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ASYMPTOTIC INTEGRATION OF SECOND-ORDER NONLINEAR DIFFERENCE EQUATIONS

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Abstract. In this work we analyse a nonlinear, second-order difference equation on an unbounded interval. We present new conditions under which the problem admits a unique solution that is of a particular linear asymptotic form. The results concern the general behaviour of solutions to the initial-value problem, as well as solutions with a given asymptote. Our methods involve establishing suitable complete metric spaces and an application of Banach's fixed-point theorem. For the solutions found in our two main theorems—fixed initial data and fixed asymptote, respectively—we establish exact convergence rates to solutions of the differential equation related to our difference equation. It turns out that for the asymptotic case there is uniform convergence for both the solution and its derivative, while in the other case the convergence is somewhat weaker. Two different techniques are utilized, and for each one has to employ ad-hoc methods for the unbounded interval. Of particular importance is the exact form of the operators and metric spaces formulated in the earlier sections.

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1. Introduction. The field of difference equations acts as a mathematical framework to study discrete processes and recursion relations. Such discrete (rather than continuous) processes arise, for example, in biology, economics and sociology, where dynamical phenomena are modelled in discrete time. Furthermore, difference equations play an important role in the numerical analysis of differential equations.

In this work we will analyse the following nonlinear, second-order difference equation on an unbounded interval:

$$\nabla\Delta x(t) + F(t, x(t), \Delta x(t)) = 0, \qquad t \in I_1, \tag{1.1}$$

where $I := [t_0, \infty) \cap \mathbb{Z}$; $I_1 := [t_0 + 1, \infty) \cap \mathbb{Z}$; $0 \leq t_0 \in \mathbb{Z}$; and for all $t \in \mathbb{Z}$ we employ the notation

$$\Delta p(t) := p(t+1) - p(t), \qquad \nabla p(t) := p(t) - p(t-1).$$

Furthermore $F\colon I_1 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous in all three variables.

Recently, the investigation **[9]** presented existence results for solutions with linear asymptotic form to the nonlinear differential equation associated with (1.1). This work may in part be considered as a discrete analogue of some of the results obtained in **[9]**, which in turn is connected to several other recent investigations on asymptotic behaviour of second-order equations, among them **[12]** and **[13]**.

In particular, we give new conditions under which (1.1) admits a unique solution x on I such that the solution is of the linear asymptotic form:

$$\lim_{t\to\infty} |x(t) - ct - m| + \lim_{t\to\infty} |\Delta x(t) - c| = 0$$

for some $c, m \in \mathbb{R}$. These results concern the general behaviour of solutions to the initial value problem, as well as solutions with a given asymptote. Our methods involve establishing suitable complete metric spaces and an application of Banach's fixed-point theorem.

Of particular significance in these types of studies is the fact that when a differential equation is discretized, surprising and interesting changes can occur in the solutions. For example, properties such as existence, uniqueness, multiplicity, oscillation and stability of solutions may not be shared between the continuous differential equation and its related discrete difference equation **[3, 14]**. In the particular case of (1.1), this is seen as an extra condition in one of the proofs (although not in the resulting theorem). To illustrate, we also investigate the backward difference equation corresponding to (1.1), where that condition does not appear.

The relation between solutions of the discrete equation (1.1) and the corresponding differential equation is of importance; Is there any type of convergence as the step-size decreases? For the solutions found in our two main theorems—fixed initial data and fixed asymptote, respectively—we establish exact convergence rates to solutions of the differential equation. It turns out that for the asymptotic case there is uniform convergence for both the solution and its derivative, while in the other case the convergence is somewhat weaker. Two different techniques are utilized, and for each one has to employ ad-hoc methods for the unbounded interval. Of particular importance is the exact form of the operators and metric spaces formulated in the earlier sections.

This paper is organized as follows. In Section 2 we introduce the weighted metrics and associated metric spaces required for the main results. In Sections 3 and 4 we state and then prove our main existence results for solutions of linear asymptotic form to (1.1), whereas in Section 5 we study the backward difference equation. Section 6 is devoted to the question of convergence, and Section 7 to some examples.

For more information on the field of difference equations, including asymptotic solutions, the reader is referred to **[1, 2, 11, 12]** and the references therein.

2. Preliminaries. Consider $C(I)$, the space of continuous functions $x\colon I \to \mathbb{R}$. Let

$$\varphi\colon I \to [m, M], \qquad 0 < m < M < \infty.$$

We introduce the space

$$X := \{x \in C(I)\colon d_\varphi(x, 0) < \infty\},$$

with the distance

$$d_\varphi(x, y) := \sup_I \left|\frac{x(t) - y(t)}{(t+1)\varphi(t)}\right| + \sup_I \left|\frac{\Delta x(t) - \Delta y(t)}{\varphi(t)}\right|, \qquad x, y \in X.$$

Then, (X, d_φ) is a complete metric space. This follows from the fact that we are working on a subset of $\mathbb{Z}$, with the induced metric being the foundation for continuity. For a different use, let us also introduce the space

$$C_{c,m}(I) := \left\{x \in C(I)\colon \lim_{t\to\infty} |x(t) - ct - m| + \lim_{t\to\infty} |\Delta x(t) - c| = 0\right\},$$

consisting of the functions on I with a bounded forward difference that asymptotically approximate the affine function $ct + m$. By endowing $C_{c,m}$ with a distance,

$$\rho_\varphi(x, y) := \sup_{t\in I} \left|\frac{x(t) - y(t)}{\varphi(t)}\right| + \sup_{t\in I} \left|\frac{\Delta x(t) - \Delta y(t)}{\varphi(t)}\right|,$$

we obtain a complete metric space $(C_{c,m}, \rho_\varphi)$. Note that though $C_{c,m}$ is not a linear subspace of $C(I)$, and $\rho_\varphi(x, 0)$ does not constitute a norm on $C_{c,m}$, still $(C_{c,m}, \rho_\varphi)$ is well defined in the setting of metric spaces. We also remark that the rescaling technique using φ as a weight dates back to [**5**].

Throughout this paper we shall assume that the following Lipschitz and convergence-type criterion holds.

CONDITION 2.1. *There exists a continuous function* $k\colon I_1 \to (0, \infty)$ *with*

$$\sum_{t\in I_1} tk(t) < \infty,$$

such that for all $t \in I_1$ *and* $p, q, u, v \in \mathbb{R}$, *we have*

$$|F(t, p, u) - F(t, q, v)| \le k(t)(|p - q| + |u - v|).$$

REMARK 2.2. Condition 2.1 is natural and encompassing, but—at least in the setting of differential equations—not necessary for the existence of asymptotically linear solutions (see, e.g., [**9**, Section 5] and [**10**]). The relation between the assumptions in those cases and Condition 2.1 is, however, not an inclusion. Note, in particular, that we give conditions for all solutions to be asymptotically linear, whereas 'weaker' conditions typically imply the existence only of some solution with the desired properties.

3. Main existence results.

We now state our main existence results.

