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Results in Mathematics

A General Simonenko Local Principle and Fredholm Condition for Isotypical Components

Alexandre Baldare

Abstract. In this paper, we derive, from a general Simonenko's local principle, Fredholm criteria for restriction to isotypical components. More precisely, we give a full proof, of the equivariant local principle for restriction to isotypical components of invariant pseudodifferential operators announced in Baldare et al. (Muenster J Math, 2021). Furthermore, we extend this result by relaxing the hypothesis made in the preceding quoted paper.

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Contents

Introduction

This paper is devoted to the proof and an application of a general Simonenko's local principle to G-invariant operators on closed manifolds. Local principles first appeared in Simonenko's work [52] and more general forms appeared in [1,28,31]. Since then local principles were intensively used to obtain Fredholm condition for singular operators, see for examples [14,37,46,51,58] and the references therein. As a consequence of the general Simonenko's local principle, we derive Fredholm conditions for restriction of G-invariant pseudodifferential operators to isotypical components.

Let G be a compact Lie group and denote by $\widehat{G}$ the set of isomorphism classes of irreducible unitary representations of G. If $P : \mathcal{H} \to \mathcal{H}'$ is a G-invariant continuous linear map between Hilbert spaces and $\alpha \in \widehat{G}$, then the operator P induces a well defined continuous linear map between the α-isotypical components

$$\pi_\alpha(P) : \mathcal{H}_\alpha \to \mathcal{H}'_\alpha.$$

In this paper, we are interested in the case where P is a pseudodifferential operator acting between sections of two vector bundles.

Assume that our compact Lie group G acts smoothly and isometrically on a compact Riemannian manifold M and on two hermitian vector bundles E_0 and E_1. Furthermore, let $P : \mathcal{C}^\infty(M, E_0) \to \mathcal{C}^\infty(M, E_1)$ be a G-invariant, classical, order m, pseudodifferential operator on M. Since P is G-invariant, its principal symbol $\sigma_m(P)$ belongs to $\mathcal{C}^\infty(T^*M \smallsetminus \{0\}; \mathrm{Hom}(E_0, E_1))^G$. Let G_ξ and G_x denote the isotropy subgroups of $\xi \in T^*_x M$ and $x \in M$, as usual. Then $G_\xi \subset G_x$ acts linearly on the fibers E_{0x} and E_{1x}. Following [4], denote by $T^*_G M$ the *G-transverse cotangent space*, see Eq. (1) and by $S^*_G M := S^*M \cap T^*_G M$ the set of unit covectors in the G-transverse cotangent space $T^*_G M$.

The previous set leads to the definition of G-transversally elliptic operators [4,54]. Recall that a G-transversally elliptic pseudodifferential operator on M is a G-invariant pseudodifferential operator whose principal symbol becomes invertible when restricted to $T^*_G M \smallsetminus \{0\}$. Since M is compact, we know that this operators are generally not Fredholm due to the lack of full ellipticity. Nevertheless, the, now well known, Atiyah-Singer's result states that if P is G-transversally elliptic then $\pi_\alpha(P)$ is Fredholm for any $\alpha \in \widehat{G}$, [4,54]. This allows directly to define an index for G-transversally elliptic operators as an element of the K-homology of C^*G, the group C^*-algebra of G. Furthermore, with little more work, Atiyah and Singer showed that this index is, in fact, a Ad-invariant distribution on G. See also [6,8,33,35,36] for related results and [7,12,44] for index theorems on G-transversally elliptic operators using equivariant cohomology. The Fredholm property of this restrictions to isotypical component was the starting point for the study carried out in [11].

We now proceed to state the main result studied in this paper but first we need few more notations and definitions from [9–11].

Assume M/G connected and let K be a minimal isotropy subgroup of G, see [16,56]. We shall say that P is transversally α-elliptic if for all $\xi \in (S_G^* M)^K$ the linear map

$$\sigma_m(P)(\xi) \otimes \mathrm{Id}_{\alpha^*} : (E_{0\xi} \otimes \alpha^*)^K \to (E_{1\xi} \otimes \alpha^*)^K$$

is invertible.

One of the main results of [11] states that P is transversally α-elliptic if, and only if, $\pi_\alpha(P)$ is Fredholm. Here, we point out that the transversal **1**-ellipticity is related with transversal ellipticity on (singular) foliations [3,25]. For G finite, this results were proved before [9,10].

In the present paper, we recall, in Definition 3.1, the notion of locally α-invertible operator at $x \in M$ introduced in [10] and we show in full generality the following result, see Theorem 3.11.

Theorem. *Assume that M is a closed, smooth manifold and that G is a compact Lie group acting smoothly on M. Let $P \in \psi^m(M; E_0, E_1)^G$ and $\alpha \in \widehat{G}$. Then the following are equivalent:*

(1) $\pi_\alpha(P) : H^s(M; E_0)_\alpha \to H^{s-m}(M; E_1)_\alpha$ is Fredholm for any $s \in \mathbb{R}$,
(2) P is transversally α-elliptic,
(3) P is locally α-invertible.

Notice that in [10], the equivalence between (1) and (2) was stated without proof under the hypothesis that $\dim G < \dim M$. Moreover, the triple equivalence was then deduced in the case of finite group using the main result of [10]. Here care is taken to state it in full generality and relax the hypothesis $\dim G < \dim M$. This proposition enlightens the results from [9–11] in the sense that it explains the local computations done.

The previous theorem, as well as intermediate results in this paper, were obtained during discussions with R. Côme, M. Lesch and V. Nistor.

We point out that the Fredholm conditions obtained in this paper are closely related to the ones in [48], for G-operators, and the ones in [18], for complexes of operators. Fredholm conditions were also investigated in different forms in [15,49,50,55] for boundary problems and in [23,29,30,32,40–42,45] using techniques of limit operators and also C^*-algebras methods. The techniques of limit operators are similar to the one used in [11] to obtain the Fredholm criterion for α-transversally elliptic operator, see also Sect. 1.3. Recent developments on singular operators including groupoid and C^*-algebras were accomplished in [2,13,19–22,24,26,38,39].

1. Preliminaries

This section is devoted to background material and results. The reader can find more details in [9–11]. The reader familiar with [9–11] can skip this section at a first reading.

1.1. Group Actions

Throughout the paper, we let G be a *compact* Lie group. Assume that G acts on a space X and that $x \in X$, then Gx is the G orbit of x and

$$G_x := \{g \in G \mid gx = x\} \subset G$$

the isotropy group of the action at x.

If $H \subset G$ is a subgroup, then $X_{(H)}$ will denote the set of elements of X whose isotropy G_x is conjugated to H in G and $G \times_H X$ the space

$$G \times_H X \,:=\, (G \times X)/\sim,$$

where $(gh, x) \sim (g, hx)$ for all $g \in G, h \in H$, and $x \in X$.

