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Analysis-Suitable T-Splines of arbitrary degree and dimension

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This paper defines analysis-suitable T-splines for arbitrary degree (including even and mixed degrees) and arbitrary dimension. We generalize the concept of anchor elements known from the two-dimensional setting, extend two existing concepts of analysis-suitability and justify their sufficiency for linear independence of the T-spline basis.

1 Introduction

T-splines were introduced in 2003 in computer-aided design as a new realization for B-splines on non-uniform meshes [1] with local mesh refinement [2]. Shortly after, Isogeometric Analysis was introduced, and T-splines were applied as ansatz functions for Galerkin schemes with promising results [3,4], but were proven to lack linear independence in certain cases [5], which actually excludes them from the application in a Galerkin method. The issue was solved in 2012 [6], proving that linear independence is guaranteed if meshline extensions at the hanging nodes, called T-junction extensions, do not intersect. This criterion is referred to as *analysis-suitability*. Still in 2012, the introduction of dual-compatibility and its equivalence to analysis-suitability [7] provided new insight on the linear independence of T-splines, and in 2013, analysis-suitability was generalized to T-splines of arbitrary polynomial degree [8], still restricted to the two-dimensional case. At that time, techniques for the construction of 3D T-spline meshes from boundary representations were introduced [9, 10], motivating the theoretical research on T-splines in three space dimensions, but in particular the linear independence of higher-dimensional T-splines was only characterized through the dual-compatibility criterion, until in 2016, a definition of T-junction extensions and a more abstract version of analysis-suitability in three dimensions [11] was introduced and, in 2017, generalized to arbitrary dimension [12], but only for odd polynomial degrees.

In this paper, we give a dimension-independent definition of analysis-suitable T-Splines of arbitrary degree. We generalize both approaches to analysis-suitability, the abstract and the geometric one, and argue that both are sufficient for linear independence of the corresponding T-splines.

The rest of this paper is organized as follows. Section 2 generalizes T-splines of arbitrary degree, in particular the concept of anchor elements, to arbitrary dimension and explains the construction of the T-spline blending functions. Section 3 gives a generalization of analysis-suitability in the sense of [12] to arbitrary degree, called abstract analysis-suitability, and a generalization of analysis-suitability in the sense of [8] to arbitrary dimension, called geometric analysis-suitability. Finally, we sketch a proof that geometric analysis-suitability is sufficient for abstract analysis-suitability, and hence that both criteria are sufficient for linearly independent T-splines. In Section 4, we give conclusions and outlook to future work.

2 Multivariate T-Splines

We consider a rectangular index domain $\Omega = \bigtimes_{k=1}^{d}[0, N_k]$, with $N_k \in \mathbb{N}$ for $k = 1, \dots, d$ and the corresponding parametric domain $\widehat{\Omega} = \bigtimes_{k=1}^{d}[\xi_0^k, \xi_{N_k}^k]$. Let $\mathcal{T}$ be a mesh of Ω, consisting of open axis-parallel boxes with integer vertices. For $k = 1, \dots, d$, we denote by $\mathcal{H}^{(k)}$ the set of k-dimensional mesh entities of $\mathcal{T}$. The union of all element boundaries $\mathrm{Sk} = \bigcup_{T\in\mathcal{T}} \partial T = \bigcup_{j=0}^{d-1} \mathcal{H}^{(j)} = \overline{\Omega} \setminus \mathcal{T}$ is called the skeleton of $\mathcal{T}$. For an index set $\kappa = \{\kappa_1, \dots, \kappa_n\}$ and a d-dimensional (volumetric) element $T = T_1 \times \dots \times T_d \in \mathcal{H}^{(d)} = \mathcal{T}$ composed from open intervals $T_1, \dots, T_d$, we denote the $(d-n)$-dimensional, κ-orthogonal interfaces by $\mathtt{H}^{(\kappa)}(T)$, i.e.

$$\mathtt{H}^{(\kappa)}(T) = \{\widetilde{T} = \widetilde{T}_1 \times \dots \times \widetilde{T}_d \mid \widetilde{T}_j \subsetneq \partial T_j \text{ for } j \in \kappa,\ \widetilde{T}_j = T_j \text{ for } j \notin \kappa\},$$

where the components $\widetilde{T}_j$ are either singleton sets or open intervals with start and end points in $\{0, \dots, N_j\}$.

