



The fine structure of the finite-size effects for the spectrum of the $OSp(n|2m)$ spin chain

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Abstract

In this paper we investigate the finite-size properties of the spectrum of quantum spin chains with local spins taken to be the fundamental vector representation of the $OSp(n|2m)$ superalgebra.

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1. Introduction

Over the years exactly solvable one-dimensional quantum magnets have been considered as suitable lattice regularization of two-dimensional space–time models of quantum field theory. In principle the respective Bethe ansatz solution offers us a non-perturbative framework to study the properties of the spectrum of the respective spin chain Hamiltonian for large system sizes. In the case of a massless theory it has been showed that the finite size corrections to the spectrum determine the conformal central charge and the anomalous dimensions of the underlying conformal field theory [1]. The study of the finite-size effects of integrable spin chains with generators on some simply laced Lie algebra G suggested that their critical behavior are governed by the properties of a field theory of Wess–Zumino–Witten type on the same group G . By way of contrast when the underlying invariance of the spin chain is based on supergroups the identification

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of the respective field theory appears to be more involved [2]. Indeed, it has been observed that the finite size spectrum of the $OSp(3|2)$ superspin chain present the unusual feature of having states with the same conformal dimension as the trivial identity operator [3,4]. Later on similar phenomena have been found to be present in a staggered $sl(2|1)$ spin chain whose degrees of freedom alternate between the fundamental and dual representations [5] as well as in staggered six-vertex model [6]. The degeneracy of many states of the spectrum was found to grow with the size of the chain and this was interpreted as the signature of the existence of non-compact degrees of freedom in the continuum limit [6].

The purpose of this paper is to study the subleading corrections to the finite-size spectrum of a number of spin chains invariant by the $OSp(n|2m)$ super Lie algebra. The results obtained here extend in a substantial way our recent analysis performed for the specific case of the $OSp(3|2)$ superalgebra [4]. In particular, we find a tower of states over the lowest energy with the same leading effective central charge c_{eff} as the size of the chain $L \rightarrow \infty$. More precisely, denoting the eigenenergies of such set of states by $E_k(L)$ we have

$$E_k(L) - Le_\infty = \frac{\pi\xi c_{\text{eff}}}{6L} + \frac{2\pi\xi}{L} \frac{\beta_k}{\log L}, \quad k = 0, 1, 2, \dots, k_\infty \quad (1.1)$$

where the integer k_∞ is typically bounded by system size L . The symbol e_∞ denotes the energy density of the ground state in the thermodynamic limit while ξ refers to the velocity of the elementary low-lying excitations. We shall notice that the amplitude β_k can be connected to a subset of the possible eigenvalues of the quadratic Casimir operator of the respective underlying $OSp(n|2m)$ superalgebra.

We recall here that the $OSp(n|2m)$ superspin chain realizes a gas of loops on the square lattice in which intersections are allowed [3]. The integer n and m parameterize the fugacity z given to every configuration of closed loops which is $z = n - 2m$. In the context of the loop model the above peculiar finite-size behavior was argued to be an indication that for $z < 2$ the crossing of loops becomes a relevant perturbation driving the system to an unusual critical phase [7]. In particular it was conjectured that the correlations functions in the loop model should be those of the Goldstone phase of the $O(z)$ sigma model. The universal behavior of the two point correlators has long been computed in [8] and it was found to decrease logarithmically with the distance. More recently this calculation has been extended to two point functions of operators composed by the product of k field components at the same point usually denominated k -leg watermelon correlators [9]. This observable measures the probability of k distinct loop segments connecting two arbitrary lattice points x and y . Here we shall argue that the asymptotic behavior of such correlation functions of the intersecting loop model can be inferred from the finite-size amplitudes β_k in analogy to the known connection among critical exponents and finite-size scaling amplitudes [1]. More precisely we observe that for large distances $r = |x - y|$ this family of correlators can be rewritten as

$$G_k(r) \sim 1/\ln(r)^{2(\beta_k - \beta_{k_0})} \quad (1.2)$$

for a suitable choice of the k_0 state.

2. The $OSp(n|2m)$ spin chain

The vertex model with rational weights which is invariant by the superalgebra $OSp(n|2m)$ was first discovered by Kulish in the context of the graded formulation of the Yang–Baxter equation [10]. The respective R-matrix $R_{ab}(\lambda)$ with spectral parameter λ can be represented as a linear combination of three basic operators,

$$R_{ab}(\lambda) = \lambda I_a \otimes I_b + P_{ab} + \frac{\lambda}{\frac{2-n+2m}{2} - \lambda} E_{ab} \tag{2.1}$$

where $R_{ab}(\lambda)$ acts on the tensor product $V_a \times V_b$ of two $(n + 2m)$ -dimensional graded vector spaces and I_a denotes the identity matrix in one of such spaces. The integers n and $2m$ stand for the number of bosonic (b) and fermionic (f) degrees of freedom.

The operator P_{ab} permutes two graded vector spaces and its expression is

$$P_{ab} = \sum_{i,j=1}^{n+2m} (-1)^{p_i p_j} e_{ij}^{(a)} \otimes e_{ji}^{(b)} \tag{2.2}$$

where $p_i = 0$ for the n bosonic basis vectors while for the $2m$ fermionic coordinates we have $p_i = 1$. The elementary matrices $e_{ij}^{(a)} \in V_a$ have only one non-vanishing element with value 1 at row i and column j .

The operator E_{ab} plays the role of a typical monoid operator which can formally be represented as,

$$E_{ab} = \sum_{i,j,l,k=1}^{n+2m} \alpha_{ij} \alpha_{lk}^{-1} e_{il}^{(a)} \otimes e_{jk}^{(b)} \tag{2.3}$$

where the non-null matrix elements α_{ij} are always ± 1 . Their precise distribution within the matrix α depends on the grading sequence we set up for the basis of the vector space. A convenient grading sequence is the basis ordering $f_1 \cdots f_m b_1 \cdots b_n f_{m+1} \cdots f_{2m}$ since it encodes in an explicit way the many $U(1)$ symmetries of the $OSp(n|2m)$ superalgebra. For this choice of grading the structure of the matrix α is [11],

$$\alpha = \begin{pmatrix} O_{n \times m} & O_{n \times m} & \mathcal{I}_{n \times n} \\ O_{m \times m} & \mathcal{I}_{m \times m} & O_{m \times n} \\ -\mathcal{I}_{m \times m} & O_{m \times m} & O_{m \times n} \end{pmatrix} \tag{2.4}$$

where $O_{N \times N}$ and $\mathcal{I}_{N \times N}$ are the null and the anti-diagonal $N \times N$ matrices, respectively. The matrix representation for other grading choices can be obtained from Eq. (2.4) by direct permutation of the vector space basis.

In the intersecting loop model realized by this superspin chain the different terms in the R -matrix (2.1) correspond to the allowed local configurations with Boltzmann weights given by the respective amplitudes, see Fig. 1. The Hamiltonian of the quantum $OSp(n|2m)$ spin chain is obtained by expanding the transfer matrix of the respective vertex model at the special value of the spectral parameter for which the R is proportional to the graded permutator. Let us denote such transfer matrix by $T(\lambda)$ on a $L \times L$ square lattice with toroidal boundary conditions. It follows that this operator can be written as the supertrace of an auxiliary operator called monodromy matrix [10],

$$T(\lambda) = \sum_{i=1}^{n+2m} (-1)^{p_i} \mathcal{T}_{ii}(\lambda) \tag{2.5}$$

where elements of the monodromy matrix \mathcal{T}_{ij} are given by an ordered product of R -matrices acting on the same auxiliary space but with distinct quantum space components,

$$\mathcal{T}(\lambda) = R_{0L}(\lambda) R_{0L-1}(\lambda) \dots R_{01}(\lambda) \tag{2.6}$$

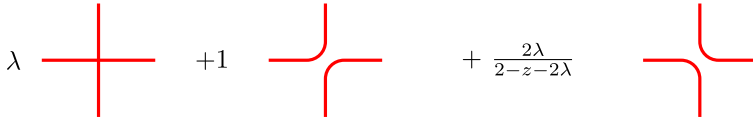


Fig. 1. The local configurations contributing to the partition function of the intersecting loop model with fugacity $z = n - 2m$ and their Boltzmann weights corresponding to the R -matrix (2.1).

