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FARTHEST POINTS AND MONOTONE OPERATORS

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Dedicated to H. Berens on the occasion of his sixtieth birthday

We apply the theory of monotone operators to study farthest points in closed bounded subsets of real Banach spaces. This new approach reveals the intimate connection between the farthest point mapping and the subdifferential of the farthest distance function. Moreover, we prove that a typical exception set in the Baire category sense is pathwise connected. Stronger results are obtained in Hilbert spaces.

1. INTRODUCTION

Let X be a real Banach space with norm $\|\cdot\|$ and dual space X^*, and let K be a nonempty closed bounded subset of X. The farthest distance function $r_K : X \to \mathbb{R}$ associated with K is defined by

$$r_K(x) = \sup\Big\{\|x-k\|\,;k \in K\Big\} \qquad (x \in X).$$

An element $k \in K$ is said to be a farthest point of $x \in X$, if $\|x-k\| = r_K(x)$. This gives rise to a set-valued mapping $Q_K : X \to 2^K$ defined by

$$Q_K(x) = \Big\{k \in K\,; \|x-k\| = r_K(x)\Big\} \qquad (x \in X),$$

which is called the farthest point mapping from X to K. A simple observation shows that each point in the domain of Q_K lies on a ray all points of which have a common farthest point. We call this phenomenon the "ray property" of Q_K (see Section 3).

The purpose of this paper is to study farthest points by monotonicity methods. Indeed, as the farthest distance function r_K is convex and continuous, its subdifferential ∂r_K is a maximal monotone operator from X to X^*. It extends the monotone operator $F\cdot\dfrac{I-Q_K}{r_K}$, where F means the duality mapping of X and $\dfrac{I-Q_K}{r_K}$ is the operator of "ray directions" related to the ray property of Q_K. If X is even a Hilbert space, a further monotone operator is available: The negative of the farthest point mapping, $-Q_K$, is then monotone and has a unique maximal monotone extension, which is the subdifferential of a continuous convex function. The latter has a counterpart for nearest points in Hilbert

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spaces. Originating from Asplund [**2**], several papers of Berens-Westphal [**9**], Berens [**8**], Westphal-Frerking [**25**], and Veselý [**24**] considered the aspect of monotony of the metric projection and its relations to best approximation in Hilbert spaces in some detail. These papers were stimulating for the present one. However, our results go definitely beyond mere analogues. In contrast to the situation of best approximation, it is possible in the context of farthest points to apply the theory of monotone operators also in non-Hilbert spaces owing to the convexity of r_K in an arbitrary Banach space. Thus, it is the subdifferential ∂r_K which is the main object of investigations in this paper.

Before describing the content of each section let us introduce some notation.

The open ball, closed ball, and sphere with centre $x \in X$ and radius $r > 0$ will be denoted by $B(x\,;r)$, $\overline{B}(x\,;r)$, and $S(x\,;r)$, respectively. The symbols $B^*(x^*\,;r)$, $\overline{B}^*(x^*\,;r)$, and $S^*(x^*\,;r)$ stand for the corresponding subsets of X^*. The closed convex hull of a set $M \subset X$ will be denoted by $\overline{\text{co}}M$, its cardinality by $|M|$.

The paper is organised as follows. In Section 2 we review some basic facts on monotone operators. Section 3 is concerned with the farthest point mapping, especially with its ray property. In Sections 4 and 5 we study the subdifferential of r_K. We start with a result on its range sharpening the well-known fact that all subgradients of r_K are contained in the closed unit ball of X^*. Then the relationship of the subdifferential ∂r_K to the farthest point mapping is analysed. For this, we distinguish between those subgradients of r_K lying on the boundary and those contained in the interior of the unit ball of X^*. Among others, we show that if X is a reflexive Banach space satisfying certain geometrical properties, then

$$\partial r_K(x) \cap S^*(0\,;1) = F\left(\frac{x - Q_{\overline{\text{co}}K}(x)}{r_K(x)}\right) \qquad (x \in X),$$

which implies that the inverse image of a subgradient of norm one is actually a ray. In Section 5 we apply the resolvent theory for monotone operators to prove that in each reflexive Banach space X the set

$$\left\{x \in X; \partial r_K(x) \cap B^*(0\,;1) \neq \emptyset\right\},$$

which is a typical exception set in the Baire category sense, is pathwise connected. This generalises a result of Balaganskiî [**3**] on the set of discontinuity points of the farthest point mapping. In Section 6 we obtain stronger results in Hilbert spaces involving the monotony of $-Q_K$.

2. Monotone operators

In this section we recall some basic results on monotone operators; for details we refer to [**6, 11, 14, 26**]. Some of the statements below, such as Proposition 2.1 and formula (2.1), are modelled for our purposes.

Let X be a real Banach space. The symbol $\langle \cdot, \cdot \rangle$ will denote the canonical bilinear form on $X \times X^*$. If X is a Hilbert space, it will be identified with its dual, and then $\langle \cdot, \cdot \rangle$ stands for the inner product of X.

Let us first recall the duality mapping of X which is actually an example of a maximal monotone operator. It is the set-valued mapping $F : X \to 2^{X^*}$ defined by

$$F(x) = \left\{ x^* \in X^* ; \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2 \right\} \qquad (x \in X).$$

F is surjective if and only if X is reflexive; in this case we identify the duality mapping of X^* with the inverse F^{-1} of F. The duality mapping reflects the metric geometry of the underlying space; see for example, the book of Cioranescu [**12**]. Indeed, X is smooth [strictly convex] if and only if F is single-valued [injective]. Moreover, the norm of X is Fréchet differentiable on $X \backslash \{0\}$ if and only if F is single-valued and norm to norm continuous. Note that the dual norm is Fréchet differentiable at each nonzero point of X^* exactly when X is reflexive, strictly convex, and satisfies the Kadec property which means that for sequences in the unit sphere weak and norm convergence agree. Such a space has been called "strongly convex" in the Russian literature.

A set-valued mapping $A : X \to 2^{X^*}$ with domain $D(A)$ and range $R(A)$ is usually identified with its graph in $X \times X^*$. It is said to be monotone, if

$$\langle x - y, x^* - y^* \rangle \geqslant 0 \qquad \forall \, (x, x^*) \, , (y, y^*) \in A.$$

A monotone mapping is called maximal monotone, if it has no proper monotone extension in $X \times X^*$.

