Families of stable sheaves

Wrong-way fibers and noncommutative deformations

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Introduction

In this thesis we want to understand properties of families of stable sheaves. First we give a quick introduction to the topic. Then we look at three specific topics regarding families of stable sheaves, each in its own section. For each topic we collect known results and describe the state-of-the-art methods to study these properties. We end each section with open questions. Then we give a quick summary of our results answering some of the open questions.

Vector bundles play an important role in many areas of mathematics and physics. It is thus a natural question to understand the structure of vector bundles. For example, Grothendieck proved that all vector bundles on the simplest smooth projective variety, the projective line $\mathbb{P}^1$, are direct sums of line bundles. Every line bundle on $\mathbb{P}^1$, that is a vector bundle of rank one, is of the form $L = \mathcal{O}_{\mathbb{P}^1}(a)$ for some $a \in \mathbb{Z}$. If E is a vector bundle of rank n on $\mathbb{P}^1$, then there is a sequence $a_1 \leqslant \ldots \leqslant a_n$ of integers such that

$$E \cong \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^1}(a_i).$$

On a different smooth projective variety X, especially if the dimension of X is greater than two, there is in general no such easy description.

A natural follow-up question is, if we can classify all vector bundles on such a variety X. For this, we first classify vector bundles by numerical invariants, for example the Hilbert polynomial P. One can quickly see that it is in general not possible, to classify these vector bundles. The problem is that many vector bundles have nontrivial automorphisms. But the answer is positive if we restrict to stable vector bundles with fixed Hilbert polynomial. There are two stability conditions which are widely used: Mumford-Takemoto stability, also called slope stability, and Gieseker stability. We will be working with slope stability, which we quickly recall (here we directly give the general definition for torsion free coherent sheaves): let (X, h) be a polarized smooth projective variety of dimension n and let E be a torsion free coherent sheaf of rank $\mathrm{rk}(E) = r$ and with first Chern class $c_1(E)$ on X, then we define the slope of E with respect of h by:

$$\mu(E) = \frac{c_1(E)h^{n-1}}{r}.$$

Then we say that E is slope (semi)stable with respect to h or h-slope (semi)stable if for any subsheaf $F \subset E$ with rank $1 \leqslant \mathrm{rk}(F) \leqslant r-1$ we have:

$$\mu(F) \underset{(}{\leq}{}_{)}\, \mu(E).$$

The modern way to construct a space that classifies stable vector bundles on a smooth projective variety X, a moduli space of stable vector bundles on X, uses the functorial point of view. To understand this point of view, we need to define families of stable vector bundles. A family of vector bundles on X parametrized by a scheme S is a coherent sheaf $\mathcal{E}$ on the product $X \times S$, which is flat over S. We say that $\mathcal{E}$ is a family of stable vector bundles with Hilbert polynomial P if for every $s \in S$ the fiber $\mathcal{E}_s$ is a stable vector bundle on the fiber X_s, which has Hilbert polynomial P. Using this notion one can define a functor:

$$\mathcal{M}_{X;P} : \mathrm{Sch}_{\mathbb{C}} \to \mathrm{Sets}$$

which sends a $\mathbb{C}$-scheme S to the set of isomorphism classes of families of stable vector bundles on X parametrized by S with fixed Hilbert polynomial P. A $\mathbb{C}$-scheme $M_{X;P}$ is a fine moduli space, if it represents this functor, that is, there is an isomorphism of functors:

$$\mathcal{M}_{X;P}(-) \cong \mathrm{Hom}_{\mathrm{Sch}_{\mathbb{C}}}(-, M_{X;P}).$$

The points of the moduli space $M_{X;P}$ describe the isomorphism classes of stable vector bundles on X with Hilbert polynomial P. More exactly, if $m \in M_{X;P}$ is a point, then there is a stable vector bundles E on X, with Hilbert polynomial P such that $m = [E]$, where $[E]$ denotes the isomorphism class of E.

These moduli spaces are interesting from two points of view. First, since they classify stable bundles on X, one can try to understand properties of the bundles by studying properties of the moduli space. Second, the moduli space $M_{X;P}$ itself is often an interesting variety, which carries a lot of geometry and other structures.

If $M_{X;P}$ is a fine moduli space, then there is canonically defined family $\mathcal{U}$ of vector bundles on X parametrized by $M_{X;P}$ corresponding to the identity map of $M_{X;P}$. The family $\mathcal{U}$ on $X \times M_{X,P}$ is called the universal family of the moduli space $M_{X,P}$. It has the following property: if $m = [E]$ is a point in $M_{X;P}$ for a stable vector bundle E on X, then there is an isomorphism

$$\mathcal{U}_m = \mathcal{U}_{[E]} \cong E$$

where $\mathcal{U}_m$ is the restriction of $\mathcal{U}$ to the fiber over $m \in M_{X;P}$. These fibers will be called the "correct" fibers of the universal family $\mathcal{U}$ in this thesis.

The first two parts of the thesis are concerned with properties of the fibers of the universal family over points $x \in X$. We call these the "wrong-way" fibers in this thesis. Thus the wrong-way fibers $\mathcal{U}_x$ are vector bundles on the moduli space $M_{X;P}$.

In the first part of this section we study the following natural question: since all correct fibers $\mathcal{U}_m$ are stable vector bundles on X, are the wrong-way fibers $\mathcal{U}_x$ stable vector bundles on $M_{X;P}$? We prove that this is indeed the case in some examples. Thus we can interpret $\mathcal{U}$ as a family of stable bundles on $M_{X;P}$ parametrized by X in these examples. We show that this identifies X with a smooth connected component of some moduli space of stable bundles on $M_{X;P}$. This iterated moduli space construction therefore produces a "kind of" duality between X and $M_{X;P}$. We start with X, construct a moduli space of stable bundle $M_{X;P}$ and get back X as (a component of) a moduli space of stable bundles on $M_{X;P}$.

The second part of this thesis deals with the derived categories of coherent sheaves on X and $M_{X;P}$. More exactly the universal family $\mathcal{U}$ gives rise to an integral functor

$$\Phi_{\mathcal{U}} : \mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}^{\mathrm{b}}(M_{X;P}).$$

In many cases this integral functor exhibits interesting relations between the derived categories. We show how to understand properties of $\Phi_{\mathcal{U}}$, for example fully faithfulness, by studying properties of the wrong-way fibers of $\mathcal{U}$.

The functor $\mathcal{M}_{X;P}$ is in general not representable, due to the existence of strictly semistable vector bundles on X. But one can show that there is a moduli space that corepresents the functor. We say that $M_{X;P}$ is a coarse moduli space for the moduli functor. This moduli space does not classify isomorphism classes of semistable vector bundles, but rather so-called S-equivalence classes of semistable vector bundles. This moduli space contains an open subset classifying stable vector bundles. But unfortunately there is no universal family in this case. There is an obstruction α in $\mathrm{H}^2(M_{X;P}, \mathcal{O}^{\times}_{M_{X;P}})$ to the existence of a universal family. One can construct a universal family $\mathcal{U}$ as a so-called family of α-twisted vector bundles. It turns out that these α-twisted vector bundles can be identified with vector bundles, which admit an action of an Azumaya algebra $\mathcal{A}$. This leads us naturally to the third part of the thesis.

In the third and last part of this thesis, we study the deformation theory of certain coherent sheaves in a noncommutative setting. Here noncommutative setting simply means that we study sheaves on X which admit an action of an associative and noncommutative algebra $\mathcal{A}$. A deformation of a coherent sheaf E is a coherent sheaf $\mathcal{E}$ on $X \times B$, flat over B, such that $\mathcal{E}_{b_0} \cong E$ for some $b_0 \in B$, in other words a deformation of E is a flat family of coherent sheaves on X with base B, such that E is a member of this family.

One can ask what properties of a coherent sheaf E are preserved by deformations and what properties can change. For example, the Hilbert polynomial is locally constant in flat families, so if B is a connected base, then the Hilbert polynomial does not change in a deformation. On the other hand, some properties of E can change, even over a connected base. For instance, we prove that it is possible to deform a torsion free coherent sheaf, which is not a vector bundle, into a vector bundle over a connected curve.

We work over the field of complex numbers $\mathbb{C}$ in this thesis. A variety is an integral scheme separated and of finite type over $\mathbb{C}$.

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1. **Stability of wrong-way fibers.**

The first non-trivial example of studying the wrong-way fibers of a family of sheaves arises in the classical study of abelian varieties. Recall that given an abelian variety A there is an associated dual abelian variety $\hat{A} := \mathrm{Pic}^0(A)$, which classifies line bundles of degree zero on A. There is the universal degree zero line bundle $\mathcal{P}$ on $A \times \hat{A}$, also known as the Poincaré bundle. It has the following properties:

i) $\mathcal{P}_{|A\times\{\alpha\}} \cong L_\alpha$ for all $\alpha \in \hat{A}$ (here $\alpha = [L_\alpha] \in \mathrm{Pic}^0(A \times \{\alpha\}) \cong \mathrm{Pic}^0(A)$).

ii) $\mathcal{P}_{|\{0_A\}\times\hat{A}}$ is trivial.