The global set of κ-orthogonal mesh entities is denoted by $\mathtt{H}^{(\kappa)} = \bigcup_{T\in\mathcal{T}} \mathtt{H}^{(\kappa)}(T) \subseteq \mathcal{H}^{(d-n)}$, with equality only if $n = 0$ or $n = d$, see Figure 1 for a 3D illustration. Note that κ may be empty, which yields $\mathtt{H}^{(\emptyset)}(T) = \{T\}$ and $\mathtt{H}^{(\emptyset)} = \mathcal{T}$.

For polynomial degrees $\mathbf{p} = (p_1, \dots, p_d) \in \mathbb{N}^d$, we split the index domain Ω into an *active region* AR and a *frame region* FR, with

$$\mathrm{AR} \coloneqq \bigtimes_{k=1}^{d} \left[\left\lfloor \frac{p_k+3}{2} \right\rfloor, N_k - \left\lfloor \frac{p_k+3}{2} \right\rfloor\right] \quad \text{and} \quad \mathrm{FR} \coloneqq \overline{\Omega \setminus \mathrm{AR}}.$$

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We restrict ourselves to certain types of index T-meshes which we call *admissible*. The index T-mesh defines T-Splines based on the knot vectors associated with the anchor elements. Since we consider p-open knot vectors in the construction, we require the following condition on the T-meshes.

Definition 2.1 (T-junctions, admissible meshes) We define for any $k = 1, \dots, d$ and $n = 0, \dots, N_j$ the slice

$$\mathrm{S}_k(n) \coloneqq \bigtimes_{j=1}^{k-1} [0, n_j] \times \{n\} \times \bigtimes_{j=k+1}^{d} [0, n_j] = \{(x_1, \dots, x_d) \in \overline{\Omega} \mid x_j = n\},$$

and we call an interface $E \in \mathcal{H}^{(d-2)}$ with $E \not\subseteq \partial\Omega$ a *hanging interface* or *T-junction* if it has valence $|\{H \in \mathcal{H}^{(d-1)} \mid E \subset \partial H\}| < 4$. Finally, a mesh $\mathcal{T}$ is called *admissible* if for $k = 1, \dots, d$

$$\mathrm{S}_k(n) \subseteq \mathrm{Sk} \quad \text{for } n = 0, \dots, \left\lfloor \frac{p_k + 3}{2} \right\rfloor \text{ and } n = N_k - \left\lfloor \frac{p_k + 3}{2} \right\rfloor, \dots, N_k, \tag{1}$$

and if there are no hanging interfaces in the frame region.

Definition 2.2 (anchors) Let $\mathbf{p} = (p_1, \dots, p_d)$ be the vector of polynomial degrees of the T-splines. The set of anchors is then defined by

$$\mathcal{A} = \{\mathbf{A} \in \mathtt{H}^{(\kappa)} \mid \mathbf{A} \subset \mathrm{AR}\} \quad \text{with } \kappa = \{\ell \mid p_\ell \text{ odd }\}.$$

Definition 2.3 (Index sets and vectors) For any mesh entity $E = E_1 \times \cdots \times E_d \in \mathtt{H}^{(\ell)}$ with an index set $\ell \subseteq \{1, \dots, d\}$, we define the index sets

$$\mathcal{I}_j(E) \coloneqq \{n \in \mathbb{N} \mid E_1 \times \cdots \times E_{j-1} \times \{n\} \times E_{j+1} \times \cdots \times E_d \subset \bigcup \mathtt{H}^{(\ell \cup \{j\})}\}.$$

