

BLOCKS WITH SMALL-DIMENSIONAL BASIC ALGEBRA

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Abstract

Linckelmann and Murphy have classified the Morita equivalence classes of p -blocks of finite groups whose basic algebra has dimension at most 12. We extend their classification to dimension 13 and 14. As predicted by Donovan's conjecture, we obtain only finitely many such Morita equivalence classes.

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1. Introduction

Let F be an algebraically closed field of characteristic $p > 0$. Donovan's conjecture (over F) states that for every finite p -group D there are only finitely many Morita equivalence classes of p -blocks of finite groups with defect group D . Since a general proof seems elusive at present, mathematicians have focused on certain families of p -groups D . This has culminated in a proof of Donovan's conjecture for all abelian 2-groups by Eaton and Livesey [4]. A different approach, introduced by Linckelmann [11], aims to classify blocks B with a given *basic algebra* A . Recall that A is the unique F -algebra (up to isomorphism) of smallest dimension which is Morita equivalent to B . Linckelmann and Murphy [11, 12] have classified all blocks B such that $\dim A \leq 12$. Since the order of a defect group is bounded in terms of $\dim A$ (see next section), one expects only finitely many such blocks up to Morita equivalence. Indeed, the list in [11] is finite. We extend their classification as follows.

THEOREM 1.1. *Let B be a block of a finite group with basic algebra A .*

- (I) *If $\dim A = 13$, then B is Morita equivalent to one of the following block algebras:*
- (a) FC_{13} ($p = 13$);
 - (b) *the principal 13-block of $\mathrm{PSL}(3, 3)$ with defect 1;*

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- (c) *the principal 17-block of $PSL(2, 16)$ with defect 1;*
 - (d) *the principal 2-block of $PGL(2, 7)$ with defect group D_{16} ;*
 - (e) *a nonprincipal 2-block of $3.M_{10}$ with defect group SD_{16} ;*
 - (f) *a nonprincipal 7-block of $6.A_7$ with defect 1.*
- (II) *If $\dim A = 14$, then B is Morita equivalent to one of the following block algebras:*
- (a) *FD_{14} ($p = 7$);*
 - (b) *the principal 5-block of S_5 with defect 1;*
 - (c) *the principal 7-block of $PSU(3, 3)$ with defect 1;*
 - (d) *the principal 19-block of $PSL(2, 37)$ with defect 1.*

The bulk of the proof is devoted to the nonexistence of a certain block with extraspecial defect group of order 27 and exponent 3. The methods are quite different from those in [12]. For some of the Brauer tree algebras occurring in [11] no concrete block algebra was given. For future reference we provide explicit examples in Table 1. Here, B_0 and B_1 denote the principal block and a suitable nonprincipal block, respectively.

TABLE 1. Morita classes.

$\dim(A)$	D	Morita classes
≤ 5	$ D = \dim(A)$	FD
6	C_3	FS_3
7	C_5	$B_0(A_5)$
	C_7	FC_7
8	C_7	$B_0(PSL(2, 13))$
	$ D = 8$	FD
9	C_9	$FC_9, B_0(PSL(2, 8))$
	$C_3 \times C_3$	$F[C_3 \times C_3], B_1(2.(S_3 \times S_3))$
10	C_5	FD_{10}
	C_{11}	$B_0(PSL(2, 32))$
11	C_7	$B_0(PSL(2, 7))$
	D_8	FS_4
	C_{11}	FC_{11}
	C_{13}	$B_0(PSL(2, 25))$
12	$C_2 \times C_2$	FA_4
13	C_7	$B_1(6.A_7)$
	C_{13}	$FC_{13}, B_0(PSL(3, 3))$
	D_{16}	$B_0(PGL(2, 7))$
	SD_{16}	$B_1(3.M_{10})$
	C_{17}	$B_0(PSL(2, 16))$
14	C_5	$B_0(S_5)$
	C_7	$FD_{14}, B_0(PSU(3, 3))$
	C_{19}	$B_0(PSL(2, 37))$

For basic algebras of dimension 15 there are still only finitely many corresponding Morita equivalence classes of blocks, but we do not know if a certain Brauer tree algebra actually occurs as a block. The details are described in Section 5.

2. Preliminaries

Before we start the proof of Theorem 1.1, we introduce a number of tools, some of which were already applied in [11]. (For more detailed definitions, see [17].)

Probably the most important Morita invariant of a block B is the *Cartan matrix* C . This is a nonnegative, integral, symmetric, positive definite and indecomposable matrix of size $l(B) \times l(B)$ where $l(B)$ denotes the number of simple modules of B . Since the simple modules of a basic algebra are one-dimensional, the sum of the entries of C equals $\dim A$ in the situation of Theorem 1.1. The largest elementary divisor of C is the order of a defect group D of B and therefore a power of p . In particular, $|D|$ is bounded in terms of $\dim A$. Another Morita invariant is the isomorphism type of the *centre* $Z(B)$ of B . In particular, in the situation of Theorem 1.1,

$$k(B) := \dim Z(B) = \dim Z(A) \leq \dim A.$$

Since we encounter many blocks of defect 1 in the sequel, we construct them first.

