Cluster tilting vs. weak cluster tilting in Dynkin type A infinity

Thorsten Holm and Peter Jørgensen

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Abstract. This paper shows a new phenomenon in higher cluster tilting theory. For each positive integer d, we exhibit a triangulated category C with the following properties.

On the one hand, the d-cluster tilting subcategories of C have very simple mutation behaviour: Each indecomposable object has exactly d mutations. On the other hand, the weakly d-cluster tilting subcategories of C which lack functorial finiteness can have much more complicated mutation behaviour: For each $0 \leq \ell \leq d - 1$, we show a weakly d-cluster tilting subcategory T_ℓ which has an indecomposable object with precisely ℓ mutations.

The category C is the algebraic triangulated category generated by a $(d + 1)$-spherical object and can be thought of as a higher cluster category of Dynkin type A_∞.

Keywords. Auslander–Reiten quiver, d-Calabi–Yau category, d-cluster tilting subcategory, Fomin–Zelevinsky mutation, functorial finiteness, left-approximating subcategory, right-approximating subcategory, spherical object, weakly d-cluster tilting subcategory.

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1 Introduction

This paper shows a new phenomenon in higher cluster tilting theory. For each integer $d \geq 1$, we exhibit a triangulated category C whose d-cluster tilting subcategories have *very simple* mutation behaviour, but whose weakly d-cluster tilting subcategories can have *much more complicated* mutation behaviour which we can control precisely.

To make sense of this, recall that if T is a full subcategory of a triangulated category, then T is called *weakly d-cluster tilting* if it satisfies the following conditions where Σ is the suspension functor:

$$t \in \mathsf{T} \quad \Longleftrightarrow \quad \operatorname{Hom}(\mathsf{T}, \Sigma t) = \cdots = \operatorname{Hom}(\mathsf{T}, \Sigma^d t) = 0,$$

$$t \in \mathsf{T} \quad \Longleftrightarrow \quad \operatorname{Hom}(t, \Sigma \mathsf{T}) = \cdots = \operatorname{Hom}(t, \Sigma^d \mathsf{T}) = 0.$$

If T is also left- and right-approximating in the ambient category in the sense of Remark 3.3, then it is called d-*cluster tilting*. These definitions are due to Iyama [6] and have given rise to an extensive homological theory, see for instance [1] and [7]. Note that if $\mathsf{T} = \operatorname{add} t$ for an object t, then T is automatically left- and right-approximating, but we will study subcategories which are not of this form since they have infinitely many isomorphism classes of indecomposable objects.

One remarkable property of d-cluster tilting theory is *mutation*. If $t \in \mathsf{T}$ is an indecomposable object, then it is sometimes possible to remove t from T and insert an indecomposable object $t^* \not\cong t$ in such a way that the subcategory remains (weakly) d-cluster tilting. This is called *mutation of* T *at* t, see [7, Section 5].

In good cases, there are exactly d different choices of t^* up to isomorphism. That is, there are d ways of mutating T at t, see [7, Section 5].

To be more precise, one hopes(!) that this happens for d-cluster tilting subcategories. Indeed, it does happen for $d = 1$ by [7, Theorem 5.3], but can fail for $d \geq 2$, see [7, Theorems 9.3 and 10.2]. The situation for weakly d-cluster tilting subcategories is less clear.

We can now explain the opening paragraph of the paper. Let us first define C which, as we will explain below, can be thought of as a d-cluster category of type A_∞.

Definition 1.1. For the rest of the paper, k is an algebraically closed field, $d \geq 1$ is an integer, and C is a k-linear algebraic triangulated category which is idempotent complete and classically generated by a $(d + 1)$-spherical object s; that is,

$$\dim_k \mathsf{C}(s, \Sigma^\ell s) = \begin{cases} 1 & \text{for } \ell = 0, d + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\mathsf{C}(-, -)$ is short for the Hom functor in C.

We prove the following three theorems about C, where Theorems A and B show *very simple*, respectively *much more complicated* mutation behaviour.

Theorem A. *Let* T *be a* d-*cluster tilting subcategory of* C *and let* $t \in \mathsf{T}$ *be indecomposable. Then* T *can be mutated at* t *in precisely* d *ways.*

Theorem B. *Let* $0 \leq \ell \leq d - 1$ *be given. Then there exists a weakly* d-*cluster tilting subcategory* T_ℓ *of* C *with an indecomposable object* t *such that* T_ℓ *can be mutated at* t *in precisely* ℓ *ways.*

Theorem C. *Let* T *be a weakly* d-*cluster tilting subcategory of* C *and let* $t \in \mathsf{T}$ *be indecomposable. Then* T *can be mutated at* t *in at most* d *ways.*
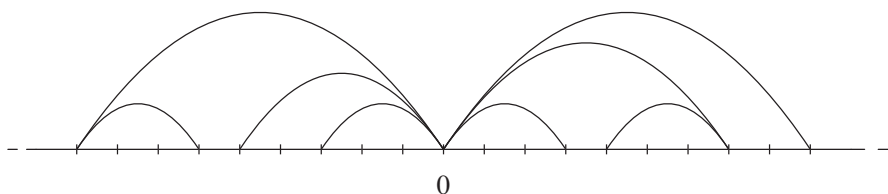
Figure 1. Part of a 4-angulation of the ∞-gon.

The interest of Theorems A and C depends on a rich supply of (weakly) d-cluster tilting subcategories in C. Indeed, such a supply exists by the following two theorems. As a prelude, note that there is a bijection between subcategories $\mathsf{T} \subseteq \mathsf{C}$ closed under direct sums and summands, and sets of d-admissible arcs $\mathfrak{T}$; see Section 2, in particular Proposition 2.4. A d-admissible arc is an arc in the upper half plane connecting two integers t, u with $u - t \geq 2$ and $u - t \equiv 1 \pmod{d}$.

Theorem D. *The subcategory* T *is weakly d-cluster tilting if and only if the corresponding set of d-admissible arcs* $\mathfrak{T}$ *is a* $(d + 2)$-*angulation of the* ∞-*gon.*

Theorem E. *The subcategory* T *is d-cluster tilting if and only if the corresponding set of d-admissible arcs* $\mathfrak{T}$ *is a* $(d + 2)$-*angulation of the* ∞-*gon which is either locally finite or has a fountain.*

We defer the definition of "$(d + 2)$-angulation of the ∞-gon" and other unexplained notions to Definition 2.3 and merely offer Figure 1 which shows part of a 4-angulation of the ∞-gon with a fountain at 0.

Note how the arcs divide the upper half plane into a collection of 'quadrangular' regions, each with four integers as 'vertices'. Some of the vertices sit at cusps.

