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Auxiliary superfields in $\mathcal{N} = 1$ supersymmetric self-dual electrodynamics

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Abstract: We construct the general formulation of $\mathcal{N}=1$ supersymmetric self-dual abelian gauge theory involving auxiliary chiral spinor superfields. Self-duality in this context is just $\mathrm{U}(N)$ invariance of the nonlinear interaction of the auxiliary superfields. Focusing on the $\mathrm{U}(1)$ case, we present the most general form of the $\mathrm{U}(1)$ invariant auxiliary interaction, consider a few instructive examples and show how to generate self-dual $\mathcal{N}$=1 models with higher derivatives in this approach.

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Contents

1 Introduction

The study of duality-invariant (or self-dual) models of nonlinear electrodynamics [1, 2]–[11] remains an active subject closely related to various issues of current interest. Recently, the attention increased further, due to the hypothetical crucial role of duality symmetries in the possible ultraviolet finiteness of $\mathcal{N}$=8 supergravity and some of its descendants (see, e.g., [12]–[15]).

Some time ago, two of us [16, 17] developed a new general formulation of U(1) duality-invariant models of nonlinear electrodynamics. This formulation involves, besides the Maxwell gauge-field strength, some auxiliary bispinor fields. The interaction in the full

Lagrangian is constructed solely from these auxiliary fields. The standard nonlinear Lagrangian of any particular duality-invariant system is recovered by eliminating these fields via their equations of motion. The main advantage of the auxiliary bispinor formulation is a linearization of the renowned self-duality condition [1–3] in this setting. Self-duality becomes simply the requirement of off-shell U(1) invariance of the interaction Lagrangian, and the U(1) duality group gets realized by linear transformations of the auxiliary fields. The auxiliary bispinor approach admits an extension to U(N) duality with N copies of the Maxwell field strength [18] as well as to the inclusion of additional scalar coset fields [19].

It was demonstrated in [20] that the so called "deformed twisted self-duality condition" recently proposed and exploited in [15, 21, 22] is in fact equivalent to the basic algebraic equation for the bispinor auxiliary fields in the formulation of [16, 17].

As suggested in [16, 17, 20], an obvious next step was to supersymmetrize the auxiliary bispinor formulation, i.e. to extend it to self-dual nonlinear $\mathcal{N}$=1 and $\mathcal{N}$=2 supergauge theories, e.g. starting from the by now standard approach of [9, 10]. This step was recently accomplished by Kuzenko [23], who showed, in particular, that the auxiliary bispinor field gets enhanced to a chiral fermionic $\mathcal{N}$=1 superfield. In this language, the U(1) duality amounts to manifest U(1) invariance of the auxiliary superfield interaction.

The purpose of the present paper is to introduce a framework which is more general than the one given in [23], concentrating on the rigid $\mathcal{N}$=1 case. Our invariant superfield density E is a function of three U(1) invariant scalar superfield variables composed of auxiliary chiral superfields U_α. By analogy with [17, 20] we also construct alternative formulations of $\mathcal{N}$=1 self-dual theories which in addition make use of auxiliary scalar superfields. Another novelty of our paper is a generalization of the $\mathcal{N}$=1 duality formulation with auxiliary superfields from the abelian situation to the case of U(N) duality.

The structure of the paper is as follows. Section 2 recapitulates the basic features of self-dual $\mathcal{N}$=1 nonlinear electrodynamics in the standard formulation. In section 3 we outline the general formulation of $\mathcal{N}$=1 supersymmetric U(1) self-dual gauge theories, employing the auxiliary chiral (but otherwise unconstrained) spinor superfield U_α in parallel with the ordinary chiral superfield strength W_α. We analyze the equation of motion for $U_\alpha(W, \bar{W})$ using a three-parametric U(1) invariant superfield density E which is most general in the case without higher derivatives. This equation is just $\mathcal{N}$=1 counterpart of the equation for the auxiliary bispinor fields of the bosonic self-dual case, which was recently rediscovered as the deformed twisted self-duality condition. By eliminating the auxiliary superfield by a recursive procedure in terms of the ordinary covariant superfield strengths $W_\alpha, \bar{W}_{\dot\alpha}$ we recover the standard representation of the general supersymmetric U(1) self-dual theory. In some particular parametrization, the interaction E depends only on a single real variable, in a more direct analogy with the bosonic U(1) self-dual theories [17] (this case was treated in [23]). In section 4 we present an alternative self-dual "M representation" which makes use of an additional scalar superfield M. It is a supersymmetric extension of the so called "μ representation" of the bosonic case [17, 20]. Like its bosonic prototype, the M representation is capable of essentially simplifying the calculations. Examples of U(1) self-dual theories are studied in section 5. We translate to our formulations the renowned $\mathcal{N}$=1 Born-Infeld theory as well as construct a new self-dual model specified by a simple

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quartic interaction of the auxiliary superfields. It is the $\mathcal{N}$=1 extension of the quartic auxiliary bosonic interaction considered firstly in [16, 17] and recently discussed in [15, 21]. We present, as a perturbative expansion, the relevant superfield actions in terms of the ordinary superfield strengths. We also show how to adjust our approach for constructing supersymmetric self-dual models containing higher derivatives and give a few examples of such models. The bosonic limit of the formulation with auxiliary spinor superfields is discussed in section 6. We give the bosonic component actions for a few examples, including those with higher derivatives. A brief account of the generalization of the new self-duality setting to the U(N) case is the subject of section 7. We present several examples of U(N) self-dual models which correspond to some particular choices of the invariant auxiliary interaction.

2 Nonlinear $\mathcal{N}$=1 electrodynamics

Here we fix our notations and sketch the superfield formalism of the $\mathcal{N}$=1 self-dual theories [6, 7, 9, 10].

Our conventions are the same as, e.g., in the book [24] and in refs. [9, 10]. We parametrize the $\mathcal{N}$=1, D=4 superspace by the coordinates

$$z = (x^m, \theta^\alpha, \bar\theta^{\dot\alpha}), \quad \theta^2 = \theta^\alpha\theta_\alpha, \quad \bar\theta^2 = \bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha} \tag{2.1}$$

and define the covariant spinor derivatives as

$$\begin{aligned} &D_\alpha = \partial_\alpha + i\bar\theta^{\dot\alpha}(\sigma^m)_{\alpha\dot\alpha}\partial_m, && D^2 = D^\alpha D_\alpha, \\ &\bar D_{\dot\alpha} = -\bar\partial_{\dot\alpha} - i\theta^\alpha(\sigma^m)_{\alpha\dot\alpha}\partial_m, && \bar D^2 = \bar D_{\dot\alpha}\bar D^{\dot\alpha}\,. \end{aligned} \tag{2.2}$$

Here α and $\dot\alpha$ are the doublet SL$(2,\mathcal{C})$ indices and $m = 0,1,2,3$ (we use the flat Minkowski metric $\eta_{mn} = \mathrm{diag}(1,-1,-1,-1)$).

The Grassmann integrals are normalized as

$$\int d^2\theta\theta^2 = -\frac{1}{4}D^2\theta^2 = 1\,, \qquad \int d^2\bar\theta\bar\theta^2 = -\frac{1}{4}\bar D^2\bar\theta^2 = 1\,, \tag{2.3}$$

$$\int d^4x d^2\theta \equiv \int d^6\zeta\,, \qquad \int d^4x d^2\theta d^2\bar\theta \equiv \int d^8z. \tag{2.4}$$

The (anti)chiral Abelian superfield strengths are defined by

$$W_\alpha = -\frac{1}{4}\bar D^2 A_\alpha = -\frac{1}{4}\bar D^2 D_\alpha V\,, \quad \bar W_{\dot\alpha} = -\frac{1}{4}D^2\bar A_{\dot\alpha} = -\frac{1}{4}D^2\bar D_{\dot\alpha}V\,, \tag{2.5}$$

where V is the gauge prepotential and $A_\alpha = D_\alpha V$, $\bar A_{\dot\alpha} = \bar D_{\dot\alpha}V$ are spinor gauge connections. The superfield strengths satisfy, besides the chirality conditions

$$\bar D_{\dot\alpha}W_\beta = 0\,, \quad D_\alpha\bar W_{\dot\alpha} = 0\,, \tag{2.6}$$

also the Bianchi identity:

$$B(W,\bar W) \equiv D^\alpha W_\alpha - \bar D_{\dot\alpha}\bar W^{\dot\alpha} = 0\ . \tag{2.7}$$

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For what follows, it will be useful to tabulate the R invariance properties of various $\mathcal{N}=1$ quantities:

$$
\begin{aligned}
R(\theta^\alpha) &= 1, \quad R(\bar\theta^{\dot\alpha}) = -1, \quad R(D_\alpha) = -1, \quad R(\bar D_{\dot\alpha}) = 1, \\
R(W_\alpha) &= 1, \quad R(\bar W_{\dot\alpha}) = -1.
\end{aligned} \tag{2.8}
$$

The "engineering" dimensions of the basic objects of the $\mathcal{N}=1$ gauge theory are (in the mass units):

$$
[V] = 0\,, \ [W_\alpha] = 3/2\,, \ [D_\beta W_\alpha] = 2\,, \tag{2.9}
$$

and the free superfield action is written as

$$
S_2(W, \bar W) = \frac{1}{4}\int d^6\zeta W^2 + \frac{1}{4}\int d^6\bar\zeta \bar W^2\,, \tag{2.10}
$$

where $W^2 \equiv W^\alpha W_\alpha$ and $\bar W^2 \equiv \bar W_{\dot\alpha}\bar W^{\dot\alpha}$.

In the nonlinear theory with one dimensionful constant f ($[f] = -2$), it is convenient to ascribe nonstandard dimensions to the basic objects,

$$
[V] = -2\,, \qquad [W_\alpha] = -1/2\,, \qquad [D_\alpha W_\beta] = 0\,, \tag{2.11}
$$

and to construct the nonlinear action as

$$
S = f^{-2}[S_2(W, \bar W) + S_{\text{int}}(W, \bar W)]\,,
$$

where

$$
S_{\text{int}}(W, \bar W) = \frac{1}{4}\int d^8z\, \mathcal{L}_{\text{int}}(W, \bar W)\,. \tag{2.12}
$$

For convenience, we will put $f = 1$ altogether.

We consider the following form of the arbitrary nonlinear interaction in the W representation:

$$
\mathcal{L}_{\text{int}} = W^2\bar W^2 \Lambda(w, \bar w, y, \bar y)\,, \tag{2.13}
$$

where the superfield density Λ depends on the dimensionless R-invariant variables

$$
w = \frac{1}{8}\bar D^2 \bar W^2\,, \quad \bar w = \frac{1}{8}D^2 W^2\,, \qquad y \equiv D^\alpha W_\alpha\,. \tag{2.14}
$$

A wide subclass of nonlinear models (including $\mathcal{N}=1$ super Born-Infeld theory) is associated with the y-independent densities $\Lambda(w, \bar w)$. In [9, 10] just this set of models was mainly addressed.

