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Limit cycle computation of self-excited dynamic systems using nonlinear modes

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A self-excited dynamic system is able to oscillate periodically by itself. Corresponding solutions of the autonomous differential equation are called limit cycles or periodic attractors. To find these solutions, a simple approach would be brute-force search for the corresponding basins of attraction. However, grid searching might become unfeasible with increasing number of degrees of freedom. Instead, solution path continuation techniques are often used to keep computational costs low. As the continuation of solution branches and their bifurcations provides only solutions which are connected to each other, isolas and detached branches are missed out. We present a method for fast limit cycle detection of self-excited systems with isolas based on nonlinear modes. A nonlinear mode, often referred to as nonlinear normal mode, is defined as a periodic motion of the undamped and unforced mechanical system. For nonconservative systems however, e.g. with friction nonlinearity, damping cannot be neglected as it is characteristic for the oscillators nonlinear dynamics. Therefore, the Extended Periodic Motion Concept (E-PMC) was proposed recently to find periodic solutions of nonconservative nonlinear systems. In this work, the E-PMC is applied to self-excited dynamic systems in order to find periodic attractors along its nonlinear modes. Zero crossings of the nonlinear damping curve indicate autonomous solutions which can be used as starting points for single parameter continuation. Thus, solutions corresponding to the main branch and detached curves in the solution space are connected by nonlinear modes. The proposed method is applied to a frictional oscillator with cubic stiffness and proves to be robust in the search for isolated periodic solutions that are already known from literature.

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1 Concept of nonlinear modes

A nonlinear mode (NM) is defined as a (nonnecessarily synchronous) periodic motion of a nonlinear system for which damping and forcing terms are neglected [2]. NMs are thus periodic solutions of the autonomous system resulting from the general equation of motion as follows

$$\underline{M}\ddot{\vec{x}} + \underline{C}\dot{\vec{x}} + \underline{K}\vec{x} + \vec{f}_{\mathrm{nl}}(\dot{\vec{x}}, \vec{x}) = \vec{F}(t) \quad , \tag{1}$$

$$\Rightarrow \quad \underline{M}\ddot{\vec{x}} + \underline{K}\vec{x} + \vec{f}_{\mathrm{nl}}(\vec{x}) = \vec{0} \quad , \tag{2}$$

where nonlinear forces are conservative. Applying Shooting or Harmonic Balance techniques, solutions of the autonomous system can be found denoted as the *Periodic Motion Concept* (PMC), cf. [1] and [3]. Like their linear counterpart, i.e. linear modes, a NM describes the region of maximum response under external forcing. In contrast to linear systems, resulting eigenfrequencies can be energy dependent which reflects the characteristic nonlinear behaviour, e.g. stiffening or softening behavior under increasing amplitudes. However, in the case of non-conservative nonlinearities neglecting damping and velocity dependent nonlinear force terms is not appropriate as fundamental nonlinear mechanisms are ignored. For periodic motion, an unforced non-conservative nonlinear system requires energy inputs to compensate the dissipative terms, as in the case of self-excitation. Thus, to analyse periodic oscillation of non-conservative systems, the *Extended Periodic Motion Concept* (E-PMC) was proposed, which introduces artificial damping in order to balance the system's energy at all times [1]. Mass proportional damping of the form $-\xi\underline{M}\dot{\vec{x}}$ is added to the unforced system yielding

$$\underline{M}\ddot{\vec{x}} + \left(\underline{C} - \xi\underline{M}\right)\dot{\vec{x}} + \underline{K}\vec{x} + \vec{f}_{\mathrm{nl}}(\dot{\vec{x}}, \vec{x}) = \vec{0} \quad . \tag{3}$$

This way, periodic solutions can be computed using the same techniques as in the conservative case. The unknown ξ is computed such that energy sources and sinks are balanced, hence achieving a periodic motion of the system in Eq. 3. Using a mass proportional negative damping term is in accordance with linear modal analysis and modal damping. Considering linear systems, the artificial damping is not affecting frequencies or mode shapes and the relation $2D\omega_0 = \xi$ holds. For nonlinear systems, mode shapes are no longer orthogonal and artificial modal coupling may be introduced. In this work, we propose to use the artificial damping as an indicator function for autonomous solutions of the true nonlinear system. Such solutions are found at points along the NM for which $\xi = 0$ hold. Here, no artificial damping is required, and as ξ vanishes, no kind of artificial modal coupling is introduced. The E-PMC is thus an excellent choice for limit cycle detection of self-excited systems with non-conservative nonlinear forces. Details considering implementation and further results can be found in [6] and [7].