We end the introduction with a few remarks about the category C which has been studied intensively in a number of recent papers [2, 4, 5, 8, 9, 11, 13]. It is determined up to triangulated equivalence by [9, Theorem 2.1]. It is a Krull–Schmidt and $(d + 1)$-Calabi–Yau category by [5, Remark 1 and Proposition 1.8], and a number of other properties can be found in [5, Sections 1 and 2]. Theorems A and E are two reasons for viewing C as a cluster category of type A_∞, since they are infinite versions of the corresponding theorems in type A_n; see [12, Theorem 3] for Theorem A and [10, Proposition 2.13] and [12, Theorem 1] for Theorem E. See also [4] for the case $d = 1$.

The paper is organised as follows: Section 2 introduces d-admissible arcs into the study of the triangulated category C and proves Theorem D. Section 3 proves Theorem E. Section 4 shows some technical results on $(d + 2)$-angulations of the ∞-gon. Section 5 proves Theorems A, B and C.

Notation 1.2. We write ind(C) for the set of isomorphism classes of indecomposable objects in C. We will follow the custom of being lax about the distinction between *indecomposable objects* and *isomorphism classes of indecomposable objects*. This makes the language a bit less precise, but avoids excessive elaborations.

The word *subcategory* will always mean *full subcategory closed under isomorphisms, direct sums, and direct summands*. In particular, a subcategory is determined by the indecomposable objects it contains.

2   The arc picture of C

Remark 2.1. By [5, Proposition 1.10], the Auslander–Reiten (AR) quiver of C consists of d components, each of which is a copy of $\mathbb{Z}A_\infty$, and Σ acts cyclically on the set of components.

Construction 2.2. We pick a component of the AR quiver of C and impose the coordinate system in Figure 2.
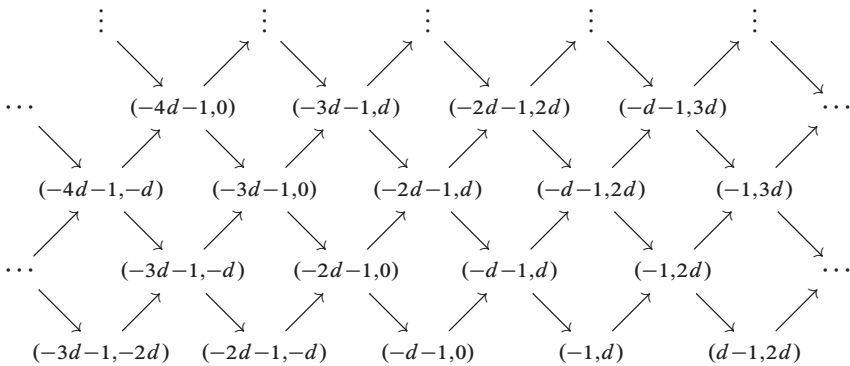

Figure 2. The coordinate system on one of the components of the AR quiver of C.

We think of coordinate pairs as indecomposable objects of C, and extend the coordinate system to the other components of the quiver by setting

$$\Sigma(t, u) = (t - 1, u - 1). \tag{2.1}$$

By [5, Proposition 1.8], the Serre functor of C is $S = \Sigma^{d+1}$. The actions of S and the AR translation $\tau = S\Sigma^{-1}$ are given on objects by

$$S(t, u) = (t - d - 1, u - d - 1), \quad \tau(t, u) = (t - d, u - d). \tag{2.2}$$

Like Σ, the Serre functor S acts cyclically on the set of components of the AR quiver. Indeed, since there are d components, the two functors have the same action on the set of components. The AR translation τ is given on each component of the AR quiver by moving one vertex to the left.

We also think of the coordinate pair (t, u) as an arc in the upper half plane connecting the integers t and u. The ensuing geometrical picture is illustrated by Figure 1. However, not all values of (t, u) are possible. Indeed, it is easy to check that the coordinate pairs which occur in Construction 2.2 are precisely the d-admissible arcs in the following definition.

Definition 2.3. A pair of integers (t, u) with $u - t \geq 2$ and $u - t \equiv 1 \pmod{d}$ is called a *d-admissible arc*.

The *length* of the arc (t, u) is $u - t$.

The arcs (r, s) and (t, u) *cross* if $r < t < s < u$ or $t < r < u < s$. Moreover, (r, s) is an *overarc* of (t, u) if $(r, s) \neq (t, u)$ and $r \leq t < u \leq s$.

Let $\mathfrak{T}$ be a set of d-admissible arcs.

We say that $\mathfrak{T}$ is a $(d + 2)$-*angulation of the ∞-gon* if it is a maximal set of pairwise non-crossing d-admissible arcs.

We say that $\mathfrak{T}$ is *locally finite* if, for each integer t, there are only finitely many arcs of the form (s, t) and (t, u) in $\mathfrak{T}$.

An integer t is a *left-fountain of $\mathfrak{T}$* if $\mathfrak{T}$ contains infinitely many arcs of the form (s, t), and t is a *right-fountain of $\mathfrak{T}$* if $\mathfrak{T}$ contains infinitely many arcs of the form (t, u). We say that t is a *fountain of $\mathfrak{T}$* if it is both a left- and a right-fountain of $\mathfrak{T}$.

The first part of the following proposition is a consequence of what we did above. The second part follows from the first because our subcategories are determined by the indecomposable objects they contain, see Notation 1.2.

Proposition 2.4. *Construction* 2.2 *above gives a bijective correspondence between* $\mathrm{ind}(\mathsf{C})$ *and the set of d-admissible arcs.*

This extends to a bijective correspondence between (i) *subcategories of* C *and* (ii) *subsets of the set of d-admissible arcs.*

Definition 2.5. Let $x \in \mathrm{ind}(\mathsf{C})$ be given. Figure 3 defines two infinite sets $F^{\pm}(x)$ consisting of vertices in the same component of the AR quiver as x. Each set contains x and all other vertices inside the indicated boundaries; the boundaries are included in the sets.

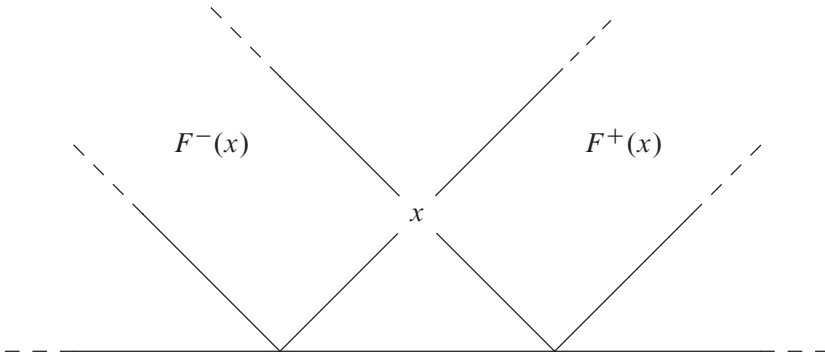Recall that $S = \Sigma^{d+1}$ is the Serre functor of C.

Figure 3. The sets $F^{\pm}(x)$.

