Finite speed of propagation and waiting time for a thin-film Muskat problem

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Propagation at a finite speed is established for non-negative weak solutions to a thin-film approximation of the two-phase Muskat problem. The expansion rate of the support matches the scale invariance of the system. Moreover, we determine sufficient conditions on the initial data for the occurrence of waiting time phenomena.

1. Introduction and main results

The Muskat problem is a complex free-boundary model that was proposed by Muskat [13] to describe the motion of two immiscible fluids with different densities and viscosities in a porous medium with impermeable bottom (such as the intrusion of water into oil). In the limit of thin fluid layers it was shown in [7] that the Muskat problem can be approximated by a strongly coupled parabolic system of equations, which, when neglecting surface tension effects, reads as follows:

$$\left.\begin{aligned} \partial_t f &= \partial_x(f\partial_x((1+R)f + Rg)), \\ \partial_t g &= R_\mu \partial_x(g\partial_x(f+g)) \end{aligned}\right\} \tag{1.1$\,a$}$$

for $(t,x) \in (0,\infty) \times \mathbb{R}$, and is supplemented with initial conditions

$$f(0) = f_0, \quad g(0) = g_0, \quad x \in \mathbb{R}. \tag{1.1$\,b$}$$

The constants R and R_μ in (1.1$\,a$), which are assumed in this paper to be positive, are defined as

$$R := \frac{\rho_+}{\rho_- - \rho_+} \quad \text{and} \quad R_\mu := \frac{\mu_-}{\mu_+}R,$$

with ρ_- and μ_- (respectively, ρ_+ and μ_+) denoting the density and viscosity of the lower fluid (respectively, upper fluid). This reduced model retains only the functions $f = f(t,x) \geqslant 0$ and $g = g(t,x) \geqslant 0$ as unknowns, where f is the thickness of the lower fluid layer and g is the thickness of the upper fluid layer, so that $f + g$ is the total height of the fluids. When $R_\mu = R$ system $(1.1\,a)$ is also a particular case of thin-film models derived in [9] in the context of seawater intrusion.

System $(1.1\,a)$ is a degenerate parabolic system with a full diffusion matrix, and can be regarded as a two-phase generalization of the porous medium equation. Among the salient features of the latter are the finite speed of propagation and waiting time phenomena. Recall that the former means that the support of solutions remains compact if it is initially compact, while a waiting time phenomenon refers to the situation in which the solution vanishes at a point on the boundary of the support of its initial condition for some time. Since system $(1.1\,a)$ is degenerate and somewhat related to the porous medium equation, it is natural to ask whether these two features also appear in the thin-film Muskat problem. It is the purpose of this paper to provide an affirmative answer to these questions.

There is a huge literature on the finite speed of propagation for degenerate parabolic equations, and various methods have been developed to investigate this issue. In particular, for second-order parabolic equations, e.g. the porous medium equation or the p-Laplacian equation, for which the comparison principle is available, this property can be derived by comparison with suitable sub- and super-solutions (see [16] and the references therein). This approach cannot, however, be extended to higher-order equations or to systems, and energy methods have been developed instead (see [2–4, 15] and the references therein). These methods were applied, in particular, to the thin-film equation, which is a fourth-order degenerate parabolic equation, and they also work for second-order equations. A few applications to systems of equations can be found in the literature: the finite speed of propagation and the occurrence of waiting time phenomena are shown in [5] for the Poisson–Nernst–Planck system, which is of diagonal type with lower-order coupling, and in [8] for the parabolic–elliptic chemotaxis Keller–Segel system, which one can view as a non-local parabolic equation.

As we shall see below, the energy method is sufficiently flexible to be adapted to study the strongly coupled degenerate parabolic system $(1.1\,a)$. Before stating our result, let us introduce the notion of a weak solution to (1.1) to be used hereafter. Let $\mathcal{K}$ denote the positive cone of the Banach space $L_1(\mathbb{R}, (1 + x^2)\,\mathrm{d}x) \cap L_2(\mathbb{R})$ defined by

$$\mathcal{K} := \{u \in L^1(\mathbb{R}, (1 + x^2)\,\mathrm{d}x) \cap L^2(\mathbb{R}) \colon u \geqslant 0\}, \tag{1.2}$$

and set $\mathcal{K}^2 := \mathcal{K} \times \mathcal{K}$.

DEFINITION 1.1. Given $(f_0, g_0) \in \mathcal{K}^2$, a pair $(f, g) \colon [0, \infty) \to \mathcal{K}^2$ is a weak solution to (1.1) if

(i) $(f, g) \in L_\infty(0, \infty; L_2(\mathbb{R}; \mathbb{R}^2))$, $(f, g) \in L_2(0, t; H^1(\mathbb{R}; \mathbb{R}^2))$ for all $t > 0$,

(ii) $(f, g) \in C([0, \infty); H^{-1}(\mathbb{R}; \mathbb{R}^2))$ with $(f, g)(0) = (f_0, g_0)$,

and (f, g) solves (1.1 a) in the following sense:

$$\left.\begin{aligned}
\int_{\mathbb{R}} f(t)\xi \, \mathrm{d}x - \int_{\mathbb{R}} f_0\xi \, \mathrm{d}x + \int_0^t \int_{\mathbb{R}} f(\sigma)[(1+R)\partial_x f + R\partial_x g](\sigma)\partial_x \xi \, \mathrm{d}x \, \mathrm{d}\sigma = 0, \\
\int_{\mathbb{R}} g(t)\xi \, \mathrm{d}x - \int_{\mathbb{R}} g_0\xi \, \mathrm{d}x + R_\mu \int_0^t \int_{\mathbb{R}} g(\sigma)(\partial_x f + \partial_x g)(\sigma)\partial_x \xi \, \mathrm{d}x \, \mathrm{d}\sigma = 0
\end{aligned}\right\} \tag{1.3}$$

for all $\xi \in C_0^\infty(\mathbb{R})$ and $t \geqslant 0$.

The existence of weak solutions to (1.1) is shown in [10] by a variational scheme. The proof relies on the observation that system (1.1 a) is a gradient flow with respect to the 2-Wasserstein metric of the energy functional

$$\mathcal{E}(f, g) := \frac{1}{2} \int_{\mathbb{R}} [f^2 + R(f+g)^2] \, \mathrm{d}x. \tag{1.4}$$

This approach actually extends to the two-dimensional setting as well as to a related fourth-order degenerate system that is also a thin-film approximation of the Muskat problem which additionally incorporates surface tension effects [11]. Note that the uniqueness of solutions to (1.1) is an open problem.

The main results of this paper are the following.

THEOREM 1.2 (finite speed of propagation). *Let (f, g) be a weak solution of (1.1). If (f, g) satisfies the local energy estimate*

$$\int_{\mathbb{R}} [f^2(T) + R(f+g)^2(T)]\zeta^2 \, \mathrm{d}x$$

$$+ \int_0^T \int_{\mathbb{R}} (f|(1+R)\partial_x f + R\partial_x g|^2 + RR_\mu g|\partial_x f + \partial_x g|^2)\zeta^2 \, \mathrm{d}x \, \mathrm{d}t$$

$$\leqslant \int_{\mathbb{R}} [f^2(0) + R(f+g)^2(0)]\zeta^2 \, \mathrm{d}x$$

$$+ 4\int_0^T \int_{\mathbb{R}} [f((1+R)f + Rg)^2 + RR_\mu g(f+g)^2]|\partial_x \zeta|^2 \, \mathrm{d}x \, \mathrm{d}t \tag{1.5}$$

for all $\zeta \in W_4^1(\mathbb{R})$ as well as for $\zeta \equiv 1$, then (f, g) has a finite speed of propagation. More precisely, if $a \geqslant 0, r_0 > 0$, and $\mathrm{supp}(f_0 + g_0) \cap (a - r_0, a + r_0) = \emptyset$, then there exists a positive constant $C_* = C_*(R, R_\mu)$ such that

$$\mathrm{supp}(f(T) + g(T)) \cap (a - \tfrac{1}{2}r_0, a + \tfrac{1}{2}r_0) = \emptyset \quad \text{for all } T \in (0, C_* r_0^{5/2}/\mathcal{E}^{1/2}(f_0, g_0)].$$

In particular, if $\mathrm{supp}(f_0 + g_0) \subset [-b_0, b_0]$, with $b_0 > 0$, then there exists a positive constant $C^* = C^*(R_\mu, R, f_0, g_0)$ such that

$$\mathrm{supp}(f(T) + g(T)) \subset [-b_0 - C^* T^{1/3}, b_0 + C^* T^{1/3}] \quad \text{for all } T > 0.$$

We note that theorem 1.2 is only valid for weak solutions that also satisfy the local energy estimate (1.5). Unfortunately, we are as yet unable to derive it for arbitrary weak solutions, and it is unclear whether it holds in particular for the weak solutions constructed in [10]. We shall show in §3 that for each initial condition

there is at least one weak solution to (1.1) satisfying the local estimate (1.5). To this end we shall adapt an approximation scheme from [6] that allows us to obtain a weak solution as a limit of classical solutions to a regularized version of the original system.