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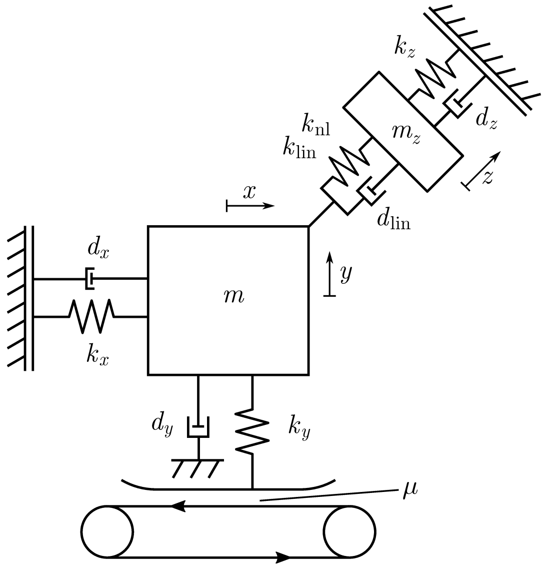

Fig. 1: 3 DOF nonlinear oscillator model, cf. [4] and [6]

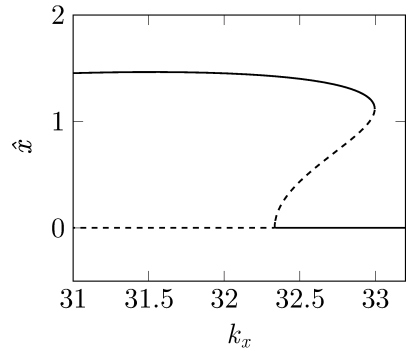

Fig. 2: Bifurcation diagram of 3 DOF model for variation of k_x

2 Limit cycle computation using nonlinear modes

Limit cycles of non-conservative nonlinear systems can be represented by 1D-submanifolds for a certain parameter variation. Fig. 2 depicts a subcritical Hopf bifurcation of the 3 DOF model from Fig. 1. The static equilibrium and a limit cycle oscillation (solid lines) are connected by an unstable branch (dashed line). However, there may be limit cycle solutions which are not connected to either of those solutions, hence forming isolas in the given parameter space. Nonlinear modes can be used here to find limit cycle solutions whithout changing a system parameter. Zero crossings of the artificial damping parameter ξ along the nonlinear mode indicate periodic solutions of the dynamical system. This complementary approach to limit cycle detection is applied to the 3 DOF nonlinear oscillator to point out its benefits.

3 3 DOF nonlinear oscillator limit cycles

The 3 DOF system shown in Fig. 1 will be analysed with respect to limit cycles, cf. [4]. The mass m_z is connected to the base oscillator via a linear and an additional cubic nonlinear spring as well as a linear dashpot damper. Energy is fed in the system through the frictional contact with a continuously moving belt. Fig. 2 shows the bifurcation plot for variation of stiffness k_x exhibiting limit cycle oscillation with typical bifurcation. Bifurcation diagrams are computed using orthogonal collocation implemented in the software package MatCont [5]. A nonlinear modal analysis is performed applying a Harmonic Balance approach with path continuation within the E-PMC with artificial damping. For the 3 DOF system there are 3 nonlinear