Let A be maximal monotone. Then it is demiclosed, that is, if $(x_n, x_n^*) \in A$ such that $x_n \to x$ and $x_n^* \rightharpoonup x^*$ (in the weak* topology of X^*), then $(x, x^*) \in A$. Moreover, for each $x \in D(A)$, Ax is a convex and weak* closed subset of X^*. Finally, A is norm to weak* upper semicontinuous on the interior of its domain.

Now, let X be a reflexive space. Then a mapping $A : X \to 2^{X^*}$ is maximal monotone if and only if its inverse $A^{-1} : X^* \to 2^X$ is so. In this case, $\overline{D(A)}$ and $\overline{R(A)}$ are convex subsets of X and X^*, respectively. Moreover, every monotone operator $B : X \to 2^{X^*}$ has a maximal monotone extension A such that $\overline{D(A)} = \overline{\text{co}} D(B)$. If B is densely defined on X and the domain of A is all of X, then one even gets uniqueness of the maximal monotone extension. For later reference we state the following proposition.

Proposition 2.1. *Let X be a reflexive Banach space, and let $B : X \to 2^{X^*}$ be a monotone operator such that $\overline{D(B)} = X$. If there exists a maximal monotone extension A of B such that $D(A) = X$, then, for each $x \in X$, the set Ax can be represented by*

$$Ax = \bigcap_{\delta > 0} \overline{\text{co}} \left\{ By \, ; \|y - x\| < \delta \right\},$$

and A is the unique maximal monotone extension of B. Moreover, $\overline{R(A)} = \overline{\text{co}} R(B)$.

Now, in addition to reflexivity, suppose that X is strictly convex and smooth. If $A : X \to 2^{X^*}$ is a maximal monotone operator, then, for each positive number λ, its resolvent

$$J_\lambda := \left(I + \lambda F^{-1} \cdot A\right)^{-1}$$

as well as its Yosida approximation

$$A_\lambda := F \cdot \frac{I - J_\lambda}{\lambda}$$

are single-valued operators and defined on all of X. The range of J_λ is $D(A)$, and

$$(J_\lambda x, A_\lambda x) \in A \qquad \forall x \in X.$$

Moreover, for each $x \in \overline{D(A)}$,

$$\lim_{(y,\lambda)\to(x,0+)} J_\lambda y = x. \tag{2.1}$$

Concerning continuity, the resolvent $J_\lambda : X \to D(A)$ is norm to weak continuous, and if, in addition, the norm of X^* is Fréchet differentiable, then J_λ is even norm to norm continuous. The Yosida approximation $A_\lambda : X \to X^*$ is a maximal monotone mapping and hence norm to weak* continuous; if, in addition, the norm of X is Fréchet differentiable, then A_λ is continuous with respect to the norm topology of both X and X^*.

If X is a Hilbert space, maximal monotone operators are characterised by Minty's theorem which says that an operator $A : X \to 2^X$ is maximal monotone if and only if, for each $\lambda > 0$, the resolvent $J_\lambda = (I + \lambda A)^{-1}$ is a contraction defined on all of X.

An important class of maximal monotone operators is given by the subdifferentials of proper, lower semicontinuous, convex functions. If $\varphi : X \to (-\infty, \infty]$ is such a function, then its subdifferential at $x \in X$ is defined by

$$\partial\varphi(x) := \left\{x^* \in X^* ; \varphi(x) + \langle y - x, x^* \rangle \leqslant \varphi(y) \quad \forall y \in X\right\},$$

the elements $x^* \in \partial\varphi(x)$ being called the subgradients of φ at x. In this paper we are concerned with real-valued convex functions which are continuous on the whole space X. In this case, the domain $D(\partial\varphi) = \left\{x \in X ; \partial\varphi(x) \neq \emptyset\right\}$ is all of X.

Note that the subdifferential of the function $x \mapsto \|x\|^2/2$ is just the duality mapping of X.

3. The farthest point mapping Q_K

This section contains some basic facts on the farthest point mapping Q_K which will be used in the sequel. In particular, we discuss what we call the "ray property" of Q_K as far as it is important for investigating the subdifferential of the farthest distance function r_K.

Suppose K is a nonempty closed bounded subset of a Banach space X. Obviously, the farthest distance function of K and its closed convex hull always coincide, that is

$$r_{\overline{\mathrm{co}}K} = r_K.$$

The corresponding result for the farthest point mapping is true under assumptions which guarantee that every farthest point in $\overline{\mathrm{co}}K$ is a strongly exposed point of $\overline{\mathrm{co}}K$ (see [**17, 21**]). For instance, if the norm of X^* is Fréchet differentiable, then

$$Q_{\overline{\mathrm{co}}K} = Q_K. \tag{3.1}$$

Extending results of Edelstein [**16**] and Asplund [**1**], Lau [**21**] showed that for any weakly compact subset K of an arbitrary Banach space X the domain of Q_K contains a dense G_δ set of X. This implies that in a reflexive space $Q_{\overline{\mathrm{co}}K}$ is densely defined on X. Moreover, if the norm of X^* is Fréchet differentiable, then by (3.1) $\overline{D(Q_K)} = X$. For further results on generic existence of farthest points see [**15, 27**].

Concerning the range of Q_K, one has

$$\overline{\mathrm{co}}R(Q_K) = \overline{\mathrm{co}}K, \tag{3.2}$$

if the norms of both X and X^* are Fréchet differentiable. This result goes back to Edelstein [**16**]; see also Lau [**21**].

Continuity and single-valuedness of the farthest point mapping were studied for example, by Blatter [**10**] and Zhivkov [**27, 28**]; for a systematic discussion of this subject and its relation to differentiability of the farthest distance function see Fitzpatrick [**18**].

