Sasakian quiver gauge theories and instantons on cones over lens 5-spaces

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Abstract

We consider SU(3)-equivariant dimensional reduction of Yang–Mills theory over certain cyclic orbifolds of the 5-sphere which are Sasaki–Einstein manifolds. We obtain new quiver gauge theories extending those induced via reduction over the leaf spaces of the characteristic foliation of the Sasaki–Einstein structure, which are projective planes. We describe the Higgs branches of these quiver gauge theories as moduli spaces of spherically symmetric instantons which are SU(3)-equivariant solutions to the Hermitian Yang–Mills equations on the associated Calabi–Yau cones, and further compare them to moduli spaces of translationally-invariant instantons on the cones. We provide an explicit unified construction of these moduli spaces as Kähler quotients and show that they have the same cyclic orbifold singularities as the cones over the lens 5-spaces.

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1. Introduction and summary

Sasaki–Einstein 5-manifolds M^5 have played a prominent role in developments in string theory. For example, type IIB string theory on $AdS_5 \times M^5$ is conjecturally dual to the 4-dimensional $\mathcal{N} = 1$ superconformal worldvolume field theory on a stack of D3-branes placed at the apex singularity of the 6-dimensional Calabi–Yau cone $C(M^5)$ over M^5 [1–6]. They have moreover served as interesting testing grounds for the suggestion that maximally supersymmetric Yang–Mills theory in 5 dimensions contains all degrees of freedom of the 6-dimensional $(2, 0)$ superconformal theory compactified on a circle [7,8]. Metrics on the non-compact spaces $C(M^5)$ are also known explicitly [9–11], in contrast to the compact examples of Calabi–Yau string compactifications.

In this paper we derive new quiver gauge theories via equivariant dimensional reduction over M^5 and describe their vacua in terms of moduli spaces of generalised instantons on the cones $C(M^5)$[1]; such instantons play a central role in supersymmetric compactifications of heterotic string theory [14]. This extends the constructions of [15] which dealt with the case of 3-dimensional Sasaki–Einstein manifolds, wherein these field theories were dubbed as "Sasakian" quiver gauge theories. The only Sasaki–Einstein 3-manifolds are the ADE orbifolds S^3/Γ of the 3-sphere by a discrete subgroup Γ of SU(2). They have natural extensions as ADE orbifolds $M^5 = S^5/\Gamma$ of the 5-sphere which preserve $\mathcal{N} = 2$ supersymmetry [2,1]. In the following we shall be interested in generalisations of these orbifolds to cases where Γ is instead a finite subgroup of SU(3). The corresponding affine cones $C(S^5/\Gamma)$ play a central role in the McKay correspondence for Calabi–Yau 3-folds [16,17]. Moreover, the BPS configurations in the worldvolume gauge theories on D-branes located at points of Calabi–Yau manifolds which are resolutions of the orbifolds $\mathbb{C}^3/\Gamma$ [18,19] are parameterised by moduli spaces of translationally-invariant solutions of Hermitian Yang–Mills equations on $\mathbb{C}^3/\Gamma$, which coincide with Calabi–Yau spaces that are partial resolutions of these orbifolds [20,21]. Drawing from the situation in the 3-dimensional case [15], it is natural to expect the same sort of similarities between these moduli spaces and those of "spherically symmetric" instantons on cones over any Sasaki–Einstein 5-manifold, where the generalised instanton equations can also be reduced to generalised Nahm equations of the form considered in [22].

On general grounds, any quasi-regular Sasaki–Einstein 5-manifold M^5 is a U(1) V-bundle over a 4-dimensional Kähler–Einstein orbifold M^4. In this paper we consider the special case where $M^5 = S^5/\Gamma$ with $\Gamma = \mathbb{Z}_{q+1} \subset$ SU(3) a cyclic group. Then $M^4 = \mathbb{C}P^2$ and we can exploit the constructions from [23] which thoroughly studies SU(3)-equivariant dimensional reduction over the Kähler coset space $\mathbb{C}P^2 \cong$ SU(3)/$S($U(2) $\times$ U(1)). We shall construct the relevant quiver bundles and study the corresponding quiver gauge theories in detail; these quivers are new and we relate them explicitly to those arising from dimensional reduction over the leaf spaces $\mathbb{C}P^2$ of the characteristic foliation of $S^5/\mathbb{Z}_{q+1}$. In particular, we will compare the moduli spaces of spherically symmetric and translationally-invariant instantons on the cones $C(S^5/\mathbb{Z}_{q+1}) \cong \mathbb{C}^3/\mathbb{Z}_{q+1}$, and show that they contain the same orbifold singularities $\mathbb{C}^3/\mathbb{Z}_N$ (where N is the rank of the gauge group) analogously to the cases of [15]. In analogy to the interpretations of [15], our constructions thereby shed light on the interplay between the Higgs branches of the worldvolume quiver gauge theories on Dp-branes which probe a set of D$(p + 6)$-branes wrapping a (partial) resolution of $C(S^5/\mathbb{Z}_{q+1})$, and BPS states of the quiver

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[1] The analogous instanton moduli spaces were studied by [12] for the 3-dimensional case and by [13] in arbitrary (odd) dimension.

gauge theories on pairs of D$(p+4)$-branes wrapping $C(S^5/\mathbb{Z}_{q+1})$ which transversally intersect D$(p+6)$-branes at the apex of the cone $C(S^5/\mathbb{Z}_{q+1})$. In this scenario, it is the codimensionality of the D-brane bound states which selects both the quiver type and the abelian category in which the quiver representation is realised; in particular, the arrows of the quivers keep track of the massless bifundamental transverse scalars stretching between constituent fractional D-branes at the vertices.

The outline of the remainder of this paper is as follows. In Section 2 we provide a detailed description of the geometry of the orbifold $S^5/\mathbb{Z}_{q+1}$ using its realisation as both a coset space and as a Sasaki–Einstein manifold. In Section 3 we give a detailed description of the quiver gauge theory induced via SU(3)-equivariant dimensional reduction over $S^5/\mathbb{Z}_{q+1}$, including explicit constructions of the quiver bundles and their connections as well as the form of the action functional. We then describe the Higgs branch vacuum states of quiver gauge theories on the cone $C(S^5/\mathbb{Z}_{q+1})$ as SU(3)-equivariant solutions to the Hermitian Yang–Mills equations in Section 4 and as translationally-invariant solutions in Section 5. In Section 6 we compare the two quiver gauge theories in some detail, including a contrasting of their quiver bundles and explicit universal constructions of their instanton moduli spaces as Kähler quotients. Four appendices at the end of the paper contain technical details and results which are employed in the main text.

2. Sasaki–Einstein geometry

In this section we shall introduce the basic geometrical constructions that we shall need throughout this paper.

2.1. Preliminaries

Sasakian manifolds M^{2n+1} of dimension $2n + 1$ are contact manifolds which form a natural bridge between two different Kähler spaces M^{2n} and M^{2n+2} of dimensions $2n$ and $2n + 2$, respectively. On the one hand, the metric cone over a Sasakian manifold M^{2n+1} gives a Kähler space $M^{2n+2} = C(M^{2n+1})$. On the other hand, the Reeb vector field on M^{2n+1} defines a foliation of M^{2n+1} and the transverse space M^{2n} is also Kähler. For further details, see for example [24].

A Riemannian manifold is called *Einstein* if its Ricci tensor is a scalar multiple of its metric. A Sasakian manifold which is also Einstein is called a *Sasaki–Einstein manifold* [24]. Since the cone over an Einstein manifold is also an Einstein space, the metric cone over a Sasaki–Einstein manifold is a Calabi–Yau space and in this case the transverse space M^{2n} is Kähler–Einstein. Because of the $\mathbb{R}_{>0}$ scaling action on the cones we can write the Calabi–Yau metric as

$$\mathrm{d}s^2_{C(M^{2n+1})} = \mathrm{d}r^2 + r^2\,\mathrm{d}s^2_{M^{2n+1}}\,, \qquad (2.1)$$

where $r \in \mathbb{R}_{\geq 0}$ and the tensor $\mathrm{d}s^2_{M^{2n+1}}$ defines a metric on the intersection M^{2n+1} of the cone with the unit sphere in $\mathbb{C}^{n+2}$.

Given a Riemannian manifold M and a finite group Γ acting isometrically on M, one can, loosely speaking, define the Riemannian space of Γ-orbits M/Γ, which is called an *orbifold* or sometimes *V-manifold*, see for instance [24]. The notion of fibre bundle can be adapted to the category of orbifolds, and we follow [24] in calling them *V-bundles*. Any quasi-regular Sasaki–Einstein manifold M^{2n+1} is a principal U(1) V-bundle over its transverse space M^{2n}. In this case the Sasaki–Einstein metric can be expressed as

$$\mathrm{d}s^2_{M^{2n+1}} = \mathrm{d}s^2_{M^{2n}} + \eta \otimes \eta\,, \qquad (2.2)$$

where $\mathrm{d}s^2_{M^{2n}}$ is the (pullback of the) Kähler–Einstein metric of M^{2n}, and η is the contact 1-form which is a connection on the fibration $M^{2n+1} \to M^{2n}$ of curvature $\mathrm{d}\eta = -2\omega_{M^{2n}}$ with $\omega_{M^{2n}}$ the Kähler form of the base M^{2n}.

2.2. Sphere S^5

The 5-dimensional sphere S^5 has two realisations: Firstly, as the coset space $S^5 = \mathrm{SU}(3)/\mathrm{SU}(2)$ and, secondly, as a principal U(1)-bundle over the complex projective plane $\mathbb{C}P^2$. As such, we have the chain of principal bundles

$$\mathrm{SU}(3) \xrightarrow{\mathrm{SU}(2)} S^5 \xrightarrow{\mathrm{U}(1)} \mathbb{C}P^2 . \tag{2.3}$$

Our description of S^5 will be based on the principal U(1)-bundle over $\mathbb{C}P^2$, and we will construct a flat connection on the principal SU(2)-bundle over S^5 by employing this feature.

2.2.1. Connections on $\mathbb{C}P^2$

Let us consider a local section U over a patch $\mathcal{U}_0$ of $\mathbb{C}P^2$ for the principal bundle $\mathrm{SU}(3) \to \mathbb{C}P^2$. For this, let $\mathrm{G} = \mathrm{SU}(3)$ and $\mathrm{H} = S(\mathrm{U}(2) \times \mathrm{U}(1)) \subset \mathrm{G}$, and consider the principal bundle associated to the coset G/H given by

$$\mathrm{G} = \mathrm{SU}(3) \xrightarrow{\mathrm{H}=S(\mathrm{U}(2)\times\mathrm{U}(1))} \mathrm{G/H} \cong \mathbb{C}P^2 . \tag{2.4}$$

By the definition of the complex projective plane

$$\mathbb{C}P^2 = \mathbb{C}^3 \,/\sim \,=\, \left\{ [z^1 : z^2 : z^3] \in \mathbb{C}^3 \,:\, [z^1 : z^2 : z^3] \sim [\lambda z^1 : \lambda z^2 : \lambda z^3] , \; \lambda \in \mathbb{C}^* \right\} , \tag{2.5}$$

one introduces on the patch $\mathcal{U}_0 = \{[z^1 : z^2 : z^3] \in \mathbb{C}P^2 \,:\, z^3 \neq 0\}$ the coordinates

$$Y := \begin{pmatrix} y^1 \\ y^2 \end{pmatrix} \sim \begin{pmatrix} z^1/z^3 \\ z^2/z^3 \end{pmatrix} . \tag{2.6}$$

Define a local section on $\mathcal{U}_0$ of the principal bundle (2.4) via [23]

$$\begin{aligned} U : \mathcal{U}_0 &\longrightarrow \mathrm{SU}(3) \\ Y &\longmapsto U(Y) := \frac{1}{\gamma} \begin{pmatrix} \bar{\Lambda} & \bar{Y} \\ -\bar{Y}^\dagger & 1 \end{pmatrix} , \end{aligned} \tag{2.7}$$

with the definitions

$$\bar{\Lambda} := \gamma \, \mathbb{1}_2 - \frac{1}{\gamma + 1} \bar{Y} \bar{Y}^\dagger \quad \text{and} \quad \gamma := \sqrt{1 + Y^\dagger Y} . \tag{2.8}$$

From these two definitions, one observes the properties

$$\bar{\Lambda}^\dagger = \bar{\Lambda} , \quad \bar{\Lambda}^2 = \gamma^2 \, \mathbb{1}_2 - \bar{Y} \bar{Y}^\dagger , \quad \bar{\Lambda} \bar{Y} = \bar{Y} \quad \text{and} \quad \bar{Y}^\dagger \bar{\Lambda} = \bar{Y}^\dagger . \tag{2.9}$$

It is immediate from (2.9) that U as defined in (2.7) is SU(3)-valued.

One can define a flat connection A_0 on the bundle (2.4) via

$$A_0 = U^\dagger \, \mathrm{d}U \equiv \begin{pmatrix} B & \bar{\beta} \\ -\beta^\top & -a \end{pmatrix} , \tag{2.10}$$

with the definitions

$$B := \frac{1}{\gamma^2} \left(\bar{\Lambda} \, d\bar{\Lambda} + \bar{Y} \, d\bar{Y}^\dagger - \frac{1}{2} \mathbb{1}_2 \, d\!\left(Y^\dagger Y\right) \right), \tag{2.11a}$$

$$\bar{\beta} := \frac{1}{\gamma^2} \bar{\Lambda} \, d\bar{Y} \quad \text{and} \quad \beta^\top := \frac{1}{\gamma^2} \, d\bar{Y}^\dagger \, \bar{\Lambda}, \tag{2.11b}$$

$$a := -\frac{1}{2\gamma^2} \left(\bar{Y}^\dagger \, d\bar{Y} - d\bar{Y}^\dagger \, \bar{Y} \right) = -\bar{a}. \tag{2.11c}$$

That $U \in \mathrm{SU}(3)$ directly implies the vanishing of the curvature 2-form $F_0 = dA_0 + A_0 \wedge A_0$, which is equivalent to the set of relations

$$dB + B \wedge B = \bar{\beta} \wedge \beta^\top \quad \text{and} \quad da = -\beta^\top \wedge \bar{\beta} = \beta^\dagger \wedge \beta, \tag{2.12a}$$

$$d\bar{\beta} + B \wedge \bar{\beta} = \bar{\beta} \wedge a \quad \text{and} \quad d\beta^\top + \beta^\top \wedge B = a \wedge \beta^\top. \tag{2.12b}$$

As elaborated in [23,25], B can be regarded as a $\mathfrak{u}(2)$-valued connection 1-form and a as a $\mathfrak{u}(1)$-valued connection. Consequently, one can introduce an $\mathfrak{su}(2)$-valued connection $B_{(1)}$ by removing the trace of B. An explicit parametrisation yields

$$B_{(1)} := B - \frac{1}{2} \operatorname{tr}(B) \, \mathbb{1}_2 \equiv \begin{pmatrix} B_{11} & \bar{B}_{12} \\ -B_{12} & -B_{11} \end{pmatrix} \quad \text{with} \quad \operatorname{tr}(B) = a, \; B_{11} = -\bar{B}_{11}. \tag{2.13}$$

The geometry of $\mathbb{C}P^2$ including the properties of the SU(3)-equivariant 1-forms β^i, the instanton connection $B_{(1)}$ and the monopole connection a are described in Appendix A.

2.2.2. Connections on S^5

Consider now the principal SU(2)-bundle

$$G = \mathrm{SU}(3) \xrightarrow{\; K = \mathrm{SU}(2)\;} G/K = S^5, \tag{2.14}$$

where $K \subset G$. Then the section U from (2.7) can be modified as

$$\hat{U} : \mathcal{U}_0 \times [0, 2\pi) \longrightarrow \mathrm{SU}(3)$$

$$(Y, \varphi) \longmapsto \hat{U}(Y, \varphi) := U(Y) \begin{pmatrix} e^{i\varphi} \mathbb{1}_2 & 0 \\ 0 & e^{-2i\varphi} \end{pmatrix} \equiv U(Y) \, Z(\varphi), \tag{2.15}$$

which is a local section of the bundle (2.14) on the patch $\mathcal{U}_0 \times [0, 2\pi)$ with coordinates $\{y^1, y^2, \varphi\}$. Note that $Z^{-1} = Z^\dagger = \bar{Z}$ and $\det(Z) = 1$, and furthermore $Z(\varphi) Z(\psi) = Z(\psi) Z(\varphi) = Z(\psi + \varphi)$, which implies that Z realises the embedding $\mathrm{U}(1) \hookrightarrow \mathrm{SU}(3)$; this also shows that $\hat{U} \in \mathrm{SU}(3)$. The modified (flat) connection $\hat{A}$ on the bundle (2.14) and the corresponding curvature $\hat{F}$ are given as

$$\hat{A} := \hat{U}^\dagger \, d\hat{U} = \mathrm{Ad}(Z^{-1}) A_0 + Z^\dagger \, dZ = \begin{pmatrix} B + i \mathbb{1}_2 \, d\varphi & \bar{\beta} \, e^{-3i\varphi} \\ -\beta^\top e^{3i\varphi} & -(a + 2i \, d\varphi) \end{pmatrix}, \tag{2.16a}$$

$$\hat{F} := d\hat{A} + \hat{A} \wedge \hat{A} = \mathrm{Ad}(Z^{-1}) F_0$$

$$= \begin{pmatrix} dB + B \wedge B - \bar{\beta} \wedge \beta^\top & \left(d\bar{\beta} + B \wedge \bar{\beta} - \bar{\beta} \wedge a\right) e^{-3i\varphi} \\ -\left(d\beta^\top + \beta^\top \wedge B - a \wedge \beta^\top\right) e^{3i\varphi} & -da - \beta^\top \wedge \bar{\beta} \end{pmatrix} = 0. \tag{2.16b}$$

Again the flatness of $\hat{A}$ yields the same set of identities (2.12), because $\hat{F}$ differs from F only by the adjoint action of Z^{-1}.

2.2.3. Contact geometry of S^5

By construction, the base space of (2.14) is a 5-sphere. The aim now is to choose a basis of the cotangent bundle T^*S^5 over the patch $\mathcal{U}_0 \times [0, 2\pi)$ such that one recovers the Sasaki–Einstein structure on S^5. For this, we start with the identifications

$$\beta_\varphi^1 := \beta^1 \, e^{3\,i\,\varphi} \equiv e^1 + i\,e^2 \,, \quad \beta_\varphi^2 := \beta^2 \, e^{3\,i\,\varphi} \equiv e^3 + i\,e^4 \quad \text{and} \quad \kappa\,e^5 := \tfrac{1}{2}\,a + i\,d\varphi \,, \quad (2.17)$$

where $\kappa \in \mathbb{C}$ is a constant to be determined and the 1-forms β^i originate from the complex cotangent space $T^*_{(Y,\bar{Y})}\mathbb{C}P^2$ at a point $(Y, \bar{Y}) \in \mathcal{U}_0 \subset \mathbb{C}P^2$. Next we define the forms

$$\omega_1 := e^{14} + e^{23} \,, \quad \omega_2 := e^{31} + e^{24} \,, \quad \omega_3 := e^{12} + e^{34} \quad \text{and} \quad \eta := e^5 \,, \quad (2.18)$$

where generally $e^{a_1 \cdots a_k} = e^{a_1} \wedge \cdots \wedge e^{a_k}$. In the basis (2.17), one obtains

$$\omega_1 = \tfrac{1}{2i} \left(\beta_\varphi^1 \wedge \beta_\varphi^2 - \bar{\beta}_\varphi^1 \wedge \bar{\beta}_\varphi^2 \right) \,, \quad \omega_2 = -\tfrac{1}{2} \left(\beta_\varphi^1 \wedge \beta_\varphi^2 + \bar{\beta}_\varphi^1 \wedge \bar{\beta}_\varphi^2 \right) \quad \text{and}$$

$$\omega_3 = -\tfrac{1}{2i} \left(\beta_\varphi^1 \wedge \bar{\beta}_\varphi^1 + \beta_\varphi^2 \wedge \bar{\beta}_\varphi^2 \right) \,. \quad (2.19)$$

Note that ω_3 coincides (up to a normalisation factor) with the Kähler form on $\mathbb{C}P^2$, cf. Appendix A. The exterior derivatives of β_φ^i and $\bar{\beta}_\varphi^i$ are given as

$$\mathrm{d}\beta_\varphi^i = e^{3\,i\,\varphi}\,\mathrm{d}\beta^i - 3\,i\,\beta_\varphi^i \wedge \mathrm{d}\varphi \quad \text{and} \quad \mathrm{d}\bar{\beta}_\varphi^i = e^{-3\,i\,\varphi}\,\mathrm{d}\bar{\beta}^i + 3\,i\,\bar{\beta}_\varphi^i \wedge \mathrm{d}\varphi \,. \quad (2.20)$$

The distinguished 1-form η is taken to be the contact 1-form dual to the Reeb vector field of the Sasaki–Einstein structure. At this stage, the choice of the quadruple $(\eta, \omega_1, \omega_2, \omega_3)$ defines an SU(2)-structure on the 5-sphere [26]. For it to be Sasaki–Einstein, one needs the relations

$$\mathrm{d}\omega_1 = 3\eta \wedge \omega_2 \,, \quad \mathrm{d}\omega_2 = -3\eta \wedge \omega_1 \quad \text{and} \quad \mathrm{d}\eta = -2\omega_3 \quad (2.21)$$

to hold [27]. Employing (2.12) one arrives at

$$\mathrm{d}\omega_1 = 6\,i\,\kappa\,\eta \wedge \omega_2 \quad \text{and} \quad \mathrm{d}\omega_2 = -6\,i\,\kappa\,\eta \wedge \omega_1 \,, \quad (2.22a)$$

$$\mathrm{d}\eta = \tfrac{i}{\kappa}\,\omega_3 \quad \text{and} \quad \mathrm{d}\omega_3 = 0 \,. \quad (2.22b)$$

Consequently, the coframe $\{\eta, \beta_\varphi^1, \beta_\varphi^2\}$ yields a Sasaki–Einstein structure on S^5 if and only if $\kappa = -\tfrac{i}{2}$, and from now on this will be the case.

2.3. Orbifold $S^5/\mathbb{Z}_{q+1}$

Our aim is to now construct a principal V-bundle over the orbifold $S^5/\mathbb{Z}_{q+1}$ by the following steps: Take the principal SU(2)-bundle $\pi : G = SU(3) \to SU(3)/SU(2) \cong S^5$, which is SU(2)-equivariant. Embed $\mathbb{Z}_{q+1} \hookrightarrow U(1) \subset SU(3)$ such that U(1) commutes with SU(2) $\subset$ SU(3), and define a $\mathbb{Z}_{q+1}$-action γ on S^5. The action $\gamma : \mathbb{Z}_{q+1} \times S^5 \to S^5$ can be lifted to an action $\widetilde{\gamma} : \mathbb{Z}_{q+1} \times G \to G$ with an isomorphism on the SU(2) fibres induced by this action. The crucial point is that the fibre isomorphism is trivial as SU(2) commutes with $\mathbb{Z}_{q+1}$ by construction. Hence one can consider the $\mathbb{Z}_{q+1}$-projection of G to the principal SU(2) V-bundle $\widetilde{G}$, which is schematically given as

$$\begin{array}{ccc} G & \xrightarrow{\widetilde{\gamma}} & \widetilde{G} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \widetilde{\pi}} \\ S^5 & \xrightarrow{\gamma} & S^5/\mathbb{Z}_{q+1} \end{array} \qquad (2.23)$$

With an abuse of notation, we will denote the V-bundles obtained via $\mathbb{Z}_{q+1}$-projection by the same symbols as the fibre bundles they originate from; only $\mathbb{Z}_{q+1}$-equivariant field configurations survive this orbifold projection.

A section $\widetilde{U}$ of the principal V-bundle (2.23) is obtained by a (further) modification of the section (2.7) as

$$\widetilde{U} : \mathcal{U}_0 \times \left[0, \tfrac{2\pi}{q+1}\right) \longrightarrow SU(3)$$

$$\left(Y, \tfrac{\varphi}{q+1}\right) \longmapsto \widetilde{U}\left(Y, \tfrac{\varphi}{q+1}\right) := U(Y) \begin{pmatrix} e^{\frac{i\varphi}{q+1}} \mathbb{1}_2 & 0 \\ 0 & e^{-2\frac{i\varphi}{q+1}} \end{pmatrix} \equiv U(Y)\, Z_{q+1}(\varphi)\,.$$

(2.24)

Here $\varphi \in [0, 2\pi)$ is again the local coordinate on the S^1-fibration $S^5 \xrightarrow{U(1)} \mathbb{C}P^2$; hence $e^{\frac{i\varphi}{q+1}} \in S^1/\mathbb{Z}_{q+1}$. Analogously to the $q = 0$ case of S^5 above, one can prove that Z_{q+1} realises the embedding $S^1/\mathbb{Z}_{q+1} \hookrightarrow U(1) \subset SU(3)$, and $\widetilde{U} \in SU(3)$. As before, one computes the connection 1-form $\widetilde{A}$ and the curvature $\widetilde{F}$ of the flat connection on the V-bundle (2.23). This yields

$$\widetilde{A} := \widetilde{U}^\dagger \, d\widetilde{U} = \mathrm{Ad}(Z_{q+1}^{-1})A_0 + Z_{q+1}^\dagger \, dZ_{q+1} = \begin{pmatrix} B + \mathbb{1}_2 \frac{i\, d\varphi}{q+1} & \bar{\beta}\, e^{-3\frac{i\varphi}{q+1}} \\ -\beta^\top e^{3\frac{i\varphi}{q+1}} & -\left(a + 2\frac{i\, d\varphi}{q+1}\right) \end{pmatrix}, \quad (2.25a)$$

$$\widetilde{F} := d\widetilde{A} + \widetilde{A} \wedge \widetilde{A} = \mathrm{Ad}(Z_{q+1}^{-1})F_0$$

$$= \begin{pmatrix} dB + B \wedge B - \bar{\beta} \wedge \beta^\top & \left(d\bar{\beta} + B \wedge \bar{\beta} - \bar{\beta} \wedge a\right) e^{-3\frac{i\varphi}{q+1}} \\ -\left(d\beta^\top + \beta^\top \wedge B - a \wedge \beta^\top\right) e^{3\frac{i\varphi}{q+1}} & -da - \beta^\top \wedge \bar{\beta} \end{pmatrix} = 0\,.$$

(2.25b)

Again the flatness of the connection $\widetilde{A}$ yields the very same relations (2.12).

2.3.1. Local coordinates

Our description of the orbifold $S^5/\mathbb{Z}_{q+1}$ follows [15]. The key idea is the embedding $S^5 = SU(3)/SU(2) \hookrightarrow \mathbb{R}^6 \cong \mathbb{C}^3$ via the relation

$$r^2 = \delta_{\hat{\mu}\hat{\nu}}\, x^{\hat{\mu}} x^{\hat{\nu}} = |z^1|^2 + |z^2|^2 + |z^3|^2 \tag{2.26}$$

where $x^{\hat{\mu}}$ ($\hat{\mu} = 1, \dots, 6$) are coordinates of $\mathbb{R}^6$ and z^α ($\alpha = 1, 2, 3$) are coordinates of $\mathbb{C}^3$; here $r \in \mathbb{R}_{>0}$ gives the radius of the embedded 5-sphere. In general, on the coordinates z^α the $\mathbb{Z}_{q+1}$-action is realised by a representation $h \mapsto (h^\alpha{}_\beta)$ such that

$$z^\alpha \longmapsto h^\alpha{}_\beta\, z^\beta \quad \text{and} \quad \bar{z}^\alpha \longmapsto \bar{h}^\alpha{}_\beta\, \bar{z}^\beta = (h^{-1})^\alpha{}_\beta\, \bar{z}^\beta\,, \tag{2.27}$$

where h is the generator of the cyclic group $\mathbb{Z}_{q+1}$. In this paper the action of the finite group $\mathbb{Z}_{q+1}$ is chosen to be realised by the embedding of $\mathbb{Z}_{q+1}$ in the fundamental 3-dimensional complex representation $\underline{C}^{1,0}$ of $SU(3)$ given by

$$(h^\alpha{}_\beta) = \begin{pmatrix} \zeta_{q+1} & 0 & 0 \\ 0 & \zeta_{q+1} & 0 \\ 0 & 0 & \zeta_{q+1}^{-2} \end{pmatrix} \in SU(3) \quad \text{with} \quad \zeta_{q+1}^l := e^{\frac{2\pi i}{q+1} l}\,. \tag{2.28}$$

Since $\mathbb{C}P^2$ is naturally defined via a quotient of $\mathbb{C}^3$, see (2.5), one can deduce the $\mathbb{Z}_{q+1}$-action on the local coordinates (y^1, y^2) of the patch $\mathcal{U}_0$ to be

$$y^\alpha \longmapsto \frac{\zeta_{q+1} z^\alpha}{\zeta_{q+1}^{-2} z^3} = \zeta_{q+1}^3 y^\alpha \quad \text{and} \quad \bar{y}^\alpha \longmapsto \frac{\zeta_{q+1}^{-1} \bar{z}^\alpha}{\zeta_{q+1}^2 \bar{z}^3} = \zeta_{q+1}^{-3} \bar{y}^\alpha \quad \text{for} \quad \alpha = 1, 2 \,. \tag{2.29}$$

Next consider the action of $\mathbb{Z}_{q+1}$ on the S^1 coordinate φ. By (2.28) one naturally has

$$\mathrm{e}^{\mathrm{i}\frac{\varphi}{q+1}} \xrightarrow{\mathbb{Z}_{q+1}} \mathrm{e}^{\mathrm{i}(\frac{\varphi}{q+1} + \frac{2\pi l}{q+1})} = \mathrm{e}^{\mathrm{i}\frac{\varphi}{q+1}} \zeta_{q+1}^l \quad \text{for} \quad l \in \{0, 1, \ldots, q\} \,, \tag{2.30}$$

i.e. the transformed coordinate $\mathrm{e}^{\mathrm{i}(\frac{\varphi}{q+1} + \frac{2\pi l}{q+1})}$ lies in the $\mathbb{Z}_{q+1}$-orbit $\left[\mathrm{e}^{\mathrm{i}\frac{\varphi}{q+1}}\right]$ of $\mathrm{e}^{\mathrm{i}\frac{\varphi}{q+1}}$.

2.3.2. Lens spaces

The spaces $S^5/\mathbb{Z}_{q+1}$ are known as lens spaces, see for instance [24]. For this, one usually embeds S^5 into $\mathbb{C}^3$ and chooses the action of $p \in \{0, 1, \ldots, q\}$ as

$$\mathbb{Z}_{q+1} \times \mathbb{C}^3 \longrightarrow \mathbb{C}^3$$
$$\left(p, (z^1, z^2, z^3)\right) \longmapsto p \cdot (z^1, z^2, z^3) := \left(\mathrm{e}^{\frac{2\pi \mathrm{i} p}{q+1}} z^1, \mathrm{e}^{\frac{2\pi \mathrm{i} p}{q+1} r_1} z^2, \mathrm{e}^{\frac{2\pi \mathrm{i} p}{q+1} r_2} z^3\right) \tag{2.31}$$

where the integers r_1 and r_2 are chosen to be coprime to $q+1$. The coprime condition is necessary for the $\mathbb{Z}_{q+1}$-action to be free away from the origin of $\mathbb{C}^3$. The quotient space $S^5/\mathbb{Z}_{q+1}$ with the action (2.31) is called the lens space $L(q+1, r_1, r_2)$ or $L^2(q+1, r_1, r_2)$. It is a 5-dimensional orbifold with fundamental group $\mathbb{Z}_{q+1}$.