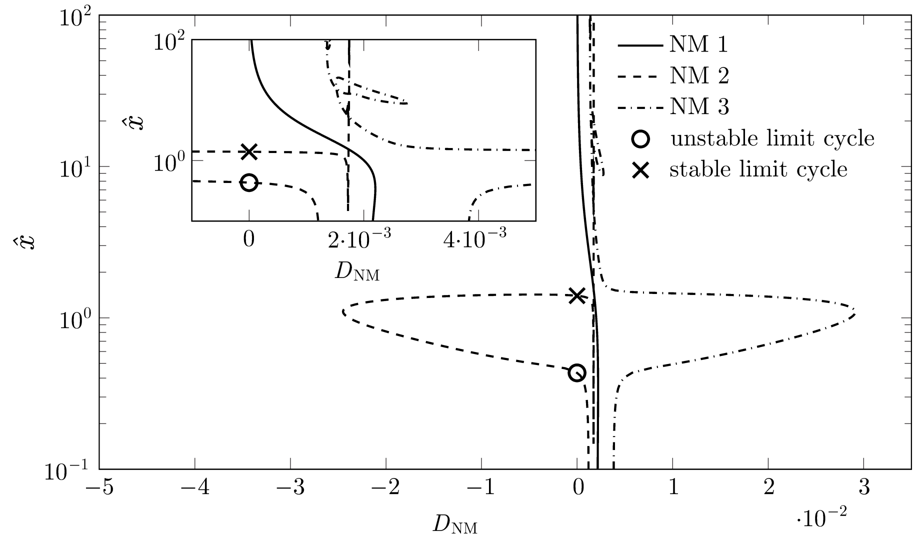

Fig. 3: Nonlinear modes of the 3 DOF oscillator at $k_x = 32.5$ with marked limit cycle solutions for which $D_{\mathrm{NM}} = 0$, cf. [6].

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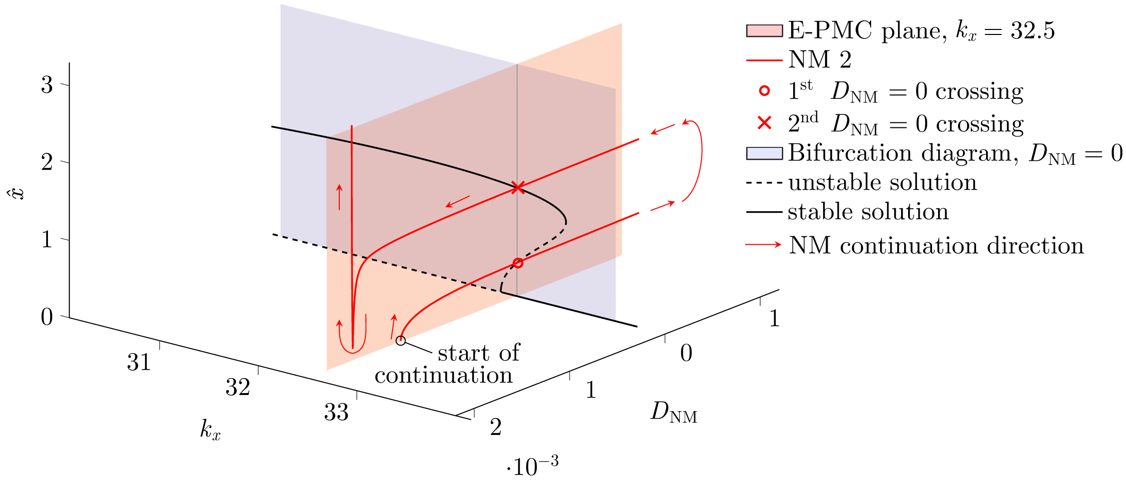

Fig. 4: Combined results of bifurcation analysis and nonlinear mode NM 2 computed for $k_x = 32.5$, cf. [6].

modes which coincide with their linear counterpart for small amplitudes, see Fig. 3. The nonlinear modes are plotted over the equivalent damping ratio D_{NM} which is defined by

$$D_{NM} = \frac{\xi}{2\omega_0} \quad , \tag{4}$$

with energy dependent eigenfrequency $\omega_0 > 0$ of the corresponding nonlinear mode. At $k_x = 32.5$, there are two zero crossings of the equivalent damping ratio, which is zero for $\xi = 0$, i.e. two periodic solutions of the oscillator are detected by the proposed approach.