Note also that theorem 1.2 gives no clue regarding the finite speed of propagation for each component when taken separately.

REMARK 1.3.

(a) It is shown in [12] that system (1.1 *a*) has self-similar solutions of the type

$$[(t, x) \mapsto (1+t)^{-1/3}(F, G)((1+t)^{-1/3}x)] \quad \text{for } t \geqslant 0 \text{ and } x \in \mathbb{R},$$

with compactly supported profiles $(F, G) \in H^1(\mathbb{R}, \mathbb{R}^2) \cap \mathcal{K}^2$. Hence, the estimate on the growth rate of the support obtained in theorem 1.2 matches that of the self-similar solutions and is likely to be optimal.

(b) The constant C^* in the last statement of theorem 1.2 depends only on f_0 and g_0 through the energy $\mathcal{E}(f_0, g_0)$ and the second moments of f_0 and g_0.

Due to [4], a direct consequence of the local energy estimate (1.5) is the occurrence of waiting time phenomena.

THEOREM 1.4 (waiting time phenomena). *Let* (f, g) *be a weak solution of (1.1) such that (1.5) holds for all* $\zeta \in W_4^1(\mathbb{R})$. *Let* $x_0 \in \mathbb{R} \setminus \text{supp}(f_0 + g_0)$ *be such that*

$$\limsup_{r \to 0} \frac{1}{r^5} \int_{x_0-r}^{x_0+r} [f_0^2 + R(f_0 + g_0)^2] \, \mathrm{d}x < \infty.$$

Then there exists a positive time T_* *such that* $x_0 \in \overline{\mathbb{R} \setminus \text{supp}(f(T) + g(T))}$ *for all* $T \in (0, T_*)$.

Let us now describe the content of this paper. Section 2 is devoted to the proof of the main results. While theorem 1.4 is a straightforward consequence of (1.5) and [4, theorem 1.2], the proof of theorem 1.2 requires several steps and is inspired by [3], which deals with the thin-film equation. We note that fewer estimates are available for (1.1) than for the equation studied in [3]. The last section is devoted to the existence of weak solutions to (1.1) that satisfy the local energy estimate (1.5).

2. Finite speed of propagation

Throughout this section, (f, g) is a weak solution of (1.1) that satisfies the local energy estimate (1.5) and

$$w := [f^2 + R(f + g)^2]^{3/4}. \tag{2.1}$$

The function w inherits some regularity properties of (f, g) as shown in the following result.

LEMMA 2.1. *Given non-negative functions* $u, v \in H^1(\mathbb{R})$, *let*

$$z := (u^2 + Rv^2)^{3/4}.$$

Then $z \in H^1(\mathbb{R})$ and

$$\partial_x z = \frac{3}{2} \frac{u \partial_x u + R v \partial_x v}{\mathbf{1}_{\{0\}}(z) + (u^2 + R v^2)^{1/4}},$$

where $\mathbf{1}_E$ is the characteristic function of the set E.

Proof. We choose positive functions $u_n, v_n \in C^\infty(\mathbb{R}) \cap H^1(\mathbb{R}), n \geqslant 1$, such that $u_n \to u$ and $v_n \to v$ in $H^1(\mathbb{R})$ and set

$$z_n := (u_n^2 + R v_n^2)^{3/4}.$$

Obviously, $z_n^{4/3} \to z^{4/3}$ in $L_1(\mathbb{R})$ and it follows from the Hölder continuity of the function $[x \mapsto |x|^{3/4}]$ that

$$|z_n - z|^{4/3} \leqslant |z_n^{4/3} - z^{4/3}| \quad \text{for all } n \in \mathbb{N}.$$

Hence, $z_n \to z$ in $L_{4/3}(\mathbb{R})$. We next note that the sequence $(z_n)_n$ is bounded in $H^1(\mathbb{R})$ so that it has a subsequence which converges weakly in $H^1(\mathbb{R})$ towards a limit coinciding with z almost everywhere. Consequently, z belongs to $H^1(\mathbb{R})$ and the formula for $\partial_x z$ follows by standard arguments. $\qquad\square$

We now derive from (1.5) a local energy estimate for the function w, defined in (2.1), which is at the heart of our analysis.

LEMMA 2.2. *The function w defined in (2.1) satisfies*

$$\int_{\mathbb{R}} w^{4/3}(T) \zeta^2 \, \mathrm{d}x + C_1 \int_0^T \int_{\mathbb{R}} |\partial_x w|^2 \zeta^2 \, \mathrm{d}x \, \mathrm{d}t$$

$$\leqslant \int_{\mathbb{R}} w^{4/3}(0) \zeta^2 \, \mathrm{d}x + C_2 \int_0^T \int_{\mathbb{R}} w^2 |\partial_x \zeta|^2 \, \mathrm{d}x \, \mathrm{d}t \qquad (2.2)$$

for all $T > 0$ and all $\zeta \in W_4^1(\mathbb{R})$. The constants C_1 and C_2 depend only on R and R_μ.

Proof. By lemma 2.1 the function w belongs to $H^1(\mathbb{R})$ and

$$|\partial_x w|^2 = \left| \frac{f[(1+R)\partial_x f + R \partial_x g] + Rg(\partial_x f + \partial_x g)}{\mathbf{1}_{\{0\}}(w) + (f^2 + R(f+g)^2)^{1/4}} \right|^2$$

$$\leqslant \frac{2f}{\mathbf{1}_{\{0\}}(w) + (f^2 + R(f+g)^2)^{1/2}} f |(1+R)\partial_x f + R \partial_x g|^2$$

$$+ \frac{2R^2 g}{\mathbf{1}_{\{0\}}(w) + (f^2 + R(f+g)^2)^{1/2}} g |\partial_x f + \partial_x g|^2$$

$$\leqslant 2 \max\left\{ 1, \frac{\sqrt{R}}{R_\mu} \right\} [f|(1+R)\partial_x f + R \partial_x g|^2 + R R_\mu g |\partial_x f + \partial_x g|^2]. \quad (2.3)$$

In addition, since $w^{4/3} \geqslant \max\{(1+R)f^2, Rg^2\}$, it holds that

$$f((1+R)f + Rg)^2 + R R_\mu g (f+g)^2 \leqslant 2 \max\left\{ \frac{R_\mu}{\sqrt{R}}, \sqrt{1+R} \right\} w^2.$$

Combining these two inequalities with (1.5) gives the claim. $\qquad\square$

We next recall that the Gagliardo–Nirenberg inequality [14, theorem 1] states that there exists a constant $C > 0$ such that

$$\|v\|_2 \leqslant C\|\partial_x v\|_2^{1/5}\|v\|_{4/3}^{4/5} + C\|v\|_{4/3} \quad \text{for all } v \in H^1((-1,1)).$$

Using a scaling argument, we deduce from the inequality above that, for $r > 0$,

$$\|v\|_2 \leqslant C\|\partial_x v\|_2^{1/5}\|v\|_{4/3}^{4/5} + Cr^{-1/4}\|v\|_{4/3} \quad \text{for all } v \in H^1((-r,r)). \qquad (2.4)$$

A consequence of the Gagliardo–Nirenberg inequality (2.4) is the following interpolation inequality in the spirit of [3, lemma 10.1].

