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SU(2|1) supersymmetric mechanics on curved spaces

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Abstract: We present SU(2|1) supersymmetric mechanics on n-dimensional Riemannian manifolds within the Hamiltonian approach. The structure functions including prepotentials entering the supercharges and the Hamiltonian obey extended curved WDVV equations specified by the manifold's metric and curvature tensor. We consider the most general $u(2)$-valued prepotential, which contains both types (with and without spin variables), previously considered only separately. For the case of real Kähler manifolds we construct all possible interactions. For isotropic ($so(n)$-invariant) spaces we provide admissible prepotentials for any solution to the curved WDVV equations. All known one-dimensional SU(2|1) supersymmetric models are reproduced.

Keywords: Extended Supersymmetry, Field Theories in Lower Dimensions, Space-Time Symmetries

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Contents

1 Introduction

One of the interesting features of $\mathcal{N}{=}4$ supersymmetric mechanics is its relation with the Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations [1, 2]. The most natural appearance of the WDVV equations is seen at the component level. As was first demonstrated in [3], on the $(2n{+}4n)$-dimensional phase space $\{x^i, p_j, \psi^{ai}, \bar\psi^j_b\}$, with $i, j = 1, \dots, n$ and $a, b = 1, 2$, the simplest ansatz for the $\mathcal{N}{=}4$ supercharges Q^a and $\overline{Q}_a$,

$$Q^a = p_i \psi^{ia} + \mathrm{i} F^{(0)}_{ijk} \psi^{ib} \psi^j_b \bar\psi^{ka} \quad \text{and} \quad \overline{Q}_a = p_i \bar\psi^i_a + \mathrm{i} F^{(0)}_{ijk} \bar\psi^i_b \bar\psi^{jb} \psi^k_a \,, \tag{1.1}$$

yields the WDVV equations

$$F^{(0)}_{ijm} \delta^{nm} F^{(0)}_{kln} - F^{(0)}_{ilm} \delta^{nm} F^{(0)}_{kjn} = 0 \qquad \text{with} \qquad F^{(0)}_{ijk} = \partial_i \partial_j \partial_k F^{(0)}(x) \tag{1.2}$$

for totally symmetric structure functions $F^{(0)}_{ijk}$, if one requires the supercharges to obey the $\mathcal{N}{=}4$ super Poincaré algebra

$$\{Q^a, \overline{Q}_b\} = \frac{\mathrm{i}}{2} \delta^a_b H \,, \quad \{Q^a, Q^b\} = 0 \,, \quad \{\overline{Q}_a, \overline{Q}_b\} = 0 \,. \tag{1.3}$$

The evaluation of the brackets in (1.3) assumed the standard Dirac brackets between the basic variables,

$$\{x^i, p_j\} = \delta^i_j \quad \text{and} \quad \{\psi^{ia}, \bar\psi^j_b\} = \frac{\mathrm{i}}{2} \delta^a_b \delta^{ij} \,. \tag{1.4}$$

The simplest form (1.1) of the supercharges does not produce (classically) any potential term in the Hamiltonian H. To generate physically interesting systems, the supercharges have to be extended by terms linear in the fermionic variables. Such linear terms come with new structure functions, so-called prepotentials, which obey differential equations extending the WDVV ones. Prepotentials come in two variants, called W and U. The first one is associated with a $u(1)$ subalgebra of the $u(2)$ R-symmetry algebra, the second one with an $su(2)$ subalgebra. The latter requires the introduction of semi-dynamical spin variables [4]. Examples of such constructions can be found in [3, 5–8] and references therein.

So far we discussed $\mathcal{N}{=}4$ supersymmetric mechanics on the Euclidian space $\mathbb{R}^n$. Recently [9, 10], the structure given by (1.1)–(1.4) was generalized to $\mathcal{N}{=}4$ supersymmetric mechanics on arbitrary Riemannian spaces, rendering it covariant under general coordinate transformations. In this case, the WDVV equations (1.2) are superseded by the 'curved WDVV equations' [9]

$$\nabla_i F_{jkm} = \nabla_j F_{ikm} \quad \text{and} \quad F_{ikp} g^{pq} F_{jmq} - F_{jkp} g^{pq} F_{imq} + R_{ijkm} = 0 \tag{1.5}$$

involving the Riemann tensor R^p_{jkm} of the Riemannian manifold. Simultaneously, the conditions on the prepotentials W or U entering the supercharges have been covariantized [10].