The *index vectors* $\mathrm{v}_j(\mathbf{A})$ for an anchor $\mathbf{A} = A_1 \times \cdots \times A_d$ are defined as subsets of the index sets $\mathcal{I}_j(\mathbf{A})$ given by:

- If p_j is odd, then $\mathrm{v}_j(\mathbf{A}) \in \mathbb{N}^{p_j+2}$ consists of the $p_j + 2$ consecutive indices $\ell_0, \dots, \ell_{p_j+1}$ in $\mathcal{I}_j(\mathbf{A})$, such that $A_j = \{\ell_{(p_j+1)/2}\}$ is the middle element.
- If p_j is even, then $\mathrm{v}_j(\mathbf{A}) \in \mathbb{N}^{p_j+2}$ consists of the $p_j + 2$ consecutive indices $\ell_0, \dots, \ell_{p_j+1}$ in $\mathcal{I}_j(\mathbf{A})$, such that $A_j = (\ell_{p_j/2}, \ell_{p_j/2+1})$ is the interval bounded by the two middle elements.

An example is given in Figure 2. With $\mathbf{p} = (3, 2)$, the set of anchors is given by all the vertical line segments in the active region AR. In Figure 2 the anchor $\mathbf{A} = \{3\} \times (2, 3)$ is marked by a solid dot. The index sets are given by fixing one coordinate and checking for which integer it is in the set of vertices, hence $\mathcal{I}_1(\mathbf{A}) = \{0, 1, 2, 3, 5, 6, 7\}$ and $\mathcal{I}_2(\mathbf{A}) = \{0, 1, 2, 3, 4, 5\}$. In the first coordinate we have $p_1 = 3$ (odd), thus we choose the $p_1 + 2 = 5$ indices $\ell_0^1, \dots, \ell_4^1$, s.t. $\ell_2^1 = 3$ and we get $\mathrm{v}_k(\mathbf{A}) = (1, 2, 3, 5, 6)$. In the other direction we have $p_2 = 2$ (even), and we choose again the $p_2 + 2 = 4$ indices

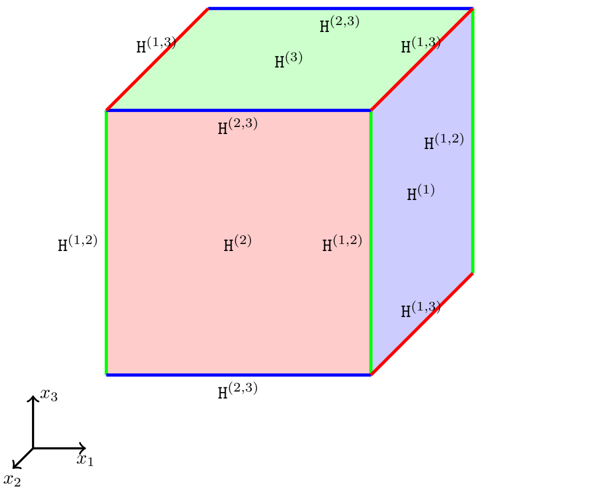

Fig. 1: Visualization of the different entities. The lines $\mathtt{H}^{(i,j)}$ have coordinates x_i and x_j fixed, and the planes (resp. faces) $\mathtt{H}^{(k)}$ have coordinate x_k fixed. For sake of simplicity, the vertices have been left out in this example.