We choose the $\mathbb{Z}_{q+1}$-action to be given by (2.28), i.e. $r_1 = 1$ and $r_2 = -2$. Then r_1 is always coprime to $q + 1$, but r_2 is coprime to $q + 1$ only if q is even. Thus for $q + 1 \in 2\mathbb{Z}_{\geq 0} + 1$ the only singular point in $\mathbb{C}^3/\mathbb{Z}_{q+1}$ is the origin, and its isotropy group is $\mathbb{Z}_{q+1}$. However, for $q + 1 \in 2\mathbb{Z}_{\geq 0}$ there is a singularity at the origin and also along the circle $\{z^1 = z^2 = 0, |z^3| = 1\} \subset S^5$ of singularities with isotropy group $\left\{0, \frac{q+1}{2}\right\} \cong \mathbb{Z}_2 \subset \mathbb{Z}_{q+1}$. Hence for the chosen action (2.28) we are forced to take $q \in 2\mathbb{Z}_{\geq 0}$ in all considerations.

2.3.3. Differential forms

Similarly to the previous case, one can construct locally a basis of differential forms. However, one has to work with a *uniformising system* of local charts on the orbifold $S^5/\mathbb{Z}_{q+1}$ instead of local charts for the manifold S^5. Choosing the identifications

$$\beta_{q+1}^1 := \beta^1 \mathrm{e}^{\frac{3\mathrm{i}\varphi}{q+1}} \equiv e^1 + \mathrm{i}\,e^2 \,, \quad \beta_{q+1}^2 := \beta^2 \mathrm{e}^{\frac{3\mathrm{i}\varphi}{q+1}} \equiv e^3 + \mathrm{i}\,e^4 \quad \text{and} \quad \eta := e^5 \equiv \mathrm{i}\,a - \frac{2\,\mathrm{d}\varphi}{q+1} \tag{2.32}$$

and by means of the relations imposed by the flatness of (2.25a), one can study the geometry of $S^5/\mathbb{Z}_{q+1}$. Defining the three 2-forms

$$\omega_1 := \frac{1}{2\mathrm{i}} \left(\beta_{q+1}^1 \wedge \beta_{q+1}^2 - \bar{\beta}_{q+1}^1 \wedge \bar{\beta}_{q+1}^2\right), \qquad \omega_2 := -\frac{1}{2} \left(\beta_{q+1}^1 \wedge \beta_{q+1}^2 + \bar{\beta}_{q+1}^1 \wedge \bar{\beta}_{q+1}^2\right)$$
$$\text{and} \quad \omega_3 := -\frac{1}{2\mathrm{i}} \left(\beta_{q+1}^1 \wedge \bar{\beta}_{q+1}^1 + \beta_{q+1}^2 \wedge \bar{\beta}_{q+1}^2\right) \tag{2.33}$$

and employing (2.12) implied by the flatness of $\widetilde{A}$, one obtains the correct Sasaki–Einstein relations (2.21).

2.3.4. $\mathbb{Z}_{q+1}$-action on 1-forms

Consider the $\mathbb{Z}_{q+1}$-action on the forms β_{q+1}^i, $\bar{\beta}_{q+1}^i$, and η. Firstly, recall the definitions (2.32) and (A.1), from which one sees that

$$\beta_{q+1}^i \xrightarrow{\mathbb{Z}_{q+1}} \zeta_{q+1}^3 \beta_{q+1}^i \quad \text{and} \quad \bar{\beta}_{q+1}^i \xrightarrow{\mathbb{Z}_{q+1}} \zeta_{q+1}^{-3} \bar{\beta}_{q+1}^i . \tag{2.34}$$

This follows directly from the transformation (2.29). Moreover, it agrees with the monodromy of β_{q+1}^i and $\bar{\beta}_{q+1}^i$ along the S^1 fibres, i.e.

$$\beta_{q+1}^i = \beta^i \, e^{3 \frac{i\varphi}{q+1}} \xrightarrow{\varphi \mapsto \varphi + 2\pi} \beta_{q+1}^i \, \zeta_{q+1}^3 . \tag{2.35}$$

Secondly, for the 1-form η from (2.32) we know that a is a U(1) connection. As any U(1) connection is automatically U(1)-invariant, due to the embedding $\mathbb{Z}_{q+1} \hookrightarrow U(1)$ one also has $\mathbb{Z}_{q+1}$-invariance of a.[2] We conclude that

$$\eta \xrightarrow{\mathbb{Z}_{q+1}} \eta . \tag{2.36}$$

From the definition (2.26) of the radial coordinate, one observes that r is invariant under $\mathbb{Z}_{q+1}$. The same is true for the corresponding 1-form, so that

$$dr \xrightarrow{\mathbb{Z}_{q+1}} dr . \tag{2.37}$$

Following [15], let T be a $\mathbb{Z}_{q+1}$-invariant 1-form on the metric cone $C(S^5/\mathbb{Z}_{q+1})$ which is locally expressed as

$$T = T_\mu \, e^\mu + T_r \, dr \equiv W_i \, \beta_{q+1}^i + \overline{W}_i \, \bar{\beta}_{q+1}^i + W_5 \, e^5 + W_r \, dr \tag{2.38}$$

with $i = 1, 2$ and $\mu = 1, \ldots, 5$, where $W_1 = \frac{1}{2}(T_1 - i\,T_2)$, $W_2 = \frac{1}{2}(T_3 - i\,T_4)$, $W_5 = T_5$ and $W_r = T_r$. This induces a representation π of $\mathbb{Z}_{q+1}$ in $\Omega^1\big(C(S^5)\big)$ as

$$W_i \longmapsto \pi(h)(W_i) = \zeta_{q+1}^{-3} \, W_i , \qquad \overline{W}_i \longmapsto \pi(h)(\overline{W}_i) = \zeta_{q+1}^3 \, \overline{W}_i , \tag{2.39a}$$

$$W_5 \longmapsto \pi(h)(W_5) = W_5 , \qquad W_r \longmapsto \pi(h)(W_r) = W_r . \tag{2.39b}$$

3. Quiver gauge theory

In this section we define quiver bundles over a d-dimensional manifold M^d via equivariant dimensional reduction over $M^d \times S^5/\mathbb{Z}_{q+1}$, and derive the generic form of a G-equivariant connection. For this, we recall some aspects from the representation theory of G = SU(3), and exemplify the relation between quiver representations and homogeneous bundles over $S^5/\mathbb{Z}_{q+1}$. Then we extend our constructions to G-equivariant bundles over $M^d \times S^5/\mathbb{Z}_{q+1}$, which will furnish a quiver representation in the category of vector bundles instead of vector spaces. We shall also derive the dimensional reduction of the pure Yang–Mills action on $M^d \times S^5$ to obtain a Yang–Mills–Higgs theory on M^d from our twisted reduction procedure (for the special case $q = 0$).

---

[2] Alternatively, one can work out the transformation behaviour of a directly from the explicit expression (2.11c).

3.1. Preliminaries

3.1.1. Cartan–Weyl basis of $\mathfrak{su}(3)$

Our considerations are based on certain irreducible representations of the Lie group $G = SU(3)$, which are decomposed into irreducible representations of the subgroup $H = SU(2) \times U(1) \subset SU(3)$. For this, we recall the root decomposition of the Lie algebra $\mathfrak{su}(3)$. There is a pair of simple roots α_1 and α_2, and the non-null roots are given by $\pm\alpha_1$, $\pm\alpha_2$, and $\pm(\alpha_1 + \alpha_2)$. The Lie algebra $\mathfrak{su}(3)$ is 8-dimensional and has a 2-dimensional Cartan subalgebra spanned by H_{α_1} and H_{α_2}. We distinguish one $\mathfrak{su}(2)$ subalgebra, which is spanned by H_{α_1} and $E_{\pm\alpha_1}$ with the commutation relations

$$\left[H_{\alpha_1}, E_{\pm\alpha_1}\right] = \pm 2 E_{\pm\alpha_1} \quad \text{and} \quad \left[E_{\alpha_1}, E_{-\alpha_1}\right] = H_{\alpha_1} \,. \tag{3.1a}$$

The element H_{α_2} generates a $\mathfrak{u}(1)$ subalgebra that commutes with this $\mathfrak{su}(2)$ subalgebra, i.e.

$$\left[H_{\alpha_2}, H_{\alpha_1}\right] = 0 \quad \text{and} \quad \left[H_{\alpha_2}, E_{\pm\alpha_1}\right] = 0 \,. \tag{3.1b}$$

In the *Cartan–Weyl basis*, the remaining generators $E_{\pm\alpha_2}$ and $E_{\pm(\alpha_1+\alpha_2)}$ satisfy non-vanishing commutation relations with the $\mathfrak{su}(2)$ generators given by

$$\left[H_{\alpha_1}, E_{\pm\alpha_2}\right] = \mp E_{\pm\alpha_2} \quad \text{and} \quad \left[E_{\pm\alpha_1}, E_{\pm\alpha_2}\right] = \pm E_{\pm(\alpha_1+\alpha_2)} \,, \tag{3.1c}$$

$$\left[H_{\alpha_1}, E_{\pm(\alpha_1+\alpha_2)}\right] = \pm E_{\pm(\alpha_1+\alpha_2)} \quad \text{and} \quad \left[E_{\pm\alpha_1}, E_{\mp(\alpha_1+\alpha_2)}\right] = \mp E_{\mp\alpha_2} \,, \tag{3.1d}$$

with the $\mathfrak{u}(1)$ generator given by

$$\left[H_{\alpha_2}, E_{\pm\alpha_2}\right] = \pm 3 E_{\pm\alpha_2} \quad \text{and} \quad \left[H_{\alpha_2}, E_{\pm(\alpha_1+\alpha_2)}\right] = \pm 3 E_{\pm(\alpha_1+\alpha_2)} \,, \tag{3.1e}$$

and amongst each other given by

$$\left[E_{\alpha_2}, E_{-\alpha_2}\right] = \tfrac{1}{2}\left(H_{\alpha_2} - H_{\alpha_1}\right) \quad \text{and} \quad \left[E_{\alpha_1+\alpha_2}, E_{-\alpha_1-\alpha_2}\right] = \tfrac{1}{2}\left(H_{\alpha_1} + H_{\alpha_2}\right) \,, \tag{3.1f}$$

$$\left[E_{\pm\alpha_2}, E_{\mp(\alpha_1+\alpha_2)}\right] = \pm E_{\mp\alpha_1} \,. \tag{3.1g}$$

3.1.2. Skew-Hermitian basis of $\mathfrak{sl}(3, \mathbb{C})$

Equivalently, we introduce the complex basis given by

$$I_1 := E_{\alpha_1+\alpha_2} - E_{-\alpha_1-\alpha_2} \,, \qquad I_2 := -\mathrm{i}\left(E_{\alpha_1+\alpha_2} + E_{-\alpha_1-\alpha_2}\right) \,, \tag{3.2a}$$

$$I_3 := E_{\alpha_2} - E_{-\alpha_2} \,, \qquad I_4 := -\mathrm{i}\left(E_{\alpha_2} + E_{-\alpha_2}\right) \,, \tag{3.2b}$$

$$I_5 := -\tfrac{\mathrm{i}}{2} H_{\alpha_2} \,, \tag{3.2c}$$

$$I_6 := E_{\alpha_1} - E_{-\alpha_1} \,, \qquad I_7 := -\mathrm{i}\left(E_{\alpha_1} + E_{-\alpha_1}\right) \,, \tag{3.2d}$$

$$I_8 := \mathrm{i} H_{\alpha_1} \,, \tag{3.2e}$$

which reflects the splitting $\mathfrak{su}(3) = \mathfrak{su}(2) \oplus \mathfrak{m}$ in which

$$I_i \in \mathfrak{su}(2) \quad \text{for} \quad i = 6, 7, 8 \quad \text{and} \quad I_\mu \in \mathfrak{m} \quad \text{for} \quad \mu = 1, \ldots, 5 \,. \tag{3.3}$$

This representation of generators is skew-Hermitian, i.e. $I_\mu = -I_\mu^\dagger$ for $\mu = 1, \ldots, 5$ and $I_i = -I_i^\dagger$ for $i = 6, 7, 8$, in contrast to the Cartan–Weyl basis. The chosen Cartan subalgebra is spanned by I_5 and I_8, and $[I_5, I_i] = 0$. From the commutation relations (3.1) one can infer the non-vanishing structure constants of these generators as

$$f_{67}{}^8 = -2 \qquad \text{plus cyclic} , \tag{3.4a}$$

$$f_{63}{}^1 = f_{64}{}^2 = f_{71}{}^4 = f_{73}{}^2 = f_{82}{}^1 = f_{83}{}^4 = 1 \qquad \text{plus cyclic} , \tag{3.4b}$$

$$f_{12}{}^5 = f_{34}{}^5 = 2 , \tag{3.4c}$$

$$f_{25}{}^1 = -f_{15}{}^2 = f_{45}{}^3 = -f_{35}{}^4 = \tfrac{3}{2} . \tag{3.4d}$$

The Killing form $K_{AB} := f_{AC}{}^D f_{DB}{}^C$ (with $A, B, \ldots = 1, \ldots, 8$) associated to this basis is diagonal but not proportional to the identity, and is given by

$$
\begin{aligned}
K_{ab} &= 12\,\delta_{ab} \quad \text{for} \quad a, b = 1, 2, 3, 4, \\
K_{55} &= 9 \quad \text{and} \quad K_{ij} = 12\,\delta_{ij} \quad \text{for} \quad i, j = 6, 7, 8 .
\end{aligned}
\tag{3.5}
$$

Introducing the 't Hooft tensors η_{ab}^{α} for $a, b = 1, 2, 3, 4$ and $\alpha = 1, 2, 3$ one has

$$f_{ab}{}^5 = 2\,\eta_{ab}^3 \quad \text{and} \quad f_{a5}{}^b = -\tfrac{3}{2}\,\eta_{ab}^3 . \tag{3.6}$$

3.1.3. Biedenharn basis

The irreducible SU(3)-representations $\underline{C}^{k,l}$ are labelled by a pair of non-negative integers (k, l) and have (complex) dimension

$$p_0 := \dim\left(\underline{C}^{k,l}\right) = \tfrac{1}{2}\,(k + l + 2)\,(k + 1)\,(l + 1) . \tag{3.7}$$

We decompose $\underline{C}^{k,l}$ with respect to the subgroup $H = SU(2) \times U(1) \subset G$, just as in [23]. A particularly convenient choice of basis for the vector space $\underline{C}^{k,l}$ is the *Biedenharn basis* [28–30], which is defined to be the eigenvector basis given by

$$H_{\alpha_1} \left| {n \atop q}\, m \right\rangle = q \left| {n \atop q}\, m \right\rangle , \quad L^2 \left| {n \atop q}\, m \right\rangle = n\,(n+2) \left| {n \atop q}\, m \right\rangle \quad \text{and} \quad H_{\alpha_2} \left| {n \atop q}\, m \right\rangle = m \left| {n \atop q}\, m \right\rangle , \tag{3.8}$$

where $L^2 := 2\left(E_{\alpha_1} E_{-\alpha_1} + E_{-\alpha_1} E_{\alpha_1} \right) + H_{\alpha_1}^2$ is the isospin operator of $\mathfrak{su}(2)$. Define the representation space (n, m) as the eigenspace with definite isospin $n \in \mathbb{Z}_{\geq 0}$ and magnetic monopole charge $\tfrac{m}{2}$ for $m \in \mathbb{Z}$. Then the SU(3)-representation $\underline{C}^{k,l}$ decomposes into irreducible $SU(2) \times U(1)$-representations (n, m) as

$$\underline{C}^{k,l} = \bigoplus_{(n,m) \in Q_0(k,l)} (n, m) , \tag{3.9}$$

where $Q_0(k, l)$ parameterises the set of all occurring representations (n, m). In Appendix B.1 we summarise the matrix elements of all generators in the Biedenharn basis.

3.1.4. Representations of $\mathbb{Z}_{q+1}$

As the cyclic group $\mathbb{Z}_{q+1}$ is abelian, each of its irreducible representations is 1-dimensional. There are exactly $q + 1$ inequivalent irreducible unitary representations ρ_l given by

$$\rho_l : \begin{array}{ccc} \mathbb{Z}_{q+1} & \longrightarrow & S^1 \subset \mathbb{C}^* \\ p & \longmapsto & e^{\frac{2\pi i\,(p+l)}{q+1}} \end{array} \quad \text{for} \quad l = 0, 1, \ldots, q . \tag{3.10}$$

3.2. Homogeneous bundles and quiver representations

Consider the groups $G = SU(3)$, $H = SU(2) \times U(1)$, $K = SU(2)$, $\widetilde{K} = SU(2) \times \mathbb{Z}_{q+1} \subset H$ and a finite-dimensional K-representation $\underline{R}$ which descends from a G-representation. Associate to the principal bundle (2.14) the K-equivariant vector bundle $\mathcal{V}_R := G \times_K \underline{R}$. Due to the embedding $\mathbb{Z}_{q+1} \hookrightarrow U(1) \subset SU(3)$ and the origin of $\underline{R}$ from a G-module, it follows that $\underline{R}$ is also a $\mathbb{Z}_{q+1}$-module. Consequently, as in Section 2.3, the $\mathbb{Z}_{q+1}$-action $\gamma : \mathbb{Z}_{q+1} \times S^5 \to S^5$ can be lifted to a $\mathbb{Z}_{q+1}$-action $\widetilde{\gamma} : \mathbb{Z}_{q+1} \times \mathcal{V}_R \to \mathcal{V}_R$ wherein the linear $\mathbb{Z}_{q+1}$-action on the fibres is trivial. Thus one can define the corresponding $\widetilde{K}$-equivariant vector V-bundle $\widetilde{\mathcal{V}}_R$ by suitable $\mathbb{Z}_{q+1}$-projection as[3]

$$
\begin{array}{ccc}
\mathcal{V}_R & \xrightarrow{\ \widetilde{\gamma}\ } & \widetilde{\mathcal{V}}_R \\
{\scriptstyle \pi}\Big\downarrow & & \Big\downarrow{\scriptstyle \widetilde{\pi}} \\
S^5 & \xrightarrow[\ \gamma\ ]{} & S^5/\mathbb{Z}_{q+1}
\end{array}
\tag{3.11}
$$

and again we denote the vector V-bundle $\widetilde{\mathcal{V}}_R$ by the same symbol $\mathcal{V}_R$ whenever the context is clear.

It is known [31] that the category of such holomorphic homogeneous vector bundles $\mathcal{V}_R$ is equivalent to the category of finite-dimensional representations of certain quivers with relations. We use this equivalence to associate quivers to homogeneous bundles related to an irreducible $SU(3)$-representation $\underline{R} = \underline{C}^{k,l}$, which is evidently a finite-dimensional (and usually reducible) representation of $SU(2) \times \mathbb{Z}_{q+1} \hookrightarrow SU(2) \times U(1)$.

3.2.1. Flat connections

Inspired by the structure of the flat connection (2.25a) on the V-bundle (2.23), one observes that it can be written as[4]

$$
\mathcal{A}_0 = \left[B_{11} H_{\alpha_1} + B_{12} E_{\alpha_1} - \left(B_{12} E_{\alpha_1} \right)^{\dagger} \right] - \tfrac{i}{2} \eta H_{\alpha_2} + \bar{\beta}_{q+1}^1 E_{\alpha_1 + \alpha_2} + \bar{\beta}_{q+1}^2 E_{\alpha_2}
$$
$$
- \beta_{q+1}^1 E_{-\alpha_1 - \alpha_2} - \beta_{q+1}^2 E_{-\alpha_2},
\tag{3.12a}
$$

or equivalently

$$
\mathcal{A}_0 = \Gamma + I_\mu \, e^\mu
\tag{3.12b}
$$

with the coframe $\{e^\mu\}_{\mu=1,\dots,5}$ defined in (2.32) and the definition

$$
\Gamma := \Gamma^i \, I_i \quad \text{with} \quad \Gamma^6 = \tfrac{i}{2} \left(B_{12} - \bar{B}_{12} \right), \quad \Gamma^7 = \tfrac{1}{2} \left(B_{12} + \bar{B}_{12} \right), \quad \Gamma^8 = -i \, B_{11}.
\tag{3.13}
$$

Note that Γ is an $\mathfrak{su}(2)$-valued connection 1-form. The flatness of $\mathcal{A}_0$, i.e. $\mathcal{F}_0 = d\mathcal{A}_0 + \mathcal{A}_0 \wedge \mathcal{A}_0 = 0$, is encoded in the relation

$$
\mathcal{F}_0 = F_\Gamma + I_\mu \, de^\mu + \Gamma^i \left[I_i, I_\mu \right] \wedge e^\mu + \tfrac{1}{2} \left[I_\mu, I_\nu \right] e^{\mu\nu} = 0,
\tag{3.14a}
$$

$$
\mathcal{F}_0 \big|_{\mathfrak{su}(2)} = 0 : F_\Gamma = -\tfrac{1}{2} f_{\mu\nu}^{\ \ i} I_i \, e^{\mu\nu},
\tag{3.14b}
$$

$$
\mathcal{F}_0 \big|_{\mathfrak{m}} = 0 : de^\mu = -\Gamma^i f_{i\nu}^{\ \ \mu} \wedge e^\nu - \tfrac{1}{2} f_{\rho\sigma}^{\ \ \mu} e^{\rho\sigma},
\tag{3.14c}
$$

---

[3] See also the treatment in [15].
[4] Note that (2.25a) implicitly uses the fundamental representation $\underline{C}^{1,0}$ of $SU(3)$.

where $F_\Gamma = d\Gamma + \Gamma \wedge \Gamma$. The equivalent information can be cast in a set of relations starting from (3.12a) and using the Biedenharn basis; see Appendix B.2 for details.

3.2.2. $\mathbb{Z}_{q+1}$-equivariance

Consider the principal V-bundle (2.23), where the $\mathbb{Z}_{q+1}$-action is defined on S^5 as in Section 2.3. The connection (3.12) is SU(3)-equivariant by construction, but one can also check its $\mathbb{Z}_{q+1}$-equivariance explicitly. For this, one needs to specify an action of $\mathbb{Z}_{q+1}$ on the fibre $\underline{C}^{k,l}$, which decomposes as an SU(2)-module via (3.9). Demanding that the $\mathbb{Z}_{q+1}$-action commutes with the SU(2)-action on $\underline{C}^{k,l}$ forces it to act as a multiple of the identity on each irreducible SU(2)-representation by Schur's lemma. Hence we choose a representation $\gamma : \mathbb{Z}_{q+1} \to \mathrm{U}(p_0)$ of $\mathbb{Z}_{q+1}$ on $\underline{C}^{k,l}$ such that $\mathbb{Z}_{q+1}$ acts on $\underline{(n,m)}$ as

$$\gamma(h)\big|_{\underline{(n,m)}} = \zeta_{q+1}^m \, \mathbb{1}_{n+1} \in \mathrm{U}(1) \, . \tag{3.15}$$

Consider the two parts of the connection (3.12): The connection Γ and the endomorphism-valued 1-form $I_\mu \, e^\mu$. In terms of matrix elements, Γ is completely determined by the 1-forms $B_{(n,m)} \in \Omega^1\big(\mathfrak{su}(2), \mathrm{End}(\underline{n,m})\big)$ which are instanton connections on the $\widetilde{\mathrm{K}}$-equivariant vector V-bundle

$$\widetilde{\mathcal{V}}_{\underline{(n,m)}} \xrightarrow{\underline{(n,m)}} \mathrm{G}/\widetilde{\mathrm{K}} \cong S^5/\mathbb{Z}_{q+1} \quad \text{with} \quad \mathcal{V}_{\underline{(n,m)}} := \mathrm{G} \times_{\mathrm{K}} \underline{(n,m)} \, , \tag{3.16}$$

simply because they are K-equivariant by construction and $\mathbb{Z}_{q+1} \hookrightarrow \mathrm{U}(1) \subset \mathrm{SU}(3)$ commutes with this particular SU(2) subgroup (see also Appendix A). More explicitly, taking (3.15) one observes that $\mathbb{Z}_{q+1}$ acts trivially on the endomorphism part,

$$\gamma(h) \, B_{(n,m)} \, \gamma(h)^{-1} = B_{(n,m)} \, , \tag{3.17}$$

as well as on the 1-form parts Γ_i because they are horizontal in the V-bundle (2.23). For $\mathbb{Z}_{q+1}$-equivariance of the second term $I_\mu \, e^\mu$, from (3.12a) and the representation π defined in (2.39) one demands the conditions

$$\gamma(h) \, E_{\mathfrak{w}} \, \gamma(h)^{-1} = \pi(h)^{-1}(E_{\mathfrak{w}}) = \zeta_{q+1}^3 \, E_{\mathfrak{w}} \quad \text{for} \quad \mathfrak{w} = \alpha_2, \alpha_1 + \alpha_2 \, , \tag{3.18a}$$

$$\gamma(h) \, E_{-\mathfrak{w}} \, \gamma(h)^{-1} = \pi(h)^{-1}(E_{-\mathfrak{w}}) = \zeta_{q+1}^{-3} \, E_{-\mathfrak{w}} \quad \text{for} \quad \mathfrak{w} = \alpha_2, \alpha_1 + \alpha_2 \, , \tag{3.18b}$$

$$\gamma(h) \, H_{\alpha_2} \, \gamma(h)^{-1} = \pi(h)^{-1}(H_{\alpha_2}) = H_{\alpha_2} \, . \tag{3.18c}$$

One can check that these conditions are satisfied by our choice of representation (3.15), due to the explicit components of the generators (B.2). We conclude that, due to our ansatz for the connection (3.12) on the principal V-bundle (2.23) and the embedding $\mathbb{Z}_{q+1} \hookrightarrow \mathrm{U}(1) \subset \mathrm{SU}(3)$, the 1-form $\mathcal{A}_0$ is indeed $\mathbb{Z}_{q+1}$-equivariant.

3.2.3. Quiver representations

Recall from [23] that one can interpret the decomposition (3.9) and the structure of the connection (3.12) as a quiver associated to $\underline{C}^{k,l}$ as follows: The appearing H-representations $\underline{(n,m)}$ form a set $Q_0(k,l)$ of vertices, whereas the actions of the generators E_{α_2} and $E_{\alpha_1+\alpha_2}$ intertwine the H-modules. These H-morphisms, together with H_{α_2}, constitute a set $Q_1(k,l)$ of arrows $(n,m) \to (n',m')$ between the vertices. The quiver $\mathcal{Q}^{k,l}$ is then given by the pair $\mathcal{Q}^{k,l} = \big(Q_0(k,l), \, Q_1(k,l)\big)$; the underlying graph of this quiver is obtained from the weight diagram of the representation $\underline{C}^{k,l}$ by collapsing all horizontal edges to vertices, cf. [23]. See Appendix C for an explicit treatment of the examples $\underline{C}^{1,0}$, $\underline{C}^{2,0}$ and $\underline{C}^{1,1}$.

3.3. Quiver bundles and connections

In the following we will consider representations of quivers not in the category of vector spaces, but rather in the category of vector bundles. We shall construct a G-equivariant gauge theory on the product space

$$M^d \times_{\widetilde{K}} G := M^d \times G/\widetilde{K} = M^d \times S^5/\mathbb{Z}_{q+1} \tag{3.19}$$

where G and all of its subgroups act trivially on a d-dimensional Riemannian manifold M^d. The equivariant dimensional reduction compensates isometries on $G/\widetilde{K}$ with gauge transformations, thus leading to quiver gauge theories on the manifold M^d.

Roughly speaking, the reduction is achieved by extending the homogeneous V-bundles (3.11) by $\widetilde{K}$-equivariant bundles $E \to M^d$, which furnish a representation of the corresponding quiver in the category of complex vector bundles over M^d. Such a representation is called a *quiver bundle* and it originates from the one-to-one correspondence between G-equivariant Hermitian vector V-bundles over $M^d \times G/\widetilde{K}$ and $\widetilde{K}$-equivariant Hermitian vector bundles over M^d, where $\widetilde{K}$ acts trivially on the base space M^d [31].

3.3.1. Equivariant bundles

For each irreducible H-representation $\underline{(n,m)}$ in the decomposition of $\underline{C}^{k,l}$, construct the (trivial) vector bundle

$$\underline{(n,m)}_{M^d} := M^d \times_{\widetilde{K}} \underline{(n,m)} \xrightarrow{\underline{(n,m)}} M^d \tag{3.20}$$

of rank $n+1$, which is $\widetilde{K}$-equivariant due to the trivial $\widetilde{K}$-action on M^d and the linear action on the fibres. For each module $\underline{(n,m)}$ introduce also a Hermitian vector bundle

$$E_{p_{(n,m)}} \xrightarrow{\mathbb{C}^{p_{(n,m)}}} M^d \quad \text{with} \quad \text{rk}(E_{p_{(n,m)}}) = p_{(n,m)} \tag{3.21}$$

with structure group $U(p_{(n,m)})$ and a $\mathfrak{u}(p_{(n,m)})$-valued connection $A_{(n,m)}$, and with trivial $\widetilde{K}$-action. Denote the identity endomorphism on the fibres of $E_{p_{(n,m)}}$ by $\pi_{(n,m)}$. With these data one constructs a $\widetilde{K}$-equivariant bundle

$$E^{k,l} \cong \bigoplus_{(n,m) \in Q_0(k,l)} E_{p_{(n,m)}} \otimes \underline{(n,m)}_{M^d} \xrightarrow{\mathbb{C}^p} M^d \tag{3.22}$$

whose rank p is given by

$$p = \sum_{(n,m) \in Q_0(k,l)} p_{(n,m)} \dim \underline{(n,m)} = \sum_{(n,m) \in Q_0(k,l)} p_{(n,m)} (n+1) . \tag{3.23}$$

Following [23], the bundle $E^{k,l}$ is the $\widetilde{K}$-equivariant vector bundle of rank p associated to the representation $\underline{C}^{k,l}|_{\widetilde{K}}$ of $\widetilde{K}$, and (3.22) is its isotopical decomposition. This construction breaks the structure group $U(p)$ of $E^{k,l}$ via the Higgs effect to the subgroup

$$\mathcal{G}^{k,l} := \prod_{(n,m) \in Q_0(k,l)} U(p_{(n,m)})^{n+1} \tag{3.24}$$

which commutes with the SU(2)-action on the fibres of (3.22).

On the other hand, one can introduce $\widetilde{\mathrm{K}}$-equivariant V-bundles over $S^5/\mathbb{Z}_{q+1}$ by (3.16). On $\mathcal{V}_{(n,m)}$ one has the $\mathfrak{su}(2)$-valued 1-instanton connection $B_{(n,m)}$ in the $(n+1)$-dimensional irreducible representation. The aim is to establish a G-equivariant V-bundle $\mathcal{E}^{k,l}$ over $M^d \times S^5/\mathbb{Z}_{q+1}$ as an extension of the $\widetilde{\mathrm{K}}$-equivariant bundle $E^{k,l}$. By the results of [31] such a V-bundle $\mathcal{E}^{k,l}$ exists and according to [23] it is realised as

$$\mathcal{E}^{k,l} := \mathrm{G} \times_{\widetilde{\mathrm{K}}} E^{k,l} = \bigoplus_{(n,m)\in Q_0(k,l)} E_{p_{(n,m)}} \boxtimes \underline{\mathcal{V}_{(n,m)}} \xrightarrow{\;V^{k,l}\;} M^d \times S^5/\mathbb{Z}_{q+1} \,, \tag{3.25}$$

where

$$V^{k,l} = \bigoplus_{(n,m)\in Q_0(k,l)} \mathbb{C}^{p_{(n,m)}} \otimes \underline{(n,m)} \tag{3.26}$$

is the typical fibre of (3.25).

3.3.2. Generic G-equivariant connection

The task now is to determine the generic form of a G-equivariant connection on (3.25). Since the space of connections on $\mathcal{E}^{k,l}$ is an affine space modelled over $\Omega^1(\mathrm{End}(\mathcal{E}^{k,l}))^{\mathrm{G}}$, one has to study the G-representations on this vector space. Recall from [23] that the decomposition of $\Omega^1(\mathrm{End}(\mathcal{E}^{k,l}))^{\mathrm{G}}$ with respect to G yields a "diagonal" subspace which accommodates the connections $A_{(n,m)}$ on (3.21) twisted by G-equivariant connections on (3.16), and an "off-diagonal" subspace which gives rise to bundle morphisms.