Now fixing $a \in A$ we get a line bundle $\hat{L}_a = \mathcal{P}_{\{a\}\times\hat{A}}$ on $\widehat{A}$. In fact $\hat{L}_a$ is a degree zero line bundle on $\hat{A}$, that is $\hat{L}_a \in \mathrm{Pic}^0(\mathrm{Pic}^0(A)) = \hat{\hat{A}}$, which defines a classifying morphism

$$\varphi : A \to \hat{\hat{A}},\ a \mapsto \hat{L}_a.$$

The classifying morphism φ for the wrong-way fibers of the Poincaré bundle $\mathcal{P}$ shows that duality for abelian varieties is well behaved, in the sense that the bidual of an abelian variety is canonically isomorphic to the abelian variety:

Theorem 1.1. [41, Corollary, p.132] *For any abelian variety A, the canonical morphism $A \to \hat{\hat{A}}$ defined by the Poincaré bundle $\mathcal{P}$ on $A \times \hat{A}$ (regarded as a family of line bundles on $\hat{A}$ parameterized by A) is an isomorphism.*

The stability of the wrong-way fibers of $\mathcal{P}$ follows immediately from the fact that the wrong-way fibers are line bundles, which are stable by definition.

The next example is that of vector bundles of rank $r \geqslant 2$ on curves of genus $g \geqslant 2$.

Remark 1.2. Almost all papers on moduli spaces of stable vector bundles on curves need to assume $g \geqslant 2$, since most of the techniques used need this assumption. The case $g = 0$ is settled by Grothendieck's description of vector bundles on $\mathbb{P}^1$: the stable vector bundles on $\mathbb{P}^1$ are exactly the line bundles. The strictly semistable bundles on $\mathbb{P}^1$ are of the form

$$E = \mathcal{O}_{\mathbb{P}^1}(a)^{\oplus n} \text{ with } n \in \mathbb{N}, n \geqslant 2 \text{ and } a \in \mathbb{Z}.$$

The case of elliptic curves, that is $g = 1$, was extensively studied in [52]. So from now on we can restrict to $g \geqslant 2$.

Pick a smooth projective curve C of genus $g \geqslant 2$ and fix a line bundle $L \in \mathrm{Pic}(C)$ of degree d. Denote by $M_C(r, L)$ the moduli space of S-equivalence classes of semistable vector bundles of rank r and with determinant L. In the case $\gcd(r, d) = 1$ it is known that every semistable vector bundle is stable implying that $M_C(r, L)$ is a smooth projective variety of dimension $(r^2 - 1)(g - 1)$, so in particular $M_C(r, L)$ is irreducible. Furthermore $\gcd(r, d) = 1$ also implies that the moduli space $M_C(r, L)$ is fine, that is there is a universal family $\mathcal{U}$ on $C \times M_C(r, L)$.

In the following the vector bundle of endomorphisms of $\mathcal{U}$ of trace zero is denoted by $\mathrm{ad}(\mathcal{U})$ and $\mathrm{ad}_c(\mathcal{U})$ denotes the vector bundle on $M_C(r, L)$ obtained by restricting $\mathrm{ad}(\mathcal{U})$ to $\{c\} \times M_C(r, L)$.

Before stating the next result, recall that given a smooth projective variety X, it is well known that the deformation theory of X is encoded in the cohomology of the tangent bundle T_X. For example $\mathrm{H}^0(X, T_X)$ contains information about infinitesimal automorphisms of X, whereas $\mathrm{H}^1(X, T_X)$ parametrizes first order deformations of X. Finally $\mathrm{H}^2(X, T_X)$ contains the obstructions to extending first order deformations of X, see [49].

In [46] Narasimhan and Ramanan study the deformation theory of the smooth and projective variety $M_C(r, L)$. This leads them naturally to look at the wrong-way fibers of $\mathcal{U}$. One of their main results is [46, Theorem 2] (which was later completed by Fonarev and Kuznetsov in [18, Proposition 2.14] as well as Belmans and Mukhopadhyay in [8, Proposition 7]):

Theorem 1.3. *a) The infinitesimal deformation map $T_cC \to \mathrm{H}^1(M_C(r, L), \mathrm{ad}_c(\mathcal{U}))$ of the bundle $\mathcal{U}$, considered as a family of bundles on $M_C(r, L)$ parametrized by C, is injective.*

b) For any $c \in C$, $\mathrm{H}^0(M_C(r,L), \mathrm{ad}_c(\mathcal{U})) = 0$. Moreover

$$\dim \mathrm{H}^1(M_C(r,L), \mathrm{ad}_c(\mathcal{U})) = 1$$
$$\dim \mathrm{H}^2(M_C(r,L), \mathrm{ad}_c(\mathcal{U})) = 0$$

These results imply, for example, that there is an isomorphism

$$H^1(M_C(r,L), T_{M_C(r,L)}) \cong H^1(C, T_C),$$

which means that $M_C(r,L)$ has the same number of moduli as the curve C.

We want to restate the results of Narasimhan and Ramanan in a form, which will be more useful for us in the following. For this we remark that the vector bundle $\mathrm{ad}_c(\mathcal{U})$ sits in the exact sequence:

$$0 \longrightarrow \mathrm{ad}_c(\mathcal{U}) \longrightarrow \mathcal{E}nd_{M_C(r,L)}(\mathcal{U}_c) \xrightarrow{\mathrm{tr}} \mathcal{O}_{M_C(r,L)} \longrightarrow 0.$$

The trace can be split by the identity morphism of $\mathcal{U}_c$, so that the exact sequence gives an isomorphism

$$\mathcal{E}nd_{M_C(r,L)}(\mathcal{U}_c) \cong \mathrm{ad}_c(\mathcal{U}) \oplus \mathcal{O}_{M_C(r,L)}.$$

To compute the cohomology of $\mathcal{O}_{M_C(r,L)}$, we use the fact that $M_C(r,L)$ is unirational, see [45, §2]. Hence we get

$$\mathrm{H}^i(M_C(r,L), \mathcal{O}_{M_C(r,L)}) = \begin{cases} \mathbb{C} & i = 0 \\ 0 & i \geqslant 1 \end{cases}.$$

Remark 1.4. The moduli space $M_C(r,L)$ is actually rational for $\gcd(r,d) = 1$. This long standing conjecture was proven by King and Schofield, see [27, Theorem 1.2]. If $\gcd(r,d) \geqslant 2$ the rationality of the moduli space $M_C(r,L)$ is only known for curves of genus $g = 2$ with $r = 2$ and $L = \mathcal{O}_C$. In this case one has $M_C(2, \mathcal{O}_C) \cong \mathbb{P}^3$, see [44, Theorem 7.2]

We are finally ready to reformulate Theorem 1.3:

Theorem 1.5. *For every closed point $c \in C$ the vector bundle $\mathcal{U}_c$ on $M_C(r,L)$ is simple and the infinitesimal deformation map of the family $\mathcal{U}$ (seen as a family of vector bundles on $M_C(r,L)$ classified by C)*

$$T_cC \to \mathrm{Ext}^1_{M_C(r,L)}(\mathcal{U}_c, \mathcal{U}_c)$$

is an isomorphism.

Since $\mathcal{U}_c$ is simple, one may ask if it is also slope stable. To study the question of slope stability of $\mathcal{U}_c$, we need a polarization on $M_C(r,L)$. It is well known that

$$\mathrm{Pic}(M_C(r,L)) = \mathbb{Z}\Theta,$$

where Θ is an ample line bundle on $M_C(r,L)$ such that $\omega_{M_C(r,L)} = \Theta^{\otimes -2}$, see [16, Théorème B, Théorème F]. In particular slope stability on $M_C(r,L)$ is independent of the chosen polarization.

Remark 1.6. This also shows that $M_C(r,L)$ is a $(r^2-1)(g-1)$-dimensional Fano variety of index two.

Using Θ as a polarization, Balaji, Brambila-Paz and Newstead proved:

Theorem 1.7. [5, Proposition 2.4] *For any closed point $c \in C$ the vector bundle $\mathcal{U}_c$ is slope semistable with respect to Θ.*

But even more is true as Lange and Newstead showed:

Theorem 1.8. [31, Proposition 2.1, Theorem] *For any closed point $c \in C$ the vector bundle $\mathcal{U}_c$ is slope stable with respect to Θ. If $c_1, c_2 \in C$ is a pair of closed points with $c_1 \neq c_2$, then $\mathcal{U}_{c_1} \ncong \mathcal{U}_{c_2}$.*

These results show that the universal family $\mathcal{U}$ on $C \times M_C(r,L)$ can also be interpreted as a family of stable vector bundles on $M_C(r,L)$ parametrized by C. Denote by $\mathcal{M}$ the moduli space of stable sheaves on $M_C(r,L)$ with the same numerical invariants as $\mathcal{U}_c$, for some $c \in C$ and hence for all $c \in C$. The family $\mathcal{U}$ thus gives rise to a classifying morphism

$$C \to \mathcal{M},\ c \mapsto [\mathcal{U}_c]\,.$$

The results of Lange and Newstead show that this morphism is injective on closed points. It is also well known that there is an isomorphism

$$T_{[\mathcal{U}_c]}\mathcal{M} \cong \mathrm{Ext}^1_{M_C(r,L)}(\mathcal{U}_c,\mathcal{U}_c)$$

so that the computations of Narasimhan and Ramanan show that the tangent space of $\mathcal{M}$ has dimension one at every point of the image of the classifying morphism. It follows that the classifying morphism identifies C with a smooth connected component of $\mathcal{M}$.