As usual considering the logarithmic derivatives of $T(\lambda)$ around the regular point $\lambda = 0$ we obtain the local integrals of motion. The first non-trivial charge turns to be the Hamiltonian whose expression is

$$H = \epsilon \sum_{i=1}^L \left[P_{i,i+1} + \frac{2}{2-n+2m} E_{i,i+1} \right], \tag{2.7}$$

where periodic boundary conditions for both bosonic and fermionic degrees of freedom is assumed. The anti-ferromagnetic regime for $n - 2m < 2$ requires the choice $\epsilon = -1$ while for $n - 2m > 2$ we need to take $\epsilon = +1$.

The spectrum of this Hamiltonian can be studied by Bethe ansatz methods and is parameterized by solutions to a set of algebraic Bethe equations. Since these Bethe equations depend on the particular choice of the grading their root configurations are grading dependent. We can however infer on the infinite volume properties of such superspin chain without the need of choosing a specific Bethe ansatz solution [3]. This can be done establishing certain a functional relation for the largest eigenvalue of the transfer matrix usually by means of the matrix inversion method [12,13]. In our case this identity can be derived combining the unitarity property of the R -matrix (2.1) together with its crossing symmetry under translation $\lambda \rightarrow (2 - n + 2m)/2 - \lambda$ of the spectral parameter. Let us denote by $[\Lambda_0(\lambda)]^L$ the largest eigenvalue which dominates the partition function of the vertex model per site in the thermodynamic limit. We find that $\Lambda_0(\lambda)$ satisfies the following constraint,

$$\Lambda_0(\lambda)\Lambda_0\left(\lambda + \frac{2-n+2m}{2}\right) = \frac{(\lambda^2 - 1)\left(\lambda + \frac{2-n+2m}{2}\right)}{\lambda} \tag{2.8}$$

Using unitarity $\Lambda_0(\lambda)\Lambda_0(-\lambda) = (1 - \lambda^2)$ we can solve the above functional relation under the assumption of analyticity in the region $0 \leq \lambda < |2 - n + 2m|/2$. The final result is

$$\begin{aligned} \Lambda_0(\lambda) &= \left[\frac{(2-n+2m)^2}{\frac{|2-n+2m|}{2} - \lambda} \right] \\ &\times \frac{\Gamma\left(1 + \frac{\lambda}{|2-n+2m|}\right) \Gamma\left(\frac{1}{2} + \frac{1}{|2-n+2m|} + \frac{\lambda}{|2-n+2m|}\right) \Gamma\left(\frac{3}{2} - \frac{\lambda}{|2-n+2m|}\right)}{\Gamma\left(\frac{1}{2} + \frac{\lambda}{|2-n+2m|}\right) \Gamma\left(\frac{1}{|2-n+2m|} + \frac{\lambda}{|2-n+2m|}\right) \Gamma\left(1 - \frac{\lambda}{|2-n+2m|}\right)} \\ &\times \frac{\Gamma\left(1 + \frac{1}{|2-n+2m|} - \frac{\lambda}{|2-n+2m|}\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{|2-n+2m|} - \frac{\lambda}{|2-n+2m|}\right)} \end{aligned} \tag{2.9}$$

where $\Gamma(x)$ is the Euler’s integral of the second kind.

The ground state energy per site e_∞ of the $OSp(n|2m)$ spin chain (2.7) is obtained by taking the logarithmic derivative of $\Lambda_0(\lambda)$ at the spectral point $\lambda = 0$. After some simplifications we find

$$e_\infty = -\frac{2}{|2-n+2m|} \left[\psi \left(\frac{1}{2} + \frac{1}{|2-n+2m|} \right) - \psi \left(\frac{1}{|2-n+2m|} \right) + 2 \ln(2) \right] + 1 \tag{2.10}$$

where $\psi(x) = \frac{d \ln \Gamma(x)}{dx}$ is the Euler psi function.

The same reasoning as above can be used to obtain the dispersion relation for the low-lying excitations, see for instance Ref. [14]. These states correspond to next largest eigenvalues of the transfer matrix and their ratios with the ground state $[\Lambda_0(\lambda)]^L$ defines the excitation function $\gamma(\lambda)$. Considering that Eq. (2.8) applies also for the excitations such function is expected to satisfy the constraint $\gamma(\lambda)\gamma(\lambda + \frac{2-n+2m}{2}) = 1$. This means that $\gamma(\lambda)$ has the real period $|2-n+2m|$ and consequently it can be expressed in terms of product of trigonometric functions. We can now follow the reasoning discussed in [14] and conclude that the dispersion relation $e(p)$ for the low-lying excitations with momenta p is

$$e(p) = \frac{2\pi}{|2-n+2m|} \sin(p) \tag{2.11}$$

and therefore the speed of sound is $\xi = \frac{2\pi}{|2-n+2m|}$.

We would like to note that for the results so far it has implicitly been assumed that $n - 2m \neq 2$. For $n - 2m = 2$ we can not derive the Hamiltonian from the R-matrix (2.1) since there is no point λ_0 such that $R_{ab}(\lambda_0) \sim P_{ab}$. These are the cases in which the Killing form of the $OSp(n|2m)$ superalgebra is degenerated. One way to circumvent this problem is to scale the spectral parameter $\lambda \rightarrow \lambda(2-n+2m)/2$ and afterwards take the limit $n - 2m \rightarrow 2$ in Eq. (2.1) to obtain

$$\bar{R}_{ab}(\lambda) = P_{ab} + \frac{\lambda}{1-\lambda} E_{ab} \quad \text{for } n - 2m = 2 \tag{2.12}$$

which is the R-matrix of the so-called Temperley–Lieb model with $E_{ab}^2 = 2E_{ab}$ [13]. We note that in this case the respective loop model realization does not permit configurations involving intersecting paths since the identity operator is not present in the R-matrix (2.12). We further recall that for $n = 2$ and $m = 0$ the vertex model corresponds to the isotropic six-vertex model. The expression of the respective anti-ferromagnetic Hamiltonian is

$$\bar{H} = -\sum_{i=1}^L E_{i,i+1} \quad \text{for } n - 2m = 2 \tag{2.13}$$

The inversion method can also provide us with exact results for the vertex model with weights based on the R-matrix (2.12). It turns out that the respective partition function per site is

$$\bar{\Lambda}_0(\lambda) = \frac{2}{1-\lambda} \frac{\Gamma(1 + \frac{\lambda}{2}) \Gamma(\frac{3}{2} - \frac{\lambda}{2})}{\Gamma(1 - \frac{\lambda}{2}) \Gamma(\frac{1}{2} + \frac{\lambda}{2})} \quad \text{for } n - 2m = 2 \tag{2.14}$$

while the ground state energy and dispersion relation associated to the Hamiltonian (2.13) are

$$\bar{e}_\infty = -2 \ln(2) \quad \text{and} \quad \bar{e}(p) = \pi \sin(p) \quad \text{for } n - 2m = 2 \tag{2.15}$$

From the above results we conclude that the bulk behavior depends only on the loop model fugacity $z = n - 2m$. In next sections we shall present evidences that this feature still remains

valid for the central charge and for the compact part of the critical exponents of the underlying conformal field theory.

