $m \in n$ then $m \subset n$.

ii) $(n, \underline{\in})$ is a well-ordered set (where $m \underline{\in} k :\iff m \in k$ or $m = k$).

The class of all ordinals is denoted by $\mathcal{O}$. If we transfer the set-theoretical definition of a well-order to class theory, $\mathcal{O}$ is a "well-ordered" class. In particular, we can write

$$n \leq m \text{ whenever } n \subseteq m, \text{ for } n, m \in \mathcal{O}.$$

1.2.2 Examples

We start with the finite ordinals:

i) $\mathbf{0} = \emptyset$,

ii) $\mathbf{1} = \{\mathbf{0}\} = \{\emptyset\}$,

iii) $\mathbf{2} = \{\mathbf{0}, \mathbf{1}\} = \{\emptyset, \{\emptyset\}\}$.

In general, for an integer $n \geq 1$,

iv) $\mathbf{n} = \{\mathbf{0}, \mathbf{1}, ..., \mathbf{n-1}\}$.

The smallest infinite ordinal is

v) $\omega = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, ...\}$

1.2.3 Definition

A **limit ordinal** is an ordinal n which has the property that

$$n = \bigcup \{m \in \mathcal{O} : m < n\}.$$

An equivalent description of a limit ordinal n is, that for each $m < n$ there exists an ordinal k such that $m < k < n$.

An ordinal n that is not a limit ordinal is called **successor ordinal**.

1.2.4 Definition

A **cardinal** is an ordinal n with the property that there is no one-to-one correspondence between n and any ordinal m such that $m < n$.

1.2.5 Examples

i) All finite ordinals are cardinals, as is ω.

ii) The successor of ω, which can be written as

$$\omega + 1 := \omega \cup \{\omega\},$$

is not a cardinal.

1.2.6 Definition

Given a cardinal κ, we write $X \subseteq_\kappa Y$, if $X \subseteq Y$ and contains strictly less than κ elements. For a finite subset X of Y this is written as $X \subseteq_\omega Y$, since ω is the smallest infinite cardinal. All sets with less than ω elements are therefore finite.

A cardinal is called **regular**, if from $\mathcal{X} \subseteq_\kappa \mathcal{P}Y$ and $X \subseteq_\kappa Y$ for each $X \in \mathcal{X}$ it follows that $\bigcup \mathcal{X} \subseteq_\kappa Y$.

A cardinal which is not regular is called **singular**.

1.2.7 Examples

i) 2 is a regular cardinal since the union of less than two sets with less than two elements contains less than two elements. Similarly, 0 and 1 are also regular cardinals.

ii) However, 3 and every larger finite cardinal is singular. E.g., two disjoint sets with two elements each have a union with four elements.

iii) ω is again a regular cardinal. Less than ω elements means a finite number of elements, and the union of a finite number of finite sets contains only a finite number of elements.

iv) For examples of infinite singular cardinals, see e.g. [33].

1.3 Semilattices and Lattices

Semilattices and lattices are among the main objects which we consider and use in this work. The following section introduces only the main notions and gives some very basic introduction. For a deeper theory and most proofs see e.g. [5, 10, 20].

1.3.1 Definition

A $\wedge$-**semilattice** is a poset S, in which any two elements $x, y \in S$ have an infimum. A poset, in which any two elements always have a supremum is called

$\vee$-**semilattice**.

The existence of binary meets (or joins) ensures the existence of meets (resp. joins) for all non-empty finite subsets of S.

1.3.2 Definition

A **semilattice** $(S, \diamond)$ or **semilattice algebra** is a commutative semigroup, in which every element is idempotent, i.e. $x \diamond x = x$.

1.3.3 Theorem

i) Every $\wedge$-semilattice ($\vee$-semilattice) $(S, \leq)$ is a semilattice with respect to the binary infimum (supremum), and we have

$$x \leq y \iff x \wedge y = x \ (\iff x \vee y = y).$$

ii) Every semilattice $(S, \diamond)$ is a $\wedge$-semilattice with respect to the order

$$x \leq y :\iff x \diamond y = x.$$

iii) Every semilattice $(S, \diamond)$ is a $\vee$-semilattice with respect to the order

$$x \leq y :\iff x \diamond y = y.$$

iv) There exists a bijection between semilattices, $\wedge$-semilattices and $\vee$-semilattices.

Proof: see e.g. [20]

1.3.4 Definition

A **κ-($\bigwedge$-)semilattice** is a poset S, in which every κ-subset $X \subseteq_\kappa S$ has a meet in S, i.e.

$$X \subseteq_\kappa S \Rightarrow \bigwedge X \text{ exists in } S.$$

We call S a **(∞-)$\bigwedge$-semilattice**, if every subset $X \subseteq S$ has a meet in S. Note that $X \subseteq_\infty S \iff X \subseteq S$.

κ-$\bigvee$-semilattices are defined dually.

From now on, κ is either an infinite cardinal or ∞.

The definition of κ-semilattices forces the existence of a top element of S for every κ-semilattice, since $\bigwedge \emptyset = \top_S$.

Hence, an ω-$\bigwedge$-semilattice is a $\wedge$-semilattice with a top element.

1.3.5 Definition

A **lattice** is a $\wedge$- and $\vee$-semilattice. A **lattice algebra** is a set with two binary operations $\triangledown$ and $\triangle$, which are associative, commutative and connected via the absorption laws

$$x \triangle (x \triangledown y) = x, \quad x \triangledown (x \triangle y) = x.$$

1.3.6 Theorem

$(X, \leq) \mapsto (X, \vee, \wedge)$ is a bijection between lattices and lattice algebras. We have

$$x \vee y = y \iff x \wedge y = x$$

for every $x, y \in X$.

Conversely, if $(X, \vee)$ and $(X, \wedge)$ are semilattices which satisfy this equation, then $(X, \vee, \wedge)$ is a lattice.

Proof: see [20]

1.3.7 Definition

A lattice L is called **complete**, if every subset $K \subseteq L$ has a meet and a join.

1.3.8 Theorem

For a poset X, the following properties are equivalent:

i) X is a complete lattice.

ii) Every subset of X has a meet.

iii) Every subset of X has a join.

iv) $X \neq \emptyset$, and every non-empty subset of X has a meet and a join.

Proof: see [20]

1.3.9 Remark

Note that a $\bigwedge$-semilattice S is a complete lattice. The supremum of a set $K \subseteq S$ is given by

$$\bigvee K = \bigwedge K^{\uparrow},$$

where $K^{\uparrow} = \{x \in S \mid \forall\, k \in K : k \leq x\}$.

1.4 Atomicity

1.4.1 Definition

In a lattice L with a bottom element $\bot$, the upper covers of $\bot$ are called **atoms**.

A lattice L is called **atomic**, if for every $x \in L \setminus \{\bot\}$ there exists an atom $a \in L$ such that $a \leq x$.

L is called **atomistic**, if every element $x \in L$ is a join of atoms.

L is called **weakly atomic**, if every interval $[a, b]$ for $a, b \in L$ with $a < b$ contains a covering pair $u \prec v$, i.e. $a \leq u \prec v \leq b$.

L is called **strongly atomic**, if every interval $[a, b]$ is atomic.

These notions can all be dualised. The lower covers of $\top$ are called **coatoms**, therfore the duals of the other properties are **coatomic**, **coatomistic** and **strongly coatomic**.

The notion of weak atomicity need not be dualised, as it is self-dual.

1.4.2 Remark

A strongly atomic lattice with a bottom element is atomic, every strongly atomic lattice is weakly atomic, and every atomistic lattice is atomic, but no other implication between these properties is true in general.

1.4.3 Example

i) The power set of a set X, ordered by inclusion, is atomistic and strongly atomic. The atoms are all singletons $\{x\}$ for $x \in X$, and every subset $Y \subseteq X$ is the union of all singletons contained in Y.

ii) The lattice $(\mathbb{N}, |)$ is strongly atomic, but not atomistic. The atoms are the prime numbers, and every natural number has at least one prime factor, but 4 is not a join of atoms, i.e. prime numbers.

iii) The integers $(\mathbb{Z}, \leq)$ are strongly atomic, but not atomic, since there are no atoms.

iv) The lattice of all cofinite subsets of $\mathbb{N}$ with $\mathbb{N}$ as top element is weakly atomic and strongly coatomic, but neither atomic nor strongly atomic, as atoms do not exist.

1.5 Distributivity and Modularity

Modular and distributive lattices have been the object of interest of mathematicians for quite some time. The first steps were made in the second half of the 19th century by Dedekind [12] and Schröder [45], Birkhoff contributed heavily to this area in the thirties and forties (see e.g. [10]) and scores of papers and books dealing with distributivity or some variation thereof have been published since.

In this section, we list a number of variations of the distributive or modular law, but we will restrict us to those which we will later consider in connection with convex geometries.

1.5.1 Lemma

In a lattice L with elements $x, y, z \in L$, the following identities are equivalent:

i) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$,

ii) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$,

iii) $(x \wedge y) \vee (x \wedge z) \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \wedge (y \vee z)$.

Proof: see e.g. [13]

1.5.2 Definition

A lattice is called **distributive**, if it satisfies one, and therefore all identities given in 1.5.1.

1.5.3 Definition

A lattice is called **modular**, if it satisfies

$$x \leq z \Rightarrow x \vee (y \wedge z) = (x \vee y) \wedge z.$$

1.5.4 Examples

Typical separating examples for distributive and modular lattices are the next two non-distributive lattices with five elements.

i) We start with the lattice M_3, which is not distributive, but modular.

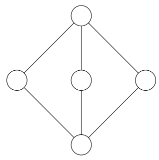


Figure 1.1: M_3

ii) The second example N_5 is neither distributive nor modular.

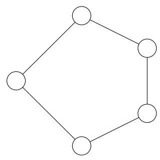


Figure 1.2: N_5

The reason for these two (counter-)examples is simple. Both modular and distributive lattices can be characterised using these two as forbidden sublattices.

1.5.5 Theorem

A lattice L is non-modular if and only if it contains a sublattice isomorphic to Figure 1.1.

Proof: For first appearance see [13]

1.5.6 Theorem

A lattice L is non-distributive if and only if it contains a sublattice isomorphic to Figure 1.1 or Figure 1.2.

Proof: First proved by Birkhoff, see [10]

1.5.7 Definition

A complete lattice L is called a **frame**, if it satisfies the following infinite distributive law

$$a \wedge \bigvee B = \bigvee (a \wedge B)$$

for all $a \in L$ and $B \subseteq L$.

We use $\bigvee (a \wedge B)$ as an an abbreviation for $\bigvee \{a \wedge b \,|\, b \in B\}$.

A **coframe** or **dual frame** is defined dually. The term "frame" was introduced around 1960, for some early publications see [15] and the articles cited there.

1.5.8 Definition

We consider a poset X. We call a non-empty set $D \subseteq X$ **directed**, if for every $x, y \in D$ there exists a $z \in D$ such that $x \leq z$ and $y \leq z$. In other words, non-empty finite sets have an upper bound in D.

1.5.9 Definition

A complete lattice L is called **meet-continuous**, if it satisfies

$$a \wedge \bigvee D = \bigvee (a \wedge D)$$

for every $a \in L$ and every directed subset $D \subseteq L$.

Note that the condition of directed subsets can be replaced by chains, without weakening the result. For a proof, see e.g. [14].

L is called **join-continuous**, if its dual is join-continuous. One of the first appearances of the name "meet-continuous lattice" was in the third edition of [10].

If in a complete lattice the meet is compatible with finite joins, i.e. distributive, and with directed joins, i.e. meet-continous, the meet is already compatible with arbitrary joins.

1.5.10 Theorem

For a complete lattice L, the following conditions are equivalent:

i) L is a frame,

ii) L is meet-continuous and distributive.

Proof: See [30]

At one point a little bit later in this section we will make use of a weak form of meet-continuity. This property was introduced by V. Diercks in [14].

1.5.11 Definition

A complete lattice L is called **weakly meet-continuous**, if for $a, b \in L$ such that $b \prec a$ and for every directed subset $D \subseteq \uparrow b$ we have

$$a \wedge \bigvee D = \bigvee (a \wedge D).$$

Weakly join-continuous lattices are defined dually.

Distributivity and modularity are very nice properties, but there are many lattices, which do not have these properties. However, quite a few lattices have properties which have distributive-like qualities.

This led to the introduction of weak forms of distributivity and modularity, some of which will be of interest to us and which we will therefore list here.

1.5.12 Definition

Consider a lattice L. We call L **meet-semidistributive**, if for all $a, b, c \in L$

$$a \wedge b = a \wedge c \Rightarrow a \wedge b = a \wedge (b \vee c).$$

A complete lattice is **completely meet-semidistributive**, if every subset $B \subseteq L$ and every element $a \in L$ satisfy

$$(\forall\, b \in B:\ x = a \wedge b) \Rightarrow x = a \wedge \bigvee B.$$

The properties of **join-semidistributivity** and of **complete join-semidistributivity** are defined dually. An early application of semidistributivity was by B. Jónsson in [37].

1.5.13 Remark

A distributive lattice is both meet- and join-semidistributive, but the converse is not true in general, not even for finite lattices.

Similarly, a frame is completely meet-semidistributive, but the converse is not true in general.

1.5.14 Example

The (complete) lattice N_5 shown in Figure 1.2 is not distributive, but both (completely) join- and meet-semidistributive.

1.5.15 Definition

A lattice L is called **lower semimodular**, if for all $a, b \in L$

$$a \prec a \vee b \Rightarrow a \wedge b \prec b.$$

Upper semimodularity is defined dually. Semimodularity is one of the oldest variations of modularity, Birkhoff writes in [10] that he was apparently the first who considered semimodular lattices in [9].

1.5.16 Remark

A modular lattice is both upper and lower semimodular, but the converse need not be true. For special cases however, upper and lower semimodularity together suffice to show modularity. The most general theorem in which this has been dealt with and of which I am aware of is by J. Reinhold.

1.5.17 Theorem

A strongly atomic $\wedge$-continuous lattice L is modular if and only if it is upper and lower semimodular.

Proof: See [43]

Special cases of strongly atomic $\wedge$-continuous lattices are lattices, in which every chain is finite, and certainly all finite lattices.

1.5.18 Example

The lattice S_7 is upper semimodular, but not lower semimodular and therefore not modular. The dual lattice S_7^* is lower but not upper semimodular.

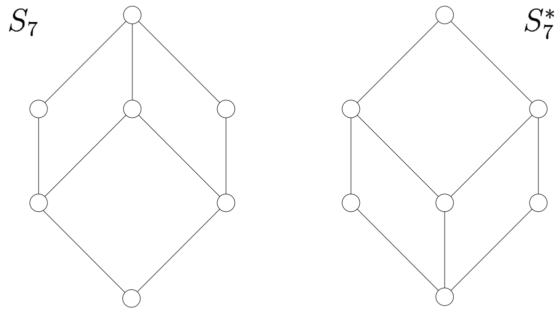


Figure 1.3: S_7 and S_7^*

1.5.19 Definition

A lattice L is called **lower locally Boolean**, if the interval $[\bigwedge\{a \mid a \prec b\}, b]$ is Boolean for every $b \in L$.

L is called **lower locally distributive**, if the interval $[\bigwedge\{a \mid a \prec b\}, b]$ is distributive for every $b \in L$.

Upper locally Boolean or **distributive** lattices are defined dually.

1.5.20 Example

For finite lattices, or even for lattices of finite length, lower locally distributive lattices are already lower locally Boolean, the converse is true in general.

Consider the cofinite topology on $\mathbb{N}$, i.e. the cofinite subsets of $\mathbb{N}$ ordered by inclusion, with the empty set as bottom element. This lattice is lower locally distributive, even a frame, but certainly not lower locally Boolean.

1.5.21 Remark

Rarely has a single property been given so many different names as in the case of locally Boolean / distributive. In the very first issue of *Order*, B. Monjardet in [41] lists a number of different names this property was given and the mathematicians who did so, as well as naming many situations in which locally Boolean / distributive lattices occur.

The final properties of this section are fairly new, one of them was first introduced in [24], the other is once again the dual version.

1.5.22 Definition

A complete lattice L is called a **weak frame**, if for $a, b \in L$ such that $b \prec a$ and every subset $D \subseteq \uparrow b$ we have

$$a \wedge \bigvee D = \bigvee (a \wedge D).$$

A complete lattice with the dual property is called a **weak coframe**. It is satisfied, if for $a, b \in L$ with $a \prec b$ and every subset $D \subseteq \downarrow b$ we have

$$a \vee \bigwedge D = \bigwedge (a \vee D).$$

An alternative description of weak coframes is that lower covers are $\bigwedge$-prime in the downset generated by their upper covers, i.e. if $a \prec b$ and $D \subseteq \downarrow b$, then $\bigwedge D \leq a$ implies that $d \leq a$ for some $d \in D$.

If you compare this definition with the definition of weakly join-continuous lattices as the dual of 1.5.11, you notice that the only difference between these two properties is the restriction to directed subsets $D \subseteq \downarrow b$ in 1.5.11. Just like 1.5.11 was a weak form of meet- and join-continuity respectively, the definitions here give us weak forms of the frame and coframe law.

If we restrict D to finite sets in the formulae given above, we obtain a weak form of semidistributivity. In finite lattices, this is already equivalent to the weak frame or coframe property, respectively.

Note that weak coframes were called local coframes in [24], but we want to use the name weak coframes to show the relation to weakly join-continuous lattices. In the next chapter, we will use weak coframes extensively, that is why we will neglect weak frames and concentrate on weak coframes.

It is interesting, that in finite lattices the weak coframe property is implied by lower local distributivity. However, the property of a lattice to be lower locally distributive or Boolean is rarely satisfied in infinite lattices. That is why we need variations of it if we want to extend results for finite lattices to infinite ones. But the connection of weak coframes to other variations of distributivity does not end with lower local distributivity, it is also related to completely $\vee$-semidistributivity.

1.5.23 Lemma

i) A finite lower locally distributive lattice L is a weak coframe.

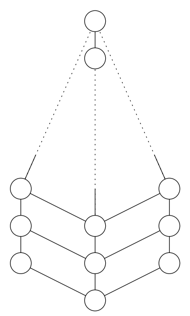
ii) A lower locally Boolean complete lattice need not be a weak coframe, nor does a weak coframe need to be lower locally distributive.

iii) A completely $\vee$-semidistributive complete lattice is a weak coframe.

Proofs: i) Suppose $a \prec b$ and $D \subseteq \downarrow b$ such that $a \not\leq d$ for all $d \in D$, but $\bigwedge D \leq a$. We make use of the fact that every finite loweer locally distribuitve lattice can be embedded in a Boolean lattice preserving rank and meet (Lemma 3.1 in [16]). We embed the interval $[\bot, b]$ in a Boolean lattice B in this way, i.e. b is the top element of B, and a a coatom. We can find an atom $\overline{a}$ which is the complement of a. Since $a \not\leq d$ for all $d \in D$, we get $\overline{a} \leq d$ for all $d \in D$, and therefore $\overline{a} \leq \bigwedge D$. But we started with $\bigwedge D \leq a$, which results in $\overline{a} \leq a$, a contradiction. Hence L is a weak coframe.

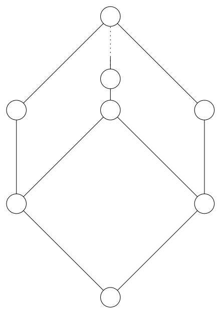
ii) For this we give two examples, one a lower locally Boolean lattice which is not a weak coframe, and one weak coframe which is not lower locally distributive.

As an example for a lower locally Boolean lattice which is not a weak coframe we can use the following lattice.



The unique coatom is not $\bigwedge$-prime, but lower covers always generate Boolean sublattices.

A weak coframe which is not lower locally distributive is e.g. the following lattice.



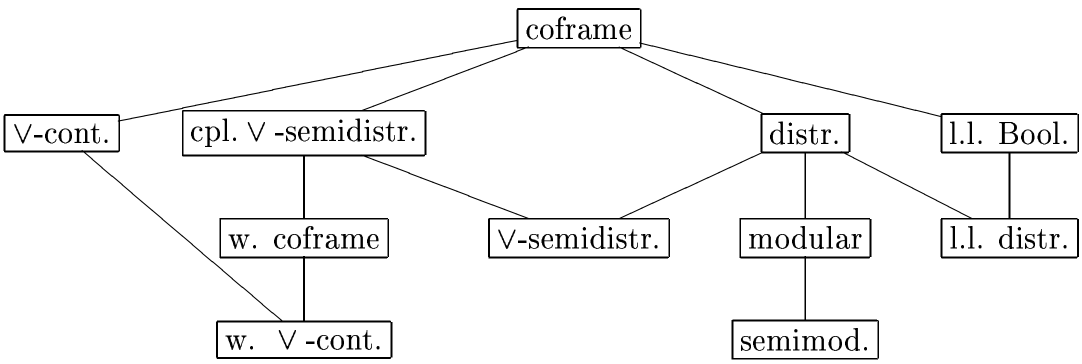
The meet of the coatoms is the bottom element, i.e. for the lattice to be lower locally distributive it would have to be distributive, but it contains an N_5 as a sublattice. However, lower covers are always $\bigwedge$-prime in the relevant lower sets, hence it is a weak coframe.

iii) Assume that L is completely $\vee$-semidistributive and consider $a, b \in L$ with $a \prec b$ and $D \subseteq\downarrow b$. If we have $a \vee d = b$ for all $d \in D$, then complete $\vee$-semidistributivity gives us $\bigwedge(a \vee D) = a \vee \bigwedge D$, i.e. the required equality.

If there is at least one $d \in D$ such that $a \vee d \neq b$, then this gives us $a \vee d = a$, as $a \prec b$. From this we get both $a \vee \bigwedge D = a$ and $\bigwedge(a \vee D) = a$, and we once more get the required equality, L is a weak coframe. $\square$

We close this section with a diagram of the variations of distributivity and modularity we discussed in this section.

1.5.24 Diagram



This diagram does not contain all possible combinations of properties and is not a lattice of properties. A weakly $\vee$-continuous and lower semimodular complete lattice need not be a coframe etc.

1.6 Prime, irreducible and compact elements

In lattice theory, there are a number of classes of elements which are very interesting for decompositions, i.e. for generating every element of a lattice as a join (or meet) of elements from a subset of the lattice. Knowledge of existence and distribution of these elements can give us a good insight into the structure of a lattice. The classical types are listed in the first definition of this section.

1.6.1 Definition

Let L be a complete lattice. We call $a \in L$

i) $\vee$-**irreducible**, if $a \neq \bot$ and $\forall\, x, y \in L : a = x \vee y \Rightarrow a = x$ or $a = y$,

ii) $\bigvee$-**isolated**, if $\forall\, X \subseteq L, X$ directed : $a = \bigvee X \Rightarrow a \in X$,

iii) $\bigvee$-**irreducible**, if $\forall\, X \subseteq L : a = \bigvee X \Rightarrow a \in X$,

iv) $\vee$-**prime**, if $a \neq \bot$ and $\forall\, x, y \in L : a \leq x \vee y \Rightarrow a \leq x$ or $a \leq y$,

v) **compact**, if $\forall\, X \subseteq L, X$ directed : $a \leq \bigvee X \Rightarrow \exists\, x \in X : a \leq x$,

vi) $\bigvee$-**prime**, if $\forall\, X \subseteq L : a \leq \bigvee X \Rightarrow \exists\, x \in X : a \leq x$.

The properties $\wedge$-**irreducible** etc. are defined dually, the dual of compact is called **dually compact** or **cocompact**.

For $\bigvee$-irreducible and -prime elements exist alternative definitions, which are equivalent to the ones we gave (see e.g. [24]) and very useful when we work with these elements.

An element $a \in L$ is $\bigvee$-irreducible, if and only if

$$a_{\vee} := \bigvee \{b \in L \mid b < a\}$$

is (the greatest element) smaller than a, and thus the unique lower cover of a.

An element $a \in L$ is $\bigvee$-prime, if and only if

$$a^{\vee} := \bigvee \{b \in L \mid a \not\leq b\}$$

is (the greatest element) not greater or equal than a.

Dual definitions produce a unique upper cover $m^{\wedge}$ for a $\bigwedge$-irreducible element $m \in L$, and the least element $m_{\wedge}$ not less than or equal to a $\bigwedge$-prime $m \in L$.

We will denote the set of all $\bigvee$-irreducible elements of X by $\mathcal{J}(X)$.

To this list we want to add one other property, which is not new [27], but has so far been used only in a very limited environment and as an additional property of $\bigvee$-irreducible elements. However, it can be introduced for arbitrary elements of arbitrary lattices.

1.6.2 Definition

Consider a lattice L. We call $a \in L$ ($\vee$-)**resistant**, if for all $b, c \in L$

$$(c < a \text{ and } a \not\leq b) \Rightarrow a \not\leq b \vee c.$$

U. Faigle called $\bigvee$-irreducible elements with this property strong, but we will use the word resistant.

A legitimation for this definiton in this section is given in the next lemma.

1.6.3 Lemma

For a lattice L and an element $a \in L$ we have the following implications:

$$a \text{ is } \vee\text{-prime} \Rightarrow a \text{ is } \vee\text{-resistant} \Rightarrow a \text{ is } \vee\text{-irreducible.}$$

Proof: This is straightforward since $a \in L$ is $\vee$- prime iff $a \not\leq b, c \Rightarrow a \not\leq b \vee c$, and a is $\vee$- irreducible iff $b, c < a \Rightarrow a \not\leq b \vee c$. □

1.6.4 Remark

For a $\bigvee$-irreducible element a of a lattice L, resistance can be formulated as

$$\forall\, x \in L : a \leq x \vee a_\vee \Rightarrow a \leq x$$

where $a_\vee$ is the unique lower cover of a.

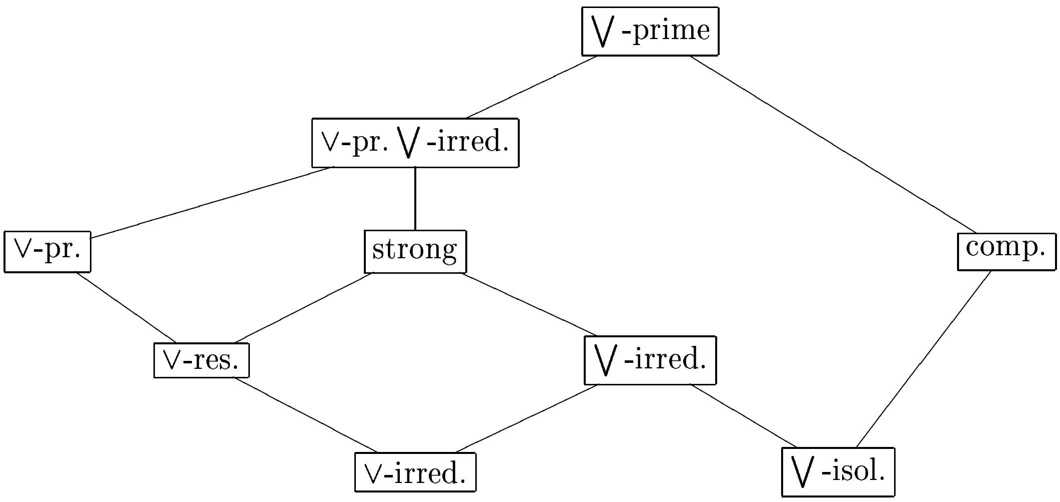
This is (essentially) the definition of Faigle. In the next chapter we will repeatedly consider lattices which are $\bigvee$-generated by resistant $\bigvee$-irreducible elements. That is why we give these specific elements a name.

1.6.5 Definition

Consider a complete lattice L. If $a \in L$ is a resistant $\bigvee$-irreducible element, we call it **strong**.

As for variations of distributivity, we give a diagram of the different types of elements we discussed here. The diagram does not cover all possible combinations of properties. In the next chapter we will e.g. consider lattices in which strong elements that are not $\bigvee$-prime will play a prominent role.

1.6.6 Diagram



1.7 Decompositions

We already introduced lattices in which every element is a join of atoms. This section will deal with variations of this property, as we look at lattices in which every element is a join or meet of irreducible, prime or compact elements.

1.7.1 Definition

Consider a complete lattice L. We say that L is

$\bigvee$-irreducibly generated, if every element of L is a join of $\bigvee$-irreducible elements,

$\vee$-irreducibly generated, if every element of L is a join of $\vee$-irreducible elements,

strongly generated, if every element of L is a join of strong elements,

algebraic, if every element of L is a join of compact elements.

$\bigwedge$**- and** $\wedge$**- irreducibly generated lattices**, as well as **dually algebraic lattices** are defined dually.

1.7.2 Remark

Since atoms have the bottom element $\bot$ as unique lower cover, they are certainly $\bigvee$-irreducible, hence atomistic lattices are a special case of $\bigvee$-irreducibly generated lattices. But note that $\bigvee$-irreducibly generated lattices are not algebraic in general.

1.7.3 Definition

Consider a complete lattice L and an element $x \in L$. If x is the join of a set of $\bigvee$-irreducible elements J, we call J a **(join) decomposition** of x. J is called **irredundant**, if no proper subset of J is a decomposition of x, i.e.

$$\forall K \subseteq J : x = \bigvee K \Rightarrow K = J.$$

If x is a meet of $\bigwedge$-irreducible elements, we speak of a meet decomposition.

The theory of decompositions in lattices is a very active field, for some recent results see e.g. [46], [47]. Probably the best known result is Theorem 6.1 in [11] for algebraic lattices. Several authors generalised this result and were able to weaken the restrictions on the lattice. The most general result to my knowledge is by Diercks [14], which is much older than [46], [47].

1.7.4 Theorem

In a complete lattice L, the following implications are true:

	(i)	*L is algebraic.*
$\Rightarrow$	*(ii)*	*L is weakly atomic and meet-continuous.*
$\Rightarrow$	*(iii)*	*L is weakly atomic and weakly meet-continuous.*
$\Rightarrow$	*(iv)*	*Every element $x \in L$ has a meet decomposition.*

Proof: see [14]

None of these implications can be extended to an equivalence in general (see [14]). There is, however, a large class of lattices, in which the first three properties coincide.

1.7.5 Theorem

In a complete and strongly generated lattice L, the following properties are equivalent:

i) L is algebraic,

ii) L is meet-continuous,

iii) L is weakly meet-continuous.

Proof: We only need to prove the implication (iii) $\Rightarrow$ (i). For this we show that every $\bigvee$-irreducible element is compact. Suppose $a \in L$ is $\bigvee$-irreducible but not compact, i.e. there is a $K \subseteq L$ such that $a \leq \bigvee K$ but for all $F \subseteq_\omega K$ we have $a \not\leq \bigvee F$. We denote the unique lower cover of a by $a_\vee$. Since a is resistant, we can enlarge F by $a_\vee$ and still get $a \not\leq \bigvee(F \cup \{a_\vee\})$. We define the directed set $D := \{\bigvee(F \cup \{a_\vee\}) \mid F \subseteq_\omega K\} \subseteq\uparrow a_\vee$, for which we have $\bigvee D = \bigvee K$. For each $d \in D$ we have $a \wedge d = a_\vee$.

We can now apply the weak meet-continuity for a and D:

$$a = a \wedge \bigvee K = a \wedge \bigvee D = \bigvee (a \wedge D) = \bigvee \{a_\vee\} = a_\vee \prec a,$$

which is a contradiction. Thus, every $\bigvee$-irreducible element is compact, and therefore every element is a join of compact element, i.e. L is algebraic. □

To illustrate that this construction is not too artificial we give two examples of classes, in which every $\bigvee$-irreducible element is resistant. More sophisticated examples will be given in the following chapters. There we will also see, that the existence of meet decompositions is still strictly weaker than the three properties given in the previous theorem.

1.7.6 Examples

i) In distributive lattices every $\bigvee$-irreducible element is prime and therefore resistant. Thus, we can apply Theorem 1.7.5 to complete, distributive and $\bigvee$-irreducibly generated lattices.

ii) Atoms are always resistant. Therefore, complete and atomistic lattices are algebraic if and only if they are (weakly) meet-continuous.

Chapter 2

Convex Geometries and the Anti-Exchange Property

2.1 Basic Notation and Terminology

In the early eighties, Paul Edelman and Robert Jamison studied the structure of lattices of convex sets, and developed an abstraction of these lattices, which were called *convex geometries* [16], [18]. They discovered a number of structural properties satisfied by these systems, many of which are even equivalent to their definition of a convex geometry. Their theory, however, was restricted exclusively to finite sets, and many of the theorems they obtained fail to be true for infinite sets in general.