We now turn to the "ray property" of Q_K. For this, suppose that $|K| \geqslant 2$. If $x \in D(Q_K)$ and $k \in Q_K(x)$, set $u := \dfrac{x-k}{r_K(x)}$. Obviously, k is also a farthest point of all points on the ray originating from x and running in the direction of u, that is, $k \in Q_K(k+tu)$ for each $t \geqslant r_K(x)$. Let

$$t_0 := \inf\{t > 0\,; k \in Q_K(k+tu)\}.$$

As $|K| \geqslant 2$, t_0 is a positive number and even a minimum. Hence, given $k \in R(Q_K)$, the set ${Q_K}^{-1}(k)$ of all elements having k as a farthest point is the union of pairwise disjoint rays of the form

$$\{k + tu\,; t \geqslant t_u\}, \tag{3.3}$$

where u is a unit vector indicating the direction of the ray and t_u is a positive real number. The set of all ray directions u arising in this way from the elements of $R(Q_K)$ is the range of the set-valued mapping $\dfrac{I - Q_K}{r_K}$, which assigns to each $x \in X$ the set $\dfrac{x - Q_K(x)}{r_K(x)}$. The following proposition states that if X is strictly convex, then any two rays in $D(Q_K)$ of the form (3.3) fail to be parallel.

Proposition 3.1. *Let X be a strictly convex Banach space. If $u \in R\left(\frac{I-Q_K}{r_K}\right)$, then there exists a unique element $k \in K$ such that $k \in Q_K(k+tu)$ for all sufficiently large t. Furthermore,*

$$\left(\frac{I-Q_K}{r_K}\right)^{-1}(u) = \{k+tu\,;t \geqslant t_0\},$$

where $t_0 := \min\{t > 0\,; k \in Q_K(k+tu)\}$. If $t > t_0$, then $k = Q_K(k+tu)$.

Proof: By assumption, there exists $(x,k) \in Q_K$ such that $u = \frac{x-k}{r_K(x)}$. Set $t_0 := \min\Big\{t > 0\,; k \in Q_K(k+tu)\Big\}$ and $y_t := k+tu$ for $t \geqslant t_0$. We know that $k \in Q_K(y_t)$ for all $t \geqslant t_0$ and thus

$$\{y_t\,; t \geqslant t_0\} \subset \left(\frac{I-Q_K}{r_K}\right)^{-1}(u).$$

Now let $t > t_0$ and $k' \in Q_K(y_t)$. Then

$$\begin{aligned} r_K(y_t) &= \|y_{t_0} - k' + (t-t_0)u\| \\ &\leqslant \|y_{t_0} - k'\| + (t-t_0)\,\|u\| \\ &\leqslant \|y_{t_0} - k\| + (t-t_0) = r_K(y_t). \end{aligned} \tag{3.4}$$

Thus, equality holds throughout these estimates, implying, in particular,

$$\|y_{t_0} - k'\| = \|y_{t_0} - k\| = r_K(y_{t_0}).$$

Hence, $k' \in Q_K(y_{t_0})$. Finally, by the strict convexity of X, we obtain from (3.4) that $k' = k$. Thus, if $t > t_0$, the ray points y_t have the unique farthest point k.

To conclude the proof, it suffices to show that if $(x_1, k_1) \in Q_K$ such that $u = \frac{x_1 - k_1}{r_K(x_1)}$ then $k_1 = k$. Indeed, for some $t > t_0$ we have

$$\begin{aligned} r_K(y_t) + r_K(x_1) &= \|y_t - k + x_1 - k_1\| \\ &\leqslant \|y_t - k_1\| + \|x_1 - k\| \leqslant r_K(y_t) + r_K(x_1). \end{aligned}$$

This yields $\|y_t - k_1\| = r_K(y_t)$, that is, $k_1 \in Q_K(y_t)$. But k is the unique farthest point of y_t. Hence, $k_1 = k$. ∎

4. The subdifferential of the farthest distance function

Let K be a nonempty closed bounded subset of a Banach space X. It is well-known that the subdifferential ∂r_K of the farthest distance function r_K is a maximal monotone operator whose domain $D(\partial r_K)$ is all of X and whose range $R(\partial r_K)$ is contained in the closed unit ball of X^*; moreover, if X is reflexive, then $\overline{R(\partial r_K)}$ is a convex set. If, in addition, X is a smooth space, we can even show that $R(\partial r_K)$ is dense in $\overline{B}^*(0;1)$. For this we need the following simple lemma on the asymptotic behaviour of $\partial r_K(x)$ for $\|x\| \to \infty$.

Lemma 4.1. *Let $((x_n, x_n^*))_{n\in\mathbb{N}}$ be a sequence in ∂r_K such that $\lim_{n\to\infty} \|x_n\| = \infty$. Then, for each $y \in X$,*

$$\lim_{n\to\infty} \left\langle \frac{x_n - y}{\|x_n - y\|}, x_n^* \right\rangle = \lim_{n\to\infty} \|x_n^*\| = 1. \tag{4.1}$$

Thus, for each $\delta \in (0,1)$, the set $(\partial r_K)^{-1}(\overline{B}^*(0;1-\delta))$ is bounded.

Proof: If $y \in X$, then we have, for each $n \in \mathbb{N}$,

$$\|x_n - y\| \leqslant r_K(x_n) + r_K(y)$$

and

$$r_K(x_n) + \langle y - x_n, x_n^* \rangle \leqslant r_K(y).$$

Combining these two inequalities and dividing them by $\|y - x_n\|$ yields

$$1 - \frac{2r_K(y)}{\|x_n - y\|} \leqslant \frac{r_K(x_n) - r_K(y)}{\|x_n - y\|} \leqslant \left\langle \frac{x_n - y}{\|x_n - y\|}, x_n^* \right\rangle \leqslant \|x_n^*\| \leqslant 1,$$

from which (4.1) follows as $n \to \infty$. ∎

Theorem 4.2. *If X is reflexive and smooth, then the range $R(\partial r_K)$ of ∂r_K is a convex set satisfying*

$$B^*(0;1) \subset R(\partial r_K) \text{ and } \overline{B}^*(0;1) = \overline{R(\partial r_K)}.$$

Proof: We first deduce the last assertion. By the convexity of $\overline{R(\partial r_K)}$ it suffices to show that $S^*(0\ 1) \subset \overline{R(\partial r_K)}$.