In other words, K-equivariance alone introduces only the connections $A_{(n,m)}$ on each bundle (3.21) as well as the SU(2)-connections $B_{(n,m)}$ on the V-bundles (3.16). On the other hand, G-equivariance additionally requires one to introduce a set of bundle morphisms

$$\phi^{\pm}_{(n,m)} \in \mathrm{Hom}\big(E_{p_{(n,m)}}, E_{p_{(n\pm 1,m+3)}}\big) \tag{3.27a}$$

and their adjoint maps

$$(\phi^{\pm})^{\dagger}_{(n,m)} \in \mathrm{Hom}\big(E_{p_{(n\pm 1,m+3)}}, E_{p_{(n,m)}}\big) \,, \tag{3.27b}$$

for all $(n,m) \in Q_0(k,l)$; one further introduces the bundle endomorphisms

$$\psi_{(n,m)} \in \mathrm{End}\big(E_{p_{(n,m)}}\big) \tag{3.27c}$$

at each vertex $(n,m) \in Q_0(k,l)$ with $m \neq 0$. The morphisms $\phi^{\pm}_{(n,m)}$ and $\psi_{(n,m)}$ are collectively called *Higgs fields*, and they realise the G-action in the same way that the generators I_{μ} (or more precisely the 1-forms $\bar{\beta}^{\pm}_{(n,m)}$ and $\frac{\mathrm{i}m}{2}\,\eta\,\Pi_{(n,m)}$) do in the case of the flat connection (3.12). The "new" Higgs fields $\psi_{(n,m)}$ implementing the vertical connection components on the (orbifold of the) Hopf bundle $S^5 \to \mathbb{C}P^2$ must be Hermitian, i.e. $\psi_{(n,m)} = \psi^{\dagger}_{(n,m)}$, by construction in order for the connection to be $\mathfrak{u}(p)$-valued.

3.3.3. Ansatz for connection

The ansatz for a G-equivariant connection on the equivariant V-bundle (3.25) is given by

$$\mathcal{A} = \widehat{A} + \widehat{\Gamma} + X_{\mu}\, e^{\mu} \tag{3.28}$$

wherein the $\mathfrak{u}(p_{(n,m)})$-valued connections $A_{(n,m)}$ and the $\mathfrak{su}(2)$-valued connection Γ are extended as

$$\widehat{A} := \bigoplus_{(n,m)} A_{(n,m)} \otimes \Pi_{(n,m)} \equiv A \otimes \mathbb{1} \quad \text{and} \quad \widehat{\Gamma} := \bigoplus_{(n,m)} \pi_{(n,m)} \otimes \Gamma^i \, I_i^{(n,m)} = \Gamma^i \, \widehat{I}_i \equiv \mathbb{1} \otimes \Gamma \,,$$

(3.29)

together with $\widehat{I}_i = \bigoplus_{(n,m)} \pi_{(n,m)} \otimes I_i^{(n,m)}$. The matrices X_μ are required to satisfy the *equivariance condition* [22,32]

$$\left[\widehat{I}_i, X_\mu \right] = f_{i\mu}{}^\nu X_\nu \quad \text{for} \quad i = 6,7,8 \quad \text{and} \quad \mu = 1,\ldots,5 \,.$$

(3.30)

As explained in [32], the equivariance condition ensures that X_μ are frame-independently defined endomorphisms that are the components of an endomorphism-valued 1-form, which is here given as the difference $\mathcal{A} - (\widehat{A} + \widehat{\Gamma})$.

The general solution to (3.30) expresses X_μ in terms of Higgs fields and generators as

$$\tfrac{1}{2}(X_1 + \mathrm{i}\,X_2) = \bigoplus_{\pm,(n,m)} \phi^\pm_{(n,m)} \otimes E^{\pm\,(n,m)}_{\alpha_1+\alpha_2} \,,$$

$$\tfrac{1}{2}(X_1 - \mathrm{i}\,X_2) = - \bigoplus_{\pm,(n,m)} (\phi^\pm)^\dagger_{(n,m)} \otimes E^{\pm\,(n,m)}_{-\alpha_1-\alpha_2} \,,$$

(3.31a)

$$\tfrac{1}{2}(X_3 + \mathrm{i}\,X_4) = \bigoplus_{\pm,(n,m)} \phi^\pm_{(n,m)} \otimes E^{\pm\,(n,m)}_{\alpha_2} \,,$$

$$\tfrac{1}{2}(X_3 - \mathrm{i}X_4) = - \bigoplus_{\pm,(n,m)} (\phi^\pm)^\dagger_{(n,m)} \otimes E^{\pm\,(n,m)}_{-\alpha_2} \,,$$

(3.31b)

$$X_5 = -\frac{\mathrm{i}}{2} \bigoplus_{(n,m)} \psi_{(n,m)} \otimes H^{(n,m)}_{\alpha_2} \,.$$

(3.31c)

Altogether the G-equivariant connection takes the form

$$\mathcal{A} = \bigoplus_{(n,m)\in Q_0(k,l)} \Big(A_{(n,m)} \otimes \Pi_{(n,m)} + \pi_{(n,m)} \otimes B_{(n,m)} - \psi_{(n,m)} \otimes \tfrac{\mathrm{i}m}{2} \, \eta \, \Pi_{(n,m)}$$
$$+ \phi^+_{(n,m)} \otimes \bar{\beta}^+_{(n,m)} + \phi^-_{(n,m)} \otimes \bar{\beta}^-_{(n,m)} - (\phi^+)^\dagger_{(n,m)} \otimes \beta^+_{(n,m)} - (\phi^-)^\dagger_{(n,m)} \otimes \beta^-_{(n,m)} \Big) \,.$$

(3.32)

3.3.4. $\mathbb{Z}_{q+1}$-equivariance

One needs to extend the $\mathbb{Z}_{q+1}$-representation γ of (3.15) to act on the fibres (3.26) of the equivariant V-bundle (3.25). Since by construction $\widetilde{K} = \mathrm{SU}(2) \times \mathbb{Z}_{q+1}$ acts trivially on the fibres of the bundles (3.21), one ends up with the representation $\gamma : \mathbb{Z}_{q+1} \to \mathrm{U}(p)$ given by

$$\gamma(h) = \bigoplus_{(n,m)\in Q_0(k,l)} \mathbb{1}_{p_{(n,m)}} \otimes \gamma(h)\big|_{\underline{(n,m)}} = \bigoplus_{(n,m)\in Q_0(k,l)} \mathbb{1}_{p_{(n,m)}} \otimes \zeta^m_{q+1} \mathbb{1}_{n+1} \,.$$

(3.33)

To prove $\mathbb{Z}_{q+1}$-equivariance of (3.28) one again needs to show two things. Firstly, the connections $A \otimes \mathbb{1}$ and $\mathbb{1} \otimes \Gamma$ have to be $\mathbb{Z}_{q+1}$-equivariant. This can be seen as follows: For $A \otimes \mathbb{1}$ the representation γ of (3.33) acts trivially on each bundle $E_{p_{(n,m)}}$, and thus

$$\gamma(h)\,(A \otimes \mathbb{1})\,\gamma(h)^{-1} = A \otimes \mathbb{1} \,.$$

(3.34)

Furthermore, $\mathbb{1} \otimes \Gamma$ is $\mathbb{Z}_{q+1}$-equivariant because Γ is by (3.17), and hence the connection $A \otimes \mathbb{1} + \mathbb{1} \otimes \Gamma$ satisfies the equivariance conditions.

Secondly, the endomorphism-valued 1-form $X_\mu e^\mu = \mathcal{A} - \widehat{A} - \widehat{\Gamma}$ needs to be $\mathbb{Z}_{q+1}$-equivariant as well. Due to its structure, one needs to consider a combination of the adjoint action of γ from (3.33) and the $\mathbb{Z}_{q+1}$-action on forms from (2.39). As γ acts trivially on each bundle $E_{p_{(n,m)}}$, the $\mathbb{Z}_{q+1}$-equivariance conditions

$$\gamma(h)\, X_\mu \,\gamma(h)^{-1} = \pi(h)^{-1}(X_\mu) \quad \text{for} \quad \mu = 1, \ldots, 5 \tag{3.35}$$

hold also for the quiver connection $\mathcal{A}$ just as they hold for the flat connection $\mathcal{A}_0$ by (3.18).

Thus the chosen representations (2.39) and (3.33) render the quiver connection (3.28) equivariant with respect to the action of $\mathbb{Z}_{q+1}$. On each irreducible representation $\underline{(n,m)}$ the generator h of $\mathbb{Z}_{q+1}$ is represented by $\zeta_{q+1}^m \mathbb{1}_{n+1}$ which depends on the U(1) monopole charge but not on the SU(2) isospin. This comes about as follows: The bundle morphisms associated to β_{q+1}^i map between bundles $E_{p_{(n,m)}} \otimes \underline{(n,m)}_{M^d}$ that differ in m by -3 (from source to target vertex), but differ in n by either $+1$ or -1. Thus the representation γ should only be sensitive to m and not to n. We shall elucidate this point further in Section 6.1.

3.3.5. Curvature

The curvature $\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ of the connection (3.28) is given by

$$\mathcal{F} = F_A \otimes \mathbb{1} + \mathbb{1} \otimes F_\Gamma + \big(\mathrm{d}X_\mu + \big[\widehat{A}, X_\mu\big]\big) \wedge e^\mu + X_\mu \, \mathrm{d}e^\mu + \big[\widehat{\Gamma}, X_\mu\big] \wedge e^\mu$$
$$+ \tfrac{1}{2} \big[X_\mu, X_\nu\big] e^{\mu\nu}, \tag{3.36a}$$

where $F_A = \mathrm{d}A + A \wedge A$. Employing the relations (3.14) then yields

$$\mathcal{F} = F_A \otimes \mathbb{1} + \big(\mathrm{d}X_\mu + \big[\widehat{A}, X_\mu\big]\big) \wedge e^\mu + \Gamma^i \big(\big[\widehat{I_i}, X_\mu\big] - f_{i\mu}{}^\nu X_\nu\big) \wedge e^\mu$$
$$+ \tfrac{1}{2} \big(\big[X_\mu, X_\nu\big] - f_{\mu\nu}{}^\rho X_\rho - f_{\mu\nu}{}^i \widehat{I_i}\big) e^{\mu\nu}. \tag{3.36b}$$

Since the matrices X_μ satisfy the equivariance relation (3.30), the final form of the curvature reads

$$\mathcal{F} = F_A \otimes \mathbb{1} + (DX)_\mu \wedge e^\mu + \tfrac{1}{2} \big(\big[X_\mu, X_\nu\big] - f_{\mu\nu}{}^\rho X_\rho - f_{\mu\nu}{}^i \widehat{I_i}\big) e^{\mu\nu}, \tag{3.36c}$$

where we defined the bifundamental covariant derivatives as

$$(DX)_\mu := \mathrm{d}X_\mu + \big[\widehat{A}, X_\mu\big]. \tag{3.36d}$$

Inserting the explicit form (3.31) for the scalar fields X_μ leads to the curvature components in the Biedenharn basis; the detailed expressions are summarised in Appendix B.3.

3.3.6. Quiver bundles

Let us now exemplify and clarify how the equivariant bundle $E^{k,l} \to M^d$ from (3.22) realises a quiver bundle from our constructions above. Recall that the quiver consists of the pair $\mathcal{Q}^{k,l} = \big(Q_0(k,l),\, Q_1(k,l)\big)$, with vertices $(n,m) \in Q_0(k,l)$ and arrows $(n,m) \to (n',m') \in Q_1(k,l)$ between certain pairs of vertices which are here determined by the decomposition (3.9). We consider a representation $\widetilde{\mathcal{Q}}^{k,l} = \big(\widetilde{Q}_0(k,l),\, \widetilde{Q}_1(k,l)\big)$ of this quiver in the category of complex vector bundles. The set of vertices is

$$\widetilde{Q}_0(k,l) = \big\{E_{p_{(n,m)}} \longrightarrow M^d, \quad (n,m) \in Q_0(k,l)\big\}, \tag{3.37}$$

i.e. the set of Hermitian vector bundles each equipped with a unitary connection $A_{(n,m)}$. The set of arrows is

$$\widetilde{Q}_1(k,l) = \left\{ \phi^{\pm}_{(n,m)} \in \mathrm{Hom}\big(E_{p(n,m)}, E_{p(n\pm 1,m+3)}\big), \quad (n,m) \in Q_0(k,l) \right\}$$

$$\cup \left\{ \psi_{(n,m)} \in \mathrm{End}\big(E_{p(n,m)}\big), \quad (n,m) \in Q_0(k,l), \; m \neq 0 \right\}, \tag{3.38}$$

which is precisely the set of bundle morphisms, i.e. the Higgs fields. These quivers differ from those considered in [23] by the appearance of vertex loops corresponding to the endomorphisms $\psi_{(n,m)}$. See Appendix C for details of the quiver bundles based on the representations $\underline{C}^{1,0}$, $\underline{C}^{2,0}$ and $\underline{C}^{1,1}$.

These constructions yield representations of quivers without any relations. We will see later on that relations can arise by minimising the scalar potential of the quiver gauge theory (see Section 3.4) or by imposing a generalised instanton equation on the connection $\mathcal{A}$ (see Section 4).

3.4. Dimensional reduction of the Yang–Mills action

Consider the reduction of the pure Yang–Mills action from $M^d \times S^5$ to M^d. On S^5 we take as basis of coframes $\{\beta^j_\varphi, \bar{\beta}^j_\varphi\}_{j=1,2}$ and $e^5 = \eta$, and as metric

$$\mathrm{ds}^2_{S^5} = R^2 \left(\beta^1_\varphi \otimes \bar{\beta}^1_\varphi + \bar{\beta}^1_\varphi \otimes \beta^1_\varphi + \beta^2_\varphi \otimes \bar{\beta}^2_\varphi + \bar{\beta}^2_\varphi \otimes \beta^2_\varphi \right) + r^2 \, \eta \otimes \eta \,. \tag{3.39}$$

The Yang–Mills action is given by

$$S = -\frac{1}{4\tilde{g}^2} \int\limits_{M^d \times S^5} \mathrm{tr}\, \mathcal{F} \wedge \star \mathcal{F}, \tag{3.40}$$

with coupling constant $\tilde{g}$ and $\star$ the Hodge duality operator corresponding to the metric on $M^d \times S^5$ given by

$$\mathrm{ds}^2 = \mathrm{ds}^2_{M^d} + \mathrm{ds}^2_{S^5} \,. \tag{3.41}$$

We denote the Hodge operator corresponding to the metric $\mathrm{ds}^2_{M^d}$ on M^d by $\star_{M^d}$. The reduction of (3.40) proceeds by inserting the curvature (3.36c) and performing the integrals over S^5, which can be evaluated by using (3.39) and the identities of Appendix D.2. One finally obtains for the reduced action

$$S = -\frac{2\pi^3 r R^4}{\tilde{g}^2} \left(\int\limits_{M^d} \mathrm{tr}\big(F_A \wedge \star_{M^d} F_A\big) \otimes \mathbb{1} \right.$$

$$+ \frac{1}{2R^2} \int\limits_{M^d} \sum_{a=1}^4 \mathrm{tr}\,(DX)_a \wedge \star_{M^d} (DX)_a + \frac{1}{r^2} \int\limits_{M^d} \mathrm{tr}\,(DX)_5 \wedge \star_{M^d} (DX)_5$$

$$+ \frac{1}{8R^4} \int\limits_{M^d} \star_{M^d} \sum_{a,b=1}^4 \mathrm{tr}\big([X_a, X_b] - f_{ab}{}^5 X_5 - f_{ab}{}^i \widehat{I}_i\big)^2$$

$$+ \frac{1}{8R^2 r^2} \int\limits_{M^d} \star_{M^d} \sum_{a=1}^4 \mathrm{tr}\big([X_a, X_5] - f_{a5}{}^b X_b\big)^2 \right). \tag{3.42}$$

Here the explicit structure constants (3.4), i.e. $f_{ab}{}^c = f_{a5}{}^5 = f_{a5}{}^i = 0$, have been used. One may detail this action further by inserting the G-equivariant solution (3.31) for the scalar

fields X_μ in the Biedenharn basis, which allows one to perform the trace over the SU(2) × U(1)-representations (n, m). The explicit but lengthy formulas are given in Appendix D.3.

3.4.1. Higgs branch

On the Higgs branch of the quiver gauge theory where all connections $A_{(n,m)}$ are trivial and the Higgs fields are constant, the vacuum is solely determined by the vanishing locus of the scalar potential. The vanishing of the potential gives rise to holomorphic F-term constraints as well as non-holomorphic D-term constraints which read as

$$[X_a, X_b] = f_{ab}{}^5 X_5 + f_{ab}{}^i \widehat{I}_i \quad \text{and} \quad [X_a, X_5] = f_{a5}{}^b X_b \,, \tag{3.43}$$

for $a, b = 1, 2, 3, 4$. The equivariance condition (3.30) implies that X_μ lie in a representation of the $\mathfrak{su}(2)$ Lie algebra. Hence the BPS configurations of the gauge theory X_μ, together with $\widehat{I}_i$, furnish a representation of the Lie algebra $\mathfrak{su}(3)$ in the representation space of the quiver in $\mathfrak{u}(p)$. These constraints respectively give rise to a set of relations and a set of stability conditions for the corresponding quiver representation. The details can be read off from the explicit expressions in Appendix D.3.

4. Spherically symmetric instantons

In this section we specialise to the case where the Riemannian manifold $M^d = M^1$ is 1-dimensional. We investigate the Hermitian Yang–Mills equations on the product $M^1 \times S^5/\mathbb{Z}_{q+1}$ for the generic form of G-equivariant connections derived in Section 3.3.

4.1. Preliminaries

Consider the product manifold $M^1 \times S^5/\mathbb{Z}_{q+1}$ with $M^1 = \mathbb{R}$ such that $M^1 \times S^5/\mathbb{Z}_{q+1} \cong C(S^5/\mathbb{Z}_{q+1})$ is the metric cone over the Sasaki–Einstein space $S^5/\mathbb{Z}_{q+1}$, which is an orbifold of the Calabi–Yau manifold $C(S^5)$. The Calabi–Yau space $C(S^5)$ is conformally equivalent to the cylinder $\mathbb{R} \times S^5$ with the metric

$$\mathrm{d}s^2_{C(S^5)} = \mathrm{d}r^2 + r^2 \, \mathrm{d}s^2_{S^5} = r^2 \left(\mathrm{d}\tau^2 + \mathrm{d}s^2_{S^5} \right) = \mathrm{e}^{2\tau} \left(\mathrm{d}\tau^2 + \delta_{\mu\nu} \, e^\mu \otimes e^\nu \right) \tag{4.1}$$

where $\tau = \log r$. The Kähler 2-form is given by

$$\omega_{C(S^5)} = \mathrm{e}^{2\tau} \left(\omega_3 + \eta \wedge \mathrm{d}\tau \right) \,. \tag{4.2}$$

4.1.1. Connections

As $\mathbb{R}$ is contractible, each bundle $E_{p_{(n,m)}} \to \mathbb{R}$ is necessarily trivial and hence one can gauge away the (global) connection 1-forms $A_{(n,m)} = A_{(n,m)}(\tau) \, \mathrm{d}\tau$; explicitly, there is a gauge transformation $g : \mathbb{R} \to \mathcal{G}^{k,l}$ such that

$$\tilde{A}_{(n,m)} = \mathrm{Ad}(g^{-1}) A_{(n,m)} + g^{-1} \frac{\mathrm{d}g}{\mathrm{d}\tau} = 0 \quad \text{with} \quad g = \exp\left(-\int A_{(n,m)}(\tau) \, \mathrm{d}\tau \right). \tag{4.3}$$

The ansatz for the connection on the equivariant V-bundle then reads

$$\mathcal{A} = \mathbb{1} \otimes \Gamma + X_\mu \, e^\mu \,, \tag{4.4}$$

where the Higgs fields $\phi^\pm_{(n,m)}$ and $\psi_{(n,m)}$ depend only on the cone coordinate τ (compare also with [32, Section 4.1]). The curvature of this connection can be read off from (3.36c) and is

evaluated to

$$\mathcal{F} = \frac{\mathrm{d}X_\mu}{\mathrm{d}\tau}\,\mathrm{d}\tau \wedge e^\mu + \frac{1}{2}\left(\left[X_\mu, X_\nu\right] - f_{\mu\nu}{}^\rho\, X_\rho - f_{\mu\nu}{}^i\, \widehat{I}_i\right)e^{\mu\nu}\,. \tag{4.5}$$

4.2. Generalised instanton equations

The ansatz (4.4) restricts the space of all connections on the SU(3)-equivariant vector V-bundle over $C(S^5/\mathbb{Z}_{q+1})$ to SU(3)-equivariant and $\mathbb{Z}_{q+1}$-equivariant connections.

4.2.1. Quiver relations
On this subspace of connections one can further restrict to holomorphic connections, i.e. connections which allow for a holomorphic structure.[5] For this, one requires the holomorphicity condition $\mathcal{F}^{0,2} = 0 = \mathcal{F}^{2,0}$ which for the connection (4.4) is equivalent to

$$\mathcal{F}_{14} + \mathcal{F}_{23} = 0\,, \qquad \mathcal{F}_{1\tau} + \mathcal{F}_{25} = 0\,, \qquad \mathcal{F}_{3\tau} + \mathcal{F}_{45} = 0\,, \tag{4.6a}$$

$$\mathcal{F}_{13} - \mathcal{F}_{24} = 0\,, \qquad \mathcal{F}_{15} - \mathcal{F}_{2\tau} = 0\,, \qquad \mathcal{F}_{35} - \mathcal{F}_{4\tau} = 0\,. \tag{4.6b}$$

Substituting the explicit components of the curvature (4.5), one finds relations for the endomorphisms X_μ given by

$$[X_1, X_4] + [X_2, X_3] = 0 = [X_1, X_3] - [X_2, X_4] \quad \text{and}$$
$$[X_a, X_5] = f_{a5}{}^b\left(X_b + \frac{2}{3}\frac{\mathrm{d}X_b}{\mathrm{d}\tau}\right) \tag{4.7}$$

for $a = 1, 2, 3, 4$.

4.2.2. Stability conditions
By well-known theorems from algebraic geometry [33–35], a holomorphic vector bundle admits solutions to the Hermitian Yang–Mills equations if and only if it is stable. This condition can be translated into a condition on the remaining $(1, 1)$-part of the curvature $\mathcal{F}$: One demands that $\mathcal{F}$ is a primitive $(1, 1)$-form, i.e. $\omega_{C(S^5)} \lrcorner \mathcal{F} = 0$, or in components

$$\mathcal{F}_{12} + \mathcal{F}_{34} + \mathcal{F}_{5\tau} = 0\,. \tag{4.8}$$

Using the explicit components (4.5) one can deduce the matrix differential equation for X_μ given by

$$[X_1, X_2] + [X_3, X_4] = 4X_5 + \frac{\mathrm{d}X_5}{\mathrm{d}\tau}\,. \tag{4.9}$$

One can also regard the stability condition in terms of a moment map μ from the space of holomorphic connections to the dual of the Lie algebra of the gauge group [36]. The dual μ^* then acts on a connection $\mathcal{A}$ via $\mu^*(\mathcal{A}) = \omega_{C(S^5)} \lrcorner \mathcal{F}$, which is well-defined as the curvature $\mathcal{F}$ is a Lie algebra-valued 2-form. Then the stability conditions correspond to the level set of zeroes $\mu^{*-1}(0)$; we shall return to this interpretation in Section 6.2.2.

---

[5] For a Hermitian connection $\mathcal{A}$ on a complex vector bundle, the requirement for it to induce a holomorphic structure is equivalent to the $(0, 1)$-part $\mathcal{A}^{0,1}$ of $\mathcal{A}$ being integrable, i.e. the corresponding curvature $\mathcal{F}$ is of type $(1, 1)$.

4.3. Examples

We shall now apply these considerations to the three simplest examples: The quivers based on the representations $\underline{C}^{1,0}$, $\underline{C}^{2,0}$ and $\underline{C}^{1,1}$. For each example we explicitly provide the representation of the generators and the form of the matrices X_μ, followed by the quiver relations and the stability conditions.

4.3.1. $C^{1,0}$-quiver

The generators in the fundamental representation $\underline{C}^{1,0}$, which splits as in (C.1), are given as

$$
I_a = \begin{pmatrix} 0_2 & I_a^{(0,-2)} \\ -\big(I_a^{(0,-2)}\big)^\dagger & 0 \end{pmatrix} \quad \text{and} \quad I_5 = \begin{pmatrix} I_5^{(1,1)} & 0 \\ 0 & I_5^{(0,-2)} \end{pmatrix} \tag{4.10a}
$$

for $a = 1, 2, 3, 4$, with components

$$
I_1^{(0,-2)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = i\, I_2^{(0,-2)} \quad \text{and} \quad I_3^{(0,-2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = i\, I_4^{(0,-2)}\,, \tag{4.10b}
$$

$$
I_5^{(0,-2)} = i\,\mathbb{1}_2 \quad \text{and} \quad I_5^{(1,1)} = -\tfrac{i}{2}\,. \tag{4.10c}
$$

The endomorphisms X_μ read as

$$
X_a = \begin{pmatrix} 0_2 & \phi \otimes I_a^{(0,-2)} \\ -\phi^\dagger \otimes \big(I_a^{(0,-2)}\big)^\dagger & 0 \end{pmatrix} \quad \text{and} \quad X_5 = \begin{pmatrix} \psi_1 \otimes I_5^{(1,1)} & 0 \\ 0 & \psi_0 \otimes I_5^{(0,-2)} \end{pmatrix} \tag{4.11}
$$

where the Higgs fields from Appendix C give a representation of the quiver

$$\tag{4.12}$$

The $\mathbb{Z}_{q+1}$-representation (3.33) reads

$$
\gamma : h \longmapsto \begin{pmatrix} \mathbb{1}_{p_{(1,1)}} \otimes \mathbb{1}_2\, \zeta_{q+1} & 0 \\ 0 & \mathbb{1}_{p_{(0,-2)}} \otimes \zeta_{q+1}^{-2} \end{pmatrix}\,, \tag{4.13}
$$

where h is the generator of the cyclic group $\mathbb{Z}_{q+1}$.

4.3.1.1. Quiver relations
The first two equations from (4.7) are trivially satisfied without any further constraints. The second set of equations all have the same non-trivial off-diagonal component (and its adjoint) which yields

$$
2\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = -3\,\phi + 2\,\phi\,\psi_0 + \psi_1\,\phi\,. \tag{4.14}
$$

Thus for the $C^{1,0}$-quiver there are no purely algebraic quiver relations.

4.3.1.2. Stability conditions   From (4.9) we read off the two non-trivial diagonal components which yield

$$\frac{1}{4}\frac{d\psi_0}{d\tau} = -\psi_0 + \phi^\dagger\phi \quad\text{and}\quad \frac{1}{4}\frac{d\psi_1}{d\tau} = -\psi_1 + \phi\phi^\dagger\,. \tag{4.15}$$

By taking ψ_0 and ψ_1 to be identity endomorphisms, we recover the Higgs branch BPS equations from equivariant dimensional reduction over $\mathbb{C}P^2$: In this limit (4.14) implies that the scalar field ϕ is independent of τ, while (4.15) correctly reproduces the D-term constraints of the quiver gauge theory for constant matrices [23,25].

4.3.2. $C^{2,0}$-quiver

The generators in the 6-dimensional representation $\underline{C}^{2,0}$, which splits as in (C.3), are given by

$$I_a = \begin{pmatrix} 0_3 & I_a^{(1,-1)} & 0 \\ -(I_a^{(1,-1)})^\dagger & 0_2 & I_a^{(0,-4)} \\ 0 & -(I_a^{(0,-4)})^\dagger & 0 \end{pmatrix} \quad\text{and}\quad I_5 = \begin{pmatrix} I_5^{(2,2)} & 0 & 0 \\ 0 & I_5^{(1,-1)} & 0 \\ 0 & 0 & I_5^{(0,-4)} \end{pmatrix} \tag{4.16a}$$

for $a = 1, 2, 3, 4$, with components

$$I_1^{(1,-1)} = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = i\,I_2^{(1,-1)} \quad\text{and}\quad I_1^{(0,-4)} = \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix} = i\,I_2^{(0,-4)}\,, \tag{4.16b}$$

$$I_3^{(1,-1)} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & \sqrt{2} \end{pmatrix} = i\,I_4^{(1,-1)} \quad\text{and}\quad I_3^{(0,-4)} = \begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix} = i\,I_4^{(0,-4)}\,, \tag{4.16c}$$

$$I_5^{(2,2)} = -i\,\mathbb{1}_3\,, \quad I_5^{(1,-1)} = \tfrac{i}{2}\,\mathbb{1}_2 \quad\text{and}\quad I_5^{(0,-4)} = 2\,i\,. \tag{4.16d}$$

The endomorphisms X_μ read

$$X_a = \begin{pmatrix} 0_3 & \phi_1 \otimes I_a^{(1,-1)} & 0 \\ -\phi_1^\dagger \otimes (I_a^{(1,-1)})^\dagger & 0_2 & \phi_0 \otimes I_a^{(0,-4)} \\ 0 & -\phi_0^\dagger \otimes (I_a^{(0,-4)})^\dagger & 0 \end{pmatrix}\,,$$

$$X_5 = \begin{pmatrix} \psi_2 \otimes I_5^{(2,2)} & 0 & 0 \\ 0 & \psi_1 \otimes I_5^{(1,-1)} & 0 \\ 0 & 0 & \psi_0 \otimes I_5^{(0,-4)} \end{pmatrix}\,, \tag{4.17}$$

with the Higgs field content from Appendix C that furnishes a representation of the quiver

$$\tag{4.18}$$

The representation (3.33) in this case reads

$$\gamma : h \longmapsto \begin{pmatrix} \mathbb{1}_{P(2,2)} \otimes \mathbb{1}_3\,\zeta_{q+1}^2 & 0 & 0 \\ 0 & \mathbb{1}_{P(1,-1)} \otimes \mathbb{1}_2\,\zeta_{q+1}^{-1} & 0 \\ 0 & 0 & \mathbb{1}_{P(0,-4)} \otimes \zeta_{q+1}^{-4} \end{pmatrix}\,. \tag{4.19}$$

4.3.2.1. Quiver relations   Again the first two equations of (4.7) turn out to be trivial, while the second set of equations have two non-vanishing off-diagonal components (plus their conjugates) which yield

$$2\frac{d\phi_0}{d\tau} = -3\phi_0 - \psi_1\phi_0 + 4\phi_0\psi_0 \quad \text{and} \quad 2\frac{d\phi_1}{d\tau} = -3\phi_1 + \phi_1\psi_1 + 2\psi_2\phi_1 \tag{4.20}$$

and the $C^{2,0}$-quiver has no purely algebraic quiver relations either.

4.3.2.2. Stability conditions   From (4.9) one obtains three non-trivial diagonal components that yield

$$\frac{1}{4}\frac{d\psi_0}{d\tau} = -\psi_0 + \phi_0^\dagger\phi_0 , \tag{4.21a}$$

$$\frac{1}{4}\frac{d\psi_1}{d\tau} = -\psi_1 - 2\phi_0\phi_0^\dagger + 3\phi_1^\dagger\phi_1 , \tag{4.21b}$$

$$\frac{1}{4}\frac{d\psi_2}{d\tau} = -\psi_2 + \phi_1\phi_1^\dagger . \tag{4.21c}$$

Taking ψ_0, ψ_1 and ψ_2 again to be identity morphisms, from (4.20) we obtain constant matrices ϕ_0 and ϕ_1 which by (4.21c) obey the expected D-term constraints from equivariant dimensional reduction over $\mathbb{C}P^2$ [23,25].