Proposition 2.6. *Let* $x, y \in \mathrm{ind}(\mathsf{C})$. *Then*

$$\dim_k \mathsf{C}(x, y) = \begin{cases} 1 & \text{for } y \in F^+(x) \cup F^-(Sx), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. See [5, Proposition 2.2]. □

In other words, x has non-zero maps to a region $F^+(x)$ in the same component of the AR quiver as itself, and to a region $F^-(Sx)$ in the "next" component of the AR quiver. Note that if $d = 1$, then the quiver has only one component so $F^-(Sx)$ is in the same component as x.

Remark 2.7. It is not hard to check that $y \in F^+(x) \Leftrightarrow x \in F^-(y)$. So the proposition is equivalent to

$$\dim_k \mathsf{C}(x, y) = \begin{cases} 1 & \text{for } x \in F^+(S^{-1}y) \cup F^-(y), \\ 0 & \text{otherwise.} \end{cases}$$

The following proposition is simple but crucial since it leads straight to Theorem D.

Proposition 2.8. *Let* $\mathfrak{x}, \mathfrak{y}$ *be* d-*admissible arcs corresponding to* $x, y \in \mathrm{ind}(\mathsf{C})$. *Then* $\mathfrak{x}$ *and* $\mathfrak{y}$ *cross if and only if at least one of the* Hom-*spaces*

$$\mathsf{C}(x, \Sigma^1 y), \ldots, \mathsf{C}(x, \Sigma^d y)$$

is non-zero.

Proof. For $1 \leq \ell \leq d$, the condition that

$$\mathsf{C}(x, \Sigma^\ell y) \neq 0$$

is equivalent to $\Sigma^\ell y \in F^+(x)$ or $\Sigma^\ell y \in F^-(Sx)$ by Proposition 2.6. If we write $\mathfrak{x} = (r, s)$, $\mathfrak{y} = (t, u)$, then, using equations (2.1) and (2.2) and the coordinate system on the AR quiver of C, it is elementary to check that

$$\Sigma^\ell y \in F^+(x) \quad \Longleftrightarrow \quad \begin{cases} u \equiv s + \ell \pmod{d}, \\ r + \ell \leq t \leq s + \ell - d - 1, \\ s + \ell \leq u, \end{cases} \tag{2.3}$$

$$\Sigma^\ell y \in F^-(Sx) \quad \Longleftrightarrow \quad \begin{cases} u \equiv s + \ell - 1 \pmod{d}, \\ t \leq r + \ell - d - 1, \\ r + \ell \leq u \leq s + \ell - d - 1. \end{cases} \tag{2.4}$$

The condition that at least one of the Hom spaces $\mathsf{C}(x, \Sigma^1 y), \ldots, \mathsf{C}(x, \Sigma^d y)$ is non-zero is hence equivalent to the existence of at least one ℓ with $1 \leq \ell \leq d$ such that the right hand side of (2.3) or (2.4) is true. It is again elementary to check that this is equivalent to the condition that $\mathfrak{x} = (r, s)$ and $\mathfrak{y} = (t, u)$ cross. □

Proof of Theorem D. Combine the definition of weakly d-cluster tilting subcategories with Propositions 2.4 and 2.8. □

3 Left- and right-approximating subcategories

Proposition 3.1. *Let* $x, y \in \mathrm{ind}(\mathsf{C})$ *be such that* $y \in F^+(x)$.

(i) *Each morphism* $x \to y$ *is a scalar multiple of a composition of irreducible morphisms.*

(ii) *A morphism* $x \to y$ *which is a composition of irreducible morphisms is non-zero.*

Keeping x, y *as above, let* $z \in \mathrm{ind}(\mathsf{C})$ *be such that* $z \in F^+(x) \cap F^+(y)$.

(iii) *Non-zero morphisms* $x \to y$, $y \to z$ *compose to a non-zero morphism* $x \to z$.

(iv) *If* $y \xrightarrow{\psi} z$ *is a non-zero morphism, then each morphism* $x \to z$ *factors as*

$$x \to y \xrightarrow{\psi} z.$$

Proof. (i) Let $x \xrightarrow{\varphi} y$ be a morphism. If $y \cong x$, then it follows from Proposition 2.6 that φ is a scalar multiple of the identity, and then we can take the claimed composition of irreducible morphisms to be empty.
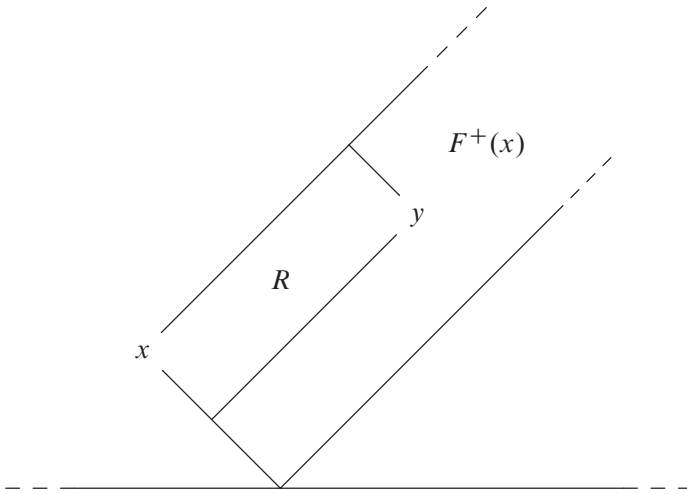
Figure 4. The rectangle R spanned by x and y.

If $y \ncong x$, then let $\tau y \to y_1 \xrightarrow{\theta} y$ be the AR triangle ending in y. Since x, y are indecomposable, the morphism φ is not a split epimorphism so it factors as

$$x \to y_1 \xrightarrow{\theta} y.$$

We can repeat this factorization process for the direct summands of y_1 to which x has non-zero morphisms, that is, the direct summands of y_1 which are in the rectangle R shown in Figure 4; cf. Proposition 2.6.

Successive repetitions show that the morphism φ is a linear combination of compositions of irreducible morphisms within R. However, the mesh relations imply that any two such compositions are scalar multiples of each other, so φ is a scalar multiple of a composition of irreducible morphisms.

(ii) By Proposition 2.6 there is a non-zero morphism $x \to y$. By part (i), it is a scalar multiple of a composition of irreducible morphisms. But as remarked in the proof of part (i), two morphisms $x \to y$ which are both compositions of irreducible morphisms are scalar multiples of each other, so it follows that any such composition is non-zero.

(iii) By part (i), each of the morphisms $x \to y$ and $y \to z$ is a scalar multiple of a composition of irreducible morphisms, so the same is true for the composition $x \to z$. But z is in $F^+(x)$, so $x \to z$ is non-zero by part (ii).