THEOREM 3.1. *Under Condition 2.1, suppose that for some* $c \in \mathbb{R}$,

$$\sum_{t\in I} |F(t, ct, c)| < \infty. \tag{3.1}$$

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Then any solution $x(t)$ *of* (1.1) *satisfies*

$$\lim_{t\to\infty}\frac{x(t)}{t}=\lim_{t\to\infty}\Delta x(t)\in\mathbb{R}.$$

Conversely, if there is such a solution, then for any $c\in\mathbb{R}$ *we have*

$$\sup_{t\in I}\left|\sum_{s=t_0}^{t}F(s,cs,c)\right|<\infty. \tag{3.2}$$

REMARK 3.2. In the case when there exists $t_1\geq t_0$, such that $F(t,ct,c)$ is of a constant sign for $t\geq t_1$, it follows from Theorem 3.1 that all solutions of (1.1) satisfy $\lim_{t\to\infty}x(t)/t=\lim_{t\to\infty}\Delta x(t)\in\mathbb{R}$ if and only if $|F(s,cs,c)|$ is summable over $s\in I$.

REMARK 3.3. The proof of Theorem 3.1 yields the existence of solutions for any initial data on a smaller interval, and then puts those solutions in a one-to-one correspondence with all solutions on I. However, without some additional assumption on $t\mapsto k(t)$ (cf. the proof of Theorem 3.1) we cannot know that the discrete initial value problem on I is solvable for any initial data $(A,B)\in\mathbb{R}^2$. Indeed there are initial data that are not extendable to the right, due to the implicit nature of (1.1).

THEOREM 3.4. *Under Condition 2.1, let* $c,m\in\mathbb{R}$ *and suppose that*

$$\sum_{s\in I}|sF(s,cs,c)|<\infty. \tag{3.3}$$

Then there exists a unique solution $x\in C_{c,m}$ *of* (1.1) *satisfying*

$$\lim_{t\to\infty}|x(t)-ct-m|+\lim_{t\to\infty}t|\Delta x(t)-c|=0.$$

Conversely, if there exists such a solution then necessarily

$$\sup_{t\in I}\left|\sum_{s=t_0}^{t}sF(s,cs,c)\right|+\sup_{t\in I}\left|\sum_{s=t_0}^{t}F(s,cs,c)\right|<\infty.$$

REMARK 3.5. In **[12]** a class of forced second-order equations are thoroughly investigated, and the discrete case handled as a special instance of Volterra–Stieltjes integro-differential equations. The comparable equation dealt with in that paper is the difference equation

$$\nabla\Delta x(t)+F(t,x(t))=0,$$

in which the nonlinearity F does not include any difference term. For the existence of a solution asymptotic to some line the authors require that $F(t,\cdot)$ is positive and non-decreasing, and that $\sum_{s\in I}F(s,cs)<\infty$ for some $c>1$ [**12**, Theorem 5.1]; to guarantee a non-negative increasing solution with $\lim_{t\to\infty}x(t)=m>0$, the nonlinearity $F=F(t,x)\geq 0$ has to fulfil Condition 2.1, as well as $\sum_{s\in I}sF(s,x(s))<m$, for all functions x with $0\leq x(t)\leq m$ [**12**, Theorem 5.2].

The first result should be compared with Theorem 3.1. The assumptions of Condition 2.1 may seem stronger than positivity and monotonicity of F but also force all solutions to be asymptotically linear. Theorem 3.4, on the other hand, is a generalization of Theorem 5.2 in [**12**]. In comparable cases Theorem 3.4 requires a bit less and provides stronger convergence, whereas the result from [**12**] yields monotonicity and non-negativity of the solution. Both results essentially rely on the same techniques and types of assumptions.

4. Main proofs.

LEMMA 4.1. *Let $g\colon I_1 \to \mathbb{R}$. For any $A, B \in \mathbb{R}$ the difference equation on I_1,*

$$\Delta\nabla x(t) = g(t), \qquad x(t_0) = A, \quad \Delta x(t_0) = B,$$

is equivalent to

$$x(t) = A + B(t - t_0) + \sum_{s=t_0+1}^{t-1} (t - s)g(s), \qquad t \in I. \tag{4.1}$$

Proof. The proof follows from direct calculation. □

4.1. Proofs for the characterization.

LEMMA 4.2. *Under Condition 2.1,*

$$\sup_{t\in I} \left| \sum_{s=t_0}^{t} F(s, x(s), \Delta x(s)) \right|$$

is finite for some $x \in X$ exactly if it is finite for all $x \in X$. The same statement holds true if we consider instead $\sum_{t\in I} |F(t, x(t), \Delta x(t))|$.

Proof. The proof can be found in [**9**, Lemma 3.5]. Since we are concerned with finiteness, we need only exchange the standard length measure ds by the point measure $d\mu := \sum_{t\in I} \delta_t$. □

LEMMA 4.3. *Under Condition 2.1, suppose* $3(t + 1)k(t) < 1$, *and that for some $c \in \mathbb{R}$, inequality* (3.1) *holds. Then, for any $A, B \in \mathbb{R}$, the map $T\colon X \to X$ defined by*

$$(Tx)(t) := A + B(t - t_0) + \sum_{s=t_0+1}^{t-1} (s - t)F(s, x(s), \Delta x(s))$$

is a contraction with respect to the metric d_φ for a suitable φ.

Proof. Define

$$\varphi(t) := \prod_{s=t_0+1}^{t} \frac{1}{1 - 3(s + 1)k(s)}, \qquad t \in I_1,$$

and let $\varphi(t_0) := 1$. Then, as can easily be verified, φ satisfies the linear discrete initial-value problem

$$\begin{aligned} \nabla\varphi(t) &= 3(t+1)k(t)\varphi(t), \qquad t \in I_1, \\ \varphi(t_0) &= 1. \end{aligned}$$

Since $3(t+1)k(t) \geq 0$ on I_1, implying $(1-3(t+1)k(t))^{-1} \geq 1$, it follows that φ is positive and non-decreasing on I. Moreover, the fact that $\sum_{t\in I_1} tk(t)$ is finite implies that φ is bounded. This can be seen in the following way:

$$\sup_{t\in I} \log(\varphi(t)) = -\sum_{t\in I_1} \log(1-3(t+1)k(t)),$$

and since every term of this series is comparable to $3(t+1)k(t)$ as $t \to \infty$, and thus also to $tk(t)$, it follows that the supremum is finite. Now, for any $x, y \in X$ and $t_1, t_2 \in I_1$ we have