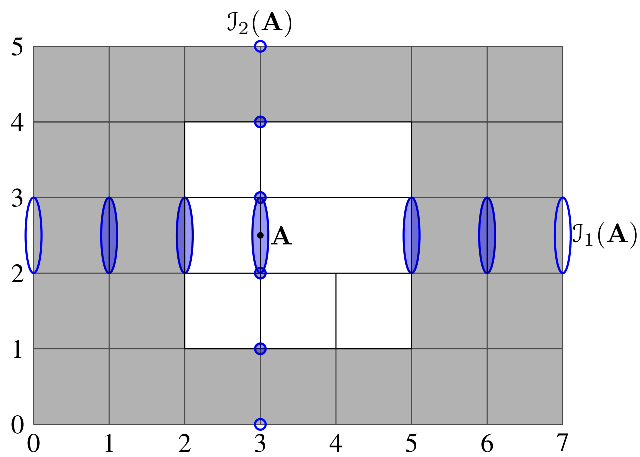

Fig. 2: Visualization of index sets and vectors for an anchor $\mathbf{A}$. The filled ellipses resp. circles correspond to the indices of the index set that fill the index vector. The marked entities correspond to the indices used for the index set, resp. vector.

$\ell_0^2, \ldots, \ell_3^2$, s.t. $(\ell_1^2, \ell_2^2) = (2, 3)$ and we get $\mathrm{v}_k(\mathbf{A}) = (1, 2, 3, 4)$. Since the index vector $\mathrm{v}_k(\mathbf{A})$ is associated to the knot vector $(\xi_{\ell_1^k}, \ldots, \xi_{\ell^k_{p_k+2}})$, the support of the T-Spline at the anchor $\mathbf{A}$ is

$$\operatorname{supp} \hat{B}_{\mathbf{A}} = [\xi_1^1, \xi_6^1] \times [\xi_1^2, \xi_4^2].$$

In a structured, uniform mesh, this construction yields the usual tensor-product B-spline basis. However, a non-uniform refinement (i.e. adaptive) results in meshes as demonstrated in Figure 2, where B-splines or NURBS cannot be applied.

Definition 2.4 (T-spline) For $p_j \in \mathbb{N}$, we denote by $B_{\mathrm{v}_j(\mathbf{A})} \colon \widehat{\Omega} \to \mathbb{R}$ the univariate B-spline function of degree p_j that is returned by the Cox-deBoor recursion with knot vector $\xi_{\mathrm{v}_j(\mathbf{A})} = (\xi_{\ell_0}, \ldots, \xi_{\ell_{p_j+1}})$. The T-spline function associated with the anchor $\mathbf{A}$ is defined as

$$B_{\mathbf{A}}(\zeta_1, \ldots, \zeta_d) \coloneqq \prod_{j=1}^{d} B_{\mathrm{v}_j(\mathbf{A})}(\zeta_j), \quad \text{for } (\zeta_1, \ldots, \zeta_d) \in \widehat{\Omega}, \tag{2}$$

and the corresponding T-spline space is given by $\mathcal{S}_{\mathrm{T},\mathcal{A}}(\widehat{\Omega}) = \operatorname{span}\{B_{\mathbf{A}} \mid \mathbf{A} \in \mathcal{A}\}$. The index support of $B_{\mathbf{A}}$ will be denoted by $\operatorname{supp}_{\Omega}(B_{\mathbf{A}}) = \bigtimes_{k=1}^{d} \operatorname{conv}(\mathrm{v}_k(\mathbf{A}))$, where $\operatorname{conv}(M)$ is the convex hull of a set M.

3 Analysis-Suitability

Definition 3.1 (Abstract T-junction extensions and analysis-suitability) We define for all $j = 1, \ldots, d$ and $n = 0, \ldots, N_j$ the *abstract T-junction extension*

$$\mathrm{ATJ}_j(n) = \mathrm{S}_j(n) \cap \bigcup_{\substack{\mathbf{A}\in\mathcal{A} \\ n\in\mathcal{I}_j(\mathbf{A})}} \operatorname{supp}_{\Omega}(B_{\mathbf{A}}) \cap \bigcup_{\substack{\mathbf{A}\in\mathcal{A} \\ n\notin\mathcal{I}_j(\mathbf{A})}} \operatorname{supp}_{\Omega}(B_{\mathbf{A}}) \quad \text{and} \quad \mathrm{ATJ}_j = \bigcup_{n=0}^{N_j} \mathrm{ATJ}_j(n).$$

We call the mesh $\mathcal{T}$ *abstractly analysis suitable (AAS)* if the abstract T-junction extensions are pairwise disjoint, i.e. if $\mathrm{ATJ}_i \cap \mathrm{ATJ}_j = \emptyset$ for $i \neq j$.

