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On the $D=4$, $N=2$ non-renormalization theorem

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Abstract

Using the harmonic superspace background field formulation for general $D=4$, $N=2$ super Yang-Mills theories, with matter hypermultiplets in arbitrary representations of the gauge group, we present the first rigorous proof of the $N=2$ non-renormalization theorem; specifically, the absence of ultraviolet divergences beyond the one-loop level. Another simple consequence of the background field formulation is the absence of the leading non-holomorphic correction to the low-energy effective action at two loops.

There are two basic formulations of the $N=2$, $D=4$ pure super Yang-Mills theory in terms of unconstrained superfields. The first (conventional) formulation, which was developed at the linearized level by Mezincescu [1] and then extended to the full nonlinear theory by Koller and Howe, Stelle and Townsend [2,3], makes use of the conventional $N=2$ superspace $\boldsymbol{R}^{4|8}$ parametrized by $z^M \equiv (x^m, \theta_i^\alpha, \bar\theta_{\dot\alpha}^i)$ where $\bar\theta^{i\dot\alpha} = \overline{\theta_i^\alpha}$. The unconstrained prepotential of this theory, $U^{ij}(z)$, is an isovector real superfield, $U^{ij} = U^{(ij)} = \overline{U_{ij}}$, taking its values in the Lie algebra of the gauge group. In this approach, the $N=2$ super Yang-Mills theory possesses a non-trivial gauge invariance with an infinite degree of reducibility [4]. The second (harmonic) formulation, developed by GIKOS [5], makes use of the $N=2$ harmonic superspace $\boldsymbol{R}^{4|8}\times S^2$. This approach extends the conventional superspace by the two-sphere $S^2 = SU(2)/U(1)$ parametrized by harmonics; that is, group elements

$$(u_i^-, u_i^+) \in SU(2)\,, \quad u_i^+ = \varepsilon_{ij}u^{+j}\,, \quad \overline{u^{+i}} = u_i^-\,, \quad u^{+i}u_i^- = 1\,. \tag{1}$$

The unconstrained prepotential of this theory is an analytic real Lie-algebra valued superfield $V^{++}(\zeta,u)$. This superfield is defined over the analytic subspace of the harmonic superspace parametrized by the variables

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$$\zeta^{\mathscr{M}} \equiv \left(x_A^m, \theta^{+\alpha}, \bar{\theta}^+_{\dot{\alpha}}, u_i^+, u_j^- \right), \tag{2}$$

where the analytic basis in the subspace is defined by

$$x_A^m = x^m - 2i\theta^{(i}\sigma^m\bar{\theta}^{j)}u_i^+u_j^-, \quad \theta_\alpha^\pm = u_i^\pm \theta_\alpha^i, \quad \bar{\theta}_{\dot{\alpha}}^\pm = u_i^\pm \bar{\theta}_{\dot{\alpha}}^i. \tag{3}$$

In this approach the $N=2$ super Yang-Mills theory is an irreducible gauge theory.

Because of the infinitely reducible gauge structure of $N=2$ super Yang-Mills theory formulated in conventional superspace, its quantization cannot be carried out using the Faddeev-Popov prescription and should be based on more powerful quantization techniques, such as the Batalin-Vilkovisky method [6] (even this latter scheme is literally applicable to finitely reducible gauge theories only). To the best of our knowledge, the Batalin-Vilkovisky quantization of the theory has never been utilized, in this context, to derive a consistent superfield effective action. Instead, the two attempts to quantize this theory, undertaken in [3,7], were based on a modified Faddeev-Popov prescription, which has not been shown to be a consistent quantization scheme. Furthermore, although the $N=2$ background field method presented in [3] has played a significant role in understanding the general structure of extended supersymmetric theories, this approach is very complicated from the technical point of view and is not suitable for carrying out actual quantum computations. It is disturbing, therefore, that the original proof of the famous $N=2$ non-renormalization theorem (see, for example, [8,9] and references therein) assumes not only the existence of an unconstrained classical formulation in conventional superspace, but also a consistent formulation of the superfield Feynman rules in this superspace which, as we have seen, has yet to be developed. An indirect proof of the $N=2$ non-renormalization theorem, based on an explicit calculation of the one-loop $N=2$ beta function and the application of anomalies considerations, was presented in [10]. A different approach to quantum calculations in $N=2$ supersymmetric theories is to reformulate them in terms of $N=1$ superspace [11], and then to use the usual $N=1$ supergraph techniques or instanton methods. Here, too, there are fundamental problems. To begin with, in this approach, the second supersymmetry is hidden. More importantly, it is far from clear that the regulators used in this approach respect the $N=2$ supersymmetry. Hence, quantum corrected Greens functions may not necessarily be $N=2$ supersymmetric. It follows that, at the very least, the inherent mechanisms of the miraculous cancellations of ultraviolet divergences are not manifest. It has also yet to be proven that this technique preserves $N=2$ supersymmetry to all loop levels in quantum corrections. We conclude that the $N=2$ non-renormalization theorem requires more careful justification than has previously appeared in the literature. Recently, the first examples of quantum calculations with manifest $N=2$ supersymmetry have been given within the context of harmonic superspace [12,13]. In this paper, we will use these new techniques to give a rigorous proof of the $N=2$ non-renormalization theorem, as well as to establish the absence of the leading finite non-holomorphic correction at the two-loop level.