Let V and W be locally convex spaces and $\mathcal{L}(V;W)$ be the set of continuous, linear maps $V \to W$. We let $\mathcal{L}(V) := \mathcal{L}(V;V)$. A representation of G on V is a continuous group morphism $G \to \mathcal{L}(V)$, where $\mathcal{L}(V)$ is equipped with the strong operator topology. Said differently, the map $G \times V \to V$ given by $(g, v) \mapsto gv$ is continuous and $v \mapsto gv$ is linear. We shall also call V a G-module.

For any two G-modules $\mathcal{H}$ and $\mathcal{H}'$, we shall denote by

$$\mathrm{Hom}_G(\mathcal{H}, \mathcal{H}') \,=\, \mathrm{Hom}(\mathcal{H}, \mathcal{H}')^G \,=\, \mathcal{L}(\mathcal{H}, \mathcal{H}')^G,$$

the set of continuous linear maps $T : \mathcal{H} \to \mathcal{H}'$ that commute with the action of G, that is, $T(gv) = gT(v)$ for all $v \in \mathcal{H}$ and $g \in G$.

Let $\mathcal{H}$ be a G-module and α an irreducible representation of G. Then p_α will denote the G-invariant projection onto the *α-isotypical component* $\mathcal{H}_\alpha$ of $\mathcal{H}$, defined as the largest (closed) G-submodule of $\mathcal{H}$ that is isomorphic to a multiple of α. In other words, $\mathcal{H}_\alpha$ is the sum of all G-submodules of $\mathcal{H}$ that are isomorphic to α. Notice that $\mathcal{H}_\alpha \simeq \alpha \otimes \mathrm{Hom}_G(\alpha, \mathcal{H})$ and

$$\mathcal{H}_\alpha \neq 0 \;\Leftrightarrow\; \mathrm{Hom}_G(\alpha, \mathcal{H}) \neq 0 \;\Leftrightarrow\; \mathrm{Hom}_G(\mathcal{H}, \alpha) \neq 0.$$

Recall that we denote by $\widehat{G}$ the set of equivalence classes of irreducible unitary representations of G. Let χ_α be the character of $\alpha \in \widehat{G}$ and $z_\alpha := \dim \alpha \chi_\alpha \in C^*G$ be the central projection associated in the group C^*-algebra C^*G of G. Then p_α is the image of z_α induced by the group action on $\mathcal{H}$. If $T \in \mathcal{L}(\mathcal{H})^G$ then $T(\mathcal{H}_\alpha) \subset \mathcal{H}_\alpha$ and we let

$$\pi_\alpha : \mathcal{L}(\mathcal{H})^G \to \mathcal{L}(\mathcal{H}_\alpha)\,, \quad \pi_\alpha(T) \,:=\, p_\alpha T|_{\mathcal{H}_\alpha}\,,$$

be the associated morphism. The morphism π_α will play an essential role in what follows.

As before, we consider a compact Lie group G and we now assume that G acts by isometries on a closed Riemannian manifold M. Let TM and T^*M be respectively the *tangent* and *cotangent bundle* on M and recall that they can be identified using the G-invariant Riemannian metric on M. Let S^*M denote the *unit cosphere bundle* of M, that is, the set of unit vectors in T^*M, as usual. Denote by $\mathfrak{g}$ the Lie algebra of G. Then any $Y \in \mathfrak{g}$ defines as usual the vector field Y_M given by $Y_M(m) = \frac{d}{dt}_{|t=0} e^{tY} \cdot m$. Denote by $\pi : T^*M \to M$ the canonical projection and let us introduce as in [4] the *G-transversal space*

$$T^*_G M := \{\xi \in T^*M \mid \xi(Y_M(\pi(\xi))) = 0, \forall Y \in \mathfrak{g}\}. \tag{1}$$

We denote by $T_G M$ the image of $T^*_G M$ in TM obtained using the Riemannian metric. In other words, $T_G M$ is the orthogonal to the orbits in TM. Finally, let $S^*_G M$ be the set of unit covectors in $T^*_G M$, that is $S^*_G M = S^*M \cap T^*_G M$.

Recall that if M/G is connected, there is a minimal isotropy subgroup K such that any isotropy subgroup of G acting on M contains a subgroup conjugated to K and $M_{(K)}$ is an open dense submanifold of M called the principal orbit bundle, see [16, Section IV. 3] and [56, Section I. 5].

1.2. Pseudodifferential Operators

Let G be a compact Lie group acting smoothly by isometries on a compact, Riemannian manifold without boundary M as before. We shall denote by $\psi^m(M;E)$ the space of order m, *classical* pseudodifferential operators on M. Let $\overline{\psi^0}(M;E)$ and $\overline{\psi^{-1}}(M;E)$ denote the respective norm closures of $\psi^0(M;E)$ and $\psi^{-1}(M;E)$. The action of G then extends to a continuous action on $\psi^m(M;E)$, $\overline{\psi^0}(M;E)$, and $\overline{\psi^{-1}}(M;E)$, see [5] for example. We shall denote by $\mathcal{K}(\mathcal{H})$ the algebra of compact operators acting on a Hilbert space $\mathcal{H}$. Of course, we have $\overline{\psi^{-1}}(M;E) = \mathcal{K}(L^2(M;E))$.

We shall denote, as usual, by $\mathcal{C}(S^*M; \mathrm{End}(E))$ the set of continuous sections of the *lift* of the vector bundle $\mathrm{End}(E) \to M$ to S^*M. We have the following well known exact sequence

$$0 \to \mathcal{K}(L^2(M;E))^G \to \overline{\psi^0}(M;E)^G \xrightarrow{\sigma_0} \mathcal{C}(S^*M; \mathrm{End}(E))^G \to 0\,.$$

See, for instance, [9, Corollary 2.7], where references are given.

Recall that a G-invariant classical pseudodifferential operator P of order m is said *elliptic* if its principal symbol is invertible on $T^*M \smallsetminus \{0\}$ and *G-transversally elliptic* if its principal symbol is invertible on $T^*_G M \smallsetminus \{0\}$ [4,5,44], see Eq. (1) for the definition of $T^*_G M$.

We may now state the classical result of Atiyah and Singer [4, Corollary 2.5].

Theorem 1.1. *(Atiyah-Singer [4, 54]) Let P be a G-transversally elliptic operator. Then, for every irreducible representation $\alpha \in \widehat{G}$, $\pi_\alpha(P) : H^s(M;E_0)_\alpha \to H^{s-m}(M;E_1)_\alpha$, is Fredholm.*

Let us recall the following fact which is a direct consequence of the fact that G acts by unitary multiplier on $\mathcal{K}(\mathcal{H})$.