Let us define

$$
M_\alpha \equiv -2i\frac{\delta S}{\delta W^\alpha}, \qquad \bar M_{\dot\alpha} \equiv 2i\frac{\delta S}{\delta \bar W^{\dot\alpha}}. \tag{2.15}
$$

Then the nonlinear equations of motion can be written in the form[1]

$$
N(W, \bar W) \equiv D^\alpha M_\alpha - \bar D_{\dot\alpha}\bar M^{\dot\alpha} = 0\,. \tag{2.16}
$$

[1]While computing the variations of the action, one should treat $W_\alpha, \bar W_{\dot\alpha}$ as unconstrained chiral superfields which are not subjected to the Bianchi identity (2.7). Correspondingly, the variables y and $\bar y$ are considered as independent. The Bianchi identity $y = \bar y$ is imposed *à posteriori*. The equations of motion (2.16) expressed through the so defined M_α and $\bar M_{\dot\alpha}$ coincide with those derived by varying S with respect to the prepotential V.

It is straightforward to find the explicit expression for the chiral superfield M_α

$$M_\alpha = -iW_\alpha \left[1 - \frac{1}{4}\bar{D}^2 \left\{\bar{W}^2 \left[\Lambda + \frac{1}{8}D^2(W^2\Lambda_{\bar{w}})\right]\right\}\right] - \frac{i}{8}\bar{D}^2 \left[\bar{W}^2 D_\alpha(W^2\Lambda_y)\right]. \quad (2.17)$$

The $O(2)$ duality transformations defined as

$$\delta W_\alpha = \omega M_\alpha(W, \bar{W}), \quad \delta M_\alpha = -\omega W_\alpha, \quad (2.18)$$

mix the equation of motion (2.16) with the Bianchi identity (2.7), leaving their set covariant. The $O(2)$ self-duality constraint ensuring the compatibility of (2.18) with the expression (2.21) for M_α and generalizing the bosonic Gaillard-Zumino condition has, in the present case, the integral form [9, 10]:

$$\mathrm{Im}\, K(W, \bar{W}) = 0, \quad (2.19)$$

$$K(W, \bar{W}) := -\int d^6\zeta\, (W^2 + M^2). \quad (2.20)$$

The funcitonal $K(W, \bar{W})$ in itself is invariant under (2.18). In view of the nilpotency property $W_\alpha W_\beta W_\gamma = 0$, for calculating $K(W, \bar{W})$ it is sufficient to know M_α in (2.21) only up to terms linear in W_α:

$$M_\alpha| = -iW_\alpha \left[1 - \frac{1}{4}\bar{D}^2 \left(\bar{W}^2\, \Gamma\right)\right] + \frac{i}{4} W^\beta \bar{D}^2 \left(\bar{W}^2 D_\alpha W_\beta \Lambda_y\right), \quad (2.21)$$

where

$$\Gamma = \Lambda + \bar{w}\Lambda_{\bar{w}} = \frac{\partial(\bar{w}\Lambda)}{\partial \bar{w}}. \quad (2.22)$$

Then, using once more the nilpotency property, this time for $\bar{W}_{\dot{\alpha}}$, we can write the functional $K(W, \bar{W})$ as an integral over the full $\mathcal{N} = 1$ superspace

$$K(W, \bar{W}) = 2\int d^8z\, W^2\bar{W}^2 \left[\Gamma - w\Gamma^2 + 2w\bar{w}\,(\Lambda_y)^2\right]. \quad (2.23)$$

All relations are simplified if $\Lambda_y = 0$. In this case

$$K(W, \bar{W}) = 2\int d^8z\, W^2\bar{W}^2\, (\Gamma - w\Gamma^2). \quad (2.24)$$

The notorious example of the $\mathcal{N}$=1 self-dual system with $\Lambda_y = 0$ is the $\mathcal{N}$=1 generalization of the BI theory [6, 7]. The function Λ in this case is known in a closed form:

$$\begin{aligned} \Lambda_{BI} &= \left[1 + \frac{1}{2}(w + \bar{w}) + \sqrt{1 + (w + \bar{w}) + \frac{1}{4}(w - \bar{w})^2}\right]^{-1} \\ &= \frac{1}{2} - \frac{1}{4}(w + \bar{w}) + \frac{1}{8}(w + \bar{w})^2 + \frac{1}{8}w\bar{w} + \ldots \end{aligned} \quad (2.25)$$

By some rather tedious work one can check the validity of the self-duality condition (2.19) for Λ_{BI}.

The $\mathcal{N}$=1 BI action can be also written in the concise form as

$$S_{BI} = \frac{1}{4}\int d^6\zeta X + \text{c.c.}\,, \tag{2.26}$$

where the auxiliary chiral superfield X satisfies the quadratic constraint [7]

$$X + \frac{1}{16}X\bar{D}^2\bar{X} = W^2. \tag{2.27}$$

The action S_{BI} is also distinguished in that it is invariant under the nonlinearly realized second $\mathcal{N}$=1 supersymmetry which extends the manifest $\mathcal{N}$=1 supersymmetry to $\mathcal{N}$=2 [7, 8].

Another possible parametrization of the interaction density in (2.13) is through the variables

$$w' = w + \frac{1}{8}\bar{y}^2\,,\ \bar{w}' = \bar{w} + \frac{1}{8}y^2\,,\ y, \bar{y}\,, \tag{2.28}$$

so that

$$\Lambda(w, \bar{w}, y, \bar{y}) = \Lambda'(w', \bar{w}', y, \bar{y})\,. \tag{2.29}$$

The invariant functional (2.23) and the self-duality condition can be rewritten in the new parametrization with the help of the relations

$$\frac{\partial\Lambda}{\partial w} = \frac{\partial\Lambda'}{\partial w'},\quad \frac{\partial\Lambda}{\partial y} = \frac{\partial\Lambda'}{\partial y} + \frac{1}{4}y\frac{\partial\Lambda'}{\partial\bar{w}'}\,. \tag{2.30}$$

3 $\mathcal{N} = 1$ self-duality with auxiliary chiral spinors

3.1 New representation for nonlinear $\mathcal{N} = 1$ electrodynamics: general setting

We introduce the auxiliary chiral spinor superfield U_α and construct the following quadratic action:

$$S_2(W, U) = \int d^6\zeta \left(UW - \frac{1}{2}U^2 - \frac{1}{4}W^2\right) + \text{c.c.}\,, \tag{3.1}$$

where $U^2 = U^\alpha U_\alpha\,, \bar{U}^2 = \bar{U}_{\dot\alpha}\bar{U}^{\dot\alpha}$. Integrating out the auxiliary superfield,

$$U_\alpha = W_\alpha\,, \tag{3.2}$$

we reproduce the standard $\mathcal{N}$=1 quadratic action (2.10). The action (3.1) is $\mathcal{N}$=1 analog of the free Maxwell action rewritten through the auxiliary bispinor fields [16, 17]. This modified Maxwell action is just the bosonic core of (3.1) (modulo auxiliary fields vanishing on shell, see section 6).

The full set of equations of motion associated with (3.1) (including (3.2)), together with the Bianchi identity (2.7), is covariant under the following U(1) duality transformations

$$\delta U_\alpha = -i\omega U_\alpha\,,\quad \delta W_\alpha = i\omega(W_\alpha - 2U_\alpha) \equiv \omega M_\alpha(U, W),\quad \text{and c.c.}\,, \tag{3.3}$$

$$\delta(W_\alpha - U_\alpha) = i\omega(W_\alpha - U_\alpha),\quad \text{and c.c.}\,. \tag{3.4}$$

Though in the free case, after substituting (3.2), $M_\alpha(U,W)$ becomes just iW_α, we will assume that the same U(1) transformations (3.3) act as the duality ones in the general interaction case too, when (3.2) is replaced by a nonlinear equation and $M_\alpha(U,W) = i(W_\alpha - 2U_\alpha)$ becomes a nontrivial functional of $W_\alpha, \bar{W}_{\dot\alpha}$.

The most general interaction of the auxiliary superfield, before imposing any self-duality constraint, can be chosen as

$$S_E(U) = \frac{1}{4}\int d^8z\, U^2\bar{U}^2\, E(u,\bar{u},g,\bar{g})\,, \tag{3.5}$$

where E is an arbitrary real function of the dimensionless Lorentz invariant superfield variables

$$u = \frac{1}{8}\bar{D}^2\bar{U}^2, \quad \bar{u} = \frac{1}{8}D^2U^2, \quad g = D^\alpha U_\alpha, \quad \bar{g} = \bar{D}_{\dot\alpha}\bar{U}^{\dot\alpha}. \tag{3.6}$$

So the total action is

$$S_{\rm tot} = S_2(W,U) + S_E(U)\,. \tag{3.7}$$

The counterpart of the free auxiliary equation (3.2) is obtained by varying (3.7) with respect to U_α

$$W_\alpha - U_\alpha = -\frac{\delta S_E}{\delta U^\alpha} = \frac{1}{8}U_\alpha\bar{D}^2\left\{\bar{U}^2\left[E + \frac{1}{8}D^2(U^2E_{\bar{u}})\right]\right\} - \frac{1}{16}\bar{D}^2[\bar{U}^2D_\alpha(U^2E_g)]\,. \tag{3.8}$$

Varying the full action with respect to the prepotential V, we obtain the dynamical equation

$$D^\alpha(W_\alpha - 2U_\alpha) + \bar{D}_{\dot\alpha}(\bar{W}^{\dot\alpha} - 2\bar{U}^{\dot\alpha}) = 0\,. \tag{3.9}$$

Comparing it with (2.16), we identify, as in the free case,

$$W_\alpha - 2U_\alpha = -iM_\alpha(W,U)\,, \quad \bar{W}_{\dot\alpha} - 2\bar{U}_{\dot\alpha} = i\bar{M}_{\dot\alpha}(W,U)\,. \tag{3.10}$$

Using the algebraic equation (3.8) and substituting its solution $U_\alpha = U_\alpha(W,\bar{W})$ into (3.9), we reproduce the initial form of the nonlinear equation of motion (2.16).