It has been known for a long time [5,14] that the conventional superfield formulation of the $N=2$ super Yang-Mills theory is simply a gauge fixed version of that theory in the harmonic superspace. More precisely, if one expresses the analytic prepotential $V^{++}(\zeta,u)$ in terms of an unconstrained superfield $U^{--}(z,u)$ over $\boldsymbol{R}^{4|8} \times S^2$ (and similarly for the analytic gauge parameter)

$$V^{++}(\zeta,u) = (D^+)^4 U^{--}(z,u), \quad U^{--}(z,u) = U^{(ij)}(z)\, u_i^- u_j^- + U^{(ijkl)}(z)\, u_i^+ u_j^+ u_k^- u_l^- + \cdots \tag{4}$$

then the original gauge freedom can be used to gauge away all but the $U^{ij}(z)$ components of $U^{--}(z,u)$; the remaining superfield $U^{ij}(z)$ being exactly Mezincescu's prepotenial. Since the harmonic formulation of the $N=2$ super Yang-Mills theory is an irreducible gauge theory, it can, unlike the conventional formulation, be properly quantized using the standard Faddeev-Popov prescription [15]. In the harmonic formulation, we simply have none of the quantization problems that are inevitable in the conventional superspace approach. Moreover, harmonic superspace allows us to describe matter hypermultiplets in arbitrary representations of the gauge group in terms of unconstrained analytic superfields [5,16]. The above remarkable features make the harmonic formulation unique and, in principle, indispensable for the study of the quantum aspects of $N=2$ super Yang-Mills theories.

In a recent paper [17], we have presented the background field method for general $N=2$ super Yang-Mills theories in harmonic superspace. The purpose of this paper is to show that this method makes it possible to develop a covariant $N=2$ diagram technique, very much like the well known $N=1$ supergraph techniques (see [8,9,18] for a review), and, for the first time, to rigorously prove the $N=2$ non-renormalization theorem. In addition, the harmonic superspace background field method allows us to obtain some important results concerning the finite structure of the low-energy effective action at higher loops.

The harmonic formulation is naturally compatible with two pictures used to describe the $N=2$ gauge supermultiplet [5], and they prove to be very useful both at the classical and quantum levels. In the first picture, called the τ-frame, the connection is u-independent. The gauge covariant derivatives read

$$\mathscr{D}_{\underline{M}} \equiv \left(\mathscr{D}_M, D^{++}, D^{--}, D^0\right), \quad \mathscr{D}_M \equiv \left(\mathscr{D}_m, \mathscr{D}^i_\alpha, \bar{\mathscr{D}}^{\dot\alpha}_i\right) = D_M + \mathrm{i}\, A_M\,, \quad A_M = A^a_M(z) T^a \tag{5}$$

and satisfy the algebra

$$\left\{\mathscr{D}^i_\alpha, \bar{\mathscr{D}}_{\dot\alpha j}\right\} = -2\mathrm{i}\,\delta^i_j \mathscr{D}_{\alpha\dot\alpha}\,, \quad \left\{\mathscr{D}^i_\alpha, \mathscr{D}^j_\beta\right\} = 2\mathrm{i}\,\varepsilon_{\alpha\beta}\varepsilon^{ij}\bar{W}\,, \quad \left\{\bar{\mathscr{D}}_{\dot\alpha i}, \bar{\mathscr{D}}_{\dot\beta j}\right\} = 2\mathrm{i}\,\varepsilon_{\dot\alpha\dot\beta}\varepsilon_{ij} W\,,$$
$$\left[D^{\pm\pm}, \mathscr{D}_M\right] = \left[D^0, \mathscr{D}_M\right] = 0\,. \tag{6}$$

Here $D_M \equiv (\partial_m, D^i_\alpha, \bar{D}^{\dot\alpha}_i)$ are the flat covariant derivatives, T^a the generators of the gauge group and the harmonic derivatives look like [15]

$$D^{\pm\pm} = u^{\pm i}\frac{\partial}{\partial u^{\mp i}}\,, \quad D^0 = u^{+i}\frac{\partial}{\partial u^{+i}} - u^{-i}\frac{\partial}{\partial u^{-i}}\,. \tag{7}$$

The covariant derivatives and a matter superfield multiplet $\Psi(z,u)$ transform under the gauge group as follows:

$$\mathscr{D}'_{\underline{M}} = e^{\mathrm{i}\tau}\,, \quad \mathscr{D}_{\underline{M}} e^{-\mathrm{i}\tau}\,, \quad \Psi' = e^{\mathrm{i}\tau}\Psi\,, \quad \tau = \tau^a(z) T^a \tag{8}$$

with τ^a being real u-independent unconstrained parameters. The existence of the second picture, called the λ-frame, follows from the algebra (6). Introducing $\mathscr{D}^\pm_\alpha = u^\pm_i \mathscr{D}^i_\alpha$ and $\bar{\mathscr{D}}^\pm_{\dot\alpha} = u^\pm_i \bar{\mathscr{D}}^i_{\dot\alpha}$, one observes that the operators $\mathscr{D}^+_\alpha$ and $\bar{\mathscr{D}}^+_{\dot\alpha}$ anticommute. Hence

$$\mathscr{D}^+_\alpha = e^{-\mathrm{i}\Omega} D^+_\alpha e^{\mathrm{i}\Omega}\,, \quad \bar{\mathscr{D}}^+_{\dot\alpha} = e^{-\mathrm{i}\Omega}\bar{D}^+_{\dot\alpha} e^{\mathrm{i}\Omega}\,, \quad \Omega = \Omega^a(z,u) T^a\,, \tag{9}$$

for some Lie-algebra valued superfield $\Omega = \Omega^a(z,u) T^a$, called the bridge. Superfield Ω has zero $U(1)$-charge, $D^0\Omega^a = 0$, and is real, $\breve{\Omega}^a = \Omega^a$, with respect to the analyticity-preserving conjugation [5], which we denote here by $\smile$. As a consequence, one can define new superfield types; that is, covariantly analytic superfields constrained by

$$\mathscr{D}^+_\alpha \Phi^{(q)} = \bar{\mathscr{D}}^+_{\dot\alpha} \Phi^{(q)} = 0\,. \tag{10}$$