$$\begin{aligned}
&\left|\frac{(Tx)(t_1)-(Ty)(t_1)}{(t_1+1)\varphi(t_1)}\right| + \left|\frac{\Delta(Tx)(t_2)-\Delta(Tv)(t_2)}{\varphi(t_2)}\right| \\
&\quad \leq \frac{1}{(t_1+1)\varphi(t_1)} \sum_{s=t_0+1}^{t_1-1} (t_1-s)\,|F(s,x(s),\Delta x(s)) - F(s,y(s),\Delta y(s))| \\
&\qquad + \frac{1}{\varphi(t_2)} \sum_{s=t_0+1}^{t_2} |F(s,x(s),\Delta x(s)) - F(s,y(s),\Delta y(s))| \\
&\quad \leq \frac{1}{(t_1+1)\varphi(t_1)} \sum_{s=t_0+1}^{t_1-1} (t_1-s)k(s)\varphi(s)\left(\frac{|x(s)-y(s)|+|\Delta x(s)-\Delta y(s)|}{\varphi(s)}\right) \\
&\qquad + \frac{1}{\varphi(t_2)} \sum_{s=t_0+1}^{t_2} k(s)\varphi(s)\left(\frac{|x(s)-y(s)|+|\Delta x(s)-\Delta y(s)|}{\varphi(s)}\right) \\
&\quad \leq \frac{1}{\varphi(t_1)} \sum_{s=t_0+1}^{t_1-1} \frac{(t_1-s)\nabla\varphi(s)}{3(t_1+1)}\left(\frac{|x(s)-y(s)|+|\Delta x(s)-\Delta y(s)|}{(s+1)\varphi(s)}\right) \\
&\qquad + \frac{1}{\varphi(t_2)} \sum_{s=t_0+1}^{t_2} \frac{\nabla\varphi(s)}{3}\left(\frac{|x(s)-y(s)|+|\Delta x(s)-\Delta y(s)|}{(s+1)\varphi(s)}\right) \\
&\quad \leq \frac{d_\varphi(x,y)}{3}\left(\frac{1}{\varphi(t_1)} \sum_{s=t_0+1}^{t_1-1} \nabla\varphi(s) + \frac{1}{\varphi(t_2)} \sum_{s=t_0+1}^{t_2} \nabla\varphi(s)\right) \\
&\quad = \frac{d_\varphi(x,y)}{3}\left(\frac{\varphi(t_1-1)-\varphi(t_0)}{\varphi(t_1)} + \frac{\varphi(t_2)-\varphi(t_0)}{\varphi(t_2)}\right) \\
&\quad \leq \frac{2}{3} d_\varphi(x,y).
\end{aligned}$$

For t_1 or t_2 equal to t_0 the same estimate trivially holds.

To prove that T maps $x \in X$ into X, first note that for all $t \in I_1$,

$$\Delta(Tx)(t) = B - \sum_{s=t_0+1}^{t} F(s, x(s), \Delta x(s)),$$

so that $\Delta(Ty)$ is bounded for $y(t) := ct \in X$, in view of (3.1). It follows that $Ty \in X$. Taking the supremum over all $t_1, t_2 \in I$ in the calculation above, we then obtain

$$d_\varphi(Tx, 0) \leq d_\varphi(Tx, Ty) + d_\varphi(Ty, 0) < d_\varphi(x, y) + d_\varphi(Ty, 0) < \infty$$

for any $x \in X$. Hence, T is a contraction on (X, d_φ). □

Proof of Theorem 3.1. In view of Condition 2.1 we see that there exists $T \geq t_0$ such that

$$3(t+1)k(t) < 1 \quad \text{for} \quad T \leq t \in I.$$

The trick now is to note that the solutions on I and on $\{t \geq T\} \cap I$, respectively, are in one-to-one correspondence with each other. Namely, if $x(t)$ solves (1.1) on I, then trivially its restriction to $\{t \geq T\} \cap I$ is a solution. Contrariwise, equation (1.1) means

$$x(t-1) = F(t, x(t), x(t+1) - x(t)) + 2x(t) - x(t+1),$$

so that, by induction, any solution on $\{t \geq T\} \cap I$ can be uniquely extended leftwards to a solution on I. Hence, there is no loss of generality in assuming $3(t+1)k(t) < 1$ on I, since we can always restrict I during the proof, and then just extend it again, without changing the solutions. (See, however, Remark 3.3 above.)

So assume that $3(t+1)k(t) < 1$. The assumptions guarantee that $y(t) := ct \in X$, so that Lemma 4.3 can be applied to yield a fixed point $x = Tx \in X$. This follows from Banach's fixed point theorem [**7**, page 10]. An easy application of Lemma 4.1 shows that x solves (1.1) for the desired initial values, and

$$\Delta x(t) = B - \sum_{s=t_0+1}^{t} F(s, x(s), \Delta x(s)).$$

In view of (3.1) and Lemma 4.2 the sum is absolutely convergent, whence $c := \lim_{t\to\infty} \Delta x(t) \in \mathbb{R}$ exists. Then,

$$\frac{x(t)}{t} - \Delta x(t) = \frac{1}{t}\left(A - Bt_0 + \sum_{s=t_0+1}^{t} sF(s, x(s), \Delta x(s))\right).$$

By considering

$$g_t(s) := \chi_{[t_0,t]} \frac{sF(s, x(s), \Delta x(s))}{t},$$

we see that for any fixed $s \in I_1$, $\lim_{t\to\infty} g_t(s) = 0$. Furthermore, $|g_t(s)| \leq |F(s, x(s), \Delta x(s))|$ which is summable over $s \in I_1$. It then follows from Lebesgue's dominated convergence theorem that

$$\lim_{t\to\infty} \frac{1}{t} \sum_{s=t_0+1}^{t} sF(s, x(s), \Delta x(s)) = 0,$$

and consequently, $x(t)/t \to c$ as $t \to \infty$.

Conversely, let x be a solution of (1.1) with a bounded forward difference Δx. It then follows that x is of the form (4.1) for some $A, B \in \mathbb{R}$. Hence, for such A, B we have that $x = Tx$, and therefore

$$\left| \sum_{s=t_0+1}^{t} F(s, x(s), \Delta x(s)) \right| = |\Delta x(t) - B| \leq \max_{t\in I_1} |\Delta x(t)| + B < \infty.$$

According to Lemma 4.2, the same inequality must hold for $y(t) := ct$, whence (3.2) holds. □

4.2. Proofs for the case of a fixed asymptote.

LEMMA 4.4. *Under Condition 2.1,*

$$\sup_{t\in I} \left| \sum_{s=t_0}^{t} s\, F(s, x(s), \Delta x(s)) \right|$$

is finite for $y(t) := ct$ *exactly if it is finite for all* $x \in \{C_{c,m}\}_{m\in\mathbb{R}}$. *The same statement holds true if we consider instead* $\sum_{t\in I} |t\, F(t, x(t), \Delta x(t))|$.

Proof. This is an alteration of Lemma 4.2, and the proof is in [**9**, Lemma 5.7]. As before, we need only substitute ds for $d\mu := \sum_{t\in I} \delta_t$. □

LEMMA 4.5. *Under Condition 2.1, let* $c, m \in \mathbb{R}$ *and suppose that* (3.3) *holds. Then the map* $S\colon C_{c,m} \to C_{c,m}$ *defined by*

$$(Sx)(t) := ct + m - \sum_{s=t+1}^{\infty} (s - t)F(s, x(s), \Delta x(s)), \qquad t \in I,$$

is a contraction with respect to ρ_φ *for a suitable* φ.

Proof. The fact that S maps $C_{c,m}$ into $C_{c,m}$ follows from (3.3) and Lemma 4.4. For example, consider the forward difference

$$\Delta(Sx)(t) = c + \sum_{s=t+1}^{\infty} F(s, x(s), \Delta x(s)).$$

Since $sF(s, cs, s)$ is absolutely summable, so is $sF(s, x(s), \Delta x(s))$ for any $x \in C_{c,m}$, and furthermore, so is $F(s, x(s), \Delta x(s))$.