Note that in practice, while solving (3.8) to restore the $W,\bar{W}$ representation of the total superfield action $S_{\rm tot}$, it is enough to consider an effective form of (3.8), in which only terms linear in U_α are kept

$$W_\alpha| = U_\alpha + \frac{1}{8}U_\alpha\bar{D}^2\,[\bar{U}^2(E + \bar{u}E_{\bar{u}})] + \frac{1}{8}U^\beta\bar{D}^2(\bar{U}^2D_\alpha U_\beta E_g). \tag{3.11}$$

This is due to the property that in (3.1) all interaction terms appear at least with the factor W^2 (or with $\bar{W}^2$ in the complex conjugated part), while in (3.5) with the factor $W^2\bar{W}^2$. It will be also useful to rewrite (3.11) as

$$W_\alpha| = U_\alpha + C_\alpha^\beta U_\beta\,, \quad C_\alpha^\beta = \frac{1}{8}\bar{D}^2(\bar{U}^2R_\alpha^\beta)\,, \quad R_\alpha^\beta := [\delta_\alpha^\beta(E + \bar{u}E_{\bar{u}}) - D_\alpha U^\beta\, E_g]\,. \tag{3.12}$$

Note that, up to the nilpotent terms $\sim \bar{U}_{\dot\alpha}$,

$$C_\alpha^\beta = uR_\alpha^\beta + O(\bar{U})\,, \tag{3.13}$$

so the matrix C_α^β, when multiplied by $\bar{U}^2$, is reduced to its "effective" form uR_α^β.

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3.2 Relation to the original formulation

The general representation for the perturbative solution of the eq. (3.11), under the assumption that all nilpotent terms can be ignored, has the form

$$U_\alpha(W,\bar{W}) \approx [\delta_\alpha^\beta + \mathcal{B}_\alpha^\beta(W,\bar{W})]W_\beta\,. \tag{3.14}$$

The chiral dimensionless matrix function $\mathcal{B}_\alpha^\beta(W,\bar{W})$ satisfies the relation

$$(\delta_\alpha^\beta + \mathcal{B}_\alpha^\beta)(\delta_\beta^\gamma + C_\beta^\gamma) = \delta_\alpha^\gamma. \tag{3.15}$$

It can be parametrized by two superfields $G(W,\bar{W})$ and $P(W,\bar{W})$

$$\mathcal{B}_\alpha^\beta \approx u[G\,\delta_\alpha^\beta + P\,D_\alpha W^\beta]\,. \tag{3.16}$$

This representation is analogous to the one used in the bosonic case [17]

$$V_{\alpha\beta} = G(F^2)F_{\alpha\beta}, \quad G(F^2) = \frac{1}{2} - \frac{\partial L}{\partial(F^2)} = \left[1 + \frac{\partial E}{\partial(V^2)}\right]^{-1}, \tag{3.17}$$

where $L(F)$ is the nonlinear Maxwell Lagrangian, and $E(V^2,\bar{V}^2)$ is the auxiliary interaction.

Expressing U_α through M_α from (3.10), we can directly compare the solution (3.14) with the formula (2.21) in the original W representation

$$U_\alpha = \frac{1}{2}W_\alpha + \frac{i}{2}M_\alpha = W_\alpha\left[1 - \frac{1}{8}\bar{D}^2[\bar{W}^2(\Lambda + \bar{w}\Lambda_{\bar{w}})]\right] + \frac{1}{8}W_\beta\bar{D}^2(\bar{W}^2 D_\alpha W^\beta \Lambda_y)\,. \tag{3.18}$$

Thus derivatives of the interaction density Λ in the W representation can be found directly from the solution (3.14) in analogy with the bosonic formalism. Examples of such calculations are considered in section 4.

Now let us derive the relation between the interaction functions in the W and W,U representations, i.e. between $\Lambda(w,\bar{w},y,\bar{y})$ and $E(u,\bar{u},g,\bar{g})$.

As the first step, we use eqs. (3.13) to find the effective relation between W^2 and U^2,

$$\begin{aligned} W^2| &= U^2\left[1 + C_\sigma^\sigma - \frac{1}{2}C_\rho^\sigma C_\sigma^\rho + \frac{1}{2}C_\sigma^\sigma C_\rho^\rho\right] \\ &= U^2\left\{1 + \frac{1}{8}\bar{D}^2\left[\bar{U}^2\left(R_\sigma^\sigma - \frac{1}{2}C_\rho^\sigma\,R_\sigma^\rho + \frac{1}{2}C_\sigma^\sigma\,R_\rho^\rho\right)\right]\right\}. \end{aligned} \tag{3.19}$$

This expression, modulo the nilpotent terms $\sim \bar{U}_{\dot{\alpha}}$, can be rewritten as

$$W^2| = U^2\,H\,, \quad H = [1 + u(E + \bar{u}E_{\bar{u}})]^2 + [1 + u(E + \bar{u}E_{\bar{u}})]ugE_g - 2\bar{u}u^2E_g^2\,. \tag{3.20}$$

It allows to find the exact relation between the 4-th order nilpotent terms

$$W^2\bar{W}^2 = U^2\bar{U}^2\,H(u,\bar{u},g,\bar{g})\bar{H}(u,\bar{u},g,\bar{g})\,. \tag{3.21}$$

As the next step, we represent $S_2(W,U)$ (3.1) as

$$S_2(W,U) = \int d^6\zeta\left[\frac{1}{4}W^2 - \frac{1}{2}(U - W)^2\right] + \text{c.c.}\,.$$

Using eq. (3.8) and the relations

$$D_\alpha U^\beta \, D^\alpha U_\beta \, U^2 = 4\bar{u} \, U^2 \ , \quad D_\alpha U^\beta \, D_\beta U^\alpha \, U^2 = (4\bar{u} + g^2) \, U^2 \, , \tag{3.22}$$

we find

$$(W-U)^2 = \frac{1}{2} U^2 (C^\sigma_\sigma C^\rho_\rho - C^\sigma_\rho C^\rho_\sigma), \tag{3.23}$$

and

$$\int d^6\zeta \, (U-W)^2 = -\frac{1}{2} \int d^8 z U^2 \bar{U}^2 T(u, \bar{u}, g, \bar{g}), \tag{3.24}$$

where

$$T(u, \bar{u}, g, \bar{g}) = u(E + \bar{u}E_{\bar{u}})^2 + ugE_g(E + \bar{u}E_{\bar{u}}) - 2u\bar{u}E_g^2. \tag{3.25}$$

Finally, we equate the actions $S_{\rm tot}$ in the W and W, U representations and obtain

$$\Lambda = \frac{1}{H\bar{H}} \Big(E + T + \bar{T} \Big) \, , \tag{3.26}$$

where H and T are the nonlinear combinations of $E, E_u, E_{\bar{u}}, E_g$ and $E_{\bar{g}}$ defined in eqs. (3.20) and (3.25).

To find the explicit form of Λ we also need to express the superfield arguments $u, \bar{u}, d, \bar{d}$ in the r.h.s. of (3.26) in terms of the original variables $w, \bar{w}, y, \bar{y}$. Because Λ appears in the action with the factor $W^2 \bar{W}^2$, we can use the effective version of the equations relating these two sets of variables

$$w| = u \, \bar{H} \, , \qquad \bar{w}| = \bar{u} \, H \, , \tag{3.27}$$

$$\begin{aligned} D_\beta W_\alpha| &= D_\beta U_\rho (\delta^\rho_\alpha + u R^\rho_\alpha), \\ y| &= g + u \, D^\alpha U_\rho R^\rho_\alpha = g + 2ug(E + \bar{u}E_{\bar{u}}) - 4u\bar{u}E_g \, . \end{aligned} \tag{3.28}$$

When solving these effective equations, one can exploit the relations analogous to (3.22)

$$D_\alpha W^\beta D^\alpha W_\beta \, W^2 = 4\bar{w} \, W^2 \ , \quad D_\alpha W^\beta \, D_\beta W^\alpha \, W^2 = (4\bar{w} + y^2) \, W^2 \, . \tag{3.29}$$

3.3 $\mathcal{N} = 1$ self-duality condition in the (W, U) representation

Now we assume that the U(1) duality transformations (3.3), (3.4) retain their form in the case with interaction (by analogy with the bosonic case) and wish to learn how the supersymmetric U(1) duality constraint (2.19), (2.20) looks in the formulation with the auxiliary spinor superfields.

The U(1) invariant (2.20) in the formulation considered becomes

$$K = -\int d^6\zeta (W^\alpha W_\alpha + M^\alpha M_\alpha) = -4 \int d^6\zeta [U^\alpha (W_\alpha - U_\alpha)] = -4 \int d^6\zeta U^\alpha \frac{\delta S_E}{\delta U^\alpha} \, . \tag{3.30}$$

Thus the supersymmetric U(1) self-duality constraint (2.19) in this formulation is none other than the condition of U(1) invariance of the auxiliary interaction (3.5):

$$K - \bar{K} = 0 \ \rightarrow \ \delta_\omega S_E = 0 \, . \tag{3.31}$$

Now it is easy to check that, for such U(1) invariant self-interactions S_E, the equation (3.10), together with the dynamical equation (3.9) and the Bianchi identities (2.7), are covariant under the transformations (3.3), (3.4).

We conclude that the whole family of self-dual models of the nonlinear $\mathcal{N}=1$ supersymmetric electrodynamics is parametrized by U(1) invariant superfunction $E_{\rm inv}(u, \bar{u}, g, \bar{g})$.

From the variables $u, \bar{u}, g, \bar{g}$ one can construct four U(1) invariants,

$$A := u\bar{u}\,, \quad C := g\bar{g}\,, \quad B := ug^2\,, \quad \bar{B} := \bar{u}\bar{g}^2\,, \tag{3.32}$$

which are connected by the relation

$$B\bar{B} = AC^2\,. \tag{3.33}$$

We are interested in such interactions which are analytic at the point $A = C = B = 0$ and so admit power series expansion in these U(1) invariant variables. With making use of the relation (3.33), it is easy to show that the most general U(1) invariant self-interaction of this type is given by the following ansatz

$$E_{\rm inv} = \mathcal{F}(B, A, C) + \bar{\mathcal{F}}(\bar{B}, A, C)\,, \tag{3.34}$$

where $\mathcal{F}(B, A, C)$ is an arbitrary analytic function. This is in contrast with the pure bosonic U(1) self-dual systems which are parametrized by a real function $\mathcal{E}$ which depends on only one real variable $a = V^2\bar{V}^2$, $V^2 := V^{\alpha\beta}V_{\alpha\beta}$, $\bar{V}^2 := \bar{V}^{\dot{\alpha}\dot{\beta}}V_{\dot{\alpha}\dot{\beta}}$ [16]. The expansions of $\mathcal{F}(B, A, C)$ and $\bar{\mathcal{F}}(\bar{B}, A, C)$ as formal series with constant coefficients look like

$$\begin{aligned} \mathcal{F} &= e_1 + e_2 A + f_1 C + f_2 C^2 + h_1 B + O(U^6)\,, \\ \bar{\mathcal{F}} &= e_1 + e_2 A + f_1 C + f_2 C^2 + \bar{h}_1 \bar{B} + O(U^6)\,. \end{aligned} \tag{3.35}$$

The derivatives of $E_{\rm inv}$ can be rewritten through the independent ones as follows[2]

$$\begin{aligned} E_{\bar{u}} &= u\mathcal{F}_A + u\bar{\mathcal{F}}_A + \bar{g}^2\bar{\mathcal{F}}_{\bar{B}}, & \bar{u}E_{\bar{u}} &= AE_A + \bar{B}\bar{\mathcal{F}}_{\bar{B}}, \\ E_g &= \bar{g}\mathcal{F}_C + \bar{g}\bar{\mathcal{F}}_C + 2ug\mathcal{F}_B, & gE_g &= CE_C + 2B\mathcal{F}_B. \end{aligned} \tag{3.36}$$

Using these formulas, all the general relations and quantities, including the matrices C_α^β and R_α^β defined in (3.12), can be easily specialized to the U(1) invariant case. In particular, the relation (3.26) for the duality-invariant case is obtained by replacing there $E \to E_{\rm inv} = \mathcal{F} + \bar{\mathcal{F}}$ and expressing the derivatives with respect to $u, \bar{u}, g, \bar{g}$ in terms of $\mathcal{F}_A, \mathcal{F}_C, \mathcal{F}_B$ (and their complex-conjugates) according to the formulas (3.36).