Here $\Phi^{(q)}(z,u)$ carries $U(1)$-charge q, $D^0\Phi^{(q)} = q\Phi^{(q)}$, and can be represented as follows:

$$\Phi^{(q)} = e^{-\mathrm{i}\Omega}\phi^{(q)}\,, \quad D^+_\alpha \phi^{(q)} = \bar{D}^+_{\dot\alpha}\phi^{(q)} = 0\,, \tag{11}$$

with $\phi^{(q)}(\zeta,u)$ being an unconstrained superfield over the analytic subspace (3). The Ω possesses a richer gauge freedom than the original τ-group. Its transformation law reads

$$e^{\mathrm{i}\Omega'} = e^{\mathrm{i}\lambda} e^{\mathrm{i}\Omega} e^{-\mathrm{i}\tau}\,, \quad \lambda = \lambda^a(\zeta,u) T^a\,, \tag{12}$$

where the unconstrained analytic gauge parameters $\lambda^a(\zeta,u)$ are real with respect to the analyticity-preserving conjugation, $\breve{\lambda}^a = \lambda^a$. The λ-frame is defined by

$$\mathscr{D}_{\underline{M}} \to \nabla_{\underline{M}} = e^{\mathrm{i}\Omega}\mathscr{D}_{\underline{M}} e^{-\mathrm{i}\Omega}\,, \quad \Psi \to \Psi_\lambda = e^{\mathrm{i}\Omega}\Psi\,. \tag{13}$$

The transformation laws of the gauge covariant derivatives and matter superfields read

$$\nabla_{\underline{M}} = e^{i\lambda}\nabla_{\underline{M}} e^{-i\lambda}, \quad \Psi'_\lambda = e^{i\lambda}\Psi_\lambda . \tag{14}$$

In the λ-frame we have

$$\nabla^+_\alpha = D^+_\alpha , \quad \bar{\nabla}^+_{\dot\alpha} = \bar{D}^+_{\dot\alpha} , \quad \nabla^0 = D^0 , \quad \nabla^{\pm\pm} = e^{i\Omega} D^{\pm\pm} e^{-i\Omega} = D^{\pm\pm} + iV^{\pm\pm} , \tag{15}$$

and the covariantly analytic superfield (11) turns into $\Phi^{(q)}_{\underline{\lambda}} = \phi^{(q)}$. The connection $V^{++} = V^{++a}T^a$ proves to be a real analytic superfield, $\breve{V}^{++a} = V^{++a}$, $D^+_\alpha V^{++} = \bar{D}^+_{\dot\alpha} V^{++} = 0$. This superfield turns out to be the single unconstrained prepotential of the pure $N=2$ SYM theory and all other objects are expressed in terms of it. In particular, the action of the theory reads [19]

$$S_{\rm SYM} = \frac{1}{g^2} {\rm tr} \int d^{12}z \sum_{n=2}^{\infty} \frac{(-i)^n}{n} \int du_1 \cdots du_n \frac{V^{++}(z,u_1) \cdots V^{++}(z,u_n)}{(u_1^+ u_2^+)(u_2^+ u_3^+) \cdots (u_n^+ u_1^+)} . \tag{16}$$

The rules of integration over $SU(2)$, as well as the properties of harmonic distributions, are given in Refs. [5,15].

In general, the gauge superfield is coupled to $N=2$ matter multiplets. They are described by the q-hypermultiplet $q^+(\zeta,u), \breve{q}^+(\zeta,u)$ and the ω-hypermultiplet $\omega(\zeta,u)$ [5], which are unconstrained analytic superfields and transform in complex R_q and real R_ω representations of the gauge group respectively. The massless hypermultiplet action is given by

$$S_{\rm MAT} = -\int du\, d\zeta^{(-4)}\, \breve{q}^+ \nabla^{++} q^+ - \tfrac{1}{2}\int du\, d\zeta^{(-4)}\, \nabla^{++}\omega^{\rm T} \nabla^{++}\omega , \tag{17}$$

where the integration is over the analytic subspace (2). The case when some hypermultiplets are massive corresponds to switching on an extra coupling to a covariantly constant $N=2$ super Yang-Mills background [12,20,21]. The hypermultiplet mass terms can also be obtained via the Scherk-Schwarz dimensional reduction from six dimensions [22,23,13].