Suppose $u^* \in S^*(0;1)$ and $u \in S(0;1)$ such that $u^* \in F(u)$. For each $n \in \mathbb{N}$, choose $x_n^* \in \partial r_K(nu)$. Then, by Lemma 4.1,

$$\lim_{n\to\infty} \langle u, x_n^* \rangle = \lim_{n\to\infty} \|x_n^*\| = 1.$$

The sequence (x_n^*) contains a subsequence which is weakly convergent to a limit that belongs to the weakly closed set $\overline{R(\partial r_K)}$ as well as to $F(u)$. Since, by the smoothness of X, $F(u)$ is a singleton, this limit has to be the element u^*. Thus $u^* \in \overline{R(\partial r_K)}$.

Now, let x^* be an element of the open unit ball $B^*(0;1)$. By what we have just proven, there exists a sequence $((x_n, x_n^*))_{n\in\mathbb{N}}$ in ∂r_K such that $\lim_{n\to\infty} x_n^* = x^*$. By Lemma 4.1, the sequence (x_n) is bounded in X, and thus has a weakly convergent subsequence with limit, say x. Since $(\partial r_K)^{-1}$ is demiclosed, we have $(x, x^*) \in \partial r_K$, which proves that x^* belongs to the range of ∂r_K.

As X^* is strictly convex, the relation $B^*(0;1) \subset R(\partial r_K) \subset \overline{B}^*(0;1)$ implies that $R(\partial r_K)$ is a convex set. ∎

To get more information on the inverse image of a single subgradient of r_K we now consider the relation between the subdifferential ∂r_K and the operator $\dfrac{I-Q_K}{r_K}$ of "ray directions".

Suppose from now on that K contains at least two elements. Then the subdifferential ∂r_K extends the monotone operator $F\cdot\dfrac{I-Q_K}{r_K}$. Indeed, if $(x,k)\in Q_K$ and $x^*\in F\Bigl(\dfrac{x-k}{r_K(x)}\Bigr)$, then $\|x^*\|=1$ and $\langle x-k,x^*\rangle = r_K(x)$. Hence we have, for each $y\in X$,

$$r_K(x)+\langle y-x,x^*\rangle = \langle y-k,x^*\rangle \leqslant \|y-k\|\ \|x^*\| \leqslant r_K(y).$$

Thus $x^*\in\partial r_K(x)$.

Since the farthest distance function of K and $\overline{\mathrm{co}}K$ coincide, the above observations remain true if Q_K is replaced by $Q_{\overline{\mathrm{co}}K}$. Furthermore, as $D(Q_{\overline{\mathrm{co}}K})$ is dense in a reflexive space X, we obtain from Proposition 2.1:

Proposition 4.3. *Suppose X is reflexive. Then ∂r_K is the unique maximal monotone extension of the operator $F\cdot\dfrac{I-Q_{\overline{\mathrm{co}}K}}{r_K}$, and for each $x\in X$,*

$$\partial r_K(x) = \bigcap_{\delta>0}\overline{\mathrm{co}}\Bigl\{F\Bigl(\frac{y-Q_{\overline{\mathrm{co}}K}(y)}{r_K(y)}\Bigr);y\in B(x;\delta)\Bigr\}. \tag{4.2}$$

Furthermore, $\overline{R(\partial r_K)} = \overline{\mathrm{co}}R\Bigl(F\cdot\dfrac{I-Q_{\overline{\mathrm{co}}K}}{r_K}\Bigr)$.

If, in addition, the norm of X^* is Fréchet differentiable, then the proposition holds true with $Q_{\overline{\mathrm{co}}K}$ replaced by Q_K.

As the range of $F\cdot\dfrac{I-Q_K}{r_K}$ and $F\cdot\dfrac{I-Q_{\overline{\mathrm{co}}K}}{r_K}$, respectively, is contained in the unit sphere of X^*, it seems reasonable to distinguish between those subgradients of r_K which lie on the boundary of the unit ball of X^* and those which are contained in its interior.

As a first result on subgradients lying on the unit sphere we have the following lemma, which extends the ray property of the farthest point mapping to the subdifferential of r_K.

Lemma 4.4. *Suppose $u^*\in\partial r_K(x)\cap S^*(0;1)$ for some $x\in X$, and let $u\in S(0;1)$ be such that $u^*\in F(u)$. Then, for each positive number t,*

$$u^*\in\partial r_K(x+tu)\subset S^*(0;1).$$

Proof: Set $x_t := x+tu$ for $t>0$. Then $\langle x_t-x,u^*\rangle = \|x_t-x\| = t$. Combining this with

$$r_K(x)+\langle x_t-x,u^*\rangle \leqslant r_K(x_t)\leqslant r_K(x)+\|x_t-x\|$$

gives $r_K(x_t)=r_K(x)+t$, whence $u^*\in\partial r_K(x_t)$ immediately follows.

Now, let $x^* \in \partial r_K(x_t)$ for some $t > 0$. Then the inequality

$$0 \leqslant r_K(x) - r_K(x_t) - \langle x - x_t, x^* \rangle = -t + t\langle u, x^* \rangle \leqslant t\big(\|x^*\| - 1\big) \leqslant 0$$

implies that $\|x^*\| = 1$. ∎

Under suitable conditions on the underlying space all subgradients of r_K that lie on the unit sphere belong to the range of the mapping $F \cdot \dfrac{I - Q_K}{r_K}$. In fact, if X is reflexive and locally uniformly convex, then

$$\partial r_K(x) \cap S^*(0\,;1) = F\left(\frac{x - Q_K(x)}{r_K(x)}\right) \qquad \forall x \in X. \tag{4.3}$$

This formula can be obtained from Asplund [**1**], though it is not explicitly stated there. We shall prove a generalised version of (4.3) under different assumptions on the Banach space X.

Proposition 4.5. *If X is reflexive and has a Fréchet differentiable norm, then, for each $x \in X$,*

$$\partial r_K(x) \cap S^*(0\,;1) = F\left(\frac{x - Q_{\overline{\text{co}}K}(x)}{r_K(x)}\right). \tag{4.4}$$

If, in addition, also the norm of X^* is Fréchet differentiable, then (4.4) holds with $Q_{\overline{\text{co}}K}$ replaced by Q_K.

Proof: We give a proof that uses the representation (4.2) for ∂r_K.