LEMMA 2.3. *There exists a $C_3 > 0$ such that, given $r > 0$ and $v \in H^1((-r,r))$,*

$$\|v\|_2^2 \leqslant C_3\|\partial_x v\|_2^{14/11}I_r^{6/11} + C_3 r^{-7/2}I_r^{3/2} \qquad (2.5)$$

holds, where

$$I_r := \int_{-r}^{r}(r - |x|)_+^2 |v(x)|^{4/3}\,\mathrm{d}x.$$

Proof. We pick $\rho \in (0,r)$ arbitrarily and infer from the Hölder inequality that

$$\int_{-r}^{r}|v|^{4/3}\,\mathrm{d}x = \int_{-\rho}^{\rho}|v|^{4/3}\,\mathrm{d}x + \int_{\{\rho < |x| < r\}}|v|^{4/3}\,\mathrm{d}x$$
$$\leqslant \frac{1}{(r - \rho)^2}\int_{-\rho}^{\rho}(r - |x|)_+^2 |v|^{4/3}\,\mathrm{d}x + 2(r - \rho)^{1/3}\|v\|_2^{4/3}.$$

We now choose $\rho \in (0,r)$ such that

$$\frac{1}{(r - \rho)^2}\int_{-\rho}^{\rho}(r - |x|)_+^2 |v|^{4/3}\,\mathrm{d}x = 2(r - \rho)^{1/3}\|v\|_2^{4/3},$$

and obtain

$$\|v\|_{4/3} \leqslant 4\|v\|_2^{6/7}I_r^{3/28}. \qquad (2.6)$$

Using (2.4) and (2.6) yields

$$\|v\|_2^2 \leqslant C\|\partial_x v\|_2^{2/5}(\|v\|_2^2)^{24/35}I_r^{6/35} + Cr^{-1/2}(\|v\|_2^2)^{6/7}I_r^{3/14},$$

and thus

$$(\|v\|_2^2)^{11/35} \leqslant C\|\partial_x v\|_2^{2/5}I_r^{6/35} + Cr^{-1/2}(\|v\|_2^2)^{6/35}I_r^{3/14}.$$

By Young's inequality we get

$$\|v\|_2^2 \leqslant C\|\partial_x v\|_2^{14/11}I_r^{6/11} + Cr^{-35/22}(\|v\|_2^2)^{6/11}I_r^{15/22}$$
$$\leqslant C\|\partial_x v\|_2^{14/11}I_r^{6/11} + \tfrac{1}{2}\|v\|_2^2 + Cr^{-7/2}I_r^{3/2},$$

and the proof is complete. $\qquad\qquad\square$

We now introduce additional notation. For $r > 0$ and $T > 0$ we set

$$u_k(r,T) := \int_0^T \int_{-r}^r |\partial_x w(t,x)|^2 (r - |x|)_+^k \, \mathrm{d}x \, \mathrm{d}t \quad \text{for } k \in \{0,1,2\},$$

$$I(r,T) := \sup_{t \in (0,T)} \int_{-r}^r w^{4/3}(t,x)(r - |x|)_+^2 \, \mathrm{d}x,$$

where w is defined in (2.1). We first derive from (2.2) an inequality relating $I(r,T)$, $u_0(r,T)$ and $u_2(r,T)$ under suitable constraints on r and T.

LEMMA 2.4. *Consider $r_0 > 0$ such that* $\mathrm{supp}(f_0 + g_0) \cap (-r_0, r_0) = \emptyset$. *There are positive constants C_4 and C_5 such that, if $T_0 > 0$ is such that*

$$C_4 T_0 (\tfrac{1}{2} r_0)^{-7/2} I^{1/2}(r_0, T_0) \leqslant \tfrac{1}{2}, \tag{2.7}$$

then

$$\tfrac{1}{3} I(r,T) + u_2(r,T) \leqslant C_5 T^{4/5} u_0^{7/5}(r,T) \tag{2.8}$$

for all $\tfrac{1}{2} r_0 \leqslant r \leqslant r_0$ and $0 < T \leqslant T_0$.

Proof. Let $T \in (0, T_0]$ and $r \in (0, r_0]$. Setting $\zeta(x) := (r - |x|)_+$, $x \in \mathbb{R}$, we observe that the assumptions on $f_0 + g_0$ guarantee that $\zeta^2(x) w^{4/3}(0,x) = 0$ for $x \in \mathbb{R}$, and we infer from (2.2) that

$$\int_{\mathbb{R}} w^{4/3}(t) \zeta^2 \, \mathrm{d}x + C_1 \int_0^t \int_{\mathbb{R}} |\partial_x w|^2 \zeta^2 \, \mathrm{d}x \, \mathrm{d}s \leqslant C_2 \int_0^t \int_{-r}^r w^2 \, \mathrm{d}x \, \mathrm{d}s$$

for all $t \in (0,T)$. Hence, there exists a constant $C > 0$ such that

$$I(r,T) + u_2(r,T) \leqslant C \int_0^T \int_{-r}^r w^2 \, \mathrm{d}x \, \mathrm{d}s.$$

Using (2.5) and the Hölder inequality, it follows that

$$I(r,T) + u_2(r,T) \leqslant C T r^{-7/2} I^{3/2}(r,T) + C I^{6/11}(r,T) \int_0^T \|\partial_x w(s)\|_{L_2((-r,r))}^{14/11} \, \mathrm{d}s$$

$$\leqslant C_4 T r^{-7/2} I^{3/2}(r,T) + C T^{4/11} I^{6/11}(r,T) u_0^{7/11}(r,T).$$

Since $I(r,T)$ is a non-decreasing function in both variables r and T, the property (2.7), together with Young's inequality and the above inequality, leads us to

$$I(r,T) + u_2(r,T) \leqslant \tfrac{1}{2} I(r,T) + \tfrac{1}{6} I(r,T) + C_5 T^{4/5} u_0^{7/5}(r,T)$$

for all $\tfrac{1}{2} r_0 \leqslant r \leqslant r_0$ and $0 < T \leqslant T_0$. This completes the proof. $\qquad\square$

We are now in a position to prove our main results.

Proof of theorem 1.2. Since (1.1) is invariant with respect to translation, we may assume that $a = 0$, so that $\mathrm{supp}(f_0 + g_0) \cap (-r_0, r_0) = \emptyset$. Then $w^{4/3}(0,x) = 0$ for $x \in (-r_0, r_0)$ and $I(r_0, t) \to 0$ as $t \to 0$ by (2.2). Consequently, there exists a $T_0 > 0$ such that the condition (2.7) in lemma 2.4 is satisfied.

Let $T \in (0, T_0)$. In view of

$$[x \mapsto \|\partial_x w(\cdot, x)\|_{L_2(0,T)}^2] \in L_1(\mathbb{R}),$$

we have

$$\partial_r u_2(r, T) = 2u_1(r, T), \qquad \partial_r u_1(r, T) = u_0(r, T) \tag{2.9}$$

for almost every (a.e.) $r \in (0, r_0)$, and the Hölder inequality yields

$$u_1(r, T) \leqslant u_2^{1/2}(r, T) u_0^{1/2}(r, T).$$

This inequality, together with (2.8), which is valid here due to the choice of T_0, gives

$$u_1(r, T) \leqslant C T^{2/5} u_0^{6/5}(r, T) = C T^{2/5} (\partial_r u_1(r, T))^{6/5} \quad \text{for a.e. } r \in (\tfrac{1}{2}r_0, r_0).$$

Equivalently,

$$u_1^{5/6}(r, T) \leqslant \frac{1}{6C_6} T^{1/3} \partial_r u_1(r, T) \quad \text{for a.e. } r \in (\tfrac{1}{2}r_0, r_0). \tag{2.10}$$

Taking a smaller value of T_0 if necessary, we further assume that

$$T_0^{1/3} u_1^{1/6}(r_0, T_0) \leqslant \tfrac{1}{2} C_6 r_0. \tag{2.11}$$

Let $T \in (0, T_0]$ and assume for the following argument by contradiction that $u_1(\tfrac{1}{2}r_0, T) > 0$. Together with the monotonicity properties of u_1 this implies that $u_1(r, T) > 0$ for all $r \in [\tfrac{1}{2}r_0, r_0]$. Due to this positivity property, we infer from (2.10) that

$$C_6 T^{-1/3} \leqslant \partial_r(u_1^{1/6})(r, T) \quad \text{for a.e. } r \in (\tfrac{1}{2}r_0, r_0).$$

After integration we end up with

$$C_6 T^{-1/3}(r_0 - r) \leqslant u_1^{1/6}(r_0, T) - u_1^{1/6}(r, T),$$

or, equivalently,

$$T^{1/3} u_1^{1/6}(r, T) \leqslant T^{1/3} u_1^{1/6}(r_0, T) - C_6(r_0 - r), \quad r \in [\tfrac{1}{2}r_0, r_0]. \tag{2.12}$$

Taking $r = \tfrac{1}{2}r_0$ in (2.12) gives

$$0 < T_0^{1/3} u_1^{1/6}(r_0, T_0) - \tfrac{1}{2} C_6 r_0,$$

and contradicts (2.11). Therefore, $u_1(\tfrac{1}{2}r_0, T) = 0$ and it follows from (2.9) that $u_0(\tfrac{1}{2}r_0, T) = 0$ for all $T \in (0, T_0]$. Recalling (2.8), we find that $I(\tfrac{1}{2}r_0, T) = 0$ for all $T \in (0, T_0]$.