So similar to the Poincaré bundle $\mathcal{P}$ on $A \times \hat{A}$ for an abelian variety A, the wrong-way fibers of the universal family $\mathcal{U}$ on $C \times M_C(r,L)$ give new examples of stable vector bundles on the moduli space of stable vector bundles on C. Furthermore their classifying morphism gives an isomorphism between C and a smooth connected component of some moduli space of stable sheaves on $M_C(r,L)$.

The same phenomenon can also be observed on surfaces:

- In [24, Example 5.3.7] Huybrechts and Lehn pick a very general quartic hypersurface X in $\mathbb{P}^3$, i.e. a K3 surface with $\mathrm{NS}(X) = \mathbb{Z}h$ and $h^2 = 4$. They study the moduli space $M_h(v)$ of h-slope stable sheaves with the Mukai vector $v = (2,-h,1)$. This moduli space is a two dimensional fine moduli space which classifies only vector bundles. In fact there is an isomorphism

$$X \xrightarrow{\cong} M_h(v),\ x \mapsto [F_x]\,,$$

  where F_x is the h-slope stable vector bundle which is given as the kernel of the canonical, in this case surjective, evaluation morphism

$$\mathrm{eval} : \mathrm{H}^0(X, I_x(1)) \otimes \mathcal{O}_X \to I_x(1).$$

  Huybrechts and Lehn construct a universal family $\mathcal{U}$ on $X \times X$, where the second copy of X is just $M_h(v)$ and prove that the correct fibers $\mathcal{U}_{|X\times\{x\}} \cong F_x$ are defined by

$$0 \longrightarrow F_x \longrightarrow \mathrm{H}^0(X, I_x(1)) \otimes \mathcal{O}_X \xrightarrow{\mathrm{eval}} I_x(1) \longrightarrow 0.$$

  while the wrong-way fibers also satisfy $\mathcal{U}_{|\{x\}\times X} \cong F_x$ given by

$$0 \longrightarrow F_x \longrightarrow \Omega_{\mathbb{P}^3}(1)_{|X} \longrightarrow I_x \longrightarrow 0.$$

  Thus the universal family $\mathcal{U}$ on $X \times X$ identifies each factor as the moduli space of the other.
- A slightly more involved example (but still with two dimensional moduli space) can be found in [40, Theorem 1.2]: Mukai starts with a general polarized K3 surface (X,h) of degree $h^2 = 2rs$ with $\gcd(r,s) = 1$ and the Mukai vector $v = (r,h,s)$. By general results the moduli space $Y := M_h(v)$ of h-slope stable sheaves is fine and again a K3 surface, as $v^2 = 0$. All sheaves classified by $M_h(v)$ are locally free, that is the universal family $\mathcal{U}$ on $X \times Y$ is also locally free. Mukai then proves that there is a canonically defined ample divisor $\hat{h}$ on Y such that for every closed point $x \in X$ the wrong-way fiber $\mathcal{U}_x$ is a vector bundle on Y, slope stable with respect to $\hat{h}$. The bundle $\mathcal{U}_x$ belongs to a moduli space $M_{\hat{h}}(w)$ for some Mukai vector w with $w^2 = 0$. Furthermore the classifying morphism of the wrong-way fibers

$$X \to M_{\hat{h}}(w),\ x \mapsto [\mathcal{U}_x]$$

  is an isomorphism.

Motivated by the above examples, one can formulate the following question in a more general setting:

Question 1.9. Let X be a smooth projective variety and M a projective fine moduli space of stable sheaves on X with universal family $\mathcal{U}$ on $X \times M$. Then

i) Is $\mathcal{U}$ also a flat family sheaves on M parametrized by X?

ii) Given a closed point $x \in X$: is the wrong-way fiber $\mathcal{U}_x$ a stable sheaf on M?

iii) If so, does the classifying map identify X with a smooth connected component of some moduli space of stable sheaves on M?

2. **Wrong-way fibers and integral functors.**

To every smooth projective variety X one can associate its bounded derived category of coherent sheaves $\mathrm{D}^{\mathrm{b}}(X)$. The derived category is a fundamental invariant and contains a lot of geometric information about X. In some cases one can even recover X from $\mathrm{D}^{\mathrm{b}}(X)$ but there are also examples of non-isomorphic varieties with equivalent derived categories. We refer to [22] for a very good introduction into derived categories and more.

The wrong-way fibers of a universal family $\mathcal{U}$ on $X \times M$ for a moduli space M of stable sheaves on a smooth projective variety X also appear in the study of integral functors between the derived categories $\mathrm{D}^{\mathrm{b}}(X)$ and $\mathrm{D}^{\mathrm{b}}(M)$. More generally the wrong-way fibers of any family $\mathcal{E}$ on $X \times S$, which is flat over X, may appear in the study of integral functors between $\mathrm{D}^{\mathrm{b}}(X)$ and $\mathrm{D}^{\mathrm{b}}(S)$, as we will explain below. In this section, we will interpret a coherent sheaf F on a smooth projective variety Y as an element in $\mathrm{D}^{\mathrm{b}}(Y)$ by thinking of F as a complex concentrated in degree zero.

In general one is interested in breaking the derived category $\mathrm{D}^{\mathrm{b}}(X)$ up into smaller, more accessible, subcategories. We want to find a so-called semi-orthogonal decomposition of $\mathrm{D}^{\mathrm{b}}(X)$. We quickly recall that a semi-orthogonal decomposition is given by a sequence $\mathcal{A}_1, \ldots, \mathcal{A}_n$ of full admissible triangulated subcategories of $\mathrm{D}^{\mathrm{b}}(X)$ such that

i) If $A_i \in \mathcal{A}_i$ and $A_j \in \mathcal{A}_j$ then
$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(X)}(A_i, A_j[l]) = 0 \text{ for } i > j \text{ and all } l,$$

ii) the $\mathcal{A}_i$ generate $\mathrm{D}^{\mathrm{b}}(X)$, that is, the smallest triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(X)$ containing all the $\mathcal{A}_i$ is already $\mathrm{D}^{\mathrm{b}}(X)$.

In this case we write
$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle .$$
Furthermore given any subcategory $\mathcal{A}$ of $\mathrm{D}^{\mathrm{b}}(X)$ we define
$$\mathcal{A}^{\perp} := \left\{ B \in \mathrm{D}^{\mathrm{b}}(X) \mid \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(X)}(A, B[l]) = 0 \text{ for all } A \in \mathcal{A} \text{ and } l \in \mathbb{Z} \right\}.$$

Two typical ways of finding semi-orthogonal decompositions are exceptional objects and fully faithful integral functors.

Recall that an object $E \in \mathrm{D}^{\mathrm{b}}(X)$ is called exceptional if
$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(X)}(E, E[i]) = \begin{cases} \mathbb{C} & i = 0 \\ 0 & i \neq 0 \end{cases}$$
Given an exceptional object $E \in \mathrm{D}^{\mathrm{b}}(X)$, the smallest subcategory of $\mathrm{D}^{\mathrm{b}}(X)$ containing E (which is isomorphic to $\mathrm{D}^{\mathrm{b}}(\mathrm{Spec}(\mathbb{C}))$) is also denoted by by E and gives rise to the semi-orthogonal decomposition
$$\mathrm{D}^{\mathrm{b}}(X) = \left\langle E^{\perp}, E \right\rangle .$$

Iterating this process leads to the notion of an exceptional sequence. This is a sequence $E_1, E_2, \ldots, E_n$ of exceptional objects in $\mathrm{D}^{\mathrm{b}}(X)$ such that
$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(X)}(E_i, E_j[l]) = \begin{cases} \mathbb{C} & l = 0, i = j \\ 0 & i > j \text{ or } l \neq 0, i = j \end{cases}$$
In this case, we have a semi-orthogonal decomposition
$$\mathrm{D}^{\mathrm{b}}(X) = \left\langle (E_1, \ldots, E_n)^{\perp}, E_1, \ldots, E_n \right\rangle .$$
The sequence is called full if $(E_1, \ldots, E_n)^{\perp} = 0$.

Remark 2.1. An exceptional collection is called strong if
$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(X)}(E_i, E_j[l]) = 0 \text{ for all } i, j \text{ and } l \neq 0$$
For a strong full exceptional sequence $E_1, \ldots, E_n$ define
$$E := \bigoplus_{i=1}^{n} E_i \text{ and } A := \mathrm{End}(E).$$

Bondal proved that in this situation there is an equivalence of derived categories

$$\mathrm{D^b}(X) \cong \mathrm{D^b}(\mathrm{mod}{-}A),$$

where $\mathrm{D^b}(\mathrm{mod}{-}A)$ is the bounded derived category of right finite-dimensional modules over the algebra A, see [10, Theorem 6.2]. This result connects the geometry of X with the representation theory of the algebra A.