If $x \in X$ and $u^* \in \partial r_K(x) \cap S^*(0\,;1)$, let $u \in S(0\,;1)$ be such that $u^* = F(u)$. For $n \in \mathbb{N}$ define

$$U_n := \left\{ F\left(\frac{y - Q_{\overline{\text{co}}K}(y)}{r_K(y)}\right); y \in B\Big(x\,;\frac{1}{n}\Big)\right\}.$$

Then U_n has nonempty intersection with the half-space

$$H_n := \left\{ x^* \in X^*\,; \langle u, x^* \rangle > 1 - \frac{1}{n}\right\}.$$

For, otherwise the complementary half-space $X^* \backslash H_n$ would contain the set U_n and consequently also its closed convex hull, implying $u^* \in X^* \backslash H_n$, which contradicts $\langle u, u^* \rangle = 1$. Hence, for each $n \in \mathbb{N}$, we can choose $u_n^* \in H_n \cap U_n$. Then $\lim\limits_{n\to\infty} \langle u, u_n^* \rangle = 1$. As the norm of X is Fréchet differentiable at u with derivative u^*, the sequence (u_n^*) strongly converges to u^*. Moreover, since $u_n^* \in U_n$, there exists $(y_n, z_n) \in Q_{\overline{\text{co}}K}$ such that $\|y_n - x\| < 1/n$ and $u_n^* \in F\left(\dfrac{y_n - z_n}{r_K(y_n)}\right)$. The sequence (y_n) converges to x, and (z_n) has a subsequence which is weakly convergent to a limit, say z, in $\overline{\text{co}}K$. Hence, $\langle x - z, u^* \rangle = r_K(x)$ and $z \in Q_{\overline{\text{co}}K}(x)$, which proves that

$$u^* \in F\left(\frac{x - Q_{\overline{\text{co}}K}(x)}{r_K(x)}\right).$$ ∎

If we pass to the inverse operators in (4.4), we obtain, under the assumptions of Proposition 4.5,

$$(\partial r_K)^{-1}(u^*) = \left(\frac{I - Q_{\overline{\text{co}}K}}{r_K}\right)^{-1} \cdot F^{-1}(u^*) \qquad \forall u^* \in S^*(0;1).$$

If, in addition, X is strictly convex and $u^* \in R(\partial r_K) \cap S^*(0;1)$, we can apply Proposition 3.1 to $F^{-1}(u^*) \in R\left(\frac{I - Q_{\overline{\text{co}}K}}{r_K}\right)$. This shows that the set of elements having u^* as a subgradient of r_K is actually a ray. This result is stated in the first part of the following theorem. For convenience, we sharpen the assumptions on the space X such that the farthest point mapping of the set K itself, instead that of its closed convex hull, is involved. The second part of the theorem concerns those subgradients of r_K which lie in the open unit ball of X^*.

Theorem 4.6.

(i) *Suppose either X is reflexive and locally uniformly convex or both X and X^* have a Fréchet differentiable norm. If $u^* \in R(\partial r_K) \cap S^*(0;1)$, then there exists a unique element $k \in K$ such that $k \in Q_K(k + tF^{-1}(u^*))$ for all sufficiently large t, and $(\partial r_K)^{-1}(u^*)$ is the ray*

$$(\partial r_K)^{-1}(u^*) = \left\{k + tF^{-1}(u^*) \, ; t \geqslant t_0\right\},$$

where $t_0 := \min\left\{t > 0 \, ; k \in Q_K(k + tF^{-1}(u^*))\right\}$.

(ii) *If X is uniformly convex and smooth, then the restriction of the mapping $(\partial r_K)^{-1}$ to the open ball $B^*(0;1)$ is single-valued and continuous with respect to the norm topologies of X and X^*.*

Proof: By the remarks preceding the theorem it is enough to prove the second part. Thus, let $x^* \in B^*(0;1)$. By Theorem 4.2, the set $(\partial r_K)^{-1}(x^*)$ is nonempty. If it were not a singleton, it would contain a line segment with endpoints, say x_0 and x_1. Set $x_\alpha := (1-\alpha)x_0 + \alpha x_1$ for $0 \leqslant \alpha \leqslant 1$. By the subgradient property, we obtain, for each $\alpha \in [0,1]$,

$$r_K(x_\alpha) - r_K(x_0) = \langle x_\alpha - x_0, x^*, \rangle \tag{4.5}$$

whence

$$r_K(x_\alpha) = (1-\alpha) r_K(x_0) + \alpha r_K(x_1)$$

follows. Now choose α such that $(1-\alpha) r_K(x_0) = \alpha r_K(x_1)$, and let (k_n) be a maximising sequence in K for the corresponding x_α, that is, $\lim_{n\to\infty} \|x_\alpha - k_n\| = r_K(x_\alpha)$. This implies

$$\lim_{n\to\infty} \left\| \frac{x_0 - k_n}{r_K(x_0)} + \frac{x_1 - k_n}{r_K(x_1)} \right\| = 2.$$

By the uniform convexity of X, it follows that

$$\lim_{n\to\infty}\left(\frac{x_0-k_n}{r_K(x_0)}-\frac{x_1-k_n}{r_K(x_1)}\right)=0. \tag{4.6}$$

This shows that $x_0 = x_1$, if $r_K(x_0) = r_K(x_1)$. Otherwise, (4.6) implies that the sequence (k_n) strongly converges to a limit $k \in K$ and that

$$\frac{x_0-k}{r_K(x_0)}=\frac{x_1-k}{r_K(x_1)}=: u,$$

where $u \in \overline{B}(0;1)$. Then

$$x_1-x_0=\big(r_K(x_1)-r_K(x_0)\big)u.$$

Combining this with (4.5) for $\alpha = 1$, we obtain $\langle u, x^*\rangle = 1$, which contradicts the fact that $\|x^*\| < 1$.