(iv) By Proposition 2.6 there is a non-zero morphism

$$x \xrightarrow{\varphi} y.$$

The composition $x \xrightarrow{\psi\varphi} z$ is non-zero by part (iii). But the space $\mathsf{C}(x, z)$ is 1-dimensional by Proposition 2.6, so any morphism $x \to z$ can be factored as $\psi \circ \alpha\varphi$ with α a scalar. □

Proposition 3.2. *Let* $x, y, z \in \mathrm{ind}(\mathsf{C})$ *be such that* $x \in F^+(S^{-1}y) \cap F^+(S^{-1}z)$ *and* $z \in F^+(y)$. *If*

$$y \xrightarrow{\psi} z$$

is a non-zero morphism, then each morphism $x \to z$ *factors as*

$$x \to y \xrightarrow{\psi} z.$$

Proof. We must show that $\mathsf{C}(x, \psi) : \mathsf{C}(x, y) \to \mathsf{C}(x, z)$ is surjective. By Serre duality, it is equivalent to show that $\mathsf{C}(\psi, Sx) : \mathsf{C}(z, Sx) \to \mathsf{C}(y, Sx)$ is injective. For this it is enough to show that $\mathsf{C}(\psi, Sx)$ is non-zero, since the Hom spaces $\mathsf{C}(z, Sx)$ and $\mathsf{C}(y, Sx)$ have dimension 0 or 1 over the ground field k by Proposition 2.6.

We must hence show that if $z \to Sx$ is non-zero, then so is the composition

$$y \xrightarrow{\psi} z \to Sx.$$

This holds by Proposition 3.1 (iii) since we have

$$z \in F^+(y) \quad \text{and} \quad Sx \in F^+(y) \cap F^+(z);$$

the latter condition holds because it is equivalent to

$$x \in F^+(S^{-1}y) \cap F^+(S^{-1}z).$$ □

Remark 3.3. Recall that if S is a subcategory of C and $x \in \mathsf{C}$ is an object, then a right-S-approximation of x is a morphism $s \xrightarrow{\sigma} x$ with $s \in \mathsf{S}$ such that each morphism $s' \to x$ with $s' \in \mathsf{S}$ factors through σ.

If each $x \in \mathsf{C}$ has a right-S-approximation, then S is called right-approximating. There are dual notions with "left" instead of "right".

The following is a generalization of [4, Theorem 4.4] and [11, Theorem 2.2], and we follow the proofs of those results.

Proposition 3.4. *Let* S *be a subcategory of* C *and let* $\mathfrak{S}$ *be the corresponding set of* d-*admissible arcs. The following conditions are equivalent:*

(i) *The subcategory* S *is right-approximating.*

(ii) *Each right-fountain of* $\mathfrak{S}$ *is a left-fountain of* $\mathfrak{S}$.

Proof. For $d = 1$ this is [11, Theorem 2.2] so assume $d \geq 2$.

Recall the notion of a slice: If (t, u) is a vertex on the base line of the AR quiver of C, then the slice starting at (t, u) is $(t, *)$; that is, it consists of the vertices with coordinates of the form (t, u'). The slice ending at (t, u) is $(*, u)$.

This means that $t \in \mathbb{Z}$ is a right-fountain of $\mathfrak{S}$ if and only if S has infinitely many indecomposable objects on the slice $(t, *)$ starting at $(t, t + d + 1)$. Likewise, t is a left-fountain of $\mathfrak{S}$ if and only if S has infinitely many indecomposable objects on the slice $(*, t)$ ending at $(t - d - 1, t) = S(t, t + d + 1)$. Hence (ii) is equivalent to the following condition on S.

(ii') Let $v \in \text{ind}(\mathsf{C})$ be on the base line of the AR quiver of C. If S has infinitely many indecomposable objects on the slice starting at v, then it has infinitely many indecomposable objects on the slice ending at Sv.

Strictly speaking, we should say "infinitely many isomorphism classes of indecomposable objects" but as mentioned in Notation 1.2 we are lax about this.

(i) $\Rightarrow$ (ii') Let $v \in \text{ind}(\mathsf{C})$ be on the base line of the AR quiver. Note that v and Sv are in different components of the AR quiver since there are $d \geq 2$ components and S moves vertices to the "next" component; cf. Construction 2.2. Figure 5 shows the components of the quiver containing v and Sv. As indicated, b is the slice starting at v and a the slice ending at Sv. Assume that (i) holds and that $\text{ind}(\mathsf{S}) \cap b$ is infinite. To show (ii'), we must show that $\text{ind}(\mathsf{S}) \cap a$ is infinite.

Let z be an indecomposable object on a with right-S-approximation $s \xrightarrow{\sigma} z$. If s_1 is an object on b, then as shown by outlines in the figure we have $z \in F^-(Ss_1)$. Hence there is a non-zero morphism $s_1 \to z$ by Proposition 2.6. So each of the infinitely many objects in $\text{ind}(\mathsf{S}) \cap b$ has a non-zero morphism to z, and each such morphism factors through σ because σ is a right-S-approximation. Since C is a Krull–Schmidt category, this implies that there is an indecomposable direct summand s' of s such that the component

$$s' \xrightarrow{\sigma'} z$$

of σ is non-zero and such that there are infinitely many objects $s_1, s_2, s_3, \ldots$ in $\text{ind}(\mathsf{S}) \cap b$ which have non-zero morphisms to s'. Note that $s' \in \mathsf{S}$ since S is closed under direct summands by assumption.

We claim that this forces s' to be on a, higher up than z. Hence, by moving z upwards we obtain infinitely many objects in $\text{ind}(\mathsf{S}) \cap a$.

To prove the claim, note that since

$$s' \xrightarrow{\sigma'} z$$

is non-zero, Remark 2.7 gives $s' \in F^+(S^{-1}z) \cup F^-(z)$. The sets $F^+(S^{-1}z)$ and $F^-(z)$ are outlined in Figure 5. And $s' \in F^+(S^{-1}z)$ is impossible because there
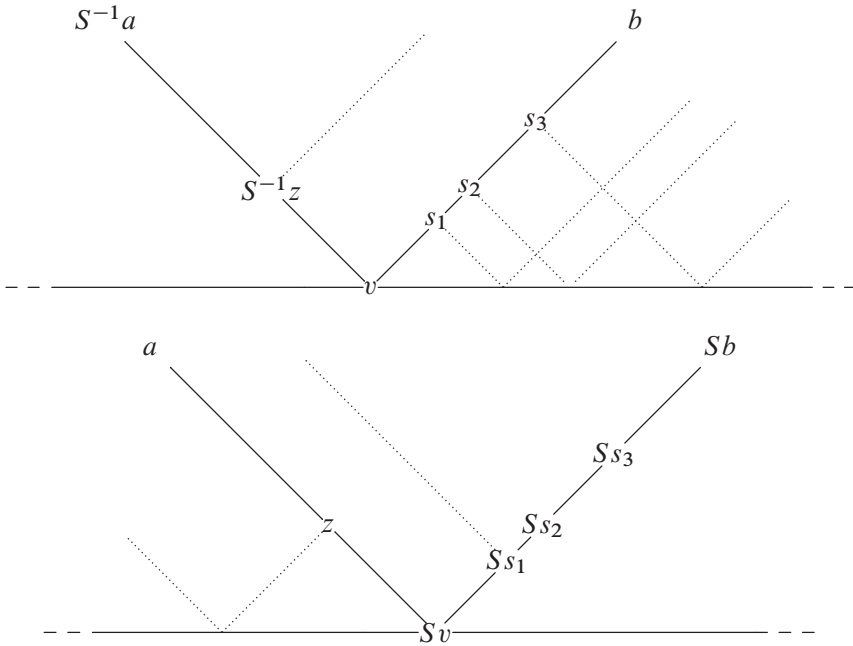
Figure 5. Objects in two components of the AR quiver.

would not be infinitely many objects in $\mathrm{ind}(\mathsf{S}) \cap b$ with a non-zero morphism to s', as one sees by considering the sets $F^+(s_i)$ which are also outlined in the figure.