As an explicit example we state Beĭlinson's famous result:

Theorem 2.2. [9] *The sequence $\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1), \ldots, \mathcal{O}_{\mathbb{P}^n}(n)$ is a strong full exceptional collection in $\mathrm{D^b}(\mathbb{P}^n)$, that is we have*

$$\mathrm{D^b}(\mathbb{P}^n) = \langle \mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1), \ldots, \mathcal{O}_{\mathbb{P}^n}(n) \rangle$$

For $n = 1$ this result shows that the derived category of $\mathbb{P}^1$ admits a semi-orthogonal decomposition by the full exceptional collection $\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(1)$. In the case of curves this is a very special case. Okawa proves in [47, Theorem 1.1] that the derived category $\mathrm{D^b}(C)$ of a smooth projective curve C with genus $g \geqslant 1$ has no nontrivial semi-orthogonal decompositions. We say that $\mathrm{D^b}(C)$ is indecomposable.

A different class of smooth projective varieties with indecomposable derived category are varieties X with $\omega_X = \mathcal{O}_X$. This applies, for instance, to K3 surfaces. This observation is due to Bridgeland, see [12, Definition 3.1, Example 3.2].

Another method to find semi-orthogonal decompositions, besides exceptional objects, uses fully faithful integral functors. First we recall the definition of integral functors: every element $\mathcal{K} \in \mathrm{D^b}(X \times Y)$ gives rise to two integral functors $\Phi_{\mathcal{K}}$ and $\widehat{\Phi}_{\mathcal{K}}$, in opposite directions, defined as follows:

Definition 2.3. The integral functors with kernel $\mathcal{K} \in \mathrm{D^b}(X \times Y)$ are defined by

$$\Phi_{\mathcal{K}} : \mathrm{D^b}(X) \to \mathrm{D^b}(Y),\ E \mapsto Rp_*(q^*E \otimes^L \mathcal{K})$$

as well as

$$\widehat{\Phi}_{\mathcal{K}} : \mathrm{D^b}(Y) \to \mathrm{D^b}(X),\ F \mapsto Rq_*(p^*F \otimes^L \mathcal{K})$$

where $p : X \times Y \to Y$ and $q : X \times Y \to X$ are the projections.

Integral functors give a way to compare the derived categories of X and Y. More exactly we are interested in those integral functors, which are fully faithful.

Remark 2.4. Recall that a functor $F : \mathcal{A} \to \mathcal{B}$ between two categories $\mathcal{A}$ and $\mathcal{B}$ is fully faithful if the induced map

$$F_{A,B} : \mathrm{Hom}_{\mathcal{A}}(A, B) \to \mathrm{Hom}_{\mathcal{B}}(F(A), F(B)).$$

is bijective for every pair $A, B \in \mathcal{A}$

If the integral functor $\Phi_{\mathcal{K}} : \mathrm{D^b}(X) \to \mathrm{D^b}(Y)$ is fully faithful then we get a semi-orthogonal decomposition

$$\mathrm{D^b}(Y) = \Big\langle (\Phi_{\mathcal{K}}(\mathrm{D^b}(X))^{\perp}, \Phi_{\mathcal{K}}(\mathrm{D^b}(X)) \Big\rangle$$

We can thus identify $\mathrm{D^b}(X)$ with a component of $\mathrm{D^b}(Y)$.

So the question becomes: given a kernel $\mathcal{K} \in \mathrm{D^b}(X \times Y)$, how can we decide if $\Phi_{\mathcal{K}}$ is fully faithful? Fortunately there is the following criterion due to Bondal and Orlov, see [11, Theorem 1.1] or [22, Proposition 7.1]:

Theorem 2.5. *The integral functor $\Phi_{\mathcal{K}}$ is fully faithful if and only if for any two closed points $x, y \in X$ one has*

$$\mathrm{Hom}_{\mathrm{D^b}(Y)}(\Phi_{\mathcal{K}}(\mathcal{O}_x), \Phi_{\mathcal{K}}(\mathcal{O}_y)[i]) = \begin{cases} \mathbb{C} & x = y \text{ and } i = 0 \\ 0 & x \neq y \text{ or } i \notin [0, \dim(X)] \end{cases}$$

Here $\mathcal{O}_x$ denotes the skyscraper sheaf of $x \in X$.

If the kernel $\mathcal{K}$ is in fact a sheaf $\mathcal{F}$, which is flat over X, then by [22, Example 5.4 vi)] we have an isomorphism

$$\Phi_{\mathcal{F}}(\mathcal{O}_x) = \mathcal{F}_x$$

for every closed point $x \in X$. In this case the criterion of Bondal and Orlov reads

Corollary 2.6. *Assume $\mathcal{F}$ is a coherent sheaf on $X \times Y$ flat over X, then $\Phi_{\mathcal{F}}$ is fully faithful if and only if*

i) For any closed point $x \in X$ one has $\mathrm{Ext}^i_Y(\mathcal{F}_x, \mathcal{F}_x) = \begin{cases} \mathbb{C} & i = 0 \\ 0 & i > \dim(X) \end{cases}$

ii) For any pair of closed points $x, y \in X$ with $x \neq y$ one has $\mathrm{Ext}^i_Y(\mathcal{F}_x, \mathcal{F}_y) = 0$ for all i

Thus if we have a smooth projective variety X with some fine moduli space of stable sheaves M, which is also a smooth projective variety, and $\mathcal{U}$ denotes the universal family then one can decide if the integral functor $\Phi_{\mathcal{U}}$ realizes $\mathrm{D^b}(X)$ as a subcategory of $\mathrm{D^b}(M)$, given that one has a good understanding of the cohomology groups of the wrong-way fibers $\mathcal{U}_x$.

Besides giving the start of a semi-orthogonal decomposition, fully faithful integral functors have another valuable property, which basically follows from the definition:

Corollary 2.7. *Assume the integral functor $\Phi_{\mathcal{K}}$ is fully faithful, then there are isomorphisms:*

$$\mathrm{Ext}^i_Y(\Phi_{\mathcal{K}}(E), \Phi_{\mathcal{K}}(F)) \cong \mathrm{Ext}^i_X(E, F) \textit{ for any pair } E, F \in \mathrm{D^b}(X).$$

This property allows to reduce computations on Y to computations on X, if one knows that the objects in question are images under the integral functor.

The first interesting example of a fully faithful integral functor, with kernel a universal family, was found by Krug and Sosna:

Theorem 2.8. [28, Theorem 1.2] *Let S be any smooth projective surface with $p_g = q = 0$. Denote the Hilbert scheme of length n subschemes of S by $S^{[n]}$ and the ideal sheaf of the universal length n subscheme $\mathcal{Z} \hookrightarrow S \times S^{[n]}$ by $\mathcal{I}_{\mathcal{Z}}$. Then the integral functor $\Phi_{\mathcal{I}_{\mathcal{Z}}}$ is fully faithful (hence $\mathrm{D^b}(S)$ can be identified subcategory of $\mathrm{D^b}(S^{[n]})$).*

To prove this result, Krug and Sosna do not use the Bondal-Orlov criterion but rather compute the right adjoint functor $(\Phi_{\mathcal{I}_{\mathcal{Z}}})^R$ of $\Phi_{\mathcal{I}_{\mathcal{Z}}}$ and show that in this case the composition

$$(\Phi_{\mathcal{I}_{\mathcal{Z}}})^R \circ \Phi_{\mathcal{I}_{\mathcal{Z}}}$$

is isomorphic to the identity functor of $\mathrm{D^b}(S)$, hence $\Phi_{\mathcal{I}_{\mathcal{Z}}}$ must be fully faithful.

Remark 2.9. In the case of surfaces the integral functor $\Phi_{\mathcal{I}_{\mathcal{Z}}}$ is fully faithful if and only if $p_g = q = 0$ by [7, Theorem A]. This excludes other interesting surfaces like K3 surfaces or abelian surfaces. But in these cases the functor is a so-called $\mathbb{P}^{n-1}$-functor, which implies, similar to Corollary 2.7, that we have an isomorphism of graded vector spaces

$$\mathrm{Ext}^*_M(\Phi_{\mathcal{I}_{\mathcal{Z}}}(E), \Phi_{\mathcal{I}_{\mathcal{Z}}}(F)) \cong \mathrm{Ext}^*_X(E, F) \otimes \mathrm{H}^*(\mathbb{P}^{n-1}, \mathbb{C}) \text{ for any } E, F \in \mathrm{D^b}(X).$$

These functors were introduced by Addington in a very general setting, see [1, §4].

The integral functor $\Phi_{\mathcal{I}_{\mathcal{Z}}}$ is a $\mathbb{P}^{n-1}$-functor in the following examples:

a) X is a K3 surface and $M = X^{[n]}$ is the Hilbert scheme of n points, see [1, Theorem 3.1].
b) X is an abelian surface and $M = \mathrm{Kum}_n(X)$ is the generalized Kummer variety (for $n \geqslant 2$), see [39, Theorem 4.1].
c) X is a K3 surface with Picard number $\rho(X) = 1$ and M a fine moduli space of stable torsion sheaves of pure dimension one on X, see [2, Theorem A].