Another generalization of $\mathcal{N}{=}4$ supersymmetric mechanics has been proposed by Smilga [11], by adding R-symmetry generators in the right-hand side of the basic commutators $\{Q^a, \overline{Q}_b\} = \frac{\mathrm{i}}{2}\delta^a_b H$. This step deforms the $\mathcal{N}{=}4$ super Poincaré algebra to an $su(2|1)$ algebra [11]. A systematic study of one-dimensional SU(2|1) supersymmetric mechanics has been conducted in [12–15] using the superspace approach.

Our main goal is to construct n-dimensional SU(2|1) supersymmetric mechanics with a $(2n{+}4n)$-dimensional phase space over an arbitrary Riemannian manifold within the Hamiltonian approach.[1] In section 2 we introduce generalized Poisson brackets which are general coordinate covariant, write down the most general ansatz for the supercharges (linear and cubic in the fermionic variables), and analyze the conditions on the structure functions. These determine the structure functions and the explicit structure of the Hamiltonian. In section 3 the known solutions [11, 12, 15] for one-dimensional SU(2|1) mechanics are reproduced. Section 4 specializes on two examples corresponding to real Kähler and isotropic spaces. For the first one, we provide exact supercharges and Hamiltonian for so-called real Kähler spaces, generalizing the results of [18, 19] to SU(2|1) supersymmetry. The second example, which relates to isotropic spaces, extends the solutions found in [10] as well as gives explicit solutions for spheres and pseudospheres. A few comments and remarks conclude the paper.

2 Supercharges and Hamiltonian

Our goal is to realize the $su(2|1)$ superalgebra

$$\begin{aligned}
&\{Q^a,\overline{Q}_b\}=\frac{\mathrm{i}}{2}\delta^a_b H-\mu I^a_b+\mathrm{i}\mu I_0\delta^a_b\,, \quad \{Q^a,Q^b\}=\{\overline{Q}_a,\overline{Q}_b\}=0\,, \\
&\{Q^a,H\}=\{\overline{Q}_a,H\}=0\,, \qquad \{I_0,Q^a\}=\frac{\mathrm{i}}{2}Q^a\,, \qquad \{I_0,\overline{Q}_a\}=-\frac{\mathrm{i}}{2}\overline{Q}_a\,, \\
&\{I^{ab},I^{cd}\}=-\epsilon^{ac}I^{bd}-\epsilon^{bd}I^{ac}\,, \quad \{I^{ab},Q^c\}=-\frac{1}{2}\left(\epsilon^{ac}Q^b+\epsilon^{bc}Q^a\right), \quad \{I^{ab},\overline{Q}_c\}=\frac{1}{2}\left(\delta^a_c\overline{Q}^b+\delta^b_c\overline{Q}^a\right),
\end{aligned} \tag{2.1}$$

[1] Particular cases of $\mathcal{N}{=}2,4$ supersymmetric mechanics with weak supersymmetry and $(4n{+}4n)$-dimensional phase spaces have been considered in [16, 17].

with a constant deformation parameter μ on the $(2n{+}4n)$-dimensional phase space given by n coordinates x^i and momenta p_i, with $i=1,\ldots,n$, each of which is accompanied by four fermionic ones ψ^{ia} and $\bar{\psi}^j_b = (\psi^{jb})^\dagger$. On the cotangent bundle over an n-dimensional Riemannian manifold, the Poisson brackets between the basic variables are defined as

$$\begin{aligned}&\{x^i, p_j\} = \delta^i_j\,, \quad \{\psi^{ai}, \bar{\psi}^j_b\} = \frac{\mathrm{i}}{2}\delta^a_b\, g^{ij}\,, \quad \{p_i, \psi^{aj}\} = \Gamma^j_{ik}\psi^{ak}\,, \quad \{p_i, \bar{\psi}^j_a\} = \Gamma^j_{ik}\bar{\psi}^k_a\,,\\ &\{p_i, p_j\} = -2\mathrm{i}R_{ijkm}\psi^{ak}\bar{\psi}^m_a\,. \end{aligned} \tag{2.2}$$

Here, Γ^i_{jk} and R^i_{jkl} are the components of the Levi-Civita connection and curvature of the metric $g_{ij}(x)$ defined in a standard way as

$$\Gamma^k_{ij} = \frac{1}{2}\, g^{km}\left(\partial_i g_{jm} + \partial_j g_{im} - \partial_m g_{ij}\right) \quad \text{and} \quad R^i{}_{jkl} = \partial_k\Gamma^i_{jl} - \partial_l\Gamma^i_{jk} + \Gamma^m_{jl}\,\Gamma^i_{mk} - \Gamma^m_{jk}\,\Gamma^i_{ml}\,. \tag{2.3}$$