Proposition 1.2. *We have natural isomorphisms*

$$p_\alpha \overline{\psi^{-1}}(M;E)^G \simeq \pi_\alpha(\overline{\psi^{-1}}(M;E)^G)$$

$$= \pi_\alpha(\mathcal{K}(L^2(M;E))^G) = \mathcal{K}(L^2(M;E)_\alpha)^G\,,$$

where the first isomorphism map is simply π_α *and*

$$\mathcal{K}(L^2(M;E))^G = \overline{\psi^{-1}}(M;E)^G \simeq \oplus_{\alpha\in\widehat{G}}\mathcal{K}(L^2(M;E)_\alpha)^G\,.$$

Proof. See, for example, [9, Section 3] for a proof. □

1.3. α-transversally Elliptic Operators

Let G be a compact Lie group acting smoothly by isometries on a compact, Riemannian manifold without boundary M as before. Let $P : \mathcal{C}^\infty(M, E_0) \to \mathcal{C}^\infty(M, E_1)$ be a G-invariant pseudodifferential operator. Let $p : M \to M/G$ be the projection. Let $M/G = \bigsqcup_{i\in I} C_i$ be the decomposition into connected components of M/G. Notice that I is finite and let $K_i \subset G$ be a minimal isotropy group for $M_i := p^{-1}(C_i)$. Denote by $(S^*_G M_i)^{K_i}$ the subset of K_i-invariant elements of $S^*_G M_i$, see Eq. (1).

Definition 1.3. [11] We shall say that $P \in \psi^m(M; E_0, E_1)^G$ is *transversally α-elliptic* if for any $i \in I$, and $\xi \in (S^*_G M_i)^{K_i}$,

$$\sigma_m(P)(\xi) \otimes \mathrm{Id}_{\alpha^*} : (E_{0\xi} \otimes \alpha^*)^{K_i} \to (E_{1\xi} \otimes \alpha^*)^{K_i}$$

is invertible.

Let us recall the main result of [11], see also [9,10] for finite groups.

Theorem 1.4. *[11] Let $m \in \mathbb{R}$, $P \in \psi^m(M; E_0, E_1)^G$ and $\alpha \in \widehat{G}$. Then*

$$\pi_\alpha(P) : H^s(M; E_0)_\alpha \;\to\; H^{s-m}(M; E_1)_\alpha$$

is Fredholm if, and only if P is transversally α-elliptic.

We now briefly relate Definition 1.3 with the notion of limit operators, see [23,29,30,32,40–42,45]. In order to simplify notations, let us assume $\alpha = \mathbf{1}$ and M/G connected and let K be the minimal isotropy subgroup. We follow [11]. Let $(x_0, \xi) \in S^*_G M_{(K)}$ and assume $G_{x_0} = K$. Let $U \subset (T_G M)_{x_0} \simeq (T^*_G M)_{x_0}$ be a slice at x_0, let $W = G\exp_{x_0}(U) \cong G/K \times U$ be the associated tube around x_0, and let

$$\eta \in E^K_{x_0} \;\text{ and }\; f \in \mathcal{C}^\infty_c(U)\,,\; f(x_0) = 1\,.$$

Notice that $(S^*_K U)_{x_0} = S^*_{x_0} U$, because $x_0 \in M_{(K)}$ and hence $\xi \in S^*_{x_0} U$. Let us define $s_\eta \in \mathcal{C}^\infty_c(W; E)^G$ and $e_t \in \mathcal{C}^\infty(W)^G$ by $s_\eta(g\exp_{x_0}(y)) := f(y)g\eta$ and $e_t(g\exp_{x_0}(y)) := e^{\imath t\langle y,\xi\rangle}$, $t \in \mathbb{R}$. In other words, they are the functions on W extending the functions $y \mapsto f(y)\eta$ and $y \mapsto e^{\imath t\langle y,\xi\rangle}$ defined on $U \subset T_{x_0}U = (T_K U)_{x_0}$ by G-invariance via $W = G\exp_{x_0}(U)$. Using oscillatory testing techniques, see, for instance [34,57], the following proposition can be shown, see [11].

Proposition 1.5. *Assume that* $0 \neq \eta \in E^K_{x_0}$. *Then, for every* $P \in \psi^0(M; E)$, *we have* $\lim_{t\to\infty} P(e_t s_\eta)(x_0) = \sigma_0(P)(\xi)\eta$. *In particular, if* $P \in \psi^0(M;E)^G$, *then*

$$\lim_{t\to\infty} \pi_{\mathbf{1}}(P)(e_t s_\eta)(x_0) = \sigma_0(P)(\xi)\eta =: \pi_{(\xi,\mathbf{1})}\big(\sigma_0(P)\big)\eta\,.$$

Remark 1.6. Let $t > 0$ and $V_0 = \mathrm{Id} : E^K_{x_0} \to E^K_{x_0}$. Let $V_t : E^K_{x_0} \to C^\infty(M, E)^G$ be the map given by $V_t(\eta) = e_t s_\eta$ and let $V_{-t} = \mathrm{ev}_{x_0} : C^\infty(M, E)^G \to E^K_{x_0}$ be the evaluation map at x_0. Then we have $V_{-t}V_t = V_0 = \mathrm{Id}_{E^K_{x_0}}$ and

$$\sigma_0(P)(\xi) = \lim_{t\to\infty} V_{-t}\pi_{\mathbf{1}}(P)V_t : E^K_{x_0} \to E^K_{x_0}. \tag{2}$$

Equation (2) is similar to the definition of limit operators, see [23,29,30,32,40–42,45].

2. Simonenko's General Localization Principle

In this section, we recall the essentials of the usual Simonenko's localization principle [52], see also [53]. The results of this section are well-know from experts, we shall include proofs for the convenience of the reader. We refer in particular to [45, Chapter 2] and [47, Chapter 2], where more general situations are treated. The general localization principle of this section will be used in the sequel to deduce Fredholm conditions for restriction to isotypical components of invariant operators on closed manifolds.

Throughout this section, we let T be a compact Hausdorff topological space and $\mathcal{C}(T)$ be the C^*-algebra of complex valued continuous functions on T. Let A be a unital C^*-algebra and assume that $\mathcal{C}(T)$ identifies with a unital sub-C^*-algebra in A, meaning, in particular, that the image of the unit $1_{\mathcal{C}(T)}$ of $\mathcal{C}(T)$ is the unit 1_A of A.

Definition 2.1. An element $a \in A$ is said to *have the strong Simonenko local property* with respect to $\mathcal{C}(T)$ if, for every $\phi, \psi \in \mathcal{C}(T)$ with compact disjoint supports, we have $\phi a \psi = 0$.

The following lemma follows for example from similar arguments as in [45, Theorem 2.1.6] and [47, Theorem 2.5.6].

Lemma 2.2. *The set* $B \subset A$ *of elements* $a \in A$ *satisfying the strong Simonenko local property is the set of elements of* A *commuting with* $\mathcal{C}(T)$.

Proof. We are going to show that the set of elements $a \in A$ with the strong Simonenko local property is a C^*-algebra B containing $\mathcal{C}(T)$ and that every irreducible representation of B restricts to a scalar valued representation on $\mathcal{C}(T)$, and hence that $\mathcal{C}(T)$ commutes with B.