4.3.3. $C^{1,1}$-quiver

The decomposition of the adjoint representation $\underline{C}^{1,1}$, which splits as given in (C.5), yields

$$I_a = \begin{pmatrix} 0_2 & I_a^{(0,0)} & I_a^{(2,0)} & 0 \\ -\left(I_a^{(0,0)}\right)^\dagger & 0 & 0 & I_a^{-\,(1,-3)} \\ -\left(I_a^{(2,0)}\right)^\dagger & 0 & 0_3 & I_a^{+\,(1,-3)} \\ 0 & -\left(I_a^{-\,(1,-3)}\right)^\dagger & -\left(I_a^{+\,(1,-3)}\right)^\dagger & 0_2 \end{pmatrix} , \tag{4.22a}$$

$$I_5 = \begin{pmatrix} I_5^{(1,3)} & 0 & 0 & 0 \\ 0 & I_5^{(0,0)} & 0 & 0 \\ 0 & 0 & I_5^{(2,0)} & 0 \\ 0 & 0 & 0 & I_5^{(1,-3)} \end{pmatrix} \tag{4.22b}$$

for $a = 1, 2, 3, 4$, with components

$$I_1^{(0,0)} = \begin{pmatrix} \sqrt{\frac{3}{2}} \\ 0 \end{pmatrix} = i\,I_2^{(0,0)} \quad \text{and} \quad I_1^{(2,0)} = \begin{pmatrix} 0 & -\sqrt{\frac{1}{2}} & 0 \\ 0 & 0 & -1 \end{pmatrix} = i\,I_2^{(2,0)}, \tag{4.22c}$$

$$I_1^{-\,(1,-3)} = \begin{pmatrix} 0 & -\sqrt{\frac{3}{2}} \end{pmatrix} = i\,I_2^{-(1,-3)} \quad \text{and} \quad I_1^{+\,(1,-3)} = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\frac{1}{2}} \\ 0 & 0 \end{pmatrix} = i\,I_2^{+(1,-3)}, \tag{4.22d}$$

$$I_3^{(0,0)} = \begin{pmatrix} 0 \\ \sqrt{\frac{3}{2}} \end{pmatrix} = i\,I_4^{(0,0)} \quad \text{and} \quad I_3^{(2,0)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{\frac{1}{2}} & 0 \end{pmatrix} = i\,I_4^{(2,0)} , \tag{4.22e}$$

$$I_3^{-\,(1,-3)} = \begin{pmatrix} \sqrt{\frac{3}{2}} & 0 \end{pmatrix} = i\,I_4^{-\,(1,-3)} \quad \text{and} \quad I_3^{+\,(1,-3)} = \begin{pmatrix} 0 & 0 \\ \sqrt{\frac{1}{2}} & 0 \\ 0 & 1 \end{pmatrix} = i\,I_4^{+\,(1,-3)} , \tag{4.22f}$$
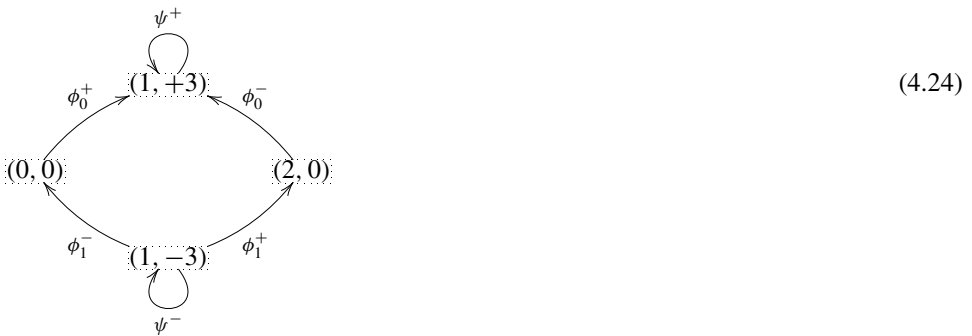
$$I_5^{(1,3)} = -\tfrac{3i}{2}\, 1\!\!1_2\,, \quad I_5^{(0,0)} = 0\,, \quad I_5^{(2,0)} = 0_3 \quad \text{and} \quad I_5^{(1,-3)} = \tfrac{3i}{2}\, 1\!\!1_2\,. \tag{4.22g}$$

The matrices X_μ are given by

$$X_a = \begin{pmatrix} 0_2 & \phi_0^+ \otimes I_a^{(0,0)} & \phi_0^- \otimes I_a^{(2,0)} & 0 \\ -(\phi_0^+)^\dagger \otimes (I_a^{(0,0)})^\dagger & 0 & 0 & \phi_1^- \otimes I_a^{-(1,-3)} \\ -(\phi_0^-)^\dagger \otimes (I_a^{(2,0)})^\dagger & 0 & 0_3 & \phi_1^+ \otimes I_a^{+(1,-3)} \\ 0 & -(\phi_1^-)^\dagger \otimes (I_a^{-(1,-3)})^\dagger & -(\phi_1^+)^\dagger \otimes (I_a^{+(1,-3)})^\dagger & 0_2 \end{pmatrix}, \tag{4.23a}$$

$$X_5 = \begin{pmatrix} \psi^+ \otimes I_5^{(1,3)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0_3 & 0 \\ 0 & 0 & 0 & \psi^- \otimes I_5^{(1,-3)} \end{pmatrix}. \tag{4.23b}$$

This example involves the collection of Higgs fields from Appendix C which furnish a representation of the quiver

$$\tag{4.24}$$

In this case the $\mathbb{Z}_{q+1}$-representation (3.33) has the form

$$\gamma : h \longmapsto \begin{pmatrix} 1\!\!1_{P_{(1,3)}} \otimes 1\!\!1_2\, \zeta_{q+1}^3 & 0 & 0 & 0 \\ 0 & 1\!\!1_{P_{(0,0)}} \otimes 1 & 0 & 0 \\ 0 & 0 & 1\!\!1_{P_{(2,0)}} \otimes 1\!\!1_3 & 0 \\ 0 & 0 & 0 & 1\!\!1_{P_{(1,-3)}} \otimes 1\!\!1_2\, \zeta_{q+1}^{-3} \end{pmatrix}. \tag{4.25}$$

4.3.3.1. Quiver relations For this 8-dimensional example, one finds that the first two equations of (4.7) have the same single non-trivial off-diagonal component (plus its adjoint) which yields

$$\phi_0^+ \phi_1^- = \phi_0^- \phi_1^+\,. \tag{4.26}$$

This equation is precisely the anticipated algebraic relation for the $C^{1,1}$-quiver expressing equality of paths between the vertices $(1, \pm 3)$, cf. [23]. The second set of equations have four non-trivial off-diagonal components (plus their conjugates) which yield

$$\frac{2}{3} \frac{d\phi_0^\pm}{d\tau} = -\phi_0^\pm + \psi^+ \phi_0^\pm \quad \text{and} \quad \frac{2}{3} \frac{d\phi_1^\pm}{d\tau} = -\phi_1^\pm + \phi_1^\pm \psi^-\,. \tag{4.27}$$

4.3.3.2. Stability conditions   From (4.9) one computes four non-vanishing diagonal components that yield

$$(\phi_0^{\pm})^{\dagger}\,\phi_0^{\pm} = \phi_1^{\mp}\,(\phi_1^{\mp})^{\dagger}\,, \tag{4.28a}$$

$$\frac{1}{4}\frac{\mathrm{d}\psi^{+}}{\mathrm{d}\tau} = -\psi^{+} + \frac{1}{2}\left(\phi_0^{+}\,(\phi_0^{+})^{\dagger} + \phi_0^{-}\,(\phi_0^{-})^{\dagger}\right)\,, \tag{4.28b}$$

$$\frac{1}{4}\frac{\mathrm{d}\psi^{-}}{\mathrm{d}\tau} = -\psi^{-} + \frac{1}{2}\left((\phi_1^{-})^{\dagger}\,\phi_1^{-} + (\phi_1^{+})^{\dagger}\,\phi_1^{+}\right)\,. \tag{4.28c}$$

We thus obtain two non-holomorphic purely algebraic conditions, which coincide with D-term constraints of the quiver gauge theory for the $C^{1,1}$-quiver, and two further differential equations which for identity endomorphisms $\psi^{\pm}$ reproduce the remaining stability equations for constant matrices $\phi_0^{\pm}$ and $\phi_1^{\pm}$ in equivariant dimensional reduction over $\mathbb{C}P^2$ [23,25].

5. Translationally-invariant instantons

In this section we study translationally-invariant instantons on a trivial vector V-bundle over the orbifold $\mathbb{C}^3/\mathbb{Z}_{q+1}$. In contrast to the G-equivariant Hermitian Yang–Mills instantons of Section 4, the generic form of a translationally-invariant connection is determined by $\mathbb{Z}_{q+1}$-equivariance alone and is associated with a different quiver.

5.1. Preliminaries

Consider the cone $C(S^5)/\mathbb{Z}_{q+1} \cong \mathbb{C}^3/\mathbb{Z}_{q+1}$, with the $\mathbb{Z}_{q+1}$-action given by (2.28), and the (trivial) vector V-bundle

$$\mathfrak{E}^{k,l} \xrightarrow{V^{k,l}} \mathbb{C}^3/\mathbb{Z}_{q+1} \tag{5.1}$$

of rank p which is obtained by suitable $\mathbb{Z}_{q+1}$-projection from the trivial vector bundle $\mathbb{C}^3 \times V^{k,l} \to \mathbb{C}^3$. The fibres of (5.1) can be regarded as representation spaces

$$V^{k,l} = \bigoplus_{(n,m)\in Q_0(k,l)} \mathbb{C}^{p(n,m)} \otimes \underline{(n,m)} \cong \bigoplus_{(n,m)\in Q_0(k,l)} \left(\mathbb{C}^{p(n,m)} \otimes \mathbb{C}^{n+1}\right) \otimes V_m\,. \tag{5.2}$$

Here V_m is the $[m]$-th irreducible representation $\rho_{[m]}$ of $\mathbb{Z}_{q+1}$ (cf. (3.10)), with $[m] \in \{0,1,\ldots,q\}$ the congruence class of $m \in \mathbb{Z}$ modulo $q+1$, and the vector space $\mathbb{C}^{p(n,m)} \otimes \mathbb{C}^{n+1}$ serves as the multiplicity space of this representation. The structure group of the bundle $\mathfrak{E}^{k,l}$ is

$$\mathfrak{G}^{k,l} := \prod_{(n,m)\in Q_0(k,l)} \mathrm{U}\big(p_{(n,m)}\,(n+1)\big)\,, \tag{5.3}$$

because the fibres are isomorphic to (5.2) and hence it carries a natural complex structure J; this complex structure is simply multiplication with i on each factor V_m. Consequently, the structure group is reduced to the stabiliser of J.

On the base the canonical Kähler form of $\mathbb{C}^3$ is given by

$$\omega_{\mathbb{C}^3} = \tfrac{\mathrm{i}}{2}\,\delta_{\alpha\beta}\,\mathrm{d}z^{\alpha} \wedge \mathrm{d}\bar{z}^{\beta}\,. \tag{5.4}$$

This Kähler form is compatible with the standard metric $\mathrm{d}s_{\mathbb{C}^3}^2 = \tfrac{1}{2}\,\delta_{\alpha\beta}\,(\mathrm{d}z^{\alpha} \otimes \mathrm{d}\bar{z}^{\beta} + \mathrm{d}\bar{z}^{\alpha} \otimes \mathrm{d}z^{\beta})$ and the complex structure $J(\mathrm{d}z^{\alpha}) = \mathrm{i}\,\mathrm{d}z^{\alpha}$, $J(\mathrm{d}\bar{z}^{\alpha}) = -\mathrm{i}\,\mathrm{d}\bar{z}^{\alpha}$.

5.1.1. Connections

Consider a connection 1-form

$$\mathcal{A} = W_\alpha \, \mathrm{d}z^\alpha + \overline{W}_\alpha \, \mathrm{d}\bar{z}^\alpha \tag{5.5}$$

on $\mathfrak{E}^{k,l}$, and impose translational invariance along the space $\mathbb{C}^3$. For the coordinate basis $\{\mathrm{d}z^\alpha, \mathrm{d}\bar{z}^\alpha\}$ of $T^*_{(z,\bar{z})}\mathbb{C}^3$ at any point $(z, \bar{z}) \in \mathbb{C}^3$, this translates into the condition

$$\mathrm{d}W_\alpha = 0 = \mathrm{d}\overline{W}_\alpha \quad \text{for} \quad \alpha = 1, 2, 3 \,. \tag{5.6}$$

Thus the curvature $\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ simplifies to

$$\mathcal{F} = \mathcal{A} \wedge \mathcal{A} = \tfrac{1}{2}\big[W_\alpha, W_\beta\big]\mathrm{d}z^\alpha \wedge \mathrm{d}z^\beta + \big[W_\alpha, \overline{W}_\beta\big]\mathrm{d}z^\alpha \wedge \mathrm{d}\bar{z}^\beta + \tfrac{1}{2}\big[\overline{W}_\alpha, \overline{W}_\beta\big]\mathrm{d}\bar{z}^\alpha \wedge \mathrm{d}\bar{z}^\beta \,. \tag{5.7}$$

5.1.2. $\mathbb{Z}_{q+1}$-action

As before one demands $\mathbb{Z}_{q+1}$-invariance due to the projection from the trivial vector bundle $\mathbb{C}^3 \times V^{k,l} \to \mathbb{C}^3$ to the trivial V-bundle $\mathfrak{E}^{k,l} \to \mathbb{C}^3/\mathbb{Z}_{q+1}$. Again one needs to choose a representation of $\mathbb{Z}_{q+1}$ on the fibres (5.2). For reasons that will become clear later on (see Section 6.1), this time one chooses

$$\gamma(h) = \bigoplus_{(n,m)\in Q_0(k,l)} \mathbb{1}_{p_{(n,m)}} \otimes \zeta_{q+1}^n \, \mathbb{1}_{n+1} \; \in \mathrm{Center}\big(\mathfrak{G}^{k,l}\big) \,. \tag{5.8}$$

One immediately sees that all elements of $\mathfrak{G}^{k,l}$ commute with the action of $\mathbb{Z}_{q+1}$ given by (5.8), i.e. $\gamma(\mathbb{Z}_{q+1}) \subset \mathrm{Center}\big(\mathfrak{G}^{k,l}\big)$. The action of $\mathbb{Z}_{q+1}$ on the coordinates z^α defined in (2.28) induces a representation π of $\mathbb{Z}_{q+1}$ in $\Omega^1(\mathbb{C}^3)$, which on the generator h of $\mathbb{Z}_{q+1}$ is given by

$$\pi(h)(W_\alpha) = \begin{cases} \zeta_{q+1}^{-1} W_i \,, & i = 1, 2 \\ \zeta_{q+1}^2 W_3 \end{cases} \quad \text{and} \quad \pi(h)(\overline{W}_\alpha) = \begin{cases} \zeta_{q+1} W_i \,, & i = 1, 2 \\ \zeta_{q+1}^{-2} W_3 \end{cases} \,. \tag{5.9}$$

The requirement of $\mathbb{Z}_{q+1}$-equivariance of the connection $\mathcal{A}$ reduces to conditions similar to (3.35), i.e. the equivariance conditions read as

$$\gamma(h) \, W_\alpha \, \gamma(h)^{-1} = \pi(h)^{-1}(W_\alpha) \quad \text{and} \quad \gamma(h) \, \overline{W}_\alpha \, \gamma(h)^{-1} = \pi(h)^{-1}(\overline{W}_\alpha) \tag{5.10}$$

for $\alpha = 1, 2, 3$, but this time with different $\mathbb{Z}_{q+1}$-actions γ and π.

5.1.3. Quiver representations

For a decomposition of the endomorphisms

$$W_\alpha = \bigoplus_{(n,m),(n',m')} (W_\alpha)_{(n,m),(n',m')},$$

$$(W_\alpha)_{(n,m),(n',m')} \in \mathrm{Hom}\big(\mathbb{C}^{p_{(n,m)}} \otimes \underline{(n, m)}, \, \mathbb{C}^{p_{(n',m')}} \otimes \underline{(n', m')}\big) \tag{5.11}$$

as before, the equivariance conditions imply that the allowed non-vanishing components are given by

$$\Phi^i_{(n,m)} := (W_i)_{(n,m),(n',m')} \neq 0 \quad \text{for} \quad n' - n = 1 \pmod{q+1}, \tag{5.12a}$$

$$\Psi_{(n,m)} := (W_3)_{(n,m),(n',m')} \neq 0 \quad \text{for} \quad n' - n = -2 \pmod{q+1}, \tag{5.12b}$$

for $i = 1, 2$, together with the analogous conjugate decomposition for $\overline{W}_\alpha$; in each instance m' is implicitly determined by n and m via the requirement $(n', m') \in Q_0(k, l)$. The structure of these endomorphisms thus determines a representation of another quiver $\mathfrak{Q}^{k,l}$ with the same vertex set $Q_0(k, l)$ as before for the quiver $\mathcal{Q}^{k,l}$ but with new arrow set consisting of allowed components $(n, m) \to (n', m')$.

5.2. Generalised instanton equations

Similarly to Section 4.2, the Hermitian Yang–Mills equations on the complex 3-space $\mathbb{C}^3/\mathbb{Z}_{q+1}$ can be regarded in terms of holomorphicity and stability conditions.

5.2.1. Quiver relations

The condition that the connection $\mathcal{A}$ defines an integrable holomorphic structure on the bundle (5.1) is, as before, equivalent to the vanishing of the $(2, 0)$- and $(0, 2)$-parts of the curvature $\mathcal{F}$, i.e. $\mathcal{F}^{0,2} = 0 = \mathcal{F}^{2,0}$, which in the present case is equivalent to

$$[W_\alpha, W_\beta] = 0 \quad \text{and} \quad [\overline{W}_\alpha, \overline{W}_\beta] = 0 . \tag{5.13}$$

The general solutions (5.12) to the equivariance conditions allow for a decomposition of the generalised instanton equations (5.13) into components given by

$$(W_1)_{(n,m),(n+1,m')} (W_2)_{(n-1,m''),(n,m)} = (W_2)_{(n,m),(n+1,m')} (W_1)_{(n-1,m''),(n,m)} , \tag{5.14a}$$

$$(W_i)_{(n,m),(n+1,m')} (W_3)_{(n+2,m''),(n,m)} = 0 = (W_3)_{(n,m),(n-2,m')} (W_i)_{(n-1,m''),(n,m)} , \tag{5.14b}$$

for $(n, m) \in Q_0(k, l)$ and $i = 1, 2$, together with their conjugate equations. Note that in (5.14a) both combinations are morphisms between the same representation spaces and hence the commutation relation $[W_1, W_2] = 0$ requires only that their difference vanish. On the other hand, in (5.14b) the two terms are morphisms between different spaces and so the relation $[W_i, W_3] = 0$ implies that they each vanish individually; in particular, in the generic case the solution has $W_3 = 0$.

5.2.2. Stability conditions

For invariant connections there is a peculiarity involved in formulating stability of a holomorphic vector bundle, see for example [20]. On a $2n$-dimensional Kähler manifold with Kähler form ω, the stability condition is usually formulated through the identity

$$\mathcal{F} \wedge \omega^{n-1} = (\omega \,\lrcorner\, \mathcal{F}) \, \omega^n \tag{5.15}$$

by demanding that $\omega \,\lrcorner\, \mathcal{F} \in \text{Center}(\mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of the structure group. For generic connections the center of $\mathfrak{g}$ is trivial and the usual stability condition $\omega \,\lrcorner\, \mathcal{F} = 0$ follows. However, for invariant connections the structure group is smaller and the center can be non-trivial. This implies that there are *several* moduli spaces of translationally-invariant (and $\mathbb{Z}_{q+1}$-equivariant) instantons depending on a choice of element in Center$(\mathfrak{g})$.

Analogously to Section 4.2, the stability condition is associated to the moment map on the space of translationally-invariant and $\mathbb{Z}_{q+1}$-equivariant connections as we elaborate on in Section 6.2.3. In this case one can use any gauge-invariant element

$$\Xi := \bigoplus_{(n,m) \in Q_0(k,l)} \mathbb{1}_{p_{(n,m)}} \otimes i \, \xi_{(n,m)} \, \mathbb{1}_{n+1} \in \text{Center}(\mathfrak{g}^{k,l}) \tag{5.16}$$

from the center of the Lie algebra

$$\mathfrak{g}^{k,l} := \bigoplus_{(n,m)\in Q_0(k,l)} \mathfrak{u}\big(p_{(n,m)}\,(n+1)\big)\,, \tag{5.17}$$

where $\xi_{(n,m)} \in \mathbb{R}$ are called *Fayet–Iliopoulos parameters*. Thus the remaining instanton equations $\omega_{\mathbb{C}^3} \lrcorner \mathcal{F} = -\mathrm{i}\,\Xi$ read

$$\big[W_1, \overline{W}_1\big] + \big[W_2, \overline{W}_2\big] + \big[W_3, \overline{W}_3\big] = -\mathrm{i}\,\Xi\,. \tag{5.18}$$

Again by substituting the general solutions (5.12a) and (5.12b) to the equivariance conditions we can decompose the generalised instanton equation (5.18) explicitly into component equations

$$\sum_{i=1}^{2} \Big((W_i)_{(n,m),(n+1,m')}\,(\overline{W}_i)_{(n+1,m'),(n,m)} - (\overline{W}_i)_{(n,m),(n-1,m')}\,(W_i)_{(n-1,m'),(n,m)} \Big)$$
$$+ (W_3)_{(n,m),(n-2,m')}\,(\overline{W}_3)_{(n-2,m'),(n,m)} - (\overline{W}_3)_{(n,m),(n+2,m')}\,(W_3)_{(n+2,m'),(n,m)}$$
$$= \mathbb{1}_{p_{(n,m)}} \otimes \mathbb{1}_{n+1}\,\xi_{(n,m)} \tag{5.19}$$

for $(n, m) \in Q_0(k, l)$.

5.3. Examples

We shall now elucidate this general construction for the three examples $\underline{C}^{1,0}$, $\underline{C}^{2,0}$ and $\underline{C}^{1,1}$. In each case we highlight the non-vanishing components of the matrices W_α and the representation (5.8).

5.3.1. $C^{1,0}$-quiver

The decomposition of the fundamental representation $\underline{C}^{1,0}$ into irreducible SU(2)-representations is given by (C.1). The non-vanishing components can be read off to be $(W_i)_{(0,-2),(1,1)}$ and their adjoints $(\overline{W}_i)_{(1,1),(0,-2)}$. Thus there are two complex Higgs fields

$$\Phi_i := (W_i)_{(0,-2),(1,1)} \quad \text{for} \quad i = 1, 2\,, \tag{5.20}$$

which determine a representation of the 2-Kronecker quiver

$$(0,-2) \underset{\Phi_2}{\overset{\Phi_1}{\rightrightarrows}} (1,1) \tag{5.21}$$

By (5.8) the representation of the generator h is given by

$$\gamma : h \longmapsto \begin{pmatrix} \mathbb{1}_{p_{(1,1)}} \otimes \mathbb{1}_2\,\zeta_{q+1} & 0 \\ 0 & \mathbb{1}_{p_{(0,-2)}} \otimes 1 \end{pmatrix}\,. \tag{5.22}$$

5.3.1.1. Quiver relations The mutual commutativity of the matrices W_α is trivial in this case, and thus there are no quiver relations among the arrows of (5.21).

5.3.1.2. Stability conditions Choosing Fayet–Iliopoulos parameters $\xi_0, \xi_1 \in \mathbb{R}$, the requirement of a stable quiver bundle yields non-holomorphic matrix equations given by

$$\Phi_1\,\Phi_1^\dagger + \Phi_2\,\Phi_2^\dagger = \mathbb{1}_{p_{(1,1)}} \otimes \xi_0 \quad \text{and} \quad \Phi_1^\dagger\,\Phi_1 + \Phi_2^\dagger\,\Phi_2 = \mathbb{1}_{p_{(0,-2)}} \otimes \mathbb{1}_2\,\xi_1\,. \tag{5.23}$$

5.3.2. $C^{2,0}$-quiver

The representation $\underline{C}^{2,0}$ is decomposed according to (C.3). The non-vanishing components can be determined as before to be $(W_i)_{(0,-4),(1,-1)}$, $(W_i)_{(1,-1),(2,2)}$ and $(W_3)_{(2,2),(0,-4)}$, together with their adjoints $(\overline{W}_i)_{(1,-1),(0,-4)}$, $(\overline{W}_i)_{(2,2),(1,-1)}$ and $(\overline{W}_3)_{(0,-4),(2,2)}$. Thus there are five complex Higgs fields

$$\Phi_i := (W_i)_{(0,-4),(1,-1)} , \quad \Phi_{i+2} := (W_i)_{(1,-1),(2,2)} \quad \text{and} \quad \Psi := (W_3)_{(2,2),(0,-4)} , \tag{5.24a}$$

for $i = 1, 2$, which can be encoded in a representation of the quiver

$$\tag{5.25}$$
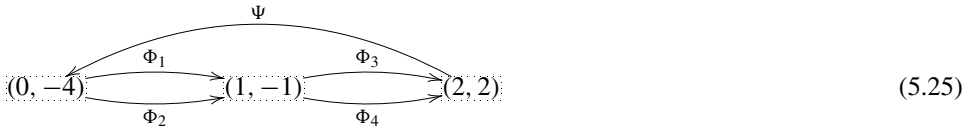
As before the representation (5.8) for this example is

$$\gamma : h \longmapsto \begin{pmatrix} \mathbb{1}_{p_{(2,2)}} \otimes \mathbb{1}_3 \, \zeta_{q+1}^2 & 0 & 0 \\ 0 & \mathbb{1}_{p_{(1,-1)}} \otimes \mathbb{1}_2 \, \zeta_{q+1} & 0 \\ 0 & 0 & \mathbb{1}_{p_{(0,-4)}} \otimes 1 \end{pmatrix} . \tag{5.26}$$

5.3.2.1. Quiver relations

The holomorphicity condition yields

$$\Phi_i \, \Psi = 0 , \quad \Psi \, \Phi_{i+2} = 0 \quad \text{and} \quad \Phi_3 \, \Phi_2 = \Phi_4 \, \Phi_1 \tag{5.27}$$

for $i = 1, 2$, plus the conjugate equations. The first two sets of quiver relations of (5.27) each describe the vanishing of a path of the quiver (5.25); an obvious trivial solution of these equations is $\Psi = 0$. The last relation expresses equality of two paths with source vertex $(0, -4)$ and target vertex $(2, 2)$.

5.3.2.2. Stability conditions

Choosing Fayet–Iliopoulos parameters $\xi_0, \xi_1, \xi_2 \in \mathbb{R}$, the stability conditions yield

$$\Phi_1^\dagger \, \Phi_1 + \Phi_2^\dagger \, \Phi_2 - \Psi \, \Psi^\dagger = \mathbb{1}_{p_{(0,-4)}} \otimes \xi_0 , \tag{5.28a}$$

$$\Phi_1 \, \Phi_1^\dagger + \Phi_2 \, \Phi_2^\dagger - \Phi_3^\dagger \, \Phi_3 - \Phi_4^\dagger \, \Phi_4 = \mathbb{1}_{p_{(1,-1)}} \otimes \mathbb{1}_2 \, \xi_1 , \tag{5.28b}$$

$$\Phi_3 \, \Phi_3^\dagger + \Phi_4 \, \Phi_4^\dagger - \Psi^\dagger \, \Psi = \mathbb{1}_{p_{(2,2)}} \otimes \mathbb{1}_3 \, \xi_2 . \tag{5.28c}$$

5.3.3. $C^{1,1}$-quiver

The decomposition of the adjoint representation $\underline{C}^{1,1}$ is given by (C.5). The non-vanishing components are $(W_i)_{(0,0),(1,3)}$, $(W_i)_{(0,0),(1,-3)}$, $(W_i)_{(1,3),(2,0)}$, $(W_i)_{(1,-3),(2,0)}$ and $(W_3)_{(2,0),(0,0)}$, together with their adjoint maps $(\overline{W}_i)_{(1,3),(0,0)}$, $(\overline{W}_i)_{(1,-3),(0,0)}$, $(\overline{W}_i)_{(2,0),(1,3)}$, $(\overline{W}_i)_{(2,0),(1,-3)}$ and $(\overline{W}_3)_{(0,0),(2,0)}$. Thus there are nine complex Higgs fields

$$\Phi_i^\pm := (W_i)_{(0,0),(1,\pm 3)} , \quad \Phi_{i+2}^\pm := (W_i)_{(1,\pm 3),(2,0)} \quad \text{and} \quad \Psi := (W_3)_{(2,0),(0,0)} , \tag{5.29}$$

for $i = 1, 2$, which can be assembled into a representation of the quiver

$$\begin{array}{c}
\Phi_1^+, \Phi_2^+ \quad (1, +3) \quad \Phi_3^+, \Phi_4^+ \\
(0,0) \xleftarrow{\quad \Psi \quad} (2,0) \\
\Phi_1^-, \Phi_2^- \quad (1, -3) \quad \Phi_3^-, \Phi_4^-
\end{array} \qquad (5.30)$$

In this example the generator h of $\mathbb{Z}_{q+1}$ has the representation

$$\gamma : h \longmapsto \begin{pmatrix} \mathbb{1}_{p_{(1,3)}} \otimes \mathbb{1}_2\, \zeta_{q+1} & 0 & 0 & 0 \\ 0 & \mathbb{1}_{p_{(0,0)}} \otimes 1 & 0 & 0 \\ 0 & 0 & \mathbb{1}_{p_{(2,0)}} \otimes \mathbb{1}_3\, \zeta_{q+1}^2 & 0 \\ 0 & 0 & 0 & \mathbb{1}_{p_{(1,-3)}} \otimes \mathbb{1}_2\, \zeta_{q+1} \end{pmatrix}. \qquad (5.31)$$

5.3.3.1. Quiver relations In this case the holomorphicity condition yields the relations

$$\Phi_i^\pm \Psi = 0, \quad \Psi\, \Phi_{i+2}^\pm = 0 \quad \text{and} \quad \Phi_3^+ \Phi_2^+ + \Phi_3^- \Phi_2^- = \Phi_4^+ \Phi_1^+ + \Phi_4^- \Phi_1^- \qquad (5.32)$$

for $i = 1, 2$. Again the first two sets of relations of (5.32) each describe the vanishing of a path in the associated quiver (5.30) (with the obvious trivial solution $\Psi = 0$), while the last relation equates two sums of paths.

5.3.3.2. Stability conditions Introducing Fayet–Iliopoulos parameters $\xi_1^\pm, \xi_2, \xi_3 \in \mathbb{R}$, from the stability conditions one obtains

$$(\Phi_1^+)^\dagger \Phi_1^+ + (\Phi_2^+)^\dagger \Phi_2^+ + (\Phi_1^-)^\dagger \Phi_1^- + (\Phi_2^-)^\dagger \Phi_2^- - \Psi\,\Psi^\dagger = \mathbb{1}_{p_{(0,0)}} \otimes \xi_0, \qquad (5.33a)$$

$$\Phi_1^\pm (\Phi_1^\pm)^\dagger + \Phi_2^\pm (\Phi_2^\pm)^\dagger - (\Phi_3^\pm)^\dagger \Phi_3^\pm - (\Phi_4^\pm)^\dagger \Phi_4^\pm = \mathbb{1}_{p_{(1,\pm3)}} \otimes \mathbb{1}_2\, \xi_1^\pm, \qquad (5.33b)$$

$$\Phi_3^+ (\Phi_3^+)^\dagger + \Phi_4^+ (\Phi_4^+)^\dagger + \Phi_3^- (\Phi_3^-)^\dagger + \Phi_4^- (\Phi_4^-)^\dagger - \Psi^\dagger \Psi = \mathbb{1}_{p_{(2,0)}} \otimes \mathbb{1}_3\, \xi_2. \qquad (5.33c)$$

6. Quiver gauge theories on cones: comparison

In Sections 4 and 5 we defined Higgs branch moduli spaces of vacua of two distinct quiver gauge theories on the Calabi–Yau cone over the orbifold $S^5/\mathbb{Z}_{q+1}$. In this section we shall explore their constructions in more detail, and describe their similarities and differences.