So we have $s' \in F^-(z)$. We already know $s' \in F^-(Ss_i)$ for each i. Hence, as one sees in Figure 5, we have s' on a. Finally, since there is a non-zero morphism

$$s' \xrightarrow{\sigma'} z,$$

it follows that s' is higher up on a than z.

(ii') $\Rightarrow$ (i) Assume that (ii') holds and that $z \in \mathrm{ind}(\mathsf{C})$ is given. We will show (i) by constructing a right-S-approximation

$$s \xrightarrow{\sigma} z.$$

We must ensure that each morphism $s' \to z$ with $s' \in \mathsf{S}$ factors through σ, and we will do so by considering the possibilities for s' and building up σ accordingly.

We only need to consider those $s' \in \mathrm{ind}(\mathsf{S})$ which have non-zero morphisms to z. By Remark 2.7 there are the cases $s' \in F^-(z)$ and $s' \in F^+(S^{-1}z)$, see Figure 6. Note that z and $S^{-1}z$ are in different components of the AR quiver; cf. the previous part of the proof.
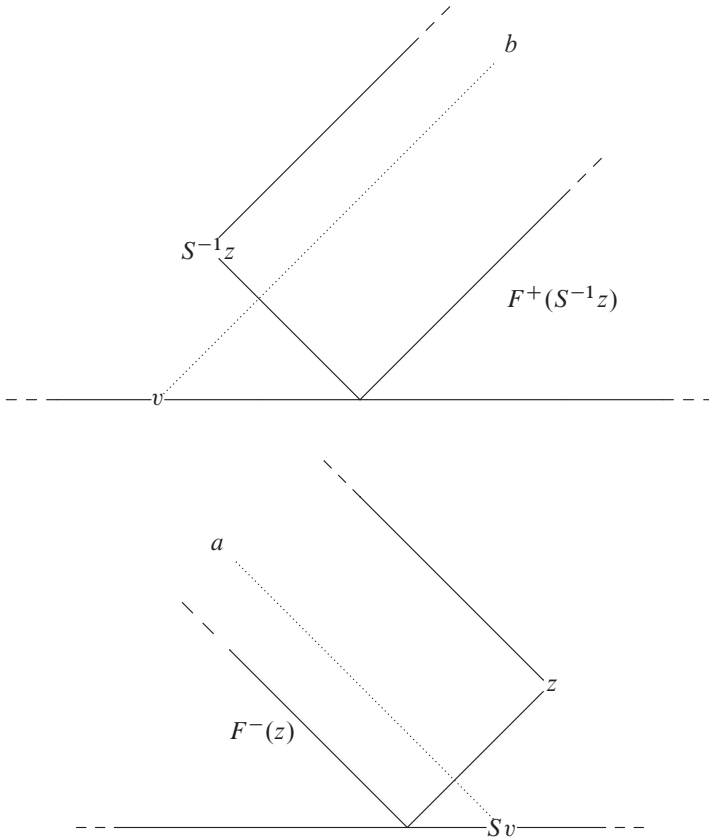
Figure 6. Another view of objects in two components of the AR quiver.

First, assume $s' \in \mathrm{ind}(\mathsf{S}) \cap F^-(z)$. The slice a in Figure 6 determines a half line $F^-(z) \cap a$. If there are objects of S on this half line, then let s_a be the one which is closest to the base line of the quiver and let

$$s_a \xrightarrow{\sigma_a} z$$

be a non-zero morphism. If s' is on a, then it is above s_a and Proposition 3.1 (iv) implies that each morphism $s' \to z$ factors through σ_a. There are only finitely many slices a intersecting $F^-(z)$. Including the corresponding morphisms σ_a as components of σ ensures that each morphism $s' \to z$ with $s' \in \mathrm{ind}(\mathsf{S}) \cap F^-(z)$ factors through σ.

Secondly, assume $s' \in \mathrm{ind}(\mathsf{S}) \cap F^+(S^{-1}z)$. The slice b in Figure 6 determines a half line $F^+(S^{-1}z) \cap b$, and we split into two cases.

The case where S has finitely many objects on $F^+(S^{-1}z) \cap b$: Let s_b be the direct sum of these objects and let each component of the morphism

$$s_b \xrightarrow{\sigma_b} z$$

be non-zero. If s' is on b, then s' is one of the direct summands of s_b and each morphism $s' \to z$ factors through σ_b since each non-zero Hom space in C is 1-dimensional.

The case where S has infinitely many objects on $F^+(S^{-1}z) \cap b$: Then $\mathrm{ind}(\mathsf{S}) \cap b$ is infinite. If b is the slice starting at v and a the slice ending at Sv, then condition (ii') says that $\mathrm{ind}(\mathsf{S}) \cap a$ is also infinite. In particular it is non-empty so we have already included the non-zero morphism

$$s_a \xrightarrow{\sigma_a} z$$

as a component of σ in the previous part of the proof. If s' is on b, then it is straightforward to use Proposition 3.2 to check that each morphism $s' \to z$ factors through σ_a.

As above, there are only finitely many slices b intersecting $F^+(S^{-1}z)$. Including the relevant morphisms σ_b as components of σ ensures that each morphism $s' \to z$ with $s' \in \mathrm{ind}(\mathsf{S}) \cap F^+(S^{-1}z)$ factors through σ.   □

A similar proof establishes the following dual result.

Proposition 3.5. *Let S be a subcategory of C and let $\mathfrak{S}$ be the corresponding set of d-admissible arcs. The following conditions are equivalent:*

  (i) *The subcategory S is left-approximating.*

 (ii) *Each left-fountain of $\mathfrak{S}$ is a right-fountain of $\mathfrak{S}$.*

Proof of Theorem E. Given a subcategory T of C and the corresponding set of d-admissible arcs $\mathfrak{T}$, Theorem D says that T is weakly d-cluster tilting if and only if $\mathfrak{T}$ is a $(d+2)$-angulation of the ∞-gon. It is not hard to see that since $\mathfrak{T}$ is a set of pairwise non-crossing d-admissible arcs, it is locally finite or has a fountain if and only if it satisfies conditions (ii) in Propositions 3.4 and 3.5. By the propositions, this happens if and only if T is left- and right-approximating.   □

4 Arc combinatorics

Construction 4.1. Let $\mathfrak{T}$ be a $(d+2)$-angulation of the ∞-gon and let $p_0 \in \mathbb{Z}$ be given. We define integers $p_1, p_2, \ldots$ inductively as follows: If p_ℓ has already

been defined, then:

- if $\mathfrak{T}$ contains no arcs of the form (p_ℓ, q), then let $p_{\ell+1} = p_\ell + 1$,

- if $\mathfrak{T}$ contains a non-zero, finite number of arcs of the form (p_ℓ, q), then let $(p_\ell, p_{\ell+1})$ be the one with maximal length,

- if $\mathfrak{T}$ contains infinitely many arcs of the form (p_ℓ, q), that is, if p_ℓ is a right-fountain of $\mathfrak{T}$, then stop the algorithm and do not define $p_{\ell+1}$.