Now let

$$\varphi(t) := \prod_{s=t}^{\infty} \frac{1}{1 + 3(t + 1 - t_0)k(t)}, \qquad t \in I,$$

so that φ is a positive and non-increasing function which satisfies the difference equation

$$\nabla\varphi(t) = -3(t+1-t_0)k(t)\varphi(t), \qquad t \in I_1.$$

For any $x, y \in X$ and any $t_1, t_2 \in I$ consider

$$\begin{aligned}
&\left|\frac{(Sx)(t_1)-(Sy)(t_1)}{\varphi(t_1)}\right| + \left|\frac{\Delta(Sx)(t_2)-\Delta(Sy)(t_2)}{\varphi(t_2)}\right| \\
&\quad \le \frac{1}{\varphi(t_1)}\sum_{s=t_1+1}^{\infty}(s-t_1)|F(s,x(s),\Delta x(s)) - F(s,y(s),\Delta y(s))| \\
&\qquad + \frac{1}{\varphi(t_2)}\sum_{s=t_2+1}^{\infty}|F(s,x(s),\Delta x(s)) - F(s,y(s),\Delta y(s))| \\
&\quad \le \frac{1}{\varphi(t_1)}\sum_{s=t_1+1}^{\infty}(s-t_1)k(s)\varphi(s)\left(\frac{|x(s)-y(s)|+|\Delta x(s)-\Delta y(s)|}{\varphi(s)}\right) \\
&\qquad + \frac{1}{\varphi(t_2)}\sum_{s=t_2+1}^{\infty}k(s)\varphi(s)\left(\frac{|x(s)-y(s)|+|\Delta x(s)-\Delta y(s)|}{\varphi(s)}\right) \\
&\quad \le \frac{1}{\varphi(t_1)}\sum_{s=t_1+1}^{\infty}\frac{-\nabla\varphi(s)(s-t_1)}{3(s+1-t_0)}\left(\frac{|x(s)-y(s)|+|\Delta x(s)-\Delta y(s)|}{\varphi(s)}\right) \\
&\qquad + \frac{1}{\varphi(t_2)}\sum_{s=t_2+1}^{\infty}\frac{-\nabla\varphi(s)}{3(s+1-t_0)}\left(\frac{|x(s)-y(s)|+|\Delta x(s)-\Delta y(s)|}{\varphi(s)}\right) \\
&\quad \le \frac{\rho_\varphi(x,y)}{3}\left(\frac{1}{\varphi(t_1)}\sum_{s=t_1+1}^{\infty}-\nabla\varphi(s) + \frac{1}{\varphi(t_2)}\sum_{s=t_2+1}^{\infty}-\nabla\varphi(s)\right) \\
&\quad = \frac{\rho_\varphi(x,y)}{3}\left(\frac{\varphi(t_1)-\lim_{t\to\infty}\varphi(t)}{\varphi(t_1)} + \frac{\varphi(t_2)-\lim_{t\to\infty}\varphi(t)}{\varphi(t_2)}\right) \\
&\quad \le \frac{2}{3}\rho_\varphi(x,y).
\end{aligned}$$

□

Proof of Theorem 3.4. It follows from Lemma 4.5 and Banach's fixed-point theorem [**7**, page 10] that there exists a unique $x \in C_{c,m}$ satisfying $x = Sx$. It is then easily seen that x is the unique solution of (1.1) in $C_{c,m}$. To see that even stronger convergence holds, consider

$$\begin{aligned}
t|\Delta x(t) - c| &= t\left|\sum_{s=t+1}^{\infty}F(s,x(s),\Delta x(s))\right| \\
&\le \sum_{s=t+1}^{\infty}|sF(s,x(s),\Delta x(s))| \to 0
\end{aligned}$$

as $t \to \infty$.

To prove the converse, note that according to Lemma 4.1 a solution of (1.1) always satisfies

$$\Delta x(t) = B + \sum_{s=t_0+1}^{t} F(s, x(s), \Delta x(s)).$$

Since, by assumption, the left-hand side has a limit, c, as $t \to \infty$, so does the right-hand side and, in effect,

$$\sup_{t \in I} \left| \sum_{s=t_0}^{t} F(s, x(s), \Delta x(s)) \right| < \infty.$$

Moreover, in view of $|x(t) - ct - m| + t|\Delta x(t) - c| \to 0$, it follows that the limit as $t \to \infty$ of

$$\begin{aligned}\sum_{s=t_0+1}^{t-1} & sF(s, x(s), \Delta x(s)) \\ &= x(t) - t \left(B + \sum_{s=t_0+1}^{t-1} F(s, x(s), \Delta x(s)) \right) - A + Bt_0\end{aligned}$$

is well defined. Thus, $\sup_{t \in I} \left| \sum_{s=t_0}^{t} sF(s, x(s), \Delta x(s)) \right| < \infty$. The assertion then follows from an argument similar to that of Lemma 4.4. □

5. The corresponding backward difference equation. For a comparison we shall consider here instead of (1.1) the corresponding backward difference equation

$$\Delta \nabla x(t) + F(t, x(t), \nabla x(t)) = 0, \qquad t \in I_1. \tag{5.1}$$

Our aim is to show that also for this equation, Theorem 3.1 holds, though the proof requires a somewhat different approach. Indeed, for the backward difference equation (5.1) there is no need to control the size of $3(t+1)k(t)$. We have the following result.

THEOREM 5.1. *Under Condition 2.1, suppose that for some $c \in \mathbb{R}$,*

$$\sum_{t \in I} |F(t, ct, c)| < \infty.$$

Then any solution $x(t)$ of (1.1) *satisfies*

$$\lim_{t \to \infty} \frac{x(t)}{t} = \lim_{t \to \infty} \nabla x(t) \in \mathbb{R}.$$

Conversely, if there is such a solution, then any $c \in \mathbb{R}$ satisfies

$$\sup_{t \in I} \left| \sum_{s=t_0}^{t} F(s, cs, c) \right| < \infty.$$

While the basic ingredients of the proof are similar to the case of (1.1), we need to redefine the metric space and its distance. We let

$$\tilde{X} := \{x \in C(I) \colon \tilde{d}_\varphi(x, 0) < \infty\},$$

for the distance

$$\tilde{d}_\varphi(x, y) := \sup_I \left| \frac{x(t) - y(t)}{(t+1)\varphi(t)} \right| + \sup_{I_1} \left| \frac{\nabla x(t) - \nabla y(t)}{\varphi(t)} \right|, \qquad x, y \in \tilde{X}.$$

Then, $(\tilde{X}, \tilde{d}_\varphi)$ is a complete metric space. We also have the following equivalent of Lemma 4.1.

LEMMA 5.2. *Let $g \colon I_1 \to \mathbb{R}$. For any $A, B \in \mathbb{R}$ the difference equation on I_1,*

$$\nabla\Delta x(t) = g(t), \qquad x(t_0) = A, \quad \nabla x(t_0 + 1) = B,$$

is equivalent to

$$x(t) = A + B(t - t_0) + \sum_{s=t_0+1}^{t-1} (t - s) g(s), \qquad t \in I.$$

The proof of the backward difference version of Lemma 4.2 is exactly the same, and we obtain the following result.

