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Calogero models and nonlocal conformal transformations

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Abstract

We propose a universal method of relating the Calogero model to a set of decoupled particles on the real line, which can be uniformly applied to both the conformal and nonconformal versions as well as to supersymmetric extensions. For conformal models the simplification is achieved at the price of a nonlocal realization of the full conformal symmetry in the Hilbert space of the resulting free theory. As an application, we construct two different $N = 2$ superconformal extensions.

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1. Introduction

The range of physical and mathematical applications of the Calogero model is impressive. Being originally formulated as an exactly solvable multi-particle quantum mechanics in one dimension [1], it played an important role in the study of matrix models [2,3], fractional statistics [4], classical and quantum integrable systems [5], the quantum Hall effect [6], superstring theory on the AdS_2 background [7], the WDVV equation [8] and BPS operators in $N = 4$ SYM theory [9] (for a recent review see [10]).

If one is concerned with only the pairwise interaction $g^2 \sum_{i<j} (x^i - x^j)^{-2}$ and disregards the harmonic potential $\omega^2 \sum_{i<j} (x^i - x^j)^2$, the Calogero model exhibits conformal symmetry [11]. This property and the fact that the isometry group of AdS_2 space is $\mathrm{SO}(1, 2)$ led the authors of [12] to conjecture that an $N = 4$ superconformal extension of the Calogero model might provide a microscopic description of the extreme Reissner–Nordström black hole in the near horizon limit, which corresponds to $AdS_2 \times S^2$ geometry. Unfortunately, a consistent $N = 4$ superconformal generalization of the Calogero model has not yet been constructed (for previous attempts see [8,13–15]). The latter problem partially motivated the present investigation.

It has been known since the original work of Calogero [1] that in the presence of harmonic forces the energy eigenvalues of the problem differ from those of decoupled oscillators only by a constant. An explicit but nonunitary similarity transformation connecting their Hamiltonians has been constructed in [16] (see also [17] for a supersymmetric extension).

When the harmonic potential is switched off one expects a similar relation between identical particles interacting via the inverse-square potential and free particles in one dimension to hold. A unitary transformation that maps the Hamiltonian of the Calogero model to that of free particles was constructed in [18]. However, the full conformal symmetry, which characterizes the case at hand, was not taken into account. Note also that the transformation considered in [18] cannot be obtained from that examined in [16] by taking the limit $\omega \to 0$. This indicates that the two approaches are essentially different.

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The purpose of this Letter is to propose a universal method of relating the Calogero model to decoupled particles, which can be uniformly applied to both the conformal and nonconformal versions as well as to supersymmetric extensions. Our approach is different from [18] in that it makes use of *all* conformal generators when constructing the transformation. In other words, we study the behaviour of the Calogero model under specific (unitary) transformations generated by the conformal algebra $\mathrm{so}(1,2)$. As shown below, although the Hamiltonian H of the Calogero model can indeed be mapped to the free Hamiltonian H_0, the generator K of special conformal transformations gets modified and keeps track of the original potential $H_{\mathrm{int}} = H - H_0$ via a nonlocal contribution,

$$K = \frac{1}{2}x^i x^i \quad \longrightarrow \quad \tilde{K} = K + \alpha^2 \mathrm{e}^{\mathrm{i}B} H_{\mathrm{int}} \mathrm{e}^{-\mathrm{i}B} \quad \text{with } H_{\mathrm{int}} = \sum_{i<j} \frac{g^2}{(x^i - x^j)^2}. \tag{1}$$

Here α is a constant, and the explicit form of the operator B is given below. A similar relation holds for an $N=2$ superconformal extension of the Calogero model, for which also the superconformal generators are modified appropriately. Thus, after applying a unitary transformation one arrives at free particles in one dimension with the (super)conformal group being realized in a nonstandard (nonlocal) way. Although quantum states look particularly simple in this framework, the action of the full conformal group in the Hilbert space proves to be rather involved.