In the framework of the background field method, one splits the gauge superfield V^{++} into background V^{++} and quantum v^{++} parts

$$V^{++} \to V^{++} + g\, v^{++} . \tag{18}$$

The theory (16) is quantized by imposing background covariant gauge conditions in order to obtain a gauge invariant effective action. This procedure has been carried out in [17]. The theory possesses two types of unconstrained analytic ghosts; the anticommuting Faddeev-Popov ghosts $\boldsymbol{b}(\zeta,u)$, $\boldsymbol{c}(\zeta,u)$ and the commuting Nielsen-Kallosh ghost $\phi(\zeta,u)$, all in the adjoint representation of the gauge group. The quantum action reads

$$S_{\rm QUANT} = S_2 + S_{\rm INT} , \tag{19}$$

where

$$S_2 = -\tfrac{1}{2}{\rm tr}\int du\, d\zeta^{(-4)}\, v^{++}\, \hat{\Box}_\lambda v^{++} + {\rm tr}\int du\, d\zeta^{(-4)}\, \boldsymbol{b}(\nabla^{++})^2 \boldsymbol{c} + \tfrac{1}{2}{\rm tr}\int du\, d\zeta^{(-4)}\, \phi(\nabla^{++})^2 \phi , \tag{20}$$

$$S_{\rm INT} = -{\rm tr}\int d^{12}z \sum_{n=3}^{\infty} \frac{(-ig)^{n-2}}{n} \int du_1 \cdots du_n \frac{v_\tau^{++}(z,u_1) \cdots v_\tau^{++}(z,u_n)}{(u_1^+ u_2^+)(u_2^+ u_3^+) \cdots (u_n^+ u_1^+)}$$
$$ - i\, g\, {\rm tr}\int du\, d\zeta^{(-4)}\, \nabla^{++} \boldsymbol{b}\, [v^{++}, \boldsymbol{c}] . \tag{21}$$

Here v_τ^{++} denotes the background τ-transform of v^{++}

$$v_\tau^{++} = e^{-i\Omega} v^{++} e^{i\Omega} , \tag{22}$$

and $\hat{\Box}_\lambda$ the λ-transform of the analytic d'Alembertian [2]

$$\hat{\Box} = -\tfrac{1}{2}(\mathscr{D}^+)^4(D^{--})^2, \tag{23}$$

which takes the second-order form

$$\hat{\Box} = \mathscr{D}^m\mathscr{D}_m + \frac{i}{2}(\mathscr{D}^{+\alpha}W)\mathscr{D}^-_\alpha + \frac{i}{2}\left(\bar{\mathscr{D}}^+_{\dot\alpha}\bar{W}\right)\bar{\mathscr{D}}^{-\dot\alpha} - \frac{i}{4}(\mathscr{D}^{+\alpha}\mathscr{D}^+_\alpha W)D^{--} + \frac{i}{8}[\mathscr{D}^{+\alpha},\mathscr{D}^-_\alpha]W + \tfrac{1}{2}\{\bar{W},W\}, \tag{24}$$

when acting on the covariantly analytic superfields. Using the Bianchi identities

$$\mathscr{D}^{+\alpha}\mathscr{D}^+_\alpha W = \bar{\mathscr{D}}^+_{\dot\alpha}\bar{\mathscr{D}}^{+\dot\alpha}\bar{W}, \quad [\mathscr{D}^{+\alpha},\mathscr{D}^-_\alpha]W = \left[\bar{\mathscr{D}}^+_{\dot\alpha},\bar{\mathscr{D}}^{-\dot\alpha}\right]\bar{W}, \tag{25}$$

one can present $\hat{\Box}$ in a slightly different form. The effective action is defined by the path integral representation [17]

$$e^{i\Gamma_{\rm SYM}} = e^{iS_{\rm SYM}}\int \mathscr{D}v^{++}\mathscr{D}\boldsymbol{b}\mathscr{D}\boldsymbol{c}\mathscr{D}\phi\left(\mathrm{Det}_{(4,0)}\hat{\Box}_\lambda\right)^{\frac{1}{2}} e^{iS_{\rm QUANT}}, \tag{26}$$

where $\mathrm{Det}_{(4,0)}\hat{\Box}_\lambda$ corresponds to the following functional integral over anticommuting analytic superfields $\rho^{(4)}(\zeta,u)$ and $\sigma(\zeta,u)$:

$$\mathrm{Det}_{(4,0)}\hat{\Box}_\lambda = \int \mathscr{D}\rho^{(4)}\mathscr{D}\sigma \exp\left\{ i\,\mathrm{tr}\int du\, d\zeta^{(-4)}\rho^{(4)}\hat{\Box}_\lambda\sigma\right\}. \tag{27}$$

The background-quantum splitting (18) should be accompanied by similar splitting for the matter superfields. Within the background field method, the effective action is described by the vacuum diagrams only, and the propagators and vertices are background dependent. The ghost superfields originate in the internal lines. In accordance with (20), the Nielsen-Kallosh ghost contributes to the one-loop effective action only. For the general $N=2$ SYM theory with classical action $S_{\rm SYM}+S_{\rm MAT}$, our strategy will consist of inserting all terms from $S_{\rm QUANT}$ with the matter background superfields into $S_{\rm INT}$.