The bifurcation diagram and the nonlinear mode computation are combined in Fig. 4. Here, the complementary computation reveals its benefits in finding periodic solutions for a fixed set of system parameters (i.e. here: fixing $k_x = 32.5$). Both, the stable and unstable limit cycle solutions are detected and coincide with the solution computed using orthogonal collocation. Fig. 5 shows the bifurcation plot from Fig. 2 for a wider range of k_x and with stepwise reduced damping values $d = d_x = d_y = d_z$. For the smaller damping values, the NMs exhibit three zero-crossings of the artificial damping term, i.e. indicating three periodic solutions of the system. The solution at low amplitude values corresponds to the stable limit cycle linked to the equilibrium, but the two intersections at high amplitude values correspond to an isolated solution branch, which is not connected to any other solution. Hence, finding this isola would have remained to a considerable amount of luck during the analysis of the system. The E-PMC adds significant value at this point, as it turns out to be a fast and robust method for finding those solutions. The additional coexisting periodic solutions found for high amlitude values $\hat{x}$ can each be set as a starting point for continuation in the $\hat{x} - k_x-$plane to compute the evolved isola. Further reduction of damping coefficients results in a larger isola which shows typical stable and unstable sections.

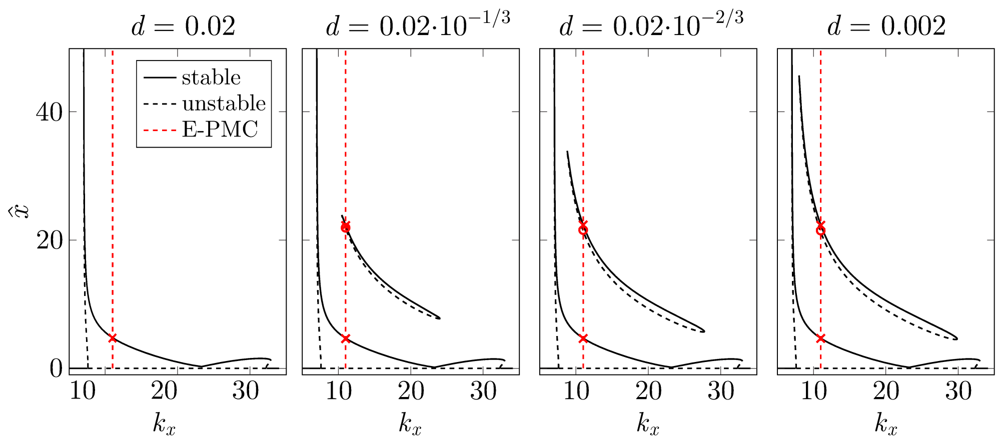

Fig. 5: Analysis results for different levels of damping with $d = d_x = d_y = d_z$ revealing an isolated branch of limit cycle solutions which is found applying the E-PMC path continuation, cf. [6].

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4 Conclusion

Nonlinear modes are a concept allowing for modal analysis of nonlinear systems motivated by its counterpart in linear modal analysis. Recent developments in nonlinear mode computation of non-conservative systems open up new possibilities beyond system characterization. In this work, the *Extended Periodic Motion Concept* (E-PMC) for non-conservative systems proposed in [1] is applied to limit cycle detection of a 3 DOF self-excited oscillator. The E-PMC is based on an artificial damping term to cancel out dissipative mechanisms and thus creating periodic motion of the frictional nonlinear system. A zero crossing of the resulting equivalent damping value D_{NM} of a nonlinear mode indicates an autonomous periodic motion which is related to a limit cycle solution of the self-excited system. This way, a limit cycle detection complementary to bifurcation analysis is established. For the 3 DOF nonlinear oscillator coexisting stable limit cycles are found one of which is located on an isolated solution path disconnected from the equilibrium. Limit cycle search using E-PMC is thus a promising supplement to classical bifurction analysis.

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