We further note that, in view of theorem 3.1(b) and (2.3),

$$u_1(r, T) \leqslant r \int_0^T \|\partial_x w(t)\|_2^2 \, dt \leqslant Cr[\mathcal{E}(f_0, g_0) - \mathcal{E}(f(T), g(T))] \leqslant Cr\mathcal{E}(f_0, g_0),$$

$$I(r, T) \leqslant r^2 \sup_{t \in [0,T]} \|w(t)\|_{4/3}^{4/3} \leqslant r^2 \sup_{t \in [0,T]} \mathcal{E}(f(t), g(t)) \leqslant r^2 \mathcal{E}(f_0, g_0),$$

so that (2.7) and (2.11) are satisfied provided that $T_0 = C_* r_0^{5/2}/\mathcal{E}^{1/2}(f_0, g_0)$ for a sufficiently small constant $C_* > 0$ depending only on R and R_μ. This proves the first claim of theorem 1.2.

Finally, let $\mathrm{supp}(f_0 + g_0) \subset [-b_0, b_0]$, with $b_0 > 0$, and let $T > 0$ be fixed. Choosing $r_0 = (T\mathcal{E}^{1/2}(f_0, g_0)/C_*)^{2/5}$, we have that, for each $a \geqslant b_0 + (T\mathcal{E}^{1/2}(f_0, g_0)/C_*)^{2/5}$ or $a \leqslant -b_0 - (T\mathcal{E}^{1/2}(f_0, g_0)/C_*)^{2/5}$,

$$\mathrm{supp}(f_0 + g_0) \cap (a - r_0, a + r_0) = \emptyset.$$

We then infer from the first statement of theorem 1.2 that $\mathrm{supp}(f(T) + g(T)) \cap (a - \frac{1}{2}r_0, a + \frac{1}{2}r_0) = \emptyset$, from which it follows that

$$\mathrm{supp}(f(T) + g(T)) \subset \left[-b_0 - \frac{T^{2/5}\mathcal{E}^{1/5}(f_0, g_0)}{2C_*^{2/5}}, b_0 + \frac{T^{2/5}\mathcal{E}^{1/5}(f_0, g_0)}{2C_*^{2/5}} \right]$$
$$\text{for all } T > 0. \quad (2.13)$$

Consequently, $(f(T) + g(T))$ is compactly supported for each $T \geqslant 0$ and we set

$$\beta(T) := \max\{b_0, \sup(\mathrm{supp}(f(T) + g(T)))\}.$$

It then follows that $\beta(T) \to \beta(0) = b_0$ as $T \to 0$. Since problem $(1.1\,a)$ is autonomous, the estimate (2.13) yields

$$\beta(T_2) - \beta(T_1) \leqslant \frac{(T_2 - T_1)^{2/5}\mathcal{E}^{1/5}(f(T_1), g(T_1))}{2C_*^{2/5}} \quad \text{for all } T_2 > T_1. \quad (2.14)$$

Besides, we know from [12, theorem 4.1(iv)] (after rescaling) that

$$\mathcal{E}(f(t), g(t)) \leqslant (1 + t)^{-1/3} \left[\mathcal{E}(f_0, g_0) + \frac{1}{6} \int_{\mathbb{R}} \left(f_0 + \frac{R}{R_\mu} g_0 \right) x^2 \, \mathrm{d}x \right]$$
$$\leqslant C t^{-1/3} \quad \text{for all } t > 0. \quad (2.15)$$

Combining (2.14) and (2.15) yields

$$\beta(T_2) - \beta(T_1) \leqslant C(T_2 - T_1)^{2/5} T_1^{-1/15} \quad \text{for all } T_2 > T_1 > 0.$$

We are now in the position to apply [3, lemma 7.6] to the above functional inequality and conclude that there exists a positive constant C^* depending only on R, R_μ, f_0, and g_0 such that

$$\beta(T) \leqslant b_0 + C^* T^{1/3} \quad \text{for all } T > 0,$$

which is the expected propagation rate. The estimate for the expansion of the left boundary of the support is derived in a similar way. □

Proof of theorem 1.4. Invoking (2.2), theorem 1.4 is a particular case of the more general result [4, theorem 1.2], which we apply with $k = 1$, $p = 2$ and $q = \frac{4}{3}$. □

3. Weak solutions satisfying the local energy estimate

As mentioned in § 1, we now check that there exists at least a weak solution to (1.1) satisfying the local energy estimate (1.5).

THEOREM 3.1 (existence of weak solutions). *Given $(f_0, g_0) \in \mathcal{K}^2$, where $\mathcal{K}^2$ is defined in (1.2), there exists at least a weak solution (f, g) to (1.1), satisfying the local energy estimate (1.5) as well as the following estimates:*

(a) $\|f(T)\|_1 = \|f_0\|_1$, $\|g(T)\|_1 = \|g_0\|_1$,

(b) $\mathcal{H}(f(T), g(T)) + \dfrac{R}{1 + 2R} \displaystyle\int_0^T \int_{\mathbb{R}} [|\partial_x f|^2 + R|\partial_x(f + g)|^2] \, \mathrm{d}x \, \mathrm{d}t \leqslant \mathcal{H}(f_0, g_0)$,

(c) $\mathcal{E}(f(T), g(T)) + \dfrac{1}{2} \displaystyle\int_0^T \int_{\mathbb{R}} [f((1 + R)\partial_x f + R\partial_x g)^2 + RR_\mu g(\partial_x f + \partial_x g)^2] \, \mathrm{d}x \, \mathrm{d}t$
$$\leqslant \mathcal{E}(f_0, g_0)$$

for all $T \in (0, \infty)$. The energy functional $\mathcal{E}$ is given by (1.4) and the entropy functional $\mathcal{H}$ is defined as

$$\mathcal{H}(f, g) := \int_{\mathbb{R}} \left(f \ln f + \frac{R}{R_\mu} g \ln g \right) \mathrm{d}x.$$

The remainder of this section is devoted to the proof of theorem 3.1. We split the proof of theorem 3.1 into two steps: we first truncate the spatial domain to a finite interval $(-L, L)$, for some arbitrary $L > 0$, and then introduce a regularized system having global classical solutions.