3. Small size results

In order to gain some insight on the spectrum properties of the $OSp(n|2m)$ superspin chain we have numerically diagonalized the respective Hamiltonians for lattice sizes $L \leq 8$. We have limited our analysis to Hamiltonians with maximum number of seven states per site $n + 2m = 7$. We find that the ground state is generically degenerated for spin chains with $n - 2m \leq 0$ while when $n - 2m \geq 1$ the ground state is always a singlet for L even. In Table 1 we present the ground state degeneracies for the $OSp(n|2m)$ spin chains studied in this paper for even and odd lattice sizes.

We have noted that for a fixed fugacity $n - 2m$ the eigenspectrum are basically the same apart degeneracies up to the size $L = 4$ for distinct values of n and m . Considerable number of new eigenvalues start to emerge for $L = 6$ but they occur at the higher energy part of the spectrum. These findings suggest that for large enough L the spectrum should satisfy the following sequence of inclusions,

$$\text{spec}[OSp(n|2m)] \subset \text{spec}[OSp(n+2|2m+2)] \subset \text{spec}[OSp(n+4|2m+4)] \subset \dots \quad (3.1)$$

such that the ground state and the low-lying excitations for a given fugacity $n - 2m$ is described by the superspin chain with the lowest possible values of the integers n and m . This feature is present in spin chains with different supergroup symmetries, e.g. for $gl(m|n)$ where a spectral embedding of models with given $m - n$ has been observed [15].

This above observation can be used in order to predict the value for the effective central charge. For $n - 2m \geq 2$ the sequence can be started with the orthogonal invariance $O(n - 2m)$ and the respective conformal field theory should be that of the Wess–Zumino–Witten model on this group see for instance [16,17]. The partition function is expected to be dominated by $n - 2m$ Ising degrees of freedom and therefore the central charge is

$$c_{\text{eff}} = (n - 2m)/2 \quad \text{for } n - 2m \geq 2 \quad (3.2)$$

On the other hand when $n - 2m < 2$ the orthogonal invariance is somehow broken and the partition function is effectively dominated by $n - 2m - 1$ bosonic degrees of freedom with effective central charge [3],

$$c_{\text{eff}} = n - 2m - 1 \quad \text{for } n - 2m < 2. \quad (3.3)$$

At this point we remark that in the context of the intersecting loop model these two regimes are distinguished by the behavior of the respective Boltzmann weights. We note that for $n - 2m < 2$ the three weights in Eq. (2.1) can be chosen positive and consequently they can be interpreted as probabilities. However when $n - 2m > 2$ one of the weights is always negative and the probability interpretation is lost. Therefore it is not a surprise that the continuum limit of these regimes are described by two different conformal field theories.

In next section we shall begin our study of the finite-size effects for large L for the superspin chains in Table 1 by using convenient grading choice for the Bethe ansatz solution. We will investigate two specific sequences of models with the same fugacity and argue that the potential extra eigenvalues does not lead to new conformal dimensions.

Table 1
Ground state energy degeneracies
for even and odd lattice sizes.

	even L	odd L
$OSp(1 2)$	3	3
$OSp(3 4)$	23	7
$OSp(2 2)$	8	4
$OSp(3 2)$	1	5
$OSp(2 4)$	16	32
$OSp(5 2)$	1	7

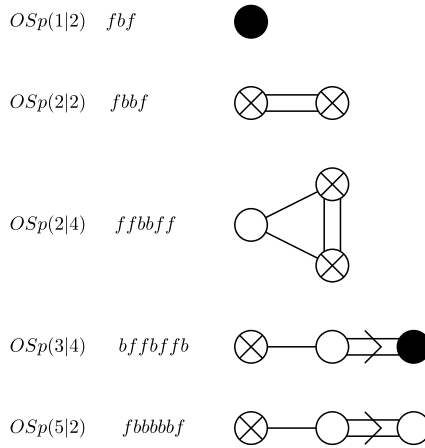


Fig. 2. The Dynkin diagram with the respective basis ordering for the superalgebras studied in this paper. The bosonic roots are represented by a white dot while the fermionic ones by a black dot or a crossed dot.

4. Finite size effects

In this section we will investigate the finite size properties of the super spin chains with the help of their Bethe ansatz solution. As mentioned above it is a common feature of integrable spin models based on super Lie algebras that the Bethe equations for the rapidities parameterizing their spectrum depend on the choice of grading. In a first step we have to choose the formulation which is most convenient for the numerical solution of the respective Bethe ansatz equations for large system sizes. By now it is well known that for rational vertex models there exists a direct connection between the form of the Bethe ansatz equations with the specific Dynkin diagram representation of the underlying superalgebra. In Fig. 2 we exhibit the diagrams with the respective grading ordering for the orthosymplectic superalgebras suitable for each super spin chain studied in this paper. The explicit form of the Bethe equations and the basic root distributions is presented in the next subsections.

Based on the numerical solution of the Bethe equations we can analyze the finite size scaling of the spectrum. For a conformally invariant theory the finite size gaps are expected to scale as [18,19]

$$X_{\text{eff}}(k; L) = \frac{L}{2\pi\xi} (E_k(L) - L\epsilon_\infty) \rightarrow X_k - \frac{c_{\text{eff}}}{12}, \tag{4.1}$$

where X_k are the scaling dimensions of the corresponding operator in the continuum limit and the effective central charge c_{eff} governs the finite size scaling of the ground state energy $E_0(L)$ of the lattice model. Similarly, from the momentum of the states the conformal spin of the corresponding operator can be determined, $s(k; L) = (L/2\pi)(P_k(L) - \mathcal{P}_0)$.

As we shall see below the spectrum of scaling dimensions of the $OSp(n|2m)$ models is highly degenerate in the thermodynamic limit. In a finite system this degeneracy is lifted by subleading corrections to scaling which can be studied in conformal perturbation theory [20,21]. With respect to the conformally invariant fixed point the lattice Hamiltonian of the isotropic $OSp(n|2m)$ superspin chains is perturbed by a marginally irrelevant operator. If the coupling constant g is initially small the effective coupling at scale L vanishes as $g(L) \sim 1/\log L$ and the corrections to scaling take the universal form

$$X(k; L) \simeq X_k + \frac{\beta(k)}{\log L}. \tag{4.2}$$

This logarithmic dependence on the system size requires information on the spectrum for large system sizes to reliably determine the scaling dimensions. Below we shall use this prediction to determine both them and the amplitudes of $\beta(k)$ extrapolating finite size data for lattice systems with up to several thousand sites based on the assumption that the corrections to scaling are rational functions of $1/\log L$.