The organization of the Letter is as follows. In Section 2 we use general properties of the $\mathrm{so}(1,2)$ algebra and construct a novel unitary transformation which maps the conformal Calogero model to a set of free particles on the real line. In Section 3 the method is applied to the nonconformal Calogero model which features an external harmonic potential for each particle. A map to a set of decoupled harmonic oscillators is constructed and shown to be much simpler than the one proposed in [16]. We then proceed to explore supersymmetric generalizations in Section 4. The $N=2$ superconformal extension of the Calogero model built in [19] is related to a set of free $N=2$ superparticles, with the $\mathrm{SU}(1,1|1)$ symmetry group being realized in a nonstandard fashion. We argue that the $N=2$ superconformal extension is not unique. Furthermore, our transformation may pave the way to constructing $N>2$ superconformal extensions of the Calogero model from a set of free superparticles. We conclude by discussing possible further developments in Section 5.

2. From the Calogero model to free particles

Our starting point is the $\mathrm{so}(1,2)$ algebra realized in the quantized n-particle Calogero model via the Weyl-ordered generators

$$H = \frac{1}{2}p_i p_i + \sum_{i<j} \frac{g^2}{(x^i - x^j)^2}, \qquad D = -\frac{1}{4}\big(x^i p_i + p_i x^i\big), \qquad K = \frac{1}{2}x^i x^i, \tag{2}$$

which satisfy

$$[H, D] = \mathrm{i}H, \qquad [H, K] = 2\mathrm{i}D, \qquad [D, K] = \mathrm{i}K. \tag{3}$$

Here, g is a dimensionless coupling constant ($[x] = [t^{\frac{1}{2}}]$), and the index i labels n identical particles (of unit mass) on the real line mutually interacting via the inverse-square potential. Putting $g=0$ yields a free-particle representation of $\mathrm{so}(1,2)$, whose generators we denote by H_0, D and K.

Each generic Lie-algebra element

$$A = \alpha H + \beta K + \gamma D, \tag{4}$$

where the real constants α and β^{-1} have the dimension of length and γ is dimensionless, determines a unitary transformation

$$(H, D, K) \quad \longmapsto \quad (H', D', K') = \big(\mathrm{e}^{\mathrm{i}A} H \mathrm{e}^{-\mathrm{i}A}, \mathrm{e}^{\mathrm{i}A} D \mathrm{e}^{-\mathrm{i}A}, \mathrm{e}^{\mathrm{i}A} K \mathrm{e}^{-\mathrm{i}A}\big) \tag{5}$$

which is an automorphism of the algebra. It is instructive to use the Baker–Campbell–Hausdorff formula

$$T' \equiv \mathrm{e}^{\mathrm{i}A} T \mathrm{e}^{-\mathrm{i}A} = \sum_{n=0}^{\infty} \frac{\mathrm{i}^n}{n!} T'_n, \qquad \text{where } T'_0 = T \text{ and } T'_n = \underbrace{\big[A, \big[A, \ldots [A, T] \ldots\big]\big]}_{n \text{ times}}, \tag{6}$$

and calculate the first three terms of the transformed Hamiltonian,

$$H'_0 = H, \qquad \mathrm{i}H'_1 = 2\beta D + \gamma H, \qquad \frac{\mathrm{i}^2}{2!}H'_2 = \left(\frac{\gamma^2}{2} - \alpha\beta\right)H + \beta^2 K + \beta\gamma D. \tag{7}$$

Apparently, the particular choice

$$\gamma = \pm 2\sqrt{\alpha\beta} \quad \text{for } \alpha\beta > 0 \tag{8}$$

produces $\frac{\mathrm{i}^2}{2!}H_2' = \beta A$ and $H_{n>2}' = 0$, terminating the series in (6) at the third step. In what follows, we always adopt this choice.

The condition (8) also terminates the series for the transformed dilatation and special conformal generators, so together we have

$$H' = (1+\gamma+\alpha\beta)H + \beta(2+\gamma)D + \beta^2 K = \kappa_\pm^2 H + 2\beta\kappa_\pm D + \beta^2 K,$$
$$D' = -\alpha\left(1+\frac{\gamma}{2}\right)H + \left(1-\frac{\gamma^2}{2}\right)D + \beta\left(1-\frac{\gamma}{2}\right)K = -\alpha\kappa_\pm H + (1-2\alpha\beta)D + \beta\kappa_\mp K,$$
$$K' = \alpha^2 H + \alpha(\gamma-2)D + \left(1-\frac{\gamma}{2}\right)^2 K = \alpha^2 H - 2\alpha\kappa_\mp D + \kappa_\mp^2 K, \tag{9}$$

where we abbreviated

$$\kappa_\pm := 1 \pm \sqrt{\alpha\beta} \quad \text{for } \alpha\beta > 0. \tag{10}$$