3.1. A regularized problem

To be more precise, given $L > 0$ and $\varepsilon \in (0, 1)$, we define the Hilbert space

$$H_{\mathcal{B}}^2 := \{u \in H^2((-L, L)) : \partial_x u(\pm L) = 0\}$$

and we note that the elliptic operator $(1 - \varepsilon^2 \partial_x^2) : H_{\mathcal{B}}^2 \to L_2((-L, L))$ is an isomorphism. Setting

$$\mathcal{R}_\varepsilon[u] := (1 - \varepsilon^2 \partial_x^2)^{-1} u \in H_{\mathcal{B}}^2 \quad \text{for } u \in L_2((-L, L)), \tag{3.1}$$

we consider the following regularized problem:

$$\left. \begin{aligned} \partial_t f_\varepsilon &= (1 + R)\partial_x(f_\varepsilon \partial_x f_\varepsilon) + R\partial_x((f_\varepsilon - \varepsilon)\partial_x \mathcal{R}_\varepsilon[g_\varepsilon]), \\ \partial_t g_\varepsilon &= R_\mu \partial_x((g_\varepsilon - \varepsilon)\partial_x \mathcal{R}_\varepsilon[f_\varepsilon]) + R_\mu \partial_x(g_\varepsilon \partial_x g_\varepsilon) \end{aligned} \right\} \tag{3.2a}$$

for $(t, x) \in (0, \infty) \times (-L, L)$, supplemented with homogeneous Neumann boundary conditions

$$\partial_x f_\varepsilon(t, \pm L) = \partial_x g_\varepsilon(t, \pm L) = 0, \quad t \in (0, \infty), \tag{3.2b}$$

and with regularized initial data

$$f_\varepsilon(0) = f_{0\varepsilon} := \mathcal{R}_\varepsilon[f_0 \mathbf{1}_{(-L, L)}] + \varepsilon, \qquad g_\varepsilon(0) = g_{0\varepsilon} := \mathcal{R}_\varepsilon[g_0 \mathbf{1}_{(-L, L)}] + \varepsilon. \tag{3.2c}$$

Clearly, the regularized initial data satisfy $(f_{0\varepsilon}, g_{0\varepsilon}) \in H_{\mathcal{B}}^2 \times H_{\mathcal{B}}^2$ and

$$f_{0\varepsilon} \geqslant \varepsilon, \qquad g_{0\varepsilon} \geqslant \varepsilon. \tag{3.3}$$

The solvability of problem (3.2) is studied in [6, theorem 2.1] with the help of the quasilinear parabolic theory developed in [1] and we recall the result now.

PROPOSITION 3.2. *The problem (3.2) has a unique non-negative classical solution*

$$f_\varepsilon, g_\varepsilon \in C([0,\infty); H^1((-L,L))) \cap C((0,\infty); H^2_\mathcal{B}) \cap C^1((0,\infty); L_2((-L,L))).$$

Moreover, we have

$$f_\varepsilon \geqslant \varepsilon, \quad g_\varepsilon \geqslant \varepsilon \quad for\ all\ (t,x) \in (0,\infty) \times (-L,L),$$

and

$$\|f_\varepsilon(t)\|_1 = \|f_{0\varepsilon}\|_1 = \|f_0 \mathbf{1}_{(-L,L)}\|_1 + 2\varepsilon L,$$
$$\|g_\varepsilon(t)\|_1 = \|g_{0\varepsilon}\|_1 = \|g_0 \mathbf{1}_{(-L,L)}\|_1 + 2\varepsilon L$$

for all $t \geqslant 0$.

The solutions constructed in proposition 3.2 enjoy additional properties (see [6, lemmas 2.4 and 2.6]).

LEMMA 3.3. *Given $T \in (0,\infty)$, the following hold:*

$$\mathcal{H}(f_\varepsilon(T), g_\varepsilon(T)) + \int_0^T \int_{-L}^L \left(\frac{|\partial_x f_\varepsilon|^2}{2} + \frac{R}{1+2R}|\partial_x g_\varepsilon|^2 \right) \mathrm{d}x\,\mathrm{d}t$$
$$\leqslant \mathcal{H}(f_\varepsilon(0), g_\varepsilon(0)) \qquad (3.4)$$

and

$$\mathcal{E}_\varepsilon(f_\varepsilon(T), g_\varepsilon(T)) + \int_0^T \int_{-L}^L [f_\varepsilon|(1+R)\partial_x f_\varepsilon + R\partial_x G_\varepsilon|^2 + RR_\mu g_\varepsilon|\partial_x(F_\varepsilon + g_\varepsilon)|^2]\,\mathrm{d}x\,\mathrm{d}t$$
$$\leqslant \mathcal{E}_\varepsilon(f_{0\varepsilon}, g_{0\varepsilon}) + \varepsilon C_2 \int_0^T \int_{-L}^L (|\partial_x f_\varepsilon|^2 + |\partial_x g_\varepsilon|^2)\,\mathrm{d}x\,\mathrm{d}t, \quad (3.5)$$

with $F_\varepsilon := \mathcal{R}_\varepsilon[f_\varepsilon]$, $G_\varepsilon := \mathcal{R}_\varepsilon[g_\varepsilon]$,

$$\mathcal{E}_\varepsilon(f_\varepsilon, g_\varepsilon) := \frac{1}{2}\left[(1+R)\|f_\varepsilon\|_2^2 + R\|g_\varepsilon\|_2^2 + R\int_{-L}^L (F_\varepsilon g_\varepsilon + G_\varepsilon f_\varepsilon)\,\mathrm{d}x \right],$$

$$\mathcal{H}(f_\varepsilon, g_\varepsilon) := \int_{-L}^L \left(f_\varepsilon \ln f_\varepsilon + \frac{R}{R_\mu} g_\varepsilon \ln g_\varepsilon \right) \mathrm{d}x.$$

As a consequence of proposition 3.2 and lemma 3.3, the following result is proved in [6].

PROPOSITION 3.4 (weak solutions on a finite interval). *There exist a sequence $\varepsilon_k \to 0$ and a pair (f, g) satisfying*

(i) *$f \geqslant 0$, $g \geqslant 0$ in $(0,\infty) \times (-L,L)$,*

(ii) *$f, g \in L_\infty(0,\infty; L_2((-L,L))) \cap L_2(0,T; H^1((-L,L)))$ for all $T > 0$,*

(iii) *$f_{\varepsilon_k} \to f$, $g_{\varepsilon_k} \to g$ in $L_2((0,T) \times (-L,L))$,*

and

$$\int_{-L}^{L} f(T)\xi \, \mathrm{d}x - \int_{-L}^{L} f_0 \xi \, \mathrm{d}x = -\int_0^T \int_{-L}^{L} f((1+R)\partial_x f + R\partial_x g)\partial_x \xi \, \mathrm{d}x \, \mathrm{d}t, \quad (3.6)$$

$$\int_{-L}^{L} g(T)\xi \, \mathrm{d}x - \int_{-L}^{L} g_0 \xi \, \mathrm{d}x = -R_\mu \int_0^T \int_{-L}^{L} g(\partial_x f + \partial_x g)\partial_x \xi \, \mathrm{d}x \, \mathrm{d}t \quad (3.7)$$

for all $\xi \in W_4^1((-L, L))$ and all $T > 0$. Moreover,

(a) $\|f(T)\|_1 = \|f_0 \mathbf{1}_{(-L,L)}\|_1$, $\|g(T)\|_1 = \|g_0 \mathbf{1}_{(-L,L)}\|_1$,

(b) $\mathcal{H}(f(T), g(T)) + \displaystyle\int_0^T \int_{-L}^{L} \left[\frac{|\partial_x f|^2}{2} + \frac{R}{1+2R}|\partial_x g|^2 \right] \mathrm{d}x \, \mathrm{d}t \leqslant \mathcal{H}(f_0, g_0),$

(c) $\mathcal{E}(f(T), g(T)) + \displaystyle\int_0^T \int_{-L}^{L} [f((1+R)\partial_x f + R\partial_x g)^2 + RR_\mu g(\partial_x f + \partial_x g)^2] \, \mathrm{d}x \, \mathrm{d}t$

$$\leqslant \mathcal{E}(f_0, g_0)$$

for all $T \in (0, \infty)$.

3.2. A local energy estimate

We now derive a local version of the inequality in proposition 3.4(c).

LEMMA 3.5. *Let (f, g) be the limit of $((f_{\varepsilon_k}, g_{\varepsilon_k}))_k$ found in proposition 3.4. Then*

$$\int_{-L}^{L} [f^2(T) + R(f+g)^2(T)]\zeta^2 \, \mathrm{d}x$$

$$+ \int_0^T \int_{-L}^{L} (f|(1+R)\partial_x f + R\partial_x g|^2 + RR_\mu g|\partial_x f + \partial_x g|^2)\zeta^2 \, \mathrm{d}x \, \mathrm{d}t$$

$$\leqslant \int_{-L}^{L} [f^2(0) + R(f+g)^2(0)]\zeta^2 \, \mathrm{d}x$$

$$+ 4\int_0^T \int_{-L}^{L} [f((1+R)f + Rg)^2 + RR_\mu g(f+g)^2]|\partial_x \zeta|^2 \, \mathrm{d}x \, \mathrm{d}t \quad (3.8)$$

for all $T > 0$ and all $\zeta \in W_4^1((-L, L))$.