The next example treats the moduli space $M_C(2, L)$ of vector bundles of rank two on a curve C of genus $g \geqslant 2$ with determinant L, where $L \in \mathrm{Pic}(C)$ is of degree one. The following results relies on work of Fonarev and Kuznetsov, see [18, Theorem 1.1], as well as Narasimhan, see [42, Theorem 1.1] and [43, Theorem 1]:

Theorem 2.10. *For a smooth projective curve C of genus $g \geqslant 2$ and a line bundle L of degree one on C the integral functor*

$$\Phi_{\mathcal{U}} : \mathrm{D^b}(C) \to \mathrm{D^b}(M_C(2, L)),$$

with kernel the universal family $\mathcal{U}$ on $C \times M_C(2,L)$, is fully faithful, hence $\mathrm{D^b}(C)$ is a subcategory of $\mathrm{D^b}(M_C(2,L))$.

Fonarev and Kuznetsov prove Theorem 2.10 explicitly for hyperelliptic curves of genus $g \geqslant 2$, i.e. curves C which admit a double cover $C \to \mathbb{P}^1$, by using the Bondal-Orlov criterion. For a hyperelliptic curve, the moduli space $M_C(2,L)$ has an equivalent description as the variety X_{g-1} of $(g-2)$-dimensional linear subspaces of $\mathbb{P}^{2g+1}$ contained in two quadrics Q_1 and Q_2 lying in this $\mathbb{P}^{2g+1}$. This result goes back to Desale and Ramanan in [14, Theorem 1]. Fonarev and Kuznetsov show that this description is equivalent to the intersection of two orthogonal isotropic Grassmannians. But the latter admit the so-called Spinor bundle and so induce a universal Spinor bundle $\mathcal{S}$ on $C \times X_{g-1}$ which can be identified (up to a twist by a line bundle) with the universal family $\mathcal{U}$ on $C \times M_C(2,L)$, using the above isomorphism. Fonarev and Kuznetsov compute all cohomology groups necessary for the Bondal-Orlov criterion by reducing the computations to the Grassmannians.

Finally they prove that if one has a relative Fourier Mukai transform over a smooth base B, then the locus of full faithfulness

$$\mathrm{FFL}(\mathcal{E}) := \{b \in B \mid \Phi_{\mathcal{E}_b} \text{ is fully faithful}\}$$

is open in B. Using the case of hyperelliptic curves they can thus conclude that $\Phi_{\mathcal{U}}$ is fully faithful for a general curve of genus $g \geqslant 2$.

On the other hand Narasimhan proves Theorem 2.10 in [42, Theorem 1.1] for all curves of genus $g \geqslant 4$ by also applying the Bondal-Orlov criterion. He can explicitly compute all necessary cohomology groups by using the so-called Hecke correspondence:

$$\begin{tikzcd}
 & \mathcal{H}(\mathcal{U},c) \arrow[dl, "q"'] \arrow[dr, "p"] & \\
M_C(2, L^{-1} \otimes \mathcal{O}_C(c)) & & M_C(2,L)
\end{tikzcd}$$

with $\mathcal{H}(\mathcal{U},c) := \mathbb{P}(\mathcal{U}_c)$. So $p : \mathcal{H}(\mathcal{U},c) \to M_C(2,L)$ is just a $\mathbb{P}^1$-bundle and the morphism $q : \mathcal{H}(\mathcal{U},c) \to M_C(2, L^{-1} \otimes \mathcal{O}_C(c))$ is the classifying morphism of a certain family $\mathcal{F}$ of semistable vector bundles of rank two and determinant $L^{-1} \otimes \mathcal{O}_C(c)$ on $C \times \mathcal{H}(\mathcal{U},c)$, see [46, §4].

In [43, Theorem 1] Narasimhan proves Theorem 2.10 for all nonhyperelliptic curves of genus $g \geqslant 3$, so especially for the missing case of nonhyperelliptic curves of genus $g = 3$. Combining the results of Fonarev and Kuznetsov and Narasimhan shows that Theorem 2.10 is true for all curves of genus $g \geqslant 2$.

Remark 2.11. Theorem 2.10 has been generalized to rank $r \geqslant 3$ for curves of large genus, see for example [8, Theorem A]. As a matter of fact, by the recent result [33, Theorem A] of Lee and Moon, the theorem is true for all pairs (r,d) with $\gcd(r,d) = 1$.

Remark 2.12. Theorem 2.10 gives evidence for the following conjectural semi-orthogonal decomposition:

$$\mathrm{D^b}(M_C(2,L)) = \left\langle \left\{ \mathrm{D^b}(C^{(k)}), \mathrm{D^b}(C^{(k)}) \right\}_{0 \leqslant k \leqslant g-2}, \mathrm{D^b}(C^{(g-1)}) \right\rangle,$$

where $C^{(k)} = C^k / \mathfrak{S}_k$ is the k-th symmetric power of C. This conjecture is due to Narasimhan and independently Belmans, Galkin and Mukhopadhyay, see [8, Remark 1]. More evidence for this conjecture can be found in [34] and [51]. We already saw that $\mathrm{D^b}(C)$ is indecomposable, by [35, Theorem 4.5] all other components are also indecomposable. So this conjecture gives a decomposition into indecomposable factors. There is also a conjecture about the semi-orthogonal decomposition of $M_C(3,L)$ for a line bundle L of degree one, see [19, Conjecture 1.9].

Remark 2.13. Theorem 2.10 answers the so-called Fano visitor problem positive for all smooth projective curves of genus $g \geqslant 2$, as $M_C(2,L)$ is Fano by Remark 1.6. The Fano visitor problem goes back to Bondal in 2011, who asked the following question:

Assume X is a smooth projective variety. Is there a smooth Fano variety Y and a fully faithful functor $F : \mathrm{D^b}(X) \to \mathrm{D^b}(Y)$?

Looking at the two examples Hilbert schemes of points on surfaces with $p_g = q = 0$ and moduli spaces of vector bundles of rank two on curves of genus $g \geqslant 2$ it is natural to ask the following question, which is also implicit in [6, Remark 30]:

Question 2.14. Let (X, h) be a polarized smooth projective surface with $p_g = q = 0$ and denote the moduli space of S-equivalence classes of h-slope semistable torsion free sheaves with rank r and Chern classes c_1 and c_2 on X by $M_X(r, c_1, c_2)$. If $M_X(r, c_1, c_2)$ is smooth, projective and fine, does the universal family $\mathcal{U}$ on $X \times M_X(r, c_1, c_2)$ induce a fully faithful integral functor

$$\Phi_{\mathcal{U}} : \mathrm{D^b}(X) \to \mathrm{D^b}(M_X(r, c_1, c_2))?$$

3. **Noncommutative surfaces and deformations.**

Let X be a smooth projective variety. A deformation of a coherent sheaf E of $\mathcal{O}_X$-modules with base B is a family $\mathcal{E}$ on $X \times B$, flat over B, such that $\mathcal{E}_{b_0} = E$ for some distinguished $b_0 \in B$. By abuse of notation we will also call the sheaf $\mathcal{E}_b$ for some $b \neq b_0$ a deformation of E.

Remark 3.1. As being torsion free is an open condition for coherent sheaves in families, see [37, Proposition 2.1], we may replace the base by an open subset. So we can always assume that for a deformation $\mathcal{E}$ of a torsion free sheaf E all sheaves $\mathcal{E}_b$ are torsion free.

Remark 3.2. Similar to the deformation theory of a smooth projective variety X, the deformation theory of the coherent sheaf E is encoded in certain cohomology groups. For example, $\mathrm{Ext}^1_X(E,E)$ parametrizes first order deformations of E, i.e. deformations over the base $B = \mathrm{Spec}(\mathbb{C}[\epsilon])$ with $\epsilon^2 = 0$. The obstructions to extend first order deformations lie in $\mathrm{Ext}^2_X(E,E)$. Furthermore there is the Kodaira-Spencer map

$$\rho : T_{b_0}B \to \mathrm{Ext}^1_X(E,E)$$

of a deformation $\mathcal{E}$ of E, which maps a tangent vector $v \in T_{b_0}B$ to the class corresponding to the first order deformation $\rho(v) \in \mathrm{Ext}^1_X(E,E)$ of the sheaf E along the direction of v.

We are mostly interested in studying locally free sheaves, meaning vector bundles, but sometimes we are just able to construct torsion free sheaves. Then one can ask: can we deform a torsion free sheaf E to a vector bundle E'? Artamkin studied the question of deforming torsion free sheaves in [4] and gave some positive results for smooth projective surfaces, which we recall:

Firstly, to measure how far away a torsion free sheaf E is from being a vector bundle, we recall that there is naturally defined vector bundle associated to a torsion free sheaf E on a smooth projective surface: the bidual $E^{**} = \mathcal{H}om_X(\mathcal{H}om_X(E, \mathcal{O}_X), \mathcal{O}_X)$.

As E is torsion free, it is naturally a subsheaf of the vector bundle E^{**} and it differs from E^{**} only in finitely many points. These facts are captured by the bidual exact sequence

$$0 \longrightarrow E \xrightarrow{\iota} E^{**} \longrightarrow T_E \longrightarrow 0, \tag{1}$$

where ι is the canonical inclusion and $T_E = E^{**}/E$ is Artinian, that is it has finite support and finite length $\ell_{\mathcal{O}_X}(T_E) < \infty$.