Our aim is to contribute to a theory of infinite convex geometries, in which the different structural properties introduced and used in the finite case are applied to infinite sets. A first paper that tries to initialise a discussion as to what an infinite convex geometry should be was published recently [3]. We use a different approach, which centres on one of the main theorems of Edelman and Jamison concerning equivalent descriptions of convex geometries. Before we give an overview of the finite theory of convex geometries, we introduce the main notions and properties, some of which have to be adjusted for an application to infinite sets.

Probably the most basic property of a system of convex sets is, that the intersection of an arbitrary number of convex sets is again a convex set. That is why we use this as the fundamental property of the objects we want to investigate.

2.1.1 Definitions

A **closure system** or **$\bigcap$-system** on S is a subset $\mathcal{C}$ of $\mathcal{P}S$ such that

$$\forall \mathcal{A} \subseteq \mathcal{C} : \bigcap \mathcal{A} \in \mathcal{C}.$$

A mapping $\Gamma : \mathcal{P}S \to \mathcal{P}S$ is a **closure operator** on S, if for all $X, Y \subseteq S$

$$X \subseteq \Gamma Y \iff \Gamma X \subseteq \Gamma Y.$$

This is the shortest definition of a closure operator one can give. It is not the one given in most books. There one will most likely find the following description of a closure operator, split up into three separate properties.

2.1.2 Lemma

$\Gamma : \mathcal{P}S \to \mathcal{P}S$ is a closure operator on S, if and only if Γ satisfies the following three conditions:

i) $X \subseteq \Gamma X$ *(extensive)*

ii) $X \subseteq Y \Rightarrow \Gamma X \subseteq \Gamma Y$ *(isotone)*

iii) $\Gamma\Gamma X = \Gamma Y$ *(idempotent).*

Proof: see [20]

Closure operators and closure systems are closely related to each other, as we can see in the next lemma.

2.1.3 Lemma

The assignments $\Gamma \mapsto \mathcal{C}_\Gamma := \{X \subseteq S : X = \Gamma X\}$ and $\mathcal{C} \mapsto \Gamma_\mathcal{C}$ defined by $\Gamma_\mathcal{C} X := \bigcap\{A \in \mathcal{C} : X \subseteq A\}$ are inverse bijections between the system of all closure operators on S and the system of all closure systems on S.

Proof: see [20]

2.1.4 Notation

Very often we will look at (usually closed) sets which are enlarged or reduced by single elements. Since terms of the form $A \cup \{x\}$ or $A \setminus \{x\}$ are rather lengthy, we will use the following abbreviations:

$A - x$ for $A\backslash\{x\}$ if $x \in A$,

$A + x$ for $A \cup \{x\}$ if $x \notin A$, and

$A(x)$ for the closure $\Gamma(A + x)$.

2.1.5 Definition

The notions of separation axioms of topological spaces can be applied to arbitrary closure systems or spaces as well. We consider a closure system $\mathcal{C}$ on a set S, the least element of $\mathcal{C}$ we will call C, given by $C = \bigcap\{A \in \mathcal{C}\}$. $\mathcal{C}$ is a

C_0**-system,** if $x \neq y$ implies $C(x) \neq C(y)$,

C_1**-system,** if $\emptyset$ and all singletons are closed,

C_D**-system,** if all sets $C(x) - x$ are closed for $x \in S \setminus C$.

We call $(S, \mathcal{C})$ a C_0-, C_1- or C_D**-space**, respectively, if $\mathcal{C}$ satisfies the corresponding separation axiom. In the case of topological closure systems, we obtain the usual T_0-, T_1 and T_D-spaces, respectively, see e.g. [4].

In a C_D-system, all point closures are $\bigvee$-irreducible. As a consequence every closed set has a join decomposition, i.e. a join representation by completely join irreducible closed sets.

This is the basis for an important connection between C_D-spaces and $\bigvee$-irreducibly generated lattices.

2.1.6 Theorem

Associating with any C_D-space $(S, \mathcal{C})$ the complete lattice $\mathcal{C}$ of all closed sets, and with each continuous map $\varphi : (S, \mathcal{C}) \to (S', \mathcal{C}')$ the join-preserving map

$$\Phi : \mathcal{C} \to \mathcal{C}',\ A \mapsto \bigcap\{B \in \mathcal{C}' \ :\ \varphi[A] \subseteq B\},$$

we obtain a categorical equivalence between C_D-spaces with continuous maps and $\bigvee$-irreducibly generated lattices with maps that preserve joins and minimal $\bigvee$-generators.

In particular, the $\bigvee$-irreducibly generated lattices are, up to isomorphism, just the C_D-systems.

Proof: see [21]

2.1.7 Definition

Motivated by the previous theorem, we will call $\bigvee$-irreducibly generated complete lattices **C_D-lattices**.

The previous theorem allows us to use C_D-lattices and diagrams thereof as examples for C_D-systems and their lattices of closed sets. Quite often we will give a diagram of a C_D-lattice instead of describing the C_D-system it represents.

We continue with a small number of other concepts frequently used in our considerations.

2.1.8 Definitions

Consider a closure operator Γ on S. We call $X \subseteq S$ a **generator** of a closed set A, if $\Gamma X = A$. A minimal generator is called a **basis**.

A point $x \in X$ is called an **extreme point of** X, if $x \notin \Gamma(X - x)$. The set of all extreme points of X is denoted by $\mathrm{E}X$, and note that $\mathrm{E}X$ is contained in every generator of ΓX.

A **copoint** of a point $x \in S$ in the closure space $(S, \mathcal{C})$ is a maximal closed set $A \in \mathcal{C}$ with $x \notin A$.

2.1.9 Remarks

For an arbitrary closure operator, neither the existence nor the uniqueness of bases are guaranteed.

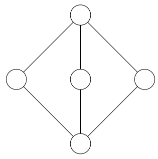
In an arbitrary closure system $\mathcal{C}$ on S, the set of extreme points of $\mathrm{E}A$ of a closed set A can be empty. This does not exclude finite systems. In the trivial closure system $\{S\}$ on S with $|S| > 1$, no set contains any extreme points.

In an arbitrary closure system, copoints of a point x need not exist nor be unique. In finite closure systems copoints and bases exist, but may not be unique.

2.1.10 Examples

For this first list of small examples, we will make use of 2.1.6, which ensures that a given $\bigvee$-irreducibly generated lattice represents a C_D-system.

i) The diamond M_3 is an example for a finite C_D-system, in which neither bases nor copoints are unique.



ii) We consider the C_D-system $\mathcal{I} := \{[a,b] \subseteq [0,1] \,:\, a,b \in [0,1]\}$ of all closed intervals of $[0,1]$. Every interval $[a,b] \neq \emptyset$ has the basis $\{a,b\}$, but if $a \neq b$ this is not a least generator as dense subsets of the interior $]a,b[$ are generators as well. Furthermore, no interval I contains extreme points except for $I = \{a\}$.

iii) In the system of all convex subsets of $[0,1]$, $\mathcal{I} := \{A \subseteq [0,1] \,:\, A \text{ convex}\}$, elements need not have bases at all. However, to every $p \in]0,1[$ there exist two copoints: $[0,p[$ and $]p,1]$, and 0 and 1 have unique copoints $]0,1]$ and $[0,1[$, respectively.

We will close this list of definitions with a property that will be a main focus of our investigations.

2.1.11 Definition

Consider a closure operator Γ on S, with corresponding closure system $\mathcal{C}$. We say $\mathcal{C}$ has the **anti-exchange property**, if

$$\forall A \in \mathcal{C}\, \forall\, x, y \in S \setminus A : A(x) = A(y) \Rightarrow x = y.$$

An equivalent definition is: $\forall\; x, y \notin A,\; x \neq y \;\&\; x \in A(y) \Rightarrow y \notin A(x)$. This explains the derivation of the name from the **exchange property**, which for two elements x, y not in a closed set A says $x \in A(y)$ if and only if $y \in A(x)$.

The anti-exchange property was first introduced by Edelman in [16] and is the central property for Edelman and Jamison's approach to general convex systems.

2.1.12 Definition

We define a quasi order on every set $S \setminus A$ for $A \in \mathcal{C}$. For $x, y \in S \setminus A$ we say

$$x \leq_A y :\Longleftrightarrow A(x) \subseteq A(y).$$

2.1.13 Lemma

If $\mathcal{C}$ is a closure system with the anti-exchange property, and $A \in \mathcal{C}$, then the relation $\leq_A$ is a partial order on $S \setminus A$. If $\mathcal{C}$ is a closure system with the exchange property, $\leq_A$ is an equivalence relation.

Proof: Reflexivity is trivial, transitivity is a direct consequence of the transitivity of $\subseteq$, and the anti-exchange property gives us anti-symmetry. In exchange-systems the exchange property gives us symmetry instead, hence we would get an equivalence relation instead of a partial order. □

2.2 Finite Convex Geometries

In this section we report the main results and ideas of [18]. For the remainder of this section, **S denotes a finite set with cardinality n**, and $\mathcal{C}$ is a closure system on S with closure operator Γ.

With the definitions given in the last section, we can formulate Theorem 2.1 of [18]. However, we will deviate a little bit from the original and speak of closed sets in contrast to convex sets as used in [18]. Since one aim of this work is to discuss and develop a theory for (infinite) convex geometries, we do not want to use the word convex at this point of time.

All skipped proofs to theorems in this section can be found in [18].

2.2.1 Theorem

Consider a closure system $\mathcal{C}$ on S. Then the following are equivalent:

i) For every closed set $A \in \mathcal{C} - S$, there exists a point $x \in S \setminus A$ such that $A + x$ is closed.

ii) For every closed set A, $A = \Gamma(E(A))$.

iii) For every closed set A and $x \notin A$, $x \in E(\Gamma(A + x))$.

iv) $\mathcal{C}$ has the anti-exchange property.

v) Every subset $X \subseteq S$ has a unique basis.

vi) For every point $x \in S$ and a copoint A of x, $A + x$ is closed.

A closure system with these properties will be called a **finite convex geometry**.

Three additional equivalent descriptions for finite convex geometries based on these notions are given in Theorems 2.2 to 2.4 of [18].

2.2.2 Theorem

The pair $(S, \mathcal{C})$ is a finite convex geometry if and only if every maximal chain of closed sets

$$\emptyset \subset A_1 \subset A_2 \subset \ldots \subset S$$

has the same length $|S| = n$.

2.2.3 Theorem

$(S, \mathcal{C})$ is a finite convex geometry if and only if the relation $\leq_A$ (as defined in 2.1.13) is a partial order on $S \setminus A$ for all $A \in \mathcal{C}$.

2.2.4 Theorem

$(S, \mathcal{C})$ is a finite convex geometry if and only if every copoint is attached at a unique point.

To this list of characterisations we will add some more descriptions of finite convex geometries.

2.2.5 Theorem

The pair $(S, \mathcal{C})$ is a finite convex geometry if and only if any covering pair $A, B \in \mathcal{C}$ with $A \prec B$ differs by exactly one element, i.e. $B \setminus A = \{x\}$.

Proof: Suppose we have $A \prec B$ and $x, y \in B \setminus A$. Then $A(x) = A(y) = B$, since B covers A. With the anti-exchange property we obtain $x = y$, i.e. $B \setminus A$ can contain only one element.

For the other direction we start with two closed sets $A \subset B$ such that $A(x) = A(y) = B$ for two elements $x, y \in B \setminus A$. Since $\mathcal{C}$ is finite, we can find a $D \in \mathcal{C}$ such that $A \subseteq D$ and $D \prec B$. Since B is the smallest closed set that contains A, x and y, $x, y \notin D$, thus $x, y \in B \setminus D$. But there can be only one element in this difference, hence $x = y$. $\square$

Note that we only need strongly coatomic lattices, which are more general than finite lattices.

This last characterisation of finite convex geometries allows us to formulate a dual version of equivalence (ii) of 2.2.1.

2.2.6 Theorem

The pair $(S, \mathcal{C})$ *is a finite convex geometry if and only if for every pair* $A, B \in \mathcal{C}$ *with* $A \subset B$ *there exists an* $x \in B \setminus A$ *such that* $B - x \in \mathcal{C}$.

Proof: Consider $A, B \in \mathcal{C}$ with $A \subset B$. Since $\mathcal{C}$ is finite, we can find a $D \in \mathcal{C}$ with $A \subseteq D \prec B$. The previous theorem tells us, that this lower cover of B satisfies $B \setminus D = \{x\}$, i.e. $D = B - x$.

If we now consider $A \in \mathcal{C}$ and $x \notin A$, we get $A \subset A(x)$. We can find a $D \in \mathcal{C}$ such that $A \subseteq D \prec A(x)$. This D cannot contain x, since $A(x)$ is the smallest closed set that contains both A and x. Thus $D = A(x) - x$, i.e. $x \in \mathrm{E}(\Gamma(A + x))$, which proves equivalence (iii) of 2.2.1. □

2.2.7 Remarks

Before we continue, we should make some comments concerning some of the properties in this quite extensive list.

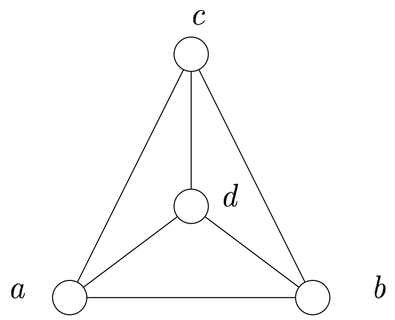
i) Equivalence (iii) simply states, that the extreme points of a closed set A form the unique basis of $A \in \mathcal{C}$ and of every subset $X \subseteq S$ with $\Gamma X = A$. Since the extreme points of a closed set A have to be contained in every generator of A, the extreme points even form the least generator of A.

ii) If the relation $\leq_A$ is a partial order for every $A \in \mathcal{C}$, then $\mathcal{C}$ has the anti-exchange property and vice versa, even if S and $\mathcal{C}$ are infinite. This is not true for the other descriptions of a finite convex geometry given in this section when applied to an infinite set/closure system.

2.2.8 Example

The first class of examples given in [18] consists of finite sets of points in $\mathbb{R}^n$. A subset $A \subseteq S$ is closed, if it is the restriction of a convex subset of $\mathbb{R}^n$ to S. We will give one specific example with four elements in $\mathbb{R}^2$.

The finite convex geometry on these four elements consists of the empty set, the four singletons, all six possible combinations of two elements, the three element sets $\{a, b, d\}$, $\{a, c, d\}$ and $\{b, c, d\}$ and the full set $S = \{a, b, c, d\}$.



The only subset of S that is not closed is the set $\{a, b, c\}$, since every convex subset of $\mathbb{R}^2$ that contains these three points always contains the point d in the interior of the triangle.

Additional classes of examples can be found in [18] and [17].

2.3 Infinite Convex Geometries

In the previous section we saw that finite convex geometries can be characterised by very different properties, e.g. by unique bases or the anti-exchange property, but also by special properties for covering pairs. In infinite systems, these properties fail to be equivalent, and in this section we want to analyse which implications between these properties still hold for infinite systems.

There are two main branches of properties which arise in this context. There is a list of anti-exchange properties, i.e. variations of the classical anti-exchange property, and there is a list of properties concerning generators or bases.

Some variations of the anti-exchange property make use of not only the closure space $\mathcal{C}$ but also of some slightly modified systems.

2.3.1 Definition

A finitely generated closed set is called a **polytope**. For a given closure system $\mathcal{C}$ we define (see [22])

$\mathcal{C}_\omega$ the system of all polytopes,

$\mathcal{C}^\uparrow$ the system of all unions of chains in $\mathcal{C}$,

$\mathcal{C}^\Uparrow$ the system of all unions of directed subsystems of $\mathcal{C}$.

$\mathcal{C}^\Uparrow$ is the least algebraic closure system which contains $\mathcal{C}$, but $\mathcal{C}^\uparrow$ need not be algebraic in general. However, each of the equations $\mathcal{C} = \mathcal{C}^\uparrow$, $\mathcal{C} = \mathcal{C}^\Uparrow$ and $\mathcal{C} = \mathcal{C}_\omega^\Uparrow$

imply algebraicity of $\mathcal{C}$.

2.3.2 Definition

We already introduced the symbol $\prec$ for the covering relation. Based on this definition we can define the **strong covering relation** $\lhd$ by

$$A \lhd B :\iff A \subset B \text{ and } B = \bigcap\{B' \in \mathcal{C} : A \subset B'\},$$

i.e. A is a $\bigwedge$-irreducible element with unique upper cover B.

It was already stated in [18], that copoints need to be $\bigwedge$-irreducible elements, and this is also true for infinite systems. The converse is also correct, every $\bigwedge$-irreducible element $A \in \mathcal{C}$ is a copoint to every $x \in B \setminus A$, where $A \lhd B$. Hence, A is a copoint of $x \in S$ if and only if $A \lhd A(x)$ (see [22]).

Using the strong covering relation, we can formulate the copoint-property 2.2.4 of the preceeding section as an anti-exchange property.

2.3.3 Lemma and Definition

In a C_D-system, the following properties are equivalent:

i) *$\mathcal{C}$ has the* **anti-exchange property for copoints**, *i.e. distinct points do not have common copoints,*

ii) *$\mathcal{C}$ has the* **anti-exchange property for strict covers**, *i.e.*

$$\forall A \in \mathcal{C}\, \forall x, y \notin A : A \lhd A(x) = A(y) \Rightarrow x = y.$$

Proof: If $A \lhd A(x) = A(y)$, then A is a maximal closed set not containing x and y, i.e. A is a copoint of both elements. This contradicts (i).

Conversely, if A is a copoint of an element x, then A cannot be a copoint of a different element y because of (ii), thus distinct points do not have common copoints. □

2.3.4 Lemma and Definition

For a C_D-system $\mathcal{C}$, the following properties are equivalent:

i) *$\mathcal{C}$ has the* **anti-exchange property for covers**, *i.e.*

$$\forall A \in \mathcal{C}\, \forall x, y \notin A : A \prec A(x) = A(y) \Rightarrow x = y,$$

ii) $\forall A, B \in \mathcal{C} : A \prec B \Rightarrow \exists x \in B \setminus A : B = A + x$.

Proof: Similar to the previous one.

This is a general principle for anti-exchange properties. If we replace the relations $\lhd$ or $\prec$ by some general relation, we obtain a universal formulation for many anti-exchange properties.

2.3.5 Definition

In general, we will say that $\mathcal{C}$ has the **anti-exchange property for** $\sqsubset$ for an arbitrary relation $\sqsubset$ on the power set of S if

$$A \sqsubset A(x) = A(y) \Rightarrow x = y$$

for all $A \in \mathcal{C}$ and $x, y \notin A$. Some examples for such relations are defined by setting $A \subset_{\mathcal{S}} B$ if and only if $A \in \mathcal{S} \subseteq \mathcal{P}S$ and $A \subset B$.

2.3.6 Definition

With these relations we can define the following properties:

i) The anti-exchange property for $\subset_{\mathcal{C}}$ is the usual anti-exchange property,

ii) the anti-exchange property for $\subset_{\mathcal{C}^{\uparrow}}$ we will call **strong anti-exchange property**, and

iii) the anti-exchange property for $\subset_{\mathcal{C}_\omega}$ we will call **finitary anti-exchange property**.

The strong anti-exchange property seems a little bit complicated, but there is an alternative description of it, see [22].

2.3.7 Lemma

In a C_D-System $\mathcal{C}$, the strong anti-exchange property is satisfied if and only if every maximal chain in $\mathcal{C}$ is a C_0-system.

This requieres the axiom of choice. For finite lattices, this is 2.2.2.

2.3.8 Lemma

In a C_D-system we have the following implications:

strong AEP ⇒ AEP ⇒ cover-AEP ⇒ copoint-AEP, and

AEP ⇒ finitary AEP.

Proof: By the definitions of the various properties.

2.4 Encoding of Anti-Exchange Properties

At the beginning of the preceding section we already mentioned that there are two main branches of anti-exchange properties. So far we dealt with the first branch, which contains properties very similar to the original anti-exchange property.

The second branch of properties concerns bases and generators. In the finite case, the uniqueness of bases for every element was one characterisation of finite convex geometries. In an infinite C_D-system, elements can have finite bases, infinite bases or just infinite generators which cannot be reduced to a basis.

Before we define the various properties of this branch, we want to introduce an encoding scheme for our extensive list of properties. This code will enable us to distinguish the properties, but it also encodes relations between them. If we know that one property implies another, this will be reflected by the code.

The basic code consists of one letter and two digits for one property. The letter is either A or B. If it is B, then every element in the corresponding systems has a basis (of $\bigvee$-irreducible elements). If it is A, then this need not be the case. Hence, a system with property Bxy implies property Axy.

The first digit represents which property of the already introduced first branch a system satisfies, the second one does so for the second branch of basis-related properties we will give now.

2.4.1 Definition and Notation

For a C_D-system $\mathcal{C}$, we say that it has property

A04, if every basis is a least generator,

A03, if every basis is unique,

A02, if bases of polytopes are unique, and finally,

A01, if finite bases are unique.

The chain of implications (A04) ⇒ (A03) ⇒ (A02) ⇒ (A01) is obvious.

Note that the last property we just gave, A01, coincides with the finitary anti-exchange property (see [22]). However, none of the other anti-exchange properties we gave earlier is implied by one of these four properties. The converse is not true in general.

2.4.2 Notation

For a C_D-system $\mathcal{C}$, we say that it has property

A10, if it satisfies the anti-exchange property for copoints,

A20, if it satisfies the anti-exchange property for covers,

A32, if it satisfies the anti-exchange property and

A42, if it satisfies the strong anti-exchange property.

This encoding has the property that a C_D-system with property Ruv has also the property Pxy if (P=A or R=B) and $(x \leq u)$ and $(y \leq v)$. To prove this we have to show that the anti-exchange property implies uniqueness of bases of polytopes. All other implications are trivial.

2.4.3 Lemma

Polytopes in a C_D-system $\mathcal{C}$ with the anti-exchange property have unique bases, i.e. $\mathcal{C}$ satisfies property A02.

Proof: Suppose the polytope $A \in \mathcal{C}$ has a finite basis $K \subseteq A$ and a different, not necessarily finite, basis $L \subseteq A$. Choose $l \in L \setminus K$ and define $B := \Gamma(L - l)$. The set $P = K \setminus B$ is a non-empty subset of K since otherwise $A = \Gamma K \subseteq B = \Gamma(L - l) \subset A$, a contradiction. We can write $P = \{p_0, ..., p_m\}$ with $m \leq |K|$.

We construct the chain $B_0 := B$, $B_{k+1} := B_k(p_k)$ for $k = 0, ..., m$. At one point of this chain we have $B_k \subset A$ but $A = B_{k+1} = B_k(p_k)$. By construction we also have $B_k(l) = A$, which implies $l = p_k$ since $\mathcal{C}$ satisfies the anti-exchange property. This contradicts $l \in L \setminus K$, hence polytopes cannot have more than one basis. □

We will soon show, that no two of these properties coincide for arbitrary infinite C_D-systems. However, before we do so, we will introduce two more properties, which are again infinite formulations of properties characterising finite convex geometries.

2.4.4 Definition

We call a C_D-system $\mathcal{C}$ **(extremally) detachable**, if

$$\forall\, A \in \mathcal{C}\,\forall\, x \in X \setminus A : A(x) - x \in \mathcal{C}.$$

We call a C_D-system **shellable**, if

$$\forall\, A, B \in \mathcal{C} : A \subset B \Rightarrow \exists\, x \in B \setminus A : B - x \in \mathcal{C}.$$

Note that the definition of a detachable closure system is very similar to the definition of C_D-systems. The next lemma contains some very important implications between certain C_D-systems. All go back to [22], but because of their importance we give proofs.

2.4.5 Lemma

i) A shellable C_D-system is detachable.

ii) A detachable C_D-system has the anti-exchange property.

iii) In a detachable C_D-system, bases are least generators.

Proofs: i) For $x \notin A$ we have $A \subset A(x)$. Shellability says that there exists an element $y \in A(x) \setminus A$ such that $A(x) - y \in \mathcal{C}$. If $y \neq x$, this would be a closed set containing A and x, but $A(x)$ is the smallest closed set that contains A and x, which gives a contradiction. Hence $y = x$ and $A(x) - x \in \mathcal{C}$.

ii) Consider a closed set A and elements $x, y \notin A$. If $A(x) = A(y)$, then $x \in A(y)$ but $x \notin A(y) - y$ since $x \in A(y) - y$ would imply $A(x) \subseteq A(y) - y = A(x) - y$. Hence $x = y$.

iii) Consider a basis K for a closed set A. For an arbitrary element $k \in K$ we define $B := \Gamma(K - k) \subset A$. Detachability gives $B(k) - k \prec B(k) = A$,i.e. $A - k \in \mathcal{C}$. Therefore the closure of every subset $L \subseteq A - k$ would be contained in $A - k$. Consequently, k is contained in every generator of A, and since k was an arbitrary element of K, the basis K must be contained in every generator and is therefore a least generator. □

Some further characterisations of detachable or shellable C_D-systems were given by M. Erné. Proofs of the next two results can be found in [22].

Recall that EA is the set of extreme points of a A.

2.4.6 Theorem

For a C_D-system $\mathcal{C}$ with closure operator Γ, the following are equivalent:

i) $\Gamma = \Gamma \circ E$.

ii) $\mathcal{C}$ *is shellable.*

iii) Each closed set has a least generator.

iv) Each closed set is generated by its extreme points.

v) $\mathcal{C}$ *is strongly coatomic and has the anti-exchange property for covers.*

2.4.7 Theorem

The following are equivalent:

i) $E = E \circ \Gamma$.

ii) $\mathcal{C}$ *is detachable.*

iii) Each principal filter of $\mathcal{C}$ *is a* C_D*-system.*

iv) For $A \in \mathcal{C}$*, the map* $x \mapsto A(x)$ *is an isomorphism between the posets* $(S \setminus A, \leq_A)$ *and* $\mathcal{J}(\uparrow A)$.

Proofs: see [22]

Before we give a diagram of all possible combinations of properties, we add one final property, but of a different nature. R. Jamison-Waldner stated in [35], that in algebraic closure systems almost all properties we introduced or cited are equivalent. An algebraic C_D-system with the anti-exchange property is detachable, but it also has the strong anti-exchange property since in this case $\mathcal{C} = \mathcal{C}^{\uparrow} = \mathcal{C}^{\Uparrow}$. However, algebraic detachable C_D-systems need not be shellable.

2.4.8 Definition

An algebraic and detachable C_D-system is called **convex geometry**.

The justification for this name is Theorem 1 in [35]. We give the theorem using our notation.

2.4.9 Theorem

In an algebraic C_D*-system* $\mathcal{C}$*, the following are equivalent:*

i) $\mathcal{C}$ is extremally detachable.

ii) $\mathcal{C}$ has the strong anti-exchange property.

iii) $\mathcal{C}$ satisfies the anti-exchange property.

iv) The relation $\leq_A$ is a partial order on $S \setminus A$ for every $A \in \mathcal{C}$.

v) Bases of elements of $\mathcal{C}$ are unique.

vi) If P is a polytope, then $P = (\Gamma \circ E)P$, i.e. $\mathcal{C}$ satisfies property A 02.

vii) $\mathcal{C}$ satisfies the anti-exchange property for copoints.

Proof: Since $\mathcal{C}$ is algebraic, it coincides with $\mathcal{C}^{\uparrow}$, hence the anti-exchange property (iii) and the strong anti-exchange property (ii) coincide. The following implications have either been shown earlier, or are trivial: (i) $\Rightarrow$ (iii) $\iff$ (ii) $\iff$ (iv) $\Rightarrow$ (vii), and (i) $\Rightarrow$ (v) $\Rightarrow$ (vi). If we show (vii) $\Rightarrow$ (i), the first chain of implications would be closed. It then suffices to show e.g. (vi) $\Rightarrow$ (iii) to close the proof.

(vii) $\Rightarrow$ (i): Assume that $A \in \mathcal{C}$ and $x \notin A$. Since $\mathcal{C}$ is algebraic, A is contained in a maximal element $Q \in \mathcal{C}$ that does not contain x, using the Lemma of Zorn. Q needs to be $\bigwedge$-irreducible, with $x \in Q^{\wedge}$, the unique upper cover of Q, since every set larger than Q must contain x. Thus $Q \lhd Q^{\wedge} = Q(x)$. The anti-exchange property for copoints gives us $Q(x) = Q+x$. Furthermore, $A(x) \subseteq Q(x) = Q+x$. The meet of $A(x)$ and Q yields the desired result: $A(x) \wedge Q = A(x) \cap Q = A(x) - x$, and therefore $A(x) - x \in \mathcal{C}$.

(vi) $\Rightarrow$ (iii): We have an algebraic closure system $\mathcal{C}$, in which every polytope is generated by its extreme points. Assume that the anti-exchange property is not satisfied, i.e. there are $A \in \mathcal{C}$ and $x, y \notin A$ such that $x \neq y$, $x \in A(y)$ and $y \in A(x)$. Since $\mathcal{C}$ is algebraic, the closure operator is finitary, therefore there exist finite subsets $E, F \subseteq_{\omega} A$ with $x \in \Gamma(E + y)$ and $y \in \Gamma(F + x)$. Using $D := E \cup F$ this leads directly to

$$B := \Gamma(D + x) = \Gamma(D + y),$$

but $x, y \notin \Gamma D$. Therefore x and y are both extreme points of B, and must both be contained in every generator of B, hence $x = y$, a contradiction. $\square$

So we can see that many of the equivalences for finite convex geometries still hold for algebraic C_D-systems, which is more general, and this class contains some very important examples. The most typical ones are given in the next section.

The next section will also show that even algebraic systems are still too restrictive, there are many examples for non-algebraic systems with various anti-exchange properties.

The next diagram contains all properties in question, and all possible combinations. The top level which deviates from the encoding we introduced earlier consists of detachable (**D**) and shellable (**S**) C_D-systems, detachable systems which satisfy the strong anti-exchange property (**SD**), shellable ones with the strong anti-exchange property (**SS**), algebraic C_D-systems with the anti-exchange property (**CG**) and shellable algebraic C_D-systems (**SG**).

To illustrate that every box represents a different class of objects, we will give an example for every box that satisfies precisely the properties of this box (and all weaker properties, of course), but no stronger one. Since this is a very tedious and time-consuming work, we use a simple construction that allows us to build examples with certain combinations of properties by combining two known systems.

Furthermore, we will make use of 2.1.6. As mentioned above, this theorem allows us to give a C_D-lattice representing a C_D-system.

2.4.10 D-Sum of C_D-systems

Consider two C_D-systems $\mathcal{A}$ and $\mathcal{B}$ on the sets A and B, respectively. Without loss of generality we assume $\bot_{\mathcal{B}} = \emptyset$ and $A \cap B = \emptyset$. We define the **D-sum** $\mathcal{C} := \mathcal{A} +_D \mathcal{B}$ on the set $A \cup B$ as

$$\mathcal{C} = \mathcal{A} \cup \{D \cup A \mid D \in \mathcal{B}\}.$$

In plain words, this puts $\mathcal{B}$ on top of $\mathcal{A}$ and amalgamates the $\top$-element of $\mathcal{A}$ and the $\bot$-element of $\mathcal{B}$. Order-theoretically this corresponds to the ordinal sum of $\mathcal{A}$ and $\mathcal{B} \setminus \bot_{\mathcal{B}}$, sometimes called the vertical sum of $\mathcal{A}$ and $\mathcal{B}$, see e.g. [29].

It is easy to see that the D-sum of two system $\mathcal{A}$ and $\mathcal{B}$ has all the properties we discussed in this section that are satisfied in both systems. The uniqueness of bases for example is not destroyed by this construction. The same is true for the various anti-exchange properties, since they can always be reduced to either elements and sets in the lower or in the upper part.

The converse is also true. If $\mathcal{A}$ or $\mathcal{B}$ does not have a property Pxy, then this is also true for their D-sum, since the D-sum contains an isomorphic copy of both systems. Hence the D-sum of two systems is an example for a system which has precisely the properties both systems share, but no stronger property.