For the construction of the supercharges Q^a and $\overline{Q}_b$ we make use of the full U(2) R-symmetry, combining the two types of prepotentials used in [10]:

$$\begin{aligned}Q^a &= p_i\psi^{ia} + \mathrm{i}W_i\psi^{ia} + J^{ac}U_i\psi^i_c + \mathrm{i}F_{ijk}\psi^{ic}\psi^j_c\bar{\psi}^{ka} + \mathrm{i}G_{ijk}\psi^{ia}\psi^{jc}\bar{\psi}^k_c\,,\\ \overline{Q}_a &= p_i\bar{\psi}^i_a - \mathrm{i}W_i\bar{\psi}^i_a - J_{ac}U_i\bar{\psi}^{ic} + \mathrm{i}F_{ijk}\bar{\psi}^i_c\bar{\psi}^{jc}\psi^k_a + \mathrm{i}G_{ijk}\bar{\psi}^i_a\bar{\psi}^j_c\psi^{ck}\,.\end{aligned} \tag{2.4}$$

Here, $\epsilon^{ac}W_i$ and $J^{ac}U_i$ are associated with the U(1) and SU(2) parts of the R-symmetry, generated by I_0 and I^{ac}, respectively. To realize the SU(2) currents J^{ac}, one needs to adjoin additional bosonic spin variables $\{u^a, \bar{u}_a |\, a=1,2\}$ [4] parameterizing an internal two-sphere and obeying the brackets

$$\{u^a, \bar{u}_b\} = -\mathrm{i}\,\delta^a_b\,, \tag{2.5}$$

in terms of which these currents read

$$J^{ab} = \frac{\mathrm{i}}{2}\left(u^a\bar{u}^b + u^b\bar{u}^a\right) \quad\Rightarrow\quad \left\{J^{ab}, J^{cd}\right\} = -\epsilon^{ac}J^{bd} - \epsilon^{bd}J^{ac}. \tag{2.6}$$

The structure functions U_i, W_i, F_{ijk} and G_{ijk} entering the supercharges (2.4) are, for the time being, arbitrary functions of the n coordinates x^i. In addition, by construction, F_{ijk} and G_{ijk} are symmetric and anti-symmetric over the first two indices, respectively:

$$F_{ijk} = F_{jik}\,, \qquad G_{ijk} = -G_{jik}\,. \tag{2.7}$$

The requirement that the supercharges (2.4) span the $su(2|1)$ superalgebra (2.1) results in the following equations:

$$G_{ijk} = 0, \qquad F_{ijk} - F_{ikj} = 0 \quad\Rightarrow\quad F_{ijk} = F_{(ijk)}\,, \tag{2.8}$$

$$\nabla_i F_{jkm} - \nabla_j F_{ikm} = 0\,, \tag{2.9}$$

$$F_{ikp}g^{pq}F_{jmq} - F_{jkp}g^{pq}F_{imq} + R_{ijkm} = 0 \tag{2.10}$$

and

$$\nabla_i W_j - \nabla_j W_i = 0 \quad\text{and}\quad \nabla_i U_j - \nabla_j U_i = 0 \;\Rightarrow\; W_i = \partial_i W \quad\text{and}\quad U_i = \partial_i U\,, \tag{2.11}$$

$$\nabla_i U_j - U_i U_j - F_{ijk}g^{km}U_m = 0\,, \tag{2.12}$$

$$\nabla_i W_j + F_{ijk}g^{km}W_m + \mu\, g_{ij} = 0\,, \tag{2.13}$$

$$g^{ij}W_i U_j - \mu = 0 \quad\text{or}\quad U_j = 0\,, \tag{2.14}$$

where, as usual,

$$\nabla_i W_j = \partial_i W_j - \Gamma^k_{ij} W_k \quad \text{and} \quad \nabla_i F_{jkl} = \partial_i F_{jkl} - \Gamma^m_{ij} F_{klm} - \Gamma^m_{ik} F_{jlm} - \Gamma^m_{il} F_{jkm} \,. \tag{2.15}$$