PROPOSITION 2.1. *Suppose that B is a p -block of a finite group with defect 1 and basic algebra A . Then $m := (p - 1)/l(B)$ is an integer, called the *multiplicity of B* . If $l(B) = 1$, then $\dim A = p$. If $l(B) = 2$, then $\dim A \in \{2p, m + 5\}$. If $l(B) = 3$, then $\dim A \in \{3p, m + 9, m + 11, 4m + 6\}$. Moreover, if $\dim A \in \{13, 14\}$, then only the blocks in Theorem 1.1 occur up to Morita equivalence.*

PROOF. By the Brauer–Dade theory, B is determined up to Morita equivalence by a planarly embedded *Brauer tree*, the multiplicity m and the position of the so-called *exceptional vertex* if $m > 1$. For precise definitions we refer to [14, Chapter 11]. If $l(B) = 1$, then B has Cartan matrix (p) and the result follows (the Brauer tree has only two vertices). We now construct the Brauer trees and Cartan matrices for $l(B) \in \{2, 3\}$. The exceptional vertex is depicted by a black dot (if $m > 1$).

(i)

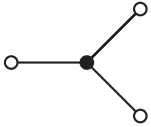
$$\circ \text{---} \bullet \text{---} \circ \qquad C = \begin{pmatrix} m+1 & m \\ m & m+1 \end{pmatrix} \qquad \dim A = 4m + 2 = 2p.$$

(ii) This case occurs for $B = FD_{2p} = A$. If $p = 7$, we get $\dim A = 14$.

$$\bullet \text{---} \circ \text{---} \circ \qquad C = \begin{pmatrix} m+1 & 1 \\ 1 & 2 \end{pmatrix} \qquad \dim A = m + 5 = \frac{p+9}{2}.$$

This case occurs for the principal block of $\text{PSL}(2, q)$ whenever p divides $q + 1$ exactly once (see [2, Section 8.4.3]). By Dirichlet’s theorem there always exists a prime $q \equiv -1 + p \pmod{p^2}$ which does the job. Choosing (p, q) in $\{(17, 16), (19, 37)\}$ yields blocks with $\dim A = 13$ and $\dim A = 14$, respectively.

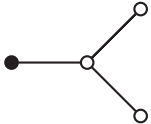
(iii)



$$C = \begin{pmatrix} m+1 & m & m \\ m & m+1 & m \\ m & m & m+1 \end{pmatrix} \quad \dim A = 9m + 3 = 3p.$$

This case occurs for $B = F[C_p \rtimes C_3] = A$. Obviously, there are no such blocks with $\dim A \in \{13, 14\}$.

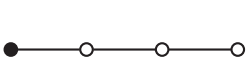
(iv)



$$C = \begin{pmatrix} m+1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \quad \dim A = m + 11 = \frac{p+32}{3}.$$

We do not know if this tree always occurs as a block algebra, but it does for a nonprincipal 7-block of the 6-fold cover $6.A_7$ (see [19]). This gives an example with $\dim A = 13$. Obviously, $\dim A = 14$ cannot occur here.

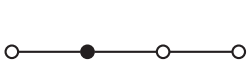
(v)



$$C = \begin{pmatrix} m+1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \quad \dim A = m + 9 = \frac{p+26}{3}.$$

By [13, Proposition 2.1], there exists a prime q such that p divides $q^3 - 1$ exactly once. Then the principal block of $GL(3, q)$ has this form by Fong and Srinivasan [6]. The principal 13-block of $PSL(3, 3)$ is an example with $\dim A = 13$. Again, $\dim A = 14$ is impossible here.

(vi)



$$C = \begin{pmatrix} m+1 & m & 0 \\ m & m+1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \quad \dim A = 4m + 6 = \frac{4p+14}{3}.$$

Again by [13, Theorem 1], there exists a prime q such that the principal block of $GU(3, q)$ has this form. The principal 7-block of $PSU(3, 3)$ is an example with $\dim A = 14$. On the other hand, $\dim A = 13$ cannot occur.

Finally, if $l(B) \geq 4$, then the trace of C is greater than or equal to 8 and we need at least six positive off-diagonal entries to ensure that C is symmetric and indecomposable. Hence, $\dim A \leq 14$ can only occur if $l(B) = 4$, $\dim A = 14$, $m = 1$ and the Brauer tree is a line. This happens for the principal 5-block of S_5 . \square

In order to investigate blocks of larger defect, we develop some more advanced methods. The decomposition matrix $Q = Q_1$ of B is nonnegative, integral and

indecomposable of size $k(B) \times l(B)$ such that $Q^t Q = C$. Given $\dim A$, there are only finitely many choices for Q . Richard Brauer introduced the so-called *contribution matrix*

$$M = M^1 := |D|QC^{-1}Q^t \in \mathbb{Z}^{k(B) \times k(B)}.$$

The heights of the irreducible characters of B are encoded in the p -adic valuation of M (see [17, Proposition 1.36]). As usual, we denote the number of irreducible characters of B of height $h \geq 0$ by $k_h(B)$. If $k_0(B) < k(B)$, then D is nonabelian according to Kessar and Malle’s [10] solution of one half of Brauer’s height-zero conjecture.