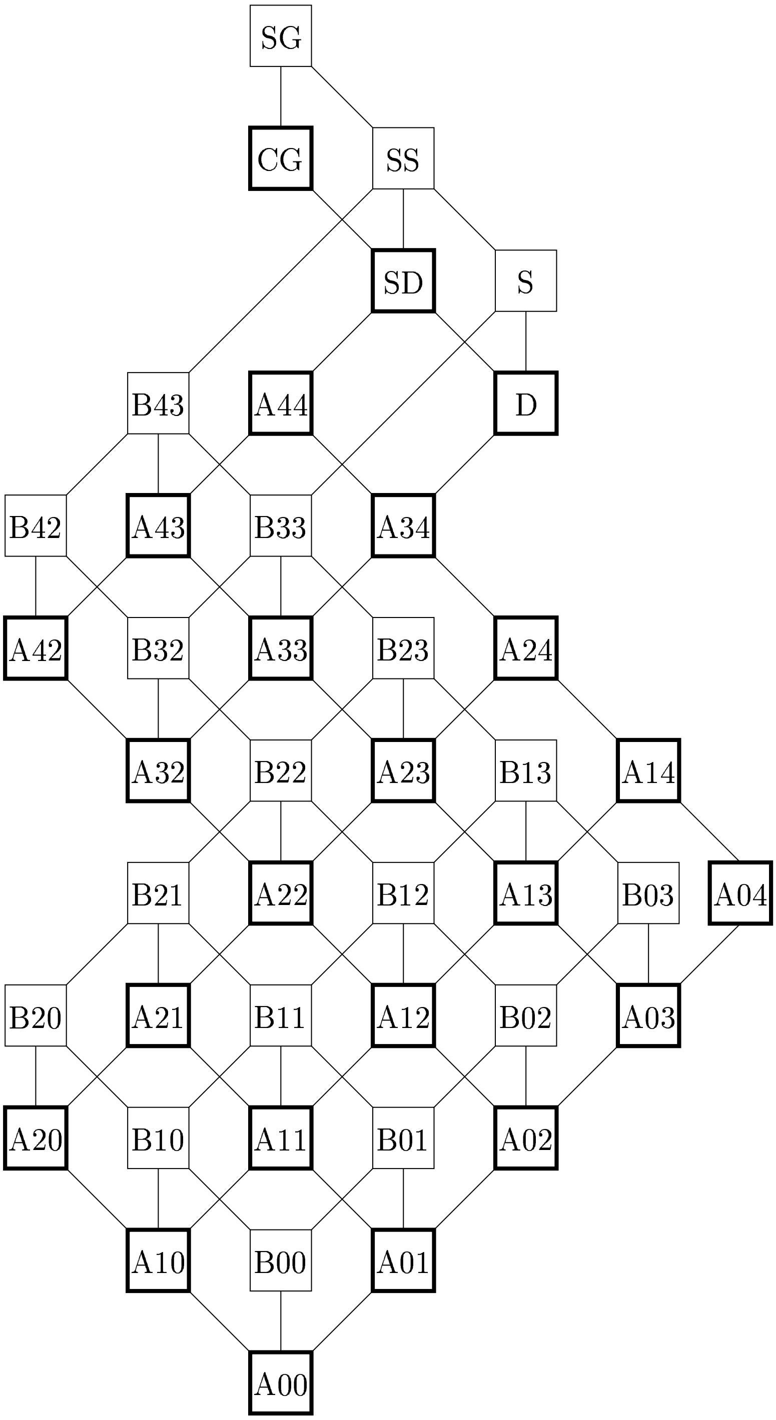


Figure 2.1: Diagram of Anti-Exchange Properties

2.4.11 Remark

Note that there is no system with one of the properties B04, B14, B24, B34 or B44. Any system in which every element has a least generator is shellable, hence the properties just mentioned cannot be weaker than shellability or strong shellability in the case of B44.

2.5 Examples

In this section we list examples for every box in our diagram, i.e. for every box we give a C_D-system which has precisely the properties of this box and satisfies no stronger property. In some cases we will use set-theoretic descriptions of the closure systems, in the other cases we show a diagram of a C_D-lattice which visualises the corresponding C_D-system on the set of $\bigvee$-irreducible elements.

We start with the strongest (combinations of) properties, since we can quite often skip a box if we can produce an example for that particular combination of properties as a D-sum of already given C_D-systems.

2.5.1 Shellable Convex Geometry (SG)

We start with the class that satisfies the strongest conditions, the shellable convex geometries, which satisfy all anti-exchange and bases properties we introduced. Objects of this type are obviously all finite convex geometries, and we can reuse example 2.2.8.

2.5.2 Convex Geometry (CG)

An example for a convex geometry, i.e. an algebraic C_D-system with the anti-exchange property, is the system of all convex sets of $\mathbb{E}^n$ [3]. Algebraicity has been known for quite a while, and detachability is easily checked. This system is certainly not shellable, as open convex subsets of $\mathbb{E}^n$ do not have extreme points.

Other examples for convex geometries are the systems of convex subsets of $\mathbb{N}$ or $\overline{\mathbb{N}}$.

Since convex sets were the initial ojects analysed in this regard, we use the name convex geometry for systems with this property. R. Jamison-Waldner, who apparently was the first to consider algebraic closure systems with the anti-exchange property in [35], initially used the term **antimatroid**, whereas P. Edelman introduced the name of convex geometries for finite systems with the AEP in [17].

When Edelman and Jamison-Waldner joined forces in [18], they kept both the name convex geometries and the restriction to finite sets.

2.5.3 Strongly Shellable System (SS)

Consider the infinite chain $(\omega^+)^{op}$.

The system of all $\bigwedge$-closed subsets of $(\omega^+)^{op}$ is shellable and satisfies the strong anti-exchange property. The crucial point is the unique limit point ω, since it is the only element that can be in the (directed) join of closed sets without being in one of the sets involved. So we see that this system is not algebraic.

2.5.4 Shellable System (S)

For this we only vary the previous example slightly. Consider the infinite chain $(2\omega^+)^{op}$ with the system of all $\bigwedge$-closed subsets of $(2\omega^+)^{op}$.

This time there are two elements, ω and 2ω, which can be in a join of closed sets without being in one of the sets involved. Thus there exist maximal chains which do not separate these two points. Simply construct a chain of sets by taking elements alternately from the upper and the lower part of $(2\omega^+)^{op}$.

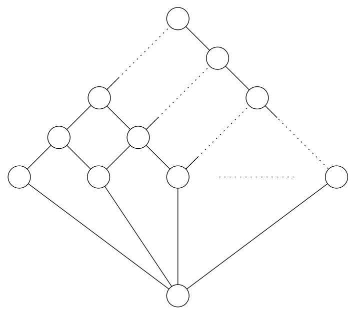
Systems of $\bigwedge$-closed subsets will be considered extensively in the next chapter, and it will contain a criteria for these systems to be shellable.

2.5.5 Detachable System (D)

The set $\mathbb{N}_0$ ordered by divisibility is a C_D-lattice which is isomorphic to a detachable C_D-system. A natural number n corresponds to the set of all prime powers p^k that divide n. E.g. the number 24 corresponds to the set $\{2, 2^2, 2^3, 3\}$. This system is neither shellable, as 0 does not have a lower cover, nor does it satisfy the strong anti-exchange property, since the chain of powers of 2 plus $\top$-element 0 is maximal but not C_0.

2.5.6 B43

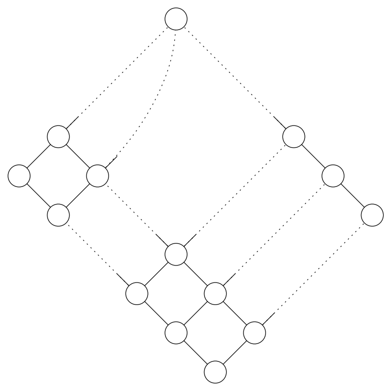
The following C_D-lattice is the visualisation for an example of a C_D-system with property B43, i.e. it satisfies the strong anti-exchange property and every element has a unique basis. Note that the basis of the top element is no least generator. The set of all atoms except the one on the right is another generator.



The system of all closed and convex intervals of ω^+ is one concrete system with this property, it is isomorphic to this lattice.

2.5.7 A44

The following C_D-lattice is (isomorphic to) an example for a C_D-system with property A44, i.e. it satisfies the strong anti-exchange property and all existing bases are least generators, but some elements (e.g. the element a) do not have a basis.



2.5.8 B32

A C_D-system with property B32, i.e. a system with the anti-exchange property in which every element has a basis, is

$$\begin{aligned} \mathcal{C} &= \{A \cup B \subseteq \mathbb{Z} \mid (A \subseteq_\omega \mathbb{N}) \wedge (B \subset \mathbb{Z} \setminus \mathbb{N})\} \\ &\cup \{A \cup B \subseteq \mathbb{Z} \mid (A \subset \mathbb{N}) \wedge (B \subseteq_\omega \mathbb{Z} \setminus \mathbb{N})\} \cup \{\mathbb{Z}\}. \end{aligned}$$

$\mathcal{C} \setminus \{\mathbb{Z}\}$ is a down-set in $\mathcal{P}\mathbb{Z}$. Every admissible subset of $\mathbb{Z}$ is a basis of itself, bases of $\mathbb{Z}$ are $\mathbb{N}$ and $\mathbb{Z} \setminus \mathbb{N}$.

2.5.9 A33

Although we can give examples of systems with A33, i.e. systems with the anti-exchange property and unique bases, by using D-sums of systems higher up in the diagram, there exists a very important class of examples for this property we want to mention here. The system of convex bodies (i.e. convex and compact subsets) of $\mathbb{E}^n$ with $\mathbb{E}^n$ as added $\top$-element has property A33, but is neither detachable nor does it have the strong anti-exchange property.

This shows that even a system which is closely related to a convex geometry need not be one itself, and not even detachable.

2.5.10 B23

The following example of a C_D-system with property B23, i.e. a system with the anti-exchange property for covers and a unique basis for every element, uses a set S which consists of three disjoint parts, an infinite set D, a countable set $C = \{c_1, c_2, ..\}$ isomorphic to ω and the two-element set $\{a, b\}$. The closure system $\mathcal{C}$ is defined as

$$\begin{array}{rcl} \mathcal{C} & = & \{T \cup \downarrow c_n \mid T \subseteq D,\, n \in \mathbb{N}\} \\ & \cup & \{F \cup \downarrow c_n \cup \{a\} \mid F \subseteq_\omega D,\, n \in \mathbb{N}\} \\ & \cup & \{F \cup \downarrow c_n \cup \{b\} \mid F \subseteq_\omega D,\, n \in \mathbb{N}\} \\ & \cup & \{S\}, \end{array}$$

where $\downarrow c_n = \{c_i \mid i = 1, .., n\}$.

The basis of the top element S is $\{a, b\}$, the bases for the other elements consist of $T \cup \{c_n\}$, $F \cup \{c_n, a\}$ or $F \cup \{c_n, b\}$. However, C is another generator of S, so property $A04$ is violated.

2.5.11 A24

An example for a system with property A24, i.e. a system with the anti-exchange property for covers in which bases are least generators, is an infinite cube with a sparse upper lid.

$$\mathcal{C} = (\omega \times \omega \times \omega) \cup (\omega \times \{0\} \times \{\omega\}) \cup (\{0\} \times \omega \times \{\omega\}) \cup (\{\omega\} \times \{\omega\} \times \{\omega\}).$$

The elements in the upper lid

$$(\omega \times \{0\} \times \{\omega\}) \cup (\{0\} \times 4\omega \times \{\omega\}) \cup (\{\omega\} \times \{\omega\} \times \{\omega\})$$

do not have a basis, but the elements in $(\omega \times \omega \times \omega)$ have their coordinates as basis.

2.5.12 B13

For an example of a C_D-system with property B13, i.e. a system with the anti-exchange property for copoints, in which every element has a unique basis, we once more use a base set S which consists of three disjoint sets, in this case ω^+ and two infinite sets C and F.

$$\begin{aligned}\mathcal{C} \;=\; & \{D \cup E \cup K \mid D \subset C,\, E \subseteq_\omega F,\\ & (K = \omega^+) \vee (K \subseteq \mathbb{N}) \vee (K \subseteq_\omega \omega) \vee (K \subseteq_\omega \mathbb{N} \cup \{\omega\})\} \cup \{S\}.\end{aligned}$$

Compare this with the example for a system with property B03. We added the two components $D \subset C$ and $E \subseteq_\omega F$ to obtain the anti-exchange property for copoints without losing existence and uniqueness of bases. The set $\mathbb{N}$ is a lower cover of ω^+, but contains two elements less than ω^+, violating the anti-exchange property for covers.

2.5.13 A14

Consider the C_D-system $\mathcal{C}$ we just used as an example for a system with property A24. The removal of a certain single element creates a system with property A14, i.e. a system with the anti-exchange property for copoints, in which bases are least generators,

$$\mathcal{C}' := \mathcal{C} \setminus (\{0\} \times \{1\} \times \{\omega\}).$$

By deleting this particular element, we do not destroy the copoint-property, but there now exists a covering pair which differs by two elements.

2.5.14 B21

Our example for a system with property B21, i.e. a system with the anti-exchange property for covers in which every element has a basis and in which finite bases are unique, uses once again a base set S of three disjoint subsets, $S = E \cup B \cup P$, in this case with the additional restriction that $2 \leq |E| < \omega$ and $\omega \leq |B|, |P|$. The following closure system is then an example for a system with property B21.

$$\mathcal{C} = \{F \cup A \cup R \mid (F \subset E) \wedge (A \subset B) \wedge (R \subseteq_\omega P)\} \cup \{S\}.$$

E is a finite basis of S, and B is an infinite basis of S, hence B03 is not satisfied.

2.5.15 B03

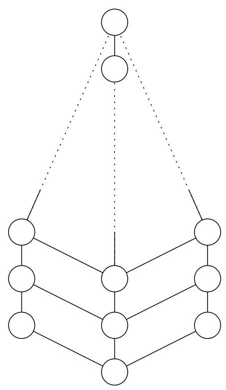
An example for a system with property B03, i.e. a system without any anti-exchange property but in which every element has a unique basis, is

$$\mathcal{C} = \{K \subseteq \omega^+ \mid (K = \omega^+) \vee (K \subseteq \mathbb{N}) \vee (K \subseteq_\omega \omega) \vee (K \subseteq_\omega \mathbb{N} \cup \{\omega\})\}.$$

The bases of an element $A \in \mathcal{C}$ are either the set A itself, or $\{0, \omega\}$ in the case of $A = \omega^+$. The anti-exchange property for copoints isnot satisfied, since $\mathbb{N}$ is both a copoint for 0 and ω.

2.5.16 A04

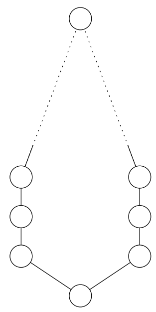
The following C_D-lattice is (isomorphic to) an example for a system with property A04, i.e. it satisfies no anti-exchange property but bases are least generators.



The unique coatom is copoint to two of the atoms, hence the anti-exchange property for copoints is not satisfied.

2.5.17 B20

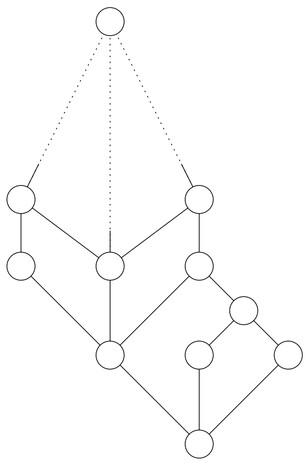
An example for a system with property B20, i.e. a system with the anti-exchange property for covers without any bases-property or the anti-exchange property is given here using a C_D-lattice.



This is the simplest example of a meet-complete ordered tree without maximal elements and an adjoined ⊤-element. All members of this class have property B20.

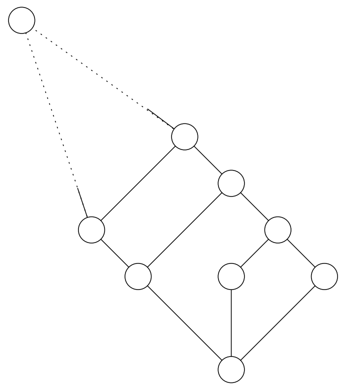
2.5.18 B10

For systems with property B10, i.e. systems with the anti-exchange property for copoints but not for covers, in which every element has a basis but where even finite bases are not unique, we can give the following example.



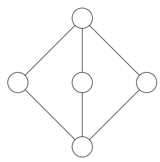
2.5.19 A10

Although we can give an example for a system with property A10, i.e. a system with the anti-exchange property for covers but without any bases-property, by taking the D-sum of examples with properties A20 and A11, we give this additional example, which is very similar to the previous one.



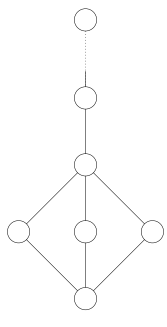
2.5.20 B00

An example for a system with property B00, i.e. a C_D-system which does not have any anti-exchange or bases-property except existence of bases for every element is given by M_3.



2.5.21 A00

Finally, a C_D-system with property A00, i.e. a C_D-system without any anti-exchange or bases property in which not every element has a basis, is given by



There is one big open question concerning this diagram, and that is whether the strong anti-exchange property implies uniqueness of bases. We have been unable to find examples for systems with the strong anti-exchange property, in which bases are not unique (i.e. examples for B42 or A42), but all attempts to prove the opposite were unsuccessful, too. Note that C_D-systems with the strong anti-exchange property are almost algebraic in the sense that the join of a chain of closed sets can only differ by one element from the union of these closed sets. Non-algebraic systems with this particular property seem to be rare, and so far we have only been able to find some in which bases are unique, see the examples for SS, B43 and A44.

Examples for the following properties can be obtained using D-sums of examples given in this section: SD, A43, B33, A34, A32, B22, A23, A22, B12, A13, A21, B11, A12, B02, A03, A20, A11, B01, A02 and A01.

2.5.22 Remarks

This extensive list of examples shows, that no two of our properties are equivalent, and that there exists a system for almost every combination of properties. The more interesting classes of objects, however, all satisfy properties rather high up in our diagram. All systems based on convex subsets of some set have the anti-exchange property, many are detachable and some even algebraic. Nevertheless, the variety of properties is a handicap to the goal of characterising "convex geometries" in a more general sense.

Note that some constructions use similar approaches. One good construction principle turned out to be to use a disjoint join of sets as base set, and to use different structures on the various parts. A good example for this is the system we gave as an example for property B13, which contains finite subsets of F, arbitrary subsets of C and certain subsets of ω^+, topped by the join of F, C and ω^+. In this way we were able to lift the example for a system with property B03 to one for B13. However, it is very difficult to combine systems with properties in the lower levels of the diagram to obtain a system with a property higher up in the diagram. A combination of two or more systems to get a new one with a specific property is at the moment only possible from above using D-sums.

2.6 Exchange and Anti-Exchange Property

In 2.1.11 we introduced both the anti-exchange and the exchange property. Although we are more interested in the anti-exchange property, we want to discuss for a second what happens if a C_D-system has both properties.

R. Jamison-Waldner dealt with the case of convex geometries (i.e. algebraic and detachable C_D-systems) with the exchange property in [35].

2.6.1 Theorem

In a convex geometry $\mathcal{C}$ on a set S with the exchange property, every subset $A \subseteq S$ that contains the least element $C \subseteq S$ of $\mathcal{C}$ is closed. If $C = \emptyset$, $\mathcal{C}$ is simply the power set of S.

Proof: see [35]

As with most of these results, the situation changes if we consider non-algebraic systems.

2.6.2 Theorem

Consider a C_D-system $\mathcal{C}$ on a set S with least element $C \subseteq S$. $\mathcal{C}$ has both the exchange and the anti-exchange property if and only if $A + x \in \mathcal{C}$ for every $A \in \mathcal{C}$ and every $x \notin A$.

Proof: If $A + x \in \mathcal{C}$ for arbitrary $A \in \mathcal{C}$ and $x \notin A$, then both the exchange and the anti-exchange property are satisfied.

Conversely, if $A \in \mathcal{C}$ and $x \notin A$, we can construct $A(x)$. For every $y \in A(x) \setminus A$ we get $A(y) = A(x)$ with the exchange property. The anti-exchange property then says $x = y$, and therefore $A(x) \setminus A = \{x\}$. □

Note that we could replace the anti-exchange property in this theorem by detachability, it too follows from the above-mentioned property. Shellability, on the other hand, is not implied by it, as can be seen in the next example.

2.6.3 Example

The easiest example for a non-algebraic C_D-system with both the exchange and the anti-exchange property is the following. Consider an infinite set S and construct the C_D-system consisting of all finite subsets of S and S as top element.

This systems trivially has both properties, but is certainly not algebraic and not the power set on S.

2.7 Lattice-Theoretical Description of Anti-Exchange Properties

In this section we want to use the categorical equivalence between the categories of C_D-systems and C_D-lattices to characterise systems with anti-exchange properties by analysing and characterising the corresponding C_D-lattice. We will see that some of these lattices have some well-known and well-studied properties.

2.7.1 Finite Convex Geometries

It is possible to characterise a finite convex geometry $(X, \mathcal{C})$ using purely lattice-theoretical properties of the lattice $\mathcal{C}$ of closed sets. The crucial property for this is lower local distributivity, which we defined earlier in 1.5.19. The following theorem was one of the first results on systems with the anti-exchange property. It is Theorem 3.3 in [16], the proof can be found there. Note that Dilworth and Edelman use the name meet-distributive for what we call lower locally distributive.

2.7.2 Theorem

A finite lattice L is isomorphic to a lattice of closed sets for some finite convex geometry if and only if L is lower locally distributive.

Two additional aspects of the lattice structure of $\mathcal{C}$ were stated in Theorem 4.2 in [18].

2.7.3 Theorem

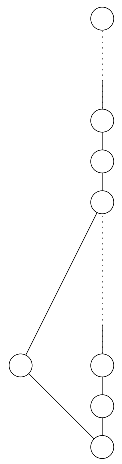
i) If C is a copoint of a finite convex geometry $(X, \mathcal{C})$ then C is a meet-irreducible element in $\mathcal{C}$.

ii) The interval $[C, K]$ in $\mathcal{C}$ is a Boolean algebra if and only if $K \setminus C \subseteq E(K)$.

We already mentioned the first part earlier, since it helps to define the anti-exchange property for copoints.

The characterisation of finite convex geometries using a single lattice-theoretical property is very nice, especially since the lower locally distributive lattices are well-studied, see [41] and the papers cited there. However, it is a property which relies heavily on the finiteness of the lattice.

The C_D-system on the set of $\bigvee$-irreducible elements of the next C_D-lattice L has not a single anti-exchange or basis property, i.e. is an example for property A00, but it is lower locally distributive, even lower locally Boolean. The sublattices generated by an element $a \in L$ and the set of all lower covers of a is always a 2-chain or the single element a, since every element has at most one lower cover.



For infinite C_D-systems with anti-exchange properties we will have to look for other properties for characterisation.

2.8 The Anti-Exchange Property for Copoints

Before we start analysing various lattice properties for our purposes, we deal with those C_D-lattices which are isomorphic to C_D-systems with the anti-exchange property for copoints.

If $\mathcal{C}$ is a C_D-system on the base set S without the anti-exchange property for copoints, it is easy to construct a system which contains the first one as a lower set and does satisfy the copoint-property. Simply consider an infinite set F disjoint from S. We then define

$$\mathcal{C}' := \{E \cup C \mid E \subseteq_\omega F,\, C \in \mathcal{C}\} \cup \{S \cup F\}.$$

The reason why this C_D-system satisfies the anti-exchange property for copoints is simply that it does not have any $\bigvee$-irreducible elements, i.e. copoints.

As a consequence, C_D-lattices corresponding to C_D-systems with the copoint-property cannot satisfy a **hereditary** lattice property unless it is satisfied by C_D-systems in general, where a hereditary property of a lattice L is one that is satisfied in every sublattice of L if it is satisfied in L. We can even weaken hereditariness to properties which are satisfied in every lower set if it is satisfied in the whole lattice. This makes it impossible to find a lattice-theoretical description of this class of C_D-systems with hereditary properties. Properties of this kind of weak hereditariness are e.g. distributivity and modularity, or existence and uniqueness of $\bigvee$-decompositions or bases.

We can give a lattice-theoretical description of the anti-exchange property for copoints using non-hereditary properties.

2.8.1 Lemma

A C_D-lattice L has the anti-exchange property for copoints if and only if

$$\forall\, j,k \in \mathcal{J}(L) : m \in L \text{ maximal in } L \setminus \uparrow j \text{ and in } L \setminus \uparrow k \Rightarrow j = k.$$

Proof: Direct translation of the anti-exchange property for C_D-systems to lattices.

From now on we will focus on C_D-systems which satisfy at least the anti-exchange property for covers for lattice-theoretical descriptions.

2.9 Narrow C_D-lattices

One of the first classes of C_D-systems with an anti-exchange property we considered was the class of systems with the anti-exchange property for covers. These are precisely those C_D-systems in which covering pairs differ in exactly one element. To describe these systems lattice-theoretically, we translate this property to C_D-lattices.

2.9.1 Definition

Consider a C_D-lattice L. We call L **narrow**, if covering pairs differ in exactly one $\bigvee$-irreducible element, i.e.

$$\forall\, a,b \in L : a \prec b \Rightarrow \exists!\, j \in \mathcal{J}(L) : j \leq b, j \nleq a.$$

2.9.2 Lemma

For a C_D-lattice L, the following are equivalent.

i) L *is narrow,*

ii) $\forall\, a \in L\, \forall\, j,k \in \mathcal{J}(L) : a \prec a \vee j = a \vee k \Rightarrow j = k.$

Recall, that $\mathcal{J}(L)$ denotes the set of $\bigvee$-irreducible elements of L.

Proof: Trivial.

So we see that the class of narrow C_D-lattices corresponds directly to the class of C_D-systems with the anti-exchange property for covers via the categorical equivalence 2.1.6.

With the notion of weak coframes (see 1.5.22) it is now possible to characterise narrow C_D-lattices.

2.9.3 Theorem

A C_D-lattice is narrow if and only if it is a lower semimodular weak coframe.

Proof: Consider a narrow lattice L. We start with semimodularity. Assume that $a \prec a \vee b$ but $a \wedge b \not\prec b$, i.e. there exists a $c \in L$ such that $a \wedge b < c < b$. Since L is $\bigvee$-irreducible generated, there exist $k, l \in \mathcal{J}(L)$ such that $k \leq c \leq b$ but $k \not\leq a \wedge b$, and $l \leq b$ but $l \not\leq c$. This implies $k \not\leq a$, $l \not\leq a$ and $l \neq k$. We have now found two $\bigvee$-irreducible elements below b and therefore below $a \vee b$ which are not below a. This contradicts that covering pairs differ by exactly one $\bigvee$-irreducible element, hence L must be lower semimodular.

We now assume that L is not a weak coframe, i.e. there are $a, b \in L$ and $D \subseteq L$ such that $a \prec b$, $D \subseteq \downarrow b$ and $\bigvee D \leq a$ but $a \notin \downarrow D$. Since L is narrow we know that there is exactly one $\bigvee$-irreducible element $j \in \mathcal{J}(L)$ such that $j \leq b$ but $j \not\leq a$.

An arbitrary $d \in D$ satisfies $a \not\leq d$, thus there exists a $k \in \mathcal{J}(L)$ with $k \leq d \leq b$ but $k \not\leq a$. Since L is narrow we get $k = j$. Since every $d \in D$ is larger than j we also get $j \leq \bigwedge D$, but we started with $\bigwedge D \leq a$, which now implies $j \leq a$, a contradiction. Consequently, L is a weak coframe.

For the second part of the equivalence we start with a lower semimodular weak coframe L, and assume that it is not narrow, i.e. that there exist $a \in L$ and $j, k \in \mathcal{J}(L)$ such that $a \prec a \vee j = a \vee k =: b$ but $j \neq k$.

Lower semimodularity applied to $a \prec a \vee j$ gives us $a \wedge j \prec j$, i.e. $a \wedge j = j_\vee$ is the unique lower cover of j. Similarly, $a \wedge k = k_\vee$ is the unique lower cover of k.

There are now two possibilities for the meet of j and k. If $j \wedge k \neq j, k$, then $j \wedge k \leq j_\vee \wedge k_\vee \leq a \wedge j \leq a$, but this contradicts that a is $\bigwedge$-prime in $\downarrow b$.

Alternatively j and k are comparable, i.e. without loss of generality, $j \wedge k = k$. This, however, implies $k \leq j_\vee = a \wedge j \leq a$, a contradiction to $a \prec a \vee k$. Since all possibilities end in contradictions, our assumption $j \neq k$ is impossible and we obtain $j = k$. Hence, L is a narrow C_D-lattice. □

In the proof we only requiered a to be $\wedge$-prime in $\downarrow b$. Compare this with the definition of semidistritbutivity in [14] and his results for semidistributive and locally distributive lattices.

Interestingly, the combination of lower semimodularity and weak coframe gives us a lower locally Boolean lattice. The converse is not true, as we have seen in the example just before 2.8.

2.9.4 Lemma

A narrow C_D-lattice L is lower locally Boolean.

Proof: The proof is easiest if you use the corresponding C_D-system $\mathcal{C}$ with anti-exchange property for covers. Consider $A \in \mathcal{C}$ and $\mathcal{B} \subseteq \mathcal{C}$ consisting of all lower covers of A, i.e.

$$\mathcal{B} = \{B \in \mathcal{C} \mid \exists\, x \in A : B = A - x\}.$$

These elements $x \in A$ such that $A - x \in \mathcal{C}$ are once again the extreme points of A. We then have $\bigwedge \mathcal{B} = \bigcap \mathcal{B} = A \setminus E(A)$, and the interval $[\bigwedge \mathcal{B}, A]$ is isomorphic to the power set of $E(A)$, and therefore Boolean. □

Now we know that the two lattice properties lower semimodularity and weak coframe together characterise all C_D-lattices which correspond to C_D-systems with the anti-exchange property for covers, and that every C_D-lattice that corresponds to an anti-exchange property higher up in our diagram has these two properties plus some others. We did not always succeed to isolate exactly these extra properties required to describe the other anti-exchange properties. However, we were successful to find the missing bit in one important case.

2.10 Persistent Lattices

2.10.1 Definition

Consider a C_D-lattice L. We call $j \in L$ **indispensible**, if it is contained in every $\bigvee$-decomposition of elements $x = a \vee j$ whenever $j \not\leq a$, i.e. j is indispensible if

$$\forall a \in L\, \forall\, K \subseteq \mathcal{J}(L) : a < a \vee j = \bigvee K \Rightarrow j \in K.$$

We call L **persistent**, if every element is a join of indispensible elements; in other words, if every $\bigvee$-irreducible element is indispensible.

This definition and the following theorem are both taken from [22].

2.10.2 Theorem

The persistent C_D-lattices are precisely those which correspond to detachable C_D-systems via the categorical equivalence 2.1.6.

Proof: First, we show that a C_D-system $\mathcal{C}$ which is a persistent C_D-lattice is detachable. Consider $A \in \mathcal{C}$ and a $\bigvee$-irreducible element $C(x) \nsubseteq A$. This gives

$$A \subset A \vee C(x) = A(x).$$

Define $\mathcal{K} \subseteq \mathcal{J}(\mathcal{C})$ by

$$\mathcal{K} = \{C(y) \mid y \in (A(x) - x) \setminus C\}.$$

By construction, $A(x) - x \subseteq \bigvee \mathcal{K} \subseteq A(x)$. Since $\mathcal{C}$ is persistent and $C(x) \notin \mathcal{K}$, we get $\bigvee \mathcal{K} \subset A(x)$, hence $\bigvee \mathcal{K} = A(x) - x$ and therefore $A(x) - x \in \mathcal{C}$.

Now we consider a detachable C_D-system $\mathcal{C}$ and show that this gives a persistent C_D-lattice. Assume $A \in \mathcal{C}$ and $C(x) \nsubseteq A$. Therefore, $A \subset A \vee C(x) = A(x)$. If $\mathcal{K}$ is a set of $\bigvee$-irreducible elements contained in A(x) without $C(x)$, then all $C(y) \in \mathcal{K}$ are contained in $A(x) - x$, since $\mathcal{C}$ is detachable. Hence,
$\bigvee \mathcal{K} \subseteq A(x) - x \subset A(x)$. This shows that $\mathcal{C}$ is persistent. □

From this we know that persistent lattices are special cases of narrow lattices. Since narrow lattices are lower semimodular weak coframes, persistent lattices need to have these properties as well. However, this does not suffice for a characterisation of persistent lattices, for this we need more properties. The missing link is given by Kung in [39].

2.10.3 Definition

A C_D-lattice is called **consistent**, if for every $a \in L$ and every $j \in \mathcal{J}(L)$ not below or equal to a, the element $a \vee j$ is $\bigvee$-irreducible in $\uparrow a$, i.e.

$$\forall\, a \in L\, \forall\, j \in \mathcal{J}(L) : a < a \vee j \Rightarrow a \vee j \in \mathcal{J}(\uparrow a).$$

The consequence of consistency is an abundance of covering pairs everywhere in the lattice. Without consistency the only covering pairs in an arbitrary C_D-lattice we know of are the $\bigvee$-irreducible elements and their lower covers. If we want to characterise detachability, we need many covering pairs, since there are many covers in the detachable C_D-system, namely those of the form $A(x) - x \prec A(x)$ for a closed set A and an element $x \notin A$.