Let us show first that B is a sub-C^*-algebra of A. Note that B is not empty since $\mathcal{C}(T) \subset B$. To show that B is a sub-C^*-algebra, the only fact that is non-trivial to prove is that $ab \in B$, for all $a, b \in B$. Let ϕ and $\psi \in \mathcal{C}(T)$

with disjoint compact supports and let θ be a function equal to 1 on $\mathrm{supp}(\psi)$ and 0 on $\mathrm{supp}(\phi)$, which exists by Urysohn's lemma. Then we have

$$\phi ab\psi = \phi a(\theta + 1 - \theta)b\psi = \phi a\theta b\psi + \phi a(1-\theta)b\psi = 0,$$

since $\phi a\theta = 0$ and $(1-\theta)b\psi = 0$.

Let $\pi : B \to \mathcal{L}(H)$ be an irreducible representation of B. First, let us show that for any $\phi, \psi \in \mathcal{C}(T)$ with disjoint support, we either have $\pi(\phi) = 0$ or $\pi(\psi) = 0$. Indeed we have $\pi(\phi)\pi(a)\pi(\psi) = 0$ since $\phi a\psi = 0$, for any $a \in B$. Assume that $\pi(\psi) \neq 0$ then there is $\eta \in H$ such that $\pi(\psi)\eta \neq 0$. Now, π is irreducible so we get that the set $\{\pi(a)\pi(\psi)\eta,\ a \in B\}$ is dense in H. Thus $\pi(\phi) = 0$ on a dense subspace of H and so on H.

Assume now that $\pi(\mathcal{C}(T)) \neq \mathbb{C}1_H$. Then there exist two distinct characters $\chi_0, \chi_1 \in \mathrm{Sp}(\pi(\mathcal{C}(T)))$. Denote by $h_\pi : \mathrm{Sp}(\pi(\mathcal{C}(T))) \to \mathrm{Sp}(\mathcal{C}(T)) = T$ the injective map adjoint to π, and choose $\phi, \psi \in \mathcal{C}(T)$ with disjoint supports such that $\phi(h_\pi(\chi_0)) = 1$ and $\psi(h_\pi(\chi_1)) = 1$. Then $\pi(\phi)(\chi_0) = 1$ and $\pi(\psi)(\chi_1) = 1$, which contradicts the fact that either $\pi(\phi) = 0$ or $\pi(\psi) = 0$. $\square$

We now fix notations and hypothesis that will remain valid until the end of this section.

Notation and hypothesis 2.3. As before, let T be a compact Hausdorff topological space and denote by $\mathcal{C}(T)$ the C^*-algebra of continuous functions on T. Let $\mathcal{G}$ be a Hilbert space and let $\mathcal{C}(T) \to \mathcal{L}(\mathcal{G})$ be a non degenerate faithful representation (*i.e.* $\mathcal{C}(T)$ identifies with its image in $\mathcal{L}(\mathcal{G})$ and the image of the constant function 1 is $\mathrm{Id} \in \mathcal{L}(\mathcal{G})$). Assume that the image of $\mathcal{C}(T)$ does not intersect $\mathcal{K}(\mathcal{G}) \smallsetminus \{0\}$. In other words, we are assuming that $\mathcal{C}(T)$ identifies with a unital sub-C^*-algebra of the Calkin algebra $\mathcal{Q}(\mathcal{G}) := \mathcal{L}(\mathcal{G})/\mathcal{K}(\mathcal{G})$. We shall denote by M_ϕ the image of a function $\phi \in \mathcal{C}(T)$ in $\mathcal{L}(\mathcal{G})$ and call it the multiplication operator by ϕ.

Remark 2.4. If X is a locally compact space and $\mathcal{G} = L^2(X, \mu)$ then the representation of $\mathcal{C}_0(X)$ is faithful if and only if μ is a strictly positive measure, *i.e.* $\mu(U) > 0$ for every open set $U \subset X$. In this case, the only compact operator in $\mathcal{C}_b(X)$ is zero, where $\mathcal{C}_b(X)$ denotes the C^*-algebra of bounded continuous function, see Lemma 3.9 below for more details.

We shall now turn to the definition of local invertibility. The definition in the present paper and in for example [47, Section 2.5] are at the first reading not the same but they describe the same property by Lemma 2.2. See also [47, Section 2.4.1].

Definition 2.5. An operator $P \in \mathcal{L}(\mathcal{G})$ is said to be *locally invertible* at $x \in T$ if there exist:

(i) a neighbourhood V_x of x and
(ii) operators Q_1^x and $Q_2^x \in \mathcal{L}(\mathcal{G})$

such that, for all $\phi \in \mathcal{C}_c(V_x)$

$$Q_1^x P M_\phi \;=\; M_\phi \;=\; M_\phi P Q_2^x \in \mathcal{L}(\mathcal{G}).$$

The operator P is said to be *locally invertible on* T if it is locally invertible at any $x \in T$.

Notation 2.6. We let $\Psi_T(\mathcal{G}) \subset \mathcal{L}(\mathcal{G})$ denote the C^*-algebra consisting of all $P \in \mathcal{L}(\mathcal{G})$ such that $M_\phi P M_\psi \in \mathcal{K}(\mathcal{G})$, for all $\phi, \psi \in \mathcal{C}(T)$ with disjoint support. We let $\mathcal{B}_T(\mathcal{G})$ denote the image of $\Psi_T(\mathcal{G})$ in the Calkin algebra $\mathcal{Q}(\mathcal{G})$.

In other words, $\mathcal{B}_T(\mathcal{G}) = q(\Psi_T(\mathcal{G}))$ where $q : \mathcal{L}(\mathcal{G}) \to \mathcal{Q}(\mathcal{G})$ is the canonical projection.

Remark 2.7. We know by Lemma 2.2 that

$$\mathcal{B}_T(\mathcal{G}) = \{P \in \mathcal{Q}(\mathcal{G}) \mid M_\phi P = P M_\phi \text{ for all } \phi \in \mathcal{C}(T)\}.$$

Said differently, $\Psi_T(\mathcal{G})$ is the essential commutant of $\mathcal{C}(T)$, that is

$$\Psi_T(\mathcal{G}) = \{P \in \mathcal{L}(\mathcal{G}),\ M_\phi P - P M_\phi \in \mathcal{K}(\mathcal{G}),\ \forall \phi \in \mathcal{C}(T)\},$$

see the relation with the work [59]. Moreover, the family of morphisms

$$\mathcal{B}_T(\mathcal{G}) \to \mathcal{B}_T(\mathcal{G}) / \ker(\mathrm{ev}_x)\mathcal{B}_T(\mathcal{G}),$$

$x \in T$ is exhaustive, see [43, Definition 3.1] for the precise definition. This follows from the definition of the central character map, see for example [10, Remark 2.11].

I would like to thank an anonymous referee for pointing out to me how to simplify the proof of the next proposition and for the reference [45, Proposition 2.2.3] where a more general situation is treated.