An important simplification occurs for

$$\alpha\beta = 1 \quad \longrightarrow \quad \kappa_+ = 2, \qquad \kappa_- = 0 \tag{11}$$

and the lower sign choice, $\gamma = -2$, namely

$$H' = \beta^2 K, \qquad D' = -D + 2\beta K, \qquad K' = 4K - 4\alpha D + \alpha^2 H. \tag{12}$$

Note that H is mapped to the free-field generator $K \equiv K_0$. For the upper sign choice one gets $K' = \alpha^2 H$ instead.

In our consideration it is only the structure of the conformal algebra which matters. So, by changing the operator A in (4) for

$$B = \lambda H_0 + \sigma K + \delta D, \tag{13}$$

analogous relations hold for a system of free particles with the generators H_0, D and K. This observation suggests (in this respect see also [18]) that one can compose the transformations generated by A and by B to map

$$H \mapsto K \equiv K_0 \mapsto H_0 \quad \text{via} \quad \alpha\beta = 1, \quad \gamma = -2, \quad \lambda\sigma = 1, \quad \delta = +2. \tag{14}$$

The second map,

$$(H_0, D, K) \quad \longmapsto \quad (H_0'', D'', K'') = \left(\mathrm{e}^{\mathrm{i}B} H_0 \mathrm{e}^{-\mathrm{i}B}, \mathrm{e}^{\mathrm{i}B} D \mathrm{e}^{-\mathrm{i}B}, \mathrm{e}^{\mathrm{i}B} K \mathrm{e}^{-\mathrm{i}B}\right), \tag{15}$$

reads

$$K'' = \lambda^2 H_0, \qquad D'' = -D - 2\lambda H_0, \qquad H_0'' = 4H_0 + 4\sigma D + \sigma^2 K. \tag{16}$$

A successive application of the two transformations then produces

$$H \mapsto \tilde{H} = H_0, \qquad D \mapsto \tilde{D} = D, \qquad K \mapsto \tilde{K} = K + \alpha^2 H_{\text{int}}'', \tag{17}$$

provided we impose the further relations

$$\beta\lambda = -1 \quad \Longrightarrow \quad \alpha\sigma = -1 \quad \text{and} \quad \alpha + \lambda = 0. \tag{18}$$

Thus, with the help of the unitary operator $\mathrm{e}^{\mathrm{i}B}\mathrm{e}^{\mathrm{i}A}$ one can transform the Hamiltonian of the Calogero model into that describing a system of free particles.

A few comments are in order. Firstly, a similar transformation of H to H_0 has been discussed in [18]. However, the authors of [18] employed (4) with $\gamma = 0$, whence their Baker–Campbell–Hausdorff series did not terminate. As was demonstrated above, our generic choices for A and B allow for a drastic simplification. Secondly, not the entire $\mathrm{so}(1,2)$ algebra was studied in [18]. According to our analysis, the operator of special conformal transformations gets modified. In fact, it effectively "hides" the interaction potential, which disappears for the Hamiltonian but gives a nonlocal contribution $\alpha^2 H_{\text{int}}'' = \alpha^2 \mathrm{e}^{\mathrm{i}B}(\sum_{i<j}\frac{g^2}{(x^i-x^j)^2})\,\mathrm{e}^{-\mathrm{i}B}$ to K. Thirdly, consistency requires the operator $\mathrm{e}^{\mathrm{i}B}\mathrm{e}^{\mathrm{i}A}$ to be independent of the remaining free parameter α, as the latter is not fixed by the formalism and has a dimension of length. In order to check this, let us differentiate $\mathrm{e}^{\mathrm{i}B}\mathrm{e}^{\mathrm{i}A}$ with respect to α and demonstrate that