The 2-blocks occurring in Theorem 1.1 are determined by the next proposition.

PROPOSITION 2.2. *Let B be a block of a finite group with Cartan matrix $C = \begin{pmatrix} 5 & 2 \\ 2 & 4 \end{pmatrix}$. Then B is Morita equivalent to the principal 2-block of $\text{PGL}(2, 7)$ or to a nonprincipal block of $3.M_{10}$. Moreover, there is no block with Cartan matrix*

$$\begin{pmatrix} 5 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 6 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

PROOF. All three matrices have largest elementary divisor 16. Therefore, $p = 2$ and a defect group D of B has order 16. For the first matrix, the possible decomposition matrices are

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ \cdot & 1 \\ \cdot & 1 \\ 1 & \cdot \\ 1 & \cdot \\ 1 & \cdot \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 \\ \cdot & 1 \\ \cdot & 1 \\ \cdot & 1 \\ 1 & \cdot \end{pmatrix}.$$

The diagonal of the contribution matrix M^1 is $(5, 5, 5, 5, 4, 4, 4)$ or $(13, 5, 5, 5, 4)$. It follows that $k_0(B) = 4$ (the first four characters have height 0). By [17, Theorem 13.6], the Alperin–McKay conjecture holds for all 2-blocks of defect 4. Thus, $k_0(B_D) = 4$ where B_D is the Brauer correspondent of B in $N_G(D)$. Now B_D dominates a block $\overline{B_D}$ of $N_G(D)/D'$ with abelian defect group D/D' . By [14, Theorem 9.23],

$$k(\overline{B_D}) = k_0(\overline{B_D}) \leq k_0(B_D) = 4.$$

Now [17, Proposition 1.31] implies $|D/D'| = 4$. Hence, D is a dihedral group, a semidihedral group or a quaternion group. A look at [17, Theorem 8.1] (the Cartan matrices in (5a) and (5b) are mixed up), tells us that $k(B) = 7$ and $D \in \{D_{16}, SD_{16}\}$. The corresponding Morita equivalence classes were computed by Erdmann [5] (see [9, Appendix] for a definitive list). Only the two stated examples occur up to Morita equivalence.

For the second matrix there is only one possible decomposition matrix and we obtain similarly that $k_0(B) = 4$ and $k(B) = 7$. By [17, Theorem 8.1], $D \cong D_{16}$. However, it can be seen from [9, Appendix] that there are no such blocks (all Cartan invariants

are positive). Nevertheless, C occurs as Cartan matrix with respect to a suitable basic set (for the principal block of $\text{PSL}(2, 17)$, for instance).

In the last case there are two feasible decomposition matrices:

$$\begin{pmatrix} 2 & \cdot & \cdot \\ 1 & 1 & \cdot \\ 1 & \cdot & \cdot \\ \cdot & 1 & 1 \\ \cdot & \cdot & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & \cdot \\ 1 & \cdot & \cdot \\ 1 & \cdot & \cdot \\ 1 & \cdot & \cdot \\ 1 & \cdot & \cdot \\ \cdot & 1 & 1 \\ \cdot & \cdot & 1 \end{pmatrix}.$$

The first matrix leads to $k_0(B) = 4$ and $k(B) = 5$. This contradicts [17, Theorem 8.1]. The second matrix reveals $k_0(B) = k(B) = 8$. Since Brauer’s height-zero conjecture holds for B by [17, Theorem 13.6], D is abelian. By [17, Theorem 8.3], D is not isomorphic to $C_4 \times C_4$. In fact, D must be elementary abelian by [18, Proposition 16], for instance. By Eaton’s classification [3], B should be Morita equivalent to the group algebra of the Frobenius group $D \rtimes C_3$. But this is a basic algebra of dimension 48. □

The local structure of B is determined by a fusion system \mathcal{F} on D (again there are only finitely many choices for \mathcal{F} when $\dim A$ is fixed). The p' -group $E := \text{Out}_{\mathcal{F}}(D)$ is called the inertial quotient of B . Recall that for every $S \leq D$ there is exactly one subpair (S, b_S) attached to \mathcal{F} (here, b_S is a Brauer correspondent of B in $C_G(S)$). After \mathcal{F} -conjugation, we may and will always assume that S is fully \mathcal{F} -normalised. Then b_S has defect group $C_D(S)$ and fusion system $C_{\mathcal{F}}(S)$. Moreover, the Brauer correspondent $B_S := b_S^{N_G(S, b_S)}$ has defect group $N_D(S)$ and fusion system $N_{\mathcal{F}}(S)$. If $S = \langle u \rangle$ is cyclic, we call $(u, b_u) := (S, b_S)$ a subsection.