This definition is applicable in the index space as well as in the parametric space, and it can be shown that AAS T-meshes generate linearly independent T-splines, see [12, Theorems 5.3.14 and 5.3.15]. However, an application in practice as a sufficient criterion for linear independence is likely to be more expensive than checking for singularity of the system matrix, including assembly. We therefore introduce a second, geometric approach to analysis-suitability which refers to the classical notion of T-junction extensions, see e.g. [6, 13, 14].

Definition 3.2 (Geometric T-junction extensions and analysis-suitability) Let $\mathtt{T} = \mathtt{T}_1 \times \cdots \times \mathtt{T}_d \in \mathcal{H}^{(d-2)}$ be a T-Junction, i.e. $\mathtt{T}$ is a hanging interface, with $\mathtt{T}_i$ and $\mathtt{T}_j$ being singletons, and let $\mathcal{I}_k(\mathtt{T})$ be its corresponding index sets. For each index k we define the extension (index) vector $v_k^e(\mathtt{T})$, to be the vector of the (p_k+1) consecutive indices of $\mathcal{I}_k(\mathtt{T})$, s.t. $\mathtt{T}_k$ is the middle element (see also the definition of index vectors for anchors). Further, let $\ell = i$ or $\ell = j$ be the index, s.t.

$$\mathrm{GTJ}_\ell(\mathtt{T}) \coloneqq \mathrm{S}_\ell(\mathtt{T}_\ell) \cap \operatorname{conv}\big([v_1^e(\mathtt{T}) \times \cdots \times v_d^e(\mathtt{T})]\big) \not\subset \mathtt{H}^{(\ell)}.$$

We then call $\mathrm{GTJ}_\ell(\mathtt{T})$ the *geometric T-Junction extension of* $\mathtt{T}$. Further, we say that $\mathcal{T}$ is *geometrically analysis-suitable (GAS)*, if for all $\mathtt{T1}, \mathtt{T2} \in \mathcal{H}^{(d-2)}$, with $\mathtt{T1}$ and $\mathtt{T2}$ hanging interfaces and corresponding direction $i \neq j$, there is $\mathrm{GTJ}_i(\mathtt{T1}) \cap \mathrm{GTJ}_j(\mathtt{T2}) = \emptyset$.

Note that ℓ is not necessary unique, however, if the mesh is geometrically analysis-suitable it is by definition.

The key ideas in both definitions are the same, i.e. the T-Junction extensions are required to be pairwise disjoint. Although the above definitions of analysis-suitability seem very different, the following theorem shows their connection, namely, that abstract T-junction extensions are neighborhoods of T-junctions.

Theorem 3.3 *All GAS T-meshes are AAS.*

Sketch of proof. We know that all uniform meshes (i.e., meshes without T-junctions and hence empty ATJs and non-existent GTJs) are AAS and GAS. Consider a non-uniform T-mesh $\mathcal{T}$ and suppose w.l.o.g. that $\mathrm{ATJ}_1 \neq \emptyset$ and hence $\mathrm{ATJ}_1(n) \neq \emptyset$ for some $n \in \{0, \ldots, N_j\}$. Then there exists

$$x \in \mathrm{ATJ}_1(n) = \mathrm{S}_1(n) \cap \bigcup_{\substack{\mathbf{A}\in\mathcal{A} \\ n\in\mathcal{I}_1(\mathbf{A})}} \operatorname{supp}_{\Omega}(B_{\mathbf{A}}) \cap \bigcup_{\substack{\mathbf{A}\in\mathcal{A} \\ n\notin\mathcal{I}_1(\mathbf{A})}} \operatorname{supp}_{\Omega}(B_{\mathbf{A}}),$$