4.1. $n - 2m = -2$: $OSp(2|4)$

For the $OSp(2|4)$ model it turns out to be most convenient to use the grading Bethe equations for the grading *ffbbff*

$$\begin{aligned} \left(\frac{\lambda_j^{(1)} + \frac{i}{2}}{\lambda_j^{(1)} - \frac{i}{2}} \right)^L &= \prod_{k=1, k \neq j}^{N_1} \frac{\lambda_j^{(1)} - \lambda_k^{(1)} + i}{\lambda_j^{(1)} - \lambda_k^{(1)} - i} \prod_{k=1}^{N_+} \frac{\lambda_j^{(1)} - \lambda_k^{(+)} - \frac{i}{2}}{\lambda_j^{(1)} - \lambda_k^{(+)} + \frac{i}{2}} \prod_{k=1}^{N_-} \frac{\lambda_j^{(1)} - \lambda_k^{(-)} - \frac{i}{2}}{\lambda_j^{(1)} - \lambda_k^{(-)} + \frac{i}{2}}, \\ j &= 1 \dots N_1, \\ \prod_{k=1}^{N_1} \frac{\lambda_j^{(+)} - \lambda_k^{(1)} + \frac{i}{2}}{\lambda_j^{(+)} - \lambda_k^{(1)} - \frac{i}{2}} &= \prod_{k=1}^{N_-} \frac{\lambda_j^{(+)} - \lambda_k^{(-)} + i}{\lambda_j^{(+)} - \lambda_k^{(-)} - i}, \quad j = 1 \dots N_+, \\ \prod_{k=1}^{N_1} \frac{\lambda_j^{(-)} - \lambda_k^{(1)} + \frac{i}{2}}{\lambda_j^{(-)} - \lambda_k^{(1)} - \frac{i}{2}} &= \prod_{k=1}^{N_+} \frac{\lambda_j^{(-)} - \lambda_k^{(+)} + i}{\lambda_j^{(-)} - \lambda_k^{(+)} - i}, \quad j = 1 \dots N_-. \end{aligned} \tag{4.3}$$

The number of Bethe roots on the three levels determine the eigenvalues of the conserved $U(1)$ charges from the Cartan subalgebra of $OSp(2|4)$. The energy of the state parameterized by a solution $\{\lambda_k^{(1)}\} \cup \{\lambda_k^{(+)}\} \cup \{\lambda_k^{(-)}\}$ to these equations is

$$E = L - \sum_{k=1}^{N_1} \frac{1}{\left(\lambda_k^{(1)}\right)^2 + \frac{1}{4}}. \tag{4.4}$$

The roots of (4.3) corresponding to the ground state and many of the low-lying excitations are found to be real with finite densities $N_1/L \rightarrow 1$, $N_{\pm}/L \rightarrow \frac{1}{2}$ in the thermodynamic limit. This fact allows to study their finite size scaling analytically based on linear integral equations

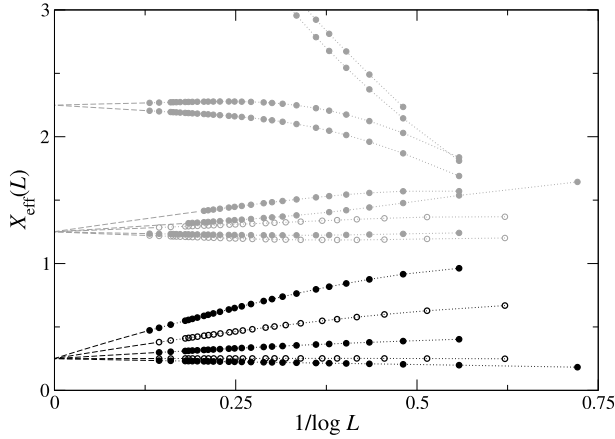


Fig. 3. Finite size spectrum of the $OSp(2|4)$ superspin chain. Displayed are the effective scaling dimensions $X_{\text{eff}}(L)$ vs. $1/\log L$. Black symbols denote levels from the lowest tower (4.8) of scaling dimensions in the sectors $(1, 2, k)$, $k = 0, 1, \dots, 4$, filled (open) symbols are data from the solution of the Bethe equations for chains of even (odd) length. Grey symbols are higher excitations. Dashed lines are extrapolations based on a rational dependence on $1/\log L$.

[22–24]. In the present case we find that the Bethe ansatz integral equations have a singular kernel, similar as in the staggered $sl(2|1)$ superspin chains and the staggered six-vertex model where this has been found to lead to a continuous spectrum of scaling dimensions [5,6,25,26]. Labeling the charge sectors of the model by the quantum numbers $n_1 = L - N_1$, $n_2 = L - N_+ - N_-$ and $n_3 = N_+ - N_-$ and the corresponding vorticities $m_{k=1,2,3}$ the resulting scaling dimensions of primary fields are

$$X_{\text{eff}}^{(2|4)}(n_k, m_k; L) \rightarrow \frac{1}{4} \left(n_1^2 + (n_1 - n_2)^2 + \epsilon n_3^2 \right) + \frac{1}{2} \left(m_1^2 + (m_1 + 2m_2)^2 + \frac{1}{\epsilon} m_3^2 \right) - \frac{1}{4} \tag{4.5}$$

and their conformal spin is $s(n_k, m_k) = \sum_k n_k m_k$. To derive (4.5) we have introduced the small parameter ϵ to regularize the singularity of the kernel. By construction the quantum numbers n_k are integers while the vorticities take integer or half-odd integer values according to the selection rules

$$m_1 \sim \frac{1}{2} n_2 \pmod{1}, \quad m_2 \sim \frac{1}{2} (n_1 - n_2 + 1) \pmod{1}. \tag{4.6}$$

m_3 is always integer. In the limit $\epsilon \rightarrow 0^+$ scaling dimensions with the same n_3 become degenerate while the vorticity m_3 is constraint to be 0 for states in the low energy spectrum (i.e. for operators with finite scaling dimension X). Taking these constraints into account we find that the conformal weights in the low energy effective theory are non-negative integers.

The ground state of the chain for even L appears in the sector with $(n_1, n_2, n_3) = (1, 2, 0)$ (and also $(1, 0, 0)$) and $m_1 = m_2 = 0$. Both from (4.5) and the extrapolation of data obtained by numerically solving (4.3) we find

$$X_{\text{eff}}^{(2|4)}(\mathbf{n} = (1, 2, 0); L) \rightarrow \frac{1}{2} - \frac{1}{4} = -\frac{c_{\text{eff}}}{12} \tag{4.7}$$

with the effective central charge $c_{\text{eff}} = -3$, in agreement with (3.3).

The lowest excitations in the sectors $(n_1, n_2, n_3) = (1, 2, k)$ with $|k| = 1, 2, 3, \dots \sim L \bmod 2$ are also described by real Bethe roots. Among these is the lowest energy state of the odd length super spin chains for $k = \pm 1$. As expected from (4.5) they exhibit the same leading finite size scaling as the ground state but different subleading corrections, see Fig. 3. From our numerical data based on the solution of the Bethe equations we find that the corrections to the scaling dimension of these states vanish as $1/\log L$, as expected from perturbative renormalization group analysis of the low temperature Goldstone phase of the loop models with $n - 2m < 2$ [7]. Analyzing the subleading corrections in detail we find

$$X^{(2|4)}((1, 2, k); L) \simeq \frac{\beta^{(2|4)}(k)}{\log L}, \quad \beta^{(2|4)}(k) = \frac{(k+1)(k-1)}{8}. \tag{4.8}$$

Similar groups of excitations parameterized by real Bethe roots appear in the sectors $(n_1, n_2, n_3) = (2, 3, k)$ and $(2, 4, k)$ with $n_2 + k \sim L \bmod 2$, see Fig. 3. The finite size analysis shows that the corresponding primaries have a scaling dimension $X_{(2,3,k)} = 1$ and $X_{(2,4,k)} = 2$, their conformal spins are $s = 1$ and $s = 0, 2$, respectively. Again, the subleading corrections to finite size scaling are found to vanish as $1/\log L$ with k -dependent amplitudes.