Proof. We set

$$U_\varepsilon := \sqrt{f_\varepsilon}\partial_x[(1+R)f_\varepsilon + RG_\varepsilon], \qquad V_\varepsilon := \sqrt{g_\varepsilon}\partial_x[F_\varepsilon + g_\varepsilon],$$

and prove first the claim (3.8) for $\zeta \in C_0^\infty((-L, L))$. We multiply the first equation of (3.2 *a*) by $((1+R)f_\varepsilon + RG_\varepsilon)\zeta^2$ and integrate over $(-L, L)$ to obtain

$$\int_{-L}^{L} \partial_t f_\varepsilon((1+R)f_\varepsilon + RG_\varepsilon)\zeta^2 \, \mathrm{d}x$$

$$= -\int_{-L}^{L} \sqrt{f_\varepsilon}U_\varepsilon \partial_x[((1+R)f_\varepsilon + RG_\varepsilon)\zeta^2] \, \mathrm{d}x + I_{1,\varepsilon} \quad (3.9)$$

with

$$I_{1,\varepsilon} := \varepsilon R \int_{-L}^{L} \partial_x G_\varepsilon \partial_x [((1+R)f_\varepsilon + RG_\varepsilon)\zeta^2]\, \mathrm{d}x.$$

Similarly, multiplying the second equation of $(3.2\,a)$ by $R(F_\varepsilon + g_\varepsilon)\zeta^2$ and integrating over $(-L, L)$ give

$$R \int_{-L}^{L} \partial_t g_\varepsilon (F_\varepsilon + g_\varepsilon)\zeta^2 \, \mathrm{d}x = -RR_\mu \int_{-L}^{L} \sqrt{g_\varepsilon} V_\varepsilon \partial_x [(F_\varepsilon + g_\varepsilon)\zeta^2]\, \mathrm{d}x + I_{2,\varepsilon} \qquad (3.10)$$

with

$$I_{2,\varepsilon} := \varepsilon RR_\mu \int_{-L}^{L} \partial_x F_\varepsilon \partial_x [(F_\varepsilon + g_\varepsilon)\zeta^2]\, \mathrm{d}x.$$

We now observe that

$$\int_{-L}^{L} \partial_t f_\varepsilon ((1+R)f_\varepsilon + RG_\varepsilon)\zeta^2 \, \mathrm{d}x + R \int_{-L}^{L} \partial_t g_\varepsilon (F_\varepsilon + g_\varepsilon)\zeta^2 \, \mathrm{d}x$$

$$= \frac{1+R}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|f_\varepsilon \zeta\|_2^2 + \frac{R}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|g_\varepsilon \zeta\|_2^2 + RJ_\varepsilon, \quad (3.11)$$

with

$$J_\varepsilon := \int_{-L}^{L} (G_\varepsilon \partial_t f_\varepsilon + F_\varepsilon \partial_t g_\varepsilon)\zeta^2 \, \mathrm{d}x$$

$$= \frac{\mathrm{d}}{\mathrm{d}t} \int_{-L}^{L} (F_\varepsilon G_\varepsilon + \varepsilon^2 \partial_x F_\varepsilon \partial_x G_\varepsilon)\zeta^2 \, \mathrm{d}x$$

$$+ 2\varepsilon^2 \int_{-L}^{L} (G_\varepsilon \partial_x \partial_t F_\varepsilon + F_\varepsilon \partial_x \partial_t G_\varepsilon)\zeta \partial_x \zeta \, \mathrm{d}x.$$

Recalling that $\zeta \in C_0^\infty((-L, L))$, we have

$$\int_{-L}^{L} (F_\varepsilon G_\varepsilon + \varepsilon^2 \partial_x F_\varepsilon \partial_x G_\varepsilon)\zeta^2 \, \mathrm{d}x$$

$$= \frac{1}{2} \int_{-L}^{L} (F_\varepsilon G_\varepsilon + \varepsilon^2 \partial_x F_\varepsilon \partial_x G_\varepsilon)\zeta^2 \, \mathrm{d}x$$

$$+ \frac{1}{2} \int_{-L}^{L} (F_\varepsilon G_\varepsilon + \varepsilon^2 \partial_x F_\varepsilon \partial_x G_\varepsilon)\zeta^2 \, \mathrm{d}x$$

$$= \frac{1}{2} \int_{-L}^{L} (F_\varepsilon G_\varepsilon - \varepsilon^2 G_\varepsilon \partial_x^2 F_\varepsilon)\zeta^2 \, \mathrm{d}x$$

$$+ \frac{1}{2} \int_{-L}^{L} (F_\varepsilon G_\varepsilon - \varepsilon^2 F_\varepsilon \partial_x^2 G_\varepsilon)\zeta^2 \, \mathrm{d}x$$

$$- \varepsilon^2 \int_{-L}^{L} G_\varepsilon \partial_x F_\varepsilon \zeta \partial_x \zeta \, \mathrm{d}x - \varepsilon^2 \int_{-L}^{L} F_\varepsilon \partial_x G_\varepsilon \zeta \partial_x \zeta \, \mathrm{d}x$$

$$= \frac{1}{2} \int_{-L}^{L} (F_\varepsilon g_\varepsilon + G_\varepsilon f_\varepsilon)\zeta^2 \, \mathrm{d}x + \varepsilon^2 \int_{-L}^{L} F_\varepsilon G_\varepsilon \partial_x (\zeta \partial_x \zeta) \, \mathrm{d}x,$$

while

$$\int_{-L}^{L} (G_\varepsilon \partial_x \partial_t F_\varepsilon + F_\varepsilon \partial_x \partial_t G_\varepsilon) \zeta \partial_x \zeta \, \mathrm{d}x = - \int_{-L}^{L} (\partial_t F_\varepsilon \partial_x G_\varepsilon + \partial_x F_\varepsilon \partial_t G_\varepsilon) \zeta \partial_x \zeta \, \mathrm{d}x$$
$$- \int_{-L}^{L} (G_\varepsilon \partial_t F_\varepsilon + F_\varepsilon \partial_t G_\varepsilon) \partial_x (\zeta \partial_x \zeta) \, \mathrm{d}x$$
$$= - \int_{-L}^{L} (\partial_t F_\varepsilon \partial_x G_\varepsilon + \partial_x F_\varepsilon \partial_t G_\varepsilon) \zeta \partial_x \zeta \, \mathrm{d}x$$
$$- \frac{\mathrm{d}}{\mathrm{d}t} \int_{-L}^{L} F_\varepsilon G_\varepsilon \partial_x (\zeta \partial_x \zeta) \, \mathrm{d}x.$$

We end up with the following formula for J_ε:

$$J_\varepsilon = \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_{-L}^{L} (F_\varepsilon g_\varepsilon + G_\varepsilon f_\varepsilon) \zeta^2 \, \mathrm{d}x - \varepsilon^2 \frac{\mathrm{d}}{\mathrm{d}t} \int_{-L}^{L} F_\varepsilon G_\varepsilon \partial_x (\zeta \partial_x \zeta) \, \mathrm{d}x$$
$$- 2\varepsilon^2 \int_{-L}^{L} (\partial_t F_\varepsilon \partial_x G_\varepsilon + \partial_x F_\varepsilon \partial_t G_\varepsilon) \zeta \partial_x \zeta \, \mathrm{d}x.$$