We also note that sometimes it is more convenient to use the equivalent set of the superfield variables (cf. eq. (2.28))

$$u' = u + \frac{1}{8}\bar{g}^2\,, \quad \bar{u}' = \bar{u} + \frac{1}{8}g^2\,, \quad g\,, \quad \bar{g}\,, \tag{3.37}$$

and, respectively,

$$E(u, g) = E'(u', g), \quad \frac{\partial E}{\partial u} = \frac{\partial E'}{\partial u'}, \quad \frac{\partial E}{\partial g} = \frac{\partial E'}{\partial g} + \frac{1}{4}g\frac{\partial E'}{\partial \bar{u}'}\,. \tag{3.38}$$

[2]For brevity, we omit the index 'inv' on E.

The new invariant variables in the self-dual theories have the form

$$A' = u'\bar{u}' \,, \quad C = g\bar{g} \,, \quad B' = u' g^2 \,, \quad \bar{B}' = \bar{u}' \bar{g}^2 \,. \tag{3.39}$$

They have an advantage of possessing a simpler component expansion.

3.4 A particular subclass of $\mathcal{N} = 1$ self-dual interactions

Now we consider the particular choice of the U(1) invariant interaction $E(A)$ involving only one real superfield variable

$$A = u\bar{u} = \frac{1}{64}(D^2U^2)(\bar{D}^2\bar{U}^2). \tag{3.40}$$

Just this case was treated in [23]. The auxiliary equation (3.8) is reduced to

$$W_\alpha - U_\alpha = \frac{1}{8}U_\alpha \bar{D}^2 \left\{ \bar{U}^2 \left[E + \frac{1}{8}D^2 \left(U^2 u\, E_A \right) \right] \right\} \tag{3.41}$$

and the effective equations take the form

$$W_\alpha| = U_\alpha[1 + u\mathcal{P}(A)], \qquad \mathcal{P}(A) = \frac{d}{dA}(AE) = E + AE_A, \tag{3.42}$$

$$\bar{w}| = \bar{u}[1 + u\mathcal{P}(A)]^2 \,, \qquad w| = u[1 + \bar{u}\mathcal{P}(A)]^2 \,. \tag{3.43}$$

These effective equations are analogous to the bosonic equations of ref. [17]

$$F_{\alpha\beta} = V_{\alpha\beta}(1 + \bar{V}^2\mathcal{E}_a), \quad F^2 = V^2(1 + \bar{V}^2\mathcal{E}_a)^2 \,, \tag{3.44}$$

where $\mathcal{E} = \mathcal{E}(a) \,, \; a = V^2\bar{V}^2$. The similarity is based on the formal correspondence

$$u \leftrightarrow \bar{V}^2, \quad A \leftrightarrow a, \quad \mathcal{P}(A) \leftrightarrow \mathcal{E}_a, \tag{3.45}$$

which can in fact be trusted by the component consideration (see section 6).

The relation (3.26) in the present case reads

$$\Lambda = \frac{E + (u + \bar{u})\mathcal{P}^2(A)}{[1 + u\mathcal{P}(A)]^2[1 + \bar{u}\mathcal{P}(A)]^2} \,. \tag{3.46}$$

It is instructive to give few first terms in the power series expansion of Λ in the W representation, starting from $\mathcal{P}(A) = e_1 + e_2A + \dots \,,\; E(A) = e_1 + \frac{1}{2}e_2A + \dots .$

For U_α we obtain the recursive equation

$$\begin{aligned} U_\alpha &= W_\alpha \frac{1}{1 + u\mathcal{P}} = W_\alpha G(w, \bar{w}) \\ &= W_\alpha[1 - e_1u - e_2u^2\bar{u} + e_1^2u^2 - e_1^3u^3 + O(U^8)], \end{aligned} \tag{3.47}$$

which implies

$$U_\alpha = W_\alpha \left[1 - e_1w + e_1^2(w^2 + 2w\bar{w}) - e_1^3(w^3 + 3w\bar{w}^2 + 8w^2\bar{w}) - e_2w^2\bar{w} + O(W^8)\right] \tag{3.48}$$

and

$$u = w - 2e_1 w\bar{w} + e_1^2\, w\bar{w}(4w + 3\bar{w}) + O(W^8)\,. \tag{3.49}$$

For Λ we have the following $(u, \bar{u})$ expansion

$$\Lambda = e_1 - e_1^2(u + \bar{u}) + \left[e_1^3(u^2 + \bar{u}^2) + \frac{1}{2}e_2 u\bar{u}\right] + (e_1^4 - e_1 e_2)(u\bar{u}^2 + u^2\bar{u})$$
$$- e_1^4(u^3 + \bar{u}^3) + O(U^8)\,, \tag{3.50}$$

which, after substituting (3.49), yields

$$\Lambda = e_1 - e_1^2(w + \bar{w}) + \left(4e_1^3 + \frac{1}{2}e_2\right) w\bar{w} + e_1^3(w^2 + \bar{w}^2)$$
$$- 2(e_1 e_2 + 5e_1^4)(w\bar{w}^2 + w^2\bar{w}) - e_1^4(w^3 + \bar{w}^3) + O(W^8)\,. \tag{3.51}$$

This perturbative solution for Λ, together with the expression (3.48), nicely agree with the effective form of eq. (3.18) for the considered case,

$$U_\alpha| = W_\alpha\left[1 - w(\Lambda + \bar{w}\Lambda_{\bar{w}})\right]. \tag{3.52}$$

4 Alternative auxiliary superfield representation

Here we construct $\mathcal{N}$=1 analog of the so called μ representation of the bosonic case. We term it "M representation". It seemingly exists only for the subclass of self-dual theories considered in [23].

Besides the chiral spinor superfields W_α and U_α we introduce a complex general scalar $\mathcal{N} = 1$ superfield M and construct the following "master" action

$$S_{\rm mast} = S_2(W, U) + S_{\rm int}(W, U, M)\,, \tag{4.1}$$

where $S_2(W, U)$ is the same as in (3.1) and

$$S_{\rm int}(W, U, M) = \frac{1}{4}\int d^8z\left[(U^2\bar{M} + \bar{U}^2 M) + M\bar{M}\, J\,(m, \bar{m})\right], \tag{4.2}$$

with

$$m = \frac{1}{8}\bar{D}^2\bar{M}\,, \quad \bar{m} = \frac{1}{8}D^2 M\,. \tag{4.3}$$

The interaction function $J(m, \bar{m})$ is real. For the special choice of the interaction function, $J_{\rm inv} = J_{\rm inv}(B)$, $B := m\bar{m}$, the action (4.2) is invariant under the duality U(1) transformations realized on the newly introduced superfield M as

$$\delta M = 2i\omega M\,, \quad \delta\bar{M} = -2i\omega\bar{M}\,, \quad \delta\bar{m} = 2i\omega\bar{m}\,, \quad \delta m = 2i\omega m\,. \tag{4.4}$$

4.1 From master action to the (W, U) formulation

Let us firstly show that, eliminating the auxiliary superfields $M\,, \bar{M}$, we will recover the particular case of the action (3.5) with $E = E(u, \bar{u})$. We assume that J starts with a constant, so the function J^{-1} is well defined at the origin.

The corresponding equations of motion are

$$M = -J^{-1}\left[U^2 + \bar{D}^2(M\bar{N}J_m)\right], \quad \bar{M} = -J^{-1}\left[\bar{U}^2 + D^2(M\bar{M}J_{\bar{m}})\right]. \tag{4.5}$$

A simple analysis show that the general solution of these equations has the form

$$M = U^2 f, \quad \bar{M} = \bar{U}^2\bar{f}\,, \tag{4.6}$$

where f is some composite superfunction.[3] Thus the integrand in (4.2) contains the nilpotent factor $U^2\bar{U}^2$ and in the subsequent manipulations we can use the effective form of various equations and relations. For f we obtain in this way the equation

$$f = -J^{-1}\left(1 + uf\bar{f}J_m\right) \quad \text{and c.c.}\,, \tag{4.7}$$

and also the relation between the variables $u, \bar{u}$ and $m, \bar{m}$

$$m = u\bar{f}\,, \quad \bar{m} = \bar{u}f\,. \tag{4.8}$$

Substituting this into (4.7), we find the simple representation for f

$$f = -\frac{1}{J + mJ_m}\,, \quad \bar{f} = -\frac{1}{J + \bar{m}J_{\bar{m}}}\,. \tag{4.9}$$

Now the integrand in (4.2) takes the same form as in (3.5), with

$$E(u, \bar{u}) = f(u, \bar{u}) + \bar{f}(u, \bar{u}) + f(u, \bar{u})\bar{f}(u, \bar{u})\,J(u, \bar{u})\,, \tag{4.10}$$

where m and $\bar{m}$ are expressed in terms of $u, \bar{u}$ by eqs. (4.8), (4.9).