As for the continuity of $(\partial r_K)^{-1}$ on $B^*(0;1)$, let (x_n^*) be a sequence in $B^*(0;1)$ which converges to a point $x^* \in B^*(0;1)$. Set $x_n := (\partial r_K)^{-1}(x_n^*)$ and $x := (\partial r_K)^{-1}(x^*)$. As $(\partial r_K)^{-1}$ is maximal monotone, we know that the sequence (x_n) weakly converges to x. We have to show that it is even strongly convergent. For each $n \in \mathbb{N}$ and each $\alpha \in [0,1]$, we have

$$r_K(x)+\alpha\langle x_n-x,x^*\rangle \leqslant r_K\big((1-\alpha)x+\alpha x_n\big)\leqslant r_K(x)+\alpha\langle x_n-x,x_n^*\rangle. \tag{4.7}$$

For $\alpha = 1$ and $n \to \infty$ this implies that $\lim_{n\to\infty} r_K(x_n) = r_K(x)$. Now we proceed similarly as in the proof of the single-valuedness. For each $n \in \mathbb{N}$, we choose $\alpha_n \in [0,1]$ such that $(1-\alpha_n)r_K(x) = \alpha_n r_K(x_n)$. Then $\lim_{n\to\infty} \alpha_n = 1/2$, and from (4.7) with α replaced by α_n, we obtain, as $n \to \infty$,

$$\lim_{n\to\infty} r_K\big((1-\alpha_n)x+\alpha_n x_n\big)=r_K(x).$$

Hence, there exists a sequence (k_n) in K such that

$$\lim_{n\to\infty}\big\|(1-\alpha_n)x+\alpha_n x_n-k_n\big\|=r_K(x)$$

implying

$$\lim_{n\to\infty}\left\|\frac{x-k_n}{r_K(x)}+\frac{x_n-k_n}{r_K(x_n)}\right\|=2,$$

and thus $\lim_{n\to\infty} x_n = x$, by the uniform convexity of X. ∎

5. The continuity set C_K and its complement

At the end of the previous section we described the inverse image of a single subgradient of r_K depending on whether it belongs to the boundary or the interior of the unit ball of X^*. With this in mind one might consider the sets of elements of X which have either at least one or all subgradients in the interior and the boundary of the unit ball of X^*, respectively. Among the four cases that can occur we are mainly interested in the set

$$C_K := \left\{x \in X\,; \partial r_K(x) \subset S^*(0\,;1)\right\}$$

and its complement

$$(\partial r_K)^{-1}\left(B^*(0\,;1)\right) = \left\{x \in X; \partial r_K(x) \cap B^*(0\,;1) \neq \emptyset\right\}.$$

Under suitable assumptions on the underlying space, C_K is a set of continuity points, see below. In any case, C_K contains the set of points at which r_K is Fréchet differentiable (see [**18**]) and thus a dense G_δ-set of X, if X is reflexive. The set C_K itself is then a dense G_δ of X. If, in addition, the norm of X is Fréchet differentiable, then C_K is exactly the set of Fréchet differentiability points of r_K (see for example [**19, 22**]). Moreover, by Proposition 4.5, C_K is contained in the domain of $Q_{\overline{\text{co}}K}$ and

$$C_K = \Bigg\{x \in X\,;\, F\left(\frac{x - Q_{\overline{\text{co}}K}(x)}{r_K(x)}\right) \text{ is a singleton and } y \mapsto F\left(\frac{y - Q_{\overline{\text{co}}K}(y)}{r_K(y)}\right) \text{ is upper semicontinuous at } x\Bigg\}.$$

If, in addition, also the norm of X^* is Fréchet differentiable, then

$$C_K = \left\{x \in X; Q_K(x) \text{ is a singleton and } Q_K \text{ is upper semicontinuous at } x\right\}, \tag{5.1}$$

see [**18**]. Thus, in this case, C_K is the set of elements x for which the optimisation problem to maximise the distance from x within the set K is well-posed in the sense that x has a unique farthest point in K, which depends on x continuously.

The set C_K has a counterpart for nearest points, say C'_K. Structural properties of the complement $X\backslash C'_K$ have been studied by several authors under various assumptions on the space X and by different methods. See for example Bartke-Berens [**7**], Balaganskiĭ [**3, 4**], Westphal-Frerking [**25**], Veselý [**24**], Konyagin [**20**], and the survey article of Balaganskiĭ-Vlasov [**5**]. Balaganskiĭ [**3**] also deals with farthest points. He shows that $X\backslash C_K$ is pathwise connected, if the norms of X and X^* are Fréchet differentiable. In the case that X is uniformly convex and smooth, it follows from Theorem 4.6 (ii) that $X\backslash C_K$ is the continuous image of a convex set and thus pathwise connected. In the following theorem we prove that this connectedness property remains true in any reflexive Banach space.

THEOREM 5.1. *If X is reflexive, then $X\backslash C_K$ is pathwise connected.*

PROOF: For abbreviation, set $A := (\partial r_K)^{-1}$ and $G := B^*(0;1)$. Let $x, y \in X\backslash C_K = A(G)$ and $x^*, y^* \in G$ such that $(x^*, x), (y^*, y) \in A$.

Since X is reflexive, we may pass to an equivalent Fréchet differentiable norm on X which induces a Fréchet differentiable dual norm on X^* (see [**13**], p. 160). This has the advantage that the resolvent $J_\lambda : X^* \to X^*$ and the Yosida approximation $A_\lambda : X^* \to X$ of A (corresponding to the new norms) are single-valued and continuous with respect to the strong topologies of X and X^*.

For each $\lambda > 0$ define $p_\lambda : [0,1] \to X^*$ by

$$p_\lambda(t) := (1-t)\big(x^* + \lambda F(x)\big) + t\big(y^* + \lambda F(y)\big).$$

Then the mapping $A_\lambda \cdot p_\lambda$ is a path in X with initial point x and endpoint y, and we shall see that if $\lambda > 0$ is small enough, then $A_\lambda p_\lambda(t)$ belongs to the set $A(G)$ for each $t \in [0,1]$. Indeed, it is sufficient to take λ such that $J_\lambda p_\lambda(t) \in G$, since $(J_\lambda p_\lambda(t), A_\lambda p_\lambda(t)) \in A$.

To obtain a suitable choice for λ, observe that the line segment

$$P := \Big\{(1-t)x^* + ty^* ; t \in [0,1]\Big\}$$

is contained in the convex set $\overline{D(A)} \cap G$ and recall that by (2.1) for each $w^* \in \overline{D(A)}$, $J_\lambda v^*$ converges to w^* if (v^*, λ) tends to $(w^*, 0+)$. Since G is an open neighbourhood of the compact set P, there are real numbers η, $\lambda_0 > 0$ such that $J_\lambda v^* \in G$ whenever $\operatorname{dist}(v^*, P) < \eta$ and $0 < \lambda < \lambda_0$.