Proposition 2.8. *Assume that $P \in \Psi_T(\mathcal{G})$ is locally invertible on T. Then P is Fredholm.*

Proof. By assumption P is locally invertible on T therefore for any $x \in T$ there are open neighborhood V_x and operators Q_1^x, Q_2^x such that for all $\phi \in \mathcal{C}_c(V_x)$,

$$Q_1^x P M_\phi \;=\; M_\phi \;=\; M_\phi P Q_2^x \in \mathcal{L}(\mathcal{G}).$$

Since T is compact, there are $x_1, \cdots, x_N$ such that $(V_{x_j})_{j=1}^N$ is a finite open cover of T. Now let $(\phi_j)_{j=1}^N$ be a partition of unity subordinated to $(V_j)_{j=1}^N$ then for all $j = 1, \cdots, N$, we have

$$Q_1^{x_j} P M_{\phi_j} \;=\; M_{\phi_j} \;=\; M_{\phi_j} P Q_2^{x_j} \in \mathcal{L}(\mathcal{G}).$$

It follows that $Q^1 := \sum_{j=1}^N Q_1^{x_j} M_{\phi_j}$ and $Q^2 := \sum_{j=1}^N M_{\phi_j} Q_2^{x_j}$ are respectively left inverse and right inverse of P modulo compact operators. Indeed, if $[A, B] = AB - BA$ denotes the commutator, we have

$$Q_1 P = \sum_{j=1}^N Q_1^{x_j} M_{\phi_j} P = \sum_{j=1}^N Q_1^{x_j} [M_{\phi_j}, P] + \sum_{j=1}^N Q_1^{x_j} P M_{\phi_j}$$

$$
\begin{aligned}
&= \sum_{j=1}^{N} Q_1^{x_j}[M_{\phi_j}, P] + \sum_{j=1}^{N} M_{\phi_j} \\
&= \sum_{j=1}^{N} Q_1^{x_j}[M_{\phi_j}, P] + \mathrm{Id},
\end{aligned}
$$

and similarly

$$
PQ_2 = \sum_{j=1}^{N} [P, M_{\phi_j}] Q_2^{x_j} + \mathrm{Id}\,.
$$

Since $P \in \Psi_T(\mathcal{G})$, we know from Remark 2.7 that $[P, M_{\phi_j}]$ is compact. Thus, $\sum_{j=1}^{N} Q_1^{x_j}[M_{\phi_j}, P]$ and $\sum_{j=1}^{N}[P, M_{\phi_j}]Q_2^{x_j}$ are compact operators and therefore P is Fredholm □

Definition 2.9. We shall say that the representation $\mathcal{C}(T) \to \mathcal{L}(\mathcal{G})$ of Notations and Hypothesis 2.3 has the property of strong convergence to 0 if for any $x \in T$ M_{χ_V} converges strongly to zero, where V runs the set of neighborhoods of x and $\chi_V \in \mathcal{C}(T, [0, 1])$ is equal to 1 on a neighborhood of x, with values in $[0, 1]$ and is supported in V. Said differently, $\mathcal{C}(T) \to \mathcal{L}(\mathcal{G})$ as the property of strong convergence to 0 if $\forall x \in T$, $\forall h \in \mathcal{G}$, $\forall \varepsilon > 0$, there is a neighborhood V' of x such that for any neighborhood V of x, if $V \subset V'$ then $\|M_{\chi_V} h\| < \varepsilon$.

Proposition 2.10. (General Simonenko's localization principle) *Let $P \in \Psi_T(\mathcal{G})$. Assume that $\mathcal{C}(T) \to \mathcal{L}(\mathcal{G})$ is as in Notation and Hypothesis 2.3 and has the property of strong convergence to* 0*, see Definition 2.9. Then P is locally invertible on T if, and only if, P is Fredholm.*

Proof. The first implication is exactly Proposition 2.8.

Let us prove the opposite implication. That is, let us assume that P is Fredholm and let us prove that P is locally invertible at $x \in T$, where x is fixed, but arbitrary. To this end, let $Q \in \mathcal{L}(\mathcal{G})$ be an inverse modulo $\mathcal{K}(\mathcal{G})$ for P, i.e. $PQ = \mathrm{Id} + K$ and $QP = \mathrm{Id} + K'$, with $K, K' \in \mathcal{K}(\mathcal{G})$. Using Lemma [27, Proposition 1.3.10], we can assume that $Q \in \Psi_T(\mathcal{G})$ if one desires. Let $\chi \in \mathcal{C}(T)$ be equal to 1 on a neighbourhood V_x of x, with values in $[0, 1]$ and supported in a neighborhood V'_x. Let $\phi \in \mathcal{C}_c(V_x)$ then

$$
\begin{aligned}
M_\phi M_\chi P Q M_\chi &= M_\phi M_\chi^2 + M_\phi M_\chi K M_\chi \qquad \text{and} \\
M_\chi Q P M_\chi M_\phi &= M_\chi^2 M_\phi + M_\chi K' M_\chi M_\phi\,.
\end{aligned}
$$

Since ϕ is supported in V_x, we have $\phi\chi = \phi$ and so

$$
M_\phi P Q M_\chi = M_\phi(1 + M_\chi K M_\chi) \quad \text{and} \quad M_\chi Q P M_\phi = (1 + M_\chi K' M_\chi) M_\phi\,.
$$

As V'_x becomes small, we have that M_χ converges strongly to 0 because $\mathcal{C}(T) \to \mathcal{L}(\mathcal{G})$ has the property of strong convergence to 0, see Definition 2.9. Since K is compact, we obtain that $\|M_\chi K M_\chi\| \to 0$. Thus, by choosing V'_x small enough, we may assume that $\|M_\chi K M_\chi\| < 1$ and $\|M_\chi K' M_\chi\| < 1$.

It follows that $(1 + M_\chi K M_\chi)$ and $(1 + M_\chi K' M_\chi)$ are invertible and this implies

$$M_\phi P\big(QM_\chi(1 + M_\chi K M_\chi)^{-1}\big) \;=\; M_\phi \qquad \text{and}$$
$$\big((1 + M_\chi K' M_\chi)^{-1} M_\chi Q\big) P M_\phi \;=\; M_\phi\,,$$

that is, P is locally invertible. This completes the second implication, and hence the proof. □

Simonenko's principle is then [52]:

Proposition 2.11. *[Simonenko's principle] Let M be a closed manifold, E a hermitian vector bundle on M, $\Psi_M(L^2(M,E))$ be as in 2.6 and let $P \in \Psi_M(L^2(M,E))$. We have that P is locally invertible on M if, and only if, it is Fredholm.*

Proof. The hypothesis of Proposition 2.10 are satisfied because if $h \in L^2(M,E)$ then $\int_V |h|^2 \mathrm{dvol}$ goes to 0 when the volume of V goes to 0, see also Remark 2.4. □

3. Equivariant Local Principle for Closed Manifolds

Let G be a compact Lie group that we assume to act smoothly by isometries on a closed Riemannian manifold M as before. We shall denote, as before, by $\widehat{G}$ the set of isomorphism classes of irreducible unitary representations of G. Let $\mathcal{H} := L^2(M,E)$ and let $\mathcal{H}_\alpha \cong \alpha \otimes (\alpha^* \otimes \mathcal{H})^G$ be the α-isotypical component associated to $\alpha \in \widehat{G}$, as in the introduction and Sect. 1.