After integration over $(0, T)$, it follows from (3.9)–(3.11) and the previous identity that

$$\tfrac{1}{2}(1 + R) \|f_\varepsilon(T)\zeta\|_2^2 - \tfrac{1}{2}(1 + R) \|f_\varepsilon(0)\zeta\|_2^2 + \tfrac{1}{2} R \|g_\varepsilon(T)\zeta\|_2^2 - \tfrac{1}{2} R \|g_\varepsilon(0)\zeta\|_2^2$$
$$+ \tfrac{1}{2} R \int_{-L}^{L} (F_\varepsilon g_\varepsilon + G_\varepsilon f_\varepsilon)(T)\zeta^2 \, \mathrm{d}x - \tfrac{1}{2} R \int_{-L}^{L} (F_\varepsilon g_\varepsilon + G_\varepsilon f_\varepsilon)(0)\zeta^2 \, \mathrm{d}x$$
$$- R\varepsilon^2 \int_{-L}^{L} (F_\varepsilon G_\varepsilon)(T) \partial_x (\zeta \partial_x \zeta) \, \mathrm{d}x + R\varepsilon^2 \int_{-L}^{L} (F_\varepsilon G_\varepsilon)(0) \partial_x (\zeta \partial_x \zeta) \, \mathrm{d}x$$
$$- 2R\varepsilon^2 \int_0^T \int_{-L}^{L} (\partial_t F_\varepsilon \partial_x G_\varepsilon + \partial_x F_\varepsilon \partial_t G_\varepsilon) \zeta \partial_x \zeta \, \mathrm{d}x \, \mathrm{d}s$$
$$= - \int_0^T \int_{-L}^{L} \sqrt{f_\varepsilon} U_\varepsilon \partial_x [((1 + R)f_\varepsilon + RG_\varepsilon)\zeta^2] \, \mathrm{d}x \, \mathrm{d}s + \int_0^T I_{1,\varepsilon} \, \mathrm{d}s$$
$$- RR_\mu \int_0^T \int_{-L}^{L} \sqrt{g_\varepsilon} V_\varepsilon \partial_x [(F_\varepsilon + g_\varepsilon)\zeta^2] \, \mathrm{d}x \, \mathrm{d}s + \int_0^T I_{2,\varepsilon} \, \mathrm{d}s.$$

Using Young's inequality we get

$$\tfrac{1}{2}(1 + R) \|f_\varepsilon(T)\zeta\|_2^2 + \tfrac{1}{2} R \|g_\varepsilon(T)\zeta\|_2^2 + \tfrac{1}{2} R \int_{-L}^{L} (F_\varepsilon g_\varepsilon + G_\varepsilon f_\varepsilon)(T)\zeta^2 \, \mathrm{d}x$$
$$+ K_\varepsilon(T) + \frac{1}{2} \int_0^T \int_{-L}^{L} [U_\varepsilon^2 + RR_\mu V_\varepsilon^2]\zeta^2 \, \mathrm{d}x \, \mathrm{d}s$$
$$\leqslant \tfrac{1}{2}(1 + R) \|f_\varepsilon(0)\zeta\|_2^2 + \tfrac{1}{2} R \|g_\varepsilon(0)\zeta\|_2^2 + \tfrac{1}{2} R \int_{-L}^{L} (F_\varepsilon g_\varepsilon + G_\varepsilon f_\varepsilon)(0)\zeta^2 \, \mathrm{d}x$$
$$+ 2 \int_0^T \int_{-L}^{L} [f_\varepsilon |(1 + R)f_\varepsilon + RG_\varepsilon|^2 + RR_\mu g_\varepsilon |F_\varepsilon + g_\varepsilon|^2] |\partial_x \zeta|^2 \, \mathrm{d}x \, \mathrm{d}s$$

$$(3.12)$$

with

$$K_\varepsilon(T) := -R\varepsilon^2 \int_{-L}^{L} (F_\varepsilon G_\varepsilon)(T)\partial_x(\zeta\partial_x\zeta)\,\mathrm{d}x + R\varepsilon^2 \int_{-L}^{L} (F_\varepsilon G_\varepsilon)(0)\partial_x(\zeta\partial_x\zeta)\,\mathrm{d}x$$

$$- 2R\varepsilon^2 \int_0^T \int_{-L}^{L} (\partial_t F_\varepsilon \partial_x G_\varepsilon + \partial_x F_\varepsilon \partial_t G_\varepsilon)\zeta\partial_x\zeta\,\mathrm{d}x\,\mathrm{d}s - \int_0^T (I_{1,\varepsilon} + I_{2,\varepsilon})\,\mathrm{d}s.$$

According to [6], the convergences of $(f_{\varepsilon_k})_k$ and $(g_{\varepsilon_k})_k$ towards f and g actually take place in stronger topologies than stated in proposition 3.4. In fact, for all $T > 0$,

$$f_{\varepsilon_k} \to f, \qquad F_{\varepsilon_k} \to f, \qquad g_{\varepsilon_k} \to g, \qquad G_{\varepsilon_k} \to g \quad \text{in } L_2(0,T;C([-L,L])),$$
(3.13)

$$f_{\varepsilon_k}(0) \to f_0, \quad F_{\varepsilon_k}(0) \to f_0, \quad g_{\varepsilon_k}(0) \to g_0, \quad G_{\varepsilon_k}(0) \to g_0 \quad \text{in } L_2((-L,L)), \quad (3.14)$$

and

$$U_{\varepsilon_k} \rightharpoonup U := \sqrt{f}((1+R)\partial_x f + R\partial_x g) \quad \text{in } L_2((0,T)\times(-L,L)), \qquad (3.15)$$
$$V_{\varepsilon_k} \rightharpoonup V := \sqrt{g}(\partial_x f + \partial_x g) \qquad\quad \text{in } L_2((0,T)\times(-L,L)). \qquad (3.16)$$

Furthermore, it follows from [6, lemmas 2.3 and 2.5] that

$$((f_{\varepsilon_k}, g_{\varepsilon_k}, F_{\varepsilon_k}, G_{\varepsilon_k}))_k \text{ are bounded in } L_\infty(0,T;L_2(-L,L)) \cap L_2(0,T;H^1(-L,L)).$$
(3.17)

We also infer from (3.13) that

$$\lim_{k\to\infty} \{\|(f_{\varepsilon_k}-f)(T)\|_2 + \|(F_{\varepsilon_k}-f)(T)\|_2 + \|(g_{\varepsilon_k}-g)(T)\|_2 + \|(G_{\varepsilon_k}-g)(T)\|_2\} = 0$$
(3.18)

for almost all $T > 0$. We may then take $\varepsilon = \varepsilon_k$ in (3.12) and let $k \to \infty$ to deduce from (3.13)–(3.18) that, for almost all $T > 0$,

$$\frac{1+R}{2}\|f(T)\zeta\|_2^2 + \tfrac{1}{2}R\|g(T)\zeta\|_2^2 + R\int_{-L}^{L}(fg)(T)\zeta^2\,\mathrm{d}x$$

$$+ \frac{1}{2}\int_0^T \int_{-L}^{L}[U^2 + RR_\mu V^2]\zeta^2\,\mathrm{d}x\,\mathrm{d}s$$

$$\leqslant \frac{1+R}{2}\|f_0\zeta\|_2^2 + \tfrac{1}{2}R\|g_0\zeta\|_2^2 + R\int_{-L}^{L}f_0 g_0\zeta^2\,\mathrm{d}x$$

$$+ 2\int_0^T \int_{-L}^{L}[f|(1+R)f + Rg|^2 + RR_\mu g|f+g|^2]|\partial_x\zeta|^2\,\mathrm{d}x\,\mathrm{d}s \qquad (3.19)$$

provided we establish that

$$\lim_{k\to\infty} K_{\varepsilon_k}(T) = 0. \qquad (3.20)$$

3.2.1. *The term $K_\varepsilon(T)$*

It remains to prove (3.20) and actually identify the behaviour of $K_\varepsilon(T)$ as $\varepsilon \to 0$. Owing to (3.14) and (3.18), it is clear that

$$\lim_{k\to\infty} R\varepsilon_k^2 \int_{-L}^{L} (F_{\varepsilon_k} G_{\varepsilon_k})(T)\partial_x(\zeta\partial_x\zeta)\,\mathrm{d}x$$
$$= \lim_{k\to\infty} R\varepsilon_k^2 \int_{-L}^{L} (F_{\varepsilon_k} G_{\varepsilon_k})(0)\partial_x(\zeta\partial_x\zeta)\,\mathrm{d}x = 0 \qquad (3.21)$$

for almost all $T > 0$. Next, it readily follows from (3.17) that

$$\lim_{\varepsilon\to 0} \int_0^T I_{1,\varepsilon}\,\mathrm{d}t = \lim_{\varepsilon\to 0} \int_0^T I_{2,\varepsilon}\,\mathrm{d}t = 0. \qquad (3.22)$$