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\mathrm{e}^{\mathrm{i}B}\mathrm{e}^{\mathrm{i}A}\right) = 0 \tag{19}$$

for our special Lie-algebra elements

$$A = \alpha H + \frac{1}{\alpha}K - 2D \quad \text{and} \quad B = -\alpha H_0 - \frac{1}{\alpha}K + 2D. \tag{20}$$

Taking into account also the commutation relations (3), which are valid for both H and H_0, one can easily verify the relations

$$\left[\frac{\mathrm{d}B}{\mathrm{d}\alpha}, B^n\right] = +2\mathrm{i}n\frac{1}{\alpha}B^n \quad\Longrightarrow\quad \frac{\mathrm{d}e^{\mathrm{i}B}}{\mathrm{d}\alpha} = \mathrm{i}\left(\frac{\mathrm{d}B}{\mathrm{d}\alpha} + \frac{1}{\alpha}B\right)e^{\mathrm{i}B} = 2\mathrm{i}\left(\frac{1}{\alpha}D - H_0\right)e^{\mathrm{i}B},$$
$$\left[\frac{\mathrm{d}A}{\mathrm{d}\alpha}, A^n\right] = -2\mathrm{i}n\frac{1}{\alpha}A^n \quad\Longrightarrow\quad \frac{\mathrm{d}e^{\mathrm{i}A}}{\mathrm{d}\alpha} = \mathrm{i}\left(\frac{\mathrm{d}A}{\mathrm{d}\alpha} - \frac{1}{\alpha}A\right)e^{\mathrm{i}A} = 2\mathrm{i}\left(\frac{1}{\alpha}D - \frac{1}{\alpha^2}K\right)e^{\mathrm{i}A}. \tag{21}$$

Together with (16) they lead to the desired result (19).

To summarize, the quantum mechanical Hamiltonian of the Calogero model can be transformed into a free Hamiltonian by applying an appropriate unitary transformation. Knowing its explicit form, the stationary states of the former model can be immediately constructed from those of the latter. This is in agreement with the claim of [4] that the quantum Calogero model hiddenly describes free particles in one dimension. It should be remembered, however, that the price paid for this change of variables is a nonlocal realization of the full conformal algebra in the Hilbert space.

3. Adding the harmonic potential

Let us now add an external harmonic potential to the model. The analysis of the previous section makes it clear that our technique can still be applied. Such a treatment of the Calogero model in the presence of a harmonic force should be much less intricate than the computation of [16], whose similarity transformation to decoupled harmonic oscillators explicitly involves the correlated ground state of the Calogero model.

Consider then the Hamiltonian

$$H_1 = \frac{1}{2}p_i p_i + \sum_{i<j}\frac{g^2}{(x^i - x^j)^2} + \frac{\omega^2}{2}x^i x^i = H + \omega^2 K. \tag{22}$$

Application of the first transformation with A as in (9) for the lower sign choice in (8) yields

$$H_1' = \left(\kappa_-^2 + \alpha^2\omega^2\right)H + \left(2\beta\kappa_- - 2\alpha\kappa_+\omega^2\right)D + \left(\beta^2 + \kappa_+^2\omega^2\right)K. \tag{23}$$

It is clear that the first term on the r.h.s. can no longer vanish for a real value of $\kappa_\pm = 1 \pm \sqrt{\alpha\beta}$. Hence, we must allow α and/or β to become complex in

$$\kappa_- = \mathrm{i}\alpha\omega \quad\Longrightarrow\quad \alpha\beta = (1 - \mathrm{i}\alpha\omega)^2, \tag{24}$$

where α remains arbitrary. This means that, as in [16], an ultimate similarity transformation is realized by a nonunitary operator. With the above relations replacing (11), the transformation specializes to

$$H' = 2\mathrm{i}\omega D + \left(\frac{1}{\alpha^2} - 4\mathrm{i}\frac{\omega}{\alpha} - 2\omega^2\right)K, \tag{25}$$

which indeed reduces to (12) for $\omega \to 0$.