Among the remaining low energy levels in the spectrum of small systems found by exact diagonalization we have identified (see Fig. 3)

- a descendent of the ground state with $X = 1, s = 1$ in the $(n_1, n_2, n_3) = (1, 2, 0)$ sector described by a Bethe root configuration containing a single 2-string of complex conjugate Bethe roots $\lambda_{0\pm}^{(1)} \simeq \lambda_0 \pm i/2$ with real λ_0 in addition to the real ones;
- two states in the sectors $(2, 4, 0)$ and $(2, 4, 2)$ disappear from the low energy spectrum as the system size is increased. Such behavior is expected for levels violating the constraint $m_3 = 0$.

4.2. $n - 2m = -1$

For the $OSp(1|2)$ model the Bethe equations are [10,27]

$$\left(\frac{\lambda_j + \frac{i}{2}}{\lambda_j - \frac{i}{2}} \right)^L = \prod_{k=1}^{L-2n} \frac{\lambda_j - \lambda_k + i}{\lambda_j - \lambda_k - i} \frac{\lambda_j - \lambda_k - \frac{i}{2}}{\lambda_j - \lambda_k + \frac{i}{2}}, \quad j = 1 \dots L - 2n. \tag{4.9}$$

Solutions of these equations parameterize highest weight states for $(4n + 1)$ -dimensional irreducible representations of $OSp(1|2)$ with superspin $J = n, n = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$ and energy

$$E = L - \sum_{j=1}^{L-2n} \frac{1}{\lambda_j^2 + \frac{1}{4}}. \tag{4.10}$$

The operator content of the effective theory describing this superspin chain at low energies is known from Ref. [27], the primary fields have scaling dimensions

$$X_{\text{eff}}^{(1|2)}(n, m; L) \rightarrow n^2 + m^2 - \frac{1}{12} \tag{4.11}$$

for states with superspin $J = n$ and vorticity m subject to the constraint $(n + m) \in \mathbb{Z} + \frac{1}{2}$.¹ Hence, for the triplet ground state $(n, m) = (\frac{1}{2}, 0)$ we find the central charge $c_{\text{eff}} = -2$. The finite size

¹ Note that there exist highly excited states for which this constraint is violated.

scaling and finite temperature properties of the $OSp(1|2)$ superspin chain have recently been studied based on a formulation of the Bethe ansatz in terms of nonlinear integral equations [28]. As a consequence of the boundary conditions used in that work the effective central charge obtained from the low temperature behavior differs from the one appearing in the finite size scaling behavior of the ground state. We note that the ground state is degenerate (up to subleading corrections to scaling) with the lowest singlet, $(n, m) = (0, \frac{1}{2})$.

As discussed above, the spectrum of the $OSp(1|2)$ superspin chain is a subset of that of the $OSp(3|4)$ model. Therefore we discuss the finite size scaling in the context of the latter based on the Bethe ansatz for grading $bffbfbb$

$$\begin{aligned} \left(\frac{\lambda_j^{(1)} + \frac{i}{2}}{\lambda_j^{(1)} - \frac{i}{2}}\right)^L &= \prod_{k=1}^{N_2} \frac{\lambda_j^{(1)} - \lambda_k^{(2)} + \frac{i}{2}}{\lambda_j^{(1)} - \lambda_k^{(2)} - \frac{i}{2}}, \quad j = 1 \dots N_1, \\ \prod_{k=1}^{N_1} \frac{\lambda_j^{(2)} - \lambda_k^{(1)} + \frac{i}{2}}{\lambda_j^{(2)} - \lambda_k^{(1)} - \frac{i}{2}} &= \prod_{k=1, k \neq j}^{N_2} \frac{\lambda_j^{(2)} - \lambda_k^{(2)} + i}{\lambda_j^{(2)} - \lambda_k^{(2)} - i} \prod_{k=1}^{N_3} \frac{\lambda_j^{(2)} - \lambda_k^{(3)} - \frac{i}{2}}{\lambda_j^{(2)} - \lambda_k^{(3)} + \frac{i}{2}}, \quad j = 1 \dots N_2, \\ \prod_{k=1}^{N_2} \frac{\lambda_j^{(3)} - \lambda_k^{(2)} + \frac{i}{2}}{\lambda_j^{(3)} - \lambda_k^{(2)} - \frac{i}{2}} &= \prod_{k=1, k \neq j}^{N_3} \frac{\lambda_j^{(3)} - \lambda_k^{(3)} + i}{\lambda_j^{(3)} - \lambda_k^{(3)} - i} \frac{\lambda_j^{(3)} - \lambda_k^{(3)} - \frac{i}{2}}{\lambda_j^{(3)} - \lambda_k^{(3)} + \frac{i}{2}}, \quad j = 1 \dots N_3, \end{aligned} \tag{4.12}$$

where the corresponding state of the superspin chain has energy

$$E = -L + \sum_{k=1}^{N_1} \frac{1}{\left(\lambda_k^{(1)}\right)^2 + \frac{1}{4}}. \tag{4.13}$$

The ground state and low lying excitations of the model have root densities $N_i/L \rightarrow 1$ in the thermodynamic limit. We label the charge sectors of the $OSP(3|4)$ model by quantum numbers $(n_1, n_2, n_3) = (N_1 - N_2 + 1, N_2 - N_3 + 1, L - N_1 - 2)$.

The lowest energy states appear in the sectors $(n_1, n_2, n_3) = (1, 1, k)$ where $k = 0, 1, 2, \dots \sim L \bmod 2$. Their Bethe roots are arranged in $(L - k - 2)/2$ complex conjugate pairs

$$\lambda_{j,\pm}^{(1)} \simeq \lambda_j^{(1)} \pm \frac{5i}{4}, \quad \lambda_{j,\pm}^{(2)} \simeq \lambda_j^{(2)} \pm \frac{3i}{4}, \quad \lambda_{j,\pm}^{(3)} \simeq \lambda_j^{(3)} \pm \frac{i}{4},$$

and real centers $\lambda_j^{(a)}$. In the thermodynamic limit these states are degenerate, see Fig. 4. The levels with $k = 0, 1$ are the lowest and their energies coincide with those of the triplet ground state and the lowest singlet excitation in the spectrum of the $OSp(1|2)$ chain. Analyzing the subleading corrections to scaling for this class of levels we conjecture $(k = 0, 1, 2, \dots)$

$$X^{(3|4)}((1, 1, k); L) \simeq \frac{\beta^{(3|4)}(k)}{\log L}, \quad \beta^{(3|4)}(k) = \frac{2k^2 + 2k - 1}{12}. \tag{4.14}$$

A second tower of primaries with spin $s = 1$ levels extrapolating to $X^{(3|4)} = 1$ is found in the $OSP(3|4)$ sectors $(n_1, n_2, n_3) = (2, 1, k)$ for $k = 0, 1, 2, \dots \sim L - 1 \bmod 2$, see Fig. 4. In the corresponding Bethe root configurations one of the $N_1 = L - 2 - k$ roots on the first level is real. The lowest of these excitations, $k = 0$, is also present as a triplet in the spectrum of the $OSp(1|2)$. In addition there are descendants of the $(1, 1, k)$ primaries with scaling dimension $X = 1$.