The one-loop correction should be investigated separately, since it is given in terms of functional determinants of special differential operators. The purely Yang-Mills part $\Gamma^{(1)}[V^{++}]$ of the one-loop effective action $\Gamma^{(1)}$ is given by

$$\Gamma^{(1)}[V^{++}] = S_{\rm SYM} + i\,\mathrm{Tr}_{R_q}\ln(\nabla^{++}) + \frac{i}{2}\mathrm{Tr}_{R_\omega}\ln(\nabla^{++})^2 - \frac{i}{2}\mathrm{Tr}_{ad}\ln(\nabla^{++})^2 + \frac{i}{2}\mathrm{Tr}_{(2,2)}\ln\hat{\Box}_\lambda - \frac{i}{2}\mathrm{Tr}_{(4,0)}\ln\hat{\Box}_\lambda. \tag{28}$$

Here the second line includes the contributions from the matter hypermultiplets and the ghost superfields, respectively. The first term in the third line comes from the functional integral

$$\left(\mathrm{Det}_{(2,2)}\hat{\Box}_\lambda\right)^{-\frac{1}{2}} = \int \mathscr{D}v^{++}\exp\left\{-\frac{i}{2}\mathrm{tr}\int du\,d\zeta^{(-4)}v^{++}\hat{\Box}_\lambda v^{++}\right\}. \tag{29}$$

One possible prescription for calculating the functional determinants in the second line of (28) has been given in our paper [12]. These one-loop contributions to the effective action contain all information about the ultraviolet

[2] We use the notation $(\mathscr{D}^+)^4 = \frac{1}{16}(\mathscr{D}^+)^2(\bar{\mathscr{D}}^+)^2$, $(\mathscr{D}^\pm)^2 = \mathscr{D}^{\pm\alpha}\mathscr{D}^\pm_\alpha$, $(\bar{\mathscr{D}}^\pm)^2 = \bar{\mathscr{D}}^\pm_{\dot\alpha}\bar{\mathscr{D}}^{\pm\dot\alpha}$ and similar notation for the flat derivatives.

divergences of the general $N=2$ SYM theory, since the one-loop supergraphs with matter external lines, as well as all the higher loop supergraphs, will be shown to be ultravioletly finite. The functional determinants in the third line of (28) can produce only ultravioletly finite corrections to the effective action. Therefore we are not going to discuss here the one-loop effective action and concentrate our attention only on higher-loop corrections to effective action.

From Eqs. (17) and (20), one can derive the superfield propagators in the λ-frame (all indices are suppressed)

$$\begin{aligned}
\langle v^{++}(1)\, v^{++}(2)\rangle &= -\frac{\mathrm{i}}{\hat{\Box}_\lambda}\overrightarrow{(D_1^+)^4}\left\{\delta^{12}(z_1-z_2)\,\delta^{(-2,2)}(u_1,u_2)\right\}\\
&= -\frac{\mathrm{i}}{\hat{\Box}_\lambda}\left\{\delta^{12}(z_1-z_2)\,\delta^{(-2,2)}(u_1,u_2)\right\}\overleftarrow{(D_1^+)^4}\,,\\
\langle q^+(1)\,\breve{q}^{\,+}(2)\rangle &= \frac{\mathrm{i}}{\hat{\Box}_\lambda}\overrightarrow{(D_1^+)^4}\left\{e^{\mathrm{i}\Omega(1)}\delta^{12}(z_1-z_2)\frac{1}{(u_1^+u_2^+)^3}e^{-\mathrm{i}\Omega(2)}\right\}\overleftarrow{(D_2^+)^4}\,,\\
\langle \omega(1)\;\omega^{\mathrm{T}}(2)\rangle &= -\frac{\mathrm{i}}{\hat{\Box}_\lambda}\overrightarrow{(D_1^+)^4}\left\{e^{\mathrm{i}\Omega(1)}\delta^{12}(z_1-z_2)\frac{(u_1^-u_2^-)}{(u_1^+u_2^+)^3}e^{-\mathrm{i}\Omega(2)}\right\}\overleftarrow{(D_2^+)^4}\,,\\
\langle \boldsymbol{c}(1)\;\boldsymbol{b}(2)\rangle &= -\frac{\mathrm{i}}{\hat{\Box}_\lambda}\overrightarrow{(D_1^+)^4}\left\{e^{\mathrm{i}\Omega(1)}\delta^{12}(z_1-z_2)\frac{(u_1^-u_2^-)}{(u_1^+u_2^+)^3}e^{-\mathrm{i}\Omega(2)}\right\}\overleftarrow{(D_2^+)^4}\,.
\end{aligned}\tag{30}$$

Here the propagators involve the background bridge Ω, which is a non-local function of the gauge superfield V^{++}. Their structure becomes much simpler in the τ-frame

$$\begin{aligned}
\langle v_\tau^{++}(1)\, v_\tau^{++}(2)\rangle &= -\frac{\mathrm{i}}{\hat{\Box}}\overrightarrow{(\mathscr{D}_1^+)^4}\left\{\delta^{12}(z_1-z_2)\,\delta^{(-2,2)}(u_1,u_2)\right\},\\
\langle q_\tau^+(1)\,\breve{q}_\tau^{\,+}(2)\rangle &= \frac{\mathrm{i}}{\hat{\Box}}\overrightarrow{(\mathscr{D}_1^+)^4}\left\{\delta^{12}(z_1-z_2)\frac{1}{(u_1^+u_2^+)^3}\right\}\overleftarrow{(\mathscr{D}_2^+)^4}\,,\\
\langle \omega_\tau(1)\;\omega_\tau^{\mathrm{T}}(2)\rangle &= -\frac{\mathrm{i}}{\hat{\Box}}\overrightarrow{(\mathscr{D}_1^+)^4}\left\{\delta^{12}(z_1-z_2)\frac{(u_1^-u_2^-)}{(u_1^+u_2^+)^3}\right\}\overleftarrow{(\mathscr{D}_2^+)^4}\,,\\
\langle \boldsymbol{c}_\tau(1)\;\boldsymbol{b}_\tau(2)\rangle &= -\frac{\mathrm{i}}{\hat{\Box}}\overrightarrow{(\mathscr{D}_1^+)^4}\left\{\delta^{12}(z_1-z_2)\frac{(u_1^-u_2^-)}{(u_1^+u_2^+)^3}\right\}\overleftarrow{(\mathscr{D}_2^+)^4}\,.
\end{aligned}\tag{31}$$

It is seen that in the τ-frame, the propagators depend on the gauge superfield V^{++} *only* via the u-independent connection A_M specifying the gauge-covariant derivatives (5). This property of the propagators in the τ-frame turns out to be very useful for the investigation of the divergence structure.