Let \mathcal{R} be a set of representatives of the \mathcal{F} -conjugacy classes of elements in D . Then a formula of Brauer asserts that

$$k(B) = \sum_{u \in \mathcal{R}} l(b_u).$$

Each b_u dominates a block $\overline{b_u}$ of $C_G(u)/\langle u \rangle$ with defect group $C_D(u)/\langle u \rangle$ and fusion system $C_{\mathcal{F}}(u)/\langle u \rangle$. If $\overline{C_u}$ is the Cartan matrix of $\overline{b_u}$, then $C_u := |\langle u \rangle| \overline{C_u}$ is the Cartan matrix of b_u . Let $Q_u := (d_{\chi\phi}^u : \chi \in \text{Irr}(B), \phi \in \text{IBr}(b_u))$ be the generalised decomposition matrix with respect to (u, b_u) . The orthogonality relations assert that $Q_u^t \overline{Q_v} = \delta_{uv} C_u$ for $u, v \in \mathcal{R}$ where δ_{uv} is the Kronecker delta and $\overline{Q_v}$ is the complex conjugate of Q_v . As above, we define the contribution matrices M^u for each $u \in \mathcal{R}$. Since the generalised decomposition numbers are algebraic integers, we may express Q_u with respect to a suitable integral basis. This yields ‘fake’ decomposition matrices \tilde{Q}_u which obey similar orthogonality relations (see [1, Section 4] for details). We call $\tilde{C}_u := \tilde{Q}_u^t \tilde{Q}_u$ the ‘fake’ Cartan matrix of b_u .

The following curious result might be of independent interest.

PROPOSITION 2.3. *Let B be a p -block of a finite group with abelian defect group D and inertial quotient E .*

- (i) *If $p = 2$, then $l(B) \equiv |E| \equiv k(E) \pmod{8}$.*
- (ii) *If $p = 3$, then $l(B) \equiv |E| \equiv k(E) \pmod{3}$.*

PROOF. We argue by induction on $|D|$. If $|D| \leq 4$, then $l(B) = |E| = k(E)$. Thus, let $|D| \geq 8$. Let $d := 8$ if $p = 2$ and $d := 3$ if $p = 3$. Let \mathcal{R} be a set of representatives of the E -orbits on D . Since E is a p' -group, we have $|C_E(u)|^2 \equiv 1 \pmod{d}$ for all $u \in D$. Hence,

$$|E| \sum_{u \in \mathcal{R}} |C_E(u)| = \sum_{u \in D} |C_E(u)|^2 \equiv |D| \equiv 0 \pmod{d}.$$

By Kessar and Malle [10] and [17, Proposition 1.31], $k(B) = k_0(B) \equiv 0 \pmod{d}$. Using Brauer’s formula and induction yields

$$l(B) = k(B) - \sum_{u \in \mathcal{R} \setminus \{1\}} l(b_u) \equiv - \sum_{u \in \mathcal{R} \setminus \{1\}} |C_E(u)| \equiv |E| \equiv \sum_{\chi \in \text{Irr}(E)} \chi(1)^2 \equiv k(E) \pmod{d}.$$

□

For the principal block B , Alperin’s weight conjecture asserts that $l(B) = k(E)$ in the situation of Proposition 2.3.

Finally, we study the elementary divisors of C via the theory of *lower defect groups*. The 1-multiplicity $m_B^{(1)}(S)$ of a subgroup $S \leq D$ is defined as the dimension of a certain section of $Z(B)$ (the precise definition in [17, Section 1.8] is not needed here). Since we are only interested in 1-multiplicities, we omit the exponent (1) from now on. Furthermore, it is desirable to attach a multiplicity to a subpair (S, b_S) instead of a subgroup. We do so by setting

$$m_B(S, b_S) := m_{B_S}(S).$$

Note that (S, b_S) is also a subpair for B_S and $m_{B_S}(S, b_S) = m_B(S, b_S)$. Now the multiplicity of an elementary divisor d of C is

$$m(d) = \sum m_B(S, b_S),$$

where (S, b_S) runs through the \mathcal{F} -conjugacy classes of subpairs with $|S| = d$. In particular, $m_B(D, b_D) = m(|D|) = 1$.

We are now in a position to investigate blocks with extraspecial defect group $D \cong 3_+^{1+2}$ of order 27 and exponent 3. The partial results on these blocks obtained by Hendren [8] are not sufficient for our purpose. We proceed in four stages. The first lemma is analogous to [17, Lemma 13.3].

LEMMA 2.4. *Let B be a block of a finite group G with defect group $D \cong C_3 \times C_3$ and inertial quotient $E \cong C_2 \times C_2$. Suppose that $l(B) = 4$. Let $D = S \times T$ with E -invariant subgroups $S \cong T \cong C_3$. Then $m_B(S, b_S) = m_B(T, b_T) = 1$.*

PROOF. By [1, Theorem 3], B is perfectly isometric to its Brauer correspondent in $N_G(D)$. It follows that the elementary divisors of the Cartan matrix of B are 1, 3, 3, 9. In particular, $m(3) = 2$. Let $U \leq D$ be of order 3 such that $S \neq U \neq T$. Then b_U is nilpotent and $l(b_U) = 1$. Since B_U has defect group D , we obtain $m_{B_U}(D) = 1$. Hence, [17, Lemma 1.43] implies $m_B(U, b_U) = m_{B_U}(U) = 0$. It follows that

$$m_B(S, b_S) + m_B(T, b_T) = m(3) = 2. \tag{2.1}$$

Similarly, b_S has defect group D and inertial quotient C_2 . Hence, $l(b_S) = 2$ by [1, Theorem 3]. This time [17, Lemma 1.43] gives