LEMMA 5.3. *Under Condition 2.1,*

$$\sup_{t \in I} \left| \sum_{s=t_0}^{t} F(s, x(s), \nabla x(s)) \right|$$

is finite for some $x \in \tilde{X}$ exactly if it is finite for all $x \in \tilde{X}$. The same statement holds true if we consider instead $\sum_{t \in I} |F(t, x(t), \nabla x(t))|$.

As we shall see, the main difference between the forward (1.1) and the backward (5.1) difference equation appears in the context of Lemma 4.3. In particular, the backward difference allows for an alternative choice of weight function φ. We now state and prove this cornerstone of Theorem 5.1.

LEMMA 5.4. *Under Condition 2.1, suppose that for some $c \in \mathbb{R}$, the inequality* (3.1) *holds. Then, for any $A, B \in \mathbb{R}$, the map $\tilde{T} \colon \tilde{X} \to \tilde{X}$ defined by*

$$(\tilde{T}x)(t) := A + B(t - t_0) + \sum_{s=t_0+1}^{t-1} (s - t) F(s, x(s), \nabla x(s)),$$

is a contraction with respect to the metric d_φ for a suitable φ.

Proof. Define

$$\varphi(t) := \prod_{s=t_0}^{t-1} \left(1 + 3(s+1)k(s)\right), \qquad t \in I_1,$$

and let $\varphi(t_0) := 1$. Then, as can easily be verified, φ satisfies the linear discrete initial value problem

$$\begin{aligned} \Delta\varphi(t) &= 3(t+1)k(t)\varphi(t), \qquad t \in I_1,\\ \varphi(t_0) &= 1. \end{aligned}$$

The function φ is positive, non-decreasing and bounded on I, where the last assertion follows as in the proof of Lemma 4.3 from the finiteness of $\sum_{t\in I} tk(t)$. Note also that we are now considering the backward difference

$$\nabla(\tilde{T}x)(t) = B - \sum_{s=t_0+1}^{t-1} F(s, x(s), \nabla x(s)),$$

so that the upper limit of summation has changed. To prove that $\tilde{T}$ is a contraction, pick any $x, y \in \tilde{X}$, $t_1, t_2 \in I_1$. Then,

$$\begin{aligned}
&\left|\frac{(\tilde{T}x)(t_1) - (\tilde{T}y)(t_1)}{(t_1+1)\varphi(t_1)}\right| + \left|\frac{\nabla(\tilde{T}x)(t_2) - \nabla(\tilde{T}v)(t_2)}{\varphi(t_2)}\right| \\
&\leq \frac{1}{(t_1+1)\varphi(t_1)} \sum_{s=t_0+1}^{t_1-1} (t_1 - s)\,|F(s, x(s), \nabla x(s)) - F(s, y(s), \nabla y(s))| \\
&\quad + \frac{1}{\varphi(t_2)} \sum_{s=t_0+1}^{t_2-1} |F(s, x(s), \nabla x(s)) - F(s, y(s), \nabla y(s))| \\
&\leq \frac{1}{(t_1+1)\varphi(t_1)} \sum_{s=t_0+1}^{t_1-1} (t_1 - s)k(s)\varphi(s) \left(\frac{|x(s) - y(s)| + |\nabla x(s) - \nabla y(s)|}{\varphi(s)}\right) \\
&\quad + \frac{1}{\varphi(t_2)} \sum_{s=t_0+1}^{t_2-1} k(s)\varphi(s) \left(\frac{|x(s) - y(s)| + |\nabla x(s) - \nabla y(s)|}{\varphi(s)}\right) \\
&\leq \frac{1}{\varphi(t_1)} \sum_{s=t_0+1}^{t_1-1} \frac{(t_1 - s)\Delta\varphi(s)}{3(t_1+1)} \left(\frac{|x(s) - y(s)| + |\nabla x(s) - \nabla y(s)|}{(s+1)\varphi(s)}\right) \\
&\quad + \frac{1}{\varphi(t_2)} \sum_{s=t_0+1}^{t_2-1} \frac{\Delta\varphi(s)}{3} \left(\frac{|x(s) - y(s)| + |\nabla x(s) - \nabla y(s)|}{(s+1)\varphi(s)}\right) \\
&\leq \frac{d_\varphi(x, y)}{3} \left(\frac{1}{\varphi(t_1)} \sum_{s=t_0+1}^{t_1-1} \Delta\varphi(s) + \frac{1}{\varphi(t_2)} \sum_{s=t_0+1}^{t_2-1} \Delta\varphi(s)\right) \\
&= \frac{d_\varphi(x, y)}{3} \left(\frac{\varphi(t_1) - \varphi(t_0+1)}{\varphi(t_1)} + \frac{\varphi(t_2) - \varphi(t_0+1)}{\varphi(t_2)}\right) \\
&\leq \frac{2}{3} d_\varphi(x, y).
\end{aligned}$$

For $t_1 = t_0$ the equivalent statement is trivial since $(\tilde{T}x)(t_0) = A$ for all $x \in \tilde{X}$. The rest of the proof is similar to that of Lemma 4.3. □

The remaining arguments needed to prove that Theorem 5.1 holds are exactly the same as in the proof of Theorem 3.1.

6. Convergence. An important and interesting question is to what extent the solutions of (1.1) approximate the solutions of the corresponding ordinary differential equation

$$x''(t) + F(t, x(t), x'(t)) = 0, \qquad t \in [t_0, \infty). \tag{6.1}$$

Above, F is a continuous function in all its variables with $[t_0, \infty) \times \mathbb{R} \times \mathbb{R}$ as its domain of definition. To make the problem precise, for any $h \in (0, 1)$ we let $I^h := t_0 + h\mathbb{N}$, and for $x \colon I^h \to \mathbb{R}$ we define

$$\Delta_h x(t) := \frac{x(t+h) - x(t)}{h}, \quad \nabla_h x(t) := \frac{x(t) - x(t-h)}{h}, \quad h > 0.$$

Then the question is whether, for fixed initial data or fixed asymptote, the solution $x(t; h) \colon I^h \to \mathbb{R}$ of

$$\nabla_h \Delta_h x(t) + F(t, x(t), \Delta_h x(t)) = 0, \quad t \in I^h, \tag{6.2}$$

converges to a solution $x(t) := x(t; 0)$ of (6.1) as $h \to 0$. For a second-order equation, a natural concept of convergence is based on the $C^1(I^h)$-metric

$$\rho(x, y; h) := \sup_{t \in I^h} |x(t) - y(t)| + \sup_{t \in I^h} |\Delta_h x(t) - \Delta_h y(t)| \,. \tag{6.3}$$

Thus, we say that $x(t, h)$ converges to $x(t)$ in C^1 if $\rho(x(t; h), x(t); h) \to 0$ as $h \to 0$.