The number $\ell_{\mathcal{O}_X}(T_E)$ is a good measure of non-locally freeness of E. In fact the bigger this number is, the further away from being a vector bundle E is, as $\ell_{\mathcal{O}_X}(T_E) = 0$ if and only if $E \cong E^{**}$, that is E is locally free.

Remark 3.3. As T_E has finite support, we have decompositions

$$T_E = \bigoplus_{i=1}^n T_{E,p_i} \text{ and } \ell_{\mathcal{O}_X}(T_E) = \bigoplus_{i=1}^n \ell_{\mathcal{O}_X}(T_{E,p_i}) \text{ where } \mathrm{supp}(T_E) = \{p_1, \ldots, p_n\}.$$

Artamkin is therefore led to make the following:

Definition 3.4. A point $p \in \mathrm{supp}(T_E)$ is said to be cancellable in the deformation $\mathcal{E}$ if $\ell_{\mathcal{O}_X}(T_{E',p}) < \ell_{\mathcal{O}_X}(T_{E,p})$ for a general fiber E' of $\mathcal{E}$. This means that for a general $b \in B$ the sheaf $E' := \mathcal{E}_b$ on X is "less" torsion free at p in deformation $\mathcal{E}$ than E and thus closer to being locally free.

To solve the deformation problem Artamkin considers the map

$$j = \iota_* \circ \delta : \mathrm{Ext}^1_X(E,E) \to \mathrm{Ext}^2_X(T_E, E^{**}). \tag{2}$$

Here δ is the connecting homomorphism in the long exact sequence associated to (1) after applying $\mathrm{Hom}_X(-,E)$:

$$\ldots \longrightarrow \mathrm{Ext}^1_X(E,E) \xrightarrow{\delta} \mathrm{Ext}^2_X(T_E,E) \longrightarrow \mathrm{Ext}^2_X(E^{**},E) \longrightarrow \ldots.$$

Applying $\mathrm{Hom}_X(T_E,-)$ to (1) gives the map ι_*:

$$\ldots \longrightarrow \mathrm{Ext}^2_X(T_E,E) \xrightarrow{\iota_*} \mathrm{Ext}^2_X(T_E,E^{**}) \longrightarrow \mathrm{Ext}^2_X(T_E,T_E) \longrightarrow 0.$$

One of the main results of Artamkin is

Theorem 3.5. [4, Corollary 1.3] *If $\xi \in \operatorname{Ext}^1_X(E,E)$ is a Kodaira-Spencer class of a deformation $\mathcal{E}$ of E over a one-dimensional base B with the property $j_p(\xi) \neq 0$, then p is cancellable in $\mathcal{E}$, where*

$$j_p : \operatorname{Ext}^1_X(E,E) \to \operatorname{Ext}^2_X(T_{E,p}, E^{**}).$$

is the appropriate direct summand of the map j from (2)*.*

Remark 3.6. Recall that a vector bundle E on a smooth projective variety $X \subset \mathbb{P}^N$ is called Ulrich bundle if

$$\mathrm{H}^i(X, E(-r)) = 0 \text{ for all } i \geqslant 0 \text{ and } 1 \leqslant r \leqslant \dim(X).$$

The result of Artamkin has recently been used to construct Ulrich bundles on surfaces, see [17] for the case of K3 surfaces and [36] for the case of surfaces of maximal Albanese dimension or with irregularity one.

Next we want to give a quick introduction to the concept of noncommutative geometry we are going to use.

There are many ways to define the notion of a "noncommutative variety", but the most geometric one is possibly given by replacing the structure sheaf $\mathcal{O}_X$ of a scheme $(X, \mathcal{O}_X)$ by a sheaf of associative $\mathcal{O}_X$-algebras $\mathcal{A}$, such that $\mathcal{A}$ is coherent and torsion free as a sheaf of $\mathcal{O}_X$-modules. Then the pair $(X, \mathcal{A})$ can be thought of as a noncommutative version of X, see [30, §2.1] for this idea and more information.

In [21] Hoffman and Stuhler consider noncommutative varieties $(X, \mathcal{A})$ with X smooth and projective and such that the stalk $\mathcal{A}_\eta$ at the generic point $\eta \in X$ is a central simple algebra over the function field $\mathbb{C}(X) = \mathcal{O}_{X,\eta}$ of X. They study the following sheaves:

Definition 3.7. A sheaf E on X is called generically simple $\mathcal{A}$-module, if

i) E is coherent and torsion free as an $\mathcal{O}_X$-module
ii) E is a left $\mathcal{A}$-module, such that the generic stalk E_η is a simple $\mathcal{A}_\eta$-module.

Remark 3.8. In the case $\mathcal{A} = \mathcal{O}_X$ generically simple $\mathcal{A}$-modules are just torsion free sheaves of rank one and locally projective ones are exactly the line bundles. Hence in general generically simple $\mathcal{A}$-modules can be considered as line bundles on the noncommutative variety $(X, \mathcal{A})$ if they are also locally projective. If $\mathcal{A}_\eta$ is a central division algebra then a generically simple $\mathcal{A}$-module E is indeed generically of rank one, that is $\mathrm{rk}_{\mathcal{A}_\eta}(E_\eta) = 1$. But $\mathrm{rk}(E) = \mathrm{rk}(\mathcal{A}) > 1$ in case $\mathcal{A} \neq \mathcal{O}_X$.

As generically simple $\mathcal{A}$-modules are simple as $\mathcal{A}$-modules, see the remark before [21, Lemma 1.2], it makes sense to ask if such sheaves have a moduli space. Indeed Hoffman and Stuhler construct a moduli scheme for these modules. To do this the authors have to define families for these $\mathcal{A}$-modules:

Definition 3.9. A family of generically simple torsion free $\mathcal{A}$-modules over a $\mathbb{C}$-scheme S is a sheaf $\mathcal{E}$ of left modules under the pullback $\mathcal{A}_S$ of $\mathcal{A}$ to $X \times S$ with the following properties:

- $\mathcal{E}$ is coherent over $\mathcal{O}_{X \times S}$ and flat over S.
- For every point $s \in S$, the fiber $\mathcal{E}_s$ is a generically simple torsion free $\mathcal{A}_{\mathbb{C}(s)}$-module.

Here $\mathbb{C}(s)$ is the residue field of S at s, and the fiber $\mathcal{E}_s$ is by definition the pullback of $\mathcal{E}$ to $X \times \mathbb{C}(s)$.

One of the main results of Hoffmann and Stuhler is:

Theorem 3.10. [21, Theorem 2.4] *There is a projective moduli scheme $M_{\mathcal{A}/X;P}$ classifying generically simple $\mathcal{A}$-modules with fixed Hilbert polynomial P.*

Remark 3.11. Theorem 3.10 shows that the moduli functor $\mathcal{M}_{\mathcal{A}/X}$ of all generically simple $\mathcal{A}$-modules has a coarse moduli space

$$M_{\mathcal{A}/X} = \coprod_P M_{\mathcal{A}/X;P}.$$

Another such decomposition is

$$M_{\mathcal{A}/X} = \coprod_{c_1,\dots,c_n} M_{\mathcal{A}/X;c_1,\dots,c_n},$$

given by fixing the Chern classes $c_i \in \mathrm{CH}^i(X)$, the Chow group of cycles modulo algebraic equivalence.

Remark 3.12. By Remark 3.8 the moduli space $M_{\mathcal{A}/X}$ can be considered as a compactification of a noncommutative Picard scheme of $(X, \mathcal{A})$.

Generically simple $\mathcal{A}$-modules behave very much like (stable) sheaves in the classical sense: for example besides being simple, they have a deformation theory governed by the $\mathrm{Ext}^i_{\mathcal{A}}$-groups, see [21, §3].

For their next result Hoffmann and Stuhler restrict further to the so-called Azumaya algebras. These are algebras locally isomorphic to a matrix algebra $\mathrm{Mat}(n \times n, \mathcal{O}_X)$ in the étale topology. In this case there is the following version of Serre duality ($n = \dim(X)$):

$$\mathrm{Ext}^i_{\mathcal{A}}(E, F) \cong \left(\mathrm{Ext}^{n-i}_{\mathcal{A}}(F, E \otimes \omega_X)\right)^*$$

Remark 3.13. Azumaya algebras on a smooth projective variety X are classified up to similarity by the so-called Brauer group $\mathrm{Br}(X)$ of X. The multiplication in this abelian group is given by the tensor product $[\mathcal{A}][\mathcal{B}] := [\mathcal{A} \otimes \mathcal{B}]$ and the inverse is given by the class of the opposite algebra $[\mathcal{A}]^{-1} = [\mathcal{A}^{op}]$.

Remark 3.14. If we denote the Brauer class of $\mathcal{A}$ by α, that is $\alpha = [\mathcal{A}] \in \mathrm{Br}(X)$, then generically simple $\mathcal{A}$-modules can also be seen as a special class of α-twisted sheaves.

The definition of an α-twisted coherent sheaf involves an appropriate analytic (or étale) open cover of X, representing the class α as a Čech 2-cocycle on this cover and then "twisting" the gluing functions of the sheaf by this 2-cocycle. This idea uses the fact that for a smooth projective variety there is an isomorphism

$$\mathrm{Br}(X) \cong \mathrm{H}^2(X, \mathcal{O}_X^\times)_{\mathrm{tor}}.$$

For an exact definition see [13, Definition 1.2.1].