We now present the proof of the $N=2$ non-renormalization theorem. Consider the loop expansion of the effective action within the context of the background field method. As is well known, the effective action in this framework is given by vacuum diagrams (that is, diagrams without external lines) with background field dependent propagators and vertices (see, for example [30]). In our case, the corresponding propagators are defined by Eqs. (30) and (31), and the vertices can be read off from Eqs. (21) and (17). It is evident that any such diagram can be expanded in terms of background fields, and leads to a set of conventional diagrams with an arbitrary number of external legs. To obtain the propagators and vertices for these conventional diagrams, we should switch off the background fields in Eqs. (30), (31), (21) and (17). As a result, we arrive at conventional harmonic supergraphs, the fundamentals of which were formulated in Ref. [15]. The third ghost ϕ completely decouples. We now discuss some useful features of the above supergraphs.

As follows from Eqs. (17) and (21), the gauge superfield vertices are given by integrals over the full superspace, while the matter vertices and the Faddeev-Popov ghosts vertices are given by integrals over the analytic subspace. Note, however, that propagators (30) and (31) contain factors of $(D^+)^4$, which can be used to transform integrals over the analytic subspace into integrals over the full superspace if we make use of the identity

$$\int du\, d\zeta^{(-4)} (D^+)^4 \mathscr{L} = \int d^{12}z\, du\, \mathscr{L}\,. \tag{32}$$

The cost of doing this is, as a rule, the removal of one of the two $(D^+)^4$-factors entering each matter and ghost propagator (30). There is, however, one special case. Let us consider a vertex with two external ω-legs, and start to transform the corresponding integral over the analytic subspace into an integral over the full superspace. To do this, we should remove the factor $(D^+)^4$ from one of the two gauge superfield propagators (30) associated with this vertex. As a result of transforming all integrals over the analytic subspace into integrals over the full superspace, each of the remaining propagators will contain, at most, one factor of $(D^+)^4$. Some applications of this procedure to the calculation of concrete harmonic supergraphs were considered in Refs. [15,12]. Thus, any supergraph contributing to the effective action is given in terms of the integrals over the full $N=2$ harmonic superspace. Since this conclusion is true for each conventional supergraph in the expansion of a given background field supergraph, we see that an arbitrary background field supergraph is also given by integrals over the full $N=2$ harmonic superspace. This is in complete analogy with $N=1$ supersymmetric field theories, where an arbitrary supergraph contributing to the effective action in the background field method contains only integrals over the full $N=1$ superspace, but not over the chiral subspace(see, for example [31,8,9,18]).

Once we have constructed the supergraphs with all vertices integrated over the full $N=2$ harmonic superspace, we can perform all but one of the integrals over the θ's, step by step and loop by loop, due to the spinor delta-functions $\delta^8(\theta_i - \theta_j)$ contained in the propagators (30). To do this, we remove the $(D^+)^4$-factors acting on the spinor delta-functions in the propagators by making an integration by parts. This allows one to obtain spinor delta-functions without $(D^+)^4$-factors. One can then perform the integrals over the θ's. We note that in the process of integration by parts, some of the $(D^+)^4$-factors can act on the external legs of the supergraph. To obtain a non-zero result in the case of an L-loop supergraph, we should remove $2L$ factors of $(D^+)^4$ attached to some of the propagators using the identity [15]

$$\delta^8(\theta_1 - \theta_2)(D_1^+)^4(D_2^+)^4\,\delta^8(\theta_1 - \theta_2) = (u_1^+ u_2^+)^4 \delta^8(\theta_1 - \theta_2)\,. \tag{33}$$

(explicit examples of this procedure can be found in Refs. [15,12]). Thus, any supergraph contributing to the effective action is given by a single integral over $d^8\theta$. We see again the complete analogy, at each step, with $N=1$ supersymmetric field theories (see, for example [31,8,9,18]).

The next step in our investigation is the calculation of the superficial degree of divergence for the theory under consideration. Let us consider an L-loop supergraph G with P propagators, N_{MAT} external matter legs and an arbitrary number of gauge superfield external legs. We denote by N_D the number of spinor covariant derivatives acting on the external legs as a result of integration by parts in the process of transformating the contributions to a single integral over $d^8\theta$. The superficial degree of divergence $\omega(G)$ of the supergraph G can readily be found

$$\omega(G) = 4L - 2P + (2P - N_{\mathrm{MAT}} - 4L) - \tfrac{1}{2}N_D = -N_{\mathrm{MAT}} - \tfrac{1}{2}N_D\,. \tag{34}$$

Here $4L$ is the contribution of the integrals over momenta, $-2P$ comes from the factors $\square^{-1}$ contained in the propagators and $2P - N_{MAT}$ is the contribution of the factors $(D^+)^4$ associated with the propagators. We should note that, at least, one of the two $(D^+)^4$-factors in each matter and ghost superfield propagator (30) was used to restore the full $N=2$ harmonic superspace measure $d^{12}zdu$. It follows that each of the propagators (30) effectively has, at most, one factor of $(D^+)^4$, leading to the contribution $2P$ in Eq. (34). The contribution $-4L$ arises from the fact that the factors $(D_1^+)^4(D_2^+)^4$ in the propagators were removed using equation Eqs. (33) in each of the L loops. However, if the supergraph has external matter legs, the actual number of $(D^+)^4$ factors in the propagator will be less then we counted above. Let us start with two examples given in Fig. 1 and Fig. 2.