In the setting of Theorem 3.1 this concept is not appropriate, however. Instead, we use a notion of convergence based on the d_φ-metric. We say that $x(t; h)$ converges to $x(t)$ in C^1_d if $d(x(t; h), x(t); h) \to 0$ as $h \to 0$, where

$$d(x, y; h) := \sup_{t \in I^h} \left| \frac{x(t) - y(t)}{t + 1} \right| + \sup_{t \in I^h} |\Delta_h x(t) - \Delta_h y(t)| \,. \tag{6.4}$$

Since we are dealing with unbounded intervals, there are some obstacles that are generally not encountered when one works on a compact set. As shall be apparent, however, there are ways of solving this problem. We shall use two different techniques, one in relation to Theorem 3.1 and one in relation to Theorem 3.4. The results are stated as Theorems 6.2 and 6.5. For that purpose we now extend Condition 2.1 to the following assumption.

CONDITION 6.1. *There exist a continuous function $k(t) \colon [t_0, \infty) \to (0, \infty)$ and a real number $\tau \geq t_0$, such that $t \mapsto tk(t)$ is non-increasing for $t \geq \tau$, with*

$$\int_{t_0}^{\infty} tk(t)\, dt < \infty,$$

and such that for all $t \geq t_0$ and $p, q, u, v \in \mathbb{R}$, we have

$$|F(t, p, u) - F(t, q, v)| \leq k(t)\,(|p - q| + |u - v|)\,.$$

6.1. Convergence of solutions as in Theorem 3.1.

THEOREM 6.2. *Let $h > 0$, assume that Condition 6.1 holds and that*

$$|F(t, ct, c)| \leq g(t),$$

where $g\colon [t_0, \infty) \to (0, \infty)$ is a continuous, non-increasing function satisfying

$$\int_{t_0}^{\infty} g(t)\, dt < \infty.$$

Then, for h small enough there exists for any initial data $(A, B) \in \mathbb{R}^2$ a solution of equation (6.2) *with $x(t_0) = A$ and $\Delta_h x(t_0) = B$. Moreover, the conclusion of Theorem 3.1 holds for this solution with I substituted for I^h. For any fixed initial data the solution of* (6.2) *converges in C_d^1 to a solution of* (6.1)*, i.e.*

$$d(x(t; h), x(t); h) \to 0, \quad \text{as} \quad h \to 0. \tag{6.5}$$

Proof. First note that the assumptions imply that Condition 2.1 and (3.1) are satisfied with I replaced by I^h. Moreover, if $3h(t+1)k(t) < 1$, then a weight function φ_h can be defined as in Lemma 4.3 on the whole of I^h. In view of Remark 3.3 we obtain the existence of a solution for any initial data.

Let us now describe the main idea of the proof. Let T_h be the map defined in Lemma 4.3 with I replaced by I^h. Recall that T_h is a contraction with contraction constant $2/3$. Similarly, we let T be the corresponding continuous version. Writing x_h for $x(t; h)$ we have

$$\begin{aligned} d_{\varphi_h}(x_h, x) &= d_{\varphi_h}(T_h x_h, Tx) \leq d_{\varphi_h}(T_h x_h, T_h x) + d_{\varphi_h}(T_h x, Tx) \\ &\leq \frac{2}{3} d_{\varphi_h}(x_h, x) + d_{\varphi_h}(T_h x, Tx). \end{aligned}$$

Hence,

$$d_{\varphi_h}(x_h, x) \leq 3 d_{\varphi_h}(T_h x, Tx).$$

The idea is to show that $d_{\varphi_h}(T_h x, Tx) \to 0$ as $h \to 0$. If this holds, we get convergence in the d_{φ_h}-sense. The convergence in C_d^1 is then a consequence of the fact that d is equivalent to d_{φ_h}. This can be seen as follows. Since φ_h is non-decreasing it is bounded from below by $\varphi_h(x_0) = 1$. On the other hand, it is bounded from above by

$$\prod_{t_0 < s \in I^h} \frac{1}{1 - 3h(s+1)k(s)}.$$

This infinite product has an upper bound which is independent of h. To see this, note that $-\log(1-y) \leq Cy$ for some $C > 0$ if $0 \leq y \leq 1/2$. Thus, if $\max_{t \geq t_0} 3h(t+1)k(t) \leq 1/2$ then we obtain that

$$\log \prod_{t_0 < s \in I^h} \frac{1}{1 - 3h(s+1)k(s)} \leq 3C \sum_{t_0 < s \in I^h} (s+1)k(s)h \leq C',$$

where C' is independent of h, due to Condition 6.1.

Let us now prove that $d_{\varphi_h}(T_h x, Tx) \to 0$ as $h \to 0$. In view of the above discussion, we can consider $d(T_h x, Tx; h)$. The distance consists of two parts. Let us look at the first part

$$\sup_{I^h} \left| \frac{(Tx)(t) - (T_h x)(t)}{t+1} \right|.$$

We have

$$\begin{aligned}
(Tx)(t) - (T_h x)(t) &= (\mathrm{I}) + (\mathrm{II}) \\
:= \int_{t_0}^{t'} (s-t) F(s, x(s), x'(s))\, ds &- \sum_{j=1}^{m} (t_j - t) F(t_j, x(t_j), \Delta_h x(t_j)) h \qquad (\mathrm{I}) \\
+ \int_{t'}^{t} (s-t) F(s, x(s), x'(s))\, ds &- \sum_{j=m+1}^{n-1} (t_j - t) F(t_j, x(t_j), \Delta_h x(t_j)) h, \qquad (\mathrm{II})
\end{aligned}$$

where $t_0 < t_1 < \cdots < t_n = t$ with $t_{j+1} - t_j = h$ for each j, and $t' = t_m$, $0 \le m \le n$, is a number depending on t. For t small we can choose $t' = t$ to make (II) vanish. For t large, we have

$$\left| \frac{1}{t+1} \int_{t'}^{t} (s-t) F(s, x(s), x'(s))\, ds \right| \le \int_{t'}^{\infty} |F(s, x(s), x'(s))|\, ds < \frac{\varepsilon}{12}$$

if t' is sufficiently large. Similarly,

$$\begin{aligned}
\left| \frac{1}{t+1} \sum_{j=m+1}^{n} (t_j - t) F(t_j, x(t_j), \Delta_h x(t_j)) h \right| &\le \sum_{j=m}^{\infty} |F(t_j, x(t_j), \Delta_h x(t_j))| h \\
&\le \sum_{j=m+1}^{\infty} g(t_j) h + \sum_{j=m}^{\infty} k(t_j)(|ct_j - x(t_j)| + |c - \Delta_h x(t_j)|) h.
\end{aligned}$$

The first sum can be made less than $\varepsilon/12$ by choosing t' sufficiently large. For the second part, we have that $|x(t)| \le c_1 t$ and $|\Delta_h x(t)| \le c_2$ for large t by the mean value theorem and by assumption, so this part can also be made less than $\varepsilon/12$ by choosing t' large according to Condition 6.1. Thus, $|(\mathrm{II})/(t+1)| < \varepsilon/4$ for all $t \in I^h$.