Any $\phi \in \mathcal{C}(M)^G$ acts by multiplication on $\mathcal{H}_\alpha$ and we shall denote also by M_ϕ the induced multiplication operator, as in Sect. 2. Furthermore, the representation of $\mathcal{C}(M/G) = \mathcal{C}(M)^G$ given by the previous multiplication operator on $\mathcal{H}$ and $\mathcal{H}_\alpha$ are non degenerate.

Definition 3.1. We shall say that $P \in \mathcal{L}(\mathcal{H})^G$ is *locally α-invertible at $x \in M$* if $\pi_\alpha(P)$ is locally invertible

at $Gx \in M/G$, see Definition 2.5.

We let $\Psi_M^G(\mathcal{H})$ denote the G-invariant elements in the C^*-algebra $\Psi_M(\mathcal{H})$, which was defined in 2.6, in the previous subsection. More precisely, using Remark 2.7

$$\Psi_M^G(\mathcal{H}) = \{P \in \mathcal{L}(\mathcal{H})^G \mid [P, M_\phi] \in \mathcal{K}(\mathcal{H}),\ \forall \phi \in \mathcal{C}(M)\}. \tag{3}$$

Before tackling the Simonenko's equivariant localization principle, let us first justify our hypothesis with the following simple example.

Example 3.2. Let $M = G$ be our manifold with its standard action by translation. In this case, $T_G^* M = G \times \{0\}$ and then every G-invariant pseudodifferential operator P is G-transversally elliptic. It follows that the restriction

$\pi_\alpha(P)$ to any isotypical component is Fredholm. Let us then consider the null operator $0 : L^2(G) \to 0$. Clearly, the restriction to the isotypical component associated with the trivial representation $\mathbf{1} = \mathbb{C}$ of G is Fredholm. In other words, $0 = \pi_{\mathbf{1}}(0) : L^2(G)^G = \mathbb{C} \to 0$ is Fredholm. But obviously, $0 = \pi_{\mathbf{1}}(0)$ is not locally $\mathbf{1}$-invertible.

This pathological example arises from the fact that there are points $x \in M$ such that the slice at x are discrete (and in fact on the whole space $M = G$ in the previous example). Nevertheless, we can extract such a pathological points using the following interesting fact.

Lemma 3.3. *Let X be a not necessarily compact G-manifold without boundary and let $x \in X$ be such that $(T^*_G X)_x = \{0\}$. Then the orbit of x is a union of connected components of X in bijection with the connected component of G/G_x.*

Proof. Since $(T^*_G X)_x = \{0\}$, we obtain that $S_x = \{x\}$ is the only slice at x. From the slice theorem, we deduce that the orbit $Gx \cong G\times_{G_x}\{x\} = G\times_{G_x} S_x$ is open but it is also compact. Therefore, Gx is a union of connected components of X in bijection with the connected components of $G/G_x \cong Gx$. □

Consider the set of points

$$\mathcal{P} := \{x \in M,\ (T^*_G M)_x = \{0\}\}. \tag{4}$$

Then M is the disjoint union of the closed manifolds $M \setminus \mathcal{P}$ and $\mathcal{P}$. Indeed, $\mathcal{P}$ is a union of clopen orbits and therefore it is also compact because M is. The same argument also implies that $M \setminus \mathcal{P}$ is a closed submanifold of M.

Remark 3.4. The set $\mathcal{P}$ will be empty for example in the following cases:

(1) if M is connected and not reduced to a single orbit,
(2) if $\dim M > \dim G$,
(3) in particular, if M/G is an orbifold of dimension > 0.

In other cases, we can use the following useful result.

Lemma 3.5. *Let $\mathcal{P} = \{x \in M,\ (T^*_G M)_x = \{0\}\}$ be the clopen introduced in Eq. (4). Let χ be the characteristic function of the clopen $M \setminus \mathcal{P}$. Let then M_χ be the multiplication operator by χ. Let $P \in \Psi_M(\mathcal{H})$ then $P = P_1 + P_2 + P_3$ with $P_1 = M_\chi P M_\chi \in \Psi_{M\setminus\mathcal{P}}(\mathcal{H})$, $P_2 = M_{1-\chi} P M_{1-\chi} \in \Psi_{\mathcal{P}}(\mathcal{H})$ and $P_3 = M_\chi P M_{1-\chi} + M_{1-\chi} P M_\chi \in \mathcal{K}(\mathcal{H})$. Furthermore, if $P \in \Psi^G_M(\mathcal{H})$ then $\pi_\alpha(P_2)$ is Fredholm for any $\alpha \in \widehat{G}$ and therefore $\pi_\alpha(P)$ is Fredohlm if, and only if, $\pi_\alpha(P_1)$ is.*

Proof. The first part is clear since M_χ and $M_{1-\chi}$ have disjoint supports. For the second part, decompose $\mathcal{P} = \sqcup_{i=1}^N \mathcal{P}_i$ into clopen orbits and let ϕ_i be the corresponding characteristic functions then as before we can write $P_2 = \sum_{i=1}^N M_{\phi_i} P_2 M_{\phi_i} + C$, where $C = \sum_{i\neq j} M_{\phi_i} P_2 M_{\phi_j}$ is compact because the supports of ϕ_i and ϕ_j are disjoint for $i \neq j$. The previous decomposition is in fact a decomposition into G-invariant operators since ϕ_i is G-invariant. Therefore, for every $\alpha \in \widehat{G}$, $\pi_\alpha(P_2)$ is Fredholm if, and only if, $\pi_\alpha(M_{\phi_i} P_2 M_{\phi_i})$ is Fredholm for any i. Now notice that $\mathcal{P}_i \cong G \times_{G_x} \{x\}$ for some $x \in \mathcal{P}$ therefore $\forall \alpha \in \widehat{G}$,

$$\begin{aligned} L^2(\mathcal{P}_i, E|_{\mathcal{P}_i})_\alpha &\cong \alpha \otimes \Big(\alpha^* \otimes L^2(\mathcal{P}_i, E|_{\mathcal{P}_i})\Big)^G \\ &\cong \alpha \otimes \Big(\alpha^* \otimes L^2(G \times_{G_x} \{x\}, G \times_{G_x} E_x)\Big)^G \\ &\cong \alpha \otimes \Big(\alpha^* \otimes E_x\Big)^{G_x} \end{aligned}$$

is finite dimensional.

It follows that there are no condition for the restriction $\pi_\alpha(P_2)$ to be Fredholm, $\forall \alpha \in \widehat{G}$. In other words, for every $\alpha \in \widehat{G}$, $\pi_\alpha(P)$ is Fredholm if, and only if, $\pi_\alpha(P_1)$ is. □

Remark 3.6. Notice that the previous proof implies that the image of $\mathcal{C}(M)^G$ in $\mathcal{L}(\mathcal{H}_\alpha)$ intersects $\mathcal{K}(\mathcal{H}_\alpha)$ when $\mathcal{P} \neq \emptyset$.