Before we give a characterisation of persistent lattices we show an interesting consequence of consistency combined with lower semimodularity.

2.10.4 Lemma

In a lower semimodular and consistent C_D-lattice L, every $j \in \mathcal{J}(L)$ is resistant (see 1.6.2).

Proof: We have to show that $j \in \mathcal{J}(L)$ with unique lower cover $j_\vee \in L$ and $b \in L$ with $j \nleq b$ implies $j \nleq b \vee j_\vee$.

Suppose there exists $j \in \mathcal{J}(L)$ and $b \in L$ such that $j \nleq b$ but $j \leq b \vee j_\vee =: a$. This implies $j < j \vee b = a$. Consistency of L says that $b \vee j$ is $\bigvee$-irreducible in $\uparrow b$, i.e. there exists an element $a' \in L$ such that $b \leq a' \prec a$. This implies $j_\vee \vee a' = a$.

But lower semimodularity applied to $a' \prec a' \vee j$ results in $a' \wedge j \prec j$. Since $j \in \mathcal{J}(L)$ this means $a' \wedge j = j_\vee$ and $j_\vee \leq a'$. From this we get $j_\vee \vee a' = a' \neq a$, a contradiction to the last paragraph. Therefore L needs to be resistant. $\square$

We now have the freedom to give two equivalent descriptions of persistent lattices.

2.10.5 Theorem

For a C_D-lattice, the following are equivalent.

i) L is persistent,

ii) L is a lower semimodular and consistent weak coframe,

iii) L is a strongly generated and consistent weak coframe.

iv) L is narrow and consistent.

Proof: For the first implication (i) $\Rightarrow$ (ii) we only need to show that persistent lattices are consistent, since they are a special case of narrow lattices. Suppose there exist $a \in L$ and $j \in \mathcal{J}(L)$ with $a < a \vee j =: x$ for which x is not $\bigvee$-irreducible in $\uparrow a$, i.e. there exists a $B \subseteq [a, x[$ such that $\bigvee B = x$. All elements $b \in B$ need to satisfy $j \nleq b$, otherwise $x = a \vee j \leq b$. We define the set $K = \{k \in \mathcal{J}(L) \,|\, \exists\, b \in B : k \leq b\}$. Since L is a C_D-lattice we have $\bigvee K = \bigvee B = x$ and $j \notin K$. This contradicts persistency which says that j is contained in every $\bigvee$-decomposition of x. Therefore L is consistent.

For the implication (ii) $\Rightarrow$ (iii), we simply use the previous lemma, the equivalence (ii) $\iff$ (iv) is 2.9.3.

For the last implication (iii) $\Rightarrow$ (i) we need a case differentiation. Suppose we have $a \in L$, $j \in \mathcal{J}(L)$ and $K \subseteq \mathcal{J}(L)$ such that $a < x := a \vee j = \bigvee K$ but $j \notin K$. Define $K' := \{k \vee a \,|\, k \in K\} \subseteq [a, x]$.

a) $\forall k' \in K' : k' < x$. This implies $x = \bigvee K \leq \bigvee K' \leq x$. This contradicts the $\bigvee$-irreducibility of x in $\uparrow a$.

b) $\exists k' \in K' : k' = x$, i.e. $\exists k \in K$ with $a \vee k = x = a \vee j$.

If j and k are comparable, e.g. $j < k$, then this violates resistance, since we get $k \leq a \vee j$ with $k \nleq a$ and $j < k$.

If j and k are incomparable, we use a similar construction as in the previous lemma. There exists a lower cover x' of x with $a \leq x' \prec x$, since x is $\bigvee$-irreducible in $\uparrow a$. This element satisfies $x' \vee j = x' \vee k = x$. The unique lower covers of j and k are again denoted by $j_\vee$ and $k_\vee$. For $j_\vee$ we have $j_\vee < x'$, since otherwise we would have $j_\vee \vee x' = x$, which would violate resistance of j. Similarly, $k_\vee < x'$. From this we get $j \wedge k = j_\vee \wedge k_\vee < x'$, but $j, k \nleq x'$, which contradicts the weak coframe property of L. □

This closes the characterisation theorem for persistent lattices. It would have been slightly shorter if we would have restricted it to equivalences (i), (ii) and (iv), but since (iii) looks slightly weaker on paper we incorporated it as well.

Kung was the first to introduce consistent lattices, but nowadays it is known that it is an alternative description of an older and well-known property, see [44].

2.10.6 Definition

Consider a C_D-lattice L. We say that L satisfies the **Kurosh-Ore Replacement Property** (or **KORP**), if for $x \in L$, $K, M \subseteq \mathcal{J}(L)$ with $x = \bigvee K = \bigvee M$, and for every $k \in K$, there exists an $m \in M$ such that $x = \bigvee((K - k) + m)$.

It suffices to consider only elements $k \in K$ for which $\bigvee(K - k) < x$.

2.10.7 Lemma

In a C_D-lattice, the KORP holds if and only if L is consistent.

Proof: See [44]

It is interesting to find a replacement or exchange property in the study of anti-exchange properties. We will come back to the KORP in a moment and look at its relation to the classical anti-exchange property.

2.11 The Anti-Exchange Property

After dealing with two variations of it, we now turn to the original anti-exchange property. Since it is stronger than the anti-exchange property for covers, lattices with the anti-exchange property are lower semimodular weak coframes, but they need not be consistent, since this would already imply detachability. A description of the anti-exchange property similar to the one for the anti-exchange property for covers, however, seems to be much more difficult. We start with a translation of the anti-exchange property for C_D-systems to a lattice-theoretical one.

2.11.1 Definition

Consider a C_D-lattice L. We say that L satisfies the **anti-exchange property** (or **AEP**), if for $a \in L$ and $j, k \in \mathcal{J}(L)$ the equality $a < a \vee j = a \vee k$ implies $j = k$.

This is a direct translation of the anti-exchange property for C_D-systems to C_D-lattices. This version, however, makes it easier to formulate equivalent descriptions of this property in lattice theory.

2.11.2 Remark

Since the concepts of persistent and narrow C_D-lattices and of the anti-exchange property as a lattice property are direct translations of properties for C_D-systems, we get the same implications between lattices as we did for closure systems. This means that persistent lattices satisfy the AEP, and lattices which satisfy the AEP are narrow.

2.11.3 Theorem

In a C_D-lattice L, the following are equivalent.

i) *L satisfies the anti-exchange property.*

ii) $\forall\, a \in L\, \forall j, k \in \mathcal{J}(L), j \neq k :$
$((a < a \vee j) \textit{ and } (k < a \vee j)) \Rightarrow a \vee k < a \vee j.$

iii) $\forall\, a \in L\, \forall\, j \in \mathcal{J}(L)\, \forall\, F \subseteq_\omega \mathcal{J}(L) : a < a \vee j = a \vee \bigvee F \Rightarrow j \in F.$

iv) $\forall\, x \in L\, \forall\, K, M \subseteq \mathcal{J}(L)\, \forall\, k \in K :$
$((x = \bigvee K = \bigvee M) \textit{ and } (\bigvee(K - k) < x))$
$\Rightarrow (\forall \in m \in M : \bigvee((K + m) - k) < x).$

Proof: (i) $\Rightarrow$ (ii): We certainly have $a \vee k \leq a \vee j$, since both a and k are below $a \vee j$. The case $a \vee k = a \vee j$ would imply $j = k$, since L satisfies the AEP, hence we have a strict inequality.

(ii) $\Rightarrow$ (i): To show the AEP, we start with $a < a \vee j = a \vee k$. If we assume that $j \neq k$, this implies $k < a \vee j$ and, using (ii), $a \vee k < a \vee j$, a contradiction.

(i) $\Rightarrow$ (iii): We use $F = \{f_1, f_2, ..., f_n\}$. We define $a := a_0$, and $a_m := a_{m-1} \vee f_m$ for $m = 1, ..., n$. This leads to a chain of inequalities:
$a_0 \leq a_1 \leq ... \leq a_n = a \vee \bigvee F$.

For one $1 \leq m \leq n$ we get $a_{m-1} < a_m = a \vee j$. But $a_m = a_{m-1} \vee f_m$ and $a \vee j = a_{m-1} \vee j$, i.e. $a_{m-1} < a_{m-1} \vee f_m = a_{m-1} \vee j$, and the AEP implies $j = f_m$, or alternatively $j \in F$.

(iii) $\Rightarrow$ (i): Follows directly if you put $F = \{k\}$.

(i) $\Rightarrow$ (iv): Suppose we have $x = \bigvee K = \bigvee M$, $\bigvee(K - k) < x$ but $\bigvee((K+m)-k) = x$. Define $a := \bigvee(K-k)$. Since $a < \bigvee((K+m)-k)$, we must have $m \notin K$. This means $\bigvee((K+m)-k) = a \vee m = x = a \vee k$, and therefore $k = m$, which is a contradiction.

(iv) $\Rightarrow$ (i): Suppose we have $a < a \vee k = a \vee m$ with $k \neq m$. We define $A := \{j \in \mathcal{J}(L) \,|\, j \leq a\}$, $K := A + k$ and $M := A + m$. This gives $x := a \vee k = \bigvee K = \bigvee M$ with $a = \bigvee(K - k) < x$, but it also implies $x = \bigvee((K + m) - k)$, a contradiction to condition (iv). □

2.11.4 Remark

Of these four equivalent characterisations of the anti-exchange property, the fourth one seems to be the most interesting one. One reason for this is its similarity to the Kurosh-Ore Replacement Property. The last term in the KORP is $x = \bigvee((K - k) + m)$, the last term here is $\bigvee((K + m) - k) < x$.

At first glance, these terms seem to oppose each other, since we have equality in one, and strict inequality in the other case. We therefore call equivalence (iv) the **Anti-Kurosh-Ore Replacement Property** (or **Anti-KORP**).

There is, however, a difference in the order in which m and k are added or subtracted, respectively. To be precise,

$$((K - k) + m) = \{j \in \mathcal{J}(L) \,|\, j \in K, j \neq k\} \cup \{m\},$$

but this only coincides with $((K + m) - k)$ if $m \neq k$. If $m = k$, then we get

$$((K + m) - k) = \{j \in \mathcal{J}(L) \,|\, j \in K, j \neq k\} = (K - k).$$

In a way, these two properties work together quite nicely. The KORP says that every element of one decomposition can be replaced by an element of a second decomposition. The Anti-KORP says that irredundant irreducible elements cannot be replaced by different elements. If both properties hold, this means that an element which is irredundant in one decomposition needs to be contained in every decomposition of the same element.

2.11.5 Corollary

A C_D-lattice L satisfies both the Kurosh-Ore Replacement Property and the Anti-Kurosh-Ore Replacement Property if and only if it is persistent.

Proof: The proof of this is simply a combination of previous results. The Anti-KORP is equivalent to the AEP, as stated in 2.11.3. According to 2.10.7, the KORP is equivalent to consistency. Hence, lattices with both the Anti-KORP and the KORP are the C_D-lattices which are lower semimodular and consistent weak coframes, i.e. persistent lattices. □

The Kurosh-Ore Replacement Property should not be confused with the exchange property (EP) given in 2.1.11. The KORP for C_D-systems says that if $K, M \subseteq S$ have the same closure $\Gamma K = \Gamma M$, then for every $k \in K$ there exists a $m \in M$ such that $\Gamma(K - k + m) = \Gamma K$. In this, we could restrict ourselves to cases in which $\Gamma(K - k) \subset \Gamma K$.

The EP applied to the same situation says that if $\Gamma(K - k) \subset \Gamma K$ and $m \in M \setminus \Gamma(K - k)$, then $\Gamma(K - k + m) = \Gamma K$. The difference is that with the EP we can take every element in M which is not contained in $\Gamma(K - k)$ to fill up $\Gamma(K - k)$ to regain ΓK, whereas the KORP only guarantees the existence of one such element. Hence, the EP is stronger than the KORP.

This does not come as a surprise, since we saw in 2.6.2 that exchange plus anti-exchange property give something very strong which implies detachability, whereas KORP and Anti-KORP (i.e. AEP) are equivalent to detachability, as seen in 2.11.5.

K. Adaricheva, V. Gorbunov and V. Tumanov approached the problem of characterising (infinite) lattices with the anti-exchange property in [3] in a different way. They restricted themselves almost exclusively to atomistic lattices, and in these lattices the AEP can be characterised by a variation of the join-semidistributivity law.

2.11.6 Definition

Consider a lattice L and a subset $A \subseteq L$. We say L satisfies $\mathbf{SD}_\vee(A)$, if $x \vee y = x \vee z$ implies $x \vee y = x \vee (y \wedge z)$ for all $x \in L$ and $y, z \in A$.

For $A = L$ this is equivalent to the classical join-semidistributivity. If we denote the set of all atoms of a lattice L by $At(L)$, we get the following results for atomistic lattices.

2.11.7 Lemma

An atomistic lattice L has the anti-exchange property if and only if is satisfies $SD_\vee(At(L))$.

Proof: See Prop. 3.1 in [3]

Although this is a nice and short characterisation of lattices with the AEP, it is inappropriate for our purposes. We consider the more general C_D-lattices, in which every element is join of $\bigvee$-irreducible elements. Atomistic lattices are a special case of C_D-lattices, but not all results for atomistic lattices are true for C_D-lattices. So instead of $SD_\vee(At(L))$ we have to ask if $SD_\vee(\mathcal{J}(L))$ suffices to describe C_D-lattices with the AEP, and the answer is no, $SD_\vee(\mathcal{J}(L))$ is strictly weaker than the AEP.

2.11.8 Lemma

i) A C_D-lattices with the AEP satisfies $SD_\vee(\mathcal{J}(L))$.

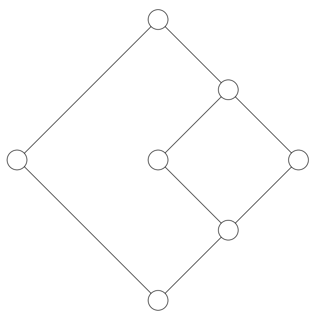
ii) A finite C_D-lattice which satisfies $SD_\vee(\mathcal{J}(L))$ need not satisfy the AEP.

iii) A narrow C_D-lattice that satisfies $SD_\vee(\mathcal{J}(L))$ must not satisfy the AEP.

iv) A C_D-lattice with the anti-exchange property for covers need not satisfy $SD_\vee(\mathcal{J}(L))$.

Proof: (i): This is obvious. Consider $a \in L$ and $j, k \in \mathcal{J}(L)$. If $a < a \vee j = a \vee k$, then the AEP implies $j = k$, and therefore $a \vee j = a \vee (j \wedge k)$.

(ii): These two properties are not even equivalent for finite lattices. The following C_D-lattice has $SD_\vee(\mathcal{J}(L))$, but not the AEP.



This finite example shows, that $SD_\vee(\mathcal{J}(L))$ does not even imply the anti-exchange property for copoints or covers. This is of course necessary, since these anti-exchange properties all coincide for finite lattices.

(iii) The lattice given as an example for property B20 is narrow and satisfies $SD_\vee(\mathcal{J}(L))$, but does not satisfy the anti-exchangeproperty. This lattice even satisfies $SD_\vee(L)$, i.e. it is semidistributive, but not the AEP.

(iv) Recall the example for property B23, in which we used a base set $S = D \cup C \cup \{a, b\}$. There we have $S = D \vee \{a\} = D \vee \{b\}$, but $D \in \mathcal{C}$ hence $S = D \vee \{a\} \neq D \vee (\{a\} \wedge \{b\}) = D$. □

We now know that $SD_\vee(\mathcal{J}(L))$ is insufficient, but it is possible to close the gap between lattices with $SD_\vee(\mathcal{J}(L))$ and those with the AEP.

2.11.9 Definition

Consider a C_D-lattice L. We call L **weakly resistant**, if it satisfies the following implication:

$$\forall a \in L\, \forall\, F \subseteq \mathcal{J}(L)\ (|F| \geq 2) : \Big(a \notin \uparrow F \Rightarrow \exists\, f \in F : (f \nleq a \vee \bigwedge F) \Big).$$

2.11.10 Lemma

Consider a C_D-lattice L. The following are equivalent:

i) L *is weakly resistant,*

ii) $\forall\, a \in L \forall F \subseteq \mathcal{J}(L),\ |F| \geq 2\, \forall\, f \in F : (f \leq a \vee \bigwedge F) \Rightarrow \exists\, f \in F : (f \leq a)$,

iii) $\forall\, a \in L \forall F \subseteq \mathcal{J}(L),\ |F| \geq 2 : \bigvee F \leq a \vee \bigwedge F \Rightarrow \exists\, f \in F : (f \leq a)$,

Proof: (ii) is just the contraposition of the implication in (i), and in (iii) we simply replaced $\forall\, f \in F : (f \leq a \vee \bigwedge F)$ by the equivalent $\bigvee F \leq a \vee \bigwedge F$. □

The name already suggests that weak resistance is a weak form of resistance, and this is indeed true.

2.11.11 Lemma

A resistant C_D-lattice is weakly resistant.

Proof: Consider a resistant C_D-lattice L, with $a \in L$ and $F \subseteq \mathcal{J}(L)$ with $|F| \geq 2$. There exists an element $f \in F$ such that $\bigwedge F < f$. If this element satisfies $f \nleq a$, then $f \nleq a \vee \bigwedge F$, since f is resistant. □

We now show that this property is what we have to add to $SD_\vee(\mathcal{J}(L))$ to obtain a characterisation of the anti-exchange property.

2.11.12 Theorem

A C_D-lattice L satisfies the anti-exchange property if and only if it is weakly resistant and satisfies $SD_\vee(\mathcal{J}(L))$.

Proof: We start with a lattice with the AEP. We already know that this implies $SD_\vee(\mathcal{J}(L))$. Suppose it is not weakly resistant, i.e. there are $a \in L$, $F \subseteq \mathcal{J}(L)$ with $|F| \geq 2$ and $f \nleq a$ for all $f \in F$, but also $f \leq a \vee \bigwedge F$ for all $f \in F$. Since trivially $a \leq a \vee \bigwedge F$ we obtain $\forall f \in F : a \vee f \leq a \vee \bigwedge F$, even $\forall f \in F : a \vee f = a \vee \bigwedge F$. For two arbitrary elements $f, g \in F$ this results in $a \vee f = a \vee \bigwedge F = a \vee g$. If we apply the AEP we get $f = g$, a contradiction to $|F| \geq 2$. Hence L is weakly resistant.

For the converse direction we start with a weakly resistant lattice L which satisfies $SD_\vee(\mathcal{J}(L))$. Consider $a \in L$ and $j, k \in J(L)$ such that $a < a \vee j = a \vee k$, i.e. $j, k \nleq a$. We put $F := \{j, k\}$. We apply $SD_\vee(\mathcal{J}(L))$ to get $a \vee j = a \vee (j \wedge k) = a \vee \bigwedge F$, similarly $a \vee k = a \vee \bigwedge F$. Since L is weakly resistant, this cannot be true for $|F| \geq 2$, thus $|F| = 1$ and we must have $j = k$. $\square$

Now we know that every lattice with the AEP satisfies a weak form of resistance, and we know that every persistent lattice is resistant. We can ask if we can replace the weak resistance in the last theorem by the full strength resistance. The next example shows that this is not possible.

2.11.13 Examples

i) Consider the following C_D-system $\mathcal{C}$ on the base set $S = \omega^+$.

$$\mathcal{C} = \{A \mid A \subseteq \omega \setminus \{0\}\} \cup \{B \mid B \subseteq_\omega \omega\} \cup \{\omega^+ \setminus \{0\}\} \cup \{\omega^+\}.$$

This system has the anti-exchange property, but the set $\{\omega^+ \setminus \{0\}\}$, which is the point closure of ω and therefore a $\bigvee$-irreducible element in the lattice, is not resistant. The join of $\omega \setminus \{0\}$ and $\{0\}$ is the full base set ω^+.

ii) We just saw that lattices (or systems) with the anti-exchange property need not be resistant. Another interesting fact is, that even narrow lattices that are resistant need not satisfy the AEP. An example for this is once again the example for property B23. It is a resistant and narrow C_D-system, i.e.

the lattice of closed sets satisfies the AEP for covers, but is does not satisfy the original anti-exchange property.

2.12 Bigenerated Lattices

Adaricheva et.al. noted in [3], that many but not all of their examples for atomistic systems with the anti-exchange property are biatomic, a property first introduced by M.K. Bennett in [6] as "additivity", then renamed to biatomicity in [7]. Since we work with the more general C_D-lattices in which every element is a join of $\bigvee$-irreducible elements instead of atoms, we have to convert this property to satisfy our demands. We also give the definition of Bennett and Birkhoff from [7].

2.12.1 Definitions

An atomistic lattice L is called **biatomic** when, given $a, b \in L - \bot$ and $p \in At(L)$ with $p \leq a \vee b$, then there are atoms $a_1, b_1 \in At(L)$ such that $a_1 \leq a$, $b_1 \leq b$ and $p \leq a_1 \vee b_1$.

A C_D-lattice is called **bigenerated** when, given $a, b \in L - \bot$ and $p \in \mathcal{J}(L)$ with $p \leq a \vee b$, then there are $j, k \in \mathcal{J}(L)$ such that $j \leq a$, $k \leq b$ and $p \leq j \vee k$.

Obviously, a lattice is biatomic, if it is atomistic and bigenerated.

The definition of a bigenerated lattice can easily be translated to bigeneration in C_D-systems.

A C_D-system $\mathcal{C}$ is called **bigenerated**, if for $A, B \in \mathcal{C}$ and $x \in A \vee B$ there are elements $a \in A$ and $b \in B$ such that $x \in C(a) \vee C(b)$.

The notions of biatomicity and bigeneration should not be confused with distributivity for $\vee$-semilattices as considered in [38] (see also [25]) which looks similar, but the conditions on the elements differ quite strongly.

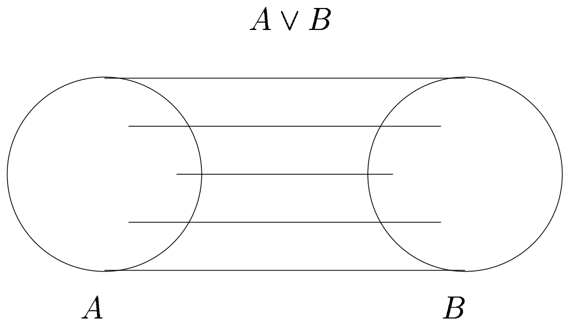
2.12.2 Remark

By definition, a C_D-system $\mathcal{C}$is bigenerated if and only if the join of two closed sets $A, B \in \mathcal{C}$ is given by

$$A \vee B = \bigcup \{C(a) \vee C(b) \mid a \in A,\, b \in B\}\,.$$

2.12.3 Examples

i) Bennett and Birkhoff already gave the most obvious and important example for a biatomic lattice, the system $Co(\mathbb{E}^2)$ of all convex subsets of $\mathbb{E}^2$, illustrated by the following picture.



ii) If we modify this example and consider only those convex subsets of $\mathbb{E}^2$ which contain the unit circle, we get a system which is still bigenerated but not biatomic, since it has no atoms.

iii) The main reason why we do not restrict our considerations to bigenerated lattices is, that the restriction of convex sets to a finite subset of $\mathbb{E}^2$ as introduced in 2.2.8 is not bigenerated, as can be seen from the example given there.

Although not all objects which are interesting to us are bigenerated lattices, it is a very nice property, and bigenerated lattices share some very good properties. It is also possible to give another characterisation of the anti-exchange property for bigenerated lattices or systems.

2.12.4 Definition

Consider a C_D-system $\mathcal{C}$, with $A \in \mathcal{C}$. We say that $a \in A$ is a **weak extreme point** of A, if we have $a \notin \Gamma\{x, y\}$ for every $x, y \in A - a$, where Γ denotes the closure operator corresponding to $\mathcal{C}$.

This is weaker than the definition of a standard extreme point. Consider e.g. the unit circle D in the C_D-system of compact bodies in $\mathbb{E}^2$. All elements on the boundary of D are weak extreme poins of D, but in this closure system they are not extreme points in the original sense. However, some authors define extreme points in this way, see e.g. [19] and [42].

2.12.5 Lemma

If the C_D-system $\mathcal{C}$ satisfies the anti-exchange property and $x \notin A$, then x is a weak extreme point of $A(x)$.

Proof: Suppose there are $y, z \in A(x) - x$ with $x \in \Gamma\{y, z\} = C(y) \vee C(z)$. At least one of these elements needs to be in $A(x) \setminus A$, since otherwise $C(y) \vee C(z) \subseteq A$.

If $y \in A$ and $z \notin A$, the AEP gives us $x \notin A(z)$, which implies that $x \notin C(y) \vee C(z) \subseteq A(z)$, a contradiction.

Thus both y and z need to be in $A(x) \setminus A$. Another application of the AEP gives us $A(y) \subset A(x)$. We define $B := A(y)$. Since we have $B(x) = A(x)$, we can now apply the AEP to B and get $B(z) \subset B(x)$, i.e. $x \notin C(y) \vee C(z) \subseteq B(z)$, another contradiction to $x \in C(y) \vee C(z)$. Thus, x is a weak extreme point of $A(x)$. $\square$

If we have a bigenerated C_D-system, we can also show the converse.

2.12.6 Theorem

Consider a bigenerated C_D-system $\mathcal{C}$, where for every $A \in \mathcal{C}$ and $x \notin A$ x is a weak extreme point of $A(x)$. Then $\mathcal{C}$ satisfies the anti-exchange property.

Proof: Suppose $A \subset A(x) = A(y)$ but $x \neq y$. Since $y \in A(x) = A \vee C(x)$ there exists an element $z \in A$ such that $y \in C(z) \vee C(x) = \Gamma\{z, x\}$. This contradicts the hypothesis that y is a weak extreme point of $A(y) = A(x)$. Therefore x needs to be equal to y, and we get the anti-exchange property. $\square$

The bigeneration cannot be dropped in this result. Consult once more the example for property B23 for a system which is not bigenerated and does not satisfy the AEP but in which every x is a weak extreme point of $A(x)$ as long as $x \notin A$.

2.13 C_D-systems and distributivity

We already mentioned in the previous chapter that distributivity is one of the most sought after properties in lattice theory. In the study of closure systems, this usually goes one step further. Here we ask for conditions for a closure system to be topological, i.e. what properties does the system have to fulfil so that finite joins and finite unions coincide.

2.13.1 Example

A distributive closure system need not be topological.

Proof: Consider the closure system $\mathcal{C} = \{\emptyset, \{a\}, \{b\}, \{a, b, x\}\}$. It is trivially distributive, even Boolean, but the join of $\{a\}$ and $\{b\}$ is not their union. Note that this system is not a C_D-system since the point closure of x is $\{a, b, x\}$, which is not $\bigvee$-irreducible. $\square$

The next result seems trivial, but it is rather fundamental.

2.13.2 Theorem

A C_D-system is distributive if and only if it is topological (hence a dual T_D-topology).

Proof: A topological closure system is always distributive, so we concentrate on the other direction of the equivalence. We have to show that, in a distributive closure system $\mathcal{C}$, the following implication is true:

$$x \in A \vee B \Rightarrow x \in A \cup B.$$

The essential part here is that $\bigvee$-irreducible elements in a distributive system are also $\vee$-prime. Therefore

$$\begin{aligned} x \in A \vee B &\iff C(x) \subseteq A \vee B \\ &\iff (C(x) \subseteq A) \text{ or } (C(x) \subseteq B) \\ &\iff x \in A \text{ or } x \in B. \quad \square \end{aligned}$$

In the case of detachable C_D-systems, we do not even have to test for distributivity if we want to know if it is topological, it suffices to test for a weaker property.

2.13.3 Theorem

A modular and detachable (C_D-)system $\mathcal{C}$ is distributive, and therefore topological.

Proof: We assume that the C_D-system $\mathcal{C}$ on the base set S is modular but not distributive, i.e. there are $A, B, D \in \mathcal{C}$ such that

$$A \vee (B \wedge D) < (A \vee B) \wedge (A \vee D).$$

Thus there exists $x \in ((A \vee B) \wedge (A \vee D)) \setminus (A \vee (B \wedge D))$, especially $x \notin B \wedge D$.

We define $P := A \vee (B \wedge D)$. We apply detachabilibty to get

$$P \leq P(x) - x \prec P(x) \leq (A \vee B) \wedge (A \vee D).$$

Since $A \leq P(x) - x \leq A \vee B$ we have the equality $A \vee B = (P(x) - x) \vee B$. We can now apply modularity:

$$\begin{aligned}(P(x) - x) \vee (B \wedge P(x)) &= ((P(x) - x) \vee B) \wedge P(x) \\ &= (A \vee B) \wedge P(x) \\ &= P(x).\end{aligned}$$

From this we get that $P(x) \wedge B \not\leq P(x) - x$, but of course $P(x) \wedge B \leq P(x)$. Therefore $x \in P(x) \wedge B \subseteq B$.

We can now repeat this argumentation with D instead of B and will obtain $x \in D$ for the same reasons. This contradicts $x \notin B \wedge D$, and $\mathcal{C}$ therefore needs to be distributive. $\square$

The two properties of modularity and detachability split even characterise T_D-systems, since the converse direction of the previous theorem is also true.

2.13.4 Lemma

A closure system $\mathcal{C}$ is modular and detachable if and only if it is a T_D-system.

Proof: We only need to show that T_D-systems are detachable, since they are always modular. For $A \in \mathcal{C}$ and $x \notin A$, we have $A \vee C(x) - x = A \cup C(x) - x \prec A \cup C(x) = A(x)$, hence $A(x) - x = A \vee C(x) - x \in \mathcal{C}$.

The next result goes one step further and allows various descriptions of what is required to get a T_D-system when we restrict ourselves to bigenerated C_D-systems.

2.13.5 Theorem

For a bigenerated detachable C_D-system $\mathcal{C}$ on S, the following are equivalent:

- *i) $\mathcal{C}$ is a topological closure system.*
- *ii) $\mathcal{C}$ is a coframe.*
- *iii) $\mathcal{C}$ is distributive.*
- *iv) $\mathcal{C}$ is modular.*
- *v) $\mathcal{C}$ is $\wedge$-semidistributive.*

vi) $\mathcal{C}$ *does not contain any sublattice* S_7^* *(see Figure 1.3).*

vii) $\forall\, a, b \in S : C(a) \vee C(b) = C(a) \cup C(b)$.

Proof: The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (vi) and (iii) $\Rightarrow$ (v) $\Rightarrow$ (vi) are trivial.

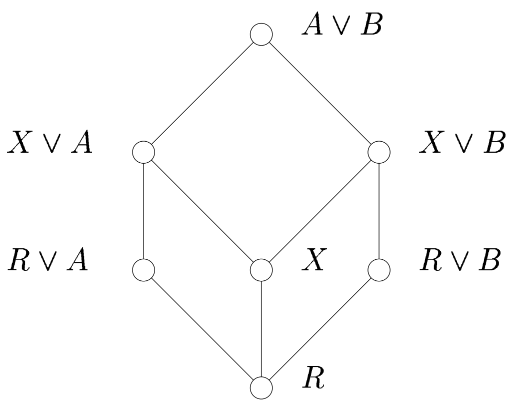
(vi) $\Rightarrow$ (vii): Suppose $x \in (C(a) \vee C(b)) \setminus (C(a) \cup C(b))$. We use $A := C(a)$, $B := C(b)$, $X = C(x)$ and $R := X - x$.

We have $R \subset R \vee A$. Equality would imply $A \subseteq R \subset X$, but since $X \subseteq A \vee B$ (and $X \nsubseteq B$) this would contradict resistance of X (see 2.10). The resistance of X is also the reason for $R \vee A \subset X \vee A$, since $R \subset X$ and $X \nsubseteq A$. Similarly, $R \subset R \vee B \subset X \vee B$.

The same arguments as above give us $X \subset X \vee A$ and $X \subset X \vee B$.

Using the anti-exchange property we get $X \vee A \subset A \vee B$ and $X \vee B \subset A \vee B$.

As a result, the seven sets R, $R \vee A$, $R \vee B$, X, $X \vee A$, $X \vee B$ and $A \vee B$ form a sublattice S_7^*, which is a contradiction.



(vii) $\Rightarrow$ (i): This requieres only one application of bigeneration:

$$\begin{aligned} A \vee B &= \bigcup \{C(a) \vee C(b) \mid a \in A, b \in B\} \\ &= \bigcup \{C(a) \cup C(b) \mid a \in A, b \in B\} \\ &= \bigcup \{C(x) \mid x \in A \cup B\} \\ &= A \cup B. \quad \square \end{aligned}$$

This theorem is the generalisation of Theorem 4.1 in [24] from intervals of κ-$\bigwedge$-closed subsets of a κ-semilattice to the more general bigenerated detachable closure systems, the proofs differ only slightly. Note that we only used the anti-exchange property and the resistance of every point closure $C(x)$, not detachability, hence we could weaken the restrictions on the closure system slightly.