As for (I), in view of the mean value theorem, we have

$$\begin{aligned}
&\sum_{j=1}^{m} \left| \int_{t_{j-1}}^{t_j} \frac{(s-t)}{t+1} F(s, x(s), x'(s))\, ds - \frac{(t_j - t)}{t+1} F(t_j, x(t_j), \Delta_h x(t_j)) h \right| \\
&= h \sum_{j=1}^{m} \left| \frac{(s_j - t)}{t+1} F(s_j, x(s_j), x'(s_j)) - \frac{(t_j - t)}{t+1} F(t_j, x(t_j), \Delta_h x(t_j)) \right|,
\end{aligned}$$

where $t_{j-1} \le s_j \le t_j$. By assumption $M := \sup_{t \in I} |F(t, x(t), x'(t))|$ is finite and consequently x' is Lipschitz continuous on $[t_0, \infty)$ with the Lipschitz constant M. By the mean value theorem we thus have

$$|\Delta_h x(t_j) - x'(t_j)| \le Mh, \qquad (6.6)$$

and consequently

$$|F(t_j, x(t_j), x'(t_j)) - F(t_j, x(t_j), \Delta x(t_j))| \leq Mh \max_{t \geq t_0} k(t).$$

Since $tF(t, x(t), x'(t))$ is uniformly continuous on $[t_0, t']$ and since $mh = t' - t_0$, it follows that there exists $h(\varepsilon)$ such that $|(\mathrm{I})/(t+1)| < \varepsilon/4$ for $t \in I^h$ if $h < h(\varepsilon)$.

As for the second part of the norm, let us point out that

$$\Delta_h(Tx)(t) = B - \int_{t_0}^{t} F(s, x(s), x'(s))\, ds + \frac{1}{h}\int_t^{t+h} (s-t-h)F(s, x(s), x'(s))\, ds.$$

The last term is bounded by hM. The difference between the other two terms and $\Delta_h(T_h x)(t)$ can be treated in the same manner as above. Thus, the second part of the norm can be made less than $\varepsilon/2$ by choosing $h < h(\varepsilon)$ for $h(\varepsilon)$ sufficiently small. Altogether we now have that $d(x_h, x; h) < \varepsilon$ if $h < h(\varepsilon)$, and we are done. □

REMARK 6.3. Let us point out that due to the second term in (6.4), (6.5) implies that $\Delta_h x(t_j) \to x'(t)$ uniformly as $h \to 0$, where $t_j \to t$ as $h \to 0$. The argument is the same as in (6.6).

REMARK 6.4. The same method applies also in the setting of Theorem 3.4. In the next section we describe a different method which has the advantage that it gives more information on the asymptotic behaviour of the solutions.

6.2. Convergence of solutions as in Theorem 3.4. The solutions found in Theorem 3.4 display a certain type of convergence on the unbounded interval I. The convergence rate can, as we shall soon see, be specified in terms of the function $F(t, ct+m, c)$. This opens up for us a classical approach: first show that convergence works on any bounded interval, and then use some *a priori* estimate for the unbounded tail. For the bounded part we essentially make use of Euler's method, while for the asymptote we utilize how the space $C_{c,m}$ was chosen.

THEOREM 6.5. *Let $h > 0$, assume that Condition 6.1 holds and that*

$$t \mapsto |tF(t, ct+m, c)|$$

is a non-increasing function for $t \geq \tau$. Then, Theorem 3.4 holds with I substituted for I^h, and (1.1) *substituted for* (6.2). *For any fixed asymptote $ct+m$ the solution of* (6.2) *converges in C^1 to a solution of* (6.1), *i.e.*

$$\rho(x(t;h), x(t); h) \to 0 \quad \text{as} \quad h \to 0. \tag{6.7}$$

Proof. First, it is basic that Theorem 3.4 holds in the context of $h\mathbb{Z}$ if it holds on $\mathbb{Z}$, since there is nothing in the proof of Theorem 3.4 that is related to the distance between points in the lattice.

Convergence on a bounded interval. For notational convenience let $t_j := t_0 + hj$,

$$x_j := x(t_j; h) \quad \text{and} \quad y_j := \Delta_h x(t_j, h) = \frac{x_{j+1} - x_j}{h},$$

all for $j \in \mathbb{N}$. Similarly, we let $y(t) := x'(t)$ for the solution $x(t)$ of the exact equation (6.1). We introduce the vector-valued *error function*

$$e_j := e(t_j; h) := [x_j, y_j] - [x(t_j), y(t_j)], \qquad I^h \to \mathbb{R}^2.$$

By definition and according to (6.2), we have $x_{j+1} = x_j + hy_j$ and $y_j = y_{j-1} - hF(t_j, x_j, y_j)$. Consequently

$$e_j - e_{j-1} = h[y_{j-1}, -F(t_j, x_j, y_j)] - h[y(t_{j-1}), -F(t_j, x(t_j), y(t_j)] - h\tau_j$$

for

$$h\tau_j := [x(t_j), y(t_j)] - [x(t_{j-1}), y(t_{j-1})] - h[y(t_{j-1}), -F(t_j, x(t_j), y(t_j))].$$

If we let $K := \max_I k(t)$, and $\|[x, y]\|_1 := |x| + |y|$ denotes the standard l^1-norm, we thus have

$$|e_{j-1}|_1 \le (1 + hK)|e_j|_1 + h|e_{j-1}|_1 + h|\tau_j|_1$$

and

$$|e_{j-1}|_1 \le \frac{1 + hK}{1 - h}|e_j|_1 + \frac{h}{1 - h}|\tau_j|_1 \le \frac{1 + hK}{1 - h}|e_j|_1 + 2h|\tau_j|_1,$$

if we choose $h \le 1/2$. Let $R := \frac{1+hK}{1-h}$. Then

$$|e_j|_1 \le R^{n-j}|e_n|_1 + 2h \sum_{i=j+1}^{n} R^{n-i}|\tau_j|_1, \quad 0 \le j \le n.$$

In view of $R = 1 + h(1 + K)/(1 - h) \le 1 + 2h(1 + K) \le e^{2h(1+K)}$, we find that for any bounded time interval $[t_0, t']$,

$$R^n \le e^{2nh(1+K)} \le e^{2t'(K+1)},$$

since n here is bounded by $t_0 + nh \le t'$. It can be seen that

$$|\tau_j|_1 \le \sup_{t_{j-1} \le t \le t_j} |x'(t) - x'(t_{j-1})| + \sup_{t_{j-1} \le t \le t_j} |x''(t) - x''(t_{j-1})|.$$

Since $x \in C^2([t_0, t'])$ we have $\max_{1 \le j \le n} |\tau_j| \to 0$ as $h \to 0$. We conclude that for $0 \le j \le n$,

$$e_j \le Me_n + o(h), \quad h \to 0, \tag{6.8}$$

where $M = M(t') > 0$ depends on t', but is independent of h. Hence, if for a fixed t' we are able to choose e_n arbitrarily small, we can then choose h small enough so that (6.7) holds on $[t_0, t']$.