6.1. Quiver bundles

6.1.1. SU(3)-equivariance

Consider the quiver bundle $\mathcal{E}^{k,l}$ over $\mathbb{R} \times S^5/\mathbb{Z}_{q+1}$ (as a special case of (3.25)). By construction the space of all connections is restricted to those which are both SU(3)-equivariant and $\mathbb{Z}_{q+1}$-equivariant. For holomorphic quiver bundles, one additionally imposes the holomorphicity condition on the allowed connections. The general solution to these constraints (up to gauge equivalence) is given by the ansatz (4.4), where the matrices X_μ satisfy the equivariance conditions (3.30) and (3.35) as well as the quiver relations (4.7). The induced quiver bundles have the following structure:

- A single morphism (arrow) $\phi_{(n,m)}^\pm$ between two Hermitian bundles (vertices) $E_{p_{(n,m)}}$ and $E_{p_{(n',m')}}$ if $n - n' = \pm 1$ and $m - m' = \pm 3$.

- An endomorphism (vertex loop) $\psi_{(n,m)}$ at each Hermitian bundle (vertex) $E_{p_{(n,m)}}$ with non-trivial monopole charge $\frac{m}{2}$.

The reason why there is precisely one arrow between any two adjacent vertices is SU(3)-equivariance, which forces the horizontal component matrices X_a for $a = 1, 2, 3, 4$ to have exactly the same Higgs fields $\phi^{\pm}_{(n,m)}$, i.e. SU(3)-equivariance intertwines the horizontal components. The vertical component X_5 can be chosen independently as it originates from the Hopf fibration $S^5 \to \mathbb{C}P^2$. No further constraints arise from $\mathbb{Z}_{q+1}$-equivariance as we embed $\mathbb{Z}_{q+1} \hookrightarrow U(1) \subset$ SU(3). These quivers are a simple extension of the quivers obtained by [23,25] from dimensional reduction over $\mathbb{C}P^2$, because the additional vertical components only contribute loops on vertices with $m \neq 0$. This structure is reminiscent of that of the quivers of [15] which arise from reduction over 3-dimensional Sasaki–Einstein manifolds.

The Hermitian Yang–Mills equations can be considered as the intersection of the holomorphicity condition (4.7) and the stability condition (4.9). In this way their form can be recognised as Nahm-type equations of the sort considered in [22]. We will come back to this point in Section 6.2.2.

6.1.2. $\mathbb{C}^3$-invariance

Consider the V-bundle $\mathfrak{E}^{k,l}$ over $\mathbb{C}^3/\mathbb{Z}_{q+1}$ from (5.1). Recall that $C(S^5) \cong \mathbb{C}^3$. In contrast to the former case, we now impose invariance under the translation group $\mathbb{C}^3$ acting on the base as well as $\mathbb{Z}_{q+1}$-equivariance. We demand that these invariant connections also induce a holomorphic structure as previously. The general solution to these constraints is given by the ansatz (5.5) where the matrices W_α are constant along the base by (5.6), they commute with each other, and they solve the $\mathbb{Z}_{q+1}$-equivariance conditions (5.12). The induced quiver representations have the following characteristic structure:

- Two morphisms (arrows) $\Phi^i_{(n,m)}$ $(i = 1, 2)$ between each pair of $\mathbb{Z}_{q+1}$-representations (vertices) $\mathbb{C}^{p(n,m)} \otimes \underline{(n, m)}$ and $\mathbb{C}^{p(n',m')} \otimes \underline{(n', m')}$ if $n - n' = \pm 1$ in $\mathbb{Z}_{q+1}$.
- One homomorphism (arrow) $\Psi_{(n,m)}$ between each pair of $\mathbb{Z}_{q+1}$-representations (vertices) $\mathbb{C}^{p(n,m)} \otimes \underline{(n, m)}$ and $\mathbb{C}^{p(n',m')} \otimes \underline{(n', m')}$ if $n - n' = \pm 2$ in $\mathbb{Z}_{q+1}$.

The reason why there are exactly two arrows between adjacent vertices is that the chosen representation (5.8) does not intertwine W_1, W_2 and acts in the same way on both of them. Thus both endomorphisms have the same allowed non-vanishing components independently of one another, which gives rise to two independent sets of Higgs fields. The next novelty, compared to the former case, is the additional arrow associated to W_3; its existence is again due to the chosen $\mathbb{Z}_{q+1}$-action. Translational invariance plus $\mathbb{Z}_{q+1}$-equivariance are weaker constraints than SU(3)-equivariance plus $\mathbb{Z}_{q+1}$-equivariance, and consequently the allowed number of Higgs fields is larger. On the other hand, holomorphicity seems to impose the constraint $W_3 = 0$ for generic non-trivial endomorphisms W_1 and W_2 as discussed in Section 5. Hence there are two arrows between adjacent vertices, i.e. with $n - n' = \pm 1$, but no vertex loops as in the former case.

It follows that the generalised instanton equations (5.13) and (5.18) give rise to non-linear matrix equations similar to those considered in [20] for moduli spaces of Hermitian Yang–Mills-type generalised instantons and in [15] for instantons on cones over 3-dimensional Sasaki–Einstein orbifolds. We will analyse these equations further in Section 6.2.3.

6.1.3. Fibrewise $\mathbb{Z}_{q+1}$-actions

We shall now explain the origin of the difference between the choices of $\mathbb{Z}_{q+1}$-representations (3.33) and (5.8). Consider the generic linear $\mathbb{Z}_{q+1}$-action on $\mathbb{C}^3$: Letting h denote the generator of the cyclic group $\mathbb{Z}_{q+1}$, and choosing $(\theta^\alpha) = (\theta^1, \theta^2, \theta^3) \in \mathbb{Z}^3$ and $(z^\alpha) = (z^1, z^2, z^3) \in \mathbb{C}^3$, one has

$$h \cdot (z^\alpha) = \left(h^\alpha{}_\beta z^\beta\right) \quad \text{with} \quad (h^\alpha{}_\beta) = \begin{pmatrix} \zeta_{q+1}^{\theta^1} & 0 & 0 \\ 0 & \zeta_{q+1}^{\theta^2} & 0 \\ 0 & 0 & \zeta_{q+1}^{\theta^3} \end{pmatrix}. \tag{6.1}$$

This defines an embedding of $\mathbb{Z}_{q+1}$ into SU(3) if and only if $\theta^1 + \theta^2 + \theta^3 = 0 \bmod q + 1$.

However, we also have to account for the representation γ of $\mathbb{Z}_{q+1}$ in the fibres of the bundles (3.25) and (5.1). These bundles are explicitly constructed from SU(3)-representations $\underline{C}^{k,l}$ which decompose under SU(2) × U(1) into a sum of irreducible representations (n, m) from (3.9). If (n, m) and (n', m') both appear in the decomposition (3.9), then there exists $(r, s) \in \mathbb{Z}_{\geq 0}^2$ such that $n - n' = \pm r$ and $m - m' = \pm 3s$.

6.1.3.1. SU(3)-equivariance

The 1-forms β_{q+1}^i transform under the generic $\mathbb{Z}_{q+1}$-action (6.1) as

$$\beta_{q+1}^i \longmapsto \zeta_{q+1}^{\theta^i - \theta^3} \beta_{q+1}^i \quad \text{for} \quad i = 1, 2, \tag{6.2}$$

while η and $d\tau$ are invariant. Thus the equivariance condition for the connection (3.28) becomes

$$\gamma(h) (X_{2i-1} - i X_{2i}) \gamma(h)^{-1} = \zeta_{q+1}^{-\theta^i + \theta^3} (X_{2i-1} - i X_{2i}) \qquad \text{for} \quad i = 1, 2, \tag{6.3a}$$

$$\gamma(h) (X_{2i-1} + i X_{2i}) \gamma(h)^{-1} = \zeta_{q+1}^{\theta^i - \theta^3} (X_{2i-1} + i X_{2i}) \qquad \text{for} \quad i = 1, 2, \tag{6.3b}$$

$$\gamma(h) X_5 \gamma(h)^{-1} = X_5. \tag{6.3c}$$

In this case the aim is to embed $\mathbb{Z}_{q+1}$ in such a way that the entire quiver decomposition (3.25) is automatically $\mathbb{Z}_{q+1}$-equivariant; hence the non-vanishing components of the matrices X_a and X_5 are already prescribed by SU(3)-equivariance. For generic θ^α it seems quite difficult to realise this embedding, because if one assumes a diagonal $\mathbb{Z}_{q+1}$-action on the fibre of the form

$$\gamma(h) = \bigoplus_{(n,m) \in Q_0(k,l)} \mathbb{1}_{p(n,m)} \otimes \zeta_{q+1}^{\gamma(n,m)} \mathbb{1}_{n+1} \quad \text{with} \quad \gamma(n,m) \in \mathbb{Z}, \tag{6.4}$$

then these equivariance conditions translate into

$$\gamma(n \pm 1, m + 3) - \gamma(n, m) = \theta^i - \theta^3 \bmod q + 1 \quad \text{for} \quad i = 1, 2 \tag{6.5}$$

on the non-vanishing components of X_a, $a = 1, 2, 3, 4$.

In this paper we specialise to the weights $(\theta^\alpha) = (1, 1, -2)$ and obtain (2.34) for the $\mathbb{Z}_{q+1}$-action on SU(3)-equivariant 1-forms. From this action we naturally obtain factors $\zeta_{q+1}^{\pm 3}$ for the induced representation $\pi(h)$. This justifies the choice of γ in (3.33), as m changes by integer multiples of 3 while n in (6.5) does not have such uniform behaviour.

6.1.3.2. $\mathbb{C}^3$-invariance The modified equivariance condition under (6.1) is readily read off to be

$$\gamma(h)\, W_\alpha\, \gamma(h)^{-1} = \zeta_{q+1}^{\theta^\alpha}\, W_\alpha \quad \text{for} \quad \alpha = 1, 2, 3 \, . \tag{6.6}$$

In contrast to the SU(3)-equivariant case above, no particular form of the matrices W_α is fixed yet, i.e. here the choice of realisation of the $\mathbb{Z}_{q+1}$-action on the fibres determines the field content. By the same argument as above, a representation of $\mathbb{Z}_{q+1}$ on the fibres of the form (6.4) allows the component $(W_\alpha)_{(n,m),(n',m')}$ to be non-trivial if and only if

$$\gamma(n', m') - \gamma(n, m) = \theta^\alpha \mod q + 1 \quad \text{for} \quad \alpha = 1, 2, 3 \, . \tag{6.7}$$

For the weights $(\theta^\alpha) = (1, 1, -2)$ we then pick up factors of $\zeta_{q+1}^{\pm 1}$ or $\zeta_{q+1}^{\pm 2}$, which excludes the choice (3.33). However, the modification to (5.8) is allowed as n changes in integer increments.

6.1.4. McKay quiver

In [15,37] the correspondence between the Hermitian Yang–Mills moduli space for translationally-invariant and $\mathbb{Z}_{q+1}$-equivariant connections and the representation moduli of the McKay quiver is employed. The McKay quiver associated to the orbifold singularity $\mathbb{C}^3/\mathbb{Z}_{q+1}$ and the weights $(\theta^\alpha) = (1, 1, -2)$ is constructed in exactly the same way as the $C^{k,l}$-quivers from Section 5, except that it is based on the regular representation of $\mathbb{Z}_{q+1}$ rather than the representations $C^{k,l}$ considered here. It is a cyclic quiver with $q + 1$ vertices labelled by the irreducible representations of $\mathbb{Z}_{q+1}$, whose underlying graph is the affine extended Dynkin diagram of type $\widehat{A}_q$, and whose arrow set coincides with those of the $C^{k,l}$-quivers. See [21,38,39] for explicit constructions of instanton moduli on $\mathbb{C}^3/\mathbb{Z}_{q+1}$ in this context.

6.2. Moduli spaces

We shall now formalise the treatment of the instanton moduli spaces. We will first present an account of the general construction following [36,40], and then discuss the individual scenarios.

6.2.1. Kähler quotient construction

Let M be a Kähler manifold of complex dimension n and $\mathcal{G}$ a compact Lie group with Lie algebra $\mathfrak{g}$. Assume that $\mathcal{G}$ acts in the cotangent bundle T^*M preserving the complex structure J and the metric g; hence $\mathcal{G}$ also preserves the Kähler form ω. Let $P = P(M, \mathcal{G})$ be a principal $\mathcal{G}$-bundle over M, $\mathcal{A}$ a connection 1-form and $\mathcal{F} = \mathcal{F}_\mathcal{A} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ its curvature.

Let $\mathrm{Ad}(P) := P \times_{\mathcal{G}} \mathcal{G}$ be the group adjoint bundle (where $\mathcal{G}$ acts on itself via the adjoint action, i.e. by the inner automorphism $h \mapsto g\, h\, g^{-1}$), and let $\mathrm{ad}(P) := P \times_{\mathcal{G}} \mathfrak{g}$ be the algebra adjoint bundle (where $\mathcal{G}$ acts on $\mathfrak{g}$ via the adjoint action, i.e. by $X \mapsto \mathrm{Ad}(g)X = g\, X\, g^{-1}$). Let $E := P \times_{\mathcal{G}} F$ be the complex vector bundle associated to a $\mathcal{G}$-representation F.

Denote the space of all connections $\mathcal{A}$ on P by $\mathbb{A} = \mathbb{A}(P)$ and note that all associated bundles E inherit their space of connections $\mathbb{A}(E)$ from P. On $\mathbb{A}(P)$ there is a natural action of the gauge group $\widehat{\mathcal{G}}$, i.e. the group of automorphisms of P which are trivial on the base M. One can identify the gauge group with the space of global sections

$$\widehat{\mathcal{G}} = \Omega^0(M, \mathrm{Ad}(P)) \tag{6.8}$$

of the group adjoint bundle, and the action is realised via the gauge transformations

$$\mathcal{A} \longmapsto g \cdot \mathcal{A} = \mathrm{Ad}(g)\mathcal{A} + g^{-1}\, \mathrm{d}g \quad \text{for} \quad g \in \Omega^0(M, \mathrm{Ad}(P)) \, . \tag{6.9}$$

The Lie algebra of the gauge group can then be identified with the space of sections

$$\widehat{\mathfrak{g}} = \Omega^0(M, \mathrm{ad}(P)) \tag{6.10}$$

of the algebra adjoint bundle, and the infinitesimal gauge transformations are given by

$$\mathcal{A} \longmapsto \delta_\chi \mathcal{A} = \mathrm{d}_\mathcal{A} \chi := \mathrm{d}\chi + [\mathcal{A}, \chi] \quad \text{for} \quad \chi \in \Omega^0(M, \mathrm{ad}(P)) \,. \tag{6.11}$$

Since $\mathbb{A}(P)$ is an affine space, its tangent space $T_\mathcal{A}\mathbb{A}$ at any point $\mathcal{A} \in \mathbb{A}$ can be canonically identified with $\Omega^1(M, \mathrm{ad}(P))$. If the structure group is a matrix Lie group, i.e. there is an embedding $\mathcal{G} \hookrightarrow \mathrm{U}(N)$ for some $N \in \mathbb{Z}_{>0}$, then $\mathfrak{g}$ is a matrix Lie algebra and the trace defines an $\mathrm{Ad}(\mathcal{G})$-invariant inner product on $\mathfrak{g}$. The induced invariant inner product on $\Omega^1(M, \mathrm{ad}(P))$ is

$$\langle X_1, X_2 \rangle := \int_M \mathrm{tr}\,(X_1 \wedge \star X_2) \quad \text{for} \quad X_1, X_2 \in \Omega^1(M, \mathrm{ad}(P)) \,, \tag{6.12a}$$

which gives rise to a gauge-invariant metric on $\mathbb{A}(P)$ via the pointwise definition

$$\boldsymbol{g}_{|\mathcal{A}}(X_1, X_2) := \langle X_1, X_2 \rangle_{|\mathcal{A}} \quad \text{for} \quad X_1, X_2 \in T_\mathcal{A}\mathbb{A} \,. \tag{6.12b}$$

The space $\mathbb{A}(P)$ moreover carries a gauge-invariant symplectic structure defined by

$$\boldsymbol{\omega}_{|\mathcal{A}}(X_1, X_2) = \int_M \mathrm{tr}\,(X_1 \wedge X_2) \wedge \omega^{n-1} \quad \text{for} \quad X_1, X_2 \in T_\mathcal{A}\mathbb{A} \,. \tag{6.13}$$

Note that the 2-form $\boldsymbol{\omega}$ is completely independent of the base point $\mathcal{A} \in \mathbb{A}$. Let D denote the exterior derivative acting on forms on $\mathbb{A}$. Then by computing

$$\begin{aligned}
\mathrm{D}\boldsymbol{\omega}_{|\mathcal{A}}(X_0, X_1, X_2) = {} & X_0\big(\boldsymbol{\omega}_{|\mathcal{A}}(X_1, X_2)\big) - X_1\big(\boldsymbol{\omega}_{|\mathcal{A}}(X_0, X_2)\big) + X_2\big(\boldsymbol{\omega}_{|\mathcal{A}}(X_0, X_1)\big) \\
& - \boldsymbol{\omega}_{|\mathcal{A}}([X_0, X_1], X_2) + \boldsymbol{\omega}_{|\mathcal{A}}([X_0, X_2], X_1) - \boldsymbol{\omega}_{|\mathcal{A}}([X_1, X_2], X_0) \,,
\end{aligned} \tag{6.14}$$

one observes that $\mathrm{D}\boldsymbol{\omega} = 0$ as $X_i\big(\boldsymbol{\omega}_{|\mathcal{A}}(X_j, X_k)\big) = 0$ due to base point independence and

$$\boldsymbol{\omega}_{|\mathcal{A}}([X_i, X_j], X_k) = \int_M \mathrm{tr}\,\big([X_i, X_j] \wedge X_k\big) \wedge \omega^{n-1} = 0 \tag{6.15}$$

as $\mathrm{tr}\,\big([X_i, X_j] \wedge X_k\big) \in \Omega^3(M)$ which renders the integrand into a form of degree larger than the top degree. It follows that $\boldsymbol{\omega}$ is a symplectic form, which promotes $\mathbb{A}$ to an infinite-dimensional Riemannian symplectic manifold $(\mathbb{A}, \boldsymbol{g}, \boldsymbol{\omega})$ equipped with a compatible $\widehat{\mathcal{G}}$-action.

6.2.1.1. Holomorphic structure   Consider now the restriction to connections on $E \to M$ which are generalised instanton connections. Recall that one part of the Hermitian Yang–Mills equations can be interpreted as holomorphicity conditions, and the corresponding subspace is

$$\mathbb{A}^{1,1} = \big\{ \mathcal{A} \in \mathbb{A}(E) : \mathcal{F}_\mathcal{A}^{0,2} = -\big(\mathcal{F}_\mathcal{A}^{2,0}\big)^\dagger = 0 \big\} \subset \mathbb{A}(E) \,. \tag{6.16}$$

This definition employs the underlying complex structure on M. As before, this condition is equivalent to the existence of a holomorphic structure on E, i.e. a Cauchy–Riemann operator $\overline{\partial}_E := \overline{\partial} + A^{0,1}$ that satisfies the Leibniz rule as well as $\overline{\partial}_E \circ \overline{\partial}_E = 0$. Thus a $\mathcal{G}$-bundle with only holomorphic connections induces a $\mathcal{G}^{\mathbb{C}}$-bundle where $\mathcal{G}^{\mathbb{C}} = \mathcal{G} \otimes \mathbb{C}$. One can show that $\mathbb{A}^{1,1}$ is an infinite-dimensional Kähler manifold, i.e. the metric $\boldsymbol{g}$ is Hermitian and the symplectic form $\boldsymbol{\omega}$ is Kähler. These tensor fields descend from $\mathbb{A}$ to $\mathbb{A}^{1,1}$ simply by restriction.

6.2.1.2. Moment map   The space $\mathbb{A}^{1,1}$ inherits a $\widehat{\mathcal{G}}$-action from $\mathbb{A}$ and since it has a $\widehat{\mathcal{G}}$-invariant symplectic form, i.e. the Kähler form $\boldsymbol{\omega}$, one can introduce a moment map

$$\mu : \mathbb{A}^{1,1} \longrightarrow \widehat{\mathfrak{g}}^* \cong \Omega^{2n}(M, \mathrm{ad}(P))$$
$$\mathcal{A} \longmapsto \mathcal{F}_\mathcal{A} \wedge \omega^{n-1} \, . \tag{6.17}$$

For this to be a moment map of the $\widehat{\mathcal{G}}$-action one needs to verify the defining properties, generalising the arguments presented in [36]. For this, note that μ is obviously $\widehat{\mathcal{G}}$-equivariant. Next let $\phi \in \Omega^0(M, \mathrm{ad}(P))$ be an element of the gauge algebra, $\phi^\sharp$ the corresponding vector field on $\mathbb{A}^{1,1}$ and $\psi \in \Omega^1(M, \mathrm{ad}(P))$ a tangent vector at the base point $\mathcal{A}$. Then the condition to verify is

$$(\phi, \mathrm{D}\mu_{|\mathcal{A}})(\psi) = \iota_{\phi^\sharp} \boldsymbol{\omega}_{|\mathcal{A}}(\psi) \, , \tag{6.18}$$

wherein ι denotes contraction and the dual pairing $(\cdot, \cdot)$ of $\widehat{\mathfrak{g}}$ with $\widehat{\mathfrak{g}}^*$ is defined via integration over M of the invariant inner product on $\mathfrak{g}$. Firstly, in the definition of μ only $\mathcal{F}_\mathcal{A}$ is base point dependent, and a standard computation gives $\mathcal{F}_{\mathcal{A}+t\,\psi} = \mathcal{F}_\mathcal{A} + t\,\mathrm{d}_\mathcal{A}\psi + \frac{1}{2}t^2\,\psi \wedge \psi$ so that $\mathrm{D}\mathcal{F}_{|\mathcal{A}} = \left(\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_{\mathcal{A}+t\,\psi}\right)_{|t=0} = \mathrm{d}_\mathcal{A}\psi$. Thus the left-hand side of (6.18) is $(\phi, \mathrm{D}\mu_{|\mathcal{A}})(\psi) = \int_M \mathrm{tr}\big((\mathrm{d}_\mathcal{A}\psi) \wedge \phi\big) \wedge \omega^{n-1}$. Secondly, the vector field $\phi^\sharp$ can be read off from (6.11) to be $\phi^\sharp_{|\mathcal{A}} = \mathrm{d}_\mathcal{A}\phi \in \Omega^1(M, \mathrm{ad}(P))$. Hence the right-hand side is $\iota_{\phi^\sharp}\boldsymbol{\omega}_{|\mathcal{A}}(\psi) = \int_M \mathrm{tr}\big((\mathrm{d}_\mathcal{A}\phi) \wedge \psi\big) \wedge \omega^{n-1}$. But from $\int_M \mathrm{d}\big(\mathrm{tr}(\psi \wedge \phi) \wedge \omega^{n-1}\big) = 0$ and $\mathrm{d}\omega = 0$ one has $\int_M \mathrm{tr}\big((\mathrm{d}_\mathcal{A}\psi) \wedge \phi\big) \wedge \omega^{n-1} = -\int_M \mathrm{tr}\big(\psi \wedge (\mathrm{d}_\mathcal{A}\phi)\big) \wedge \omega^{n-1}$, and therefore the relation (6.18) holds, i.e. μ is a moment map of the $\widehat{\mathcal{G}}$-action on $\mathbb{A}^{1,1}$.

We will use the dual moment map defined by

$$\mu^* : \mathbb{A}^{1,1} \longrightarrow \widehat{\mathfrak{g}} = \Omega^0(M, \mathrm{ad}(P))$$
$$\mathcal{A} \longmapsto \omega \lrcorner \mathcal{F}_\mathcal{A} \, , \tag{6.19}$$

which is equivalent to the definition (6.17) due to the identification $\widehat{\mathfrak{g}} \cong \widehat{\mathfrak{g}}^*$ given by (5.15) (generically by a choice of metric). Thus we will no longer explicitly distinguish between the moment map μ and its dual μ^*.

For *regular* elements $\Xi \in \widehat{\mathfrak{g}}$, the centraliser of Ξ in $\widehat{\mathcal{G}}$ is the maximal torus and $\mu^{-1}(\Xi) \subset \mathbb{A}^{1,1}$ defines a submanifold which carries a $\widehat{\mathcal{G}}$-action. The quotient of the level sets[6]

$$\mathbb{A}^{1,1} /\!\!/_\Xi \, \widehat{\mathcal{G}} := \mu^{-1}(\Xi) \big/ \widehat{\mathcal{G}} \tag{6.20}$$

is well-defined, and moreover it defines a Kähler space as the Kähler form and the complex structure descend from $\mathbb{A}^{1,1}$ by gauge-invariance. The level set of zeroes is precisely the Hermitian Yang–Mills moduli space.

6.2.1.3. Complex group action   As the $\mathcal{G}$-action in T^*M preserves the Kähler structure, one can extend it to a $\mathcal{G}^\mathbb{C}$-action in T^*M. The same is true for the extension to the complexification of the $\widehat{\mathcal{G}}$-action on $\mathbb{A}^{1,1}$, i.e. the holomorphicity conditions $\mathcal{F}^{0,2}_\mathcal{A} = 0$ are invariant under the action of the complex gauge group

$$\widehat{\mathcal{G}}^\mathbb{C} = \widehat{\mathcal{G}} \otimes \mathbb{C} \, . \tag{6.21}$$

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[6] One must in fact take $\Xi \in \mathrm{Center}(\widehat{\mathfrak{g}})$ for a well-defined quotient.

For $\mathcal{A} \in \mathbb{A}^{1,1}$ the orbit $\widehat{\mathcal{G}}_{\mathcal{A}}^{\mathbb{C}}$ of the $\widehat{\mathcal{G}}^{\mathbb{C}}$-action is given by

$$\widehat{\mathcal{G}}_{\mathcal{A}}^{\mathbb{C}} = \left\{ \mathcal{A}' \in \mathbb{A}^{1,1} : \mathcal{A}' = g \cdot \mathcal{A} , \; g \in \widehat{\mathcal{G}}^{\mathbb{C}} \right\} . \tag{6.22}$$

A point $\mathcal{A} \in \mathbb{A}^{1,1}$ is called *stable* if $\widehat{\mathcal{G}}_{\mathcal{A}}^{\mathbb{C}} \cap \mu^{-1}(\Xi) \neq \emptyset$. Denote by $\mathbb{A}_{\mathrm{st}}^{1,1}(\Xi) \subset \mathbb{A}^{1,1}$ the set of all stable points (for a given regular element Ξ). Then the Kähler quotient can be identified with the GIT quotient (see for instance [41])

$$\mathbb{A}^{1,1} /\!\!/_{\Xi} \widehat{\mathcal{G}} \cong \mathbb{A}_{\mathrm{st}}^{1,1}(\Xi) / \widehat{\mathcal{G}}^{\mathbb{C}} . \tag{6.23}$$

In the following we discuss applications of this Kähler quotient construction to SU(3)-equivariant and $\mathbb{Z}_{q+1}$-equivariant instantons on the Calabi–Yau cone $M = C(S^5/\mathbb{Z}_{q+1})$, as well as to the $\mathbb{C}^3$-invariant and $\mathbb{Z}_{q+1}$-equivariant case. These vacuum moduli spaces are special cases of those constructed above, as we do not consider generic connections but rather equivariant connections. For instance, equivariance reduces the gauge groups.

6.2.2. SU(3)-equivariance

Consider the space of SU(3)-equivariant connections $\mathbb{A}(\mathcal{E}^{k,l})$ on the bundle (3.25) (for $d = 1$), which is an affine space modelled on $\Omega^1\big(C(S^5/\mathbb{Z}_{q+1}), \mathrm{End}_{\mathrm{U}(1)}(V^{k,l})\big)$. The structure group $\mathcal{G}^{k,l}$ of the bundle (3.25) is given by (3.24). An element $X \in \Omega^1\big(C(S^5/\mathbb{Z}_{q+1}), \mathrm{End}_{\mathrm{U}(1)}(V^{k,l})\big)$ can be expressed as

$$X = X_\mu e^\mu + X_\tau \, \mathrm{d}\tau \equiv X_j \theta^j + \overline{X}_j \overline{\theta}^j , \tag{6.24}$$

once one has chosen the coframe $\{e^\mu, \mathrm{d}\tau\}$ of the conformally equivalent cylinder $\mathbb{R} \times S^5/\mathbb{Z}_{q+1}$ with $r = \mathrm{e}^\tau$. One can equivalently use the complex basis $\theta^j = e^{2j-1} + \mathrm{i}\, e^{2j}$ for $j = 1, 2, 3$, where $e^6 := \mathrm{d}\tau$; then $(X_j)^\dagger = -\overline{X}_j$. Thus once one has fixed a choice of coframe on the Calabi–Yau cone $C(S^5/\mathbb{Z}_{q+1})$, the tangent space to $\mathbb{A}(\mathcal{E}^{k,l})$ at a point $\mathcal{A}$ is described by all 6-tuples $(\{X_\mu\}, X_\tau)$ or equivalently $(\{X_j\}, \{\overline{X}_j\})$. Here X_μ and X_τ depend only on the cone coordinate τ by SU(3)-equivariance.[7]

6.2.2.1. Instanton equations

One can eliminate the linear terms in (4.7) and (4.9) via the redefinitions

$$X_a = \mathrm{e}^{-\frac{3}{2}\tau} \mathcal{X}_a \quad \text{for} \quad a = 1, 2, 3, 4 \quad \text{and} \quad X_5 = \mathrm{e}^{-4\tau} \mathcal{X}_5 , \quad X_\tau = \mathrm{e}^{-4\tau} \mathcal{X}_6 . \tag{6.25}$$

Using 't Hooft tensors the matrix equations read

$$\eta_{ab}^1 [\mathcal{X}_a, \mathcal{X}_b] = 0 \quad \text{and} \quad \eta_{ab}^2 [\mathcal{X}_a, \mathcal{X}_b] = 0 , \tag{6.26a}$$

$$\frac{\mathrm{d}\mathcal{X}_a}{\mathrm{d}s} = -\eta_{ab}^3 [\mathcal{X}_b, \mathcal{X}_5] - [\mathcal{X}_a, \mathcal{X}_6] , \tag{6.26b}$$

$$\frac{\mathrm{d}\mathcal{X}_5}{\mathrm{d}s} = -\lambda(s) \big([\mathcal{X}_1, \mathcal{X}_2] + [\mathcal{X}_3, \mathcal{X}_4]\big) - [\mathcal{X}_5, \mathcal{X}_6] , \tag{6.26c}$$

where $s := \frac{1}{4}\, \mathrm{e}^{-4\tau} \in \mathbb{R}_{>0}$ and $\lambda(s) = \left(\frac{1}{4s}\right)^{\frac{5}{4}}$. The equations (6.26) are automatically satisfied in the temporal gauge $X_\tau = 0$ by taking *constant* scalar fields X_μ for $\mu = 1, \ldots, 5$ satisfying the Higgs branch BPS equations (3.43) of the quiver gauge theory.

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[7] Recall that the equivariance condition (3.30) makes the endomorphisms X_μ base point independent on $S^5/\mathbb{Z}_{q+1}$; hence it is consistent to have solely τ-dependent matrices X_μ in any coframe.