Most of the properties given in this theorem are lattice properties. Thus it is also possible to formulate the theorem in purely lattice-theoretical terms. See also Chapter 2 in [14] for similar results using the exclusion of S_7^*.

2.13.6 Theorem

In a bigenerated and persistent C_D-lattice L, the following are equivalent:

i) L is spatial, i.e. every element is a join of $\vee$-prime elements.

ii) L is a coframe.

iii) L is distributive.

iv) L is modular.

v) L is $\wedge$-semidistributive.

vi) L does not contain a sublattice S_7^*.

vii) For all $i, j, k \in \mathcal{J}(L)$ with $i \leq j \vee k$ we have $i \leq j$ or $i \leq k$.

Proof: This proof is practically identical to the previous one, just in lattice-theoretical terms. (vii) is a lattice-theoretical translation of the corresponding equivalence in the previous theorem. This and bigeneration can be combined to show that every $\bigvee$-irreducible element is $\vee$-prime, i.e. we get the implication (vii) $\Rightarrow$ (i). $\square$

The next chapter will focus on the just mentioned intervals of subsemilattices. Since they are C_D-systems with certain anti-exchange properties, we can apply the results from this chapter to these systems.

Chapter 3

Intervals in the lattice of κ-$\bigwedge$-subsemilattices

3.1 Introduction and History

The study of lattices of ($\wedge$- or $\vee$-)subsemilattices of semilattices has a long history, quite often in connection with a general convexity theory. We give some examples which influenced this work. R. Jamison-Waldner was one of the first when he applied Tietze's convexity theorem to the system of all $\wedge$-semilattices of a $\wedge$-semilattice in [34], and he, together with P. Edelman, used lattices of finite semilattices as an example for finite convex geometries in [18]. Further articles by these authors that deal with lattices of subsemilattices, mainly as examples for general convex geometries, are [16], [17], [35] and [36].

To my knowledge, Libkin and Muchnik were the first to give a characterisation of the lattice of $\vee$-subsemilattices of a $\vee$-semilattice in [40]. They characterised these lattices as algebraic and biatomic lattices with an additional technical property. We will look at the non-technical properties given there and consider them for the more general approach we have in mind.

Some interesting results are due to K. Adaricheva. She did not only try to identify those lattices of subsemilattices with certain properties like semidistributivity in [1], but also gave a construction for finite lattices of subsemilattices in [2].

Furthermore, in addition to $\wedge$-subsemilattices of $\wedge$-semilattices, she also analysed $\bigwedge$-subsemilattices of $\bigwedge$-semilattices which, as we will show in this chapter, can show a very different behaviour than their to finite meets restricted counterparts. This was continued in a joint paper with V. Gorbunov and V. Tumanov in [3], in which they tried to start a discussion how to extend the theory of finite convex geometries and the anti-exchange property to infinite environments.

All these results and approaches have in common that only the full system of all subsemilattices of a semilattice is discussed. Furthermore, only some of these works allow infinite semilattices, and even fewer consider not only $\wedge$- but also $\bigwedge$-semilattices.

A step towards a more general approach was made in [26] by M. Erné, B. Šešelja and A. Tepavčević, in which intervals of κ-$\bigwedge$-subsemilattices of κ-$\bigwedge$-semilattices for an arbitrary regular cardinal were considered for the first time. This paper was succeeded by [24], in which M. Erné continued and intensified the study of these intervals.

We want to continue this work and extend our knowledge of intervals of κ-$\bigwedge$-subsemilattices. This includes general properties like bigeneration, similarities to other classes considered in the general theory of convexity like anti-exchange properties, but also properties which are interesing and important but not satisfied by intervals of κ-$\bigwedge$-subsemilattices in general, like e.g. distributivity or algebraicity.

In addition to this we want to consider alternative approaches to general convexity theory, and try to apply these approaches to our intervals. Even if our intervals do not completely fit into alternative theories of convexity, we hope that partial results will help us to understand the general structure of intervals of subsemilattices.

We will base our considerations on [24] and will use the notation used there. Since we continue the work started in [24], we use κ-$\bigwedge$-semilattices, but the whole theory could also be developed for κ-$\bigvee$-semilattices in the same way.

3.2 General Properties

In this chapter our main focus will be systems of subsemilattices of given semilattices, but at a very general level. We will work with κ-$\bigwedge$-semilattices, which we already defined in 1.3.4. The usual $\wedge$-semilattices are, of course, a special case of κ-$\bigwedge$-semilattices ($\kappa = \omega$, to be exact), but higher cardinals can lead to quite different properties of the systems under consideration. We already used the term subsemilattice from time to time, we will now put this term on solid grounds. Recall that we write $X \subseteq_\kappa Y$ if X is a subset of Y which contains strictly less than κ many elements.

3.2.1 Definition

Consider a κ-$\bigwedge$-semilattice S.

We define the operator

$$\Lambda_\kappa Y = \left\{ \bigwedge\nolimits_S X \mid X \subseteq_\kappa Y \right\} \quad (Y \subseteq S).$$

The fixed points of this operator are the **κ-$\bigwedge$-subsemilattices of S**. They form a closure system $\mathcal{S}_\kappa$. If κ is a regular cardinal, Λ_κ is the corresponding closure operator. If κ were a singular cardinal, we could use the least regular cardinal $\overline{\kappa}$ greater than κ to get the closure operator corresponding to $\mathcal{S}_\kappa$, so the regularity is not an insurmountable barrier. Furthermore, κ-subsemilattices are the same as $\overline{\kappa}$-subsemilattices. Therefore we can assume that κ **is always a regular cardinal or** ∞ without loss of generality, where $X \subseteq_\infty Y$ means $X \subseteq Y$.

Let C be a κ-$\bigwedge$-subsemilattice of S. We denote by

$$\mathcal{C}_\kappa := [C, S] = \{A \in \mathcal{S}_\kappa \mid C \subseteq A\}$$

the **interval of all κ-$\bigwedge$-subsemilattices of S that contain C**. This is a closure system, too, with the closure operator

$$\Gamma A = \Lambda_\kappa(A \cup C) = \{a \wedge c \mid a \in \Lambda_\kappa A,\ c \in C\} \ \ (A \subseteq S).$$

3.2.2 Lemma

For a given κ-$\bigwedge$-semilattices S, $\mathcal{S}_\kappa$ is an atomistic closure system in which every point closure is an atom, and every interval $\mathcal{C}_\kappa$ is a C_D-system. In particular, all point closures are $\bigvee$-irreducible.

Proof: The $\perp$-element of $\mathcal{S}_\kappa$ is $\{\top_S\}$, and for every $x \in S$ the set $\{x, \top_S\}$ is closed under κ-meets, i.e. it is a κ-$\bigwedge$-subsemitlattice of S and thus an atom of $\mathcal{S}_\kappa$.

The point closures of an interval $\mathcal{C}_\kappa$ are

$$C(x) = \Gamma\{x\} = C \cup \{x \wedge c \mid c \in C\}$$

for $x \notin C$.

From this it can be seen that $C(x) - x$ is also κ-$\bigwedge$-closed and the unique lower cover of $C(x)$, since x cannot be a meet of elements in $C(x) - x$. Thus $C(x) - x \in \mathcal{C}_\kappa$ and every point closure is $\bigvee$-irreducible. $\square$

The join of a non-empty subset $\mathcal{B} \subseteq \mathcal{C}_\kappa$ must contain the union of $\mathcal{B}$ and be κ-$\bigwedge$-closed. It is therfore given by

$$\bigvee\nolimits_{\mathcal{C}_\kappa} \mathcal{B} = \left\{ \bigwedge\nolimits_S X \mid X \subseteq_\kappa \bigcup \mathcal{B} \right\},$$

the special case of a join of two elements $A, B \in \mathcal{C}_\kappa$ is

$$A \vee B = \{a \wedge b \,|\, a \in A,\, b \in B\}\,.$$

We can use these first observations to give some general properties of intervals of κ-$\bigwedge$-subsemilattices.

3.2.3 Lemma

Every interval $\mathcal{C}_\kappa$ of $\mathcal{S}_\kappa$ is bigenerated.

Proof: The bigeneration property is obvious:

$$\begin{aligned} A \vee B &= \{a \wedge b \,|\, a \in A,\, b \in B\} \\ &= \bigcup \{C(a) \vee C(b) \mid a \in A,\, b \in B\}\,. \quad \Box \end{aligned}$$

3.2.4 Lemma

$\mathcal{C}_\omega$ is an algebraic C_D-system.

Proof: Consider a directed subset $\mathcal{B}$ of $\mathcal{C}_\omega$, and choose an arbitrary $b \in \bigvee \mathcal{B}$. To this b exists a $X \subseteq_\omega \bigcup \mathcal{B}$. Since X is finite, it is already contained in the join of a finite number of elements of $\mathcal{B}$, and as $\mathcal{B}$ is directed, these are all contained in a single element $B \in \mathcal{B}$, i.e. $X \subseteq_\kappa B$. Hence, $b = \bigwedge X \in B \subseteq \bigcup \mathcal{B}$. $\Box$

These lemmata cover two fundamental properties of the aforementioned characterisation of the lattice of $\wedge$-subsemilattices of a $\wedge$-semilattice given by Libkin and Muchnik in [40]. For the atomistic system $\mathcal{S}_\kappa$ bigeneration is, of course, biatomicity.

That $\mathcal{C}_\omega$ is algebraic is rather straightforward. However, for an arbitrary regular cardinal κ, $\mathcal{C}_\kappa$ is not algebraic in general. We will soon give a precise description which intervals are algebraic.

We already know that intervals of subsemilattices are C_D-systems. Our next step will be to pinpoint the position of these intervals in the big diagram of anti-exchange properties (see also [24]).

3.2.5 Theorem

Every interval $\mathcal{C}_\kappa = [C, S]$ of κ-$\bigwedge$-subsemilattices of a κ-$\bigwedge$-semilattice S is detachable, but it is neither shellable nor does it satisfy the strong anti-exchange property in general if $\kappa > \omega$.

Proof: We have to show that $A(x) - x \in \mathcal{C}_\kappa$ for $A \in \mathcal{C}_\kappa$ and $x \notin A$. We have that

$$A(x) = A \cup \{a \wedge x \,|\, a \in A\},$$

which is similar to the construction of the point closures $C(x)$. Again, x cannot be a meet of elements in $A(x) - x$, as this would either imply that $x \in A$, which is not the case, or that an element of the form $a \wedge x$ is larger than x, which is also impossible.

Hence, $A(x) - x$ is κ-$\bigwedge$-closed, i.e. in $\mathcal{C}_\kappa$, and $\mathcal{C}_\kappa$ is therefore detachable.

For a non-shellable interval of κ-$\bigwedge$-subsemilattices, we use the $\bigwedge$-semilattice $S = [0, \infty] \subseteq \mathbb{R}$ and consider the whole system $\mathcal{S}_\kappa$ of all $\bigwedge$-subsemilattices of S, i.e. $\kappa = \infty$ ($\kappa > \omega$ would suffice). We have $S \in \mathcal{S}_\kappa$, but for every $x \in S$, $S - x \notin \mathcal{S}_\kappa$, hence $\mathcal{S}_\kappa$ is not shellable.

Furthermore, this system does not have the strong anti-exchange property. Consider the chain $\mathcal{D} = \{D_n \,|\, n \in \mathbb{N}\}$ of $\bigwedge$-subsemilattices of the form $D_n := \{0, 1, 2, ..., n\}$ for $n \in \mathbb{N}$. By adding S to this chain it becomes a maximal chain which does not separate arbitrary pairs of points. All non-integers are only contained in the $\top$-element of the chain. □

Since we just showed that these intervals need not have the strong anti-exchange property, we get an already mentioned fact without need for a proof. We simply refer to our list of anti-exchange properties in Chapter 2, in which the strong anti-exchange property is always satisfied in convex geometries (algebraic and detachable closure systems).

3.2.6 Corollary

Intervals of κ-$\bigwedge$-subsemilattices are not algebraic in general.

The fact that all intervals of κ-$\bigwedge$-subsemilattices are detachable gives us a lot of information about these systems. We can use everything we derived for these systems in the previous chapter. We know that these systems, considered as C_D-lattices, are lower semimodular, resistant, consistent, lower locally Boolean and weak coframes. We know that they satisfy the anti-exchange property, and that covers differ by exactly one element.

It will be interesting to see, if we can find additional properties of this class which we have not considered yet.

3.3 Intervals of Subsemilattices and Partial Orders

In 2.1.13 we stated that every C_D-system $\mathcal{C}$ which satisfies the anti-exchange property induces a partial order $\leq_A$ on $S \setminus A$ for every $A \in \mathcal{C}$. The definition was

$$x \leq_A y :\iff A(x) \subseteq A(y)$$

for $x, y \in S \setminus A$.

We want to examine the connection between these partial orders and the original order $\leq$ on the underlying κ-$\bigwedge$-semilattice S. The first observation is straightforward.

3.3.1 Lemma

Consider an interval $\mathcal{C}_\kappa$ of κ-$\bigwedge$-subsemilattices of a κ-$\bigwedge$-semilattice S. If $x \leq_A y$ for any $A \in \mathcal{C}_\kappa$, then $x \leq y$.

Proof: $x \leq_A y \iff A(x) \subseteq A(y) \iff (\exists\, a \in A : x = a \wedge y) \Rightarrow x \leq y$. □

So we see that these partial orders are always contained in the original order $\leq$, but we cannot expect them to coincide. The biggest difference between $\leq$ and an order of the form $\leq_A$ is given in the following example.

3.3.2 Example

Consider the system $\mathcal{S}_\kappa$ of all κ-$\bigwedge$-subsemilattices of a κ-$\bigwedge$-semilattice S. The partial order induced by $A := \bot_{\mathcal{S}_\kappa} = \{\top_S\} \in \mathcal{S}_\kappa$ is simply equality on the set $S - \top_S$.

We have $x \leq_A y \iff x \in A(y) = \{y, \top_S\} \iff x = y$, since $\leq_A$ is only defined for $x, y \notin A$.

Occasionally, however, for certain $A \in \mathcal{C}_\kappa$ and $x, y \in S \setminus A$, the order on $\{x, y\}$ with respect to A must coincide with the original order.

3.3.3 Lemma

Consider an interval $\mathcal{C}_\kappa$ of κ-$\bigwedge$-subsemilattices of a κ-$\bigwedge$-semilattice S, with $A \in \mathcal{C}_\kappa$ and $x, y, z \in S \setminus A$ such that $A(x) \subseteq A(z)$ and $A(y) \subseteq A(z)$.

Then we have $x \leq y \iff x \leq_A y$.

Proof: We already know that $x \leq_A y$ implies $x \leq y$. Suppose $x \leq y$. Both elements x and y are contained in $A(z)$, i.e. there are elements $a_x, a_y \in A$ such that $x = a_x \wedge z$ and $y = a_y \wedge z$.

From this we get $x \leq a_x \wedge y$. Since $y \leq z$ we also get $y \wedge a_x \leq z \wedge a_x = x$. Thus $x = a_x \wedge y$, and therefore $x \in A(y)$, i.e. $x \leq_A y$. □

3.4 Algebraic and Coalgebraic Intervals of Subsemilattices

In this section we want to characterise those intervals of κ-$\bigwedge$-subsemilattices which are algebraic. A major element of this characterisation is already contained in 1.7.5, since intervals of subsemilattices are detachable and therefore resistant.

With the help of this theorem, we can be very specific as to which intervals are algebraic. Recall that κ is a regular cardinal with $\kappa \geq \omega$, and note that every κ-$\bigwedge$-subsemilattice of a κ-$\bigwedge$-semilattice S is also a $\wedge$-subsemilattice (i.e. a ω-$\bigwedge$-subsemilattice) of S.

3.4.1 Theorem

An interval $\mathcal{C}_\kappa$ of κ-$\bigwedge$-subsemilattices of a κ-$\bigwedge$-semilattice S is algebraic if and only if it coincides with $\mathcal{C}_\omega$, the interval of all $\wedge$-subsemilattices of S that contain $C = \bigcap \mathcal{C}_\kappa$.

Proof: We have already shown that intervals of $\wedge$-subsemilattices are algebraic, and if $\mathcal{C}_\kappa$ coincides with such an interval, it must also be algebraic.

The other direction is less trivial, but because of 1.7.5 we only have to show that $\mathcal{C}_\kappa$ is not $\wedge$-continuous if it does not coincide with $\mathcal{C}_\omega$.

Suppose that $\mathcal{C}_\kappa \neq \mathcal{C}_\omega$. Then there exists $R \in \mathcal{C}_\omega \backslash \mathcal{C}_\kappa$ for which

$$\overline{R} = \bigwedge \{A \in \mathcal{C}_\kappa \mid R \leq A\}$$

is the smallest element of $\mathcal{C}_\kappa$ properly containing R. We can find $x \in \overline{R} \setminus R$ and $K \subseteq_\kappa R$ such that

$$\left(x = \bigwedge K\right) \text{ and } \left(\forall B \subseteq_\omega R : x \neq \bigwedge B\right).$$

For an arbitrary $F \subseteq_\omega K$ we define

$$\begin{aligned} C(F) &:= \Gamma(F) \\ &= \{c \wedge f \mid c \in C,\ f \in \Lambda_\kappa(F)\} \\ &= \{c \wedge f \mid c \in C,\ f \in \Lambda_\omega(F)\}\,. \end{aligned}$$

From this we see that $C(F) \in \mathcal{C}_\omega \cap \mathcal{C}_\kappa$, thus $C(F) \subseteq R$ which implies $x \notin C(F)$.

We define $\mathcal{D} = \{C(F) \mid F \subseteq_\omega K\}$, which is a directed subset of $\mathcal{C}_\kappa$, as $C(F_1), C(F_2) \leq C(F_1 \cup F_2)$.

We have $\forall D \in \mathcal{D} : C(x) \wedge D = C(x) \wedge C(F) \leq C(x) - x$, and this implies $\bigvee(C(x) \wedge D) \leq C(x) - x$, but $C(x) \wedge \bigvee \mathcal{D} = C(x)$, as $x \in \bigvee \mathcal{D}$. This shows that $\mathcal{C}_\kappa$ is not $\wedge$-continuous. □

We could even have gone further and restricted ourselves to show that $\mathcal{C}_\kappa$ is not weakly $\wedge$-continuous, but the proof is a little bit more technical.

3.4.2 Remark

This result was very surprising, since algebraicity was one of the most basic properties in the characterisation of lattices of $\wedge$-subsemilattices. This theorem now shows that this is never true for higher cardinalities, unless all infinite meets can already be reduced to finite meets. Thus the whole arsenal of results for algebraic lattices or algebraic closure systems (e.g. [11]) cannot be applied to the general situation we have here.

A characterisation of coalgebraic intervals of κ-$\bigwedge$-subsemilattices for arbitrary regular cardinals κ has not been found yet, but there are some results in this area. What we cannot expect is an application of the dual of 1.7.5, since intervals of subsemilattices are not generated by dually resistant $\bigwedge$-irreducible elements. If that were the case, the problem would have been solved immediately, since weak coframes are trivially weakly $\vee$-continuous.

We can also give a first example of a system of $\bigwedge$-κ-subsemilattices which is not $\vee$-continuous.

3.4.3 Example

We consider $S = \mathbb{Z}^2 \cup \{\infty\}$ with $\mathcal{S}_\omega$ being the system of all $\wedge$-subsemilattices.

We choose $A = \{(-1,1), \infty\}$ and $\mathcal{D} = \{D_n \mid n \in \mathbb{N}\}$ with $D_n = \{(m,0) \mid n \leq m\} \cup \{\infty\}$. Then

$$A \vee \bigwedge \mathcal{D} = A \neq \{(-1,0), (-1,1), \infty\} = \bigwedge (A \vee \mathcal{D})\,.$$

This shows, that $\mathcal{C}_\omega$ need not be $\vee$-continuous.

The following theorem is taken from [1], an alternative and shorter proof can be found in [24].

3.4.4 Definition

By a **hook** in a $\wedge$-semilattice, we mean a subsemilattice $D \cup \{a, b\}$, where D is isomorphic to ω or its dual, and $a = b \wedge d < b, d$ for all $d \in D$.

3.4.5 Theorem

For a $\wedge$-semilattice S, the following are equivalent:

i) $\mathcal{S}_\omega$ *is coalgebraic,*

ii) $\mathcal{S}_\omega$ *is $\vee$-continuous,*

iii) $\mathcal{S}_\omega$ *does not contain any hook.*

Proof: See [24]

The absence of certain hooks in S seem to be essential for $\vee$-continuity of $\mathcal{C}_\kappa$, as we can see in the following result.

3.4.6 Theorem

For $\kappa > \omega$, $\mathcal{C}_\kappa$ is $\vee$-continuous iff $S \setminus C$ does not contain a hook $B \cup \{a, x\}$ with the following properties:

i) B *is isomorphic to* $\mathbb{N}$*, i.e.* $B = \{b_1, b_2, ...\}$ *with* $b_j < b_{j+1}$ *for* $j \in \mathbb{N}$.

ii) $\forall\, b \in B : x = a \wedge b < a, b$,

iii) $x \nleq_C a$, $\forall\, b \in B : x \nleq_C b$,

iv) $\forall\, y \in S \setminus C\, ((\exists\, n_0 \in \mathbb{N} \forall\, m \geq n_0\, \exists\, c_m \in C : y = b_m \wedge c_m) \Rightarrow x \nleq y)$.

Proof: If $S \setminus C$ contains a hook with these properties, $\mathcal{C}_\kappa$ cannot be $\vee$-continuous, since we can do the following construction.

For $n \in \mathbb{N}$ we define $B_n :=\uparrow b_n \cap B$, $D_n := \Gamma_\kappa(B_n)$, $\mathcal{D} := \{D_n | n \in \mathbb{N}\}$, and $A := C(a)$. By construction, $x \in \bigwedge (A \vee D_n)$.

We have to verify that $x \notin A \vee \bigwedge \mathcal{D}$. Otherwise, there would be an element $y \in \bigwedge \mathcal{D}$ such that $x = a \wedge y$. This y would be of the form $y = c_m \wedge b_m$ for every

b_m larger than, say b_{n_0}, and since $x \not\leq_C a$, y cannot be in C. This contradicts property (iv), hence such an element y does not exist, $\mathcal{C}_\kappa$ cannot be $\vee$-continuous.

If $\mathcal{C}_\kappa$ is not $\vee$-continuous, there are an $A \in \mathcal{C}_\kappa$ and a non-empty chain $\mathcal{D} \subseteq \mathcal{C}_\kappa$ such that $A \vee \bigwedge \mathcal{D} < \bigwedge (A \vee D)$. We choose $x \in \bigwedge (A \vee D) \setminus (A \vee \bigwedge \mathcal{D})$. Since $x \notin \bigwedge \mathcal{D}$, there is a $D \in \mathcal{D}$ with $x \notin D$. Furthermore, $\mathcal{D}$ cannot have a minimal element, thus we can find a subchain $\{D_n \mid n \in \mathbb{N}\}$ of $\mathcal{D}$ which is isomorphic to $\mathbb{N}^{op}$ and no element of which contains x.

W.l.o.g we restrict us to this subchain and call it $\mathcal{D}$. For this we have the following properties:

i) $\forall a \in A \, \forall d \in \bigwedge \mathcal{D} : x \neq a \wedge d$,

ii) $\forall \, n \in \mathbb{N} \, \exists \, d_n \in D_n \, \exists \, a_n \in A : x = a_n \wedge d_n$.

We define $a := \bigwedge a_n$, which is in A as only countable infima are involved. We also define $b_n := \bigwedge \{d_m | m \geq n\}$ for every $n \in \mathbb{N}$. The equation $b_n \in D_n$ is true for the same reasons.

The set $B := \{b_n | n \in \mathbb{N}\}$ is a chain isomorphic to $\mathbb{N}$ ($b_1 \leq b_2 \leq ...$), and all elements satisfy $x = a \wedge b_n < a, b_n$, as $x \notin D_n, A$. Note that B cannot be finite, a maximal element $\overline{b}$ of B would lead to $\overline{b} \in \bigwedge \mathcal{D}$ and thus to $x \in A \vee \bigwedge \mathcal{D}$, a contradiction.

If either a or any of the b_n were in C, x would be an element of some D_n or A, which it is not. Therefore, $B \cup \{a, x\} \subseteq S \setminus C$. The equations $x \not\leq_C a$ and $x \not\leq_C b_n$ are satisfied for the same reasons.

Furthermore, property (iv) is satisfied since otherwise $x \leq y$, $y \in \bigwedge \mathcal{D}$ and
$x \leq a \wedge y \leq a \wedge b_m = x$, which implies $x \in A \vee \bigwedge \mathcal{D}$, a contradiction. □

Thus we have shown, that the presence (or absence) of these special hooks determines $\vee$-continuity of $\mathcal{C}_\kappa$.

It is possible to give a slightly easier formulation of this result for $\mathcal{C}_\infty$.

3.4.7 Theorem

$\mathcal{C}_\infty$ is $\vee$-continuous iff $S \setminus C$ does not contain any hook $B \cup \{a, x\}$ with the following properties:

i) B is isomorphic to $\mathbb{N}$,

ii) $\forall \, b \in B : x = b \wedge a < b, a$,

iii) $\forall \, y \in B \cup \{a\} : y < c_x := \bigwedge \{c \in C \mid x < c\}$, especially $x \not\leq_C a$.

Proof: If $S\backslash C$ contains a hook of this form, construct a chain $\mathcal{D}$, where the elements $D \in \mathcal{D}$ are of the form $\Gamma(\uparrow b \cap B)$ for $b \in B$. For $z \in \bigwedge \mathcal{D} \setminus C$ we have $z = c_n \wedge b_n$ with $c_n \in C$ and $b_n \not\leq c_n$ for every $n \in \mathbb{N}$. Because of (iii), $x \not\leq c_n$. Therefore, $x \not\leq z = c_n \wedge b_n$ for every $z \in \bigwedge \mathcal{D}$. Together with $x \not\leq_C a$ this gives us $x \notin C(a) \vee \bigwedge \mathcal{D}$, but for every $D \in \mathcal{D}$ we have $x \in C(a) \vee D$ by construction, hence $x \in \bigwedge(A \vee \mathcal{D})$. Thus $\mathcal{C}$ is not $\vee$-continuous.

If $\mathcal{C}_\infty$ is not $\vee$-continuous, there is an element $A \in \mathcal{C}_\infty$ and a chain $\mathcal{D} \subseteq \mathcal{C}_\infty$ such that $\bigwedge(A \vee D) \neq A \vee \bigwedge \mathcal{D}$. This implies $A \vee \bigwedge \mathcal{D} < \bigwedge(A \vee D)$, and we can find $x \in \bigwedge(A \vee D) \backslash (A \vee \bigwedge \mathcal{D})$.

This element x has a number of properties:

i) $\forall D \in \mathcal{D}\, \exists\, d \in D\, \exists\, a \in A : x = a \wedge d$,

ii) $\forall d \in \bigwedge \mathcal{D}\, \forall a \in A : x \neq a \wedge d$, especially

iii) $x \notin A$, and

iv) $x \notin \bigwedge \mathcal{D}$, which implies

v) $\exists\, D \in \mathcal{D} : x \notin D$.

We define $a := \bigwedge(A \cap \uparrow x)$. This element satisfies $a \in A$, $x < a$ and $x \not\leq_C a$. We also define $B := \{b \in S \mid b = \bigwedge(D \cap \uparrow x) \text{ for } D \in \mathcal{D}\}$, with $x \leq b$ for all $b \in B$. Furthermore, all $b \in B$ satisfy $x = a \wedge b$, and since $x \notin \bigwedge \mathcal{D}$, at least one element $b \in B$ has $x < b$.

If any of the elements in $B \cup \{a\}$ were in C, this would contradict either $x \notin A$ or $x \notin \bigwedge \mathcal{D}$, thus $a \in S\backslash C$ and $B \subseteq S\backslash C$.

All $b \in B$ satisfy $x = a \wedge b$, and since $x \notin \bigwedge \mathcal{D}$, at least one element $b \in B$ has $x < b$.

Suppose $\overline{b} \in B$ was a maximal (hence greatest) element in B, especially $\overline{b} = \bigwedge(D \cap \uparrow x)$ for some $D \in \mathcal{D}$. If $\overline{b}$ were not in $\bigwedge \mathcal{D}$, then there is a $D' \in \mathcal{D}$ such that $\overline{b} \notin D'$. This leads to $D \subset D'$ since $\mathcal{C}$ is a chain, and $b = \bigwedge(D \cap \uparrow x) < \bigwedge(D' \cap \uparrow x) =: b' \in B$, a contradiction to the maximality of b. Hence $\overline{b} \in \bigwedge \mathcal{D}$ and $a \wedge \overline{b} = x$, which would imply $x \in A \vee \bigwedge \mathcal{D}$. This is clearly wrong, thus B does not have a maximal element. Therefore, B contains a subchain which is isomorphic to $\mathbb{N}$ and satisfies $x = a \wedge b < a, b$ for all $b \in B$.

By construction, $b = \bigwedge(D \cap \uparrow x) < \bigwedge(C \cap \uparrow x) = c_x$ as $b \in S \setminus C$ and $D \in \mathcal{C}$. Similarly, $a < c_x$, hence property (iii) is satisfied.

All in all we have constructed a hook with the required properties, closing this proof. □

3.4.8 Remarks

For the special case $\mathcal{C}_\infty = \mathcal{S}_\infty$, the condition (iii) simply requires the elements a and x to be distinct.

The technical conditions in the previous two theorems are requiered to prevent x to be in A, $\bigwedge \mathcal{D}$ or $A \vee \bigwedge \mathcal{D}$. For this we have to make restrictions on the least element $C \in \mathcal{C}$. The meet of all subchains of the form $\uparrow b_n \cap B$ for $n \in \mathbb{N}$ might be empty, but the meet of the closures of these sets, $\Gamma(\uparrow b_n \cap B)$, need not be. It is the elements in this meet that we have to restrict to prevent x from being in $A \vee \bigwedge \mathcal{D}$, and this can only be achieved with complex conditions, i.e. the above-mentioned technical ones.

Note also, that the existence of a descending chain B isomorphic to $\mathbb{N}^{op}$ does not prevent $\mathcal{C}_\kappa$ with $\kappa > \omega$ from being $\vee$-continuous. The reason is that in the construction principle we use, the meet $z = \bigwedge B$ would be contained in every $D \in \mathcal{C}$, which would result in $x = a \wedge z \in A \vee \bigwedge \mathcal{D}$. The contradiction we wanted to achieve would not materialise.

Since the previous results all require (the absence of) a chain isomorphic to $\mathbb{N}$, we can give some specific classes of semilattices which have $\vee$-continuous intervals of κ-$\bigwedge$-subsemilattices.

3.4.9 Corollary

If $C \subseteq S$ is a $\bigwedge$-subsemilattice of the $\bigwedge$-semilattice S, and $S \setminus C$ satisfies the acc or is a chain, then $\mathcal{C}_\kappa$ is $\vee$-continuous.

We gave a number of results on $\vee$-continuity in the context of intervals of subsemilattices, but only one of them also deals with coalgebraicity. As the next example will illustrate, the question whether an interval of subsemilattices is coalgebraic is not identical to the question for $\vee$-continuous intervals, and the problem of characterising coalgebraic intervals of subsemilattices for cardinals larger than ω is still unsolved.

3.4.10 Example

Consider the $\bigwedge$-semilattice $S = [0, 1] \subseteq \mathbb{R}$. Since S is a chain, $\mathcal{S}_\kappa$ is certainly $\vee$-continuous.

Assume $A \in \mathcal{S}_\infty - S$, i.e. there is a $b \in S \setminus A$. For this we define

$$\overline{b} := \bigwedge \{a \in A \,|\, b < a\} ,$$

giving $[b, \overline{b}) \subseteq S \setminus A$.

Based on these elements, we define a set $\mathcal{D}$ of $\bigwedge$-subsemilattices of S:

$$\begin{array}{rclrcl} b_1 & := & \frac{\overline{b}+b}{2}, & D_1 & := & A \cup [b_1, \overline{b}) \\ b_2 & := & \frac{\overline{b}+b_1}{2}, & D_2 & := & A \cup [b_2, \overline{b}) \\ & \vdots & & & \vdots & \\ b_n & := & \frac{\overline{b}+b_{n-1}}{2}, & D_n & := & A \cup [b_n, \overline{b}) \end{array}$$

for $n \in \mathbb{N}$.