$$m_B(S, b_S) = m_{B_S}(S) + m_{B_S}(D) - 1 \leq l(b_S) - 1 = 1,$$

and similarly $m_B(T, b_T) \leq 1$. By (2.1), we must have equality. □

We recall that every 3'-automorphism group E of $D \cong 3_+^{1+2}$ acts faithfully on $D/\Phi(D) \cong C_3 \times C_3$. This allows us to regard E as a subgroup of the semilinear group $\Gamma L(1, 9) \leq \text{GL}(2, 3)$. Note that $\Gamma L(1, 9)$ is isomorphic to the semidihedral group SD_{16} . Moreover, $C_E(Z(D)) = E \cap \text{SL}(2, 3) \leq Q_8$.

LEMMA 2.5. *Let B be a block of a finite group G with defect group $D \cong 3_+^{1+2}$ and inertial quotient $E \cong SD_{16}$. Suppose that $Z := Z(D) \trianglelefteq G$ and that $\text{IBr}(b_Z)$ contains at least four Brauer characters which are not G -invariant. Then $m_B(Z, b_Z) > 0$.*

PROOF. Since $C_E(Z) \cong Q_8$ acts regularly on D/Z , there are two subgroups, say Z and S , of order 3 in D up to \mathcal{F} -conjugation. Hence, $m(3) = m_B(Z, b_Z) + m_B(S, b_S)$. We observe that B_S has defect group $N_D(S) = SZ \cong C_3 \times C_3$ and inertial quotient $C_2 \times C_2$. By [1, Theorem 3], $l(B_S) \in \{1, 4\}$. In the first case $m_B(S, b_S) = 0$ by [17, Lemma 1.43], and in the second case $m_B(S, b_S) = m_{B_S}(S, b_S) = 1$ by Lemma 2.4. Thus, it suffices to show that $m(3) \geq 2$.

Since E acts nontrivially on Z , we have $|G : N| = 2$ where $N := C_G(Z)$. As usual, b_Z dominates a block \bar{b}_Z with defect group $D/Z \cong C_3 \times C_3$ and inertial quotient $C_E(Z) \cong Q_8$. By hypothesis, $l(b_Z) \geq 4$. By [1, Lemma 13], there exists a basic set Γ for \bar{b}_Z (which is a basic set for b_Z as well) such that G acts on Γ and the Cartan matrix of b_Z with respect to Γ is

$$3 \begin{pmatrix} 2 & 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & 1 & 2 \\ 1 & 1 & 2 & 1 & 2 \\ 1 & 1 & 1 & 2 & 2 \\ 2 & 2 & 2 & 2 & 5 \end{pmatrix} \quad \text{or} \quad 3(1 + \delta_{ij})_{i,j=1}^8.$$

We may assume that $\theta_1, \dots, \theta_4 \in \Gamma$ such that $\phi := \theta_1^G = \theta_2^G$ and $\mu := \theta_3^G = \theta_4^G$ belong to a basic set Δ of B . In order to determine the Cartan matrix C of B with respect to Δ , we introduce the projective indecomposable characters Φ_ϕ and Φ_μ (note that these are generalised characters in our setting). By [14, Theorem 8.10], $\Phi_\phi = \Phi_{\theta_1}^G$ and $\Phi_\mu = \Phi_{\theta_3}^G$.

In particular, Φ_ϕ and Φ_μ vanish outside N . We compute

$$\begin{aligned} [\Phi_\phi, \Phi_\phi] &= \frac{1}{|G|} \sum_{g \in G} |\Phi_\phi(g)|^2 = \frac{1}{2} \frac{1}{|N|} \sum_{g \in N} |\Phi_\phi(g)|^2 \\ &= \frac{1}{2} [\Phi_{\theta_1} + \Phi_{\theta_2}, \Phi_{\theta_1} + \Phi_{\theta_2}] = 9 = [\Phi_\mu, \Phi_\mu], \\ [\Phi_\phi, \Phi_\mu] &= \frac{1}{2} [\Phi_{\theta_1} + \Phi_{\theta_2}, \Phi_{\theta_3} + \Phi_{\theta_4}] = 6. \end{aligned}$$

Let $\tau \in \Delta \setminus \{\phi, \mu\}$. If τ_N is the sum of two characters in Γ , then $l(b_Z) = 8$ and

$$[\Phi_\phi, \Phi_\tau] = 6 = [\Phi_\mu, \Phi_\tau].$$

If, on the other hand, $\tau_N \in \Gamma$, then also $(\Phi_\tau)_N = \Phi_{\tau_N}$ by [14, Corollary 8.8] and

$$[\Phi_\phi, \Phi_\tau] = [\Phi_\mu, \Phi_\tau] \in \{3, 6\},$$

depending on $l(b_Z)$. In any case, C has the form

$$C = \begin{pmatrix} 9 & 6 & a_1 & \cdots & a_s \\ 6 & 9 & a_1 & \cdots & a_s \\ a_1 & a_1 & * & \cdots & * \\ \vdots & \vdots & \vdots & & \vdots \\ a_s & a_s & * & \cdots & * \end{pmatrix}$$

with $a_1, \dots, a_s \in \{3, 6\}$. By the Gauss algorithm there exist $X, Y \in GL(l(B), \mathbb{Z})$ such that

$$XCY = \begin{pmatrix} 3 & . & . \\ . & 3 & . \\ . & . & * \end{pmatrix}.$$

Since all elementary divisors of C are powers of 3, it follows that $m(3) \geq 2$ as desired. □