If the algorithm stops, then it defines a sequence with finitely many elements,

$$p_0 < \cdots < p_m.$$

If it does not stop, then it defines a sequence with infinitely many elements,

$$p_0 < p_1 < \cdots ,$$

and we set $m = \infty$. Let us sum up the properties of the sequence.

(i) If $m < \infty$, then p_m is a right-fountain of $\mathfrak{T}$.

(ii) $(p_\ell, p_{\ell+1})$ is either a pair of consecutive integers or an arc in $\mathfrak{T}$.

(iii) $p_\ell - p_0 \equiv \ell \pmod{d}$.

To see (iii), note that the length of a d-admissible arc is $\equiv 1 \pmod{d}$.

Collin Bleak proved that a triangulation of the ∞-gon has a left-fountain if and only if it has a right-fountain, and his method also works for $(d+2)$-angulations. We thank him for permitting us to provide a proof of the following lemma which establishes the "only if" direction. "If" follows by symmetry. See also [3, Lemma 4.11].

Lemma 4.2. *Let $\mathfrak{T}$ be a $(d+2)$-angulation of the ∞-gon. Suppose that p_0 is a left-fountain of $\mathfrak{T}$ and perform Construction 4.1.*

(i) *The construction gives a finite sequence $p_0 < \cdots < p_m$ with $0 \leq m \leq d$.*

(ii) *p_m is a right-fountain of $\mathfrak{T}$.*

(iii) *Let $\mathfrak{t} \in \mathfrak{T}$ and assume $\mathfrak{t} \neq (p_\ell, p_{\ell+1})$ for $\ell \in \{0, \ldots, m-1\}$. Then $\mathfrak{t}$ has an overarc $\mathfrak{r} \in \mathfrak{T}$.*

Proof. (i) Assume to the contrary that $m \geq d + 1$; this includes the possibility $m = \infty$. Construction 4.1 (iii) shows that (p_0, p_{d+1}) is a d-admissible arc. If (p_0, p_1) is a d-admissible arc, then (p_0, p_{d+1}) has strictly greater length than (p_0, p_1) and by Construction 4.1 we have $(p_0, p_{d+1}) \notin \mathfrak{T}$. If (p_0, p_1) is not a

d-admissible arc, then by Construction 4.1 there are no arcs in $\mathfrak{T}$ of the form (p_0, q) so we have $(p_0, p_{d+1}) \notin \mathfrak{T}$ again. In either case there must be an arc $(r, s) \in \mathfrak{T}$ which crosses (p_0, p_{d+1}), that is, we have $r < p_0 < s < p_{d+1}$ or $p_0 < r < p_{d+1} < s$. But p_0 is a left-fountain of $\mathfrak{T}$ so $r < p_0 < s < p_{d+1}$ is impossible since it would imply that (r, s) crossed an arc in $\mathfrak{T}$.

We must therefore have $p_0 < r < p_{d+1} < s$. However, this also leads to a contradiction: We cannot have $p_\ell < r < p_{\ell+1}$ for any $\ell \in \{0, \ldots, d\}$, for if we did, then $(p_\ell, p_{\ell+1})$ would not be consecutive integers whence $(p_\ell, p_{\ell+1}) \in \mathfrak{T}$ by Construction 4.1 (ii), but this arc would cross $(r, s) \in \mathfrak{T}$. So we must have $r = p_\ell$ for an $\ell \in \{1, \ldots, d\}$. Hence $\mathfrak{T}$ contains arcs of the form (p_ℓ, q), and by Construction 4.1 the one with maximal length is $(p_\ell, p_{\ell+1})$. But this contradicts $(r, s) \in \mathfrak{T}$ because we know $r = p_\ell$ and $p_{\ell+1} \leq p_{d+1} < s$.

(ii) See Construction 4.1 (i).

(iii) Let us write $\mathfrak{t} = (t, u)$ and search for $\mathfrak{r}$.

Since p_0 and p_m are a left-fountain and a right-fountain of $\mathfrak{T}$, we must have $t < u \leq p_0$ or $p_0 \leq t < u \leq p_m$ or $p_m \leq t < u$.

If $t < u \leq p_0$, then we can choose $\mathfrak{r} = (r, p_0) \in \mathfrak{T}$ with $r < t$. If $p_m \leq t < u$, then we can choose $\mathfrak{r} = (p_m, s) \in \mathfrak{T}$ with $u < s$.

Now assume $p_0 \leq t < u \leq p_m$.

If $t = p_\ell$ for an $\ell \in \{0, \ldots, m-1\}$, then $\mathfrak{t} \in \mathfrak{T}$ is an arc of the form (p_ℓ, q). Among the arcs in $\mathfrak{T}$ of this form, by Construction 4.1 the one with maximal length is $(p_\ell, p_{\ell+1})$. Since $\mathfrak{t} \neq (p_\ell, p_{\ell+1})$ by assumption, we get that

$$\mathfrak{r} = (p_\ell, p_{\ell+1}) \in \mathfrak{T}$$

is an overarc of $\mathfrak{t}$.

If $p_\ell < t < p_{\ell+1}$ for an $\ell \in \{0, \ldots, m-1\}$, then we must have $u \leq p_{\ell+1}$, since otherwise $(t, u) \in \mathfrak{T}$ and $(p_\ell, p_{\ell+1}) \in \mathfrak{T}$ would cross. But then

$$\mathfrak{r} = (p_\ell, p_{\ell+1}) \in \mathfrak{T}$$

is again an overarc of $\mathfrak{t} = (t, u)$. $\qquad\square$

Lemma 4.3. *Let $\mathfrak{T}$ be a $(d+2)$-angulation of the ∞-gon. Then $\mathfrak{T}$ is either locally finite or has precisely one left-fountain and one right-fountain.*

Proof. If $\mathfrak{T}$ is not locally finite, then it has a left- or a right-fountain. By Lemma 4.2 (ii) and its mirror image, it has both a left- and a right-fountain. It is easy to see that in any event, it has at most one left- and at most one right-fountain. $\quad\square$

Lemma 4.4. *Let $\mathfrak{T}$ be a $(d+2)$-angulation of the ∞-gon which is locally finite or has a fountain. Then each arc $\mathfrak{t} \in \mathfrak{T}$ has an overarc $\mathfrak{r} \in \mathfrak{T}$.*
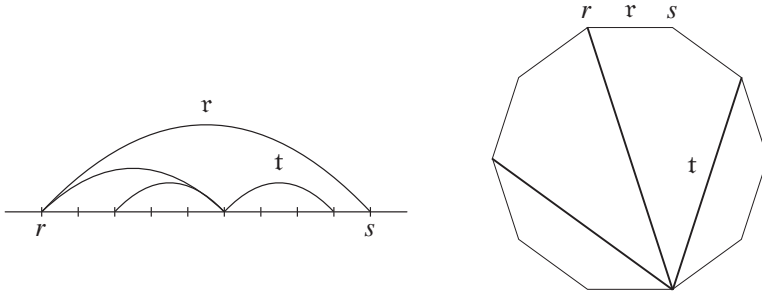
Figure 7. If $r = (r, s)$ is a d-admissible arc, then $r, r + 1, \ldots, s$ can be viewed as the vertices of a polygon R. If $t = (t, u)$ has r as an overarc, then t can be viewed as a d-admissible diagonal of R.