Then $\mathcal{D}$ has $\bigwedge \mathcal{D} = A$, but for every finite subset $\mathcal{F} \subseteq_\omega \mathcal{D}$ there is a $x \in [b, \overline{b})$ which is in $\bigwedge \mathcal{F}$, leading to $\bigwedge \mathcal{F} \nleq A$.

The result is that A is not a co-compact element in $\mathcal{S}_\infty$, but A was an arbitrary element of $\mathcal{S}_\infty \setminus S$, thus there are no co-compact elements, and $\mathcal{S}_\infty$ is certainly not co-algebraic.

This shows that $\vee$-continuous intervals of $\wedge$-subsemilattices need not be co-algebraic.

3.5 Irreducible and Prime Elements

One of the best known implications of algebraicity is the existence of $\bigwedge$-decompositions, as mentioned in 1.7.4. Since intervals of subsemilattices are neither algebraic nor coalgebraic in general, we have to look directly at $\bigwedge$- and $\bigvee$-irreducible elements.

One of the most fundamental properties of intervals of subsemilattices is that they are C_D-systems, i.e. the point closures are precisely the $\bigvee$-irreducible elements, and thus every element has a $\bigvee$-decomposition.

It is also possible to characterise the $\bigvee$-prime elements of $\mathcal{C}_\kappa$. They are exactly those point closures $C(x)$ of elements $x \in S \setminus C$ which are **κ-irreducible over C**, i.e. $x = \bigwedge X$ with $X \subseteq_\kappa S$ implies $x = b \wedge c$ for some $b \in X$ and $c \in C$ (see [24]).

The description of $\bigwedge$-irreducible or -prime elements is more difficult.

3.5.1 Lemma

i) An element $Q \in \mathcal{C}_\kappa$ is $\bigwedge$-irreducible iff $x := \bigwedge(S \setminus Q)$ is the smallest element not in Q, and $x \leq_Q y$ for all $y \in S \setminus Q$.

ii) An element $Q \in \mathcal{C}_\infty$ is $\bigwedge$-irreducible iff $Q = (S \setminus \uparrow x) \cup \uparrow \overline{x} = S \setminus (\uparrow x \setminus \uparrow \overline{x})$ for a covering pair $x \prec \overline{x} \in S$ with $x \in S \setminus C$.

iii) An element $P \in \mathcal{C}_\kappa$ is $\bigwedge$-prime iff $x := \bigwedge(S \setminus P)$ is the smallest element not in P, and $x \leq_C y$ for all $y \in S \setminus P$.

iv) An element $P \in \mathcal{S}_\infty$ is $\bigwedge$-prime iff $S \setminus P = \{x\}$ for some $x \in S$.

Proof: In all cases $Q + x$ or $P + x$ denote the unique upper covers of Q and P, respectively.

i) $\bigwedge$-irreducible elements all have unique upper covers. Since covering pairs only differ by one element, we can call this element x. As $Q(x)$ must be contained in all other upper bounds of Q, $x \leq_Q y$ for all $y \in S \setminus Q$. On the other hand, if Q is of this form, it is maximal with respect to not containing x, i.e. a copoint of x and therefore $\bigwedge$-irreducible.

ii) In the special case $\mathcal{C}_\infty$, the infimum $\bigwedge(Q \cap \uparrow x) =: y$ is in Q, and has to be an upper cover of x in S. Otherwise there would be an element $z \in S$ with $x < z < y$ such that $z = z \wedge q \neq x$ for all $q \in Q \cap \uparrow x$, i.e. $x \not\leq_Q z$, which cannot be. Therefore, irreducible elements have to have the form $Q = (S \setminus \uparrow x) \cup \uparrow \overline{x}$ for a covering pair $x \prec \overline{x} \in S$ with $x \in S \setminus C$. Conversely, all sets of the form $Q = (S \setminus \uparrow x) \cup \uparrow \overline{x}$ with $C \subseteq Q$ are in $\mathcal{C}_\infty$ and $x \prec \overline{x}$ have a unique upper cover $Q + x$ in $\mathcal{C}_\infty$. Thus they are $\bigwedge$-irreducible elements of $\mathcal{C}_\infty$.

iii) $P \in \mathcal{C}_\kappa$ is $\bigwedge$-prime iff $M = \bigwedge\{X \in \mathcal{C}_\kappa \mid X \nsubseteq P\}$ is the least element not less than or equal to P. We have that $C(x) \nsubseteq P$, but $C(x) - x \subseteq P$, i.e. $C(x)$ is a minimal element not less than or equal to P. For $C(x)$ to be the least element of this form, $C(x)$ needs to be smaller than any other $C(y)$, if $C(y) \nsubseteq P$, i.e. $x \leq_C y$ for every $y \in S \setminus P$.

iv) In the special case $P \in \mathcal{S}_\infty$, we have $x \leq_C y \iff x = y$, since $C = \{\top_S\}$ and $C(x) = \{x, \top_S\}$. Therefore, an element of P can only be $\bigwedge$-prime, if $S \setminus P = \{x\}$. $\square$

3.5.2 Example

$\bigwedge$-irreducible or $\bigwedge$-prime elements of $\mathcal{C}_\kappa$ need not exist. E.g. $[0,1] \setminus \{x\}$ is not $\bigwedge$-closed for any $x \in [0,1]$, thus $\bigwedge$-prime elements do not exist for $S = [0,1]$ and $\kappa = \infty$.

This is also an example for a system without $\bigwedge$-irreducible elements, since it is impossible to construct a $\bigwedge$-subsemilattice of S which has a unique upper cover in $\mathcal{S}_\kappa$.

3.6 Decompositions

The rather complex nature of $\bigwedge$-irreducible elements hinders the existence of $\bigwedge$-decompositions and makes it difficult to give decent characterisations of intervals of subsemilattices in which every element has a $\bigwedge$-decomposition. However, for one very important class we were able to give such a characterisation.

3.6.1 Theorem

Each element of $\mathcal{S}_\infty$ has a $\bigwedge$-decomposition iff S is strongly atomic.

Proof: We will use that the existence of $\bigwedge$-decompositions is equivalent to the following property. To $A, B \in \mathcal{S}_\infty$ with $A \nsubseteq B$ there exists a $\bigwedge$-irreducible $Q \in \mathcal{S}_\infty$ such that $A \nsubseteq Q$ and $B \subseteq Q$.

Also note that every $\bigwedge$-irreducible element of $\mathcal{S}_\infty$ is of the form $Q = (S \setminus \uparrow x) \cup (\uparrow \overline{x})$ for a covering pair $x \prec \overline{x}$ in S.

We start with a strongly atomic $\bigwedge$-semilattice S and show that $\mathcal{S}_\infty$ is $\bigwedge$-irreducibly generated. Take $A, B \in \mathcal{S}_\infty$ with $A \nsubseteq B$. Thus there exists an $x \in A \setminus B$. Define $y := \bigwedge \{b \in B \,|\, x < b\}$, which satisfies $y \in B$ and $x < y$. Since S is strongly atomic, we can find an element $\overline{x} \in S$ such that $x \prec \overline{x} \leq y$. We use this element to construct $Q := (S \setminus \uparrow x) \cup (\uparrow \overline{x})$ to obtain a $\bigwedge$-irreducible element of $\mathcal{S}_\infty$ that satisfies $A \nsubseteq Q$ and $B \subseteq Q$ as $x \in A \setminus Q$ and $B \cap \uparrow x \setminus \uparrow \overline{x} = \emptyset$.

For the converse direction, consider $a, b \in S$ with $a < b$. We construct $B := (S \setminus \uparrow a) \cup (\uparrow b)$ and $A := B + a$. It is easily verifiable that both A and B are in $\mathcal{S}_\infty$, in which they are a covering pair (by construction).

Since $A \nsubseteq B$, there exists a $\bigwedge$-irreducible $Q \in \mathcal{S}_\infty$ such that $A \nsubseteq Q$ but $B \subseteq Q$, where Q is of the form $Q = (S \setminus \uparrow x) \cup (\uparrow \overline{x})$ for some $x \prec \overline{x} \in S$. We first note that $a \notin Q$, as $a \in Q$ would imply $A \subseteq Q$, which is forbidden. Now

$$B \leq Q \Rightarrow x \notin B \Rightarrow x \in S \setminus B = \uparrow a \setminus \uparrow b \Rightarrow a \leq x,$$

and

$$a < x \Rightarrow a \in S \setminus \uparrow x \subseteq Q \Rightarrow a \in Q \Rightarrow A \subseteq Q,$$

which is false. Thus we get $a = x$. Furthermore,

$$(b \in Q) \wedge (\overline{x} \nleq b) \Rightarrow b \in S \setminus \uparrow x = S \setminus \uparrow a \Rightarrow a \nleq b,$$

but we started with $a < b$, so we get $\overline{x} \leq b$.

We can sum this up and get $a = x \prec \overline{x} \leq b$, proving the strong atomicity of S.
□

This shows that $\bigwedge$-decompositions are rare for $\kappa = \infty$, but they can exist. Note that the existence of $\bigwedge$-decompositions for intervals of $\wedge$-subsemilattices $\mathcal{C}_\kappa$ is trivial, since these intervals are algebraic (see [11]). Furthermore, if $\mathcal{C}_\kappa$ is strongly atomic, we even have existence of irredundant $\bigwedge$-decompositions.

We give one example of a lattice of $\bigwedge$-subsemilattices which has decompositions, but in which infinite meets do not coincide with finite meets.

3.6.2 Example

We consider the complete semilattice S of closed subsets of $[0, 1]$. Since sets containing a single element are closed and finite unions of closed sets are closed, S is strongly atomic.

Therefore, every element A in the system $\mathcal{S}_\infty$ of all complete subsemilattices has a $\bigwedge$-decomposition.

We define $D_n := \left\{ \left[0, \frac{1}{m}\right] \mid m \leq n,\, m \in \mathbb{N} \right\}$ for every $n \in \mathbb{N}$. Then $\mathcal{D} := \{D_n \mid n \in \mathbb{N}\}$ is a chain of elements in $\mathcal{S}_\infty$. The supremum of $\mathcal{D}$ however is $\bigvee \mathcal{D} = \bigcup \mathcal{D} \cup \{\{0\}\}$, with $\{\{0\}\} \notin D_n$ for $n \in \mathbb{N}$.

If we take $A = \{\{0\}, [0, 1]\} \in \mathcal{S}_\infty$, we get

$$A \wedge \bigvee \mathcal{D} = A \neq \{[0, 1]\} = \bigvee (A \wedge D_n),$$

confirming that $\mathcal{S}_\infty$ is not $\wedge$-continuous (and therefore $\mathcal{S}_\infty \neq \mathcal{S}_\omega$).

Although we already noted that the existence of $\bigvee$-decompositions is clear for intervals of subsemilattices, we want to mention one result on this topic which is quite interesting.

3.6.3 Lemma

The following are equivalent for any interval $\mathcal{C}_\kappa$ of κ-$\bigwedge$-subsemilattices.

i) $\mathcal{C}_\kappa$ is strongly coatomic.

ii) Each member of $\mathcal{C}_\kappa$ has a least $\bigvee$-decomposition.

iii) Each member of $\mathcal{C}_\kappa$ has a unique irredundant $\bigvee$-decomposition.

iv) Each member of $\mathcal{C}_\kappa$ has an irredundant $\bigvee$-decomposition.

Proof: See Cor. 3.2 in [24]

3.6.4 Remark

This result is not surprising if we use Chapter 2. We know that bases of elements in detachable C_D-systems are always least generators. Furthermore, we know that it is the class of shellable C_D-systems in which every element has a least generator, and shellable systems are precisely the strongly coatomic detachable systems.

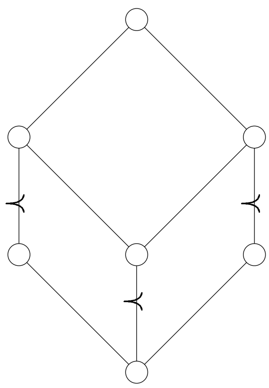
3.7 Distributivity

This section will be very short, since there are not many new results in this area. The precise description of distributive intervals of subsemilattices has been done by Erné in Theorem 4.1 of [24]. This is in a way a special case of 2.13.5, which characterises detachable and bigenerated C_D-systems which are distributive. Historically, 2.13.5 was motivated by Theorem 4.1 of [24].

However, since the original result is more precise and gives a good insight into the structure of intervals of subsemilattices, we want to repeat it and some related results without proof.

3.7.1 Definition

We say a lattice **contains a sublattice S_7^* faithfully**, if the vertical three cover relations hold not only in S_7^* but also in the entire lattice.



3.7.2 Theorem

The following statements are equivalent:

i) $\mathcal{O} = \{S \setminus B \mid B \in \mathcal{C}\}$ *is a topology (on* $S \setminus C$*).*

ii) $\mathcal{C}$ *is closed under binary unions.*

iii) $\mathcal{C}$ *is a coframe.*

iv) $\mathcal{C}$ *is distributive.*

v) $\mathcal{C}$ *is modular.*

vi) $\mathcal{C}$ *is* $\wedge$*-semidistributive.*

vii) All atoms of any subinterval $\mathcal{C}'$ *of* $\mathcal{C}$ *are* $\vee$*-prime in* $\mathcal{C}'$*.*

viii) $\mathcal{C}$ *does not contain any sublattice sublattice* S_7^* *(faithfully).*

ix) $a \wedge b \in C(a) \cup C(b)$ *for all* $a, b \in S$*.*

x) For distinct $a, b, s \in S \setminus C$ *with* $s = a \wedge b$*, there is a* $c \in C$ *with* $s = a \wedge c$ *or* $s = b \wedge c$*.*

Each of these conditions implies

xi) $\mathcal{C}$ *is upper semimodular and lower continuous.*

If $\mathcal{C}$ *is shellable, the converse implication is also true.*

A related results concerns upper semimodularity of intervals of subsemilattices. For this we say that a subset C of a meet-semilattice S is **weakly meet-dense**, if for all $a, b \in S$ with $a \not\leq b$, there is a $c \in C$ with $a \wedge b = b \wedge c$ or $a \wedge b \leq a \wedge c < a$. Note that every meet-dense subset has this property.

3.7.3 Theorem

The following are equivalent for any interval $\mathcal{C}_\kappa$ *of* κ*-*$\bigwedge$*-subsemilattices.*

i) $\mathcal{C}$ *is upper semimodular.*

ii) $A, B \in \mathcal{C}$ *and* $A + x \in \mathcal{C}$ *imply* $A \vee B + x \in \mathcal{C}$*.*

iii) $A, B \in \mathcal{C}$ *and* $x \in S \setminus A$ *imply* $A(x) \vee B - x \subseteq (A(x) - x) \vee B$*.*

iv) $C(a) \vee C(b) - a \subseteq (C(a) - a) \vee C(b)$ *for all* $a, b \in S$*.*

v) C *is weakly meet-dense in* S*.*

vi) For distinct $a, b, s \in S \setminus C$ *with* $s = a \wedge b$*, there is a* $c \in C$ *with* $s = a \wedge c$ *or* $s = b \wedge c$ *or (*$s \leq a \wedge c < a$ *and* $s \leq b \wedge c < b$*).*

We see that the characterisation of upper semimodular and distributive intervals are very similar, especially if we compare equivalence (x) of 3.7.2 and equivalence

(vi) of 3.7.3. This similarity indicates that if certain restrictions on S or C are met, then upper semimodularity already implies distributivity.

3.7.4 Corollary

Suppose the interval of κ-$\bigwedge$-subsemilattices $\mathcal{C}_\kappa$ is upper semimodular, and C or $S \setminus C$ satisfies the dcc. Then $\mathcal{C}_\kappa$ is already distributive.

Proof: We only give a short sketch of the proof. If $\mathcal{C}_\kappa$ is upper semimodular but not distributive, we can find distinct elements $a_1, b_1, s = a_1 \wedge b_1 \in S \setminus C$ such that there exists an element $c_1 \in C$ with

$$s \leq a_1 \wedge c_1 < a_1 \text{ and } s \leq b_1 \wedge c_1 < b_1,$$

but $s \neq a_1 \wedge c$ and $s \neq b_1 \wedge c$ for all $c \in C$. This produces new elements $a_2 = a_1 \wedge c_1$ and $b_2 = b_1 \wedge c_1$, both of which must be in $S \setminus C$. These elements also satisfy $a_2 \wedge b_2 = s$, hence there exists a $c_2 \in C$ with

$$s \leq a_2 \wedge c_2 < a_2 \text{ and } s \leq b_2 \wedge c_2 < b_2.$$

If we continue this procedure, we obtain three descending chains, $a_1 > a_2 > \ldots$ and $b_1 > b_2 > \ldots$ which are contained in $S \setminus C$, and $c_1 > c_2 > \ldots$ which is a subset of C.

If C or $S \setminus C$ satisfies the *dcc*, one of these chains terminates after a finite number of steps. This would generate an element $c' \in C$ such that $a \wedge c' = s$ or $b \wedge c' = s$, a contradiction to our assumption that this is not the case for all elements in C. $\square$

For some corollaries and further remarks concerning distributive intervals and related properties, see [24].

3.8 Construction I

We now turn to some of the most interesting aspects of the theory of intervals of subsemilattices, the construction. This includes both directions, the construction of the interval of all κ-$\bigwedge$-subsemilattices of a κ-$\bigwedge$-semilattice S which contain a κ-$\bigwedge$-subsemilattice C, and the reconstruction of a κ-$\bigwedge$-semilattice S with κ-$\bigwedge$-subsemilattice C from a given C_D-system $\mathcal{C}$ or C_D-lattice L. We have to note at this point that the second part has not been solved completely, we can only give partial results and explain some approaches to and problems of this task.

3.8.1 Mathematical Construction

The most basic aspect of construction has already been dealt with. The fixed points of the closure operator Γ are the members of the closure system $\mathcal{C}_\kappa$. This is independent of the size of S and C, but also of κ.

By applying Γ to every subset of $S \setminus C$ we obtain every element of the interval we are looking for, but this approach does not make use of any properties of S or $\mathcal{C}$ to decrease the amount of work we invest. Thus, it does not seem to be first choice when we want to construct a specific interval of subsemilattices with as little work as possible.

Though it is difficult or maybe even impossible to give a precise description of the elements of an infinite interval of subsemilattices, this is not the case for finite intervals.

3.8.2 Algorithmic Construction of Finite Intervals of Subsemilattices

If we have a finite $\wedge$.semilattice S with $\wedge$-subsemilattice C, it is possible to determine all elements of the corresponding interval $\mathcal{C}_\omega$ in a finite amount of time. Since we have gained quite a knowledge of the structure of these intervals, it is even possible to determine all elements in an acceptable amount of time.

The property we exploit for this endeavour is the anti-exchange property for covers, i.e. the fact that covers differ by exactly one element. It allows us to construct the elements of $\mathcal{C}_\omega$ layer by layer. We can either start with C and add those elements x of $S \setminus C$ which are minimal in the relation $\leq_C$, i.e. for which $C + x$ is closed, or we start with S and reduce it by deleting single elements x such that $S - x$ is closed. This second approach uses, of course, shellability.

It is the second approach, which we will use. It is much easier to identify those elements which we can delete from one closed set to obtain a lower cover, than to identify those which we can add to a small one to obtain an upper cover.

If $A \in \mathcal{C}$ and $A - x \in \mathcal{C}_\omega$, then x needs to be a $\wedge$-irreducible element in A, and conversely it is the $\wedge$-irreducible elements which we can remove from A and obtain a closed set. It is possible to identify $\wedge$-irreducible elements in A from the fact that they have exactly one upper cover in A, as we restrict ourselves to the finite case. So, if we know the upper covers of every element, we always know which elements we can remove in the next step.

The other alternative is much more work-intensive, since to a given closed set B we have to find those elements $x \notin B$ which satisfy $B + x \in \mathcal{C}_\omega$. For this we will have to generate $B(x)$ for possible candidates $x \notin B$ and test if they have the required form. This takes much more time than the first alternative, we will therefore not use this one.

In addition to the precise construction of the $\wedge$-subsemilattices from the given $\wedge$-semilattice S, we have to store the semilattice structure of S somehow, since it is essential to our task. One way to do so is to simply list the number of upper covers for each element, everything else can be derived from this information, and it is an easy exercise to determine all $\wedge$-irreducible elements from it.

This consideration allows us to formulate a basic algorithm, which we can implement.

3.8.3 The Basic Algorithm

We start with a semilattice $S = \{a_0, a_1, ..., a_{n-1}\}$ and a subsemilattice $C = \{a_{i_0}, a_{i_1}, ..., a_{i_m}\}$.

i) Identify all $\wedge$-irreducible elements of $S \setminus C$.

ii) Remove one $\wedge$-irreducible element x to obtain $A := S - x \in \mathcal{C}$.

iii) Apply the algorithm to the reduced semilattice A, if A contains at least one element $a_i \in A \setminus C$. Otherwise, return to the previous level and continue with the next $\wedge$-irreducible element y of $A + x$.

This is only a very basic and, if implemented in this way, inefficient algorithm, since most subsemilattices would be construced numerous times. To get a quicker algorithm, we need to remember which subsemilattices we already constructed. This can be done in the following way.

We start with S, remove x and calculate all subsemilattices of S which do not contain x by applying the algorithm to these smaller sets. Afterwards, we remove the next $\wedge$-irreducible element on our list and calculate all subsemilattices of S which do not contain y but DO contain x. In the next phase, we calculate all subsemilattice of S which do not contain a $\wedge$-irreducible element z but which contain x and y and so on.

If we do so, we will not construct the same subsemilattice twice. The amount of work required to keep track of which elements still need to be removed and which need to stay in is almost minimal.

The code of a C++-program which uses this approach is added in the appendix. The semilattice S and a subsemilattice C are decoded in a file `data.txt`. The output is written into a file `ausgabe.txt`, and consists of S, C and a list of all elements of $\mathcal{C}$ as well as their number. New results are appended to the file, so that old results are not deleted.

3.8.4 Some Remarks on the Program

We start with the initial data. The information on S given in `data.txt` consists of

- the number n of elements in S,
- the number of upper covers for the elements from 1 to $n-1$,
- the list of upper covers for each element from 1 to $n-1$,
- the number of elements in C, and
- the list of elements in C.

It might be possible to derive some of these items from the others, but by giving the size of certain sets in advance, it is easier to initialise pointers and variables.

Although the initial data is restricted to the upper covers, the program determines the upper cut or filter of every element by storing all upper covers and the upper covers of these and so on. The reason for this is that when we remove an element x from a semilattice, we have to update the upper cover relation (and the upper cuts). The lower covers of x may gain a new upper cover in the form of the former unique upper cover of x. It would be easy if it was always the case that this upper cover of x is added to the list of upper covers of former lower covers of x, but this is not true. For an easy example, use the pentagon and remove the atom which is also a coatom. The remaining subsemilattice is a chain, and the bottom element does not gain another upper cover.

The rest of the program consists of the various variables to store subsemilattices and other information as e.g. on which irreducible elements have already been removed at an earlier time, and of a number of functions required in the process of generating subsemilattices, like the above-mentioned removal of an element combined with the required update of the cover relation.

3.9 Construction II

The second part on construction is much more difficult. We have to start with a discussion on the initial data. If we start with an interval $\mathcal{C}_\kappa$ of κ-$\bigwedge$-subsemilattices, it is easy to identify the two most important sets immeadiately. The top element of $\mathcal{C}_\kappa$ is S, the bottom element is C.

If $\mathcal{C}$ is a C_D-system which is isomorphic to an interval of subsemilattices, this ceases to be true. Typical systems of this type are ones which started as intervals, but from which some or all elements of the least element C have been removed. Since these elements are contained in every set in the closure system, the structure of the C_D-system remains the same.

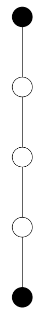
If we start with a C_D-lattice L, we can construct a C_D-system on the set of $\bigvee$-irreducible elements of L. This would only cover a part of the initial set, a set which is isomorphic to (or resembles) $S \setminus C$, but we have no information on elements which are contained in every element of $\mathcal{C}$, i.e. we get a C_D-system of the type we described in the previous paragraph.

We illustrate this problem with a small example.

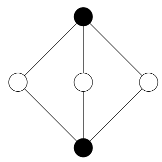
3.9.1 Example

Suppose L is isomorphic to the power set on a three-element set. Then there are at least two possibilities to find a set S with a subsemilattice C such that $\mathcal{C} = [C, S]$ is isomorphic to L. We give diagrams for these possibilities, and we use black dots to indicate elements of C.

i) The first possibility is to use a five-element chain for S, with $C = \{\top, \bot\}$. Since every subset that contains $\top$ is $\wedge$-closed, the resulting system of subsemilattices is isomorphic to the power set on three elements.



ii) The second possibility is to use a three-element antichain, to which we again add top and bottom element, the two of which make up C. Once again, the corresponding interval of subsemilattices is isomorphic to the power set on three elements.



Both possibilities have a base set of five elements and a subsemilattice containing two elements, i.e. they are very similar. The two semilattices, however, are very different from each other. This shows that even if we are able to reconstruct a semilattice from a given C_D-lattice or C_D-system, it is unlikely that we get the same semilattice which was used to construct the C_D-system.

Furthermore, the bottom element in the first possibility is superfluous, it only shows that even the same cardinalities need not have the same result. We would obtain the same result, if we drop the bottom element, hence we would only have a base set of four elements.

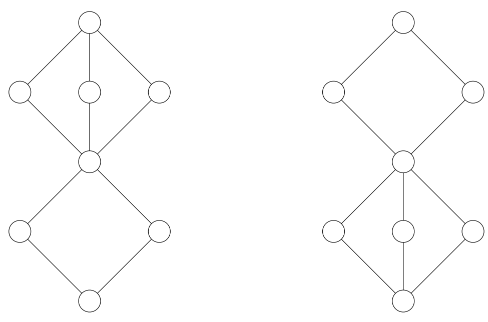
3.9.2 $\mathcal{S}_\omega$ for Finite Base Sets

The previous example shows that two very different semilattices with appropriate subsemilattices can have identical intervals of subsemilattices. This is not true, if we consider the full system of all subsemilattices of a finite semilattice.

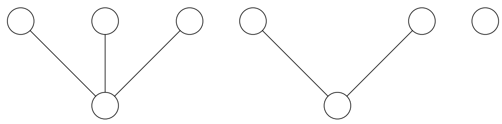
Adaricheva has shown in [2] that it is possible to construct a semilattice structure on S from the system $\mathcal{S}$. The procedure is very technical and may not be optimal, and a complete verification is hampered by occasional errors in the translation from Russian into English, but it uses some interesting ideas and observations.

Without going into the details of the construction, we want to mention at this point that this construction contains choice at some point of time, which requires us to choose a linear order on a set of elements which are completely independent of each other, Adaricheva uses the term **completely free** for these.

This is not very surprising if you note that the systems of all subsemilattices of the next two semilattices are isomorphic.



We can see that the upper and lower parts can be exchanged without changing the resulting system of subsemilattices, since meets in one part are independent of the other. Note that we actually have three parts, but the third part is simply the single element set containing the top element.



Semilattices consisting of more independent parts have even larger degrees of freedom, but the principle is the same.

3.9.3 Lemma

Consider a finite set $\mathcal{A}$ of κ-$\bigwedge$-semilattices. If S is defined as an ordinal sum of the elements of $\mathcal{A}$ with added $\top$-element, the system of all κ-$\bigwedge$-subsemilattices $\mathcal{S}_\kappa$ of S is isomorphic to the product of the systems of all κ-$\bigwedge$-subsemilattices $\mathcal{S}_A$ of every $A \in \mathcal{A}$.

This is independent of the order on $\mathcal{A}$ in the ordinal sum.

Proof: If $S := \bigoplus \mathcal{A} \oplus \{\top\}$ for some order on $\mathcal{A}$, then $B \cup \{\top\} \in \mathcal{S}_\kappa$ if and only if $B \cap A \in \mathcal{S}_A$ for every $A \in \mathcal{A}$, i.e. if and only if $\mathcal{S}_\kappa \simeq \prod_{A \in \mathcal{A}} \mathcal{S}_A$. $\square$

The previous example simply used a two-element set for $\mathcal{A}$.

An interesting aspect of the above-mentioned construction principle by K. Adaricheva is, that it only uses the lowest levels of the system $\mathcal{S}$. The bottom element of $\mathcal{S}$ is the set $\{\top_S\}$, the next level consists of the atoms, which are all of the form $\{\top_S, x\}$ for $x \in S$, i.e. represent the elements of $S - \top_S$. These two levels are trivial and do not contribute in an effort to reconstruct a semilattice structure on S. This changes dramatically with the next two levels.

3.9.4 Definition

Consider a complete lattice L. We call the upper covers of atoms **bi-toms**, and the upper covers of bi-toms are called **tri-toms**.

Sometimes bi-toms and tri-toms are called hyper-atoms.

3.9.5 Description of Bi-Toms

Every bi-tom of a system of subsemilattices $\mathcal{C}$ is of the form $A = \{a, b, \top_S\}$ with $a, b \in S - \top_S$ and either $a < b$ or $b < a$.

Equality of a and b would give an atom instead of a bi-tom, and if a and b were incomparable, A would not be $\wedge$-closed.

Thus the set of bi-toms gives a full list of which elements of S are comparable and which are incomparable, but we get no indication as to how these elements are ordered.

3.9.6 Description of Tri-Toms

Unlike bi-toms, tri-toms are not of one type, but two. A tri-tom can be of the form $B = \{a, b, c, \top_S\}$ with $a, b, c \in S - \top_S$ pairwise comparable, or we can have that $a \wedge b = c$.

It is this second type of tri-toms, which are interesting to us, since they help us to determine the structure on S. We will call these tri-toms **relevant**.

It is easy to determine whether a tri-tom is relevant or not. Relevant tri-toms are joins of two atoms, e.g. for the tri-tom given here we have $B = \{a, \top_S\} \vee \{b, \top_S\}$, since B is the smallest $\wedge$-closed subset containing a and b. If a, b, c were pairwise comparable, the join of two atoms would only generate bi-toms.

Thus the level of tri-toms, with some assistance from the atoms, gives us all incomparable pairs of elements and their meets. A repeated and interlinked analysis of tri-toms gives all information which is obtainable from the system. Larger sets in $\mathcal{S}$ do not give additional information. The procedures for extracting the relevant information and the resulting sequences of elements are called **descents** and **zig-zags** in [2].

3.9.7 Reconstruction of Semilattices for Finite Base Sets

The reconstruction of a semilattice S with subsemilattice C from a given C_D-lattice L is much more difficult. It starts with the fact that we only know the number of elements in $S \setminus C$, which correspond to the $\bigvee$-irreducible elements of L. The smallest necessary number of elements in C must be deduced from the lattice structure.

Furthermore, there might be a large number of different $\wedge$-semilattices with intervals of $\wedge$-subsemilattices isomorphic to L, as we have already mentioned.

Closely related to the problem of reconstructing the semilattice is the question whether L is a representation of some interval of subsemilattices at all. Since we cannot give a complete characterisation of these lattices at this point, we can only hope to find a representation by constructing a semilattice S with subsemilattice C based on the data from L.

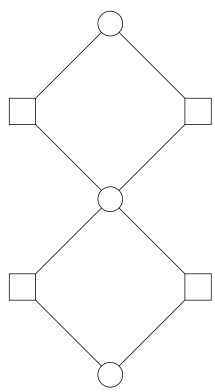
Consider for example the class of finite distributive lattices. We know that intervals of subsemilattices can be distributive, so we can ask the question whether all finite distributive lattices are isomorphic to some interval of subsemilattices. Since finite distributive lattices are algebraic and shellable they are good candidates for intervals of subsemilattices from the vantage point of the anti-exchange properties.

Starting with the smallest distributive lattices, we can show that all of them containing seven or less elements are isomorphic to some interval of subsemilattices.

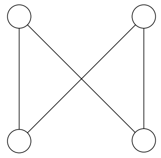
A major tool for a verification of this is lemma 3.3.3, which we will use in one example of a lattice with seven elements which is not trivial.

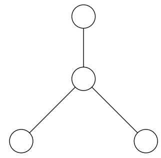
3.9.8 Example

We want to show, that the following distributive lattice is isomorphic to an interval of subsemilattices.



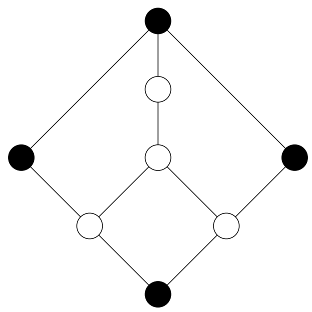
The $\bigvee$-irreducible elements are shown as boxes, since they represent the four potential point closures of the interval. Hence, $S \setminus C = \{a, b, d, e\}$. Using 3.3.3 we see that the order on these four elements must be one of the following two.