If we define

$$\tilde{E} = u\bar{u}\,E\,, \quad \tilde{J} = m\bar{m}\,J \tag{4.11}$$

it is straightforward to show that

$$\tilde{E} = \tilde{J} - m\tilde{J}_m - \bar{m}\tilde{J}_{\bar{m}}\,, \quad \tilde{J} = \tilde{E} - u\tilde{E}_u - \bar{u}\tilde{E}_{\bar{u}}\,, \tag{4.12}$$

and

$$u = -\tilde{J}_{\bar{m}}\,,\ \bar{u} = -\tilde{J}_m\,, \quad m = \tilde{E}_{\bar{u}}\,,\ \bar{m} = \tilde{E}_u\,. \tag{4.13}$$

These relations are recognized as the basic relations of the μ representation (Legendre transformation), the only difference being superfields in place of fields. Some their useful corollaries directly relating the functions J and E are

$$J + mJ_m = -\frac{1}{E + uE_u}\,, \quad \text{and c.c.}\,, \tag{4.14}$$

$$J = -\frac{E + uE_u + \bar{u}E_{\bar{u}}}{(E + uE_u)(E + \bar{u}E_{\bar{u}})}\,, \quad E = -\frac{J + mJ_m + \bar{m}J_{\bar{m}}}{(J + mJ_m)(J + \bar{m}J_{\bar{m}})}\,. \tag{4.15}$$

We would like to point out once more that all these algebraic relations are valid up to nilpotent terms vanishing under $U^2\bar{U}^2$.

[3]This can be proved, e.g., by introducing a small parameter before the second terms in the square brackets in (4.5) and representing the solution as a perturbative series in this parameter. One can show that each term of this series contains U^2 as a factor.

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4.2 New representation for the $\mathcal{N}=1$ self-dual systems

Now we will obtain a new representation for the Lagrangians of nonlinear electrodynamics in terms of the superfields W_α and M, eliminating from the master action (4.1) the spinor superfield U_α instead of M.

For U_α we obtain the expression

$$U_\alpha(1+m) = W_\alpha\,, \quad UW = W^2(1+m)^{-1}\,, \quad U^2 = W^2(1+m)^{-2}\,, \tag{4.16}$$

which, after substitution into (4.1), yields the chiral representation for the action in the (W, M) representation

$$S(W,M) = \frac{1}{4}\int d^6\zeta\left[W^2\,\frac{1-m}{1+m} - \frac{1}{8}\bar{D}^2(M\bar{M}J)\right] + \text{c.c.}\,. \tag{4.17}$$

Another form of the same action is

$$\begin{aligned} S(W,M) &= S_2(W) + S_{\rm int}(W,M)\,,\\ S_{\rm int}(W,M) &= \frac{1}{4}\int d^8z\left[\left(\frac{1}{1+m}\,W^2\bar{M} + \frac{1}{1+\bar{m}}\,\bar{W}^2 M\right) + M\bar{M}\,J\right]. \end{aligned} \tag{4.18}$$

In what follows we will be interested in the self-dual systems, with

$$J = J(B), \quad B = m\bar{m}\,.$$

The equation of motion for M is

$$M = -J^{-1}\left[\frac{1}{(1+m)^2}\,W^2 + \frac{1}{8}\bar{D}^2(M\bar{M}\,\bar{m}J_B)\right]. \tag{4.19}$$

Like in the case of eq. (4.5), the solution of (4.19) has the form

$$M = W^2\,\mathcal{B}\,, \tag{4.20}$$

where $\mathcal{B}$ is some composite superfield. Due to the appearance of the maximal nilpotent factor $W^2\bar{W}^2$ in (4.18), we can use the fully reduced effective relations, i.e. make the change $\mathcal{B} \to \mathcal{B}(w,\bar{w})$ in (4.20) and pass to the effective equation

$$\bar{m} = \mathcal{B}(w,\bar{w})\bar{w}. \tag{4.21}$$

Now from (4.19) we obtain

$$\bar{m} = -\bar{w}\,\frac{1}{(1+m)^2(J+BJ_B)}\,, \quad m = -w\,\frac{1}{(1+\bar{m})^2(J+BJ_B)}\,. \tag{4.22}$$

These equations are analogous to the basic equations in the bosonic μ representation [17, 20],

$$F^2 = -\bar{\mu}(1+\mu)^2\,I_b\,, \quad \bar{F}^2 = -\mu(1+\bar{\mu})^2\,I_b\,,\; b = \mu\bar{\mu}\,, \tag{4.23}$$

with the obvious correspondence

$$(w,\bar{w}) \;\leftrightarrow\; (\bar{F}^2, F^2)\,, \quad (m,\bar{m}) \;\leftrightarrow\; (\mu,\bar{\mu})\,, \quad (J+BJ_B) \;\leftrightarrow\; I_b\,. \tag{4.24}$$

Solving eqs. (4.22) for m and $\bar{m}$ in terms of $w, \bar{w}$ and substituting the solution into (4.18), we can find the relevant self-dual $\mathcal{N}=1$ action in terms of the superfield strengths $W_\alpha, \bar{W}_{\dot{\alpha}}$. Like in the bosonic case, only for some special superfunctions J these equations have the solution in a closed form, while in other cases one manages to obtain the action only as a power series in $w, \bar{w}$. Nevertheless, the way to the final $W, \bar{W}$ action in the (W, M) representation in some cases turns out to be easier than in the original (W, U) representation, which deals with the variables $u, \bar{u}$ instead of $m, \bar{m}$. Despite this technical difference, both representations (at least for the considered particular set of self-dual systems, with $\Lambda = \Lambda(w, \bar{w})$) are equivalent to each other. The basic objects in both representations are related to each other by the relations (4.8), (4.9), (4.14) and (4.15) specialized to the U(1) invariant case, e.g.,

$$\frac{d}{dA}(AE) = -\left[\frac{d}{dB}(BJ)\right]^{-1}, \quad A = B\left[\frac{d}{dB}(BJ)\right]^2, \quad B = A\left[\frac{d}{dA}(AE)\right]^2, \tag{4.25}$$

with $A = u\bar{u}\,,\ B = m\bar{m}\,.$

5 Examples of the $\mathcal{N}=1$ self-dual models

5.1 $\mathcal{N}=1$ Born-Infeld

The superfield action for the $\mathcal{N}$=1 BI theory can be rewritten in the polynomial form by making use of the auxiliary complex superfields X and R

$$S(W,X,R) = S_2(W,\bar{W}) + S_{\text{int}}(W,X,R), \tag{5.1}$$

$$S_{\text{int}}(W,X,R) = \frac{1}{8}\int d^6z\Big\{X\bar{X} - \bar{R}\left[X + \frac{1}{16}\bar{D}^2(X\bar{X}) - W^2\right] - R\left[\bar{X} + \frac{1}{16}D^2(X\bar{X}) - \bar{W}^2\right]\Big\}. \tag{5.2}$$

Varying it with respect to R, we obtain the constraint

$$W^2 = X + \frac{1}{16}\bar{D}^2(X\bar{X}) \sim X + \frac{1}{16}X\bar{D}^2\bar{X}\,, \tag{5.3}$$

which guarantees chirality of the solution, $\bar{D}_{\dot{\alpha}}X = 0$. Note that this constraint yields the dimensionless effective relation

$$w = x + \frac{1}{2}x\bar{x}, \quad x = \frac{1}{8}\bar{D}^2\bar{X}, \quad \bar{x} = \frac{1}{8}D^2X\,, \tag{5.4}$$

which is equivalent to the algebraic equation in the bosonic BI theory.

The superfield action (5.2) becomes

$$S(W,X) = S_2(W,\bar{W}) + \frac{1}{8}\int d^6z X\bar{X}\,, \tag{5.5}$$

which can be shown to be equivalent to (2.26).

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On the other hand, the superfield X can be eliminated via its equation of motion

$$X - R - \frac{1}{16} X \bar{D}^2 \bar{R} - \frac{1}{16} X D^2 R = X - R - \frac{1}{2} X (r + \bar{r}) = 0 \,, \tag{5.6}$$

where $r = \frac{1}{8}\bar{D}^2\bar{R}\,, \quad \bar{r} = \frac{1}{8}D^2 R$. Now we wish to solve this equation for X in terms of $R, \bar{R}$ and to finally get the R, W form of the action (5.2). The exact solution of (5.6) is as follows

$$X = \frac{R}{(1 - \frac{1}{2}r - \frac{1}{2}\bar{r})} \,. \tag{5.7}$$

Substituting this solution for X in the action (5.2) we obtain

$$S_{\rm int}(W, R) = \frac{1}{8} \int d^6 z \left[(\bar{R}W^2 + R\bar{W}^2) - \frac{R\bar{R}}{(1 - \frac{1}{2}r - \frac{1}{2}\bar{r})} \right] . \tag{5.8}$$

This action is recognized as a particular case of our representation (4.18) after redefining the auxiliary variables as

$$\frac{2M}{1 + \bar{m}} = R \,, \quad \bar{r} = \frac{2\bar{m}}{1 + \bar{m}} \,. \tag{5.9}$$

The inverse relation involves the differential operator V

$$M = \frac{1}{2(1 - V)} R \,, \quad V := \frac{1}{16} R D^2 , \quad \bar{m} = \frac{\bar{r}}{2 - \bar{r}} \,. \tag{5.10}$$

The transformation law for the superfield R follows from the U(1) transformation (4.4) of M and $\bar{m} = \frac{1}{8}D^2 M$:

$$\delta R = i\omega R(2 - \bar{r}) \,, \quad \delta \bar{r} = i\omega \bar{r}(2 - \bar{r}) \,. \tag{5.11}$$

The action (5.8) yields the auxiliary equation

$$W^2 - \frac{R}{1 - \frac{1}{2}r - \frac{1}{2}\bar{r}} - \bar{D}^2 \left[\frac{R\bar{R}}{16\,(1 - \frac{1}{2}r - \frac{1}{2}\bar{r})^2} \right] = 0 \,, \tag{5.12}$$

which is a particular case of (4.19). Using (5.7), we can bring this relation to the form of the constraint (5.3).

The $\mathcal{N}$=1 BI model in our (W, m) formalism corresponds to the choice of the invariant density

$$J^{BI} = \frac{2}{B - 1} \,, \quad J^{BI} + B J^{BI}_B = -\frac{2}{(B - 1)^2} \,, \tag{5.13}$$

$$\bar{w} = \frac{2\bar{m}(1 + m)^2}{(m\bar{m} - 1)^2} \,. \tag{5.14}$$

The effective equation for $\bar{w}$ is similar to the bosonic relation for F^2 in the μ representation of the BI theory [17, 20].