Changing to a complex basis as before and defining

$$\mathcal{Y}_j = \tfrac{1}{2}\left(\mathcal{X}_{2j-1} - \mathrm{i}\,\mathcal{X}_{2j}\right) \quad \text{and} \quad \overline{\mathcal{Y}}_j = \tfrac{1}{2}\left(\mathcal{X}_{2j-1} + \mathrm{i}\,\mathcal{X}_{2j}\right) \quad \text{for} \quad j = 1, 2, 3\,, \tag{6.27}$$

the resulting holomorphicity conditions are

$$\left[\mathcal{Y}_1, \mathcal{Y}_2\right] = 0 \quad \text{and} \quad \left[\overline{\mathcal{Y}}_1, \overline{\mathcal{Y}}_2\right] = 0\,, \tag{6.28a}$$

$$\frac{\mathrm{d}\mathcal{Y}_i}{\mathrm{d}s} = -2\mathrm{i}\left[\mathcal{Y}_i, \mathcal{Y}_3\right] \quad \text{and} \quad \frac{\mathrm{d}\overline{\mathcal{Y}}_i}{\mathrm{d}s} = 2\mathrm{i}\left[\overline{\mathcal{Y}}_i, \overline{\mathcal{Y}}_3\right] \quad \text{for} \quad i = 1, 2\,, \tag{6.28b}$$

while the stability condition yields

$$\frac{\mathrm{d}\mathcal{Y}_3}{\mathrm{d}s} + \frac{\mathrm{d}\overline{\mathcal{Y}}_3}{\mathrm{d}s} = 2\mathrm{i}\left[\mathcal{Y}_3, \overline{\mathcal{Y}}_3\right] + 2\mathrm{i}\lambda(s)\left(\left[\mathcal{Y}_1, \overline{\mathcal{Y}}_1\right] + \left[\mathcal{Y}_2, \overline{\mathcal{Y}}_2\right]\right)\,. \tag{6.28c}$$

Analogously to the generic situation, we define the subspace

$$\mathbb{A}^{1,1}\left(\mathcal{E}^{k,l}\right) = \left\{\left(\{\mathcal{Y}_j\}, \{\overline{\mathcal{Y}}_j\}\right) \in \mathbb{A}\left(\mathcal{E}^{k,l}\right) : \text{(6.28a) and (6.28b) hold}\right\}\,. \tag{6.29}$$

6.2.2.2. Real gauge group   On the space $\mathbb{A}^{1,1}(\mathcal{E}^{k,l})$ there is an action of the gauge group

$$\widehat{\mathcal{G}}^{k,l} := \Omega^0\left(\mathbb{R}_{>0}, \mathcal{G}^{k,l}\right)\,, \tag{6.30}$$

with $\mathcal{G}^{k,l} \hookrightarrow \mathrm{U}(p)$, given by[8]

$$\overline{\mathcal{Y}}_i \longmapsto \mathrm{Ad}(g)\overline{\mathcal{Y}}_i \quad \text{for} \quad i = 1, 2 \quad \text{and} \quad \overline{\mathcal{Y}}_3 \longmapsto \mathrm{Ad}(g)\overline{\mathcal{Y}}_3 + \frac{\mathrm{i}}{2}\left(\frac{\mathrm{d}g}{\mathrm{d}s}\right)g^{-1}\,, \tag{6.31a}$$

for $g \in \widehat{\mathcal{G}}^{k,l}$. One readily checks that the full set of equations (6.28) is invariant under these "real" gauge transformations. Moreover, one can always find a gauge transformation $g \in \widehat{\mathcal{G}}^{k,l}$ such that $g \cdot \mathcal{X}_6 = 0$ or equivalently $g \cdot \overline{\mathcal{Y}}_3 = g \cdot \mathcal{Y}_3$.

6.2.2.3. Complex gauge group   The space $\mathbb{A}^{1,1}(\mathcal{E}^{k,l})$ also admits an action of the complex gauge group

$$\left(\widehat{\mathcal{G}}^{k,l}\right)^{\mathbb{C}} := \Omega^0\left(\mathbb{R}_{>0}, (\mathcal{G}^{k,l})^{\mathbb{C}}\right)\,, \tag{6.32}$$

with $\left(\mathcal{G}^{k,l}\right)^{\mathbb{C}} \hookrightarrow \mathrm{GL}(p, \mathbb{C})$. However, only the equations (6.28a) and (6.28b) are invariant under the "complex" gauge transformations given by

$$\overline{\mathcal{Y}}_i \longmapsto \mathrm{Ad}(g)\overline{\mathcal{Y}}_i \quad \text{and} \quad \mathcal{Y}_i \longmapsto \mathrm{Ad}(g^{*-1})\mathcal{Y}_i \quad \text{for} \quad i = 1, 2\,, \tag{6.33a}$$

$$\overline{\mathcal{Y}}_3 \longmapsto \mathrm{Ad}(g)\overline{\mathcal{Y}}_3 + \frac{\mathrm{i}}{2}\left(\frac{\mathrm{d}g}{\mathrm{d}s}\right)g^{-1} \quad \text{and} \quad \mathcal{Y}_3 \longmapsto \mathrm{Ad}(g^{*-1})\mathcal{Y}_3 + \frac{\mathrm{i}}{2}\,g^{*-1}\left(\frac{\mathrm{d}g^{\dagger}}{\mathrm{d}s}\right)\,, \tag{6.33b}$$

where $g \in \left(\widehat{\mathcal{G}}^{k,l}\right)^{\mathbb{C}}$ and $g^{*-1} = (g^{-1})^{\dagger}$.

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[8] We assume that the paths $g(s) : (0, \infty) \to \mathcal{G}^{k,l}$ are sufficiently smooth.

6.2.2.4. Kähler structure Following the construction of Section 6.2.1, the next step is to define a Kähler structure on $\mathbb{A}^{1,1}(\mathcal{E}^{k,l})$. The tangent space $T_{\mathcal{A}}\mathbb{A}(\mathcal{E}^{k,l})$ at point $\mathcal{A}$ is $\Omega^1\big(C(S^5/\mathbb{Z}_{q+1}), \mathrm{End}_{\mathrm{U}(1)}(V^{k,l})\big)$, so a tangent vector $\boldsymbol{x} = \boldsymbol{x}_j\,\theta^j + \overline{\boldsymbol{x}}_j\,\overline{\theta}^j$ over $\mathbb{A}^{1,1}(\mathcal{E}^{k,l})$ is defined by linearisation of the holomorphicity equations (6.28a) and (6.28b) for paths $\overline{\boldsymbol{x}}_j(s)$: $(0, \infty) \to \mathrm{End}_{\mathrm{U}(1)}(V^{k,l})^{\mathbb{C}}$. The gauge transformations are given by $\overline{\boldsymbol{x}}_j \mapsto \mathrm{Ad}(g)\overline{\boldsymbol{x}}_j$ for $j = 1, 2, 3$.

A metric on $\mathbb{A}^{1,1}(\mathcal{E}^{k,l})$ can be defined from (6.12) as

$$g_{|\mathcal{A}}(\boldsymbol{x}, \boldsymbol{y}) := \frac{1}{2} \int\limits_{0^+}^{\infty} \mathrm{d}s \, \sum_{j=1}^{3} \mathrm{tr}\big(\boldsymbol{x}_j^{\dagger}\,\boldsymbol{y}_j + \boldsymbol{x}_j\,\boldsymbol{y}_j^{\dagger}\big)\,, \tag{6.34}$$

where the integral over $S^5/\mathbb{Z}_{q+1}$ drops out here as the tangent vectors at equivariant connections are independent of the coordinates of $S^5/\mathbb{Z}_{q+1}$. A symplectic form on $\mathbb{A}^{1,1}(\mathcal{E}^{k,l})$ can likewise be defined from (6.13) as

$$\omega_{|\mathcal{A}}(\boldsymbol{x}, \boldsymbol{y}) := \frac{\mathrm{i}}{2} \int\limits_{0^+}^{\infty} \mathrm{d}s \, \sum_{j=1}^{3} \mathrm{tr}\big(\boldsymbol{x}_j^{\dagger}\,\boldsymbol{y}_j - \boldsymbol{x}_j\,\boldsymbol{y}_j^{\dagger}\big)\,. \tag{6.35}$$

Both g and ω are gauge-invariant by construction. Moreover, we immediately see that for the choice of complex structure[9] $J(\overline{\boldsymbol{x}}_j) = \mathrm{i}\,\overline{\boldsymbol{x}}_j$ the symplectic form ω and the metric g are compatible, i.e. $g(\cdot, J\cdot) = \omega(\cdot, \cdot)$.

6.2.2.5. Moment map On the Kähler manifold $\mathbb{A}^{1,1}(\mathcal{E}^{k,l})$ we define a moment map by

$$\mu : \mathbb{A}^{1,1}(\mathcal{E}^{k,l}) \longrightarrow \widehat{\mathrm{End}}_{\mathrm{U}(1)}(V^{k,l})$$

$$(\{\mathcal{Y}_j\}, \{\overline{\mathcal{Y}}_j\}) \longmapsto \frac{\mathrm{d}\mathcal{Y}_3}{\mathrm{d}s} + \frac{\mathrm{d}\overline{\mathcal{Y}}_3}{\mathrm{d}s} - 2\,\mathrm{i}\,[\mathcal{Y}_3, \overline{\mathcal{Y}}_3] - 2\,\mathrm{i}\,\lambda(s)\,\big([\mathcal{Y}_1, \overline{\mathcal{Y}}_1] + [\mathcal{Y}_2, \overline{\mathcal{Y}}_2]\big)\,, \tag{6.36}$$

which readily gives us the Kähler quotient for the instanton moduli space

$$\mathcal{M}_{k,l}^{\mathrm{SU}(3)} = \mu^{-1}(0) \,\big/\, \widehat{\mathcal{G}}^{k,l} \,. \tag{6.37}$$

6.2.2.6. Stable points We can alternatively describe the moduli space $\mathcal{M}_{k,l}^{\mathrm{SU}(3)}$ via the stable points

$$\mathbb{A}_{\mathrm{st}}^{1,1}(\mathcal{E}^{k,l}) := \Big\{(\{\mathcal{Y}_j\}, \{\overline{\mathcal{Y}}_j\}) \in \mathbb{A}^{1,1}(\mathcal{E}^{k,l}) \,:\, \big(\widehat{\mathcal{G}}^{k,l}\big)_{(\{\mathcal{Y}_j\},\{\overline{\mathcal{Y}}_j\})}^{\mathbb{C}} \cap \mu^{-1}(0) \neq \emptyset\Big\}\,, \tag{6.38}$$

and by taking the GIT quotient as before to get

$$\mathcal{M}_{k,l}^{\mathrm{SU}(3)} \cong \mathbb{A}_{\mathrm{st}}^{1,1}(\mathcal{E}^{k,l}) \,\big/\, \big(\widehat{\mathcal{G}}^{k,l}\big)^{\mathbb{C}}\,. \tag{6.39}$$

We show below that it is sufficient to solve the holomorphicity equations (subject to certain boundary conditions), as the solution to the stability equation then follows automatically by a complex gauge transformation. More precisely, for every point in $\mathbb{A}^{1,1}(\mathcal{E}^{k,l})$ there exists a unique point in its complex gauge orbit which satisfies the stability equation, i.e. every point in $\mathbb{A}^{1,1}(\mathcal{E}^{k,l})$ is stable.

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[9] We essentially use the complex structure J of $\mathbb{C}^3$.

6.2.2.7. Solutions of the holomorphicity equations   Following [42] one can regard the holomorphicity equations as being locally trivial. For this, we use a complex gauge transformation (6.33) to eliminate $\overline{\mathcal{Y}}_3$ via

$$\widetilde{\overline{\mathcal{Y}}}_3 = \mathrm{Ad}(g)\overline{\mathcal{Y}}_3 + \frac{\mathrm{i}}{2}\left(\frac{\mathrm{d}g}{\mathrm{d}s}\right)g^{-1} = 0\,. \tag{6.40}$$

From the holomorphicity equations (6.28b) and (6.28a) one obtains in this gauge

$$\frac{\mathrm{d}\widetilde{\overline{\mathcal{Y}}}_i}{\mathrm{d}s} = 0 \quad\text{and}\quad \widetilde{\overline{\mathcal{Y}}}_i = \overline{\mathcal{T}}_i \quad\text{with}\quad \left[\overline{\mathcal{T}}_1,\overline{\mathcal{T}}_2\right] = 0\,, \tag{6.41}$$

where $\overline{\mathcal{T}}_i$ are *constant* for $i = 1, 2$. Consequently the general local solution of the holomorphicity equations (6.28a) and (6.28b) is

$$\overline{\mathcal{Y}}_i = \mathrm{Ad}(g^{-1})\overline{\mathcal{T}}_i \quad\text{with}\quad \left[\overline{\mathcal{T}}_1,\overline{\mathcal{T}}_2\right] = 0 \quad\text{and}\quad \overline{\mathcal{Y}}_3 = -\frac{\mathrm{i}}{2}\,g^{-1}\frac{\mathrm{d}g}{\mathrm{d}s}\,, \tag{6.42}$$

with $g \in (\widehat{\mathcal{G}}^{k,l})^{\mathbb{C}}$. A solution to the commutator constraint chooses $\overline{\mathcal{T}}_i$ for $i = 1, 2$ as elements of the Cartan subalgebra of the complex Lie algebra $\mathrm{End}_{\mathrm{U}(1)}(V^{k,l})^{\mathbb{C}}$ of the structure group (3.24).

6.2.2.8. Solutions of the stability equation   We also need to solve the stability equation (6.28c), for which we follow again [42]. Recall that the complete set of instanton equations (6.28) is $\widehat{\mathcal{G}}^{k,l}$-invariant, and for each $g \in (\widehat{\mathcal{G}}^{k,l})^{\mathbb{C}}$ define

$$h = h(g) = g\,g^{\dagger} : (0,\infty) \longrightarrow \left(\mathcal{G}^{k,l}\right)^{\mathbb{C}}/\mathcal{G}^{k,l} \hookrightarrow \mathrm{GL}(p,\mathbb{C})/\mathrm{U}(p)\,. \tag{6.43}$$

Fix a 6-tuple $\left\{\overline{\mathcal{Y}}_j, \mathcal{Y}_j\right\}_{j=1,2,3}$ and define the gauge transformed 6-tuple $\left\{\widetilde{\overline{\mathcal{Y}}}_j, \widetilde{\mathcal{Y}}_j\right\}_{j=1,2,3}$. We will study the critical points of the functional

$$\mathcal{L}_\epsilon[g] = \frac{1}{2}\int_\epsilon^{\frac{1}{\epsilon}} \mathrm{d}s \ \mathrm{tr}\left(\left|\widetilde{\mathcal{Y}}_3 + \widetilde{\overline{\mathcal{Y}}}_3\right|^2 + 2\lambda(s)\sum_{i=1}^{2}\left|\widetilde{\overline{\mathcal{Y}}}_i\right|^2\right) \quad\text{for}\quad 0 < \epsilon < 1\,. \tag{6.44}$$

As the instanton equations are invariant under $\mathrm{U}(p)$-valued gauge transformations, we can restrict g to take values in the quotient $\mathrm{GL}(p,\mathbb{C})/\mathrm{U}(p)$ which may be identified with the set of positive Hermitian $p \times p$ matrices [42]. Hence it is sufficient to consider variations with $\delta g = \delta g^{\dagger}$ around $g = \mathbb{1}_p$ (and with $\delta g \neq 0$). Then the gauge transformations (6.33) imply that

$$\delta\overline{\mathcal{Y}}_3 = \left[\delta g, \overline{\mathcal{Y}}_3\right] + \frac{\mathrm{i}}{2}\frac{\mathrm{d}\delta g}{\mathrm{d}s} \quad\text{and}\quad \delta\overline{\mathcal{Y}}_i = \left[\delta g, \overline{\mathcal{Y}}_i\right] \quad\text{for}\quad i = 1, 2\,. \tag{6.45}$$

The variation then leads to

$$\delta\mathcal{L}_\epsilon[g] = -\mathrm{i}\int_\epsilon^{\frac{1}{\epsilon}} \mathrm{d}s \ \mathrm{tr}\left(\mu(\{\mathcal{Y}_j\},\{\overline{\mathcal{Y}}_j\})\,\delta g\right)\,, \tag{6.46}$$

i.e. the critical points of (6.44) form the zero-level set of the moment map.

Now we use the solution (6.42) as an initial evaluation of $\mathcal{L}_\epsilon$. Then we obtain the functional of h given by

$$\mathcal{L}_\epsilon[h] = \frac{1}{2} \int\limits_\epsilon^{\frac{1}{\epsilon}} \mathrm{d}s \left(\frac{1}{4} \operatorname{tr} \left(h^{-1} \frac{\mathrm{d}h}{\mathrm{d}s} \right)^2 + V(h) \right), \tag{6.47}$$

where the potential $V(h) = 2\lambda(s) \sum_{i=1}^{2} \operatorname{tr}\big(h^{-1}\overline{\mathcal{T}}_i\, h\, \overline{\mathcal{T}}_i^{\,\dagger}\big)$ is positive. This implies that for any boundary values $h_\pm \in (\mathcal{G}^{k,l})^{\mathbb{C}}/\mathcal{G}^{k,l}$ there exists a continuous path[10]

$$h_\epsilon : \left[\epsilon, \tfrac{1}{\epsilon}\right] \longrightarrow \left(\mathcal{G}^{k,l}\right)^{\mathbb{C}} \big/ \mathcal{G}^{k,l} \quad \text{with} \quad h(\epsilon) = h_- \quad \text{and} \quad h(\tfrac{1}{\epsilon}) = h_+, \tag{6.48}$$

which is smooth on $\left(\epsilon, \tfrac{1}{\epsilon}\right)$ and minimises the functional $\mathcal{L}_\epsilon$. Hence for any choice of complex gauge transformation g such that $gg^\dagger = h_\epsilon$, the triple $g \cdot \left(\{\overline{\mathcal{T}}_i\}_{i=1,2}, 0\right) = \left(\{\mathrm{Ad}(g)\overline{\mathcal{T}}_i\}_{i=1,2}, \tfrac{\mathrm{i}}{2}\left(\tfrac{\mathrm{d}g}{\mathrm{d}s}\right)g^{-1}\right)$ satisfies the stability equation $\mu(\{\mathcal{Y}_j\}, \{\overline{\mathcal{Y}}_j\}) = 0$ on $\left(\epsilon, \tfrac{1}{\epsilon}\right)$ for any $0 < \epsilon < 1$.

The uniqueness of the solution h_ϵ and its extension to the limit $\epsilon \to 0$ follows from [42] similarly to the proof of [43, Lemma 3.17].[11] The gauge transformation $g_\infty = (h_\infty)^{\frac{1}{2}}$ is obtained from $h_\infty = \lim_{\epsilon \to 0} h_\epsilon$. However, the corresponding complex gauge transformation $g = g(h_\epsilon)$ is not unique. Similarly to [42,43], given a solution $\{\overline{\mathcal{Y}}_j\}_{j=1,2,3}$ of the holomorphicity equations one can define two solutions $\{\overline{\mathcal{Y}}_j{}'\}_{j=1,2,3} = \{g_1 \cdot \overline{\mathcal{Y}}_j\}_{j=1,2,3}$ and $\{\overline{\mathcal{Y}}_j{}''\}_{j=1,2,3} = \{g_2 \cdot \overline{\mathcal{Y}}_j\}_{j=1,2,3}$ of the stability equation for any $g_1, g_2 \in (\widehat{\mathcal{G}}^{k,l})^{\mathbb{C}}$. By uniqueness one has $g_1 g_1^\dagger = g_2 g_2^\dagger$; thus there exists $\tilde{g} \in \widehat{\mathcal{G}}^{k,l}$ such that $g_1(s) = g_2(s)\,\tilde{g}(s)$. This ambiguity in the choice of $g = g(h_\epsilon)$ can be removed as follows: The complete set of instanton equations is invariant under $\widehat{\mathcal{G}}^{k,l}$ and a $\widehat{\mathcal{G}}^{k,l}$ gauge transformation is sufficient to eliminate $\mathcal{X}_6$. Hence one can demand that the gauge transformation $\{\overline{\mathcal{Y}}_j{}'\}_{j=1,2,3} = \{g \cdot \overline{\mathcal{Y}}_j\}_{j=1,2,3}$ of a solution $\{\overline{\mathcal{Y}}_j\}_{j=1,2,3}$ satisfies $\overline{\mathcal{Y}}_3{}' = \mathcal{Y}_3{}'$. This fixes $g = g(h_\epsilon)$ uniquely.

6.2.2.9. Boundary conditions   A trivial solution of (6.26) is given by

$$\mathcal{X}_6(s) = 0 \quad \text{and} \quad \mathcal{X}_\mu(s) = T_\mu \quad \text{with} \quad [T_\mu, T_\nu] = 0 \quad \text{for} \quad \mu, \nu = 1, \dots, 5, \tag{6.49}$$

where T_μ are constant elements in the Cartan subalgebra $\mathfrak{u}(1)^p$ of $\mathrm{End}_{\mathrm{U}(1)}(V^{k,l})$. From the rescaling (6.25) we then see that the original scalar fields X_μ are singular at the origin $r = 0$ (corresponding to $\tau \to -\infty$ or $s \to \infty$). Following [43,44], in the generic case we choose boundary conditions for X_μ such that[12] $X_\mu(\tau) \to 0$ as $\tau \to +\infty$ for $\mu = 1, \dots, 5$. Arguing as in [43], this implies the existence of the limit of $\mathcal{X}_\mu(s)$ for $s \to 0$ and hence the solutions extend over the half-closed interval $\mathbb{R}_{\geq 0}$. Since (6.26) is a system of first order ordinary differential equations, it suffices to impose one additional boundary condition for the matrices $\mathcal{X}_\mu(s)$ on $[0, \infty)$ which we take to be

$$\lim_{s \to \infty} \mathcal{X}_\mu(s) = \mathrm{Ad}(g_0) T_\mu, \tag{6.50}$$

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[10] See for instance the note under [42, Corollary 2.13]: Since $\mathrm{GL}(p, \mathbb{C})/\mathrm{U}(p)$ satisfies all necessary conditions for the existence of a unique stationary path between any two points, the quotient $(\mathcal{G}^{k,l})^{\mathbb{C}}/\mathcal{G}^{k,l} \cong \prod_{(n,m)} \mathrm{GL}(p_{(n,m)}, \mathbb{C})/\mathrm{U}(p_{(n,m)}) \times \mathrm{GL}(n+1, \mathbb{C})/\mathrm{U}(n+1)$ inherits these properties.

[11] We omit a description of the required differential inequality as well as a treatment of potential pole contributions from $\lambda(s)$; see [13, Section 3] for a general discussion of these issues.

[12] From now on we will no longer deal with the scalar field X_6 as it can always be gauged away.

for suitable $g_0 \in \mathcal{G}^{k,l}$ ensuring compatibility with the SU(3)-equivariant structure from (3.31) (cf. Section 4 for explicit examples). Then the value of $\mathcal{X}_\mu(s)$ at $s = 0$ is completely determined by the solution.

From (6.28b) it follows that the paths $\overline{\mathcal{Y}}_i(s)$ for $i = 1, 2$ each lie respectively in the same adjoint orbits $\mathcal{O}_i$ of the complex Lie algebra $\mathrm{End}_{U(1)}(V^{k,l})^\mathbb{C}$ for all $s \in [0, \infty)$. Let $\overline{\mathcal{T}}_i = \frac{1}{2}(T_{2i-1} + \mathrm{i}\, T_{2i})$ for $i = 1, 2$, and denote by $\mathcal{O}_{\overline{\mathcal{T}}_i}$ the adjoint orbit of $\overline{\mathcal{T}}_i$ in $\mathrm{End}_{U(1)}(V^{k,l})^\mathbb{C}$. Then the boundary conditions (6.50) imply that the closures $\overline{\mathcal{O}_{\overline{\mathcal{T}}_i}}$ contain $\mathcal{O}_i$ for $i = 1, 2$. If the quintuple $\{T_\mu\}_{\mu=1,\dots,5}$ is *regular* in the Cartan subalgebra of $\mathrm{End}_{U(1)}(V^{k,l})$, i.e. the joint centraliser of T_μ in $\mathcal{G}^{k,l}$ is the maximal torus $U(1)^p$, then $\overline{\mathcal{O}_{\overline{\mathcal{T}}_i}} = \mathcal{O}_{\overline{\mathcal{T}}_i}$ are regular orbits and hence $\mathcal{O}_{\overline{\mathcal{T}}_i} = \mathcal{O}_i$ [43]. By our previous results, there exists a unique complex gauge transformation g, which is bounded and framed, such that $\{g \cdot \overline{\mathcal{Y}}_j\}_{j=1,2,3}$ satisfies (6.28c) and $g \cdot \overline{\mathcal{Y}}_3$ is skew-Hermitian. Employing (6.28a), it follows that in this case there is a map

$$\mathcal{M}_{k,l}^{SU(3)} \longrightarrow \mathcal{O}_{\overline{\mathcal{T}}_1} \times \mathcal{O}_{\overline{\mathcal{T}}_2}$$
$$\left(\{\mathcal{Y}_j(\tau)\}_{j=1,2,3}, \{\overline{\mathcal{Y}}_j(\tau)\}_{j=1,2,3}\right) \longmapsto \left(\overline{\mathcal{Y}}_1(0), \overline{\mathcal{Y}}_2(0)\right) \tag{6.51}$$

from the moduli space of solutions satisfying the boundary conditions (6.50) together with the equivariance condition imposed by our construction. Arguing as in [43], by our construction of local solutions to the complex equations, and the existence of a unique solution to the real equation within the complex gauge orbit of these elements, this map is a bijection which moreover preserves the holomorphic symplectic structure. This space is naturally a complex symplectic manifold of (complex) dimension $2 \dim(\mathcal{G}^{k,l})^\mathbb{C} - \sum_{i=1}^2 \dim(\mathcal{Z}_{\overline{\mathcal{T}}_i})$ with the product of the standard Kirillov–Kostant–Souriau symplectic forms on the orbits, where $\mathcal{Z}_{\overline{\mathcal{T}}_i} \subset (\mathcal{G}^{k,l})^\mathbb{C}$ is the subgroup that commutes with $\overline{\mathcal{T}}_i$ for $i = 1, 2$. By our general constructions it is a Kähler manifold. In the cases that SU(3)-equivariance forces $\overline{\mathcal{T}}_i = 0$ for some $i \in \{1, 2\}$, the corresponding orbit closure $\overline{\mathcal{O}_{\overline{\mathcal{T}}_i}}$ should be replaced by the nilpotent cone $\mathcal{N}$ of dimension $\dim(\mathcal{G}^{k,l})^\mathbb{C} - p$ which consists of all nilpotent elements of $\mathrm{End}_{U(1)}(V^{k,l})^\mathbb{C}$. The variety $\mathcal{N}$ has singularities corresponding to non-regular nilpotent orbits, and in particular it contains the locus of Kleinian singularities $\mathbb{C}^2/\mathbb{Z}_p$ in complex codimension 2; see [15] for further details. Thus in this case the moduli space is *singular* and by SU(3)-equivariance we expect that it contains the singular subvariety $\mathbb{C}^3/\mathbb{Z}_p$.

6.2.3. $\mathbb{C}^3$-invariance

Now we turn our attention to the space of translationally-invariant connections $\mathbb{A}(\mathfrak{E}^{k,l})$ on the bundle (5.1). The structure group $\mathfrak{G}^{k,l}$ of (5.1) (which in this case coincides with the gauge group) is given by (5.3) and its Lie algebra $\mathfrak{g}^{k,l}$ by (5.17). A generic element of the tangent space $T_\mathcal{A}\mathbb{A}(\mathfrak{E}^{k,l})$ at a point $\mathcal{A} \in \mathbb{A}(\mathfrak{E}^{k,l})$ is given by

$$W = W_\alpha \, \mathrm{d}z^\alpha + \overline{W}_\alpha \, \mathrm{d}\bar{z}^\alpha \in \Omega^1\left(\mathbb{C}^3/\mathbb{Z}_{q+1}, \mathfrak{g}^{k,l}\right), \tag{6.52}$$

with constant W_α, $\overline{W}_\alpha$ for $\alpha = 1, 2, 3$. As before, let us define a metric $\boldsymbol{g}$ on $\mathbb{A}(\mathfrak{E}^{k,l})$. Gauge transformations of tangent vectors $\boldsymbol{w} = \boldsymbol{w}_\alpha \, \mathrm{d}z^\alpha + \overline{\boldsymbol{w}}_\alpha \, \mathrm{d}\bar{z}^\alpha$ are given by $\overline{\boldsymbol{w}}_\alpha \mapsto \mathrm{Ad}(g)\overline{\boldsymbol{w}}_\alpha$ for $\alpha = 1, 2, 3$. We deduce the metric to be

$$\boldsymbol{g}_{|\mathcal{A}}(\boldsymbol{w}, \boldsymbol{v}) := \frac{1}{2} \sum_{\alpha=1}^3 \mathrm{tr}\left(\boldsymbol{w}_\alpha^\dagger \boldsymbol{v}_\alpha + \boldsymbol{w}_\alpha \boldsymbol{v}_\alpha^\dagger\right), \tag{6.53}$$

and a symplectic form via

$$\omega_{|\mathcal{A}}(\boldsymbol{w}, \boldsymbol{v}) := \frac{\mathrm{i}}{2} \sum_{\alpha=1}^{3} \operatorname{tr}\bigl(\boldsymbol{w}_\alpha^\dagger \boldsymbol{v}_\alpha - \boldsymbol{w}_\alpha \boldsymbol{v}_\alpha^\dagger\bigr) . \tag{6.54}$$

These definitions follow directly from the translationally-invariant limit of (6.12) and (6.13) (and agree with those of [20]). Evidently the metric and symplectic structure are gauge-invariant.

Define the subspace of invariant connections that satisfy the holomorphicity conditions (5.13) as

$$\mathbb{A}^{1,1}\bigl(\mathfrak{C}^{k,l}\bigr) = \Bigl\{ (\{W_\alpha\}, \{\overline{W}_\alpha\}) \in \mathbb{A}\bigl(\mathfrak{C}^{k,l}\bigr) : \bigl[\overline{W}_\alpha, \overline{W}_\beta\bigr] = 0 \quad \text{for} \quad \alpha, \beta = 1, 2, 3 \Bigr\} , \tag{6.55}$$

which is a finite-dimensional Kähler space by the general considerations of Section 6.2.1.

6.2.3.1. Moment map   The corresponding moment map can be introduced as before via

$$\mu : \mathbb{A}^{1,1}\bigl(\mathfrak{C}^{k,l}\bigr) \longrightarrow \mathfrak{g}^{k,l}$$

$$(\{W_\alpha\}, \{\overline{W}_\alpha\}) \longmapsto \mathrm{i} \sum_{\alpha=1}^{3} \bigl[W_\alpha, \overline{W}_\alpha\bigr] , \tag{6.56}$$

but in this case it is possible to choose various gauge-invariant levels Ξ from (5.16) and consequently define different moduli spaces

$$\mathcal{M}_{k,l}^{\mathbb{C}^3}(\Xi) = \mu^{-1}(\Xi) \bigm/ \mathfrak{G}^{k,l} . \tag{6.57}$$

6.2.3.2. Real gauge group   The complete set of instanton equations (5.13) and (5.18) is invariant under the action of the gauge group (5.3) with the usual transformations

$$\overline{W}_\alpha \longmapsto \mathrm{Ad}(g)\overline{W}_\alpha \quad \text{for} \quad \alpha = 1, 2, 3 \tag{6.58}$$

for $g \in \mathfrak{G}^{k,l} \hookrightarrow \mathrm{U}(p)$.

6.2.3.3. Complex gauge group   Recalling that the holomorphicity conditions allow for the introduction of a $(\mathfrak{G}^{k,l})^{\mathbb{C}}$-bundle, we find that the corresponding equations are invariant under $(\mathfrak{G}^{k,l})^{\mathbb{C}}$ gauge transformations. Again the stability equation is not invariant under the action of the complex gauge group.

6.2.3.4. Stable points   The set of stable points is defined as before to be

$$\mathbb{A}_{\mathrm{st}}^{1,1}\bigl(\mathfrak{C}^{k,l}; \Xi\bigr) := \Bigl\{ (\{W_\alpha\}, \{\overline{W}_\alpha\}) \in \mathbb{A}^{1,1}\bigl(\mathfrak{C}^{k,l}\bigr) : (\mathfrak{G}^{k,l})^{\mathbb{C}}_{(\{W_\alpha\}, \{\overline{W}_\alpha\})} \cap \mu^{-1}(\Xi) \neq \emptyset \Bigr\} , \tag{6.59}$$

and by taking the GIT quotient one obtains the Ξ-dependent moduli spaces[13]

$$\mathcal{M}_{k,l}^{\mathbb{C}^3}(\Xi) \cong \mathbb{A}_{\mathrm{st}}^{1,1}\bigl(\mathfrak{C}^{k,l}; \Xi\bigr) \bigm/ (\mathfrak{G}^{k,l})^{\mathbb{C}} . \tag{6.60}$$

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[13] This description is analogous to the quiver GIT quotients used by [21,37] to describe instanton moduli on $\mathbb{C}^3/\mathbb{Z}_{q+1}$ as representation moduli of the McKay quiver.