If we take λ such that

$$0 < \lambda < \lambda_0 \text{ and } \lambda \cdot \max\Big\{\|x\|\ , \|y\|\Big\} < \eta,$$

then we have, for each $t \in [0,1]$,

$$\Big\|p_\lambda(t) - \big((1-t)x^* + ty^*\big)\Big\| = \lambda\Big\|(1-t)F(x) + tF(y)\Big\| < \eta,$$

that is, $\operatorname{dist}(p_\lambda(t), P) < \eta$. Thus, $J_\lambda p_\lambda(t)$ is in G, as desired. ∎

Concerning the set G in the proof above, note that one only needs that G is open and convex to conclude that the image $A(G)$ under a maximal monotone operator A is pathwise connected. A slight modification of the proof yields this assertion even under the hypothesis that G is open and $G \cap \overline{D(A)}$ is connected. This extends a theorem of Veselý [**24**] for Hilbert spaces to arbitrary reflexive Banach spaces.

By Theorem 5.1 it is clear that $X\backslash C_K$ is uncountable, if it contains at least two points. It may happen, however, that $X\backslash C_K$ is a singleton. Indeed, if the closed convex hull of K is a closed ball, say, $\overline{\text{co}}K = \overline{B}(x;r)$ for some $x \in X$ and some $r \geqslant 0$, then, for each $y \in X$, $r_K(y) = r + \|y - x\|$, implying $\partial r_K(y) = F\Big(\dfrac{y-x}{\|y-x\|}\Big)$ and thus $y \in C_K$, if $y \neq x$, and $\partial r_K(x) = \overline{B}^*(0;1)$, hence $x \in X\backslash C_K$. The following proposition shows that under appropriate assumptions on X this case is the only exception.

Proposition 5.2. *If X is a reflexive, strictly convex space and its norm is Fréchet differentiable, then $X\backslash C_K$ is a singleton if and only if $\overline{\mathrm{co}}K$ is a closed ball.*

Proof: If $x \in X$ such that $X\backslash C_K = \{x\}$, then, by Theorem 4.2, $\partial r_K(x) = \overline{B}^*(0;1)$, implying $F\Big(\frac{x - Q_{\overline{\mathrm{co}}K}}{r_K(x)}\Big) = S^*(0;1)$, by Proposition 4.5. As the duality mapping F is bijective, the latter gives $Q_{\overline{\mathrm{co}}K}(x) = S(x\,;r_K(x))$, and hence $\overline{\mathrm{co}}K = \overline{B}(x\,;r_K(x))$. ∎

We conclude this section with some remarks on rays in the set C_K. If u^* is a subgradient of r_K of norm one, then by Lemma 4.4, for every $x \in (\partial r_K)^{-1}(u^*)$ and every $u \in F^{-1}(u^*)$, the ray $R := \{x + tu\,; t > 0\}$ is contained in $C_K \cap (\partial r_K)^{-1}(u^*)$. Without loss of generality we assume that $\min\big\{t \in \mathbb{R}\,; u^* \in \partial r_K(x + tu)\big\}$ is zero such that $\overline{R} = R \cup \{x\}$ is a maximal ray in $(\partial r_K)^{-1}(u^*)$. Then the question arises whether the initial point x of $\overline{R}$ also belongs to C_K or not. The following two examples show that both cases are possible. Let X be the Euclidean plane and $K = \big\{(1 - \varphi^3)(\cos\varphi, \sin\varphi)\,; 0 \leqslant \varphi \leqslant 1\big\}$. Then $u^* = u := (-1, 0)$ generates the ray $R = \big\{(-t, 0)\,; t > 0\big\}$ in C_K whose initial point $(0, 0)$ also belongs to C_K. On the other hand, if K is a closed ball with centre x in an arbitrary Banach space X and u^* is any unit vector in X^*, then every $u \in F^{-1}(u^*)$ generates the ray $R = \{x + tu\,; t > 0\}$ and $x \in X\backslash C_K$.

6. The farthest point mapping in Hilbert spaces

If X is a Hilbert space, then not only the operator $\dfrac{I - Q_K}{r_K}$ is monotone, but also $-Q_K$ itself. Indeed, if $(x_1, k_1), (x_2, k_2) \in Q_K$, then

$$2\langle x_1 - x_2, -k_1 + k_2\rangle = \|x_1 - k_1\|^2 - \|x_1 - k_2\|^2 + \|x_2 - k_2\|^2 - \|x_2 - k_1\|^2 \geqslant 0.$$

More generally, $-Q_K$ is cyclically monotone, that is, if $(x_0, k_0), (x_1, k_1), \ldots, (x_n, k_n) \in Q_K$, then

$$\sum_{j=0}^{n} \langle x_{j+1} - x_j, -k_{j+1}\rangle \geqslant 0,$$

where $(x_{n+1}, k_{n+1}) := (x_0, k_0)$. The operator $-Q_K$ has a unique maximal monotone extension, which is described in the following theorem. For this and further information see also [**23**].

Theorem 6.1. *If X is a Hilbert space, then $-Q_K$ has a unique maximal monotone extension in $X \times X$, namely the subdifferential $\partial\psi_K$ of the continuous convex function $\psi_K : X \to \mathbb{R}$, defined by*

$$\psi_K(x) := \sup\Big\{\frac{1}{2}\|k\|^2 - \langle x, k\rangle\,; k \in K\Big\} = \frac{1}{2}r_K^2(x) - \frac{1}{2}\|x\|^2.$$

The domain of $\partial\psi_K$ is the whole space X; its range satisfies $\overline{R(\partial\psi_K)} = -\overline{\mathrm{co}}K$. Moreover, for each $x \in X$,

$$x + \partial\psi_K(x) = \partial\Big(\frac{1}{2}r_K^2\Big)(x) = r_K(x)\partial r_K(x). \tag{6.1}$$

PROOF: By definition, ψ_K is the upper envelope of a family of affine functions and thus has to be convex.