LEMMA 2.6. *Let B be a block of a finite group G with defect group $D \cong 3_+^{1+2}$ and fusion system $\mathcal{F} = \mathcal{F}(J_4)$. Then B cannot have Cartan matrix $\begin{pmatrix} 7 & 1 \\ 1 & 4 \end{pmatrix}$.*

PROOF. By way of contradiction, suppose that B has the given Cartan matrix C . Then B has decomposition matrix

$$\begin{pmatrix} 2 & . \\ 1 & . \\ 1 & . \\ . & 1 \\ . & 1 \\ . & 1 \\ 1 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & . \\ 1 & . \\ 1 & . \\ 1 & . \\ 1 & . \\ . & 1 \\ . & 1 \\ . & 1 \\ 1 & 1 \end{pmatrix}.$$

The diagonal of the contribution matrix M^1 is

$$(16, 4, 4, 7, 7, 7, 9) \quad \text{or} \quad (4, 4, 4, 4, 4, 4, 7, 7, 7, 9).$$

It follows that $k_0(B) \in \{6, 9\}$ and $k_1(B) = 1$ (the last row corresponds to the character of height 1). From the Atlas, all nontrivial elements of D are \mathcal{F} -conjugate. Let (z, b_z) be a nontrivial subsection such that $z \in Z := Z(D)$. By [16, Table 1.2], B has inertial quotient SD_{16} . It follows that b_z is a block with defect group D and inertial quotient Q_8 . Moreover, $l(b_z) = k(B) - l(B) \in \{5, 8\}$. The possible Cartan matrices of b_z are given in the proof of Lemma 2.5. The generalised decomposition numbers $d_{\chi\phi}^z$ are Eisenstein integers and can be expressed with respect to the integral basis $1, e^{2\pi i/3}$. According to the action of $N_G(Z, b_z)$ on $\text{IBr}(b_z)$ there are eight possibilities for the ‘fake’ Cartan matrix \tilde{C}_z which are listed explicitly in [1, proof of Lemma 14]. In each case we apply an algorithm of Plesken [15] (implemented in GAP [7]) to determine the feasible ‘fake’ decomposition matrices \tilde{Q}_z . To this end we also take into account that the diagonal of M^z is $(11, 23, 23, 20, 20, 20, 18)$ or $(23, 23, 23, 23, 23, 23, 20, 20, 20, 18)$, since $M^1 + M^z = |D|1_{k(B)}$. It turns out that only two of the eight cases can actually occur. If $k(B) = 7$ then $N_G(Z, b_z)$ has one fixed point in $\text{IBr}(b_z)$, and if $k(B) = 10$ then $N_G(Z, b_z)$ has two fixed points in $\text{IBr}(b_z)$. Hence, in both cases the block B_Z fulfils the assumption of Lemma 2.5. Consequently, $m(3) = m_B(Z, b_Z) = m_{B_Z}(Z, b_Z) > 0$. However, the elementary divisors of C are 1 and 27, a contradiction. \square

PROPOSITION 2.7. *There does not exist a block of a finite group with Cartan matrix $\begin{pmatrix} 7 & 1 \\ 1 & 4 \end{pmatrix}$.*

PROOF. As in Lemma 2.6, any block B with the given Cartan matrix C has a defect group D of order 27. The possible decomposition matrices were also computed in the proof of Lemma 2.6. In particular, $k_0(B) \in \{6, 9\}$, $k_1(B) = 1$ and $k(B) - l(B) \in \{5, 8\}$. By Kessar and Malle [10], D is nonabelian. By [17, Theorem 8.15], D cannot have exponent 9, that is, $D \cong 3_+^{1+2}$. The fusion systems \mathcal{F} on that group were classified in Ruiz and Viruel [16]. As explained before, we regard the inertial quotient E of B as a subgroup of SD_{16} . Let \mathcal{R} be a set of representatives of the \mathcal{F} -conjugacy classes in D . For $1 \neq u \in \mathcal{R}$, we have $l(b_u) \equiv |C_E(u)| \pmod{3}$ by Proposition 2.3 (applied to $\overline{b_u}$ if $u \in Z := Z(D)$). Therefore, the residue of $k(B) - l(B)$ modulo 3 only depends on \mathcal{F} . If D contains \mathcal{F} -essential subgroups, then \mathcal{F} is the fusion system of one of the following groups H :

$$C_3^2 \rtimes \text{SL}(2, 3), \quad C_3^2 \rtimes \text{GL}(2, 3), \quad \text{PSL}(3, 3), \quad \text{PSL}(3, 3).2, \quad {}^2F_4(2)', \quad J_4.$$