Here the encircled ● means a vertex corresponding to the integral over the analytic subspace and ● means the same vertex transformed into an integral over the full superspace. The solid line corresponds to a matter superfield propagator. Fig. 1 shows that, in the process of the transformation, we removed all $(D^+)^4$-factors from the gauge propagators. Fig. 2 shows that, in process of transformation, we removed two factors of $(D^+)^4$ from the matter propagator. These examples illustrate the general situation that each two external matter legs take away one $(D^+)^4$-factor from the integrand. Indeed, let us consider a chain of propagators which ends at two external q^+- or ω-legs. Taking into account that any interaction in the theory under consideration necessarily includes gauge superfields, one observes that each of the above chains contains a number of vertices which is larger than the number of matter propagators by one. As a result, after restoring the full measure, we get the number of remaining $(D^+)^4$-factors to be equal to the number of propagators minus one. This means that the two external matter legs take away one factor of $(D^+)^4$ from the integrand. This result explains the term $-N_{MAT}$ in Eq. (34). In the process of integration by parts in order to restore the full measure, some of the spinor derivatives can act on the external legs. Hence, they can not influence the power of momentum in the integrand. This leads to the contribution $-\frac{1}{2}N_D$ in Eq. (34). We see immediately that all supergraphs with external matter legs are automatically finite. As to supergraphs with pure gauge superfield legs, they are clearly finite only if some non-zero number of spinor covariant derivatives acts on the external legs. We will now show that this is always the case beyond one loop.

The Feynman rules for $N=2$ supersymmetric field theories in the harmonic superspace approach have been formulated in the λ-frame, where the propagators are given by (30). As we have noted, all vertices in the background field supergraphs, including the vertices of matter and Faddeev-Popov ghosts superfields, can be given in a form containing integrals over the full $N=2$ harmonic superspace only. To be more precise, this property is stipulated by the identity in λ-frame

$$(D^+)^4 \,\hat{\square}_\lambda = \hat{\square}_\lambda (D^+)^4 . \tag{35}$$

This identity allows one to operate with factors $(D^+)^4$ as in case without background field, and use them to transform the integrals over the analytic subspace into integrals over the full superspace directly in background field supergraphs. Let us consider the structure of the propagators in the λ-frame (30). The background field V^{++} enters these propagators via both $\hat{\square}_\lambda$ and the background bridge Ω. The form of the propagators (30) has one drawback: if we use this form, we can not say how many spinor derivatives act on the external legs since the explicit dependence of Ω on the background field is rather complicated. To clarify the situation when a

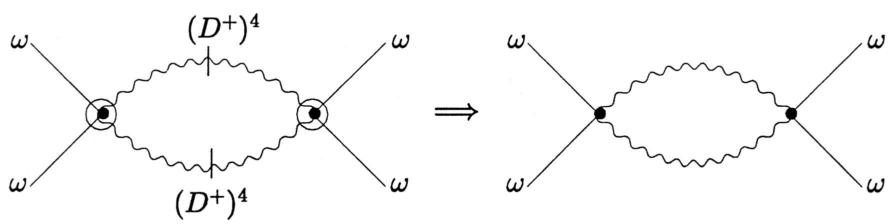

Fig. 1.

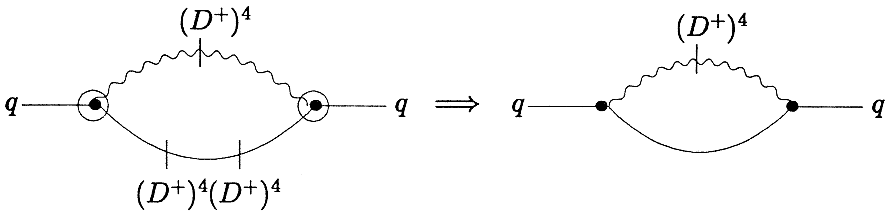

Fig. 2.

number of spinor derivatives act on external legs, we use a completely new (in comparison with conventional harmonic supergraph approach [15]) step and transform the supergraph to the τ-frame (after restoring the full superspace measure at the matter and ghost vertices). The propagators in the τ-frame are given by (31); they contain, at most, one factor of $(\mathscr{D}^+)^4$ after restoring the full superspace measure at the matter and ghost vertices. The essential feature of these propagators is that they contain the background field V^{++} only via the $\hat{\Box}$ and $\mathscr{D}^+$-factors; that is, only via the u-independent connections A_M (5) (see Eqs. (23), (24)). But all connections A_M contain at least one spinor covariant derivative acting on the background superfield V^{++} [5]. Therefore, if we expand any background field supergraph in the background superfield V^{++}, we see that each external leg must contain at least one spinor covariant derivative. Thus, the number N_D in Eq. (34) must be greater than or equal to one. As a consequence $\omega(G) \ll 0$ and, hence, all supergraphs are ultravioletly finite beyond the one-loop level. This completes the proof of the non-renormalization theorem.