The same recipe works for the B transformation, which is again found from A by replacing $H \to H_0$ and changing the overall sign,

$$A = \alpha H + \frac{1}{\alpha}(1 - \mathrm{i}\alpha\omega)^2 K - 2(1 - \mathrm{i}\alpha\omega)D, \qquad B = -\alpha H_0 - \frac{1}{\alpha}(1 - \mathrm{i}\alpha\omega)^2 K + 2(1 - \mathrm{i}\alpha\omega)D. \tag{26}$$

It is straightforward to write down the second transformation and verify that

$$\tilde{H}_1 \equiv e^{\mathrm{i}B}e^{\mathrm{i}A}H_1\, e^{-\mathrm{i}A}e^{-\mathrm{i}B} = H_0 + \omega^2 K, \tag{27}$$

which proves that we have indeed mapped the nonconformal Calogero model to decoupled harmonic oscillators, via a simple explicit albeit nonunitary similarity transformation. Clearly, the limit $\omega \to 0$ connects with the results of the previous section.

Finally, like in the previous case one can establish the independence of the transformation on the parameter α. Thus, the formalism developed in the preceding section is universal and can be applied to both the conformal and nonconformal Calogero models.

4. Superconformal extensions

The unitary transformation constructed above has many interesting applications. In particular, it allows one to address the issue of superconformal extensions of the Calogero model. Below we treat in detail the $N = 2$ case. In our setting, this amounts to adding fermionic coordinates to the free model and to properly modifying the nonlocal generator $\tilde{K}$ such as to close the superconformal algebra. The inverse unitary transformation with the standard form (20) for A and B then maps the set of free superparticles back to the desired superconformal Calogero model with the standard representation of K.

Apart from the so(1, 2) generators, the $N = 2$ superconformal algebra contains two supersymmetry generators Q and $\bar{Q}$ which are hermitian conjugates of each other, two superconformal generators S and $\bar{S}$ also related by Hermitian conjugation, and a u(1) generator J. Altogether there are four bosonic and four fermionic operators, which obey the nonvanishing commutation relations (suppressing Hermitian conjugates)

$$\begin{aligned} &[H, D] = \mathrm{i}H, \qquad [K, D] = -\mathrm{i}K, \qquad [Q, D] = \frac{\mathrm{i}}{2}Q, \qquad [S, D] = -\frac{\mathrm{i}}{2}S, \\ &[Q, J] = -\frac{1}{2}Q, \qquad [S, J] = -\frac{1}{2}S, \qquad [H, K] = 2\mathrm{i}D, \qquad [Q, K] = -\mathrm{i}S, \\ &\{Q, \bar{Q}\} = 2H, \qquad \{S, \bar{S}\} = 2K, \qquad \{Q, \bar{S}\} = -2D - 2\mathrm{i}J + \mathrm{i}C, \qquad [S, H] = \mathrm{i}Q. \end{aligned} \tag{28}$$

Here, C is a real constant which stands for a central charge. For the realization of this algebra we need to add to the coordinates x^i the same number n of canonical pairs of free fermions ψ^i and $\bar{\psi}^i$, subject to the standard anticommutation relations

$$\{\psi^i, \bar{\psi}^j\} = \delta^{ij} \quad \text{and} \quad \{\psi^i, \psi^j\} = 0 = \{\bar{\psi}^i, \bar{\psi}^j\} \quad \text{with } (\psi^i)^* = \bar{\psi}^i. \tag{29}$$

The algebra (28) suggests that $\tilde{H} = H_0 = \frac{1}{2}p^i p^i$ is accompanied by

$$\tilde{Q} = Q_0 = \psi^i p^i \quad \text{and} \quad \tilde{\bar{Q}} = \bar{Q}_0 = \bar{\psi}^i p^i \tag{30}$$

and the dilatation and u(1) generators

$$\tilde{D} = D = -\frac{1}{4}\left(x^i p_i + p_i x^i\right) \quad \text{and} \quad \tilde{J} = J = \frac{1}{4}\left(\psi^i \bar{\psi}^i - \bar{\psi}^i \psi^i\right). \tag{31}$$

The remaining (conformal) generators $\tilde{K}$, $\tilde{S}$ and $\tilde{\bar{S}}$ are nonlocal but acquire the standard form in the interacting model,

$$K = \frac{1}{2}x^i x^i \quad \text{and} \quad S = \psi^i x^i, \quad \bar{S} = \bar{\psi}^i x^i. \tag{32}$$