If $(x,k) \in Q_K$, then $\psi_K(x) = \|k\|^2/2 - \langle x,k\rangle$, and, for each $y \in X$,

$$\psi_K(x) + \langle y-x, -k\rangle = \frac{1}{2}\|k\|^2 - \langle y,k\rangle \leqslant \psi_K(y).$$

This shows that the subdifferential $\partial\psi_K$ is actually a maximal monotone extension of $-Q_K$. Its uniqueness follows from Proposition 2.1, which also gives that $\overline{R(\partial\psi_K)} = \overline{\text{co}}R(-Q_K)$, the latter being equal to $-\overline{\text{co}}K$, by (3.2).

The relation (6.1) between $\partial\psi_K$, $\partial(r_K^2/2)$, and ∂r_K is evident from the definition of ψ_K. ∎

The function ψ_K is the counterpart of a function, say φ_K, which is related to nearest points and was introduced by Asplund [**2**] in 1969 for studying the convexity of Chebyshev sets in Hilbert spaces. Indeed, if K is a closed subset of a Hilbert space, then φ_K is defined by

$$\varphi_K(x) := \sup\Big\{\langle x,k\rangle - \frac{1}{2}\|k\|^2; k \in K\Big\} = \frac{1}{2}\|x\|^2 - \frac{1}{2}{d_K}^2(x),$$

where $d_K(x) := \inf\Big\{\|x-k\|\,; k \in K\Big\}$ is the usual distance function. The subdifferential $\partial\varphi_K$ is the unique maximal monotone extension of the nearest point mapping P_K. Moreover, as was observed by Berens-Westphal [**9**], P_K is maximal monotone if and only if K is convex.

Maximal monotony of $-Q_K$ for a bounded set K is characterised by the next proposition, which is an immediate consequence of Minty's theorem.

PROPOSITION 6.2. *Suppose X is a Hilbert space. Then $-Q_K$ is maximal monotone if and only if K is a singleton.*

The interplay between the two subdifferentials we are concerned with here allows us to sharpen and to extend some of the results that are true for non-Hilbert spaces. The following proposition is an improvement of Theorem 4.6 (ii).

PROPOSITION 6.3. *If X is a Hilbert space, then, for each $\delta \in (0,1)$, the mapping $(\partial r_K)^{-1}$ satisfies a Lipschitz condition on $\overline{B}(0;1-\delta)$. Thus any two points of $X\backslash C_K$ can be joined by a Lipschitz curve completely contained in $X\backslash C_K$.*

PROOF: For fixed $\delta \in (0,1)$, let

$$M := \sup\Big\{r_K(x)\,; x \in (\partial r_K)^{-1}(\overline{B}(0;1-\delta))\Big\}.$$

For $i = 1,2$ choose $y_i \in \overline{B}(0;1-\delta)$ and set $x_i := (\partial r_K)^{-1}(y_i)$. By (6.1), we have $r_K(x_i)y_i \in x_i + \partial\psi_K(x_i)$, which implies $J(r_K(x_i)y_i) = x_i$, where J denotes the resolvent $(I+\partial\psi_K)^{-1}$ of $\partial\psi_K$. By the contraction property of J as well as of r_K, we obtain

$$\|x_1 - x_2\| \;=\; \Big\|J(r_K(x_1)y_1) - J(r_K(x_2)y_2)\Big\|$$

$$\begin{aligned}&\leqslant \left\|r_K(x_1)y_1 - r_K(x_2)y_2\right\| \\ &\leqslant r_K(x_1)\|y_1-y_2\| + \|y_2\|\,\|x_1-x_2\| \\ &\leqslant M\|y_1-y_2\| + (1-\delta)\|x_1-x_2\|\end{aligned}$$

from which

$$\|x_1 - x_2\| \leqslant \frac{M}{\delta}\|y_1-y_2\|$$

is deduced. ∎

The next theorem describes the resolvent $(I+\partial\psi_K)^{-1}$ on rays starting at the origin.

Theorem 6.4. *Suppose X is a Hilbert space and $|K| \geqslant 2$. Let J denote the resolvent $(I+\partial\psi_K)^{-1}$ of $\partial\psi_K$. Then, for each $u \in S(0;1)$, the mapping*

$$[0,\infty) \ni t \mapsto J(tu)$$

is a Lipschitz curve which starts at the Chebyshev centre of K, that is, the unique point in X at which r_K attains its infimum, and runs to infinity.

If $u \notin R(\partial r_K)$, then the curve is completely contained in the set $X\backslash C_K$.

If $u \in R(\partial r_K)$, then there is a positive number t_0 such that

$$J(tu) \in \begin{cases} X\backslash C_K & \text{if } 0 \leqslant t < t_0, \\ C_K & \text{if } t > t_0. \end{cases}$$

Proof: For $t \geqslant 0$ set $x_t := J(tu)$. Then by (6.1),

$$tu \in r_K(x_t)\partial r_K(x_t). \tag{6.2}$$

If $t = 0$, this gives $0 \in \partial r_K(x_0)$ and hence $x_0 \in X\backslash C_K$. If $x_t \in C_K$ for some $t > 0$, then (6.2) implies

$$r_K(x_t) = t \quad\text{and}\quad u \in \partial r_K(x_t). \tag{6.3}$$

Thus $x_t \in X\backslash C_K$ for each $t \geqslant 0$, if $u \notin R(\partial r_K)$.

On the other hand, if $u \in R(\partial r_K)$, then by Theorem 4.6 (i) there is a point $k \in K$ such that $(\partial r_K)^{-1}(u) = \{k + tu\,; t \geqslant t_0\}$ where $t_0 = \min\big\{t > 0\,; k \in Q_K(k+tu)\big\}$. Then for each $t \geqslant t_0$, we have $r_K(k+tu) = t$, and by (6.1), $tu \in (I+\partial\psi_K)(k+tu)$ implying $x_t = k + tu$. By Lemma 4.4, $x_t \in C_K$ for each $t > t_0$. If x_t belonged to C_K also for some $t \in (0, t_0)$, then (6.3) would hold for this t, and by the representation of $(\partial r_K)^{-1}(u)$ as a ray, $x_t = k + t_1 u$ for some $t_1 \geqslant t_0$. As $r_K(x_t) = t$ and $r_K(k + t_1 u) = t_1$, we have $t = t_1$, which is a contradiction. Hence, $x_t \in X\backslash C_K$ for each $t \in [0, t_0)$.

∎

Note that the pathwise connectedness of the set $X\backslash C_K$ in case X is a Hilbert space can be deduced also from Theorem 6.4.

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