The two lower points, a and b, need to be incomparable, since their point closures are both contained in a third point closure. A similar restriction to the two other elements, d, e, does not exist, therefore we get these two alternatives for ordering them.

While the left order cannot be extended to a semilattice which has the properties we want, the order on the right can. We have to add a number of elements, all of which are in C and shown as black dots.



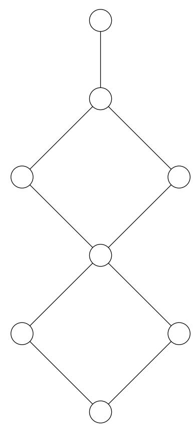
We end up with a semilattice with eight elements, four of which are in C. It is easy to see, that the interval of subsemilattices of S that contain C is isomorphic to the lattice we started with.

The problems we had realising this example with the left order on our four elements suggests that we cannot succeed if we force the upper two elements to be incomparable as well. This leads directly to a counterexample to the hypothesis that every finite distributive lattice is isomorphic to an interval of subsemilattices.

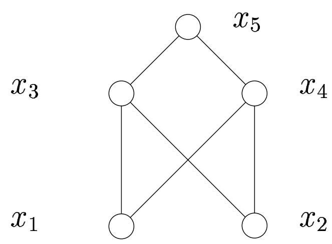
3.9.9 Lemma

There are finite distributive C_D-lattices which are not isomorphic to an interval of subsemilattices.

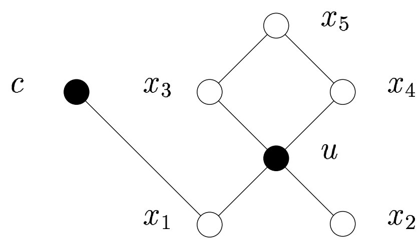
To be precise, the following lattice L cannot be isomorphic to a $\mathcal{C}_\kappa$.



Proof: We assume that L is isomorphic to some interval $\mathcal{C}_\kappa$ of subsemilattices of S that contain C. L contains five $\bigvee$-irreducible elements, i.e.
$S \setminus C = \{x_1, x_2, x_3, x_4, x_5\}$. Lemma 3.3.3 implies that the order on these five elements of S is



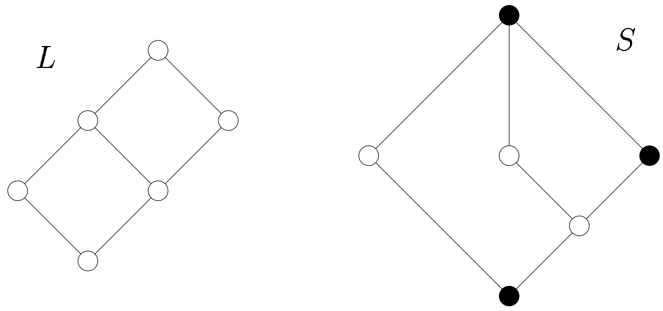
Since S is a semilattice, $u := x_3 \wedge x_4$ exists and is in C, since it cannot be x_1 or x_2 as they are incomparable. Furthermore, $C(x_1) \subseteq C(x_3)$, i.e. there exists a $c \in C$ such that $x_1 = x_3 \wedge c$. Consequently, $x_1 \leq c$ and $x_1 \leq u$. This would result in an order diagram of the following form.



The contradiction is obvious: $x_1 \leq u \wedge c \leq x_3 \wedge c = x_1$, hence $x_1 = u \wedge c$. But $u \wedge c \in C$, whereas $x_1 \in S \setminus C$, so we see that L cannot be isomorphic to an interval of subsemilattices. $\square$

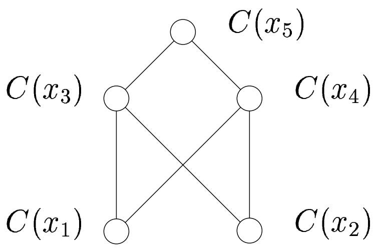
The example we give here is the only distributive lattice with 8 or fewer elements, which is not isomorphic to an interval of subsemilattices. For every other distributive lattice of this size there exists a semilattice with appropriate subsemilattice such that the lattice and interval are isomorphic.

We give one example of a distributive lattice L with six elements and a semilattice S with subsemilattice C for which $L \simeq \mathcal{C}_\omega$. Once again, elements of C are shown as black dots.



3.9.10 Corollary

No set of five point closures $C(x_i) \in \mathcal{C}_\kappa$ for $i = 1, ..., 5$ can be ordered like



unless $y = x_3 \wedge x_4 \in S \setminus C$.

The thorough inspection of the counterexample leads to some general properties concerning the arrangement of elements of C and $S \setminus C$.

3.9.11 Lemma

Consider a κ-$\bigwedge$-semilattice with κ-$\bigwedge$-subsemilattice C and $x, y \in S \setminus C$.

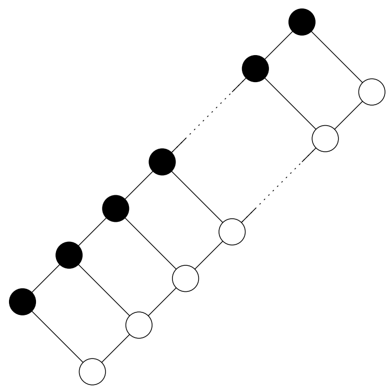
If $x \leq_C y$, then the whole interval $[x, y]$ is contained in $S \setminus C$.

Consequently, if a subset $K \subseteq_\kappa S \setminus C$ has a lower bound $x \in S \setminus C$ with respect to $\leq_C$, i.e. $x \leq_C k$ for all $k \in K$, then $\bigwedge K \in S \setminus C$.

Proof: Suppose there is a $c_1 \in C$ such that $x < c_1 < y$. Since $x \leq_C y$, there is a $c_2 \in C$ such that $x = c_2 \wedge y$, but this leads to $x \leq c_1 \wedge c_2 \leq c_2 \wedge y = x$, a contradiction just like in the counterexample. □

We now know that there are certain distributive lattices which are not isomorphic to intervals of subsemilattices, but it is not obvious at first glance, which lattices are or are not of this type, and there are some dramatic differences between the more special system $\mathcal{S}_\kappa$ and the general intervals $\mathcal{C}_\kappa$.

Systems $\mathcal{S}_\kappa$ of all κ-$\bigwedge$-subsemilattices of a κ-$\bigwedge$-semilattice S tend to have a large width, but if we choose S and C carefully, it is possible to generate very thin intervals of κ-$\bigwedge$-subsemilattices. Consider for example the next (finite) $\wedge$-semilattice S, in which the elements of the $\wedge$-subsemilattice C are marked by black dots.



The resulting interval $\mathcal{C}$ is a chain of the same length as $S \setminus C$, and it impossible to find a thinner non-empty lattice than this.

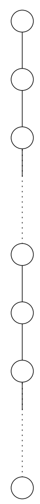
All in all, these are not very satisfactory results. We do not have a complete lattice theoretical characterisation of finite intervals of subsemilattices, and the verification of single lattices whether or not they are isomorphic to intervals can be very difficult.

Not very surprisingly, things do not get better if we allow base sets, C_D-systems or C_D-lattices to be infinite.

3.9.12 Reconstruction of Intervals for Infinite Base Sets

In the infinite case, we have to deal with some additional problems. The first one is, that there might be infinitely many C_D-lattices which are isomorphic to a given C_D-system.

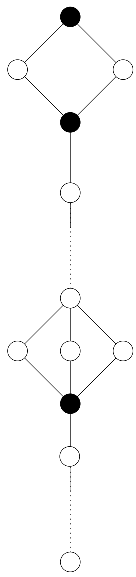
As an example, consider the $\bigwedge$-semilattice that is an ordinal sum of two infinite chains.



The system $\mathcal{S}_\infty$ of all $\bigwedge$-subsemilattices consists of almost all subsets of S. The only restriction is that if a set contains an infinite number of elements of one of the chains, it needs to contain the infimum of that chain to be $\bigwedge$-closed. In particular, $\mathcal{S}_\infty$ contains all finite subsets of S.

If we try to reconstruct the original $\bigwedge$-semilattice from the C_D-system, we see that all elements are comparable, i.e. S needs to be a chain, and we have two prominent elements which must be meets of infinite sets. Hence we get the general structure of S, but there is no restriction on a single element whether it belongs to the upper or the lower chain. Which is not surprising, since you can change the position of every element except the two infinte meets without affecting the system $\mathcal{S}_\infty$.

If we allow the system to be an interval $\mathcal{C}_\infty$ of subsemilattices, we are not restricted to chains any more. By adding elements of C at every point, where a finite number of elements have a meet, it is also possible to obtain the same C_D-system. The basic idea is shown in the next diagram.



Another problem in this context is the inability to identify meets of κ-sets. The crucial part of the Adaricheva approach was, that we were able to identify the meet of two incomparable elements through the closure of the two-element set, which contains three elements at most.

If we work with κ-$\bigwedge$-semilattices, we are interested in the meets of κ-sets. However, unless we are extremely lucky, the closure of a κ-set K will not contain just one element more than K, but many more.

Consider for example the countable set $\mathbb{Q} \cap [0,1] \subseteq [0,1]$. For $\kappa > \omega$, the closure of $\mathbb{Q} \cap [0,1]$ in the system of all κ-$\bigwedge$-subsemilattices of $[0,1]$ is $[0,1]$ itself, i.e. it even has a larger cardinality than $\mathbb{Q} \cap [0,1]$. This example is doubly perfidious, since the meet of $\mathbb{Q} \cap [0,1]$ is already contained in it, but the pure closure system does not reveal this.

Hence completely new ideas might be required to make progress in this area.

So far we have used two main approaches for an analysis of intervals of subsemilattices, direct analysis of intervals, their lattice structure etc., and the general theory on convex geometry we gave in Chapter 2. In this we considered intervals of subsemilattices as convex sets in an abstract definition of convexity. Maybe we can learn more about these intervals, if we use other definitions of convexity. This will be the central idea of the next chapter.

Chapter 4

Geometric Properties of Intervals of Subsemilattices

4.1 Introduction and Basic Notation

In our approach to construct a theory of convex geometries in Chapter 2, we focussed on closure systems with anti-exchange properties.

This is not the only possibility to define convex structures. There are quite a number of different approaches to this, some of which we want to mention at this point. These approaches share that they are based on geometric properties of the system of convex subsets of $\mathbb{E}^n$, which were generalised to more abstract objects.

Most importantly, we are interested whether intervals of subsemilattices can be considered in the presented theories, or if at least some results can be modified and applied to them.

The alternative approaches are all taken from [48] by M. van de Vel and a lecture on convex geometries [23] by M. Erné. Most of the terminology can be found in [48], which also contains numerous examples and much more on the topic of convex structures than mentioned here.

That we cannot expect to use all results from van de Vel for our purposes is obvious, since the basic objects considered in [48] are the following.

4.1.1 Definition

Consider a closure system $\mathcal{C}$ on the set S, with the closure operator given by Γ. We call $\mathcal{C}$ a **convexity alignment**, if it is algebraic.

The pair $(S, \mathcal{C})$ of a set with a convexity alignment is called a **convex structure**.

We know that intervals of κ-$\bigwedge$-subsemilattices are not algebraic in general. However, van de Vel introduces a number of notations, ideas, properties and results which are independent of the algebraicity of $\mathcal{C}$, and some can be modified to meet our requirements. We can e.g. modify the notion of arity to general closure systems.

4.1.2 Remark

Consider a closure system $\mathcal{C}$. Recall that an element $A \in \mathcal{C}$ that is generated by a finite set is called a **polytope** (see 2.3.1), i.e. there exists a finite set $F \subseteq_\kappa S$ such that $\Gamma F = A$.

Polytopes play an important role in the theory of convex structures. Algebraic closure systems all satisfy the equation

$$\Gamma B = \bigcup \{\Gamma E \mid E \subseteq_\omega B\}$$

for $B \subset S$.

That is why we can say that a convex structure is determined by its polytopes. But there are a number of systems in which this determination by finite sets can be reduced even further.

4.1.3 Definition

A polytope which can be spanned by n or less elements, with $n \in \mathbb{N}$, is called a **n-polytope**.

The smallest element of $\mathcal{C}$ is a 0-polytope, every set of the form $\Gamma\{a, b\}$ with $a \neq b$ is a 2-polytope. These are sometimes called a **segment joining a and b**.

The closure operator Γ is of **arity n**, if

$$\Gamma A = \bigcup \{\Gamma E \mid E \subseteq A,\ |E| \leq n\}$$

for $A \subseteq S$.

The closure system $\mathcal{C}$ is called **n-determined**, if

$$C \in \mathcal{C} \iff \forall E \subseteq C,\ |E| \leq n : \Gamma E \subseteq C.$$

4.1.4 Remark

Arity and determination need not coincide, but systems with unary (1-ary) closure operators are precisely the 1-determined systems. These are the A-topologies, i.e. closure systems which are closed under arbitrary unions.

Systems with finitary closure operators are precisely the finitely determined closure systems, these are the above-mentioned algebraic closure systems or convexity alignments. See [23] for details.

Furthermore, if Γ is n-ary, $\mathcal{C}$ is n-determined, but the converse need not be true in general. A n-ary operator is also $(n+1)$-ary, and a n-determined system is $n+1$-determined, but these implications cannot be extended to equivalences in general.

4.1.5 Example

Consider the system $\mathcal{C}$ of all convex subsets of $\mathbb{R}^2$. $\mathcal{C}$ is 2-determined, and Γ is ternary (3-ary), but not binary.

If we consider the system of convex subsets of $\mathbb{R}^n$ with $n \in \mathbb{N}$, we see that it is always 2-determined, and the operator is $(n+1)$-ary.

The definitions of arity and determination did not require $\mathcal{C}$ to be algebraic, but non-algebraic systems are by nature not finitary. If we generalise the definitions to infinite settings, we can apply them to non-algebraic systems.

4.1.6 Definition

Consider $\mathcal{Z} \subseteq \mathcal{P}S$ and a closure system $\mathcal{C}$ on S with corresponding closure operator Γ. We say that Γ is **$\mathcal{Z}$-ary** if

$$\Gamma A = \bigcup\{\Gamma Z \mid Z \in \mathcal{Z} \cap \mathcal{P}A\}.$$

We say that $\mathcal{C}$ is **$\mathcal{Z}$-determined**, if

$$(\forall Z \in \mathcal{Z} \cap \mathcal{P}A : \Gamma Z \subseteq A) \Rightarrow A \in \mathcal{C}.$$

Consider a cardinal κ. We define

$$\mathcal{P}_\kappa S := \{A \subseteq_\kappa S\}.$$

4.1.7 Lemma

Consider a closure system $\mathcal{C}$ on S, with corresponding closure operator Γ. Then we get

$$\Gamma \textit{ is } \mathcal{P}_\kappa\textit{-ary} \Rightarrow \mathcal{C} \textit{ is } \mathcal{P}_\kappa\textit{-determined.}$$

If κ is regular, the converse is true, too.

Proof: See [23]

4.1.8 Remark

For $\kappa < \omega$, $\mathcal{P}_\kappa$-arity and -determination coincides with $(\kappa - 1)$-arity and -determination respectively, and a $\mathcal{P}_\omega$-ary operator is precisely a finitary one.

4.1.9 Lemma

Consider a regular cardinal κ. Intervals $\mathcal{C}_\kappa$ of κ-$\bigwedge$-subsemilattices of a κ-$\bigwedge$-semilattice S are κ-determined closure systems with κ-ary closure operators.

Proof: The elements of $\mathcal{C}_\kappa$ are precisely those sets, that are closed under κ-meets. This coincides with the definition of κ-determination

$$\forall K \subseteq_\kappa A : \Gamma K \subseteq A \Rightarrow A \in \mathcal{C}_\kappa,$$

i.e. a set A is closed if for every κ-set K of A it also contains the meet of K.

The previous lemma yields κ-arity for Γ, since κ is regular. $\square$

Closely related to arity and determination is the concept of intervals and interval systems with which we start the next section.

4.2 Alternative Approaches to Convexity

4.2.1 Definition

Consider a set S and a function

$$I : S \times S \to \mathcal{P}S,$$

which is extensive, i.e. $a, b \in I(a, b)$, and symmetric, i.e. $I(a, b) = I(b, a)$.

Then I is called an **interval operator on S**, and $I(a, b)$ is the **interval between a and b**. (S, I) is called an **interval space**.

4.2.2 Examples

If $(S, \mathcal{C})$ is a closure system (e.g. a convex structure), $(a, b) \mapsto \Gamma\{a, b\}$ defines an interval operator on S.

It is also possible to construct a convex structure from a given interval operator. A subset C of S is interval convex, if $I(a, b) \subseteq C$ for all $a, b \in C$.

If I is based on a closure system $\mathcal{C}$, this leads to [48]

$$\forall\, a, b \in S: \; I(a, b) \subseteq \Gamma\{a, b\}.$$

Alternative notations for the interval $I(a, b)$ generated by $\{a, b\}$ are ab, $[a, b]$ or $\overline{ab}$.

Interval systems are precisely the 2-determined closure systems. This approach is based on one of the most classical descriptions of convexity: A set A is convex if and only if for every pair $a, b \in A$ it also contains the interval $[a, b]$.

This is how convexity is defined in school, and probably the best description of convexity for non-mathematicians. Therefore it is a little bit disappointing that intervals of subsemilattices do not have this property in general.

The definition of an interval given here is a more general notion of intervals than we used so far. Until now, an interval for us was a set of elements I between two given elements a, b, i.e. for all $i \in I : a \leq i \leq b$, and we used this notion when we spoke of intervals of subsemilattices. When we speak of interval systems, we allow for much more general structures, and to expect elements in ab to be in some way between a and b might not be the best approach to intervals in this sense.

So far we have always considered closure systems with certain additional properties, and we want to continue this and look at further interesting ones.

4.2.3 Definition

Consider a closure system $\mathcal{C}$ on S with closure operator Γ. We say $\mathcal{C}$ satisfies the **Join-hull commutativity (JHC)**, if for every non-empty $A \in \mathcal{C}$ and every $x \in S \setminus A$

$$\Gamma(A + x) = \bigcup\{\Gamma\{a, x\} \mid a \in A\}.$$

4.2.4 Corollary

Every binary closure system has JHC, and if $\mathcal{C}$ is algebraic, the Join-hull commutativity implies that $\mathcal{C}$ is 2-determined.

This gives us the impression that JHC is closely related to binary closure systems and 2-determined closure operators. To draw the conclusion that this is a property rarely satisfied by intervals of subsemilattices would be premature.

4.2.5 Lemma

Every interval $\mathcal{C}_\kappa$ of κ-$\bigwedge$-subsemilattices has the Join-hull commutativity.

Proof: JHC is a direct consequence of bigeneration:

$$\Gamma(A + x) = A \vee \Gamma\{x\} = \bigcup\{\Gamma\{a, x\} \mid a \in A\}. \quad \square$$

Some other prominent properties considered for modelling convexity are only loosely connected to a closure operator, and are motivated more by geometric properties of convex sets in euclidean space. Two examples for this are the Peano property and the Pasch property.

4.2.6 Definition and Notation

Consider a closure system $\mathcal{C}$ on S with closure operator Γ. We recall, that the interval between two elements $a, b \in S$ can be written as ab. Using this, we define the **extension** or **ray at a away from b** by

$$a/b := \{x \in S \mid a \in bx\}$$

for $a, b \in S$.

Furthermore, we extend the idea of an interval to sets and define

$$AB := \bigcup\{ab \mid a \in A,\, b \in B\}$$

for $A, B \subseteq S$.

If the closure system is the system of all κ-$\bigwedge$-subsemilattices $\mathcal{S}_\kappa$ of a κ-$\bigwedge$-semilattice S, we have

$$a/b = \begin{cases} \{a\} & \text{if } a \not\leq b \\ \{x \mid a = x \wedge b\} & \text{if } a < b \\ S & \text{if } a = b \end{cases}$$

4.2.7 Definition

With the notation given in the previous paragraph, we say that $\mathcal{C}$ satisfies the **Peano property**, if

$$a(bc) = (ab)c$$

for all $a, b, c \in S$.

We say that it satisfies the **Pasch property**, if

$$(b'/b) \cap (a'/a) \neq \emptyset \Rightarrow (a'b) \cap (b'a) \neq \emptyset$$

for every $a, a', b, b' \in S \setminus C$.

If $\mathcal{C}$ does not contain $\emptyset$ as smallest element, we have to replace it by the smallest element $C \in \mathcal{C}$ on the right hand side of this implication. (a'/a) need not be in $\mathcal{C}$, we therefore do not need to make a similar restriction on the left hand side.

Although the geometric motivations for these two properties are similar to each other, see [48] for an illustration, they differ quite strongly in our context of intervals of subsemilattices.

4.2.8 Lemma

Every interval of κ-$\bigwedge$-subsemilattices has the Peano property.

Proof: This is a consequence of the fact that $a(bc) = \Gamma\{a, b, c\}$, since

$$a(bc) = \Gamma\{a\} \vee \Gamma\{b, c\} = \left\{\bigwedge K \in S \,|\, K \subseteq \{a, b, c\}\right\} = \Gamma\{a, b, c\}. \quad \square$$

4.2.9 Lemma

i) The system of all κ-$\bigwedge$-subsemilattices $\mathcal{S}_\kappa$ of a κ-$\bigwedge$-semilattice S has the Pasch property with the restriction that $\{a', b'\} \neq \{\top_S\}$.

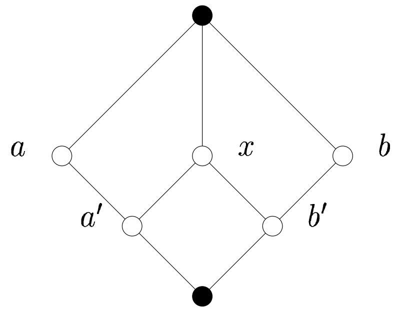
ii) Intervals $\mathcal{C}_\kappa$ in $\mathcal{S}_\kappa$ need not have the Pasch property.

Proof: (i) We start with the equation $(a'/a) \cap (b'/b) \neq \emptyset$. To show that the Pasch property is satisfied as long as $a' \neq \top \neq b'$, we need a case differentiation. One case of $x \in (a'/a) \cap (b'/b)$ is, if we have $a' = a \wedge x$ and $b' = b \wedge x$. This leads to

$$\underbrace{a' \wedge b}_{\in a'b} = a \wedge x \wedge b = \underbrace{a \wedge b'}_{\in ab'}.$$

If $a' = a$ or $b' = b$, it is trivial that the property is satisfied. All other cases can be checked to be true as well.

(ii) We give a counterexample.



We have a non-trivial smallest element $C = \{\top_S, \bot_S\}$ of $\mathcal{C}_\kappa$, with $x \in (a'/a) \cap (b'/b)$ but $(a'b) \cap (ab') = C$. Thus, $\mathcal{C}_\kappa$ does not satisfy the Pasch property. $\square$

This shows once again, that intervals of subsemilattices are more complex than the full system of subsemilattices of a semilattice. Another example for this is the next property.

4.2.10 Definition

A closure system $\mathcal{C}$ on S with closure operator Γ has the **subdivision** or **partition property**, if every subset $F \subseteq S \setminus C$ with $|F| \geq 2$ (and $C = \bot_{\mathcal{C}}$) satisfies

$$\forall\, p \in \Gamma F : \Gamma F = \bigcup \{\Gamma(F - a + p) \mid a \in F\}\,.$$

4.2.11 Lemma

Every $\mathcal{S}_\kappa$ has the subdivision property.

Proof: Suppose there are $x, p \in \Gamma F$ such that for all $a \in F - p$, $x \notin \Gamma(F - a + p)$. This leads to $x = \bigwedge F$, $x \neq p$ and $|F| < \kappa$.

Since $p \in \Gamma F$, there is a $K \subseteq_\kappa F$ such that $p = \bigwedge K$. Replacing K by p in $\bigwedge F$ we get

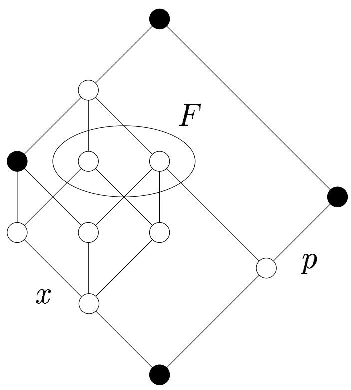
$$x = \bigwedge F = \bigwedge (F - K) \wedge p \in \Gamma(F - a + p)$$

for every $a \in K$. This is a contradiction to our assumption, hence $\mathcal{S}_\kappa$ has the subdivision property. $\square$

4.2.12 Example

An interval $\mathcal{C}_\kappa$ need not have the subdivision property.

Proof: We give an example, in which the property is violated.



In this example, $x \in \Gamma F$, but $x \notin \bigcup\{\Gamma(F - a + p) \mid a \in F\}$. $\square$

We will come back to the subdivision property in the next section, where we will encounter a very similar, slightly stronger condition.

Before we close this small collection of alternative approaches to convexity, we turn to a theory which differs from the ones we discussed so far, the **Bryant-Webster spaces**. These are defined using an axiom system based on a number of geometric properties of the convex sets of vector spaces, without requiring linear algebra.

Instead of a closure system, Bryant-Webster spaces are based on a set S with a set-valued operator $I : S \times S \to \mathcal{P}S$, similar to interval spaces.

Two axioms in this system are the already-mentioned Pasch property, and the ramification property.

4.2.13 Definition

Consider a set S with a set-valued operator $I : S \times S \to \mathcal{P}S$. We again use the notation ab for $I(a, b)$.

We say that the corresponding convex structure has the **ramification property**, if for all $a, b, c \in S$, $b \notin ac$ and $c \notin ab$ imply $ab \cap ac = \{a\}$.

If we want to apply this property to intervals of subsemilattices, we need an interval operator I. This is the closure operator, restricted to two-element sets, i.e.

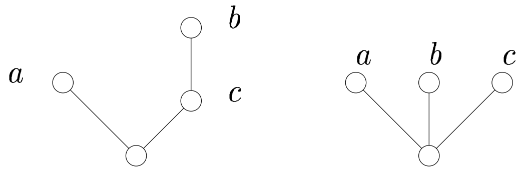
$$I : S \times S \to \mathcal{P}S,\ (a,b) \mapsto \Gamma\{a,b\}.$$

Furthermore, we have to change the implied equation $ab \cap ac = \{a\}$ to $ab \cap ac = C(a)$, since both ab and ac always contain $C(a)$.

As with interval systems, this is very restrictive and looses all information of $\mathcal{C}_\kappa$ which we obtain from infinite sets. In addition to this, there are many semilattices, regardless of size, which do not have the ramification property. We give two examples of $\wedge$-subsemilattices of a $\wedge$-semilattice S which prevent S from having this property.

4.2.14 Examples

If a $\wedge$-semilattice S contains $\wedge$-subsemilattices with lower sets of the following form, $\mathcal{S}_\kappa$ does not satisfy the ramification property.



In both cases, the conditions $b \notin \Gamma\{a,c\}$ and $c \notin \Gamma\{a,b\}$ are satisfied, but $\Gamma\{a,b\} \cap \Gamma\{a,c\} \neq \{a, \top_S\}$, as this is $C(a)$ if we consider the system $\mathcal{S}_\kappa$.

If S is a lattice which is not distributive, it contains the N_5 or the M_3 as a sublattice. Hence it contains a $\wedge$-subsemilattice with a lower set of one the forms given here which prevent $\mathcal{S}_\kappa$ from having the ramification property.

4.2.15 Corollary

If the system $\mathcal{S}$ of all κ-$\bigwedge$-subsemilattices of a κ-$\bigwedge$-semilattice S has the ramification property, S is distributive.

Now that we discussed a number of alternative approaches to convexity, and some of the properties used for these theories, we notice that these approaches differ quite strongly in some regards. Some of these share a common basis, as they use a closure operator with additional properties for their considerations, but these properties can be very different from each other.

Important or at least interesting for us is that most approaches and properties are incompatible with intervals of subsemilattices, i.e. our intervals $\mathcal{C}_\kappa$ do not satisfy the requirements of these theories.

4.3 Convex Invariants

Another topic that is often discussed when working on convexity is the class of convex invariants, such as the numbers of Helly, Radon and Carathéodory. As in the previous section, we introduce the basic constructions and ideas concerning a small number of convex invariants, with a small discussion on their relevance or interaction in the case of intervals of subsemilattices. Notation and some results are again taken from [48] and [23].

4.3.1 Definition

Consider a closure system $\mathcal{C}$ on the set S with closure operator Γ. The bottom element of $\mathcal{C}$ is denoted C. A subset $F \subseteq S \setminus C$ with $|F| \geq 2$ is called

Radon-dependent or **R-dependent**, if and only if

$$\exists\, \emptyset \neq E \subset F : \Gamma E \cap \Gamma(F \setminus E) \neq C,$$

Helly-dependent or **H-dependent**, if and only if

$$\bigcap \{\Gamma(F - a) \mid a \in F\} \neq C,$$

Carathéodory-dependent or **C-dependent**, if and only if

$$\Gamma F = \bigcup \{\Gamma(F - a) \mid a \in F\},$$

Exchange-dependent or **E-dependent**, if and only if

$$\forall\, b \in F : \Gamma(F - b) \subseteq \bigcup \{\Gamma(F - a) \mid a \in F - b\}.$$

Sets that are not R-dependent are called **R-independent**, and the same definition is used for sets which are not H-,C- or E-dependent.

It is interesting to see, that Radon- and Helly-dependence are defined using meets of closed sets, whereas Carathéodory- and Exchange-dependence are defined using joins of closed sets.

Our definition varies slightly from the one in [48], as we allow our closure system to have a non-trivial bottom element C. It is also obvious that the definitions only make sense for sets with more than one element. Sets with two elements can be dependent, and we illustrate what it means for a set $F = \{x, y\}$ to be dependent of these types.

Note that the notion of dependence or independence is not restricted to these four approaches. In [28], C.-A. Faure and A. Fröhlicher call a subset $F \subseteq S$ independent, if for every $x \in F$ one has $x \notin \Gamma(F - x)$, which is even stronger than C-independence. This form of independence is also a translation of independence of a subset of a lattice L as given in [11] to closure systems. In this work we will restrict our considerations to the four types given above.

4.3.2 Example

We use the same notation as in the previous definition.

i) By definition of R- and H-dependence, $F = \{x, y\}$ is R-dependent if and only if it is H-dependent if and only if

$$C(x) \cap C(y) \neq C.$$

ii) $F = \{x, y\}$ is E-dependent if and only if $C(x) \subseteq C(y)$ and $C(y) \subseteq C(x)$, i.e. if and only if $C(x) = C(y)$.

As a consequence we get that every two-element set $\{x, y\}$ is E-independent iff the closure system is a C_0-system. This is certainly the case, if we have C_D-systems.

iii) $F = \{x, y\}$ is C-dependent if and only if

$$C(x) \vee C(y) = C(x) \cup C(y).$$

We see that the four concepts can but need not vary. Some relations between these concepts are given in the next lemma.

4.3.3 Lemma

We use the same notation as in the previous definition.

i) If $F \subseteq S \setminus C$ is R-dependent, it is H-dependent.

ii) If $\mathcal{C}$ satisfies the subdivision property and $F \subseteq S \setminus C$ is H-dependent, then it is E-dependent.

iii) If $\mathcal{C}$ has the property JHC, and $F \subseteq S \setminus C$ is E-dependent, then F is C-dependent.

iv) If $\mathcal{C}$ is the system of convex subsets of a $\mathbb{K}$-vector space X, then the four conditions for dependence are equivalent.

Proof: See [23] or [48]

We can use these results for the study of our intervals of subsemilattices.

4.3.4 Lemma

We consider the C_D-system $\mathcal{S}_\kappa$ of all κ-$\bigwedge$-subsemilattices of a κ-$\bigwedge$-semilattice S, an interval $\mathcal{C}_\kappa$ in $\mathcal{S}_\kappa$, and subsets $E \subseteq S - \top_S$ and $F \subseteq S \setminus C$ with $|E|, |F| \geq 2$.

i) The following implications are true in $\mathcal{S}_\kappa$:
E R-dependent $\Rightarrow$ E H-dependent $\Rightarrow$ E E-dependent $\Rightarrow$ E C-dependent.

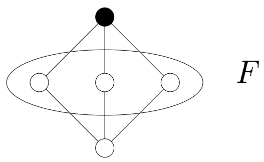
No implication can be extended to an equivalence in general.

ii) The following implications are true in $\mathcal{C}_\kappa$:
F R-dependent $\Rightarrow$ F H-dependent, and
F E-dependent $\Rightarrow$ F C-dependent.