The last case was excluded in Lemma 2.6. In the remaining cases we can compare with the principal block of H to derive the contradiction

$$2 \equiv k(B) - l(B) \equiv k(B_0(H)) - l(B_0(H)) \not\equiv 2 \pmod{3}.$$

Hence, there are no \mathcal{F} -essential subgroups, that is, $\mathcal{F} = \mathcal{F}(D \rtimes E)$. Suppose that $E \leq Q_8$. Then $N_G(Z, b_Z) = C_G(Z)$ and $b_Z = B_Z$ has fusion system \mathcal{F} as well. If $E = 1$, then B is nilpotent in contradiction to $l(B) = 2$. Thus, let $E \neq 1$. Let $\overline{B_Z}$ be the block

with defect group D/Z dominated by B_Z . By [1, Theorem 3] and Proposition 2.3, $l(B_Z) = l(\overline{B_Z}) \geq 2$. Since E acts semiregularly on D/Z , the Cartan matrix of $\overline{B_Z}$ has elementary divisors 1 and 9 (see [17, Proposition 1.46]). Hence, 3 is an elementary divisor of the Cartan matrix of B_Z . Since $Z \leq Z(C_G(Z))$, it follows that $m(3) \geq m_B(Z, b_Z) = m_{B_Z}(Z, b_Z) > 0$ by [17, Lemma 1.44], a contradiction.

We are left with the situation $E \not\cong Q_8$. Here, $\mathcal{R} \cap Z = \{1, z\}$. The case $E \cong C_2 \times C_2$ is impossible by a comparison with the principal block of $D \rtimes E$ as above. We summarise the remaining cases (the second column refers to the small groups library in GAP).

E	realising group	$l(b_z)$	$\sum_{u \notin Z(D)} l(b_u)$
C_2	54 : 5	1	$2 + 2 + 1 + 1 + 1$
C_8	216 : 86	4	1
D_8	216 : 87	4	$2 + 2$

In the first case, $k_0(B) = 9$ and there exists $u \in \mathcal{R} \setminus Z$ such that u and u^{-1} are \mathcal{F} -conjugate. Then $l(b_u) = 1$ and the Cartan matrix of b_u is (9). The generalised decomposition matrix Q_u is integral, since $Q_u = Q_{u^{-1}} = \overline{Q_u}$. The only choice up to signs is $Q_u = (\pm 1, \dots, \pm 1, 0)^t$ where the last character has height 1. However, Q_u cannot be orthogonal to the decomposition matrix of B as computed in Lemma 2.6. Next let $E \cong C_8$. Here $k_0(B) = 6$ and the generalised decomposition matrix Q_u has the form $Q_u = (\pm 2, \pm 1, \dots, \pm 1, 0)^t$. More precisely, Q_1 and Q_u can be arranged as

$$(Q_1, Q_u) = \begin{pmatrix} 2 & . & 1 \\ 1 & . & -1 \\ 1 & . & -1 \\ . & 1 & 2 \\ . & 1 & -1 \\ . & 1 & -1 \\ 1 & 1 & . \end{pmatrix}.$$

So we compute the diagonal of the contribution matrix M^z as (8, 20, 20, 8, 17, 17, 18). By [1, Proposition 7] (applied to the dominated block with defect group $C_3 \times C_3$), there exists a basic set Γ for b_z such that the Cartan matrix becomes $3(2 + \delta_{ij})_{i,j=1}^4$ and $N_G(Z, b_Z)$ acts on Γ . There are three such actions. In each case we may compute the ‘fake’ Cartan matrix \tilde{C}_z and apply Plesken’s algorithm. It turns out that none of those cases leads to a valid configuration.

Finally, let $E \cong D_8$ and $u \in \mathcal{R}$ such that $l(b_u) = 2$. We check that u is \mathcal{F} -conjugate to u^{-1} . The Cartan matrix of b_u is $3\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ up to basic sets. Let $U := \langle u \rangle$. If $N_G(U, b_u)$ interchanges the Brauer characters of b_u , then the ‘fake’ Cartan matrix becomes $\tilde{C}_u = \begin{pmatrix} 5 & 1 \\ 1 & 2 \end{pmatrix}$ (see [1, proof of Lemma 14]). But then $k_0(B) \leq 6$, which is not the case. Therefore, $N_G(U, b_u)$ fixes the Brauer characters of b_u and B_U satisfies $l(B_U) = 4$ by Clifford theory. By Lemma 2.4, we conclude that $m(3) \geq m_B(U, b_U) = m_{B_U}(U, b_U) = 1$. This is the final contradiction. □

Since the contribution matrix does not depend on basic sets, the proof shows more generally that $\begin{pmatrix} 7 & 1 \\ 1 & 4 \end{pmatrix}$ cannot be the Cartan matrix of a block with respect to any basic set. This is in contrast to the main result of [12] where the authors showed that $\begin{pmatrix} 5 & 1 \\ 1 & 2 \end{pmatrix}$ is not the Cartan matrix of a block with defect group $C_3 \times C_3$, although a transformation of basic sets results in the Cartan matrix

$$\begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 5 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

of the Frobenius group $C_3^2 \rtimes C_2$.