Finally, the other generators of the $su(2|1)$ superalgebra acquire the form

$$\begin{aligned} H = {} & g^{ij} p_i p_j + g^{ij} \partial_i W \partial_j W + \frac{1}{2} J^2 g^{ij} \partial_i U \partial_j U + 4\big(\epsilon^{cd} \nabla_i \partial_j W - \mathrm{i} J^{cd} \nabla_i \partial_j U\big) \psi^i_c \bar\psi^j_d \\ & - 4\big(\nabla_m F_{ijk} + R_{ijkm}\big) \psi^{ic} \bar\psi^j_c \, \psi^{kd} \bar\psi^m_d \,, \end{aligned} \tag{2.16}$$

$$I^{ab} = J^{ab} + \mathrm{i} g_{ij} \big(\psi^{ia} \bar\psi^{jb} + \psi^{ib} \bar\psi^{ja}\big) \quad \text{and} \quad I_0 = g_{ij} \psi^{ci} \bar\psi^j_c \,, \tag{2.17}$$

where the Casimir $J^2 = J^{cd} J_{cd}$ plays the role of a coupling constant. The equation (2.9) qualifies F_{ijk} as a so-called third-rank Codazzi tensor [20], while (2.10) is the curved WDVV equations [9], and (2.11)–(2.14) are the deformed analogs of the curved equations considered in [10] and of the flat potential equations discussed in [6] and [8].

Two limiting cases are noteworthy. First, putting $W = 0$ implies via (2.13) that $\mu = 0$, bringing us back to the standard $\mathcal{N}{=}4, d{=}1$ super Poincaré algebra — the case considered in detail in [10]. The converse is not true: $\mu = 0$ admits the simultaneous presence of both U and W, as long as their gradients are orthogonal to each other. Second, putting $U = 0$ solves (2.12) and (2.14), and it removes the spin variables together with their currents J^{ab} from the supercharges, the Hamiltonian and the R-currents.

Summarizing, to construct SU(2|1) supersymmetric n-dimensional mechanics on a Riemannian manifold with metric g_{ij}, one has to

- solve the curved WDVV equations (2.9), (2.10) for the fully symmetric function F_{ijk},
- find the admissible prepotentials W and U as solutions to the equations (2.11)–(2.14).

In the following we shall use this procedure. To begin with, let us demonstrate how the known particular cases of one-dimensional SU(2|1) mechanics fit into our scheme. Then we shall investigate two special geometries allowing for explicit solutions of the curved WDVV equations.

3 One-dimensional SU(2|1) mechanics

In the distinguished case of a one-dimensional space the metric is always flat and can be fixed to $g_{11} = 1$ without loss of generality. Therefore, the curved WDVV equations become trivial and put no restrictions on the single remaining component F_{111}.

The $n = 1$ variant of (2.12)–(2.14) reads

$$U'' - F_{111} U' - U'^2 = 0\,, \quad W'' + F_{111} W' + \mu = 0\,, \quad W' U' - \mu = 0 \quad \text{or} \quad U' = 0\,, \tag{3.1}$$

where $'$ means differentiation with respect to the single variable $x^1 = x$. These three equations are not independent. For $U' \neq 0$, the two second-order equations follow from each other via $W'U' = \mu$. In this generic situation, we have the freedom to freely dial one function. The choice of any one structure function determines the other two:

$$F_{111} = -\frac{W''+\mu}{W'} = \frac{U''-U'^2}{U'} \quad \text{and} \quad U' = \mu/W' \qquad \text{or} \qquad W' = \mu/U' \,, \tag{3.2}$$

$$W' = -\mu\, \mathrm{e}^{-F_{11}} \int^x \mathrm{e}^{F_{11}} \qquad \text{and} \quad U' = -\mathrm{e}^{F_{11}} \Big/ \int^x \mathrm{e}^{F_{11}} \quad \text{with} \quad F'_{11} = F_{111} \,. \tag{3.3}$$

The Hamiltonian reads

$$H = p^2 + \big(W'\big)^2 + \frac{1}{2} J^2 \big(U'\big)^2 + 4\big(\epsilon^{cd} W'' - \mathrm{i} J^{cd} U''\big) \psi_c \bar\psi_d - 4F'_{111} \psi^c \bar\psi_c \, \psi^d \bar\psi_d \,, \tag{3.4}$$

which may be expressed purely in terms of either W', U', or F_{11} via (3.2) or (3.3).