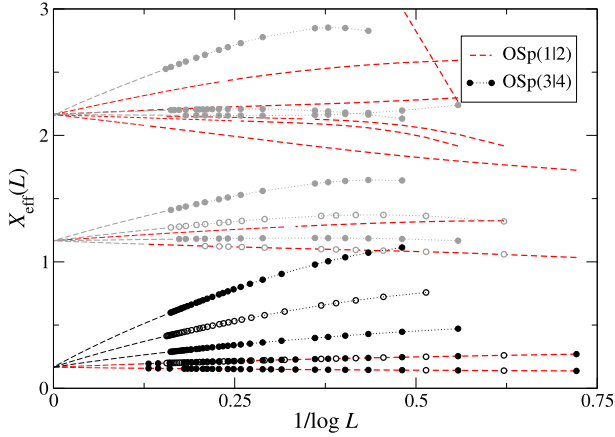


Fig. 4. Finite size spectrum of the $OSp(3|4)$ (symbols) superspin chain. Levels already present in the $OSp(1|2)$ model due to the inclusion (3.1) are marked by red dashed lines. Displayed are the effective scaling dimensions $X_{\text{eff}}(L)$ vs. $1/\log L$. Black symbols denote levels from the lowest tower (4.14) of scaling dimensions in the $OSp(3|4)$ sectors $(1, 1, k)$, $k = 0, 1, \dots, 4$. Filled (open) symbols are data chains of even (odd) length. Also shown are extrapolations based on a rational dependence on $1/\log L$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

The next excitations, both of the $OSp(1|2)$ and the $OSp(3|4)$ chain, for which we have determined the Bethe root configurations correspond to fields with scaling dimension $X = 2$. Their conformal spin is $s = 0$ or 2 .

4.3. $n - 2m = 0$

We study the spectrum of the $OSp(2|2)$ superspin chain using the Bethe equations in the grading $fbbf$:

$$\left(\frac{\lambda_j^{(1)} + \frac{i}{2}}{\lambda_j^{(1)} - \frac{i}{2}}\right)^L = \prod_{k=1}^{N_-} \frac{\lambda_j^{(1)} - \lambda_k^{(2)} + i}{\lambda_j^{(1)} - \lambda_k^{(2)} - i} \quad j = 1 \dots N_+, \tag{4.15}$$

$$\left(\frac{\lambda_j^{(2)} + \frac{i}{2}}{\lambda_j^{(2)} - \frac{i}{2}}\right)^L = \prod_{k=1}^{N_+} \frac{\lambda_j^{(2)} - \lambda_k^{(1)} + i}{\lambda_j^{(2)} - \lambda_k^{(1)} - i} \quad j = 1 \dots N_-.$$

Solutions to these equations parameterize states with energy

$$E = L - \sum_{k=1}^{N_+} \frac{1}{\left(\lambda_k^{(1)}\right)^2 + \frac{1}{4}} - \sum_{k=1}^{N_-} \frac{1}{\left(\lambda_k^{(2)}\right)^2 + \frac{1}{4}}. \tag{4.16}$$

The ground state and low energy excitations of the $OSp(2|2)$ superspin chain are described by real roots of (4.15) with densities $N_k/L \rightarrow 1/2$ in the thermodynamic limit, see [29]. Similarly as for the $OSp(2|4)$ chain above we use this fact to analytically compute the scaling dimensions of primary fields from the finite size spectrum. Introducing quantum numbers $n_1 = L - N_+ - N_-$ and $n_2 = N_+ - N_-$ for the $U(1)$ charges and regularizing the singularity of the Bethe ansatz kernel we find that the scaling dimensions of primaries are

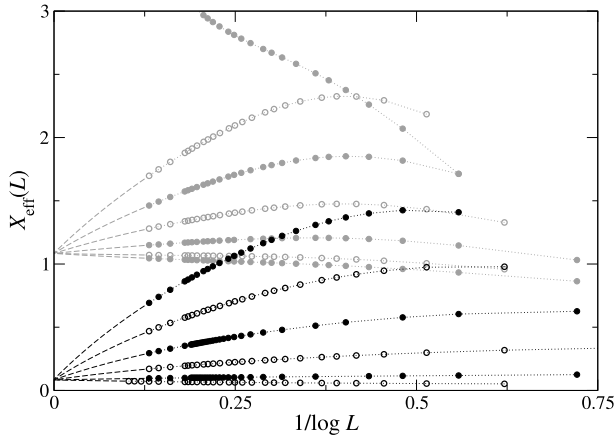


Fig. 5. As Fig. 3 but for the $OSp(2|2)$ superspin chain. Black symbols denote the levels from the lowest tower (4.20) of scaling dimensions in the sectors $(1, k)$, $k = 0, 1, \dots, 4$.

$$X_{\text{eff}}^{(2|2)}(n_k, m_k; L) \rightarrow \frac{1}{4} \left(n_1^2 + \epsilon n_2^2 \right) + \frac{1}{4} \left(m_1^2 + \frac{1}{\epsilon} m_2^2 \right) - \frac{1}{6}. \tag{4.17}$$

Their conformal spin is $s = n_1 m_1 + n_2 m_2$. Here, the charges $n_{1/2}$ are integers, the corresponding vorticities take values according to the selection rules

$$m_1 \sim L - n_1 \pmod{2}, \quad m_2 \in \mathbb{Z}. \tag{4.18}$$

For levels from the low energy spectrum in the thermodynamic limit (where the regularization constant $\epsilon \rightarrow 0^+$) the vorticity m_2 is constrained to be 0.

The lowest energy states appear in the sectors $n_1 = 1, m_1 = 0$ such that

$$X_{\text{eff}}^{(2|2)}(n_1 = 1, m_1 = 0; L) \rightarrow \frac{1}{4} - \frac{1}{6} = -\frac{c_{\text{eff}}}{12} \tag{4.19}$$

with the effective central charge of the $OSp(2|2)$ superspin chain, $c_{\text{eff}} = -1$. Scaling dimension and spin of the corresponding primary are $X = 0, s = 0$. The degeneracy of the scaling dimensions for levels with different n_2 is lifted for finite system sizes, see Fig. 5. Analyzing the subleading corrections to scaling of the $(n_1, n_2) = (1, k)$ states, $k = 0, 1, 2, \dots \sim L - 1 \pmod{2}$, we find a tower of levels:

$$X^{(2|2)}(\mathbf{n} = (1, k); L) \simeq \frac{\beta_k(2|2)}{\log L}, \quad \beta_k(2|2) = \frac{2k^2 - 1}{8}. \tag{4.20}$$

A tower of spin $s = 1$ excitations extrapolating to $X = 1$ is found in the sectors $(n_1, n_2) = (2, k)$ with $k = 0, 1, 2, \dots \sim L \pmod{2}$. A state with spin $s = 2$ in the sector $(2, 4)$ disappears from the low energy spectrum as the system size is increased since the restriction $m_2 = 0$ is violated.