The goal is to construct the interacting-model Hamiltonian H and supercharges Q and $\bar{Q}$ by working our way back from the free model with the help of the algebra (28). To this end, we begin with the special conformal generator and parametrize as before

$$\tilde{K} = K + \alpha^2 \mathrm{e}^{\mathrm{i}B} H_{\text{int}} \mathrm{e}^{-\mathrm{i}B} \quad \text{but with } H_{\text{int}} = \sum_{i<j} \frac{g^2}{(x^i - x^j)^2} + V, \tag{33}$$

allowing for a new contribution V due to the fermions. The algebra commutators (28) then consistently fix the form of the superconformal generator $\tilde{S}$,

$$[\tilde{Q}, \tilde{K}] = -\mathrm{i}\tilde{S} \quad \Longrightarrow \quad \tilde{S} = S - \mathrm{i}\alpha \mathrm{e}^{\mathrm{i}B}[S, V]\mathrm{e}^{-\mathrm{i}B}. \tag{34}$$

Hermitian conjugation produces $\tilde{\bar{S}}$. Other structure relations of the superconformal algebra (28) yield the following restrictions on the form of V:

$$\begin{aligned} &[K, V] = 0, \qquad [D, V] = -\mathrm{i}V, \qquad [J, V] = 0, \\ &[Q, H_{\text{int}}] + \mathrm{i}\big[H_0 + V, [S, V]\big] = 0, \qquad \big\{S, [\bar{S}, V]\big\} = C, \\ &\big\{[S, V], [\bar{S}, V]\big\} + \mathrm{i}\big\{Q, [\bar{S}, V]\big\} + \mathrm{i}\big\{\bar{Q}, [S, V]\big\} + 2H_{\text{int}} = 0, \end{aligned} \tag{35}$$

plus their Hermitian conjugates.

Let us define an $N = 2$ Calogero model by finding a solution to Eqs. (35). The first line in (35) implies that the potential V is a homogeneous function of the x^i of degree -2. Being u(1) neutral, it involves an equal number of ψ^i and $\bar{\psi}^i$. Thus, it is natural to take the simplest ansatz

$$V = V_{ij}(x)\,\psi^i \bar{\psi}^j = \frac{1}{2}V_{ii}(x) + \frac{1}{2}V_{ij}(x)\big[\psi^i, \bar{\psi}^j\big] \tag{36}$$

with unknown functions $V_{ij}(x)$. Substituting this form into the remaining (anti)commutators in (35) one obtains a system of partial differential equations,

$$\begin{aligned} &-2V_{ij} = \partial_i\big(V_{jp}x^p\big) + \partial_j\big(V_{ip}x^p\big), \qquad V_{ij} + \partial_j\big(V_{ip}x^p\big) = 0, \qquad \partial_p V_{ij} = \partial_i V_{pj}, \\ &\partial_i\big(V_{ip}x^p\big) + \big(V_{ip}x^p\big)\big(V_{is}x^s\big) - 2\sum_{i<j} \frac{g^2}{(x^i - x^j)^2} = 0, \qquad V_{ij}x^i x^j = C. \end{aligned} \tag{37}$$

The first equation implies that $V_{ij} = V_{ji}$. Then the second restriction gives the condition

$$\partial_i\big(V_{jp}x^p\big) - \partial_j\big(V_{ip}x^p\big) = 0 \quad \Longrightarrow \quad V_{ip}x^p = \partial_i \Phi \tag{38}$$

with some scalar function Φ. The remaining equations in (37) imply that

$$V_{ij} = -\partial_i \partial_j \Phi \tag{39}$$

and constrain Φ to obey the partial differential equations

$$\partial_i \partial_i \Phi + (\partial_i \Phi)(\partial_i \Phi) = 2\sum_{i<j} \frac{g^2}{(x^i - x^j)^2} \quad \text{and} \quad x^i \partial_i \Phi = C. \tag{40}$$

Any solution Φ to these equations will give rise to an $N = 2$ superconformal extension of the Calogero model.