Eq. (5.14) and its conjugate can be solved for $m, \bar{m}$ in terms of $w, \bar{w}$, which finally reproduces the $\mathcal{N}=1$ BI action (2.25)

$$\bar{m} = \frac{Q - 1 + \frac{1}{2}(w - \bar{w})}{Q + 1 - \frac{1}{2}(w - \bar{w})}\,, \tag{5.15}$$

$$Q(w, \bar{w}) = \sqrt{1 + w + \bar{w} + (1/4)(w - \bar{w})^2}\,, \tag{5.16}$$

$$G = 1 - w\Lambda - w\bar{w}\Lambda_{\bar{w}} = \frac{1}{1 + \bar{m}} = \frac{1}{2Q}\left[Q + 1 - \frac{1}{2}(w - \bar{w})\right]. \tag{5.17}$$

The $\mathcal{N}=1$ BI theory in the original (W, U) representation corresponds to the choice

$$\mathcal{P}(A) = \frac{d}{dA}(AE^{BI}) = \frac{1}{2}(B - 1)^2, \quad A = \frac{4B}{(1 - B)^4}\,, \tag{5.18}$$

$$2\mathcal{P} = [1 - A\mathcal{P}^2]^2\,, \tag{5.19}$$

whence $E^{BI}(A) = \frac{1}{2} - \frac{1}{8}A + \frac{3}{32}A^2 + O(A^3)\,.$

5.2 Other examples

The $\mathcal{N}$=1 analog of the bosonic simplest interaction model [16, 20] corresponds to the choice $J = -2$ or $E = \frac{1}{2}$. The corresponding algebraic equations read

$$W_\alpha = U_\alpha + \frac{1}{16}U_\alpha \bar{D}^2\bar{U}^2 = U_\alpha\left(1 + \frac{1}{2}u\right), \tag{5.20}$$

$$W^2 = U^2\left(1 + \frac{1}{2}u\right)^2, \quad \bar{w}| = \bar{u}\left(1 + \frac{1}{2}u\right)^2. \tag{5.21}$$

The perturbative solution for $u(w, \bar{w})$ is completely similar to the corresponding bosonic solution in the simplest interaction model [20]

$$\Lambda_{SI} = \frac{1}{2} - \frac{1}{4}(w + \bar{w}) + \frac{1}{2}w\bar{w} + \frac{1}{8}(w^2 + \bar{w}^2) \tag{5.22}$$

$$-\frac{5}{8}w\bar{w}(w + \bar{w}) - \frac{1}{16}(w^3 + \bar{w}^3) + O(W^8)\,. \tag{5.23}$$

By analogy with bosonic invariant interaction $I_b = \frac{2}{b-1}$ [20] we can obtain the exact formula for $\Lambda(w, \bar{w})$ for the choice

$$J + BJ_B = \frac{2}{B - 1}\,. \tag{5.24}$$

In this case the effective equation (4.22) is reduced to the solvable cubic equation for the real superfield $\hat{r}$

$$m = \hat{r} - \frac{1}{4}(w - \bar{w})\,. \tag{5.25}$$

We can also study the simple example of the invariant interaction with the additional variable g

$$E = \frac{1}{2} + f_1\, g\bar{g}\,, \tag{5.26}$$

where f_1 is a real constant. The basic recursive auxiliary equation in this case is

$$U_\alpha = W_\alpha - \frac{1}{16} U_\alpha \bar{D}^2 \bar{U}^2 + \frac{f_1}{8} U_\beta \bar{D}^2 \, (\bar{U}^2 D_\alpha U^\beta \bar{g}) \,. \tag{5.27}$$

The 5-th order term in its perturbative solution has the form

$$U_\alpha^{(5)} = \frac{1}{2} W_\alpha \left(w\bar{w} + \frac{1}{2}\, w^2 \right) + f_1 \, w\bar{y} \, W_\beta D_\alpha W^\beta . \tag{5.28}$$

Using eq. (3.18), we obtain the corresponding term in Λ:

$$\Lambda_y^{(2)} = f_1 \bar{y}, \quad \Lambda^{(2)} = -\frac{1}{4}(w + \bar{w}) + f_1 y\bar{y} \,. \tag{5.29}$$

Thereby we restore the nonlinear superfield action up to the 6-th order in $W_\alpha, \bar{W}_{\dot{\alpha}}$.

5.3 $\mathcal{N} = 1$ self-dual models with higher derivatives

In supersymmetric self-dual models with higher derivatives we still can use our basic bilinear action $S_2(W, U)$ (3.1), which corresponds to the usual free equations without additional derivatives. This choice guarantees the U(1) duality of the entire equations of motion, if we construct the U(1) invariant interaction with higher derivatives solely in terms of auxiliary superfields. In the W representation the action will involve powers of some basic coupling constant c of dimension -2, as well as plenty of additional dimensionless coupling constants.

Thus the self-dual interactions with higher derivatives can be naturally introduced via the modification of the auxiliary invariant interaction. The possible bilinear interaction

$$c a_1 \int d^8 z \bar{U}^{\dot{\alpha}} \partial_{\alpha\dot{\alpha}} U^\alpha \,, \tag{5.30}$$

drastically changes the status of the superfield U_α which will become propagating. In components, this would mean, in particular, that the scalar component field $v(x)$ of U_α (and, respectively, $D(x)$ in the W representation) will propagate. This effect disappears in the limit $a_1 \to 0$. Such self-dual deformations of the bilinear bosonic action were considered, e.g., in [15, 22], and the interactions of the type (5.30) would give $\mathcal{N} = 1$ self-dual superextensions of these deformed actions. Further in this subsection we will focus on the deformations which do not change the free action.

As an example, let us consider the U(1) invariant quartic interaction of the auxiliary superfields with higher derivatives

$$\frac{1}{4} c \int d^8 z \, b_1 \, \partial^m U^2 \partial_m \bar{U}^2 \,, \tag{5.31}$$

where b_1 is a dimensionless coupling constant. The auxiliary equation has the form

$$W_\alpha = U_\alpha + \frac{1}{8} c b_1 U_\alpha \Box \bar{D}^2 \bar{U}^2 = U_\alpha \, (1 + c b_1 \, \Box u) \,, \tag{5.32}$$

$$\bar{w}| = \bar{u} \, (1 + c b_1 \Box u)^2 \ . \tag{5.33}$$

Its perturbative solution is as follows

$$U_\alpha| = W_\alpha \left[1 - cb_1\Box w + c^2 b_1^2(\Box w)^2 + 2c^2 b_1^2 \Box(w\Box\bar{w}) + \ldots\right] , \tag{5.34}$$

where $\Box = \partial^m\partial_m$. Using this solution, we can construct the corresponding self-dual nonlinear action with higher derivatives in the W representation.

We can also study an example of the 6-th order invariant interaction with higher derivatives

$$\frac{1}{4}cb_2 \int d^8 z U^2 \bar{U}^2 \partial_m g \partial^m \bar{g} \tag{5.35}$$

and find the power-series solution of the corresponding auxiliary equation.

Further generalizations involve U(1) invariant interactions with derivatives multiplied by some polynomials in the dimensionless variables $u,\ \bar{u},\ g,\ \bar{g}$.

Using the invariant interactions of the type

$$\frac{1}{4}c^2 \int d^8 z\, b_4(\Box U^2)(\Box\bar{U}^2) , \tag{5.36}$$

we arrive at the self-dual theories with the growing powers of higher derivatives. In this case, the auxiliary equation has the form

$$W_\alpha = U_\alpha + \frac{1}{8}c^2 b_4 U_\alpha \Box^2 \bar{D}^2 \bar{U}^2 = U_\alpha(1 + c^2 b_4 \Box^2 u) . \tag{5.37}$$

To summarize, the principle that the higher-derivative actions in the $\mathcal{N}$=1 electrodynamics models are generated by some higher-derivative U(1) invariant interactions of the auxiliary spinor superfields automatically yields the self-dual nonlinear actions in the W representation.

6 Bosonic limit

In this section we consider the bosonic component Lagrangians corresponding to some superfield ones considered above. Our conventions on the bosonic component fields are

$$\begin{aligned} W_\alpha &= 2iF_\alpha{}^\beta\theta_\beta - \theta_\alpha D + \frac{i}{2}\theta^2(\sigma^m\bar{\theta})_\beta(\delta^\beta_\alpha\partial_m D - 2i\partial_m F_\alpha{}^\beta) , \quad F_\alpha{}^\beta = \frac{1}{4}(\sigma^m\bar{\sigma}^n)^\beta_\alpha F_{mn} , \\ U_\alpha &= 2iV_\alpha{}^\beta\theta_\beta - \theta_\alpha v + \frac{i}{2}\theta^2(\sigma^m\bar{\theta})_\beta(\delta^\beta_\alpha\partial_m v - 2i\partial_m V_\alpha{}^\beta) . \end{aligned} \tag{6.1}$$

Here, the symmetric complex bispinor field $V_{\alpha\beta}(x) = V_{\beta\alpha}(x)$ and the complex scalar field $v(x)$ are not subject to any constraints off shell. We also have $F_{mn} = \partial_m A_n - \partial_n A_m$ and $D = \bar{D}$ in virtue of the superfield Bianchi identity (2.7).

Various superfield objects constructed from W_α and U_α have the following bosonic limits

$$\begin{aligned} &W^2 \to (-2\varphi + D^2)\theta^2 ,\ U^2 \to (-2\nu + v^2)\theta^2 ,\ UW \to (-2VF + vD)\theta^2 , \\ &w \to (\bar{\varphi} - \frac{1}{2}D^2) ,\ y, \bar{y} \to 2D ,\ u \to (\bar{\nu} - \frac{1}{2}\bar{v}^2) ,\ g \to 2v , \end{aligned} \tag{6.2}$$

where

$$\varphi = F^{\alpha\beta}F_{\alpha\beta} = \frac{1}{4}F^{mn}F_{mn} + \frac{i}{4}F^{mn}\tilde{F}_{mn}\,, \qquad \nu = V^{\alpha\beta}V_{\alpha\beta}$$

as in [20]. In the second line of (6.2) we took into account that the relevant superfield arguments appear in the superfield actions always with the nilpotent factors $W^2\bar{W}^2$ or $U^2\bar{U}^2$ and so, after integration over $\theta, \bar{\theta}$ and taking the bosonic limit, are reduced to their lowest $\theta = \bar{\theta} = 0$ components. The study of the component bosonic equations can give us a further insight into the properties of the superfield equations for $U^2(W)$ and $D^\alpha U_\alpha(W)$ (3.27), (3.28).

The free actions (2.10) and (3.1) are reduced to

$$S_2(W) \to \int d^4x \left[-\frac{1}{2}(\varphi + \bar{\varphi}) + \frac{1}{2}D^2\right] = \int d^4x \left(-\frac{1}{4}F^{mn}F_{mn} + \frac{1}{2}D^2\right), \tag{6.3}$$

$$S_2(W,U) \to \int d^4x \left(-2VF + \nu + \frac{1}{2}\varphi + vD - \frac{1}{2}v^2 - \frac{1}{4}D^2 + \text{c.c.}\right). \tag{6.4}$$

After integrating out the auxiliary fields $V_{\alpha\beta}, v$ from (6.4) we recover (6.3).