3. Basic algebras of dimension 13

Suppose that B is a block with basic algebra A of dimension 13 and Cartan matrix C . We discuss the various possibilities for C . If $l(B) = 1$ then $C = (13)$, $p = 13$ and B has defect 1. This is covered by Proposition 2.1. For $l(B) = 2$ we obtain the following possibilities for C up to labelling of the simple modules:

$$\begin{array}{cccccccc} C & \begin{pmatrix} 9 & 1 \\ 1 & 2 \end{pmatrix} & \begin{pmatrix} 8 & 1 \\ 1 & 3 \end{pmatrix} & \begin{pmatrix} 7 & 1 \\ 1 & 4 \end{pmatrix} & \begin{pmatrix} 6 & 1 \\ 1 & 5 \end{pmatrix} & \begin{pmatrix} 7 & 2 \\ 2 & 2 \end{pmatrix} & \begin{pmatrix} 6 & 2 \\ 2 & 3 \end{pmatrix} & \begin{pmatrix} 5 & 2 \\ 2 & 4 \end{pmatrix} & \begin{pmatrix} 4 & 3 \\ 3 & 3 \end{pmatrix} \\ \det C & 17 & 23 & 27 & 29 & 10 & 14 & 16 & 3 \end{array}$$

The determinants 10 and 14 are not prime powers. If $\det C$ is a prime, then the result follows from Proposition 2.1. The remaining cases with $\det C \in \{16, 27\}$ were handled in Proposition 2.2 and Proposition 2.7, respectively.

We now turn to $l(B) = 3$. Up to labelling, the following possibilities may arise:

$$\begin{array}{cccccc} C & \begin{pmatrix} 5 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix} & \begin{pmatrix} 5 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} & \begin{pmatrix} 4 & 1 & 1 \\ 1 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix} & \begin{pmatrix} 4 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix} & \begin{pmatrix} 4 & 0 & 1 \\ 0 & 3 & 1 \\ 1 & 1 & 2 \end{pmatrix} \\ \det C & 16 & 13 & 19 & 18 & 17 \end{array}$$

$$\begin{array}{cccc} C & \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 0 \\ 1 & 0 & 3 \end{pmatrix} & \begin{pmatrix} 3 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} & \begin{pmatrix} 3 & 2 & 0 \\ 2 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} & \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix} \\ \det C & 21 & 7 & 1 & 2 \end{array}$$

The determinants 1, 18 and 21 are impossible and the prime determinants are settled by Proposition 2.1. The remaining case was dealt with in Proposition 2.2.

If $l(B) \geq 4$, then the trace of C is greater than or equal to 8. Since C is symmetric and indecomposable, we need at least six more nonzero entries. But then $\dim A \geq 8 + 6 = 14$.

4. Basic algebras of dimension 14

In this section, B is a block with basic algebra A of dimension 14. Since 14 is not a prime power, $l(B) \geq 2$. In view of Proposition 2.1, we only list the possible Cartan matrices C such that $\det C$ is a prime power, but not a prime:

$$\begin{array}{rccccc}
 C & \begin{pmatrix} 6 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} & \begin{pmatrix} 5 & 1 & 1 \\ 1 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix} & \begin{pmatrix} 4 & 2 & 0 \\ 2 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} & \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 2 \\ 0 & 2 & 2 \end{pmatrix} & \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix} \\
 \det C & 16 & 25 & 4 & 4 & 4
 \end{array}$$

The 2-blocks of defect 2 were classified by Erdmann [5]. The Morita equivalence classes are represented by FD , FA_4 and $B_0(A_5)$. Only the last block did not already appear in Linckelmann’s list. It is easy to check that $B_0(A_5)$ has a basic algebra of dimension 18. The case $\det C = 16$ was dealt with in Proposition 2.2. Now let $\det C = 25$ and $p = 5$. Since $l(B) = 3$ does not divide $p - 1 = 4$, D is elementary abelian of order 25. The decomposition matrix is

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In particular, $k(B) - l(B) = 5$. Let $E \leq GL(2, 5)$ be the inertial quotient of B . Every nontrivial subsection (u, b_u) satisfies $l(b_u) = |C_E(u)|$ by Brauer–Dade theory. In particular, $k(B) - l(B)$ only depends on the action of E on D . An inspection of [1, Theorem 5] shows that $k(B) - l(B) = 5$ never occurs. Hence, this case is impossible as well.

5. The next challenge

While classifying blocks B with basic algebra of dimension 15, only the following Cartan matrices are hard to deal with:

$$\begin{pmatrix} 5 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 6 & 0 & 1 \\ 0 & 3 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

The first matrix belongs to a Brauer tree algebra and could potentially arise from a 13-block of defect 1 (see Proposition 2.1). David Craven has informed me that such a block most likely does not exist (assuming the classification of finite simple groups).

The second matrix leads, once again, to a defect group D of order 27. Moreover, $k(B) = k_0(B) \in \{6, 9\}$. Arguing along the lines of Proposition 2.7, it can be shown with some effort that D is abelian. Now the block is ruled out by Proposition 2.3.

Finally, for basic algebras of dimension 16, a 3×3 Cartan matrix with largest elementary divisor 32 shows up. We made no attempt to say something about such blocks.

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