The general solution to (40) can be put in the form

$$\Phi = \mu \sum_{i<j} \ln\big|x^i - x^j\big| + \nu \ln\sqrt{x^2} + \Lambda\left(\left\{\frac{x^i}{x^1}\right\}\right), \tag{41}$$

where μ and ν are dimensionless constants, $x^2 \equiv x^i x^i$, and Λ is a general function of coordinate ratios. Putting for simplicity $\Lambda \equiv 0$ and inserting (41) into (40), we find the conditions

$$\mu(\mu - 1) = g^2 > -\frac{1}{4} \quad \text{and} \quad \nu\big(\nu + n(n-1)\mu + n - 2\big) = 0, \tag{42}$$

which give four solutions for the pair $\mu(n, g)$ and $\nu(n, g)$. The central charge is fixed at

$$C(n, g) = \frac{n(n-1)}{2}\mu + \nu. \tag{43}$$

Differentiating twice as in (39) and inserting in (36) yields

$$V = \sum_{i<j} \frac{\mu}{(x^i - x^j)^2} - \frac{n-2}{2}\frac{\nu}{x^2} + \frac{1}{2}\sum_{i\neq j} \frac{\mu}{(x^i - x^j)^2}\big[\psi^i, \bar\psi^i - \bar\psi^j\big] - \frac{1}{2}\sum_{i,j} \frac{\nu}{x^2}\frac{x^2\delta^{ij} - 2x^i x^j}{x^2}\big[\psi^i, \bar\psi^j\big] \tag{44}$$

and, hence, with (42) the interaction Hamiltonian

$$H_{\text{int}} = \sum_{i<j} \frac{\mu^2}{(x^i - x^j)^2} + \frac{1}{2}\sum_{i\neq j} \frac{\mu}{(x^i - x^j)^2}\big[\psi^i, \bar\psi^i - \bar\psi^j\big] - \frac{n-2}{2}\frac{\nu}{x^2} - \frac{1}{2}\sum_{i,j} \frac{\nu}{x^2}\frac{x^2\delta^{ij} - 2x^i x^j}{x^2}\big[\psi^i, \bar\psi^j\big] \tag{45}$$

but also $\tilde K$ and $\tilde S$. The original Calogero coupling g^2 has been replaced by μ^2, of which ν is a function via (42). By the very construction, this $H = H_0 + H_{\text{int}}$ along with D and K from (33) furnish a representation of so$(1, 2)$. Therefore, they can be used to construct the inverse transformation $\mathrm{e}^{-\mathrm{i}A}\mathrm{e}^{-\mathrm{i}B}$ and hence the supercharge, which for $\nu = 0$ reads

$$Q = \mathrm{e}^{-\mathrm{i}A}\mathrm{e}^{-\mathrm{i}B}\big(\psi^i p^i\big)\mathrm{e}^{\mathrm{i}B}\mathrm{e}^{\mathrm{i}A} = \psi^i p^i + \mathrm{i}[V, S] = \psi^i\left(p^i + \mathrm{i}\sum_{k(\neq i)} \frac{\mu}{x^i - x^k}\right). \tag{46}$$

It may be checked that the same transformation maps $\tilde S$ of (34) back to S as it should.

Beautifully enough, with $\nu = 0$ we have reproduced precisely the $N = 2$ superextension constructed by Freedman and Mende [19] in the framework of supersymmetric quantum mechanics. For the other solution to (42), $\nu = 2 - n - n(n-1)\mu$, we have apparently found an alternative superextension (see also [20]).

5. Concluding remarks

In this Letter we have constructed a simple unitary transformation relating the conformal Calogero model to a system of free particles on the real line. The simplification was achieved at a price of a highly nontrivial and, in particular, nonlocal realization of the full conformal symmetry in the resulting free theory. The transformation was shown to be universal and applicable to the nonconformal Calogero model as well as to $N = 2$ supersymmetric extensions. In the latter case we reconstructed not only the model of Freedman and Mende but found a second variant.

Turning to possible further developments, first to mind comes the $N = 4$ superconformal extension of the Calogero model, which seems crucial for testing a conjecture of Gibbons and Townsend [12]. The construction realized for the su$(1, 1|1)$ superalgebra in Section 4 can literally be generalized to the su$(1, 1|2)$ superalgebra. This project is under way. Another interesting point is to employ our transformation for deriving the propagator of the Calogero model starting from the free propagator. Finally, it may be worthwhile to generalize the analysis of Section 3 to the case of a harmonic pair potential.

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