Recall that if $x \in M$ then $W_x \cong G \times_{G_x} U_x$ denotes a tube around x and U_x a slice at x, see [56, Section I. 5]. Moreover, we have a G-equivariant isomorphism of vector bundles $E \cong G \times_{G_x} (U_x \times E_x)$. The next lemma could also be deduced from [17, Corollary 1.5].

Lemma 3.7. *Let $\alpha \in \widehat{G}$ and let $\mathcal{H}_\alpha = L^2(M,E)_\alpha \cong \alpha \otimes L^2(M, E \otimes \alpha^*)^G$. The subset*

$$\mathcal{N}_\alpha := \big\{x \in M,\ \exists W_x,\ \textit{such that } L^2(W_x, E)_\alpha = \{0\}\big\} \tag{5}$$

is a G-invariant clopen.

Proof. Replacing E with $E \otimes \alpha$, we see that we can assume that α is the trivial representation and therefore that $L^2(W_x, E)_\alpha = L^2(W_x, E)^G$. Clearly, $\mathcal{N}_\alpha$ is G-invariant. We shall denote simply $\mathcal{N}_\alpha$ by $\mathcal{N}$ is this proof since we consider the trivial representation.

Notice now that

$$\mathcal{N} = \big\{x \in M,\ \forall W_x,\ L^2(W_x, E)^G = \{0\}\big\} \tag{6}$$

because if W_x and W'_x are two tubes around $x \in M$ then

$$L^2(W_x, E)^G \cong L^2(U_x, E_x)^{G_x} \cong L^2(U'_x, E_x)^{G_x} \cong L^2(W'_x, E_x)^G.$$

Let us show that $\mathcal{N}$ is open. Let $x \in \mathcal{N}$. By definition, there is W_x such that $L^2(W_x, E)^G = \{0\}$. Let $y \in W_x$ and assume that there is a tube W_y around y such that $L^2(W_y, E)^G \neq \{0\}$. By G-invariance of W_x and W_y, we see that we can assume W_y small enough such that $W_y \subset W_x$. But then

$\{0\} \neq L^2(W_y, E)^G \subset L^2(W_x, E)^G$ which contradicts the fact that $x \in \mathcal{N}$. Therefore, $W_x \subset \mathcal{N}$ and $\mathcal{N}$ is open.

We now show that $\mathcal{N}$ is closed. Let $x \in M \setminus \mathcal{N}$. By Eq. (6), there is W_x such that $L^2(W_x, E)^G = L^2(U_x, E_x)^{G_x} \neq \{0\}$. Let K be a minimal isotropy subgroup for the linear action of G_x on U_x, see [56, Section 5]. Notice that K acts trivially on U_x by minimality. Then we have $\{0\} \neq L^2(U_x, E_x)^{G_x} \subset L^2(U_x, E_x)^K = L^2(U_x, E_x^K)$. It follows that there is $v \in E_x^K \setminus \{0\}$. Let $y \in U_x$ and denote by $W'_y \cong G_x \times_{G_y} U'_y \subset U_x$ a tube around y in U_x. Denote by $W'_{y(K)}$ the principal orbit bundle of W'_y that is the dense open subset of W'_y given by the points with stabilizer conjugated with K in G_x. Each point of $W'_{y(K)}$ has a neighborhood of the form $G_x/K \times V$ with K acting trivially on V. Let $s \in \mathcal{C}(G_x/K \times V, E_x)$ be given by $s([g], z) = gv$. The section s does not depend on the representative of $[g]$ in G_x because $v \in E_x^K$ and is clearly G_x-invariant. If $f \in \mathcal{C}_c(G_x/K \times V)^{G_x}$ is any compactly supported function then $fs \in \mathcal{C}_c(G_x/K \times V, E_x)^{G_x} \subset L^2(W'_y, E_x)^{G_x}$. Now if $W_y \cong G \times_{G_y} U_y$ is a tube around y in W_x then assuming U'_y small enough we have $G \times_{G_x} W'_y \subset G \times_{G_y} U_y$ and thus $0 \neq L^2(W'_y, E_x)^{G_x} \cong L^2(G \times_{G_x} W'_y, E)^G \subset L^2(W_y, E)^G$. It follows that $y \in M \setminus \mathcal{N}$ and therefore $W_x \subset M \setminus \mathcal{N}$. In other words, $M \setminus \mathcal{N}$ is open. This complete the proof. □

Remark 3.8. Let $\mathcal{N}_\alpha = \{x \in M,\ \exists W_x,\ \text{such that } L^2(W_x, E)_\alpha = \{0\}\}$ be the clopen defined in Lemma 3.7. We have $L^2(\mathcal{N}_\alpha, E)_\alpha = \{0\}$ and therefore there is no condition on $L^2(\mathcal{N}_\alpha, E)_\alpha$ for an operator to be Fredholm. By a discussion similar to the one of Lemma 3.5, we see that $P \in \Psi_M(\mathcal{H})^G$ is such that $\pi_\alpha(P)$ is Fredholm if, and only, if $\chi_{M \setminus \mathcal{N}_\alpha} P \chi_{M \setminus \mathcal{N}_\alpha}$ is Fredholm, where $\chi_{M \setminus \mathcal{N}_\alpha}$ denotes the characteristic function of $M \setminus \mathcal{N}_\alpha$.

Moreover, we see that if $\mathcal{N}_\alpha$ is not empty then the image of $\mathcal{C}(\mathcal{N}_\alpha)^G$ in $\mathcal{L}(\mathcal{H}_\alpha)$ is 0, i.e. $M_\phi = 0$, $\forall \phi \in \mathcal{C}(\mathcal{N}_\alpha)^G$. It follows that the image of $\mathcal{C}(M)^G$ is the same as the image of $\mathcal{C}(M \setminus \mathcal{N}_\alpha)^G$.

For example, let $G = SO(3)$, let $M = S_1^2 \sqcup S_1^2$ be the disjoint union of two spheres $S^2 \subset \mathbb{R}^3$ with a trivial action on S_1^2 and the induced action from $\mathbb{R}^3$ on S_2^2. Let then $E = M \times \mathbb{C}^3$ with the natural action on $\mathbb{C}^3$. Then $L^2(M, E)^G = L^2(S_2^2, \mathbb{C}^3)^G \neq 0$. Indeed, $(\mathbb{C}^3)^G = 0$ because $\mathbb{C}^3$ is an irreducible representation of $SO(3)$ therefore $L^2(S_1^2, \mathbb{C}^3)^G = L^2(S_1^2, (\mathbb{C}^3)^G) = 0$. Moreover, the function $f(x) = x \in \mathbb{R}^3 \subset \mathbb{C}^3$ is G-invariant and belongs to $L^2(S_2^2, \mathbb{C}^3)^G$. Now if $\chi_{S_1^2}$ is the characteristic function of S_1^2 then $M_{\chi_{S_1^2}} : L^2(M, E)^G \to L^2(M, E)^G$ is zero.