Convergence on some unbounded interval. We now move on to prove that given $\varepsilon > 0$, we can find an unbounded interval $[t', \infty)$ on which

$$\rho(x(t;h), ct + m; h) < \varepsilon/2 \tag{6.9}$$

for all h including $h = 0$, i.e. $x(t)$. Because if so, then the triangle inequality implies that

$$\rho(x(t;h), x(t); h) < \varepsilon \quad \text{whenever} \quad t \geq t'.$$

What we need to do is to show that the map S defined in Lemma 4.5 is well defined on a certain subset of $C_{c,m}$, and that all functions in this subset fulfil (6.9). To proceed, let $t' \geq \max\{t_0, 1, \tau\}$ be a number such that

$$\int_{t'}^{\infty} tk(t)\, dt \leq \frac{1}{6}$$

and define $r(t)\colon I^h \cap [t', \infty) \to [0, \infty)$ by

$$r(t) := 3h \sum_{s \in t + h\mathbb{Z}^+} |sF(s, cs + m, c)| \leq 3 \int_t^{\infty} |sF(s, cs + m, c)|\, ds.$$

The inequality follows from the fact that, by assumption, $|tF(t, ct + m, c)|$ is a non-increasing function for $t \geq t' \geq \tau$. Consider then

$$C^r_{c,m} := \{x \in C(I^h)\colon |x(t) - ct - m| + |\Delta_h x(t) - c| \leq r(t) \text{ for } t \geq t'\},$$

which is a closed subset of $C_{c,m}$, hence a complete metric space. The crucial fact here is that the map S defined in Lemma 4.5 preserves $C^r_{c,m}$. To show this we first observe that the sums

$$\sum_{s \in t + h\mathbb{Z}^+} (s - t)|F(s, x(s), \Delta_h x(s))| \quad \text{and} \quad \sum_{s \in t + h\mathbb{Z}^+} |F(s, x(s), \Delta_h x(s))|$$

can both be bounded from above by

$$\sum_{s \in t + h\mathbb{Z}^+} s|F(s, x(s), \Delta_h x(s))|.$$

So when $t \geq t'$,

$$\begin{aligned}
&h \sum_{s \in t + h\mathbb{Z}^+} s|F(s, x(s), \Delta_h x(s))| \\
&\quad \leq h \sum_{s \in t + h\mathbb{Z}^+} s\,(|F(s, x(s), \Delta_h x(s)) - F(s, cs + m, c)| + |F(s, cs + m, c)|) \\
&\quad \leq h \sum_{s \in t + h\mathbb{Z}^+} s(k(s)(|x(s) - cs - m| + |\Delta_h x(s) - c|) + |F(s, cs + m, c)|) \\
&\quad \leq h \sum_{s \in t + h\mathbb{Z}^+} sk(s)r(s) + \frac{r(t)}{3} \leq r(t) \left(h \sum_{s \in t + h\mathbb{Z}^+} sk(s) + \frac{1}{3} \right) \\
&\quad \leq r(t) \left(\int_t^{\infty} sk(s)\, ds + \frac{1}{3} \right) \leq \frac{r(t)}{2}
\end{aligned}$$

implies that $Sx \in C^r_{c,m}$ whenever $x \in C^r_{c,m}$.

Hence, we pick $t' \in I^h$ such that $r(t') < \varepsilon/2$, by the very construction of $C^r_{c,m}$ that guarantees the validity of (6.7) on $[t', \infty)$, independently of h. For any h we then have $e_n < \varepsilon$, if n satisfies $t_n = t_0 + nh = t'$. Thus, e_n as in (6.8) is bounded above by ε, so that for some possibly smaller ε there exists $h(\varepsilon)$ with the property that $\rho(x(t; h), x(t); h) < \varepsilon$ for all $h < h(\varepsilon)$. In conclusion, (6.9) holds on all of I. □

7. Examples.

7.1. A linear equation. As an example we consider the difference equation

$$\nabla\Delta x(t) + a(t)\Delta x(t) + b(t)x(t) = 0, \qquad t \in \mathbb{Z}^+. \tag{7.1}$$

We identify $F(t, x, \Delta x) = a(t)\Delta x + b(t)x$, and set

$$k(t) := |a(t)| + |b(t)|.$$

Then, $|F(t, p, u) - F(t, q, v)| \leq k(t)\,(|u - v| + |p - q|)$, and for Condition 2.1 to hold we require that

$$\sum_{t\in\mathbb{Z}^+} t\,(|a(t)| + |b(t)|) < \infty. \tag{7.2}$$

Solutions are asymptotically linear. The prerequisites of Theorem 3.1 are fulfilled, and for every solution of (7.1) there exists a real constant c, such that

$$\frac{x(t)}{t} \to c \quad \text{as} \quad t \to \infty.$$

Prescribed linear asymptotes. If in addition to (7.2) we have that

$$\sum_{t\in\mathbb{Z}^+} t^2\,|b(t)| < \infty,$$

then, for every given pair $(c, m) \in \mathbb{R}^2$, there is according to Theorem 3.4 a unique solution $x \in C_{c,m}$ (cf. Section 2) such that

$$|x(t) - ct - m| + t|\Delta x(t) - c| \to 0 \quad \text{as} \quad t \to \infty. \tag{7.3}$$

Asymptotic convergence. Now suppose that $a, b \in C([1, \infty), \mathbb{R})$, and consider the differential counterpart of (7.1):

$$x''(t) + a(t)x'(t) + b(t)x(t) = 0, \qquad t \geq 1. \tag{7.4}$$

Say that we could find $k \in C([1, \infty), \mathbb{R}^+)$ with $t \mapsto tk(t)$ non-increasing for large t,

$$|a(t)| + |b(t)| \leq k(t) \quad \text{and} \quad \int_1^\infty tk(t)\,dt < \infty. \tag{7.5}$$

(This is the case for example if $t \mapsto t(|a(t)| + |b(t)|)$ is non-increasing and integrable, or if $a, b \in \mathcal{O}\left(t^{-2-\varepsilon}\right)$.) Then we have convergence of solutions of the discrete difference equation (7.1) to solutions of the continuous differential equation (7.4) in the sense of

Theorem 6.2 (i.e. in the topology given by (6.5)).

Uniform convergence. If in addition to (7.5) the function

$$t \mapsto t\,|ca(t) + (m + ct)b(t)|$$

is non-increasing, then the solutions of the difference equation (7.1) converge uniformly in C^1 towards corresponding solutions of the differential equation (7.4), as described in Theorem 6.5.

7.2. A nonlinear example. Consider now the nonlinear equation

$$\nabla\Delta x(t) + \frac{k(t)\sin(\Delta x(t))}{(1 + x^2(t))^s} = 0, \qquad s > 0. \tag{7.6}$$

The functions sin and $\xi \mapsto (1 + \xi^2)^{-s}$ are bounded with bounded derivatives, whence equation (7.6) fulfils Condition 2.1 whenever $\sum_{t\in\mathbb{Z}^+} tk(t) < \infty$. We may then apply Theorems 3.1 and 3.4: all solutions are asymptotically linear, and for every given asymptote $y(t) = ct + m$, there is a solution satisfying (7.3). In the case when $t \mapsto tk(t)$ can be extended to a non-increasing continuous function all the assumptions of Theorems 6.2 and 6.5 are also satisfied, and the solutions of the difference equation (7.6) converge uniformly in C^1 to solutions of the corresponding differential equation as the mesh size vanishes.

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