Proof. The case where $\mathfrak{T}$ has a fountain at p_0: Then we must have $m = 0$ in Lemma 4.2, and Lemma 4.2 (iii) implies the present result.

The case where $\mathfrak{T}$ is locally finite: Let us write $t = (t, u)$ and search for r. We can assume that, among the arcs in $\mathfrak{T}$ of the form (t, v), the one of maximal length is (t, u), since otherwise there is obviously an overarc. Let $p_0 = t$ and perform Construction 4.1; then $(p_0, p_1) = (t, u)$. Since $\mathfrak{T}$ is locally finite, it has no right-fountain, so Construction 4.1 (i) implies $m = \infty$. Construction 4.1 (iii) implies that (p_0, p_{d+1}) is a d-admissible arc. It has strictly greater length than (p_0, p_1), so $(p_0, p_{d+1}) \notin \mathfrak{T}$ follows.

There must hence be an arc $(r, s) \in \mathfrak{T}$ which crosses (p_0, p_{d+1}), that is, we have $r < p_0 < s < p_{d+1}$ or $p_0 < r < p_{d+1} < s$.

First, assume $r < p_0 < s < p_{d+1}$. Note that we cannot have $p_0 < s < p_1$ since then $(r, s) \in \mathfrak{T}$ and $(p_0, p_1) \in \mathfrak{T}$ would cross. So $p_1 \leq s$ whence $r = (r, s) \in \mathfrak{T}$ is an overarc of $(p_0, p_1) = (t, u)$.

Secondly, assume $p_0 < r < p_{d+1} < s$. This leads to a contradiction in the same way as in the second paragraph of the proof of Lemma 4.2. □

Construction 4.5. Let $\mathfrak{T}$ be a $(d + 2)$-angulation of the ∞-gon and let

$$r = (r, s) \in \mathfrak{T}.$$

We can view $\{r, \ldots, s\}$ as the vertices of an $(s - r + 1)$-gon R. Each pair $(r, r+1)$, $(r + 1, r + 2), \ldots, (s - 1, s)$ is viewed as an edge of R, and so is the arc (r, s); that is, r and s are viewed as consecutive vertices of R. Each d-admissible arc t of which $r = (r, s)$ is an overarc is viewed as a d-admissible diagonal of R. See Figure 7. In particular, the set

$$\mathfrak{R} = \{t \in \mathfrak{T} \mid r \text{ is an overarc of } t\}$$

is a $(d + 2)$-angulation of R.

Observe that $\mathfrak{R}$ divides R into $(d+2)$-gons, and that one of these $(d+2)$-gons, say T, has r and s among its vertices. We can write the whole set of vertices of T as

$$r < t_1 < \cdots < t_d < s,$$

and hence each of

$$(r, t_1), (t_1, t_2), \ldots, (t_{d-1}, t_d), (t_d, s)$$

is either a pair of consecutive integers or a diagonal in $\mathfrak{R}$, that is, an arc in $\mathfrak{T}$.

Lemma 4.6. *Let $\mathfrak{T}$ be a $(d+2)$-angulation of the ∞-gon and let $\mathsf{t} \in \mathfrak{T}$.*

(i) *If $\mathfrak{U}$ is a set of d-admissible arcs not in $\mathfrak{T} \setminus \mathsf{t}$ such that $(\mathfrak{T} \setminus \mathsf{t}) \cup \mathfrak{U}$ is a $(d+2)$-angulation of the ∞-gon, then $\mathfrak{U} = \{\mathsf{t}^*\}$ for a single d-admissible arc t^*.*

(ii) *If t has an overarc in $\mathfrak{T}$, then there are $d+1$ choices of t^*.*

(iii) *If t has no overarc in $\mathfrak{T}$, then there are $\leq d$ choices of t^*.*

Proof. Suppose that t has the overarc $\mathsf{r} \in \mathfrak{T}$. We will establish (i) and (ii) for t.

The set of all arcs in $\mathfrak{T}$ of which $\mathsf{r} \in \mathfrak{T}$ is an overarc can be viewed as a $(d+2)$-angulation $\mathfrak{R}$ of a polygon R by Construction 4.5. When $(\mathfrak{T} \setminus \mathsf{t}) \cup \mathfrak{U}$ is a $(d+2)$-angulation of the ∞-gon, r is an overarc of each arc in $\mathfrak{U}$ since removing t does not create any room above its overarc r. It follows that $\mathfrak{U}$ can be viewed as a set of d-admissible diagonals of R such that $(\mathfrak{R} \setminus \mathsf{t}) \cup \mathfrak{U}$ is a $(d+2)$-angulation of R. Then it is well known that, as desired, $\mathfrak{U}$ has one element which can be chosen in $d+1$ different ways.

Now suppose that t has no overarc in $\mathfrak{T}$. We will establish (i) and (iii) for t.

Lemma 4.4 shows that $\mathfrak{T}$ is not locally finite and does not have a fountain. Lemma 4.3 shows that $\mathfrak{T}$ has a left-fountain p_0 which is not a right-fountain. We can perform Construction 4.1. By Lemma 4.2 (i–ii) this gives a sequence

$$p_0 < \cdots < p_m$$

with $m \leq d$ where p_m is a right-fountain of $\mathfrak{T}$. Note that $1 \leq m$ since p_0 is not a right-fountain. By Construction 4.1 (ii), each $(p_\ell, p_{\ell+1})$ is either a pair of consecutive integers or an arc in $\mathfrak{T}$, and it follows from Lemma 4.2 (iii) that $\mathsf{t} = (p_j, p_{j+1})$ for a $j \in \{0, \ldots, m-1\}$.