Lemma 3.9. *Let $\mathcal{P}$ be the clopen introduced in Eq. (4) and let $\mathcal{N}_\alpha$ be the clopen introduced in Eq. (5). Let $\alpha \in \widehat{G}$ and let $\mathcal{H}_\alpha = L^2(M, E)_\alpha \cong \alpha \otimes L^2(M, E \otimes \alpha^*)^G$. Let $f \in \mathcal{C}(M \setminus (\mathcal{P} \cup \mathcal{N}_\alpha))^G$ then $M_f \in \mathcal{K}(\mathcal{H}_\alpha)$ if, and only, if $f = 0$.*

Proof. We may assume α to be the trivial representation $\mathbf{1} \in \widehat{G}$. Let $f \in \mathcal{C}(M \setminus (\mathcal{P} \cup \mathcal{N}_\alpha))^G$ be non zero such that M_f is compact on $\mathcal{H}_{\mathbf{1}} = L^2(M, E)^G$.

Let then $W_x \cong G \times_{G_x} U_x$ be a tube on which $|f| > \varepsilon > 0$. Denote by χ_x the characteristic function of W_x and let $\widetilde{M_f}$ be the restriction of M_f to $L^2(W_x, E)^G \cong L^2(U_x, E_x)^{G_x}$. Then $\widetilde{M_f}$ is invertible with inverse $M_{f\chi_x}$. Thus by Banach open mapping theorem $\widetilde{M_f}$ is open but also compact therefore $L^2(U_x, E_x)^{G_x}$ is finite dimensional. Notice that $L^2(U_x, E_x)^{G_x} \neq 0$ since $x \in M \smallsetminus \mathcal{N}_\alpha$. Let then $s \in L^2(U_x, E_x)^{G_x}$ be non zero. For any $n \in \mathbb{N}$, we can certainly find $n+1$ disjoint G_x-invariant annulus in U_x such that the restriction of s to each of this annulus is non zero because the action is isometric and $\dim U_x > 0$ since $x \in M \smallsetminus \mathcal{P}$. By considering the characteristic functions χ_i of this $n+1$ annulus, we get $n+1$ linearly independent functions $\chi_i s$, thus a contradiction. □

Recall that we denote by $\mathcal{H} = L^2(M, E)$. Let $\alpha \in \widehat{G}$, let $P \in \mathcal{L}(\mathcal{H})^G$ and recall that $\pi_\alpha(P) : \mathcal{H}_\alpha \to \mathcal{H}_\alpha$ is the restriction of P to the α-isotypical component $\mathcal{H}_\alpha \cong \alpha \otimes (\alpha^* \otimes \mathcal{H})^G$ of $\mathcal{H}$.

Proposition 3.10. (Simonenko's equivariant localization principle) *Let M be a closed G-manifold as before. Let $\mathcal{P}$ be as in Eq. (4) and let $\alpha \in \widehat{G}$. Let $P \in \Psi_M^G(\mathcal{H}) = \{P \in \mathcal{L}(\mathcal{H})^G | [P, M_\phi] \in \mathcal{K}(\mathcal{H}), \forall \phi \in \mathcal{C}(M)\}$. Then P is locally α-invertible on $M \smallsetminus \mathcal{P}$ (see Definition 3.1) if, and only if, $\pi_\alpha(P)$ is Fredholm.*

Proof. Let $\mathcal{N}_\alpha$ be as in Eq. (5). Notice that any operator is locally α-invertible at $x \in \mathcal{N}_\alpha$ as operator between the null vector space. Similarly, on $\mathcal{N}_\alpha$ the operator $\pi_\alpha(P)$ is Fredholm.

By Lemma 3.5 and Remark 3.8, we may replace M with $M \smallsetminus (\mathcal{P} \cup N)$ and assume that for any $x \in M$, $(T_G^* M)_x$ is not reduce to $\{0\}$ and that there is a tube W_x around x such that $L^2(W_x, E_x)_\alpha \neq \{0\}$. Under this hypothesis, we have that $\mathcal{C}(M/G) \to \mathcal{L}(\mathcal{H}_\alpha)$ is faithful, non degenerate and does not intersect $\mathcal{K}(\mathcal{H}_\alpha) \smallsetminus \{0\}$, see Lemma 3.9 and Notation and Hypothesis 2.3. Moreover, $\mathcal{C}(M/G) \to \mathcal{L}(\mathcal{H}_\alpha)$ has the property of strong convergence to 0, see Definition 2.9. Indeed, this is equivalent to say that the volume of the slice at x goes to zero when it becomes small. Let us now introduce the C^*-algebra $\Psi_{M/G}(\mathcal{H}_\alpha)$ defined in 2.6. Clearly, $\pi_\alpha(\Psi_M^G(\mathcal{H}))$ is a sub-C^*-algebra of $\Psi_{M/G}(\mathcal{H}_\alpha)$. Therefore, $\pi_\alpha(P) \in \pi_\alpha(\Psi_M^G(\mathcal{H})) \subset \Psi_{M/G}(\mathcal{H}_\alpha)$ is Fredholm if, and only if, it is locally invertible on M/G. By definition, P is locally α-invertible at $x \in M$ if, and only, if $\pi_\alpha(P)$ is locally invertible at $Gx \in M/G$, thus the result follows from Proposition 2.10. □

Theorem 3.11. *Let M be a closed G-manifold as before and let $\mathcal{P}$ be as in Eq. (4). Let $P \in \psi^m(M; E_0, E_1)^G$ and $\alpha \in \widehat{G}$. Then the following are equivalent:*

1. *$\pi_\alpha(P) : H^s(M; E_0)_\alpha \to H^{s-m}(M; E_1)_\alpha$ is Fredholm for any $s \in \mathbb{R}$,*
2. *P is transversally α-elliptic (see Definition 1.3),*
3. *P is locally α-invertible on $M \smallsetminus \mathcal{P}$ (see Definition 3.1).*

Proof. The first equivalence is given by Theorem 1.4. Now Proposition 3.10 implies that *(1)* is equivalent to *(3)*. □

In particular, we obtain the following consequence of the localization principle.

Corollary 3.12. *Let* $P \in \psi^m(M;E)^G$ *be a* G*-transversally elliptic operator, see Sect. 1.2. Then* P *is locally* α*-invertible on* $M \setminus \mathcal{P}$ *for any* $\alpha \in \widehat{G}$*, as in Definition 3.1.*

Proof. Using Theorem 1.1 we obtain that $\pi_\alpha(P)$ is Fredholm. Therefore by Proposition 3.10 P is locally α-invertible. □

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Declarations

Conflicts of interest The author declare that he has no conflict of interest.

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Alexandre Baldare
Institut für Analysis
Welfengarten 1
30167 Hannover
Germany
e-mail: alexandre.baldare@math.uni-hannover.de
URL: https://baldare.github.io/Baldare.Alexandre/

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