Three different limits can be taken. First, $W' = 0$ yields $\mu = 0$. However, $\mu = 0$ admits two disjoint solutions,

$$W' = 0 \quad \text{and} \quad U' = -\mathrm{e}^{F_{11}} \Big/ \int^x \mathrm{e}^{F_{11}} \qquad \text{or} \qquad U' = 0 \quad \text{and} \quad W' \sim \mathrm{e}^{-F_{11}} \,. \tag{3.5}$$

Second, $U' = 0$ removes the spin variables, and the Hamiltonian reduces to

$$H = p^2 + \big(W'\big)^2 + 4\, W'' \psi^a \bar\psi_a + 4 \left(\frac{W''+\mu}{W'}\right)' \psi^a \bar\psi_a \, \psi^b \bar\psi_b \,, \tag{3.6}$$

which has been constructed in [11, 12]. Third, $F_{111} = 0$ leads to

$$W' = -\mu\, (x{-}x_0) \quad \text{and} \quad U' = -1/(x{-}x_0) \,, \tag{3.7}$$

which has been found in [15]. In this case the supercharges become linear in the fermions.

4 Examples of n-dimensional mechanics with potentials

Once we start to consider the n-dimensional mechanics, the first problem is to solve the curved WDVV equations (2.8)–(2.10). The general solution of these equations is unknown, but in some exceptional cases the solution can easily be constructed. Solving thereafter a system of differential equations (2.11)–(2.14), one can explicitly find the corresponding potentials.

4.1 Real Kähler spaces

The first example of multidimensional mechanics concerns the so-called 'real Kähler spaces' [18, 19], which are defined by a metric of the form

$$g_{ij} = \frac{\partial^2 G}{\partial x^i \partial x^j} \quad \Rightarrow \quad \Gamma_{ijk} = \frac{1}{2} \frac{\partial^3 G}{\partial x^i \partial x^j \partial x^k} \tag{4.1}$$

determined by a scalar function G. It is rather easy to check that two solutions of the curved WDVV equations for such a metric are

$$F^{(1)}_{ijk} = \Gamma_{ijk} \quad \text{and} \quad F^{(2)}_{ijk} = -\Gamma_{ijk} \,. \tag{4.2}$$

With this input the equations (2.11)–(2.14) drastically simplify and can be solved explicitly as

$$W^{(1)} = -\mu\, G + \lambda_i x^i \qquad \text{and} \qquad U^{(1)} = -\log\big(\sigma^i \partial_i G\big)\,, \tag{4.3}$$

$$W^{(2)} = -\mu\,\big(x^i\partial_i G - G\big) + \lambda^i \partial_i G \quad \text{and} \quad U^{(2)} = -\log\big(\sigma_i x^i\big)\,, \tag{4.4}$$

where λ_i and σ^j are constants subject to the condition

$$\sigma_i \lambda^i = 0\,. \tag{4.5}$$

Thus, we have a family of n-dimensional SU(2|1) mechanics defined on any real Kähler space.

4.2 Isotropic spaces

The second example relates to the n-dimensional mechanics with potentials on the isotropic spaces. Let us remind that in [9] a large class of solutions to the curved WDVV equations (2.8)–(2.10) has been constructed on isotropic spaces. The metric of such a manifold is SO(n) invariant, i.e. it admits $\frac{1}{2}n(n-1)$ Killing vectors and can be written in the form

$$g_{ij} = \frac{1}{f(r)^2}\,\delta_{ij} \quad \text{with} \quad r^2 = \delta_{ij}x^i x^j \quad \Rightarrow \quad \Gamma^k_{ij} = -\frac{f'}{rf}\Big(x_i\delta^k_j + x_j\delta^k_i - x^k\delta_{ij}\Big) \tag{4.6}$$

with a positive real function f, where (in this subsection) $'$ means differentiation with respect to r. The ansatz

$$F_{ijk} = a(r)\, x^i x^j x^k + b(r)\big(\delta_{ij}x^k + \delta_{jk}x^i + \delta_{ik}x^j\big) + f(r)^{-2}F^{(0)}_{ijk} \tag{4.7}$$

extending an arbitrary solution $F^{(0)}_{ijk}$ of the flat WDVV equations (1.2) obeys the curved WDVV equations if $x^i F^{(0)}_{ijk} = \delta_{jk}$,

$$a = \frac{2f\big(f-rf'\big) \pm \big(2f^2 - 3rff' + r^2(f')^2 + r^2ff''\big)}{r^4 f^3\big(f-rf'\big)} \quad \text{and} \quad b = -\frac{f \pm \big(f-rf'\big)}{r^2 f^3}\,. \tag{4.8}$$