4.4. $n - 2m = +1$

The finite size spectrum of the $OSp(3|2)$ superspin chain has been studied extensively using its solution by means of the algebraic Bethe ansatz in Ref. [4]. The ground state displays no

finite size corrections from it has been concluded that $c_{\text{eff}} = 0$. The excitations considered in that work can be grouped into towers extrapolating to integer scaling dimensions $X = 0, 1, 2, \dots$ or disappear from the low energy spectrum in the thermodynamic limit. The degeneracies in the spectrum of scaling dimensions is lifted for finite system sizes: with the exception of the ground state the finite size gaps show strong logarithmic corrections to scaling. For the levels in the $X = 0$ tower these corrections have been found to scale as

$$X_{\text{eff}}^{(3|2)}(k; L) \simeq 0 + \frac{\beta_k(3|2)}{\log L}, \quad \beta_k(3|2) = \frac{k(k-1)}{2} \tag{4.21}$$

4.5. $n - 2m = +3$

As mentioned above the continuum limit of the $OSp(n|2m)$ superspin chain for $n - 2m > 2$ is expected to be different from that for the cases discussed so far. Here first insights into the finite size spectrum can be obtained from the spectral inclusion $\text{spec}[O(3)] \subset \text{spec}[OSp(5|2)] \subset \dots$, see (3.1). The integrable $O(3)$ spin chain (or the spin $S = 1$ Takhtajan–Babujian model [30, 31]) is known to be a lattice realization of the $SU(2)$ Wess–Zumino–Witten–Novikov (WZNW) model at level $k = 2$ with central charge $c_{\text{eff}} = 3/2$ and spectrum of conformal weights $h \in \{j(j+1)/4 : j = 0, 1/2, 1\}$. Its primaries can be written as composite operators built from an Gaussian representing the Kac–Moody algebra with topological charge $k = 2$ and an Ising field [32]. The lowest scaling dimensions appearing in the lattice model of length L are

$$X^{O(3)} \in \begin{cases} \{\frac{3}{8}, 1, \dots\} & \text{for } L \text{ even} \\ \{\frac{3}{8}, \frac{1}{2}, 1, \dots\} & \text{for } L \text{ odd} \end{cases} \tag{4.22}$$

and higher descendents thereof. We note that the level with $X^{O(3)} = 1/2$ has conformal spin $s = 1/2$ and is therefore not realized in the spectrum of the even L chain. The corrections to scaling due to the marginally irrelevant perturbation of the conformal fixed point present in the lattice Hamiltonian have been computed in perturbation theory [33]. For the ground state this leads to logarithmic corrections to the central charge

$$c_{\text{eff}}^{O(3)}(L) \simeq \frac{3}{2} + \frac{3}{2(\log L)^3}, \tag{4.23}$$

while the finite size gap of the lowest triplet and singlet excitations are

$$\begin{aligned} X_{S=1}^{O(3)}(L) &\simeq \frac{3}{8} - \frac{1}{4} \frac{1}{\log L}, \\ X_{S=0}^{O(3)}(L) &\simeq \frac{3}{8} + \frac{3}{4} \frac{1}{\log L}. \end{aligned} \tag{4.24}$$

(Note that the singlet is realized in the spectrum of the $O(3)$ spin chain with an odd number of sites only.)

The additional levels in the spectrum of the $OSp(5|2)$ superspin chain can be studied based on its solutions by Bethe ansatz. For the grading $fbbbbb$ the Bethe equations read

$$\begin{aligned}
 \left(\frac{\lambda_j^{(1)} + \frac{i}{2}}{\lambda_j^{(1)} - \frac{i}{2}}\right)^L &= \prod_{k=1}^{N_2} \frac{\lambda_j^{(1)} - \lambda_k^{(2)} + \frac{i}{2}}{\lambda_j^{(1)} - \lambda_k^{(2)} - \frac{i}{2}}, & j = 1 \dots N_1, \\
 \prod_{k=1, k \neq j}^{N_2} \frac{\lambda_j^{(2)} - \lambda_k^{(2)} + i}{\lambda_j^{(2)} - \lambda_k^{(2)} - i} &= \prod_{k=1}^{N_1} \frac{\lambda_j^{(2)} - \lambda_k^{(1)} + \frac{i}{2}}{\lambda_j^{(2)} - \lambda_k^{(1)} - \frac{i}{2}} \prod_{k=1}^{N_3} \frac{\lambda_j^{(2)} - \lambda_k^{(3)} + \frac{i}{2}}{\lambda_j^{(2)} - \lambda_k^{(3)} - \frac{i}{2}}, & j = 1 \dots N_2, \\
 \prod_{k=1, k \neq j}^{N_3} \frac{\lambda_j^{(3)} - \lambda_k^{(3)} + \frac{i}{2}}{\lambda_j^{(3)} - \lambda_k^{(3)} - \frac{i}{2}} &= \prod_{k=1}^{N_2} \frac{\lambda_j^{(3)} - \lambda_k^{(2)} + \frac{i}{2}}{\lambda_j^{(3)} - \lambda_k^{(2)} - \frac{i}{2}}, & j = 1 \dots N_3.
 \end{aligned}
 \tag{4.25}$$

Solutions to these equations parameterize $OSp(5|2)$ highest weight states with energy

$$E = -L + \sum_{j=1}^{N_1} \frac{1}{(\lambda_j^{(1)})^2 + \frac{1}{4}}. \tag{4.26}$$

Unlike in the $OSp(n|2m)$ models with $n - 2m < 2$ discussed above the ground state of the superspin chain remains a unique singlet indicating the absence of a symmetry breaking transition into a low temperature phase of the loop models in this regime [7]. Labeling the charge sectors of the $OSp(5|2)$ chain with quantum numbers $(n_1, n_2, n_3) = (L - N_1, N_1 - N_2, N_2 - N_3)$ the lowest excitations above the ground state of the $O(3)$ chain are found in the sector with $(k, 0, 0)$, $k = 1, 2, 3, \dots \sim L \bmod 2$. The corresponding Bethe root configuration consists of $(L - k)/2$ pairs of complex conjugate roots on each level with complex parts²

$$\lambda_{j,\pm}^{(1)} \simeq \lambda_j^{(1)} \pm \frac{3i}{4}, \quad \lambda_{j,\pm}^{(2)} \simeq \lambda_j^{(2)} \pm \frac{i}{4}, \quad \lambda_{j,\pm}^{(3)} \simeq \lambda_j^{(3)} \pm \frac{i}{4},$$

and real centers $\lambda_j^{(a)}$, $a = 1, 2, 3$. The states with $k = 1$ and 2 appear as the triplet excitations in the spectrum of the $O(3)$ model for chains of odd and even length, respectively, and the state with $k = 3$ has the energy of the $O(3)$ singlet. The energy levels for $k > 3$ are not in the $O(3)$ part of the spectrum. In the thermodynamic limit, $L \rightarrow \infty$, they degenerate, see Fig. 6. For large but finite systems we find that this degeneracy is lifted as

$$X_{\text{eff}}^{(5|2)}((k, 0, 0); L) \simeq \frac{3}{8} + \frac{\beta_k(5|2)}{\log L}, \quad \beta_k(5|2) = \frac{2k^2 - 6k + 3}{4} \tag{4.27}$$

matching the known behavior (4.24) for the $O(3)$ levels $k = 1, 2, 3$.

Higher energy excitations for even length superspin chains have been found extrapolating to scaling dimensions $X = 1, 3/8 + 1, 2$, and $3/8 + 2$, see Fig. 6. The first of these is in the sector with $(n_1, n_2, n_3) = (1, 1, 0)$ and its energy is that of the zero-spin field with scaling dimensions $X = 1$ in the $O(3)$ model. Its root configuration differs from the one for the lowest tower by one root $\lambda^{(1)} = 0$ on the first level. The energy is that of the zero-spin field with scaling dimensions $X = 1$ in the $O(3)$ model.

We have investigated the corrections to finite size scaling of mostly spin zero levels in the $OSp(5|2)$ chain of even length up to some energy cutoff which including the first states extrapolating to $X = 3/8 + 2$. Among these we find no evidence for the existence of towers of dimensions except those starting at the descendents of the field with $X = 3/8$. This resembles

² For the level with $k = 1$ the pairs with real part closest to the origin are strongly deformed.