The pair (X, α) is sometimes called a twisted variety. Huybrechts and Stellari have studied properties of twisted $K3$-surfaces in detail, see [25] and [26].

All results can be rephrased in terms of α-twisted sheaves. The approach of using Azumaya algebras avoids working with open covers and gluing functions. On the other hand, this approach forces the ranks of the sheaves involved to be considerably larger.

The second main result of Hoffmann and Stuhler is:

Theorem 3.15. [21, Theorem 3.6] *Let X be an abelian or K3 surface, and let $\mathcal{A}$ be a sheaf of Azumaya algebras over X. Suppose $\mathcal{A}_\eta \cong \mathrm{Mat}(n \times n; D)$ for a central division algebra D of dimension r^2 over the function field $\mathbb{C}(X)$.*

i) The moduli space $M_{\mathcal{A}/X}$ of generically simple $\mathcal{A}$-modules E is smooth.
ii) There is a nowhere degenerate alternating 2-form on the tangent bundle of $M_{\mathcal{A}/X}$.
iii) If $r \geqslant 2$, then the open locus $M^{lp}_{\mathcal{A}/X}$ of locally projective $\mathcal{A}$-modules E is dense in $M_{\mathcal{A}/X}$.
iv) If we fix the Chern classes $c_1 \in \mathrm{NS}(X)$ and $c_2 \in \mathbb{Z}$ of E, then

$$\dim M_{\mathcal{A}/X;c_1,c_2} = \Delta/(nr)^2 - c_2(A)/n^2 - r^2\chi(\mathcal{O}_X) + 2$$

where $\Delta = 2r^2nc_2 - (r^2n - 1)c_1^2$ is the discriminant of E.

In iii) the authors prove that for every generically simple $\mathcal{A}$-module E there is a deformation $\mathcal{E}$ over a one dimensional base B, such that for general $b \in B$ the fiber $\mathcal{E}_b$ is a locally projective generically simple $\mathcal{A}$-module. To achieve this results, the authors show that there is an element $\xi \in \mathrm{Ext}^1_{\mathcal{A}}(E, E)$ such that $j_p(\xi) \neq 0$ for all $p \in \mathrm{supp}(T_E)$ for the noncommutative version of Artamkin's map

$$j_p : \mathrm{Ext}^1_{\mathcal{A}}(E, E) \to \mathrm{Ext}^2_{\mathcal{A}}(T_{E,p}, E^{**}).$$

The class ξ is given by a first order deformation of E. Using the smoothness of $M_{\mathcal{A}/X}$ this deformation can be extended to a deformation $\mathcal{E}$ of E over a smooth connected curve B, whose Kodaira-Spencer class is exactly ξ. A study of the forgetful functor $\mathrm{Coh}(X,\mathcal{A}) \to \mathrm{Coh}(X,\mathcal{O}_X)$ shows that the map

$$\mathrm{Ext}^2_{\mathcal{A}}(T_E, E^{**}) \to \mathrm{Ext}^2_X(T_E, E^{**})$$

is injective, which reduces the problem to the use of Artamkin's result by considering the class ξ as an element in $\mathrm{Ext}^1_X(E,E)$.

Remark 3.16. The fact that a torsion free generically simple $\mathcal{A}$-module E can be deformed into a locally projective one (over a connected base B) is a new phenomenon. In the classical case $\mathcal{A} = \mathcal{O}_X$ locally projective and just torsion free generically simple $\mathcal{O}_X$-modules lie in different connected components of the moduli space. The reason for this phenomenon is that the latter satisfy the valuative criterion for properness, while former do not, see [21, Remark 1.6].

Looking at the results in this section, one can ask the following questions:

Question 3.17. i) Let $(X,\mathcal{A})$ be a noncommutative K3 surface with an Azumaya algebra $\mathcal{A}$. By i) and ii) of Theorem 3.15 the moduli space $M_{\mathcal{A}/X;c_1,c_2}$ is smooth and has a symplectic structure. Is it a hyperkähler variety, in other words an irreducible holomorphic symplectic manifold? If yes, what is its deformation class?

ii) Are there other noncommutative surfaces $(X,\mathcal{A})$, possibly with $\mathcal{A}$ not necessarily an Azumaya algebra, such that $M_{\mathcal{A}/X;c_1,c_2}$ is smooth and such that every generically simple $\mathcal{A}$-module can be deformed into a locally projective one?

This thesis consists of the following eight articles:

[R1] Fabian Reede and Ziyu Zhang. Examples of smooth components of moduli spaces of stable sheaves. *Manuscripta Math.*, 165(3-4):605–621, 2021.
[R2] Fabian Reede and Ziyu Zhang. Stability of some vector bundles on Hilbert schemes of points on K3 surfaces. *Math. Z.*, 2021, online first.
[R3] Fabian Reede and Ziyu Zhang. Stable vector bundles on generalized Kummer varieties. *Forum Math.*, 34(4):1015–1031, 2022.
[R4] Fabian Reede. Smooth components on special iterated Hilbert schemes. *C. R. Math. Acad. Sci. Paris*, 360:425–429, 2022.
[R5] Fabian Reede. The Fourier-Mukai transform of a universal family of stable vector bundles. *Internat. J. Math.*, 32(2):Paper No. 2150007, 13, 2021.
[R6] Fabian Reede. The symplectic structure on the moduli space of line bundles on a noncommutative Azumaya surface. *Beitr. Algebra Geom.*, 60(1):67–76, 2019.
[R7] Norbert Hoffmann and Fabian Reede. Torsion-free rank one sheaves over del Pezzo orders. *J. Algebra*, 493:251–266, 2018.
[R8] Fabian Reede. Rank one sheaves over quaternion algebras on Enriques surfaces. *Adv. Geom.*, 2021, online first.

These are eight out of my ten articles written between 2015 and 2021, after finishing my PhD in 2013. Four of the articles are written with coauthors. All eight articles are published in peer-reviewed journals.

We will now give a quick summary of the main results of each paper with some ideas of the proofs.

1. **Examples of smooth components of moduli spaces of stable sheaves.**

In the article [R1] we study Question 1.9. A positive answer to all questions, especially when X is of low dimension and M is of higher dimension, would be interesting from two perspectives. First of all, examples of stable sheaves on higher dimensional varieties are in general difficult to construct. For example on higher dimensional irreducible holomorphic symplectic manifolds it is very hard to prove stability of a given vector bundle as these varieties have Picard number $\rho(X) \geqslant 2$. So even finding appropriate polarizations, which are needed to check stability, is hard to do. One important class of stable vector bundles are the tautological bundles on Hilbert schemes of points $S^{[n]}$ for a surface S, whose stability was studied by Schlickewei, Wandel and Stapleton, see [48, 54, 55, 50]. Here the tautological bundle $E^{[n]}$ on the Hilbert scheme $S^{[n]}$ associated to a vector bundle E on S is defined as follows: the Hilbert scheme comes with the universal length n subscheme $\mathcal{Z} \subset S \times S^{[n]}$, then one defines

$$E^{[n]} := p_*(q^*E \otimes \mathcal{O}_{\mathcal{Z}}),$$

where p and q are the projections from $S \times S^{[n]}$ to $S^{[n]}$ and S respectively. Question 1.9 provides another natural approach for finding new examples.

Secondly, moduli spaces of stable sheaves on higher dimensional varieties are in general badly behaved, they satisfy Murphy's law, see [53, Theorem 1.1 M6]. A positive answer to Question 1.9 would allow us to identify some nicely behaved components of such moduli spaces, and at the same time give an explicit description of a complete family of stable sheaves over these components.

The main result of this article is:

Theorem. *All subquestions in Question 1.9 have a positive answer in the following four examples*

a) X is a smooth projective variety, $M = X^{[2]}$ the Hilbert scheme of two points and $\mathcal{U} = \mathcal{I}_{\mathcal{Z}}$ the universal ideal sheaf
b) X is a K3 surface, $M = X^{[n]}$ is the Hilbert scheme of n points for any $n \geqslant 1$ and $\mathcal{U} = \mathcal{I}_{\mathcal{Z}}$ is the universal ideal sheaf
c) X is a abelian surface and $M = \mathrm{Kum}_n(X)$ is the generalized Kummer variety for any $n \geqslant 2$ and $\mathcal{U} = \mathcal{I}_{\mathcal{Z}}$ is the universal ideal sheaf
d) X is a K3 surface with Picard number one, M is a fine moduli space of stable torsion sheaves of pure dimension one on X and $\mathcal{U}$ is the universal family of this moduli space.