The background field formulation allows us to prove some important properties of the quantum corrections to those parts of the effective action which depend on the pure $N=2$ Yang-Mills superfield V^{++}. As in conventional quantum field theory, we can suppose that $\Gamma[V^{++}]$ is described in terms of effective Lagrangians. That is

$$\Gamma[V^{++}] = \left(\int d^4x d^4\theta \, \mathscr{L}^{(c)}_{\text{eff}} + \text{c.c.} \right) + \int d^4x d^8\theta \, \mathscr{L}_{\text{eff}} \,, \tag{36}$$

where $\mathscr{L}^{(c)}_{\text{eff}}$ can be called the chiral effective Lagrangian and $\mathscr{L}_{\text{eff}}$ can be called the general effective Lagrangian. If the theory under consideration is formulated within the background field method, the effective Lagrangians $\mathscr{L}^{(c)}_{\text{eff}}$ and $\mathscr{L}_{\text{eff}}$ should be constructed only from field strengths W and $\overline{W}$ and their covariant derivatives. Therefore, the effective Lagrangians can be written as follows: $\mathscr{L}_{\text{eff}} = H(W,\overline{W}) +$ terms depending on covariant derivatives of W and $\overline{W}$ and $\mathscr{L}^{(c)}_{\text{eff}} = F(W) +$ terms depending on covariant derivatives of the strengths and preserving chirality, with holomorphic $F(W)$ and hermitian $H(W,\overline{W})$ functions of the superfield strengths. The chiral effective Lagrangian of the form $\mathscr{L}^{(c)}_{\text{eff}} = F(W)$ is associated with the leading low-energy behaviour of the effective action and defines the vacuum structure of the theory [24–26,11]. We note that the effective holomorphic Lagrangian $\mathscr{L}^{(c)}$ is analogous to the chiral effective Lagrangian in $N=1$ theories [27]. The general effective Lagrangian of the form $\mathscr{L}_{\text{eff}} = H(W,\overline{W})$ defines the first non-leading corrections to the effective dynamics [28,11,12,29].

A simple consequence of the background field formulation is that there are no quantum corrections to $H(W,\overline{W})$ at two loops in the pure $N=2$ super Yang-Mills theory without matter. All two-loop supergraphs contributing to the effective action within the background field method are given in Fig. 3. Here the wavy line

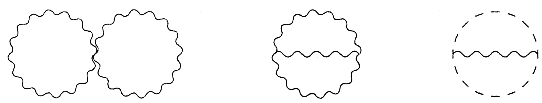

Fig. 3.

corresponds to the super Yang-Mills propagator and the dotted line to the ghost propagator. These propagators are given by Eqs. (30) and (31).

As we have noted, in order to get a non-zero result in two-loop supergraphs, we should use Eq. (33) twice. This implies that we should have 16 spinor covariant derivatives to reduce the θ-integrals over the full superspace to a single one. All these spinor derivatives come from the propagators (30) and (31). After we use one $(D^+)^4$-factor from the ghost propagator to restore the full superspace measure, we see that the propagators of both gauge and ghost superfields have at most a single factor $(D^+)^4$. It is evident that the number of these D-factors is not sufficient to form all 16 D-factors we need in two-loop supergraphs. However, there is another source of D-factors in supergraphs. Extra D-factors can come from the expansion of the inverse analytic d'Alembertian (24) in a power series of the field-strengths W and $\overline{W}$. As can be seen from (24), the spinor covariant derivatives enter the analytic d'Alembertian always multiplied by the derivatives of W and $\overline{W}$. If we omit these derivatives, the operator $\hat{\Box}$ in (24) takes the form $\hat{\Box} = \mathscr{D}^m \mathscr{D}_m + \frac{1}{2}\{\overline{W}, W\}$, and does not contain the spinor covariant derivatives. Therefore, the two-loop supergraphs given in Fig. 3 do not contribute to the effective action if the covariant derivatives of W and $\overline{W}$ are switched off. Thus, there are no two-loop quantum corrections to the non-holomorphic effective Lagrangian $H(W,\overline{W})$. It is worth pointing out that this result is simply a consequence of the $N=2$ background field method and does not demand any direct calculation of the supergraphs. Moreover, this result will be true even if we take into account the two-loop matter contribution to the effective action depending only on V^{++}. This is almost obvious since, after restoring the full superspace measure, the matter superfield propagators have effectively the same structure as the gauge and ghost superfield propagators. Another consequence of the $N=2$ background field method is a very simple proof of the known result concerning the absence of corrections to $F(W)$ beyond one loop. We will consider this last statement in a forthcoming paper.

To conclude, we have presented a rigorous and simple proof of the $N=2$ non-renormalization theorem according to which the divergences in $N=2$ super Yang-Mills theory with matter are absent beyond one loop. Our proof was based on two key details. The first is the formulation of the theory in harmonic superspace in terms of unconstrained superfields. As a result, we have no quantization problems, as compared to the formulations in conventional $N=2$ superspace. The Feynman rules have a simple structure analogous to those in $N=1$ supersymmetric theories. Second, the background field method [17] allows one to formulate a manifestly $N=2$ supersymmetric and gauge invariant perturbation procedure for calculating the effective action. The most important point of our proof was the transformation to the τ-frame, where the entire dependence of the propagators on the background gauge superfield was contained in the covariant derivatives.

The background field method gives the possibility to investigate the structure of the effective action in a very clear and simple manner. In particular, we have shown, without the necessity of a direct calculation, that there are no two-loop corrections to the effective Lagrangian $H(W,\overline{W})$.

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