The interaction (3.5) is reduced to

$$S_{\text{int}}(W,U) \;\Rightarrow\; \int d^4x \left(\nu - \frac{1}{2}v^2\right)\left(\bar{\nu} - \frac{1}{2}\bar{v}^2\right) E\left[\left(\nu - \frac{1}{2}v^2\right), \left(\bar{\nu} - \frac{1}{2}\bar{v}^2\right), 2v, 2\bar{v}\right]. \tag{6.5}$$

Now it becomes clear why using of the alternative superfield variables (3.37), (3.39) looks more preferable: these quantities have a simpler bosonic limit

$$u' \to \bar{\nu}\,, \quad \bar{u}' \to \nu\,, \tag{6.6}$$

$$E \to E'(\nu, \bar{\nu}, v, \bar{v})\,, \quad \Lambda \to \Lambda'(\varphi, \bar{\varphi}, D)\,. \tag{6.7}$$

Correspondingly, for the invariant interactions we have

$$A' \to a := \nu\bar{\nu}\,, \quad B' \to \bar{\nu}v^2\,, \quad \bar{B}' \to \nu\bar{v}^2\,, \tag{6.8}$$

$$\mathcal{F}(A,B,C) \to \mathcal{F}'(\nu\bar{\nu}, v^2\bar{\nu}, v\bar{v})\,. \tag{6.9}$$

Just this choice of E is most convenient for examining the role of the scalar auxiliary fields $v, \bar{v}, D$. For these fields we obtain the following equations of motion:

$$\delta D: \quad D = v + \bar{v}\,, \tag{6.10}$$

$$\delta v: \quad D - v - v(\bar{\nu} - \frac{1}{2}\bar{v}^2)E' + (\nu - \frac{1}{2}v^2)(\bar{\nu} - \frac{1}{2}\bar{v}^2)E'_v = 0\,. \tag{6.11}$$

For the self-dual case E'_v is proportional to v or $\bar{v}$, so eq. (6.11) and its conjugate, after eliminating D by eq. (6.10), are reduced to a system of two homogeneous equations for $v, \bar{v}$, such that the determinant of the 2×2 matrix of the coefficients is non-vanishing at the origin. This means that the main perturbative solution of (6.11) in the duality-invariant case is

$$v = \bar{v} = 0 \;\Rightarrow\; D = 0\,. \tag{6.12}$$

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To make sure, we analyzed the nonlinear equation (6.11) for the simple interaction $E' = e_1 + f_1\, g\bar{g} \;\Rightarrow\; e_1 + 4 f_1\, v\bar{v}$ and did not find any nontrivial analytic solution $v \neq 0$. Nevertheless, the existence of some non-perturbative non-trivial solutions for $v, \bar{v}$ (and D) under some special choices of E (or E') cannot be excluded.

For the basic solution (6.12), $S_{\rm int}(W,U)$ in the bosonic limit becomes

$$S_{\rm int}(W,U) \;\to\; \int d^4x\, a\, E(a)\,. \tag{6.13}$$

Comparing it with the general auxiliary self-interaction in the tensorial auxiliary field formulation of the bosonic self-dual Maxwell models [18, 20], we identify

$$aE(a) = \mathcal{E}(a)\,. \tag{6.14}$$

Interactions with higher derivatives change eq. (6.11). In particular, the superfield interaction (5.35) yields the component term

$$4cb_2 \int d^4x \left(\nu - \frac{1}{2}v^2\right)\left(\bar{\nu} - \frac{1}{2}\bar{v}^2\right)\partial^m v \partial_m \bar{v} \tag{6.15}$$

in addition to the bosonic action (6.4). Once again, solving recursively the equations for $v, \bar{v}$, we find the trivial perturbative solution (6.12) as the unique one.

The Fayet-Iliopoulos (FI) term ξD softly breaks the U(1) duality and deforms the δD equation to $\xi + D = v + \bar{v}$. The models with this term added provide nontrivial solutions for the auxiliary fields v and D, depending on the parametrization of E'.[4]

The bilinear invariant interaction (5.30) gives the bosonic Lagrangian $\sim ca_1 \partial^m v \partial_m \bar{v}$ and so radically affects the component equation (6.11), yet preserving self-duality. The former auxiliary fields $v, \bar{v}$ become propagating in this case, while elimination of D produces "mass terms" for these fields.

It is natural to treat both this bilinear interaction and the FI term as a kind of non-perturbative effects generating nontrivial solutions for the auxiliary fields. So in the presence of such terms the dependence of the auxiliary interaction E on the additional superfield variables $(g, \bar{g}) \sim (v, \bar{v})$ can prove very essential.

7 U(N) duality for $\mathcal{N} = 1$

7.1 Auxiliary chiral U(N) superfields

Let us consider N Abelian superfield strengths

$$W^i_\alpha = -\frac{1}{4}\bar{D}^2 A^i_\alpha = -\frac{1}{4}\bar{D}^2 D_\alpha V^i\,, \quad \bar{W}^i_{\dot\alpha} = -\frac{1}{4}D^2 \bar{A}^i_{\dot\alpha} = -\frac{1}{4}D^2 \bar{D}_{\dot\alpha} V^i\,,\; i = 1, \ldots N\,, \tag{7.1}$$

with V^i being N real gauge prepotentials. By definition, all superfields are transformed by the vector representation of the group $O(N)$:

$$\delta W^i_\alpha = \xi^{ik} W^k_\alpha\,, \quad \delta \bar{W}^i_{\dot\alpha} = \xi^{ik} \bar{W}^k_{\dot\alpha}\,, \tag{7.2}$$

[4] A nontrivial FI-term deformed solution for the field D was considered in [25] for the case of $\mathcal{N}$=1 BI theory.

where $\xi^{ik} = -\xi^{ki}$ are real group parameters. The free action of this set of superfields is

$$S_2(W^i, \bar{W}^i) = \frac{1}{4} \int d^6\zeta (W^i W^i) + \text{c.c.}\,, \tag{7.3}$$

where $(W^i W^i) := W^{i\alpha} W^i_\alpha$.

Further, we introduce the notation

$$W^{kl} = (W^{k\alpha} W^l_\alpha), \quad \bar{W}^{kl} = (\bar{W}^k_{\dot\alpha} \bar{W}^{l\dot\alpha}) \tag{7.4}$$

and consider the particular parametrization of the nonlinear $O(N)$ and R invariant superfield interaction

$$S_\Lambda(W^{kl}, \bar{W}^{kl}) = \frac{1}{4} \int d^8 z W^{kl} \bar{W}^{rs} \Lambda_{kl,rs}(w, \bar{w})\,, \tag{7.5}$$

where $\Lambda_{kl,rs}(w, \bar{w})$ is a function of the dimensionless Lorentz invariant matrix variables

$$w^{kl} = \frac{1}{8} \bar{D}^2 \bar{W}^{kl}, \quad \bar{w}^{kl} = \frac{1}{8} D^2 W^{kl}\,. \tag{7.6}$$

The integral conditions of U(N) duality have the following form:

$$\text{Im} \int d^6\zeta \left[(W^i M^k) - (W^k M^i) \right] = 0\,, \tag{7.7}$$

$$\text{Im} \int d^6\zeta \left[(W^i W^k) + (M^i M^k) \right] = 0\,, \tag{7.8}$$

where

$$M^k_\alpha \equiv -2i \frac{\delta S}{\delta W^{k\alpha}}\,, \tag{7.9}$$

and $(W^i W^k), (W^i M^k)$ and $(M^i M^k)$ are defined similarly to (7.4). The antisymmetric in i and k condition (7.7) means the off-shell $O(N)$ symmetry, so the nontrivial constraint in the general case is the nonlinear condition (7.8).

Consider the following transformations

$$\delta_\eta W^i_\alpha = \eta^{ik} M^k_\alpha\,, \quad \delta_\eta M^i_\alpha = -\eta^{ik} W^k_\alpha\,, \tag{7.10}$$

where $\eta^{ik} = \eta^{ki}$ are real parameters. The complex combination

$$U^k_\alpha = \frac{1}{2}(W^k_\alpha + i M^k_\alpha) \tag{7.11}$$

transform linearly in the group U(N) according to the fundamental representation of the latter:

$$\delta U^k_\alpha = (\xi^{kl} - i\eta^{kl}) U^l_\alpha\,. \tag{7.12}$$

The covariance of the set of Bianchi identities for W^i_α together with the superfield equations of motion following from the action $S_2 + S_\Lambda$ under the coset U(N)/$O(N)$ transformations (7.10) is the correct generalization of the notion of U(1) self-duality to the considered case. The condition (7.8) ensures compatibility of this duality covariance with the definition (7.9).

The basic steps in generalizing the U(1) self-duality setting with the single auxiliary superfield U_α to the U(N) case is to interpret U^k_α defined in (7.11) as auxiliary chiral superfields ($R(U^i_\alpha) = 1$) and to replace (7.3) by the following bilinear action:

$$S_2(W^k, U^k) = \int d^6\zeta \left[(W^kU^k) - \frac{1}{4}(W^kW^k) - \frac{1}{2}(U^kU^k)\right] + \text{c.c.}\,. \tag{7.13}$$

The corresponding auxiliary interaction $S_E(U)$ is chosen as an arbitrary $O(N)$ and R invariant functional of the auxiliary superfields U^k and $\bar{U}^k$ (and perhaps of their derivatives). The basic equation for the superfield U^k_α is

$$W^k_\alpha = U^k_\alpha - \frac{\delta S_E(U)}{\delta U^{k\alpha}}\,. \tag{7.14}$$

Like in the U(1) case, it is straightforward to show that the U(N) duality conditions (7.8) amount to the U(N)-invariance of $S_E(U)$:

$$\int d^6\zeta U^k_\alpha \frac{\delta S_E(U)}{\delta U^l_\alpha} - \int d^6\bar{\zeta}\bar{U}^k_{\dot\alpha}\frac{\delta S_E(U)}{\delta \bar{U}^l_{\dot\alpha}} = 0\,. \tag{7.15}$$

A particular parametrization of S_E is through independent dimensionless R-invariant Lorentz scalars

$$u^{kl} = \frac{1}{8}\bar{D}^2\bar{U}^{kl}\,, \qquad \bar{u}^{kl} = \frac{1}{8}D^2U^{kl}\,, \tag{7.16}$$

$$\delta_\eta u^{kl} = -i\eta^{kr}u^{rl} - i\eta^{lr}u^{rk}, \qquad \delta_\eta \bar{u}^{kl} = i\eta^{kr}\bar{u}^{rl} + i\eta^{lr}\bar{u}^{rk} \tag{7.17}$$

(these are analogs of the variables u and $\bar{u}$ of the U(1) case). Then in the self-dual theory we can consider the following particular R- and U(N) invariant interaction of the auxiliary superfields

$$S_E = \frac{1}{4}\int d^8z (U^lU^k)(\bar{U}^r\bar{U}^s)E_{kl,rs}\,, \tag{7.18}$$

where $E_{kl,rs}$ is the U(N) covariant dimensionless superfield density composed out of the variables (7.16).