The moment map (6.56) transforms under $g \in (\mathfrak{G}^{k,l})^{\mathbb{C}}$ as

$$\mu\big(\{W_\alpha\}, \{\overline{W}_\alpha\}\big) = \mathrm{i} \sum_{\alpha=1}^{3} \big[W_\alpha, \overline{W}_\alpha\big]$$

$$\longmapsto \mathrm{i}\, \mathrm{Ad}(g) \sum_{\alpha=1}^{3} \big[h^{-1}\, W_\alpha\, h, \overline{W}_\alpha\big], \qquad (6.61)$$

where we introduced $h = h(g) = g^\dagger g \in (\mathfrak{G}^{k,l})^{\mathbb{C}}/\mathfrak{G}^{k,l}$. Similarly to before, h can be identified with a positive Hermitian $p \times p$ matrix. Moreover, $\mathrm{Ad}(g')\,\Xi = \Xi$ for any $g' \in \mathfrak{G}^{k,l}$. By the embedding $\mathfrak{G}^{k,l} \hookrightarrow \mathrm{U}(p)$ and the polar decomposition of an element $g \in (\mathfrak{G}^{k,l})^{\mathbb{C}}$ into $g = h' \exp(\mathrm{i}\, X)$ for Hermitian $h' \in \mathfrak{G}^{k,l}$ and skew-adjoint $X \in \mathfrak{g}^{k,l}$, we have

$$\mathrm{Ad}(g)\,\Xi = \mathrm{Ad}(h')\big(\mathrm{Ad}(\exp(\mathrm{i}\, X))\,\Xi\big) = \mathrm{Ad}(h')\big(\Xi + \mathrm{i}\,[X, \Xi]\big) = \mathrm{Ad}(h')\,\Xi = \Xi, \qquad (6.62)$$

where we used the Baker–Campbell–Hausdorff formula and the fact that Ξ is central in $\mathfrak{g}^{k,l}$. It follows that $\mathrm{Center}\big(\mathfrak{g}^{k,l}\big) \subset \mathrm{Center}\big((\mathfrak{g}^{k,l})^{\mathbb{C}}\big)$. Hence a point $\big(\{W_\alpha\}, \{\overline{W}_\alpha\}\big) \in \mathbb{A}^{1,1}(\mathfrak{E}^{k,l})$ is stable if and only if there exists a positive Hermitian matrix h (modulo unitary transformations) satisfying the equation

$$\sum_{\alpha=1}^{3} \big[h^{-1}\, W_\alpha\, h, \overline{W}_\alpha\big] = -\mathrm{i}\,\Xi. \qquad (6.63)$$

By our general constructions the moduli spaces $\mathcal{M}_{k,l}^{\mathbb{C}^3}(\Xi)$ are Kähler spaces, which however are generically *not* smooth manifolds but have a complicated scheme structure with branches of varying dimension that should be analysed within the context of a perfect obstruction theory; such an analysis is beyond the scope of the present paper. Generally, the canonical map $\mathcal{M}_{k,l}^{\mathbb{C}^3}(\Xi) \to \mathcal{M}_{k,l}^{\mathbb{C}^3}(0)$ is a partial resolution of singularities for generic Ξ. For example, in the case $p_{(n,m)} = 1$ for all $(n, m) \in Q_0(k, l)$ (so that $V^{k,l} \cong \underline{C}^{k,l}$ and $p = p_0$), for generic levels $\Xi \neq 0$ the moduli spaces $\mathcal{M}_{k,l}^{\mathbb{C}^3}(\Xi)$ are schemes akin to the $\mathbb{Z}_p$-Hilbert scheme of $p = \dim(\underline{C}^{k,l})$ points on $\mathbb{C}^3$ for the $\mathbb{Z}_p$-action given by (2.28) (with $q = p - 1$), which are partial resolutions of the singular spaces $\mathcal{M}_{k,l}^{\mathbb{C}^3}(0)$ that correspond to configurations of p points of $\mathbb{C}^3$ given as unions of $\mathbb{Z}_p$-orbits (cf. [21,20] for the case of the McKay quiver)[14]; these are precisely the same types of singularities encountered in the moduli spaces $\mathcal{M}_{k,l}^{\mathrm{SU}(3)}$ above.

Acknowledgements

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[14] In the special case $q = 1$ the $\mathbb{Z}_2$-Hilbert scheme is the product $M_1 \times \mathbb{C}$ where M_1 is the total space of the canonical line bundle $\mathcal{O}_{\mathbb{C}P^1}(-2) \to \mathbb{C}P^1$ (Eguchi–Hanson space), whereas for $q = 2$ the $\mathbb{Z}_3$-Hilbert scheme is the total space of the canonical line bundle $\mathcal{O}_{\mathbb{C}P^2}(-3) \to \mathbb{C}P^2$ (local del Pezzo surface of degree 0).

Appendix A. Bundles on $\mathbb{C}P^2$

A.1. Geometry of $\mathbb{C}P^2$

A.1.1. SU(3)-equivariant 1-forms

Consider the row vector $\beta^\top = (\beta^1, \beta^2)$. The relations (2.11) and (2.12) dictate the explicit form of the 1-forms β^i and their exterior derivatives as

$$\beta^i = \frac{1}{\gamma}\,\mathrm{d}y^i - \frac{1}{\gamma^2\,(\gamma+1)}\,y^i \sum_{j=1}^{2} \bar{y}^j\,\mathrm{d}y^j\,, \qquad \bar{\beta}^i = \frac{1}{\gamma}\,\mathrm{d}\bar{y}^i - \frac{1}{\gamma^2\,(\gamma+1)}\,\bar{y}^i \sum_{j=1}^{2} y^j\,\mathrm{d}\bar{y}^j\,, \tag{A.1a}$$

$$\mathrm{d}\beta^1 = -\beta^1 \wedge \left(B_{11} + \tfrac{3}{2}\,a\right) + \beta^2 \wedge \bar{B}_{12}\,, \qquad \mathrm{d}\beta^2 = -\beta^1 \wedge B_{12} + \beta^2 \wedge \left(B_{11} - \tfrac{3}{2}\,a\right), \tag{A.1b}$$

$$\mathrm{d}\bar{\beta}^1 = -\left(B_{11} + \tfrac{3}{2}\,a\right) \wedge \bar{\beta}^1 - B_{12} \wedge \bar{\beta}^2\,, \qquad \mathrm{d}\beta^2 = \bar{B}_{12} \wedge \bar{\beta}^1 + \left(B_{11} - \tfrac{3}{2}\,a\right) \wedge \bar{\beta}^2\,. \tag{A.1c}$$

One can regard β^i as a basis for the $(1, 0)$-forms and $\bar{\beta}^i$ as a basis for the $(0, 1)$-forms of the complex cotangent bundle over the patch $\mathcal{U}_0$ of $\mathbb{C}P^2$ with respect to an almost complex structure J. The canonical 1-forms $\mathrm{d}y^i$ and $\mathrm{d}\bar{y}^i$ could equally well be used for a holomorphic decomposition with respect to J, but the forms β^i and $\bar{\beta}^i$ are SU(3)-equivariant.

A.1.2. Hermitian Yang–Mills equations

The canonical Kähler 2-form on the patch $\mathcal{U}_0$ is given by

$$\omega_{\mathbb{C}P^2} = -\mathrm{i}\,R^2\,\beta^\top \wedge \bar{\beta} = \mathrm{i}\,R^2\,\left(\beta^1 \wedge \bar{\beta}^1 + \beta^2 \wedge \bar{\beta}^2\right)\,, \tag{A.2}$$

where R is the radius of the linearly embedded projective line $\mathbb{C}P^1 \subset \mathbb{C}P^2$. The 1-form $B_{(1)}$ is then an instanton connection by the following argument: Locally, one can define a $(2, 0)$-form Ω proportional to $\beta^1 \wedge \beta^2$. The Hermitian Yang–Mills equations for a curvature 2-form F are

$$\Omega \wedge F = 0 \quad \text{and} \quad \omega_{\mathbb{C}P^2} \lrcorner F = 0\,, \tag{A.3}$$

which translate to $F = F^{1,1}$ being a $(1, 1)$-form for which $\mathrm{tr}(F^{1,1}) = 0$; here the contraction $\lrcorner$ between two forms η and η' is defined as $\eta \lrcorner \eta' := \star\left(\eta \wedge \star\eta'\right)$. The curvature $F_B = \mathrm{d}B + B \wedge B = \bar{\beta} \wedge \beta^\top$ is a $(1, 1)$-form which is $\mathfrak{u}(2)$-valued, i.e. $\mathrm{tr}(F_B) = 2a \neq 0$. However $F_a = \mathrm{d}a = \beta^\dagger \wedge \beta$ is also a $(1, 1)$-form. Thus the curvature of the connection $B_{(1)} = B - \tfrac{1}{2}\,a\,\mathbb{1}_2$ given by $F_{B_{(1)}} = F_B - \tfrac{1}{2}\,F_a\,\mathbb{1}_2$ is a $(1, 1)$-form and by construction traceless; hence $B_{(1)}$ is an $\mathfrak{su}(2)$-valued connection satisfying the Hermitian Yang–Mills equations, i.e. it is an instanton connection.

A.2. Hopf fibration and associated bundles

Consider the principal U(1)-bundle $S^5 = \mathrm{SU}(3)/\mathrm{SU}(2) \to \mathbb{C}P^2$. One can associate to it a complex vector bundle whose fibres carry any representation of the structure group U(1), i.e. a complex vector space V together with a group homomorphism $\rho : \mathrm{U}(1) \to \mathrm{GL}(V)$. Then the associated vector bundle E is given as $E := S^5 \times_\rho V \to \mathbb{C}P^2$. In particular, one can choose $V = \underline{m}$ to be the one-dimensional irreducible representation of highest weight $m \in \mathbb{Z}$. Following [25], one then generates associated complex line bundles $L_{\frac{m}{2}} := (L^{\otimes m})^{\frac{1}{2}}$.

A.2.1. Chern classes and monopole charges

Using the conventions of [25] for $\mathbb{C}P^2$, there is a normalised volume form

$$\beta_{\text{vol}} := \frac{1}{2\pi^2} \beta^1 \wedge \bar{\beta}^1 \wedge \beta^2 \wedge \bar{\beta}^2 \quad \text{with} \quad \int_{\mathbb{C}P^2} \beta_{\text{vol}} = 1 \, , \tag{A.4}$$

and the canonical Kähler 2-form (A.2) with

$$\omega_{\mathbb{C}P^2} \wedge \omega_{\mathbb{C}P^2} = -\left(2\pi R^2\right)^2 \beta_{\text{vol}} \, . \tag{A.5}$$

Consider the connection a from (2.11c) on the line bundle L associated to the Hopf bundle $S^5 \to \mathbb{C}P^2$ and the fundamental representation. Since its curvature is $F_a = \frac{\mathrm{i}}{R^2} \omega_{\mathbb{C}P^2}$, the total Chern character of the monopole bundle L is

$$\text{ch}(L) = \exp\left(\tfrac{\mathrm{i}}{2\pi} F_a\right) = \exp(\xi) \tag{A.6}$$

where $\xi := -\frac{1}{2\pi R^2} \omega_{\mathbb{C}P^2}$. Then one immediately reads off the first Chern class

$$c_1(L) = \xi \quad \text{with} \quad \int_{\mathbb{C}P^2} \xi \wedge \xi = -1 \, . \tag{A.7}$$

Since $[\xi] = [c_1(L)]$ generates $H^2(\mathbb{C}P^2, \mathbb{Z}) \cong \mathbb{Z}$ [23], this identifies the first Chern number of L as -1. Thus $L \equiv L_1$ exists globally, and the dual bundle L_{-1} has first Chern class $c_1(L_{-1}) = -c_1(L)$ and hence first Chern number $+1$. For all other bundles $L_{\frac{m}{2}}$ one takes the connection to be $\frac{m}{2} a$, which changes the first Chern class accordingly to

$$c_1\left(L_{\frac{m}{2}}\right) = \tfrac{m}{2} \xi \, , \tag{A.8}$$

and the first Chern number to $-\frac{m}{2}$. Hence only for even values of m do the line bundles $L_{\frac{m}{2}}$ exist globally in the sense of conventional bundles. On the other hand, for odd values of m the line bundles $L_{\frac{m}{2}}$ (and also the instanton bundles I_n for odd values of the isospin n [25]) are examples of *twisted bundles*. The obstruction to the global existence of these bundles is the failure of the cocycle condition for transition functions on triple overlaps of patches, which is given by a non-trivial integral 3-cocycle representing the Dixmier–Douady class of an abelian gerbe; see for example [45] for more details. As argued in [25], the Chern number $\frac{m}{2}$ of the line bundle $L_{-\frac{m}{2}}$ should be taken as the monopole charge rather than the H_{α_2}-eigenvalue m in the Biedenharn basis.

Appendix B. Representations

B.1. Biedenharn basis

Let us summarise the relevant details we need concerning the Biedenharn basis [28–30], which is defined as the basis of eigenvectors according to (3.8); we follow [23,25] for the presentation and notation.

B.1.1. Generators
The remaining generators of $\mathfrak{su}(3)$ act on this eigenvector basis as

$$E_{\pm\alpha_1} \left| \begin{matrix} n \\ q \end{matrix} m \right\rangle = \frac{1}{2} \sqrt{(n \mp q)(n \pm q + 2)} \left| \begin{matrix} n \\ q \pm 2 \end{matrix} m \right\rangle, \tag{B.1a}$$

$$E_{\alpha_2} \left| \begin{matrix} n \\ q \end{matrix} m \right\rangle = \sqrt{\frac{n-q-2}{2(n+1)}} \, \Lambda^+_{k,l}(n,m) \left| \begin{matrix} n+1 \\ q-1 \end{matrix} m+3 \right\rangle + \sqrt{\frac{n+q}{2(n+1)}} \, \Lambda^-_{k,l}(n,m) \left| \begin{matrix} n-1 \\ q-1 \end{matrix} m+3 \right\rangle, \tag{B.1b}$$

$$E_{\alpha_1+\alpha_2} \left| \begin{matrix} n \\ q \end{matrix} m \right\rangle = \sqrt{\frac{n+q+2}{2(n+1)}} \, \Lambda^+_{k,l}(n,m) \left| \begin{matrix} n+1 \\ q+1 \end{matrix} m+3 \right\rangle + \sqrt{\frac{n-q}{2(n+1)}} \, \Lambda^-_{k,l}(n,m) \left| \begin{matrix} n-1 \\ q+1 \end{matrix} m+3 \right\rangle, \tag{B.1c}$$

with $E^\dagger_{\alpha_2} = E^\top_{\alpha_2} = E_{-\alpha_2}$ and $E^\dagger_{\alpha_1+\alpha_2} = E^\top_{\alpha_1+\alpha_2} = E_{-(\alpha_1+\alpha_2)}$. It is convenient to express the generators as

$$E^{+\,(n,m)}_{\alpha_1+\alpha_2} = \sum_{q\in Q_n} \sqrt{\frac{n+q+2}{2(n+1)}} \, \Lambda^+_{k,l}(n,m) \left| \begin{matrix} n+1 \\ q+1 \end{matrix} m+3 \right\rangle \left\langle \begin{matrix} n \\ q \end{matrix} m \right|, \tag{B.2a}$$

$$E^{-\,(n,m)}_{\alpha_1+\alpha_2} = \sum_{q\in Q_n} \sqrt{\frac{n-q}{2(n+1)}} \, \Lambda^-_{k,l}(n,m) \left| \begin{matrix} n-1 \\ q+1 \end{matrix} m+3 \right\rangle \left\langle \begin{matrix} n \\ q \end{matrix} m \right|, \tag{B.2b}$$

$$E^{+\,(n,m)}_{\alpha_2} = \sum_{q\in Q_n} \sqrt{\frac{n-q-2}{2(n+1)}} \, \Lambda^+_{k,l}(n,m) \left| \begin{matrix} n+1 \\ q-1 \end{matrix} m+3 \right\rangle \left\langle \begin{matrix} n \\ q \end{matrix} m \right|, \tag{B.2c}$$

$$E^{-\,(n,m)}_{\alpha_2} = \sum_{q\in Q_n} \sqrt{\frac{n+q}{2(n+1)}} \, \Lambda^-_{k,l}(n,m) \left| \begin{matrix} n-1 \\ q-1 \end{matrix} m+3 \right\rangle \left\langle \begin{matrix} n \\ q \end{matrix} m \right|, \tag{B.2d}$$

where $Q_n := \{-n, -n+2, \ldots, n-2, n\}$ and

$$\Lambda^+_{k,l}(n,m) = \frac{1}{\sqrt{n+2}} \sqrt{\left(\frac{k+2l}{3} + \frac{n}{2} + \frac{m}{6} + 2\right)\left(\frac{k-l}{3} + \frac{n}{2} + \frac{m}{6} + 1\right)\left(\frac{2k+l}{3} - \frac{n}{2} - \frac{m}{6}\right)}, \tag{B.3a}$$

$$\Lambda^-_{k,l}(n,m) = \frac{1}{\sqrt{n}} \sqrt{\left(\frac{k+2l}{3} - \frac{n}{2} + \frac{m}{6} + 1\right)\left(\frac{l-k}{3} + \frac{n}{2} - \frac{m}{6}\right)\left(\frac{2k+l}{3} + \frac{n}{2} - \frac{m}{6} + 1\right)}, \tag{B.3b}$$

with $\Lambda^-_{k,l}(0,m) := 0$ [25]. The identity operator $\Pi_{(n,m)}$ of the representation $\underline{(n,m)}$ is given by

$$\Pi_{(n,m)} = \sum_{q\in Q_n} \left| \begin{matrix} n \\ q \end{matrix} m \right\rangle \left\langle \begin{matrix} n \\ q \end{matrix} m \right|. \tag{B.4}$$

B.1.2. Fields

The 1-instanton connection (2.13) is represented in the Biedenharn basis by

$$B_{(1)} = B_{11} H_{\alpha_1} + B_{12} E_{\alpha_1} - \left(B_{12} E_{\alpha_1}\right)^\dagger$$

$$= \sum_{n,q,m} \left(B_{11}\, q \left| \begin{matrix} n \\ q \end{matrix} m \right\rangle \left\langle \begin{matrix} n \\ q \end{matrix} m \right| + \frac{1}{2} B_{12} \sqrt{(n-q)(n+q+2)} \left| \begin{matrix} n \\ q+2 \end{matrix} m \right\rangle \left\langle \begin{matrix} n \\ q \end{matrix} m \right| \right.$$

$$\left. - \frac{1}{2} \bar{B}_{12} \sqrt{(n+q)(n-q+2)} \left| \begin{matrix} n \\ q-2 \end{matrix} m \right\rangle \left\langle \begin{matrix} n \\ q \end{matrix} m \right| \right)$$

$$\equiv \bigoplus_{(n,m)\in Q_0(k,l)} B_{(n,m)}, \tag{B.5}$$

where $B_{(n,m)} \in \Omega^1\big(\mathfrak{su}(2), \mathrm{End}\big(\underline{(n,m)}\big)\big)$. One further introduces matrix-valued 1-forms given by

$$\bar{\beta}_{q+1} = \bar{\beta}^1_{q+1}\, E_{\alpha_1+\alpha_2} + \bar{\beta}^2_{q+1}\, E_{\alpha_2} \equiv \bigoplus_{(n,m)\in Q_0(k,l)} \left(\bar{\beta}^+_{(n,m)} + \bar{\beta}^-_{(n,m)}\right), \tag{B.6a}$$

with the morphism-valued 1-forms

$$\bar{\beta}^\pm_{(n,m)} \in \Omega^1\left(S^5/\mathbb{Z}_{q+1}\, , \, \mathrm{Hom}\big(\underline{(n,m)}\, , \, \underline{(n\pm 1, m+3)}\big)\right), \tag{B.6b}$$

and the corresponding adjoint maps

$$\beta^\pm_{(n,m)} \in \Omega^1\left(S^5/\mathbb{Z}_{q+1}\, , \, \mathrm{Hom}\big(\underline{(n\pm 1, m+3)}\, , \, \underline{(n,m)}\big)\right). \tag{B.6c}$$

They have the explicit form

$$\bar{\beta}^\pm_{(n,m)} = \frac{\Lambda^\pm_{k,l}(n,m)}{\sqrt{2(n+1)}} \sum_{q\in Q_n} \left(\sqrt{n\pm q+1\pm 1}\ \bar{\beta}^1_{q+1} \left|\begin{matrix} n\pm 1 \\ q+1 \end{matrix}\ m+3\right\rangle\left\langle\begin{matrix} n \\ q \end{matrix}\ m\right|\right.$$

$$\left. + \sqrt{n\mp q+1\pm 1}\ \bar{\beta}^2_{q+1} \left|\begin{matrix} n\pm 1 \\ q-1 \end{matrix}\ m+3\right\rangle\left\langle\begin{matrix} n \\ q \end{matrix}\ m\right|\right). \tag{B.7}$$

B.1.3. Skew-Hermitian basis

Similarly to [46], for a given representation $\underline{C}^{k,l}$ of the generators I_i and I_μ defined in (3.2) the decomposition into the Biedenharn basis yields

$$I_1 = \bigoplus_{(n,m)} I_1^{(n,m)} = \bigoplus_{\pm,\,(n,m)} \left(E^{\pm\,(n,m)}_{\alpha_1+\alpha_2} - E^{\pm\,(n,m)}_{-\alpha_1-\alpha_2}\right), \tag{B.8a}$$

$$I_2 = \bigoplus_{(n,m)} I_2^{(n,m)} = -\mathrm{i} \bigoplus_{\pm,\,(n,m)} \left(E^{\pm\,(n,m)}_{\alpha_1+\alpha_2} + E^{\pm\,(n,m)}_{-\alpha_1-\alpha_2}\right), \tag{B.8b}$$

$$I_3 = \bigoplus_{(n,m)} I_3^{(n,m)} = \bigoplus_{\pm,\,(n,m)} \left(E^{\pm\,(n,m)}_{\alpha_2} - E^{\pm\,(n,m)}_{-\alpha_2}\right), \tag{B.8c}$$

$$I_4 = \bigoplus_{(n,m)} I_4^{(n,m)} = -\mathrm{i} \bigoplus_{\pm,\,(n,m)} \left(E^{\pm\,(n,m)}_{\alpha_2} + E^{\pm\,(n,m)}_{-\alpha_2}\right), \tag{B.8d}$$

$$I_5 = \bigoplus_{(n,m)} I_5^{(n,m)} = -\frac{\mathrm{i}}{2} \bigoplus_{(n,m)} H^{(n,m)}_{\alpha_2}. \tag{B.8e}$$

The commutation relations $[I_i, I_a] = f_{ia}{}^b\, I_b$ and $[I_i, I_5] = 0$ induced by (3.4) respectively imply relations among the components given by

$$I_i^{(n',m')}\, I_a^{(n,m)} = I_a^{(n,m)}\, I_i^{(n,m)} + f_{ia}{}^b\, I_b^{(n,m)}, \tag{B.9a}$$

$$I_i^{(n,m)}\, I_5^{(n,m)} = I_5^{(n,m)}\, I_i^{(n,m)}, \tag{B.9b}$$

where $i \in \{6, 7, 8\}$, $a \in \{1, 2, 3, 4\}$, $I_i = \bigoplus_{(n,m)} I_i^{(n,m)}$ and $(n', m') = (n\pm 1, m+3)$.

B.2. Flat connections

One can compute the matrix elements of $\mathcal{A}_0$ from (3.12) with respect to the Biedenharn basis. By choosing an SU(3)-representation $\underline{C}^{k,l}$, which induces an SU(2)-representation by restriction, one induces a connection $\mathcal{A}_0$ on the vector V-bundle

$$\widetilde{\mathcal{V}}_{\underline{C}^{k,l}} \xrightarrow{\underline{C}^{k,l}} \mathrm{G}/\widetilde{\mathrm{K}} \quad \text{with} \quad \mathcal{V}_{\underline{C}^{k,l}} := \mathrm{G} \times_{\mathrm{K}} \underline{C}^{k,l} \tag{B.10}$$

associated to the principal V-bundle (2.23). Then the connection $\mathcal{A}_0$ can be decomposed into morphism-valued 1-forms

$$\mathcal{A}_0 = \bigoplus_{(n,m)\in Q_0(k,l)} \left(B_{(n,m)} - \frac{\mathrm{i}\,m}{2}\,\eta\,\Pi_{(n,m)} + \bar{\beta}^+_{(n,m)} + \bar{\beta}^-_{(n,m)} - \beta^+_{(n,m)} - \beta^-_{(n,m)} \right) \tag{B.11}$$

with respect to this basis. The computation of the vanishing curvature $\mathcal{F}_0 = 0$ yields relations between the different matrix elements given by

$$\mathrm{d}B_{(n,m)} + B_{(n,m)} \wedge B_{(n,m)} - \tfrac{\mathrm{i}\,m}{2}\,\mathrm{d}\eta\,\Pi_{(n,m)}$$
$$= \bar{\beta}^+_{(n-1,m-3)} \wedge \beta^+_{(n-1,m-3)} + \bar{\beta}^-_{(n+1,m-3)} \wedge \beta^-_{(n+1,m-3)}$$
$$+ \beta^+_{(n,m)} \wedge \bar{\beta}^+_{(n,m)} + \beta^-_{(n,m)} \wedge \bar{\beta}^-_{(n,m)} \,, \tag{B.12a}$$

$$0 = \mathrm{d}\bar{\beta}^\pm_{(n,m)} + B_{(n+1,m+3)} \wedge \bar{\beta}^\pm_{(n,m)} + \bar{\beta}^\pm_{(n,m)} \wedge B_{(n,m)} - \tfrac{3\,\mathrm{i}}{2}\,\eta\,\Pi_{(n\pm1,m+3)} \wedge \bar{\beta}^\pm_{(n,m)}, \tag{B.12b}$$

$$0 = \bar{\beta}^+_{(n,m)} \wedge \bar{\beta}^-_{(n+1,m-3)} + \bar{\beta}^-_{(n+2,m)} \wedge \bar{\beta}^+_{(n+1,m-3)} \,, \tag{B.12c}$$

$$0 = \bar{\beta}^+_{(n,m)} \wedge \bar{\beta}^-_{(n,m)} + \beta^-_{(n+1,m+3)} \wedge \bar{\beta}^+_{(n-1,m+3)} \,, \tag{B.12d}$$

$$0 = \bar{\beta}^\pm_{(n,m)} \wedge \bar{\beta}^\pm_{(n\mp1,m-3)} \,, \tag{B.12e}$$

plus their conjugate equations.

B.3. Quiver connections

One can also compute the matrix elements of the curvature (3.36c) in the Biedenharn basis. For this, the curvature $\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ is arranged into components

$$(\mathcal{F})_{(n,m),(n',m')} \in \Omega^2\left(\mathcal{E}^{k,l},\,\mathrm{End}\big(E_{p_{(n,m)}}, E_{p_{(n',m')}}\big) \otimes \mathrm{End}\big(\underline{(n,m)},\,\underline{(n',m')}\big)\right), \tag{B.13}$$

which can be simplified by using the relations (B.12). We denote the curvature of the connection $A_{(n,m)}$ on the bundle (3.21) by

$$F_{(n,m)} := \mathrm{d}A_{(n,m)} + A_{(n,m)} \wedge A_{(n,m)} \tag{B.14a}$$

and the bifundamental covariant derivatives of the Higgs fields as

$$D\phi^\pm_{(n,m)} := \mathrm{d}\phi^\pm_{(n,m)} + A_{(n\pm1,m+3)}\,\phi^\pm_{(n,m)} - \phi^\pm_{(n,m)}\,A_{(n,m)} \,, \tag{B.14b}$$

$$D\psi_{(n,m)} := \mathrm{d}\psi_{(n,m)} + A_{(n,m)}\,\psi_{(n,m)} - \psi_{(n,m)}\,A_{(n,m)} \,. \tag{B.14c}$$

Then the non-zero curvature components read as

$$
\begin{aligned}
(\mathcal{F})_{(n,m),(n,m)} = {}& F_{(n,m)} \otimes \Pi_{(n,m)} - D\psi_{(n,m)} \wedge \tfrac{\mathrm{i}\,m}{2}\,\eta\,\Pi_{(n,m)} \\
& - \big(\psi_{(n,m)} - \mathbb{1}_{p_{(n,m)}}\big) \otimes \tfrac{\mathrm{i}\,m}{2}\,\mathrm{d}\eta\,\Pi_{(n,m)} \\
& + \big(\mathbb{1}_{p_{(n,m)}} - \phi^+_{(n-1,m-3)}\,(\phi^+)^\dagger_{(n-1,m-3)}\big) \otimes \bar\beta^+_{(n-1,m-3)} \wedge \beta^+_{(n-1,m-3)} \\
& + \big(\mathbb{1}_{p_{(n,m)}} - \phi^-_{(n+1,m-3)}\,(\phi^-)^\dagger_{(n+1,m-3)}\big) \otimes \bar\beta^-_{(n+1,m-3)} \wedge \beta^-_{(n+1,m-3)} \\
& + \big(\mathbb{1}_{p_{(n,m)}} - (\phi^+)^\dagger_{(n,m)}\,\phi^+_{(n,m)}\big) \otimes \beta^+_{(n,m)} \wedge \bar\beta^+_{(n,m)} \\
& + \big(\mathbb{1}_{p_{(n,m)}} - (\phi^-)^\dagger_{(n,m)}\,\phi^-_{(n,m)}\big) \otimes \beta^-_{(n,m)} \wedge \bar\beta^-_{(n,m)}\,,
\end{aligned}
\tag{B.15a}
$$

$$
\begin{aligned}
(\mathcal{F})_{(n,m),(n\pm1,m+3)} = {}& D\phi^\pm_{(n,m)} \wedge \bar\beta^\pm_{(n,m)} - \big((m+3)\,\psi_{(n\pm1,m+3)}\,\phi^\pm_{(n,m)} \\
& - m\,\phi^\pm_{(n,m)}\,\psi_{(n,m)} - 3\phi^\pm_{(n,m)}\big) \otimes \tfrac{\mathrm{i}}{2}\,\eta\,\Pi_{(n\pm1,m+3)} \wedge \bar\beta^\pm_{(n,m)}\,,
\end{aligned}
\tag{B.15b}
$$

$$
\begin{aligned}
&(\mathcal{F})_{(n+1,m-3),(n+1,m+3)} \\
&\quad = \big(\phi^+_{(n,m)}\,\phi^-_{(n+1,m-3)} - \phi^-_{(n+2,m)}\,\phi^+_{(n+1,m-3)}\big) \otimes \bar\beta^+_{(n,m)} \wedge \bar\beta^-_{(n+1,m-3)}\,,
\end{aligned}
\tag{B.15c}
$$

$$
\begin{aligned}
&(\mathcal{F})_{(n-1,m+3),(n+1,m+3)} \\
&\quad = -\big(\phi^+_{(n,m)}\,(\phi^-)^\dagger_{(n,m)} - (\phi^-)^\dagger_{(n+1,m+3)}\,\phi^+_{(n-1,m+3)}\big) \otimes \bar\beta^+_{(n,m)} \wedge \beta^-_{(n,m)}\,,
\end{aligned}
\tag{B.15d}
$$

which are accompanied by the anti-Hermiticity conditions

$$
(\mathcal{F})_{(n',m'),(n,m)} = -\big((\mathcal{F})_{(n,m),(n',m')}\big)^\dagger\,.
\tag{B.15e}
$$

By setting $\psi_{(n,m)} = \mathbb{1}_{p_{(n,m)}}$ for all $(n,m) \in Q_0(k,l)$, these curvature matrix elements correctly reproduce those computed in [25] for equivariant dimensional reduction over $\mathbb{C}P^2$.