By Construction 4.5 applied to $\mathsf{t} = (p_j, p_{j+1})$, there is a sequence of integers

$$p_j < q_1 < \cdots < q_d < p_{j+1}$$

such that each of $(p_j, q_1), (q_1, q_2), \ldots, (q_{d-1}, q_d), (q_d, p_{j+1})$ is either a pair of consecutive integers or an arc in $\mathfrak{T}$. We hence have a sequence of integers

$$p_0 < p_1 < \cdots < p_j < q_1 < \cdots < q_d < p_{j+1} < \cdots < p_m \qquad (4.1)$$

where each pair of neighbouring elements is either a pair of consecutive integers or an arc in $\mathfrak{T} \setminus t$. In particular, each pair of neighbouring elements has a difference which is $\equiv 1 \pmod{d}$.

Now consider a d-admissible arc

$$t^* = (v, w) \notin \mathfrak{T} \setminus t$$

which crosses no arc in $\mathfrak{T} \setminus t$.

We cannot have $w \leq p_0$. For if we did, then t^* would not cross $t = (p_j, p_{j+1})$, and hence t^* would cross no arc in $\mathfrak{T}$ whence $t^* \in \mathfrak{T}$. Since $t^* \notin \mathfrak{T} \setminus t$, this would force $t^* = t$, but this contradicts $v < w \leq p_0 \leq p_j$. We also cannot have $v < p_0 < w$ because p_0 is a left-fountain of $\mathfrak{T}$. Similarly, we cannot have $p_m \leq v$ or $v < p_m < w$.

We conclude that

$$p_0 \leq v < w \leq p_m.$$

We claim that, in fact, v and w must be among the elements of the sequence (4.1).

Namely, assume that at least one of v and w is not an element of the sequence. Then it is strictly between two such elements. For the sake of argument, say $q_\ell < v < q_{\ell+1}$. Then we cannot have $q_{\ell+1} < w$, for then $t^* = (v, w)$ and $(q_\ell, q_{\ell+1}) \in \mathfrak{T} \setminus t$ would cross. So we must have

$$v < w \leq q_{\ell+1}.$$

Hence $(q_\ell, q_{\ell+1})$ is an overarc of $t^* = (v, w)$.

However, the set of all arcs in $\mathfrak{T}$ of which $(q_\ell, q_{\ell+1}) \in \mathfrak{T} \setminus t$ is an overarc can be viewed as a $(d + 2)$-angulation $\mathfrak{R}'$ of a polygon R' by Construction 4.5, and t^* can be viewed as a d-admissible diagonal of this polygon. Note that $(q_\ell, q_{\ell+1})$ is not an overarc of $t = (p_j, p_{j+1})$ and so $\mathfrak{R}' \subseteq \mathfrak{T} \setminus t$. Hence the assumption that t^* crosses none of the arcs in $\mathfrak{T} \setminus t$ means that it crosses none of the diagonals in $\mathfrak{R}'$. But then $t^* \in \mathfrak{R}'$ whence $t^* \in \mathfrak{T} \setminus t$ which is a contradiction.

So we have $t^* = (v, w)$ with v, w elements in the sequence (4.1). However, we saw that each pair of neighbouring elements in this sequence has a difference which is $\equiv 1 \pmod{d}$. Hence, for t^* to be a d-admissible arc, v and w must either be neighbours in the sequence, or $nd + 1$ steps apart for an integer $n \geq 1$. But they cannot be neighbours for then we would have $t^* \in \mathfrak{T} \setminus t$, so v and w must be $nd + 1$ steps apart in the sequence.

Since $m \leq d$ by Lemma 4.2 (i), the sequence (4.1) has $m + 1 + d \leq 2d + 1$ elements. It follows that $n = 1$ and hence two different choices of t^* must cross each other; this shows part (i) of the present lemma. It also follows that there are at most $(2d + 1) - (d + 1) = d$ different choices for t^*, showing part (iii) of the present lemma.   □

5 Proofs of Theorems A, B and C

Remark 5.1. Let T be a weakly d-cluster tilting subcategory of the triangulated category C and let $\mathfrak{T}$ be the corresponding $(d + 2)$-angulation of the ∞-gon.

Lemma 4.6 (i) says that if we drop one arc from $\mathfrak{T}$, then we must add precisely one other d-admissible arc to get a new $(d + 2)$-angulation.

So if we drop one indecomposable object from T, then we must add precisely one other indecomposable object to get a new weakly d-cluster tilting subcategory of C.

That is, "mutating T at t" has the expected effect of replacing t by a single other indecomposable object.

Proof of Theorem A. By Theorem E which was proved in Section 3, a d-cluster tilting subcategory T of C corresponds to a $(d + 2)$-angulation of the ∞-gon $\mathfrak{T}$ which is locally finite or has a fountain.

By Lemma 4.4, each $t \in \mathfrak{T}$ has an overarc $r \in \mathfrak{T}$.

By Lemma 4.6 (ii), this means that there are $d + 1$ different choices of a d-admissible arc t^* such that $(\mathfrak{T} \setminus t) \cup t^*$ is a $(d + 2)$-angulation of the ∞-gon.

Excluding the trivial choice $t^* = t$ leaves d choices for t^* and translating back to T shows Theorem A. □

Proof of Theorem B. Let $\ell \in \{0, \ldots, d - 1\}$ be given. Figure 8 shows part of a $(d + 2)$-angulation $\mathfrak{T}_\ell$ of the ∞-gon. It contains the arc $t = (0, d + 1)$ and has a left-fountain at 0 and a right-fountain at $d + 1 + \ell$.
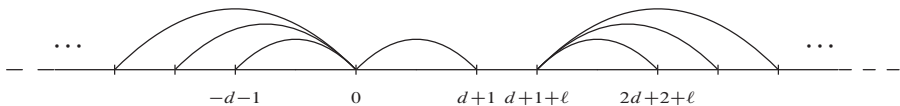

Figure 8. A $(d + 2)$-angulation $\mathfrak{T}_\ell$ of the ∞-gon where the arc $t = (0, d + 1)$ can be replaced in ℓ ways.

There are $\ell + 1$ different choices of a d-admissible arc t^* such that $(\mathfrak{T} \setminus t) \cup t^*$ is a $(d + 2)$-angulation of the ∞-gon; namely,

$$t^* = (p, p + d + 1)$$

for $p \in \{0, \ldots, \ell\}$.

Excluding the trivial choice $t^* = t$ leaves ℓ choices. By Theorem D which was proved in Section 3, the $(d + 2)$-angulation $\mathfrak{T}_\ell$ therefore corresponds to a weakly d-cluster tilting subcategory T_ℓ with the property claimed in Theorem B. □

Proof of Theorem C. Similar to the proof of Theorem A, but $t \in \mathfrak{T}$ may or may not have an overarc, so both parts (ii) and (iii) of Lemma 4.6 are needed. □

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Author information

Thorsten Holm, Institut für Algebra, Zahlentheorie und Diskrete Mathematik,
Fakultät für Mathematik und Physik, Leibniz Universität Hannover,
Welfengarten 1, 30167 Hannover, Germany.
E-mail: `holm@math.uni-hannover.de`

Peter Jørgensen, School of Mathematics and Statistics, Newcastle University,
Newcastle upon Tyne NE1 7RU, United Kingdom.
E-mail: `peter.jorgensen@ncl.ac.uk`