and there exist anchors $\mathbf{A}^{(1)} = \mathbf{A}_1^{(1)} \times \cdots \times \mathbf{A}_d^{(1)}, \mathbf{A}^{(2)} = \mathbf{A}_1^{(2)} \times \cdots \times \mathbf{A}_d^{(2)} \in \mathcal{A}$ with $n \in \mathcal{I}_1(\mathbf{A}^{(1)})$ and $n \notin \mathcal{I}_1(\mathbf{A}^{(2)})$ such that $x \in \mathrm{S}_1(n) \cap \operatorname{supp}(B_{\mathbf{A}^{(1)}}) \cap \operatorname{supp}(B_{\mathbf{A}^{(2)}})$. The definition of $\mathcal{I}_1(\bullet)$ yields that $\{n\} \times \mathbf{A}_2^{(1)} \times \cdots \times \mathbf{A}_d^{(1)} \subseteq \bigcup \mathtt{H}^{(\ell \cup \{1\})}$

and $\{n\} \times \mathbf{A}_2^{(2)} \times \cdots \times \mathbf{A}_d^{(2)} \not\subseteq \bigcup \mathtt{H}^{(\ell \cup \{1\})}$. Hence, there exist points $r, s \in \mathrm{S}_1(n) \cap \operatorname{supp}(B_{\mathbf{A}^{(1)}}) \cap \operatorname{supp}(B_{\mathbf{A}^{(2)}})$ such that $r \in \mathrm{S}_1(n) \cap \bigcup \mathtt{H}^{(\ell \cup \{1\})}$ and $s \in \mathrm{S}_1(n) \setminus \bigcup \mathtt{H}^{(\ell \cup \{1\})}$. Somewhere between r and s is a hanging interface $\mathtt{T} \subset \mathrm{S}_1(n) \cap \operatorname{supp}(B_{\mathbf{A}^{(1)}}) \cap \operatorname{supp}(B_{\mathbf{A}^{(2)}}) \cap \bigcup \mathtt{H}^{(\ell \cup \{1\})}$. The maximal distance between x and $\mathtt{T}$ are $\frac{p+1}{2}$ segments in dimensions with odd polynomial degree and $\frac{p}{2}$ segments in dimensions with even polynomial degree, except the first dimension, since both x and $\mathtt{T}$ are in $\mathrm{S}_1(n)$.

Consequently, the arbitrarily chosen $x \in \mathrm{ATJ}_1$ is always contained in some geometric T-junction extension, and hence the union of all geometric T-junction extensions in first direction is a superset of the union of the corresponding abstract T-junction extension,

$$x \in \mathrm{GTJ}_1(\mathtt{T}) \subseteq \bigcup_{\substack{\mathtt{T}' \text{ T-junction} \\ \text{in direction } 1}} \mathrm{GTJ}_1(\mathtt{T}') \quad \text{for all } x \in \mathrm{ATJ}_1 \quad \Longrightarrow \quad \mathrm{ATJ}_1 \subseteq \bigcup_{\substack{\mathtt{T}' \text{ T-junction} \\ \text{in direction } 1}} \mathrm{GTJ}_1(\mathtt{T}')$$

and analogously for other dimensions $j = 2, \ldots, d$.

If $\mathcal{T}$ is not AAS, then two ATJs intersect, so will two GTJs, and hence $\mathcal{T}$ is not GAS. This concludes the proof. □

4 Conclusions & Outlook

We have generalized analysis-suitability of T-splines to arbitrary degree in higher dimensions. In addition to the abstract notion of analysis-suitability developped in [12], a generalized version of classical geometric T-junction extensions was shown to be an appropriate tool for analysis-suitability as well. Ongoing work is the detailed elaboration of the sketched proof of Theorem 3.3 and the precise application of [12, Theorems 5.3.14 and 5.3.15] to the definitions given here, furthermore an implementation for trivariate AS T-splines and the development of anisotropic refinement schemes that preserve analysis-suitability.

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