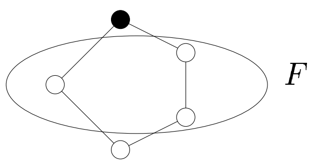
No additional implication is true in general.

Proof: We know that $\mathcal{S}_\kappa$ has the subdivision property, and that every interval $\mathcal{C}_\kappa$ has JHC. Together with the previous lemma, this yields all implications given here. To show that no other implication is true in general, we give a short list of counterexamples. As before, black dots mark elements of the bottom element of $\mathcal{C}_\kappa$, which always contains the top element of S.

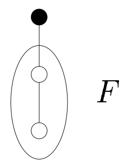
We start with a semilattice S and a subset F which is H-dependent, but not R-dependent.



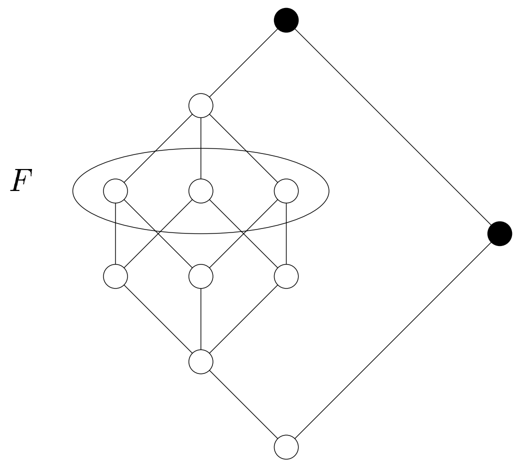
The following semilattice S contains a subset F which is E-dependent, but not H-dependent.



Next is a very simple semilattice S with a subset F which is C-dependent, but not E-dependent.



The first three examples were very basic, since we were using $\mathcal{S}_\kappa$. An example for a H-dependent but E-independet set must use a non-trivial interval $\mathcal{C}_\kappa$. Thus, the next construction is slightly more complicated.



Note that F is both R- and H-dependent, but neither E- nor C-dependent. □

We see that Radon- and Helly-dependence are closely related, and that Carathéodory- and Exchange-dependence are related to each other, two facts that are already visible in the definitions. In the general case of intervals of κ-$\bigwedge$-subsemilattices, these two pairs need not be related to each other.

One important property which all four types of dependence share when applied to intervals of κ-$\bigwedge$-subsemilattices is the following one.

4.3.5 Theorem

Consider an interval $\mathcal{C}_\kappa$ of κ-$\bigwedge$-subsemilattices of a κ-$\bigwedge$-semilattice S.

If $F \subseteq S \setminus C$ is R-dependent, and $F \subset A \subseteq S \setminus C$, then A is also R-dependent.

The same is true for H-,E- and C-dependent sets. In other words, independence (of all four kinds) is hereditary.

Proof: This is trivial for both Radon- and Helly-dependence, but not for Carathéodory- or Exchange-dependence. We show that this is true for Carathéodory-dependence, the proof for Exchange-dependent sets is similar.

Assume that F is C-dependent. A closer look at C-dependency shows that the crucial element is $\bigwedge A$. A is C-independent if and only if

$$\bigwedge A \in \Gamma A \setminus \bigcup \{\Gamma(A-a) \mid a \in A\}.$$

Since F is C-dependent, we have $x = \bigwedge F \in \Gamma(F - f)$ for some $f \in F$. We split A into two parts, and get

$$\bigwedge A = \bigwedge F \wedge \bigwedge (A \setminus F).$$

Since $\bigwedge F \in \Gamma(F-f) \subseteq \Gamma(A-f)$ and also $\bigwedge(A \setminus F) \in \Gamma(A-f)$, the meet of these two needs to be in $\Gamma(A-f)$ as well, i.e. $\bigwedge A \in \Gamma(A-f)$. This shows that A is C-dependent. □

The fact that supersets of H- or R-dependent sets are again H- or R-dependent, respectively, is true for every closure system. This is not the case for C- or E-dependence. See [48] for further information on closure systems without this property.

The fact that supersets of dependent sets are again dependent is important for the definition of the convex invariants we want to discuss in this section, as will be obvious from the next definition. Since this is not true for general closure systems, we restrict our considerations to κ-$\bigwedge$-intervals of κ-$\bigwedge$-semilattices from now on.

4.3.6 Definition

Consider an interval $\mathcal{C}_\kappa$ of κ-$\bigwedge$-subsemilattces of a κ-$\bigwedge$-semilattice S. We define the following cardinal numbers

$r(\mathcal{C}_\kappa) = \min\{m \mid m < |F| \Rightarrow F \text{ R-dependent}\}$,
the **Radon number of** $\mathcal{C}_\kappa$;

$h(\mathcal{C}_\kappa) = \min\{m \mid m < |F| \Rightarrow F \text{ H-dependent}\}$,
the **Helly number of** $\mathcal{C}_\kappa$;

$e(\mathcal{C}_\kappa) = \min\{m \mid m < |F| \Rightarrow F \text{ E-dependent}\}$,
the **exchange number of** $\mathcal{C}_\kappa$;

$c(\mathcal{C}_\kappa) = \min\{m \mid m < |F| \Rightarrow F \text{ C-dependent}\}$,
the **Carathéodory number of** $\mathcal{C}_\kappa$.

These are the definitions for these numbers as given by van de Vel in [48], but modified to suit the theory of intervals of subsemilattices.

Theorem 4.3.5 is very helpful for the determination of dependence numbers. For example if we can show that every two-element set $F = \{x, y\} \subseteq S \setminus C$ with $x \neq y$ is C-dependent, then $c(\mathcal{C}_\kappa) = 1$. More general, if all sets F with $|F| = n$ are dependent, then the corresponding dependence number is $< n$.

Without Theorem 4.3.5, we would have to show that every set with more than two elements is C-dependent, too. Otherwise we could not be certain that $c(\mathcal{C}_\kappa) = 1$, it might be much higher.

There are a number of results for these numbers. The first one for the Helly number of a finite convex geometry was mentioned by Edelman and Jamison-Waldner in [18].

4.3.7 Definition

Consider a C_D-system $\mathcal{C}$ with smallest element C. We call a set $K \in \mathcal{C}$ **free**, if every element $x \in K \setminus C$ is an extreme point of K, i.e. $K - x \in \mathcal{C}$.

4.3.8 Theorem

The Helly number of a finite convex geometry $\mathcal{C}$ is equal to the maximum cardinality of $K \setminus C$ of a free set K.

Proof: see [36]

This result cannot be extended to intervals of subsemilattices or to convex subsets of $\mathbb{E}^n$ in general, the proof requires a finite set, but a few related results exist.

4.3.9 Lemma

Consider an interval $\mathcal{C}_\kappa$ of κ-$\bigwedge$-subsemilattices of a κ-$\bigwedge$-semilattice S.

If $K \in \mathcal{C}_\kappa$ is free, then $|K \setminus C| \leq h(\mathcal{C}_\kappa)$, i.e. the Helly number must be larger or equal to the maximum cardinality of the remainder $K \setminus C$ of a free set K.

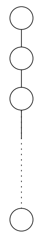
However, such a maximal free set, i.e. $K \in \mathcal{C}_\kappa$ with $|K \setminus C| = h(\mathcal{C}_\kappa)$, need not exist.

Proof: If K is free, then $K \setminus C$ is H-independent:

$$\bigcap \{\Gamma(K - x) \mid x \in K \setminus C\} = \bigcap \{K - x \mid x \in K \setminus C\} = C.$$

Hence, the Helly number must be larger or equal to the cardinality of the remainder $K \setminus C$ of a free set K.

For the second part, consider the infinite chain $S = (\omega^+)^{op}$ and the system $\mathcal{S}_\infty$ of all $\bigwedge$-subsemilattices of S.



The smallest cardinal ρ such that every subset of $S \setminus C$ with a cardinality larger than ρ is H-dependent is ω. Every finite subset B of $S \setminus C$ generates a free set $\Gamma B = B + \top_S$, but infinite sets are never free. Hence there are no maximal free sets and $h(\mathcal{S}_\kappa)$ is strictly larger than the cardinality of every free set. □

In addition to these very specific results for the Helly number, we can give some results which are true for both the Radon and the Helly number.

4.3.10 Lemma

We use the same notation as in the previous lemma. If $F \subseteq S \setminus C$ is well-ordered, then F is both Radon- and Helly-independent.

Consequently, if $\mathcal{C}_\kappa$ has a finite Helly or Radon number, $S \setminus C$ does not contain an infinite well-ordered subset.

Proof: A well-ordered set is always Helly-independent, hence also Radon-independent. A finite Radon or Helly number implies that every infinite subset $F \subseteq S \setminus C$ is R- or H-dependent, i.e. F cannot be well-ordered. □

4.3.11 Lemma

For an arbitrary interval $\mathcal{C}_\kappa$ of κ-$\bigwedge$-subsemilattices of a κ-$\bigwedge$-semilattice S, Radon- or Helly-dependent subsets are not restricted in their size, i.e. for every cardinal ρ there exists a κ-$\bigwedge$-semilattice S which has a Helly and Radon number larger than ρ.

Proof: Consider the system $\mathcal{S}_\kappa$ for $S = \sigma$, where $\rho < \sigma$ are (regular) cardinals. σ itself is Radon- and Helly-independent, hence the Radon and Helly numbers are larger than ρ. □

It is not possible to extend the previous result for Radon and Helly numbers to Exchange or Carathéodory numbers. These are always bounded by κ.

4.3.12 Lemma

Consider an interval $\mathcal{C}_\kappa$ of κ-$\bigwedge$-subsemilattices of a κ-$\bigwedge$-semilattice S. If $F \subseteq S \setminus C$ with $|F| \geq \kappa$, then F is Exchange- and Carathéodory-dependent. Consequently, $e(\mathcal{C}_\kappa), c(\mathcal{C}_\kappa) \leq \kappa$.

Proof: We only need to show that $F \subseteq S \setminus C$ with $|F| \geq \kappa$ is always E-dependent. We recall that

$$\Gamma(F-p) = \left\{ \bigwedge K \mid K \subseteq_\kappa F - p \right\}$$

for every $p \in F$. Thus if $x \in \Gamma(F-p)$, then $x \in \Gamma K$ for some $K \subseteq_\kappa F - p$. But κ-subsets contain strictly less than κ elements, i.e. K is contained in many sets of the form $\Gamma(F-p-a) \subseteq \Gamma(F-a)$ with $a \in F-p$. Thus, F is E-dependent. □

4.3.13 Example

We just saw that $c(\mathcal{C}_\kappa) \leq \kappa$. To show that this inequality cannot be strengthened to a strict inequality we give an example.

Consider a set M with a cardinality $|M| > \kappa$, where κ is an **inaccessible cardinal**, i.e. κ is regular and for every cardinal $\rho < \kappa$ there exists a cardinal ν such that $\rho < \nu < \kappa$.

We define the κ-$\bigwedge$-semilattice S as the power set $\mathcal{P}M$ of M, ordered by inclusion. If we consider the usual closure system $\mathcal{S}_\kappa$ of all κ-$\bigwedge$-subsemilattices of S, then every set $F \subseteq_\kappa S$ of co-atoms of S is C-independent, since $\bigwedge F < \bigwedge(F-x)$ for all $x \in F$. The inaccessability of κ then gives us $c(\mathcal{C}_\kappa) = \kappa$.

The previous lemma implies, that well-ordered sets are not always Exchange-independent. In fact, the converse is true.

4.3.14 Lemma

Every well-ordered set $F \subseteq S \setminus C$ with $|F| > 2$ is Exchange-dependent.

Proof: If F is well-ordered and $x \in \Gamma(F-p)$, then there exist $K \subseteq_\kappa F - p$ and $c \in C$ such that $x = \bigwedge K \wedge c$. Since F is well-ordered, the meet of K is in K, hence $x = f \wedge c$ for a $f \in K \subseteq F - p$. This gives us $x \in \bigcup \{\Gamma(F-a) \mid a \in F - p\}$, as long as $F - p$ contains at least two elements. □

Since E-dependent sets are also C-dependent, large enough well-ordered sets are also always C-dependent. In fact, we can give a very specific description of C-independent sets for $\mathcal{S}_\kappa$ and $\mathcal{C}_\kappa$. We start by translating the definition for C-dependence to a definition for C-independence for a general closure system.

4.3.15 Lemma

Consider a closure system $\mathcal{C}$ on a set S with bottom element C. A subset $F \subseteq S \backslash C$ with $|F| \geq 2$ is C-independent, if and only if

$$\exists\, x \in \Gamma F\, \forall\, a \in F : x \notin \Gamma(F - a).$$

Proof: Direct translation of the definition.

For intervals of subsemilattices, we can be more specific. We start with $\mathcal{S}_\kappa$ as it is slightly less technical.

4.3.16 Lemma

Consider the system $\mathcal{S}_\kappa$ of all κ-$\bigwedge$-subsemilattices of a κ-$\bigwedge$-semilattice S, and a subset $F \subseteq S - \top_S$ with $|F| \geq 2$. The following are equivalent:

i) F *is C-independent.*

ii) $\forall\, a \in F : \bigwedge F < \bigwedge(F - a)$.

iii) $\forall\, K, L \subseteq F : K \neq L \Rightarrow \bigwedge K \neq \bigwedge L$.

Proof: Condition (ii) is a translation of the definition to the context of subsemilattices, and the implication (iii) $\Rightarrow$ (ii) is trivial. We only need to show the implication (ii) $\Rightarrow$ (iii).

Assume that $K \nsubseteq L$ but $y = \bigwedge K = \bigwedge L$. This gives us

$$\bigwedge F = \bigwedge K \wedge \bigwedge(F \setminus K) = \bigwedge L \wedge \bigwedge(F \setminus K).$$

For an arbitrary $k \in K \setminus L$ this leads to $\bigwedge F \in \Gamma(F - k)$, since $L \cup (F \setminus K) \subseteq F - k$, a contradiction to (ii). $\square$

Compare this with the definition of the breadth of a semilattice (see e.g. [10]).

4.3.17 Definition

The ($\wedge$-)**breadth** of a $\wedge$-semilattice S is defined by

$$b_\wedge(S) = \min\left\{m \mid \forall\, F \subset S\, \exists E \subseteq_{m^+} F : \bigwedge F = \bigwedge E\right\}.$$

4.3.18 Corollary

Consider the system $\mathcal{S}_\kappa$ of all κ-$\bigwedge$-subsemilattices of a κ-$\bigwedge$-semilattice S. Then we have $c(\mathcal{S}_\kappa) = b_\wedge(\mathcal{S}_\kappa)$.

The characterisation of C-independent sets for intervals of subsemilattices differs only slightly from the one just given.

4.3.19 Lemma

Consider an interval $\mathcal{C}_\kappa$ of κ-$\bigwedge$-subsemilattices of a κ-$\bigwedge$-semilattice S, and a subset $F \subseteq S \setminus C$ with $|F| \geq 2$. The following are equivalent:

i) F *is C-independent.*

ii) $\bigwedge F \notin \bigcup \{\Gamma(F-a) \mid a \in F\}$.

iii) $\forall\, c \in C\, \forall\, a \in F : \bigwedge F < c \Rightarrow \bigwedge F < \bigwedge(F-a) \wedge c$.

iv) $\forall\, c \in C\, \forall\, K, L \subseteq F : (\bigwedge F < c$ *and* $K \neq L) \Rightarrow \bigwedge K \wedge c \neq \bigwedge L \wedge c$.

Proof: The proof is only a small variation of the previous one. The only interesting element is $\bigwedge F$, since all non-trivial subsets of F and their meets are contained in some $\Gamma(F-a)$, and if $\bigwedge F$ is contained in one such set, then so is $\bigwedge F \wedge c$ for all $c \in C$. $\square$

From these two results we can deduce a strong restrictions on F to be C-independent, and an upper bound for $c(\mathcal{C}_\kappa)$.

4.3.20 Corollary

C-independent sets of κ-$\bigwedge$-semilattices are antichains.

4.3.21 Corollary

The maximal size of anti-chains in $S \setminus C$ is an upper bound for the Carathéodory number $c(\mathcal{C}_\kappa)$.

On the other end of the possible spectrum of Carathéodory numbers for $\mathcal{C}_\kappa$ we have a different result.

4.3.22 Theorem

A interval $\mathcal{C}_\kappa$ has Carathéodory number $c(\mathcal{C}_\kappa) = 1$ if and only if $\mathcal{C}_\kappa$ is distributive.

Proof: At the beginning of this section we saw that the condition for a two-element set $\{x, y\}$ to be C-dependent is

$$C(x) \vee C(y) = C(x) \cup C(y).$$

If this is true for every set $\{x, y\} \subseteq S \setminus C$, this is precisely condition (ix) of 3.7.2, hence $\mathcal{C}_\kappa$ is distributive.

Conversely, if $\mathcal{C}_\kappa$ is distributive, then

$$C(x) \vee C(y) = C(x) \cup C(y),$$

i.e. every two-element set is C-dependent. From this we can deduce that every subset $F \subseteq S \setminus C$ with at least two elements is C-dependent, hence $\mathcal{C}_\kappa$ has a Carathéodory number of one. □

Finally we mention a consequence of distributivity in $\mathcal{C}_\kappa$ for one of the other properties sometimes considered in convexity theory.

4.3.23 Lemma

If $\mathcal{C}_\kappa$ is distributive, it has the subdivision property (see 4.2.10).

Proof: For the subdivision property we have to show that

$$\Gamma F \subseteq \bigcup \{\Gamma(F - a + p) \mid a \in F\}$$

for every $F \subseteq S \setminus C$ with $|F| \geq 2$ and every $p \in \Gamma F$. But if $\mathcal{C}_\kappa$ is distributive, then any such F is C-dependent, i.e.

$$\Gamma F \subseteq \bigcup \{\Gamma(F - a) \mid a \in F\},$$

which is obviously stronger than the condition for the subdivision property. □

The last results lead to some questions. If $c(\mathcal{C}_\kappa) = 2$, does this imply upper semimodularity, which is slightly weaker than distributivity? Or what about the subdivision property, does it imply some weak form of distributivity? All these questions have to be answered in the negative.

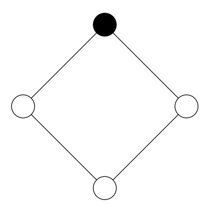
4.3.24 Lemma

i) The subdivision property does not imply upper semimodularity, and upper semimodularity does not imply the subdivision property.

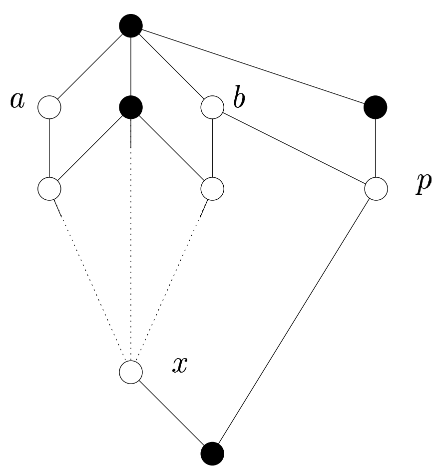
ii) If $c(\mathcal{C}_\kappa) = 2$, *then* $\mathcal{C}_\kappa$ *need not be upper semimodular nor satisfy the subdivision property.*

Proof: Each of the mentioned implications is violated in one of the following two examples of intervals of $\wedge$-subsemilattices of $\wedge$-semilattices S.

The first one is one of the most basic $\wedge$-semilattices, and we use the system $\mathcal{S}_\omega$. This system has the subdivision property and has a Carathéodory number of 2, but is not upper semimodular.



The second example is more complicated. We use a $\wedge$-semilattice S with a non-trivial $\wedge$-subsemilattice C.



C-independent sets have to be antichains in $S \setminus C$, hence $c(\mathcal{C}_\omega) \leq 3$. It is easy to see that antichains with three elements are C-dependent, but $F = \{a, b\}$ is independent since $x = a \wedge b$ is not in $C(a)$ or $C(b)$. Hence the interval of $\wedge$-subsemilattices for this example has a Carathéodory number of 2.

Furthermore, $\mathcal{C}_\omega$ is upper semimodular as C is weakly meet dense in S (see 3.7.3). It does not have the subdivision property, since $x \notin \Gamma(F - a + p) \cup \Gamma(F - b + p)$ but $x \in \Gamma F$ if we again use $F = \{a, b\}$. $\square$

Appendix A

C++-Code

A.1 The Main Program

```
// IntervalDeterminator.cxx
#include <iostream>
#include <string>
#include <set>
#include <vector>
#include <fstream>
#include <cstdlib>

ofstream result("ausgabe.txt", ios::out | ios::app);
#include "coverrel.h"
#include "subsemi.h"
using namespace std;

subsemi L;
int n, i, j, k, h, m;
int x,y,z;
vector<subsemi> Interval;
vector<int> V, Irred, B, uppercoverno;

ifstream latticestruc("data.txt");

/* Format of data.txt:
   The file data.txt contains the requiered information on S and C,
   here called L and B.

   All elements are integers, the top element is always 0, the
   elements are ordered from top to bottom, i.e. if k<l are
```

```
   positive numbers, the elements k and l are not ordered l<k. The
   elements are numbered from 0 to n-1.

   The first line contains the number of elements in S. The second
   line contains the number of upper covers of the elements 1 to
   n-1. The next n-1 lines contain the (names of the) upper covers
   of the elements 1 to n-1.

   This suffices to represent S. The second block is for the bottom
   element of the interval, C. The first line of the second block
   gives the number of elements of C, the second line lists these
   elements (by their names/numbers).
*/

void displaysemlatt( subsemi U);
void displayelements( subsemi U);
void level(subsemi U, int l);
void inter(subsemi U, int l);
subsemi filterfill(subsemi U);

bool reducecheck( subsemi U);

int main()
{
  //Initialisation of the semilattice L
  latticestruc >> n;                 // Read the number of elements.

  coverrel empty;                    // Empty coverrel for the
                                     // initialization.
  for (int j=0; j<n; j++)            // L has n elements.
    L.comps.push_back(empty);

  for (int j=0; j<n; j++)
    L.comps[j].element = j;          // The elements are added

  uppercoverno.push_back(0);         // The top element has no
                                     // upper cover.
  for (int j=1; j<n; j++)
    {
      x=0;                          // We read the number of
      latticestruc >> x;            // upper covers for the elements
      uppercoverno.push_back(x);    // 1 to n-1.
    }

  for (int j=1; j<n; j++)
```

```
    {
      for (int k=0; k< uppercoverno[j]; k++)
{
  y=0;
  latticestruc >> y;
  L.comps[j].uppercov.insert(y);
}
    }

  L.filterfilling();
                       // The filter is filled using the upper covers.


  for (int j=1; j< L.comps.size(); j++)
    L.comps[j].posupp = L.comps[j].uppercov.begin();
                         //The iterators are initialized.

// end of initialization of L

  for (int j=0; j<n; j++) V.push_back(0);      // V is initialized.

  for (int j=0; j<n; j++) L.Irred.push_back(L.comps[j].uppercov.size());
                                               // Irred is initialized.


  // The initialization of the least element of the interval

  for (int j=0; j<n; j++) B.push_back(0);       // B is initialized.

  latticestruc >> m;                   // Read the number of elements
                                       // in B.
  for (int j=0; j<m; j++)
    {
      z = 0;
      latticestruc >> z;
      B[z] = 1;
    }


  result << "A semilattice L, a subsemilattice B, and the interval [B,L]: "
  << endl;
  result << endl;

  Interval.push_back(L);
                      // The first element of Interval is the initial
```

```
                      //  subsemilattice L.

  result << "The elements of L, their upper covers and cuts: " << endl;
  Interval[0].displaysemlatt();
  result << endl;

  result << "The elements of B: ";
  for (int j=0; j<n; j++)
    if (B[j] == 1)
      result << j << " ";
  result << endl;
  result << endl;

  inter(Interval[0],1);
               // We calculate all subsemilattices of L that contain B.

  result << "Size of Interval: " << Interval.size() << endl;
               // The total number of subsemilattices incl. L and B.

  for (int k=0; k<Interval.size(); k++)
    {        // This lists all subsemilattices by giving their elements.
      result << "Elements of U[" << k << "]: ";
      Interval[k].displayelements();
      result << endl;
    }
  result << endl;
  result << "----------------------------------------------" << endl;
  result << endl;

  return 0;
}


void level (subsemi U, int l)
                     // This function finds ALL subsemilattices of U, if
{                    // U = L and l = 1;
  vector<int> index;
  subsemi U2;

  cout << "The vector V (just before next level): ";
  for (int j=0; j<n ; j++) cout << V[j];
  cout << endl;

  for (int j=0; j<U.Irred.size(); j++)
      if (U.Irred[j] == 1 && V[j] == 0 )
```

```
index.push_back(j);

  for (int j=0; j< index.size(); j++)
    {
      U2=U;
      U2.removecomp(index[j]);
      Interval.push_back(U2);
      V[index[j]]=1;

      level(U2,l+1);
    }
  for (int j=0; j<V.size(); j++)
    if (V[j] == 1) V[j] = 0;

  cout << "The vector V (just after a reduction): ";
  for (int j=0; j<n ; j++) cout << V[j];
  cout << endl;
}

void inter (subsemi U, int l)
                    // This function finds all subsemilattices of L, that
{                   // contain B.
  vector<int> index;
  subsemi U2;

  for (int j=0; j<U.Irred.size(); j++)
      if (U.Irred[j] == 1 && V[j] == 0 && B[j]==0 )
index.push_back(j);

  for (int j=0; j< index.size(); j++)
    {
      U2=U;
      U2.removecomp(index[j]);
      Interval.push_back(U2);
      V[index[j]]=1;

      inter(U2,l+1);
    }
  for (int j=0; j<V.size(); j++)
    if (V[j] == 1) V[j] = 0;
}
```

A.2 The Class subsemi

```
// subsemi.h
// This class is for creating and handling subsemilattices

#include <iostream>
#include <string>
#include <set>
#include <cstdlib>

using namespace std;

// ofstream result("ausgabe.txt", ios::out | ios::app);

typedef vector<coverrel> neighstruc;      //entries are coverrels
typedef neighstruc::iterator neighiter;

class subsemi
{

 public:
  neighstruc comps;      //The elements of subsemi.h are vectors
                         // of coverrels.
  neighiter compiter;

  vector<int> Irred;

  void removecomp(int x);        // removes the component x
  void filterfilling();
  void displaysemlatt();
  void displayelements();
};

void subsemi::removecomp(int x)  //removes component x, but also
{                                //updates upper cover relation
  int l=comps.size();
  int q = 0, p;
  int uppcov = 0;
  int neighcheck=0;
  bool covcheck, filtercheck;

  for (int j=0; j<comps.size(); j++)
                                // determine upper cover of x
    {  if  (comps[j].element == x)
      {  q = j;
```

```
                               // q is the index of x
 uppcov = *comps[j].uppercov.begin();
                               // gives upper cover of x
 if (comps[j].uppercov.size() != 1)
                               // Unchecked! Must be unique!
   { cout << "Element not irreduzible!" << endl;
             return;
           }
      }
    }

   for (int j=0; j<comps.size(); j++)
                              //goes through the components of U
      { comps[j].posupp = comps[j].uppercov.begin();
                              //pointer to first element of
                              //the current uppercovers.
        covcheck = false;

        for (int k=0; k<comps[j].uppercov.size();k++)
                              //this loop checks upper covers for x

  { if (*comps[j].posupp == x)
    covcheck = true;
      comps[j].posupp++;
  }

if (covcheck == true )
                              //if an element has x as upper cover:
  {
    comps[j].uppercov.erase(x);
                             //upper cover x is removed
    comps[j].posupp=comps[j].uppercov.begin();
    filtercheck=false;
    for (int k=0; k < comps[j].uppercov.size(); k++)
                            //this checks, if the upper cover
      {              //of x is a new upper cover of
                            //the current element, or merely
                            //larger
for (int h=0; h< comps.size(); h++)
  {
    if (comps[h].element == *comps[j].posupp)
      {
comps[h].posfilter = comps[h].filter.begin();
for (int m=0; m< comps[h].filter.size(); m++)
  {
```

```
    if (*comps[h].posfilter == uppcov)
      filtercheck = true;
    comps[h].posfilter++;
  }
      }
  }
comps[j].posupp++;
      }
    if (filtercheck == false)
                                  //uppcov is added as an upper cover
                                  //if no upper cover of the element
                                  //is smaller than uppcov
      comps[j].uppercov.insert(uppcov);

  }
      }

   compiter = &comps[q];       //compiter points to the coverrel with x
   comps.erase(compiter);      //erases the coverrel for the element x


  for (int j=0; j<comps.size(); j++)
                               //goes through the components of U
                               //and returns all pointers to startvalue.
     {
       comps[j].posupp = comps[j].uppercov.begin();
       comps[j].posfilter = comps[j].filter.begin();
     }

  for (int j=0; j<Irred.size(); j++)
                             //the vector Irred is updated in two steps
                             //first a vector of zeros.
    Irred[j]=0;

  for (int j=0; j<comps.size(); j++)
                             //then we put ones' at the positions of
    {                        //irreduzible elements.
      p = comps[j].element;
      if (comps[j].uppercov.size() == 1)
Irred[p] =1;
    }
}

void subsemi::filterfilling()
{
```

```
  int ic;

  for (int j=0; j<comps.size() ; j++)
    {
      comps[j].posupp = comps[j].uppercov.begin();

      for (int k=0; k<comps[j].uppercov.size(); k++)
{
  comps[j].filter.insert(*comps[j].posupp);
  ic = 0;
  for (int h=0; h < comps.size(); h++)
    {
      if (comps[h].element == *comps[j].posupp)
ic = h;
    }
  comps[ic].posfilter = comps[ic].filter.begin();
  for (int m=0; m< comps[ic].filter.size(); m++)
    {
      comps[j].filter.insert(*comps[ic].posfilter);
      comps[ic].posfilter++;
    }
  comps[j].posupp++;
}
    }
}

void subsemi::displaysemlatt()
{
  for (int j=0; j<comps.size() ; j++)
                                     //Prints elements and all  covers
    {
      comps[j].posupp = comps[j].uppercov.begin();
      comps[j].posfilter = comps[j].filter.begin();
      result << "Element: " << comps[j].element << endl;
      result << "Upper covers: ";
      for (int i=0; i<comps[j].uppercov.size();i++)
{
  result << *comps[j].posupp << " ";
  comps[j].posupp++;
}
      result << endl;
      result << "Filter: ";
      for (int i=0; i<comps[j].filter.size(); i++)
{
  result << *comps[j].posfilter << " ";
```

```
  comps[j].posfilter++;
}
      result << endl;
    }
}

void subsemi::displayelements()
{
  for (int j=0; j<comps.size(); j++)      //Prints only elements of U
    result << comps[j].element << " ";
}
```

A.3 The Class coverrel

```
// coverrel.h
// the cover-relation for the subsemilattices

#include <iostream>
#include <string>
#include <set>
#include <cstdlib>
using namespace std;

typedef set<int> IntSet;          //entries are integers
typedef IntSet::iterator SetIter;

class coverrel
{ public:
  int element;       // name of the element

  IntSet uppercov;   // upper covers
  IntSet filter;     // the upper cut of element

  SetIter posupp;    // position in upper cover
  SetIter posfilter;   //position in filter
};
```

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Lebenslauf

Lars Ritter
Simrockstr. 24a
30171 Hannover

geboren am 30.03.1975 in Hannover

Schulbildung

1981-1985	Friedrich-Ebert-Schule in Hannover, Badenstedt
1985-1987	Orientierungsstufe im Schulzentrum Badenstedt, Hannover
1987-1994	Humboldtschule Hannover, Abitur

Zivildienst

07/1994 - 09/1995	Zivildienst im Seniorenzentrum Ihme-Ufer, Hannover

Studium

10/1995	Studienbeginn an der Universität Hannover
1997	Vordiplom in Mathematik
09/1998 - 09/1999	Studium an der Brunel University, London, England
1999	Master of Science an der Brunel University Masterarbeit mit dem Titel: *A High Wavenumber Boundary Element Method for an Acoustic Scattering Problem*
2000	Diplom in Mathematik an der Universität Hannover Diplomarbeit mit dem Titel: *A Comparison of Galerkin and Collocation Methods on a Wave Scattering Problem*
seit 11/2000	Doktorand am Institut für Mathematik der Universität Hannover betreut von Prof. Dr. M. Erné

wissenschaftliche Tätigkeiten an der Universität Hannover

11/2000 - 10/2004	wissenschaftlicher Mitarbeiter am Institut für Mathematik
seit 11/2004	hilfswissenschaftlicher Mitarbeiter am Institut für Mathematik