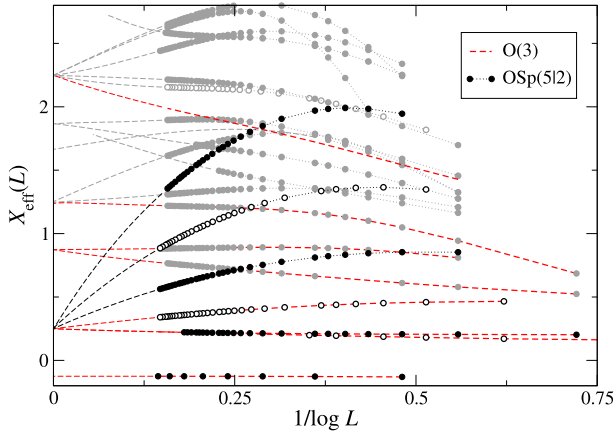


Fig. 6. As Fig. 4 but for the $OSp(5|2)$ superspin chain (symbols) and the $O(3)$ spin chain (red dashed lines). Black symbols denote the levels from the tower (4.27) of scaling dimensions in the $OSp(5|2)$ sectors $(k, 0, 0)$, $k = 1, 2, \dots, 6$.

the presence of both a continuous and a discrete part in the conformal spectrum of the $sl(2|1)$ superspin chain with alternating quark and antiquark representations and its deformation [5,25,34,35].

5. Discussion

In this paper we have studied the fine structure appearing in the finite size spectrum of the $OSp(n|2m)$ superspin chains. We find that the ground states of these models have a finite degeneracy for $n - 2m < 2$. For large finite system size L there exists a tower of scaling dimensions extrapolating to that of the identity operator, $X = 0$, and forming a continuum in the thermodynamic limit, $L \rightarrow \infty$.

For $n - 2m = 3$ (and most likely for all $n - 2m > 2$) the ground state of the superspin chain is a unique singlet, with the same energy as the ground state of the $O(3)$ spin chain. The low energy effective description of the $O(3)$ model is known to be the $SU(2)_2$ WZNW model, its lowest two excitations with scaling dimension $X = 3/8$ show strong subleading corrections to scaling (4.24) due to the presence of a marginally irrelevant perturbation in the lattice model. These levels are also present in the spectrum of the $OSp(5|2)$ chain, see Eq. (3.1). In addition, however, we have found continua of scaling dimensions to emerge starting at $X = 3/8$ and its descendents.

These observations are reminiscent of the appearance of a continuous component in the spectrum of scaling dimensions in staggered (super) spin chains [5,6,25,26]. There are some differences to our present findings though: the fine structure in the finite size spectrum of the staggered models has been argued to be a consequence of a non-compact degree of freedom in the low energy theory and the subleading gaps vanish quadratically with the inverse of $\log L$. This has to be contrasted to the linear dependence predicted from conformal field theory for the WZNW models with a marginal perturbation and observed in the towers of excitations of the $OSp(n|2m)$ models. Similarly, the corrections to scaling of the ground state of the staggered models due to a marginal perturbation by a *continuum* of excitations differ from (4.23) [36].

From the analysis of our data we have formulated conjectures for the amplitudes of the sub-leading (logarithmic) corrections to the lowest tower of excitations. For the $OSp(n|2m)$ models considered above these amplitudes are found to be quadratic functions of the single quantum

number labeling the different levels in the lowest tower. This suggests that they should be connected to the quadratic Casimir of the algebra underlying the superspin chain. This is similar to the case of the $O(3)$ model where (4.24) is derived by studying the effect of a marginal interaction of left and right Kac–Moody currents $\mathbf{J}_L \cdot \mathbf{J}_R$. In a multiplet with given left and right spin quantum numbers \mathbf{S}_L and \mathbf{S}_R this results in [33]

$$X_{\mathbf{S}_L+\mathbf{S}_R}^{O(3)} \simeq X_{WZW} - \frac{\mathbf{S}_L \cdot \mathbf{S}_R}{\log L}. \tag{5.1}$$

For the orthosymplectic algebras $OSp(n|2m)$ the eigenvalues of the Casimir operator on a highest weight vector $(\nu|\mu) = (\nu_1, \dots, \nu_{[n/2]}|\mu_1, \dots, \mu_m)$ are [37].

$$C_2^{(m|2n)}(\nu|\mu) = 2 \left\{ \sum_{a=1}^{[n/2]} \nu_a(\nu_a + n - 2m - 2a) - \sum_{\alpha=1}^m \mu_\alpha(\mu_\alpha + 2m + \ell_n - 2\alpha) \right\}, \tag{5.2}$$

where $\ell_n = 2$ (1) for n even (odd). The first term in this expression, $2\nu_1(\nu_1 + n - 2m - 2)$, is present in the Casimir for each of the algebras with given $n - 2m$ for $n > 1$. We note that, the amplitudes of the subleading logarithmic corrections measured relative to the first level with positive scaling dimension (i.e. after subtracting the smallest non-negative amplitude $\beta^{(n|2m)}$) appearing in the superspin chains with $n - 2m < 2$ for even L the amplitudes can be directly related to the corresponding Casimir eigenvalue:

$$\begin{aligned} OSp(2|4) : \quad \beta^{(2|4)}(k) - \frac{3}{8} &= \frac{(k+2)(k-2)}{8} = \frac{C_2^{(2|4)}(k+2|0,0)}{16}, \\ OSp(3|4) : \quad \beta^{(3|4)}(k) - \frac{1}{4} &= \frac{(k+2)(k-1)}{6} = \frac{C_2^{(3|4)}(k+2|0,0)}{12}, \\ OSp(2|2) : \quad \beta^{(2|2)}(k) - \frac{1}{8} &= \frac{(k+1)(k-1)}{4} = \frac{C_2^{(2|2)}(k+1|0)}{8}, \\ OSp(3|2) : \quad \beta^{(3|2)}(k) - 0 &= \frac{k(k-1)}{2} = \frac{C_2^{(3|2)}(k|0)}{4}. \end{aligned}$$

This can be compared to the case of $OSp(5|2)$ where the finite part of the scaling dimension for the lowest tower is already positive. Therefore, measuring $\beta^{(5|2)}$ relative to the smallest one we get

$$OSp(5|2) : \quad \beta^{(5|2)}(k) + \frac{1}{4} = \frac{(k-1)(k-2)}{2} = \frac{C_2^{(5|2)}(k-2|0,0)}{4}.$$

Being corrections to the scaling dimensions of primary field these amplitudes are expected to determine logarithmic corrections to correlation functions. Again, the scaling dimensions should be measured starting from the smallest non-negative one for the given model. For the $OSp(n|2m)$ models with $n - 2m < 2$ this predicts two-point functions of these fields to be

$$G_k^{(n|2m)}(r) \sim (\log r)^{-\alpha_k}, \quad \alpha_k = \frac{k(k+n-2m-2)}{2-n+2m} \tag{5.3}$$

For the correlation functions to decay at large distances k has to be restricted to $k > 2 - n + 2m$. Eq. (5.3) agrees with the exponents for k -leg watermelon correlators proposed for the Goldstone phase of intersecting loop models with fugacity $n - 2m < 2$ and studied numerically using Monte Carlo simulations for $n - 2m = 1$ [8,9].

Eq. (5.3) leads us to propose the existence of a family of fields in the $O\text{Sp}(5|2)$ model whose two-point correlation functions feature a multiplicative logarithmic correction to the algebraic behavior expected from conformal field theory, i.e.

$$G_k^{(5|2)}(r) \sim \frac{1}{r^{3/4}} (\log r)^{-k^2-k+1/2} \quad k \geq 0. \quad (5.4)$$

We note that $G_{k=0}^{(5|2)}(r)$ is the spin–spin correlation function in the $O(3)$ model [33].

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