A simple example of the action functional of this type contains the matrix $E^{lk}(A)$, which depends on the matrix argument $A^{kl} = \bar{u}^{kr}u^{rl}$:

$$S_E = \frac{1}{4}\int d^8z (\bar{U}^k\bar{U}^s)(U^sU^l)E^{lk}(A)\,. \tag{7.19}$$

We can also consider the interaction with a scalar invariant density,

$$S_E = \frac{1}{4}\int d^8z (U^kU^l)(\bar{U}^l\bar{U}^k)E(A_n)\,, \tag{7.20}$$

where the dimensionless invariant variables A_n are defined as follows

$$A_n = \frac{1}{n}\,\text{Tr}\, A^n\,. \tag{7.21}$$

Using the relations

$$\delta A_n = \frac{1}{4}D^2(\delta U^k U^r)(uA^{n-1})^{rk}\,, \quad (uA^{n-1})^{rk} = (uA^{n-1})^{kr}\,, \tag{7.22}$$

we derive the equation of motion for the auxiliary spinor superfield in this case

$$W^k_\alpha - U^k_\alpha = \frac{1}{8}U^l_\alpha \bar{D}^2\Big\{(\bar{U}^l\bar{U}^k)E(A_n) + \frac{1}{8}(\bar{U}^p\bar{U}^t)D^2[(U^tU^p)E_n u^{ls}(A^{n-1})^{sk}]\Big\}\,, \tag{7.23}$$

where $E_n = \partial E/\partial A_n$. This U(N) covariant superfield equation describes the particular class of self-dual models.

7.2 U(N) analog of the M representation

An alternative representation for the supersymmetric U(N) self-dual theories deals with W^k_α, U^k_α and, in addition, with the auxiliary general scalar superfields $M^{kl} = M^{lk}$ and their dimensionless derivatives $\bar{m}^{kl} = \frac{1}{8}D^2M^{kl}$ (as well as with the corresponding conjugated superfields). Under the duality group U(N) the new auxiliary superfields are transformed as

$$\delta_\eta \bar{M}^{kl} = i\eta^{kr}\bar{M}^{rl} + i\eta^{lr}\bar{M}^{kr}\,, \qquad \delta M^{kl} = -i\eta^{kr}M^{rl} - i\eta^{lr}M^{kr}\,, \tag{7.24}$$

$$\delta_\eta m^{kl} = i\eta^{kr}m^{rl} + i\eta^{lr}m^{kr}, \qquad \delta_\eta \bar{m}^{kl} = -i\eta^{kr}\bar{m}^{rl} - i\eta^{lr}\bar{m}^{kr}\,. \tag{7.25}$$

The general "master" action is a sum of the bilinear action $S_2(W,U)$ (7.13) and the U(N) invariant interaction

$$S_{\text{int}}(U,M) = \frac{1}{4}\int d^8z[(U^kU^l)\bar{M}^{kl} + (\bar{U}^k\bar{U}^l)M^{kl}] + S_{\text{int}}(M)\,, \tag{7.26}$$

$$S_{\text{int}}(M) = \frac{1}{4}\int d^8z\bar{M}^{kl}M^{rs}\,J_{kl,rs}\,(m,\bar{m})\,, \tag{7.27}$$

where $J_{kl,rs}$ is a dimensionless covariant density. So, the master action is

$$S(W,U,M) = S_2(W,U) + S_{\text{int}}(U,M)\,. \tag{7.28}$$

For the density $J_{kl,rs}$ we can choose, e.g., the following particular parametrization:

$$\begin{aligned} J_{kl,rs} &= \frac{1}{4}(\delta^{ks}J^{rl} + \delta^{ls}J^{rk} + \delta^{kr}J^{sl} + \delta^{lr}J^{sk})\,, \\ \delta_\eta J^{lk}(B) &= i\eta^{ls}J^{sk} - iJ^{ls}\eta^{sk}\,, \end{aligned} \tag{7.29}$$

where $J^{lk}(B)$ is a matrix function of $B^{ij} = m^{is}\bar{m}^{sj}$, for instance,

$$J^{lk} = -2\delta^{lk} + \frac{1}{2}i_2B^{lk} + \ldots\,. \tag{7.30}$$

Varying (7.28) with respect to U^k_α , we obtain the equation

$$W^k_\alpha = U^l_\alpha(\delta^{kl} + m^{kl})\,. \tag{7.31}$$

Then, eliminating the variables U^k_α, we come to the U(N) analog of the representation (4.18)

$$S(W^{kl}, M^{kl}) = S_2(W, \bar W) + S_{\rm int}(M) \\ + \frac{1}{4}\int d^8z {\rm Tr}\left[W\left(\frac{\mathbf{1}}{\mathbf{1}+m}\right)\bar M + \bar W\left(\frac{\mathbf{1}}{\mathbf{1}+\bar m}\right)M\right] \tag{7.32}$$

where $\mathbf{1}$ denotes the unit matrix. Varying (7.28) with respect to the superfields $\bar M^{kl}$, we obtain the U(N) analog of the equations (4.19). This equation can be solved perturbatively. The matrix solution $M(W, \bar W)$ yields the self-dual superfield action in the W representation.

The parametrization of $J_{kl,rs}$, which is yet simpler than (7.29), involves only one invariant function J

$$J_{kl,rs} = \frac{1}{2}(\delta_{kr}\delta_{ls} + \delta_{ks}\delta_{lr})J(m, \bar m), \quad S_{\rm int}(M) = \frac{1}{4}\int d^8z \bar M^{kr}M^{rk}J(B_n)\,. \tag{7.33}$$

The variables B_n on which the invariant function J depends are defined as

$$B_n = \frac{1}{n}{\rm Tr}B^n\,. \tag{7.34}$$

The simplest possible interaction is $J(B_1)$, and it corresponds to the special choice of the invariant density in (7.20) as $E(A_1)$.

7.3 Examples of the U(N) self-dual theories

As an example of the U(N) self-dual action, we may consider the following simplest interaction:

$$S_{SI} = \frac{1}{8}\int d^8z\, (U^kU^l)(\bar U^k\bar U^l)\,, \tag{7.35}$$

which gives us the basic algebraic spinor equations in the form

$$W^k_\alpha = U^k_\alpha + \frac{1}{16}U^l_\alpha \bar D^2(\bar U^k\bar U^l)\,. \tag{7.36}$$

Using the perturbative solution $U^k_\alpha(W, \bar W)$, we obtain an U(N) analog of our model (5.23)

$$S_\Lambda = \frac{1}{8}\int d^8z {\rm Tr}\left[W\bar W - \frac{1}{2}(W\bar w\bar W + Ww\bar W) + O(W^4)\right]. \tag{7.37}$$

The U(N) supersymmetric generalization of the BI model is based on the following matrix algebraic relation [4, 5, 9, 10]

$$X^{kl} + \frac{1}{16}X^{kj}\bar D^2\bar X^{lj} = W^{kl}\,, \tag{7.38}$$

where $X^{kl} \neq X^{lk}$ are the auxiliary chiral superfields, and $\bar X^{jl}$ are the conjugated antichiral superfields. The nonsymmetric matrix W^{kl} corresponds to the so-called U(N) $\times$ U(N) duality.

In our formulation this model corresponds to the U(N) invariant representation (7.29) with

$$J^{lk} = 2\left(\frac{\mathbf{1}}{B-\mathbf{1}}\right)^{lk}. \tag{7.39}$$

The equivalence of the two formulations can be checked by comparing the perturbative expansions of the relevant actions $S_\Lambda(W, \bar{W})$.

In our formalism we can also consider alternative U(N) self-dual generalizations of the supersymmetric U(1) BI model. For instance, we can use (7.33) with the one-parameter invariant interaction

$$J(B_1) = \frac{2}{B_1 - 1}\,, \quad B_1 = m^{kl}\bar{m}^{lk}\,. \tag{7.40}$$

8 Conclusions

In this paper, we constructed the most general $\mathcal{N}$=1 superextension of the auxiliary bispinor field formulation of the U(1) duality-invariant nonlinear electrodynamics models which was proposed in [17, 20]. The auxiliary bispinor fields are accommodated by the auxiliary spinor superfield and the full set of self-dual $\mathcal{N}$=1 models is parametrized by U(1) duality-invariant self-interactions of this superfield. The conventional nonlinear action in terms of the Maxwell superfield strengths is reproduced as a result of elimination of the auxiliary superfield by its equation of motion.

As compared with the recent paper [23] devoted to the same issue of $\mathcal{N}$=1 supersymmetrizing of the formulation with bispinor fields, we allow for the most general dependence of the auxiliary superfield Lagrangians on the U(1) duality-invariant superfield arguments. Though the dependence on the extra U(1) invariant superfield variables gets seemingly inessential on shell, when considering the "pure" $\mathcal{N}$=1 self-dual systems which deal with the Maxwell superfield strengths only, it can become essential and capable to provide new models in the cases of various deformations of such systems, e.g., through adding the Fayet-Iliopoulos term to the action [25] or turning on the couplings to the charged chiral matter. Other new results of our study is the construction of $\mathcal{N}$=1 generalization of the so called μ version of the approach of [17, 20] (which significantly simplify various computations), and finding out how to generate self-dual $\mathcal{N}$=1 systems with higher derivatives from the appropriate modifications of the U(1) invariant auxiliary interaction. In the latter case the extra superfield variables we have introduced can play an essential role: they indeed considerably enlarge the set of possible U(1) invariant interactions, and these additional interactions are not trivialized on shell in some conceivable cases.

We also presented a few explicit examples of generating duality-invariant $\mathcal{N}$=1 superfield systems in the approach with auxiliary spinor superfield, as well as gave the bosonic component Lagrangians for the general case, with the auxiliary fields being kept, and compared these Lagrangians with those derived in [17, 20] within the non-supersymmetric setting.

We gave a brief account of the formalism of auxiliary superfields for the $\mathcal{N} = 1$ supersymmetric models with the U(N) duality, generalizing the similar formulation of the bosonic case [18]. A few examples of the U(N) self-dual models were presented.

As for further perspectives, it seems important to extend the formulation with auxiliary superfields to the more general case with the Sp($2N, R$) duality symmetry supported by the additional scalar chiral superfields living in the coset Sp($2N, R$)/U(N). Also it would be interesting to elaborate on the $\mathcal{N} = 1$ version of the proposal of ref. [20] about the possibility to deal, at all steps including quantization, with the off-shell auxiliary (super)field representation of self-dual (super)electrodynamics without explicitly eliminating these auxiliary objects.

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