Finally, since

$$\partial_t f_\varepsilon = \partial_x(\sqrt{f_\varepsilon}U_\varepsilon - R\varepsilon\partial_x G_\varepsilon),$$

the boundedness (3.17) of $(U_\varepsilon)_\varepsilon$ in $L_2((0,T)\times(-L,L))$, and the boundedness of $(f_\varepsilon)_\varepsilon$ in $L_\infty(0,T;L_2(-L,L))$ imply the boundedness of $(\sqrt{f_\varepsilon}U_\varepsilon - R\varepsilon\partial_x G_\varepsilon)_\varepsilon$ in $L_2(0,T;L_{4/3}(-L,L))$. As a consequence, the sequence $(\partial_t f_\varepsilon)_\varepsilon$ is bounded in $L_2(0,T;(W_4^1(-L,L))')$, and so, by a similar argument, is $(\partial_t g_\varepsilon)_\varepsilon$. Owing to the properties of $(1 - \varepsilon^2\partial_x^2)^{-1}$, we conclude that

$$(\partial_t F_\varepsilon)_\varepsilon, (\partial_t G_\varepsilon)_\varepsilon \quad \text{are bounded in } L_2(0,T;(W_4^1)') \qquad (3.23)$$

(see [6, lemma 3.1] for a similar result). Now, since $\zeta \in C_0^\infty((-L,L))$ and $W_4^1(-L,L)$ is an algebra, we infer from (3.23) that

$$2R\varepsilon^2 \left| \int_0^T \int_{-L}^{L} (\partial_t F_\varepsilon\partial_x G_\varepsilon + \partial_x F_\varepsilon\partial_t G_\varepsilon)\zeta\partial_x\zeta\,\mathrm{d}x\,\mathrm{d}s \right|$$
$$\leqslant C\varepsilon^2 \int_0^T (\|\partial_t F_\varepsilon\|_{(W_4^1)'}\|\partial_x G_\varepsilon\|_{W_4^1} + \|\partial_t G_\varepsilon\|_{(W_4^1)'}\|\partial_x F_\varepsilon\|_{W_4^1})\|\zeta\partial_x\zeta\|_{W_4^1}\,\mathrm{d}s$$
$$\leqslant C(\zeta,T)\varepsilon^2 \left[\int_0^T (\|\partial_x G_\varepsilon\|_{W_4^1}^2 + \|\partial_x F_\varepsilon\|_{W_4^1}^2)\,\mathrm{d}s \right]^{1/2}. \qquad (3.24)$$

Now, owing to (3.1), for almost all $t \in (0,T)$ the function $\partial_x F_\varepsilon(t)$ solves

$$\partial_x F_\varepsilon - \varepsilon^2\partial_x^2\partial_x F_\varepsilon = \partial_x f_\varepsilon \quad \text{in } (-L,L), \qquad \partial_x F_\varepsilon(\pm L) = 0,$$

which implies that

$$\|\partial_x F_\varepsilon\|_2^2 + \varepsilon^2\|\partial_x^2 F_\varepsilon\|_2^2 + \varepsilon^4\|\partial_x^3 F_\varepsilon\|_2^2 \leqslant C\|\partial_x f_\varepsilon\|_2^2.$$

These estimates, along with the Gagliardo–Nirenberg inequality [14, theorem 1], give

$$\|\partial_x F_\varepsilon\|_{W_4^1} \leqslant C(\|\partial_x F_\varepsilon\|_4 + \|\partial_x^2 F_\varepsilon\|_4)$$
$$\leqslant C(\|\partial_x^2 F_\varepsilon\|_2^{1/4}\|\partial_x F_\varepsilon\|_2^{3/4} + \|\partial_x^3 F_\varepsilon\|_2^{1/4}\|\partial_x^2 F_\varepsilon\|_2^{3/4} + \|\partial_x^2 F_\varepsilon\|_2)$$
$$\leqslant C\varepsilon^{-5/4}\|\partial_x f_\varepsilon\|_2.$$

As a similar estimate is valid for $\|\partial_x G_\varepsilon\|_{W_4^1}$ with $\|\partial_x g_\varepsilon\|_2$ instead of $\|\partial_x f_\varepsilon\|_2$, we deduce from (3.17) and (3.24) that

$$
\begin{aligned}
2R\varepsilon^2 \bigg| \int_0^T \int_{-L}^L (\partial_t F_\varepsilon \partial_x G_\varepsilon + \partial_x F_\varepsilon \partial_t G_\varepsilon) \zeta \partial_x \zeta \,\mathrm{d}x \,\mathrm{d}s \bigg| & \\
\leqslant C(\zeta, T)\varepsilon^{3/4} \bigg[\int_0^T (\|\partial_x f_\varepsilon\|_2^2 + \|\partial_x g_\varepsilon\|_2^2) \,\mathrm{d}s \bigg]^{1/2} & \\
\leqslant C(\zeta, T)\varepsilon^{3/4}. & \quad (3.25)
\end{aligned}
$$

Combining (3.21), (3.22) and (3.25) gives the claim (3.20) and completes the proof of (3.8) for $\zeta \in C_0^\infty(\mathbb{R})$, its validity for all $T > 0$ being obtained by a lower semicontinuity argument. According to the regularity of (f, g), the extension of lemma 3.5 to all functions $\zeta \in W_4^1((-L, L))$ follows by a density argument. $\qquad\square$

3.3. The limit $L \to \infty$

For each positive L, we denote the couple found in proposition 3.4 by (f^L, g^L). The family $((f^L, g^L))_L$ satisfies the same bounds as the family $((f_\varepsilon, g_\varepsilon))_\varepsilon$, so that taking the limit as $L \to \infty$ may be done as $\varepsilon \to 0$, the only difference being the unboundedness of the domain that one has to cope with. To this end we derive the following lemma, which controls the behaviour of (f^L, g^L) at infinity.

LEMMA 3.6. *It holds that*

$$
\int_{-L/2}^{L/2} \left(f^L + \frac{R}{R_\mu} g^L \right)(T) x^2 \,\mathrm{d}x \leqslant \int_{-L}^L \left(f_0 + \frac{R}{R_\mu} g_0 \right) x^2 \,\mathrm{d}x + T\mathcal{E}(f_0, g_0) \quad (3.26)
$$

for all $T > 0$.

Proof. We define the function

$$
\Phi(x) = \begin{cases} -2Lx - x^2 - \frac{3}{4}L^2, & -L \leqslant x \leqslant -\frac{1}{2}L, \\ x^2, & -\frac{1}{2}L \leqslant x \leqslant \frac{1}{2}L, \\ 2Lx - x^2 - \frac{3}{4}L^2, & \frac{1}{2}L \leqslant x \leqslant L. \end{cases}
$$

We take $\xi = \Phi$ in (3.6) and $\xi = R\Phi/R_\mu$ in (3.7). Then, using integration by parts and the bound $\Phi'' \leqslant 2$, we obtain

$$
\begin{aligned}
\int_{-L}^L \left(f^L + \frac{R}{R_\mu} g^L \right)(T)\Phi \,\mathrm{d}x &= \int_{-L}^L \left(f_0 + \frac{R}{R_\mu} g_0 \right)\Phi \,\mathrm{d}x \\
&\quad + \frac{1}{2} \int_0^T \int_{-L}^L ((f^L)^2 + R(f^L + g^L)^2)\Phi'' \,\mathrm{d}x \,\mathrm{d}t \\
&\leqslant \int_{-L}^L \left(f_0 + \frac{R}{R_\mu} g_0 \right)\Phi \,\mathrm{d}x + T\mathcal{E}(f_0, g_0).
\end{aligned}
$$

In addition,

$$
x^2 \mathbf{1}_{[-L/2, L/2]} \leqslant \Phi(x) \leqslant x^2 \quad \text{for } x \in [-L, L],
$$

and the claim follows. $\qquad\square$

Using lemma 3.6 we may argue as in the proof of proposition 3.4 (see [6]), to take the limit as $L \to \infty$ and complete the proof of theorem 3.1. In particular, we use lemma 3.6 to establish the entropy inequality in theorem 3.1(b) as well as the conservation of mass in theorem 3.1.

Acknowledgements

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