If we choose the minus sign in the above expressions, i.e. for

$$a = \frac{ff' - r(f')^2 - rff''}{r^3 f^3\,(f-rf')} \quad \text{and} \quad b = -\frac{f'}{rf^3}\,, \tag{4.9}$$

then a prepotential W solving (2.13) is easily constructed,

$$W = w(r,\mu) + W^{(0)} \qquad \text{with} \qquad w'(r,\mu) = \frac{\alpha(f^2)' - \mu\, r}{2\,f\,(f-rf')}\,, \tag{4.10}$$

where α is some constant and $W^{(0)}$ obeys the flat equation

$$\partial_i\partial_j W^{(0)} + F^{(0)}_{ijm}\delta^{mn}\partial_n W^{(0)} = 0 \qquad \text{subject to} \quad x^i\partial_i W^{(0)} = \alpha\,. \tag{4.11}$$

This extends the prepotential solution found in [10] to $\mu \neq 0$. To this configuration one may add a simple solution to (2.12) for a prepotential U respecting also (2.14),

$$U = \log\frac{\mu\, f^2}{\mu\, r^2 - 2\alpha f^2}\,. \tag{4.12}$$

The prepotentials W and U above generate in the Hamiltonian the bosonic potential

$$V = f^2\,\partial_i W^{(0)} \partial_i W^{(0)} + \frac{(\mu r - 2\alpha f f')(\mu r^2 - 4\alpha f^2 + 2\alpha r f f')}{4r\,(f - rf')^2} + 2J^2\,\frac{r^2\mu^2(f - rf')^2}{(\mu r^2 - 2\alpha f^2)^2}\,. \tag{4.13}$$

An interesting case is the (pseudo)sphere, $f = 1 + \epsilon\, r^2$ with $\epsilon = \pm 1$. For this manifold, the potential reads

$$V_{\text{sphere}} = (1{+}\epsilon r^2)^2\,\partial_i W^{(0)} \partial_i W^{(0)} + \frac{(\mu{-}8\epsilon\alpha)^2}{16\epsilon}\,V_{\text{Higgs}} - \frac{\mu^2}{16\epsilon} + \frac{8\epsilon\mu^2\,J^2\,(V_{\text{Higgs}}{-}1)}{\big(8\alpha\epsilon\,V_{\text{Higgs}} + \mu\,(1{-}V_{\text{Higgs}})\big)^2} \tag{4.14}$$

with the Higgs-oscillator potential [21, 22]

$$V_{\text{Higgs}} = \left(\frac{1+\epsilon r^2}{1-\epsilon r^2}\right)^2. \tag{4.15}$$

For $J^2 = 0$ or $\mu = 8\alpha\epsilon$, simplifications occur,

$$V_{\text{sphere}}\big|_{\mu=8\alpha\epsilon} = (1{+}\epsilon r^2)^2\,\partial_i W^{(0)} \partial_i W^{(0)} - 4\alpha^2\epsilon + 8\epsilon\,J^2\,(V_{\text{Higgs}}{-}1)\,. \tag{4.16}$$

5 Conclusions

We extended the previous analysis [10] of $\mathcal{N}{=}4$ supersymmetric mechanics on arbitrary Riemannian spaces to systems from $\mathcal{N}{=}4, d{=}1$ super Poincaré symmetry to SU(2|1) supersymmetry. The extension is parametrized by a deformation parameter μ, which only enters in the equation determining the prepotential W and relating it with the prepotential U. All other equations, in particular the curved WDVV equations [9], and the form of the supercharges, R-currents and Hamiltonian are unchanged.

A novel feature in our consideration is the presence of both types of prepotentials, W and U, associated with the U(1) and SU(2) parts of the R-symmetry, respectively.[2]

Two special geometries have been considered in detail. Real Kähler spaces admit an explicit solution for all structure functions. On isotropic spaces, we constructed admissible structure functions for any conformally invariant solution to the flat structure equations. As an application, a Hamiltonian potential for SU(2|1) supersymmetric mechanics on a (pseudo)sphere was presented. All known one-dimensional systems enjoying SU(2|1) supersymmetry [12, 15] can be easily reproduced in our framework.

One future task even on flat space is a classification of admissible potentials when both prepotentials, W and U, are present. At the moment we can do this only for the special case when one of them depends on r only. Another interesting question is whether there exist other geometries besides the real Kähler case which admit a fully explicit solution. Since the real Kähler spaces unambiguously arise in the superfield approach [18, 19], it seems compelling to perform a superspace description of the mechanics presented here. To this end, it is unclear whether the standard superspace is sufficient or whether we have to employ the deformed one introduced and advocated in [12, 15].

[2] This is actually also possible in the super Poincaré limit, but requires their gradients to be mutually orthogonal.

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