Appendix C. Quiver bundle examples

C.1. $C^{1,0}$-quiver

Consider the fundamental 3-dimensional representation $\underline{C}^{1,0}$ of $G = SU(3)$. Its decomposition into irreducible $SU(2)$-representations is given by

$$
\underline{C}^{1,0}\big|_{SU(2)} = \underline{(0,-2)} \oplus \underline{(1,1)}\,,
\tag{C.1}
$$

wherein $\underline{(0,-2)}$ is the $SU(2)$-singlet and $\underline{(1,1)}$ is the $SU(2)$-doublet. Using the general quiver construction of Section 3.3, the G-action dictates the existence of bundle morphisms

$$
\phi := \phi^+_{(0,-2)} \in \mathrm{Hom}\big(E_{p_{(0,-2)}}, E_{p_{(1,1)}}\big), \quad \phi^\dagger := (\phi^+)^\dagger_{(0,-2)} \in \mathrm{Hom}\big(E_{p_{(1,1)}}, E_{p_{(0,-2)}}\big),
\tag{C.2a}
$$

$$
\psi_0 := \psi_{(0,-2)} \in \mathrm{End}\big(E_{p_{(0,-2)}}\big)\,, \qquad \psi_1 := \psi_{(1,1)} \in \mathrm{End}\big(E_{p_{(1,1)}}\big)\,.
\tag{C.2b}
$$

C.2. $C^{2,0}$-quiver

The 6-dimensional representation $\underline{C}^{2,0}$ of $SU(3)$ splits under restriction to $SU(2)$ as

$$
\underline{C}^{2,0}\big|_{SU(2)} = \underline{(2,2)} \oplus \underline{(1,-1)} \oplus \underline{(0,-4)}\,.
\tag{C.3}
$$

The SU(3)-action intertwines the irreducible SU(2)-modules and the corresponding bundles. The actions of $E_{\alpha_1+\alpha_2}$ and E_{α_2} respectively yield Higgs fields

$$\phi_0 := \phi_{(0,-4)}^+ \in \mathrm{Hom}\big(E_{p_{(0,-4)}}, E_{p_{(1,-1)}}\big)\,, \quad \phi_1 := \phi_{(1,-1)}^+ \in \mathrm{Hom}\big(E_{p_{(1,-1)}}, E_{p_{(2,2)}}\big)\,. \quad \text{(C.4a)}$$

Due to the non-zero restrictions of H_{α_2} to its eigenspaces $\underline{(0,-4)}$, $\underline{(1,-1)}$ and $\underline{(2,2)}$, one further has three bundle endomorphisms

$$\begin{aligned}
&\psi_0 := \psi_{(0,-4)} \in \mathrm{End}\big(E_{p_{(0,-4)}}\big)\,, \qquad \psi_1 := \psi_{(1,-1)} \in \mathrm{End}\big(E_{p_{(1,-1)}}\big), \\
&\psi_2 := \psi_{(2,2)} \in \mathrm{End}\big(E_{p_{(2,2)}}\big)\,.
\end{aligned} \qquad \text{(C.4b)}$$

C.3. $C^{1,1}$-quiver

The 8-dimensional adjoint representation of SU(3) splits under restriction to SU(2) as

$$\underline{C}^{1,1}\big|_{\mathrm{SU}(2)} = \underline{(1,3)} \oplus \underline{(0,0)} \oplus \underline{(2,0)} \oplus \underline{(1,-3)}\,. \qquad \text{(C.5)}$$

The action of SU(3) implies the existence of the following bundle morphisms: The actions of $E_{\alpha_1+\alpha_2}$ and E_{α_2} translate into the Higgs fields

$$\phi_1^+ := \phi_{(1,-3)}^+ \in \mathrm{Hom}\big(E_{p_{(1,-3)}}, E_{p_{(2,0)}}\big), \qquad \phi_1^- := \phi_{(1,-3)}^- \in \mathrm{Hom}\big(E_{p_{(1,-3)}}, E_{p_{(0,0)}}\big), \qquad \text{(C.6a)}$$

$$\phi_0^+ := \phi_{(0,0)}^+ \in \mathrm{Hom}\big(E_{p_{(0,0)}}, E_{p_{(1,3)}}\big), \qquad \phi_0^- := \phi_{(2,0)}^- \in \mathrm{Hom}\big(E_{p_{(2,0)}}, E_{p_{(1,3)}}\big), \qquad \text{(C.6b)}$$

whereas the action of H_{α_2} generates

$$\psi^\pm := \psi_{(1,\pm 3)} \in \mathrm{End}\big(E_{p_{(1,\pm 3)}}\big)\,. \qquad \text{(C.6c)}$$

Note that H_{α_2} neither introduces endomorphisms on $\underline{(0,0)}$ and $\underline{(2,0)}$ nor does it intertwine these SU(2)-multiplets. This follows from the fact that these representations are subspaces of the kernel of H_{α_2}, and that H_{α_2} commutes with the entire Lie algebra $\mathfrak{su}(2)$.

Appendix D. Equivariant dimensional reduction details

D.1. 1-form products on $\mathbb{C}P^2$

The metric on $M^d \times \mathbb{C}P^2$ is given as

$$\mathrm{d}s^2 = \mathrm{d}s_{M^d}^2 + \mathrm{d}s_{\mathbb{C}P^2}^2\,, \qquad \text{(D.1)}$$

where

$$\mathrm{d}s_{M^d}^2 = G_{\mu'\nu'}\,\mathrm{d}x^{\mu'} \otimes \mathrm{d}x^{\nu'} \qquad \text{(D.2)}$$

with $(x^{\mu'})$ local real coordinates on the manifold M^d and $\mu', \nu', \ldots = 1, \ldots, d$. The metric on $\mathbb{C}P^2$ is written as

$$g_{\mathbb{C}P^2} := \mathrm{d}s_{\mathbb{C}P^2}^2 = R^2 \left(\beta^1 \otimes \bar\beta^1 + \bar\beta^1 \otimes \beta^1 + \beta^2 \otimes \bar\beta^2 + \bar\beta^2 \otimes \beta^2\right)\,. \qquad \text{(D.3)}$$

This metric is compatible with the Kähler form (A.2), and by defining the complex structure J via $\omega_{\mathbb{C}P^2}(\cdot, \cdot) = g_{\mathbb{C}P^2}(\cdot, J\,\cdot)$ on the cotangent bundle of $\mathbb{C}P^2$ one obtains $J\beta^i = \mathrm{i}\,\beta^i$ and $J\bar\beta^i =$

$-i\,\bar{\beta}^i$ for $i = 1, 2$. The corresponding Hodge operator is denoted $\star_{\mathbb{C}P^2}$, with the non-vanishing 1-form products

$$\star_{\mathbb{C}P^2} 1 = R^4\, \beta^1 \wedge \bar{\beta}^1 \wedge \beta^2 \wedge \bar{\beta}^2 = 2\big(\pi\,R^2\big)^2\, \beta_{\text{vol}}\,, \tag{D.4a}$$

$$\bar{\beta}^1 \wedge \star_{\mathbb{C}P^2} \beta^1 = \bar{\beta}^2 \wedge \star_{\mathbb{C}P^2} \beta^2 = \beta^1 \wedge \star_{\mathbb{C}P^2} \bar{\beta}^1 = \beta^2 \wedge \star_{\mathbb{C}P^2} \bar{\beta}^2 = 2\pi^2\,R^2\, \beta_{\text{vol}}\,, \tag{D.4b}$$

$$\star_{\mathbb{C}P^2} \bar{\beta}^1 \wedge \beta^1 = \beta^2 \wedge \bar{\beta}^2\,, \qquad \star_{\mathbb{C}P^2} \bar{\beta}^2 \wedge \beta^2 = \beta^1 \wedge \bar{\beta}^1\,, \tag{D.4c}$$

$$\star_{\mathbb{C}P^2} \bar{\beta}^1 \wedge \beta^2 = \bar{\beta}^1 \wedge \beta^2\,, \qquad \star_{\mathbb{C}P^2} \bar{\beta}^2 \wedge \beta^1 = \bar{\beta}^2 \wedge \beta^1\,. \tag{D.4d}$$

For later use we shall also need to compute various products involving matrix-valued 1-forms. Firstly, we have[15]

$$\operatorname{tr} \frac{\beta^{\pm}_{(n,m)} \wedge \star_{\mathbb{C}P^2} \bar{\beta}^{\pm}_{(n,m)}}{\Lambda^{\pm}_{k,l}(n,m)^2} = 2\pi^2\,R^2\,(n+1\pm1)\,\beta_{\text{vol}}\,, \tag{D.5a}$$

$$\operatorname{tr} \frac{\beta^{\pm}_{(n,m)} \wedge \bar{\beta}^{\pm}_{(n,m)} \wedge \star_{\mathbb{C}P^2}\big(\beta^{\pm}_{(n,m)} \wedge \bar{\beta}^{\pm}_{(n,m)}\big)^{\dagger}}{\Lambda^{\pm}_{k,l}(n,m)^4} = 2\pi^2\,(n+1\pm1)\,\beta_{\text{vol}}\,, \tag{D.5b}$$

$$\operatorname{tr} \frac{\bar{\beta}^{\pm}_{(n,m)} \wedge \beta^{\pm}_{(n,m)} \wedge \star_{\mathbb{C}P^2}\big(\bar{\beta}^{\pm}_{(n,m)} \wedge \beta^{\pm}_{(n,m)}\big)^{\dagger}}{\Lambda^{\pm}_{k,l}(n,m)^4} = 2\pi^2\,\frac{(n+1\pm1)^2}{n+1}\,\beta_{\text{vol}}\,, \tag{D.5c}$$

$$\operatorname{tr} \frac{\bar{\beta}^{+}_{(n,m)} \wedge \bar{\beta}^{-}_{(n+1,m-3)} \wedge \star_{\mathbb{C}P^2}\big(\bar{\beta}^{+}_{(n,m)} \wedge \bar{\beta}^{-}_{(n+1,m-3)}\big)^{\dagger}}{\Lambda^{+}_{k,l}(n,m)^2\,\Lambda^{-}_{k,l}(n+1,m-3)^2} = 2\pi^2\,\frac{n+1}{3}\,\beta_{\text{vol}}\,, \tag{D.5d}$$

$$\operatorname{tr} \frac{\bar{\beta}^{+}_{(n,m)} \wedge \beta^{-}_{(n,m)} \wedge \star_{\mathbb{C}P^2}\big(\bar{\beta}^{+}_{(n,m)} \wedge \beta^{-}_{(n,m)}\big)^{\dagger}}{\Lambda^{+}_{k,l}(n,m)^2\,\Lambda^{-}_{k,l}(n,m)^2} = 2\pi^2\,\frac{n\,(n+2)}{n+1}\,\beta_{\text{vol}}\,. \tag{D.5e}$$

The trace formulas (D.5) will have to be supplemented by

$$\operatorname{tr} \frac{\beta^{+}_{(n,m)} \wedge \bar{\beta}^{+}_{(n,m)} \wedge \star_{\mathbb{C}P^2}\big(\beta^{-}_{(n,m)} \wedge \bar{\beta}^{-}_{(n,m)}\big)^{\dagger}}{\Lambda^{+}_{k,l}(n,m)^2\,\Lambda^{-}_{k,l}(n,m)^2} = 2\pi^2\,\frac{2n\,(n+2)}{3(n+1)}\,\beta_{\text{vol}}\,, \tag{D.6a}$$

$$\operatorname{tr} \frac{\bar{\beta}^{+}_{(n-1,m-3)} \wedge \beta^{+}_{(n-1,m-3)} \wedge \star_{\mathbb{C}P^2}\big(\bar{\beta}^{-}_{(n+1,m-3)} \wedge \beta^{-}_{(n+1,m-3)}\big)^{\dagger}}{\Lambda^{+}_{k,l}(n-1,m-3)^2\,\Lambda^{-}_{k,l}(n+1,m-3)^2} = 2\pi^2\,\frac{2(n+1)}{3}\,\beta_{\text{vol}}\,, \tag{D.6b}$$

$$\operatorname{tr} \frac{\beta^{\pm}_{(n,m)} \wedge \bar{\beta}^{\pm}_{(n,m)} \wedge \star_{\mathbb{C}P^2}\big(\bar{\beta}^{\pm}_{(n\mp1,m-3)} \wedge \beta^{\pm}_{(n\mp1,m-3)}\big)^{\dagger}}{\Lambda^{\pm}_{k,l}(n,m)^2\,\Lambda^{\pm}_{k,l}(n\mp1,m-3)^2} = -2\pi^2\,\frac{n\,(n+2)}{n+1\mp1}\,\beta_{\text{vol}}\,, \tag{D.6c}$$

$$\operatorname{tr} \frac{\beta^{\pm}_{(n,m)} \wedge \bar{\beta}^{\pm}_{(n,m)} \wedge \star_{\mathbb{C}P^2}\big(\bar{\beta}^{\mp}_{(n\pm1,m-3)} \wedge \beta^{\mp}_{(n\pm1,m-3)}\big)^{\dagger}}{\Lambda^{\pm}_{k,l}(n,m)^2\,\Lambda^{\mp}_{k,l}(n\pm1,m-3)^2}$$
$$= 2\pi^2\,\Big(\frac{n\,(n+2)}{3(n+1\pm1)} - (n+1)\Big)\,\beta_{\text{vol}} \tag{D.6d}$$

---

[15] The expressions (D.5) correct the trace formulas from [25, eq. (B.7)].

and one additionally needs the traces

$$\mathrm{tr}\,\frac{\beta^\pm_{(n,m)} \wedge \bar\beta^\pm_{(n,m)}}{\Lambda^\pm_{k,l}(n,m)^2} = -\frac{\mathrm{i}}{2R^2}\,(n+1\pm1)\,\omega_{\mathbb{C}P^2} = \star_{\mathbb{C}P^2}\,\mathrm{tr}\,\frac{\left(\beta^\pm_{(n,m)} \wedge \bar\beta^\pm_{(n,m)}\right)^\dagger}{\Lambda^\pm_{k,l}(n,m)^2}\,,\qquad (D.7a)$$

$$\mathrm{tr}\,\frac{\bar\beta^\pm_{(n\mp1,m-3)} \wedge \beta^\pm_{(n\mp1,m-3)}}{\Lambda^\pm_{k,l}(n\mp1,m-3)^2} = \frac{\mathrm{i}}{2R^2}\,(n+1)\,\omega_{\mathbb{C}P^2} = \star_{\mathbb{C}P^2}\,\mathrm{tr}\,\frac{\left(\bar\beta^\pm_{(n\mp1,m-3)} \wedge \beta^\pm_{(n\mp1,m-3)}\right)^\dagger}{\Lambda^\pm_{k,l}(n\mp1,m-3)^2}\,.$$
$$(D.7b)$$

D.2. 1-form products on S^5

Let us write the metric (3.39) in the forms

$$\mathrm{ds}^2_{S^5} = g_{ij}\left(\beta^i_\varphi \otimes \bar\beta^j_\varphi + \bar\beta^j_\varphi \otimes \beta^i_\varphi\right) + g_{55}\,\eta \otimes \eta = 2R^2\,\delta_{ab}\,e^a \otimes e^b + r^2\,e^5 \otimes e^5\,,\qquad (D.8)$$

for $i,j = 1,2$ and $a,b = 1,2,3,4$, where r is the radius of the S^1-fibre of the Hopf bundle $S^5 \to \mathbb{C}P^2$; the corresponding Hodge operator is denoted $\star_{S^5}$. Define the normalised volume form η_{vol} on S^5 as

$$\star_{S^5}1 = -(2\pi)^3\,r\,R^4\,\eta_{\mathrm{vol}}\quad\text{with}\quad \beta_{\mathrm{vol}} \wedge \eta = -4\pi\,\eta_{\mathrm{vol}} = -\frac{2}{\pi^2}\,e^{12345}\quad\text{and}$$

$$\int_{S^5} \eta_{\mathrm{vol}} = 1\,.\qquad (D.9)$$

In the computation of the reduced action (3.42) we use the identities

$$e^\mu \wedge \star_{S^5} e^\nu = \sqrt{g}\,g^{\mu\nu}\,e^{12345} = \begin{cases} 4\pi^3\,r\,R^2\,\eta_{\mathrm{vol}}\,, & \mu=\nu=a\,,\\ \frac{(2\pi)^3\,R^4}{r}\,\eta_{\mathrm{vol}}\,, & \mu=\nu=5\,,\\ 0\,, & \mu\neq\nu\,, \end{cases}\qquad (D.10a)$$

$$e^{\mu\nu} \wedge \star_{S^5} e^{\rho\sigma} = \begin{cases} \sqrt{g}\,g^{\mu\rho}\,g^{\nu\sigma}\,e^{12345}\,, & \mu=\rho,\,\nu=\sigma\,,\\ -\sqrt{g}\,g^{\mu\sigma}\,g^{\nu\rho}\,e^{12345}\,, & \mu=\sigma,\,\nu=\rho\,,\\ 0\,, & \text{otherwise}\,, \end{cases}\qquad (D.10b)$$

$$e^{ab} \wedge \star_{S^5} e^{ab} = 2\pi^3\,r\,\eta_{\mathrm{vol}}\quad\text{and}\quad e^{a5} \wedge \star_{S^5} e^{a5} = \frac{4\pi^3\,R^2}{r}\,\eta_{\mathrm{vol}}\,.\qquad (D.10c)$$

We can reduce the action of the Hodge operator in 5 dimensions to the action of $\star_{\mathbb{C}P^2}$ from Appendix D.1 to get

$$\star_{S^5}\beta^i_\varphi = r\left(\star_{\mathbb{C}P^2}\beta^i_\varphi\right) \wedge \eta\,,\quad \star_{S^5}\bar\beta^i_\varphi = r\left(\star_{\mathbb{C}P^2}\bar\beta^i_\varphi\right) \wedge \eta\,,\qquad (D.11a)$$

$$\star_{S^5}\left(\beta^i_\varphi \wedge \bar\beta^j_\varphi\right) = r\left(\star_{\mathbb{C}P^2}\beta^i_\varphi \wedge \bar\beta^j_\varphi\right) \wedge \eta\,,\quad \star_{S^5}\left(\beta^i_\varphi \wedge \beta^j_\varphi\right) = r\left(\star_{\mathbb{C}P^2}\beta^i_\varphi \wedge \beta^j_\varphi\right) \wedge \eta\,,$$
$$(D.11b)$$

$$\star_{S^5}\left(\eta \wedge \beta^i_\varphi\right) = \frac{1}{r}\,\star_{\mathbb{C}P^2}\beta^i_\varphi\,,\quad \star_{S^5}\left(\eta \wedge \bar\beta^i_\varphi\right) = \frac{1}{r}\,\star_{\mathbb{C}P^2}\bar\beta^i_\varphi\,,\qquad (D.11c)$$

$$\star_{S^5}\eta = \frac{2(\pi\,R^2)^2}{r}\,\beta_{\mathrm{vol}}\,,\quad \eta \wedge \star_{S^5}\eta = -\frac{(2\pi)^3\,R^4}{r}\,\eta_{\mathrm{vol}}\,.\qquad (D.11d)$$

We can additionally compute

$$d\eta = -2\omega_3 = i\left(\beta_\varphi^1 \wedge \bar\beta_\varphi^1 + \beta_\varphi^2 \wedge \bar\beta_\varphi^2\right) = -\frac{1}{R^2}\,\omega_{\mathbb{C}P^2}\,, \tag{D.12a}$$

$$\star_{S^5}d\eta = -\frac{1}{R^2}\,\star_{S^5}\omega_{\mathbb{C}P^2} = -\frac{r}{R^2}\,(\star_{\mathbb{C}P^2}\omega_{\mathbb{C}P^2}) \wedge \eta = \frac{r}{R^2}\,\omega_{\mathbb{C}P^2} \wedge \eta\,, \tag{D.12b}$$

$$d\eta \wedge \star_{S^5}d\eta = -2(2\pi)^3\,r\,\eta_{\mathrm{vol}}\,, \tag{D.12c}$$

wherein we used $\star_{\mathbb{C}P^2}\omega_{\mathbb{C}P^2} = -\omega_{\mathbb{C}P^2}$ and (A.5). Note that due to the structure of the extension from $\mathbb{C}P^2$ to S^5, the matrices accompanying contributions from η or $d\eta$ are always proportional to the identity operators $\Pi_{(n,m)}$; thus their inclusion does not alter the trace formulas of Appendix D.1.

D.3. Yang–Mills action

The reduction of (3.40) proceeds by writing

$$\mathrm{tr}\,\mathcal{F} \wedge \star \mathcal{F} = -\sum_{(n,m)\in Q_0(k,l)} \mathrm{tr}\big(\mathcal{F} \wedge \star \mathcal{F}^\dagger\big)_{(n,m),(n,m)}\,. \tag{D.13}$$

We insert the explicit non-vanishing components (B.15), rescale the horizontal Higgs fields

$$\phi_{(n,m)}^\pm \longrightarrow \frac{1}{\Lambda_{k,l}^\pm(n,m)}\,\phi_{(n,m)}^\pm \tag{D.14}$$

as in [25] (but *not* the vertical Higgs fields $\psi_{(n,m)}$), and evaluate the traces over the representation spaces $\underline{(n,m)}$ for each weight $(n,m)\in Q_0(k,l)$ using the matrix products from Appendix D.1 and the relations of Appendix D.2. Finally, one then integrates over S^5 using the unit volume form η_{vol} introduced in Appendix D.2. The dimensionally reduced Yang–Mills action on M^d then reads as[16]

$$
\begin{aligned}
S = \frac{2\pi^3\,r\,R^4}{\tilde g^2} \int\limits_{M^d} d^d x\,\sqrt{G} \sum_{(n,m)\in Q_0(k,l)} \mathrm{tr}\Bigg(&(n+1)\big(F_{(n,m)}\big)^\dagger_{\mu'\nu'}\big(F_{(n,m)}\big)^{\mu'\nu'}\\
&+\frac{n+2}{R^2}\big(D_{\mu'}\phi_{(n,m)}^+\big)^\dagger D^{\mu'}\phi_{(n,m)}^+ + \frac{n+1}{R^2}D_{\mu'}\phi_{(n-1,m-3)}^+\big(D^{\mu'}\phi_{(n-1,m-3)}^+\big)^\dagger\\
&+\frac{n}{R^2}\big(D_{\mu'}\phi_{(n,m)}^-\big)^\dagger D^{\mu'}\phi_{(n,m)}^- + \frac{n+1}{R^2}D_{\mu'}\phi_{(n+1,m-3)}^-\big(D^{\mu'}\phi_{(n+1,m-3)}^-\big)^\dagger\\
&+\frac{n+2}{R^4}\big(\Lambda_{k,l}^+(n,m)^2\,\mathbb{1}_{p_{(n,m)}} - (\phi^+)^\dagger_{(n,m)}\,\phi_{(n,m)}^+\big)^2\\
&+\frac{n}{R^4}\big(\Lambda_{k,l}^-(n,m)^2\,\mathbb{1}_{p_{(n,m)}} - (\phi^-)^\dagger_{(n,m)}\,\phi_{(n,m)}^-\big)^2\\
&+\frac{n+1}{n\,R^4}\big(\Lambda_{k,l}^+(n-1,m-3)^2\,\mathbb{1}_{p_{(n,m)}} - \phi_{(n-1,m-3)}^+\,(\phi^+)^\dagger_{(n-1,m-3)}\big)^2\\
&+\frac{(n+1)^2}{(n+2)\,R^4}\big(\Lambda_{k,l}^-(n+1,m-3)^2\,\mathbb{1}_{p_{(n,m)}} - \phi_{(n+1,m-3)}^-\,(\phi^-)^\dagger_{(n+1,m-3)}\big)^2
\end{aligned}
$$

---

[16] By setting $\psi_{(n,m)} = \mathbb{1}_{p_{(n,m)}}$ for all $(n,m)\in Q_0(k,l)$ and $r = \frac{1}{4\pi}$ in (D.15) we obtain the quiver gauge theory action for equivariant dimensional reduction over the complex projective plane $\mathbb{C}P^2$; this reduction eliminates the last nine lines of (D.15) and the resulting expression corrects [25, eq. (3.5)].

$$+ \frac{2(n+3)}{3R^4} \left| \phi^+_{(n,m)} \phi^-_{(n+1,m-3)} \right.$$

$$- \frac{\Lambda^+_{k,l}(n,m)\, \Lambda^-_{k,l}(n+1,m-3)}{\Lambda^+_{k,l}(n+1,m-3)\, \Lambda^-_{k,l}(n+2,m)} \phi^-_{(n+2,m)} \phi^+_{(n+1,m-3)} \Big|^2$$

$$+ \frac{2n(n+2)}{(n+1)R^4} \left| \phi^+_{(n,m)} (\phi^-)^\dagger_{(n,m)} \right.$$

$$- \frac{\Lambda^+_{k,l}(n,m)\, \Lambda^-_{k,l}(n,m)}{\Lambda^+_{k,l}(n-1,m+3)\, \Lambda^-_{k,l}(n+1,m+3)} (\phi^-)^\dagger_{(n+1,m+3)} \phi^+_{(n-1,m+3)} \Big|^2$$

$$+ \frac{4n(n+2)}{3(n+1)R^4} \left(\left(\Lambda^+_{k,l}(n,m)^2\, \mathbb{1}_{p_{(n,m)}} - (\phi^+)^\dagger_{(n,m)} \phi^+_{(n,m)} \right) \right.$$

$$\times \left(\Lambda^-_{k,l}(n,m)^2\, \mathbb{1}_{p_{(n,m)}} - (\phi^-)^\dagger_{(n,m)} \phi^-_{(n,m)} \right) \Big)$$

$$- \frac{2(n+2)}{R^4} \left(\left(\Lambda^+_{k,l}(n,m)^2\, \mathbb{1}_{p_{(n,m)}} - (\phi^+)^\dagger_{(n,m)} \phi^+_{(n,m)} \right) \right.$$

$$\times \left(\Lambda^+_{k,l}(n-1,m-3)^2\, \mathbb{1}_{p_{(n,m)}} - \phi^+_{(n-1,m-3)} (\phi^+)^\dagger_{(n-1,m-3)} \right) \Big)$$

$$+ \frac{2}{R^4} \left(\frac{n}{3} - n - 1 \right) \left(\left(\Lambda^+_{k,l}(n,m)^2\, \mathbb{1}_{p_{(n,m)}} - (\phi^+)^\dagger_{(n,m)} \phi^+_{(n,m)} \right) \right.$$

$$\times \left(\Lambda^-_{k,l}(n+1,m-3)^2\, \mathbb{1}_{p_{(n,m)}} - \phi^-_{(n+1,m-3)} (\phi^-)^\dagger_{(n+1,m-3)} \right) \Big)$$

$$+ \frac{2}{R^4} \left(\frac{n+2}{3} - n - 1 \right) \left(\left(\Lambda^-_{k,l}(n,m)^2\, \mathbb{1}_{p_{(n,m)}} - (\phi^-)^\dagger_{(n,m)} \phi^-_{(n,m)} \right) \right.$$

$$\times \left(\Lambda^+_{k,l}(n-1,m-3)^2\, \mathbb{1}_{p_{(n,m)}} - \phi^+_{(n-1,m-3)} (\phi^+)^\dagger_{(n-1,m-3)} \right) \Big)$$

$$- \frac{2n}{R^4} \left(\left(\Lambda^-_{k,l}(n,m)^2\, \mathbb{1}_{p_{(n,m)}} - (\phi^-)^\dagger_{(n,m)} \phi^-_{(n,m)} \right) \right.$$

$$\times \left(\Lambda^-_{k,l}(n+1,m-3)^2\, \mathbb{1}_{p_{(n,m)}} - \phi^-_{(n+1,m-3)} (\phi^-)^\dagger_{(n+1,m-3)} \right) \Big)$$

$$+ \frac{4(n+1)}{3R^4} \left(\left(\Lambda^+_{k,l}(n-1,m-3)^2\, \mathbb{1}_{p_{(n,m)}} - \phi^+_{(n-1,m-3)} (\phi^+)^\dagger_{(n-1,m-3)} \right) \right.$$

$$\times \left(\Lambda^-_{k,l}(n+1,m-3)^2\, \mathbb{1}_{p_{(n,m)}} - \phi^-_{(n+1,m-3)} (\phi^-)^\dagger_{(n+1,m-3)} \right) \Big)$$

$$+ \frac{(n+1)m^2}{4r^2} D_{\mu'} \psi_{(n,m)} \left(D^{\mu'} \psi_{(n,m)} \right)^\dagger + \frac{2(n+1)m^2}{R^4} \left(\psi_{(n,m)} - \mathbb{1}_{p_{(n,m)}} \right)^2$$

$$- \frac{m(n+2)}{R^4} \left(\left(\Lambda^+_{k,l}(n,m)^2\, \mathbb{1}_{p_{(n,m)}} - (\phi^+)^\dagger_{(n,m)} \phi^+_{(n,m)} \right) \left(\psi_{(n,m)} - \mathbb{1}_{p_{(n,m)}} \right) \right)$$

$$- \frac{m\,n}{R^4} \left(\left(\Lambda^-_{k,l}(n,m)^2\, \mathbb{1}_{p_{(n,m)}} - (\phi^-)^\dagger_{(n,m)} \phi^-_{(n,m)} \right) \left(\psi_{(n,m)} - \mathbb{1}_{p_{(n,m)}} \right) \right)$$

$$+ \frac{m(n+1)}{R^4}$$

$$\times \left(\left(\Lambda^+_{k,l}(n-1,m-3)^2\, \mathbb{1}_{p_{(n,m)}} - \phi^+_{(n-1,m-3)} (\phi^+)^\dagger_{(n-1,m-3)} \right) \left(\psi_{(n,m)} - \mathbb{1}_{p_{(n,m)}} \right) \right)$$

$$+ \frac{m(n+1)}{R^4}$$

$$\times \left(\left(\Lambda^-_{k,l}(n+1,m-3)^2\, \mathbb{1}_{p_{(n,m)}} - \phi^-_{(n+1,m-3)} (\phi^-)^\dagger_{(n+1,m-3)} \right) \left(\psi_{(n,m)} - \mathbb{1}_{p_{(n,m)}} \right) \right)$$

$$+ \frac{n+1}{4R^2\,r^2}\left| m\,\psi_{(n,m)}\,\phi^+_{(n-1,m-3)} - (m-3)\,\phi^+_{(n-1,m-3)}\,\psi_{(n-1,m-3)} - 3\phi^+_{(n-1,m-3)} \right|^2$$

$$+ \frac{n+1}{4R^2\,r^2}\left| m\,\psi_{(n,m)}\,\phi^-_{(n+1,m-3)} - (m-3)\,\phi^-_{(n+1,m-3)}\,\psi_{(n+1,m-3)} - 3\phi^-_{(n+1,m-3)} \right|^2$$

$$+ \frac{n+2}{4R^2\,r^2}\left| (m+3)\,\psi_{(n+1,m+3)}\,\phi^+_{(n,m)} - m\,\phi^+_{(n,m)}\,\psi_{(n,m)} - 3\phi^+_{(n,m)} \right|^2$$

$$+ \frac{n}{4R^2\,r^2}\left| (m+3)\,\psi_{(n-1,m+3)}\,\phi^-_{(n,m)} - m\,\phi^-_{(n,m)}\,\psi_{(n,m)} - 3\phi^-_{(n,m)} \right|^2 \Bigg) . \qquad \text{(D.15)}$$

Note that while the trace in (3.42) is taken over the full fibre space $V^{k,l}$ of the equivariant vector bundle (3.22), in (D.15) the trace over the $\mathrm{SU}(2) \times \mathrm{U}(1)$-representations $\underline{(n, m)}$ has already been evaluated.

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