In fact in all cases the main problem was to establish the flatness of $\mathcal{U}$ over X. In example a), b) and c) it is enough to prove the flatness of the universal family $\mathcal{Z}$ over X. In a) and c) we use the so-called miracle flatness theorem:

Theorem. [38, Theorem 23.1, Corollary] *Let $f : X \to Y$ be a morphism between projective varieties such that X is Cohen-Macaulay, Y is smooth and all fibers of f have the same dimension, then f is flat.*

In both examples we can check all assumptions of the theorem, especially we find that all fibers of the projection $\mathcal{Z} \to X$ have the same dimension. In case b) the flatness of $\mathcal{Z}$ over X was proved in [29, Theorem 2.1] using different methods. In all three cases the proofs use rather elemetary methods, as we have an explicit description of the universal family. But in case d) we need to use some deep results about Cohen-Macaulay sheaves due to Arinkin, see [3, §2], as well as a flatness result in the derived category, see for example [22, Lemma 3.31].

The stability of the wrong-way fibers $\mathcal{U}_x$ follows immediately from the fact that in all four cases these sheaves have rank one, so there can be no destabilizing subsheaves.

To realize X as a smooth connected component of some moduli space $\mathcal{M}$ on M we need to be able to compute certain cohomology groups on M. In case a) we find these groups by elementary calculations, again using the explicit description of $\mathcal{Z}$ in this case. In the cases b), c) and d) we use the fact that the wrong-way fibers can be described as images of skyscraper sheaves of the integral functor $\Phi_{\mathcal{U}}$, that is

$$\mathcal{U}_x = \Phi_{\mathcal{U}}(\mathcal{O}_x).$$

We can then use Addington's and Meachan's results about the $\mathbb{P}^{n-1}$-functor $\Phi_{\mathcal{U}}$, see Remark 2.9, to reduce cohomology computations from M to X, which imply the desired result.

2. **Stability of some vector bundles on Hilbert schemes of points on K3 surfaces.**

The article [R2] continues the construction of new stable sheaves on hyperkähler varieties. The main result in this article is

Theorem. *All subquestions in Question 1.9 have a positive answer in the following two examples:*

a) X is a K3 surface with $\mathrm{NS}(X) = \mathbb{Z}h$ such that $h^2 = 4k$ for any $k \geqslant 1$, M is the fine moduli space $M_h(v)$ of h-slope stable sheaves on X with Mukai vector $v = (k+1, -h, 1)$ and $\mathcal{U}$ is the universal family of this moduli space.

b) X is a K3 surface with $\mathrm{NS}(X) = \mathbb{Z}e \oplus \mathbb{Z}f$ such that $e^2 = -2k$, $f^2 = 0$ and $ef = 2k+1$ for any $k \geqslant 2$, M is the fine moduli space $M_h(v)$ of h-slope stable sheaves with Mukai vector $v = (2k-1, h, 2k)$ for $h = e + (2k-1)f$ and $\mathcal{U}$ is the universal family of this moduli space.

In both cases, there is an isomorphism

$$X^{[k]} \xrightarrow{\cong} M_h(v), \; [I_Z] \mapsto [E_Z]$$

which is given in case a) by the spherical twist $T_{\mathcal{O}_X}$ around $\mathcal{O}_X$ (up to a shift), while in case b) it is given by the inverse spherical twist $T_{\mathcal{O}_X}^{-1}$ (up to twists with line bundles). Recall that the spherical twist

$$T_{\mathcal{O}_X} : \mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}^{\mathrm{b}}(X)$$

is a nontrivial autoequivalence of the derived category, see [22, §8] for more information.

A first step is to check that the image E_Z of an ideal sheaf I_Z under this equivalence is actually a sheaf on X and not just a complex in $\mathrm{D}^{\mathrm{b}}(X)$. This is indeed the case and even more is true: every E_Z is a vector bundle and slope stable with respect to h. Thus the moduli spaces $M_h(v)$ classify only vector bundles, which implies that the universal family $\mathcal{U}$ is itself locally free over $X \times M_h(v)$, hence flat over X.

To study the stability of the wrong-way fibers $\mathcal{U}_x$ we explicitly construct the universal families in both examples. Using this construction, we get an explicit description of the wrong-way fibers. We can then use the method developed by Stapleton in [50]: the idea is to do computations on the product X^k with $\mathfrak{S}_k$-equivariant sheaves instead of computations on $X^{[k]}$. This idea is a slightly modified version of the Bridgeland-King-Reid equivalence:

$$\mathrm{D^b}(X^{[k]}) \xrightarrow{\cong} \mathrm{D}^b_{\mathfrak{S}_k}(X^k).$$

These computations show that the wrong-way fibers are slope stable with respect to a natural nef divisor on $X^{[k]}$. Using a perturbation argument one can then prove that for each $\mathcal{U}_x$ there is an ample divisor H on $X^{[k]}$ such that $\mathcal{U}_x$ is H-slope stable.

Remark. Here we understand slope stability with respect to a nef divisor as slope stability with respect to a movable curve class. This stability was studied in detail by Greb, Kebekus and Peternell in [20]. They prove that this notion of stability has the same properties as stability with respect to an ample class.

The results so far show that each wrong-way fiber is slope stable with respect to some ample class. We generalize Stapleton's perturbation argument to actually show that there is one ample class H such that all wrong-way fibers $\mathcal{U}_x$ for $x \in X$ are slope stable with respect to H. Then we prove that the wrong-way fibers have a description as images under the integral functor $\Phi_{\mathcal{I}_\mathcal{Z}}$. For example, in case a) we have:

$$\mathcal{U}_x \cong \Phi_{\mathcal{I}_\mathcal{Z}}(I_x(h)),$$

where I_x is the ideal sheaf of $x \in X$. In example b) this description is slightly more involved. Anyway, in both cases this description allows to use Addington's $\mathbb{P}^n$-functor results to reduce the necessary cohomology computation from $X^{[k]}$ to X. These computations then show that we can find X as a smooth connected component on some moduli space of stable sheaves $\mathcal{M}$ on $M_h(v)$.

3. **Stable vector bundles on generalized Kummer varieties.**

In [R3] we replace the Hilbert scheme of k-points $X^{[k]}$ on a K3 surface X by the generalized Kummer variety $\mathrm{Kum}_n(A)$ for an abelian surface A. The article consists of two sections.

In the first section we generalize Stapleton's ideas to the generalized Kummer variety $\mathrm{Kum}_n(A)$. Recall that the generalized Kummer variety is defined as the fiber of the Albanese morphism of the Hilbert scheme $A^{[n+1]}$:

$$\mathrm{alb} : A^{[n+1]} \to A.$$

It can be factored to give a more geometric description:

$$\mathrm{alb} : A^{[n+1]} \xrightarrow{\mathrm{HC}} A^{(n+1)} \xrightarrow{\sum} A$$

where $\mathrm{HC} : A^{[n+1]} \to A^{(n+1)}$ is the Hilbert-Chow morphism, which is a resolution of singularities of the symmetric power $A^{(n+1)}$ and $\sum : A^{(n+1)} \to A$ is the summation morphism coming from the group law on A. The generalized Kummer variety is defined to be the fiber over 0_A:

$$\mathrm{Kum}_n(A) := \mathrm{alb}^{-1}(0_A).$$

Mimicking the construction in the case of the Hilbert scheme, instead of doing computations on the generalized Kummer variety we do computations with $\mathfrak{S}_{n+1}$-equivariant sheaves on

$$P_n(A) = \left\{ (a_0, \ldots, a_n) \in A^{n+1} \mid \sum_{i=0}^{n} a_i = 0_A \right\}$$

based on the equivalence

$$\mathrm{D^b}(\mathrm{Kum}_n(A)) \xrightarrow{\cong} \mathrm{D}^b_{\mathfrak{S}_{n+1}}(P_n(A)).$$

Using this method the first theorem of this article is

Theorem. *Assume (A, H) is a polarized abelian surface, then*

i) for any vector bundle $E \neq \mathcal{O}_A$, slope stable with respect to H, the associated tautological vector bundle $E^{[n]}$ is slope stable with respect to an ample class on $\mathrm{Kum}_n(A)$.

ii) there is one ample class D on $\mathrm{Kum}_n(A)$ such that all bundles $F^{[n]}$ for F in the same moduli as E are slope stable with respect to D.
iii) if a moduli space $M_H(v)$ only classifies stable vector bundles on A, then it can be found as a smooth connected component on a moduli space $\mathcal{M}$ of stable vector bundles on $\mathrm{Kum}_n(A)$.

Result iii) shows that smooth connected components on moduli spaces of stable vector bundles on higher dimensional hyperkähler varieties need not be hyperkähler, contrary to the case of K3 surfaces, see [23, Theorem 10.3.10]. In this case, $M_H(v)$ is holomorphic symplectic, but not simply connected.

In the second section we study again wrong-way fibers of universal families. For this we recall that the Albanese morphism of a moduli space $M_H(v)$ is given by

$$\mathrm{alb}_v : M_H(v) \to A \times \hat{A},$$

see [56, §4] for an explicit description of alb_v. By [56, Theorem 0.2] the fiber

$$K_H(v) = \mathrm{alb}_v^{-1}(0_A, 0_{\hat{A}})$$

is a hyperkähler variety deformation equivalent to $\mathrm{Kum}_n(A)$. We then follow the construction in [R2]: Firstly, there is an isomorphism $A^{[n+1]} \times A \cong M_H(v)$ for a certain moduli space of torsion free rank one sheaves on A. We apply the classical Fourier-Mukai transform:

$$\Phi_{\mathcal{P}} : \mathrm{D^b}(A) \to \mathrm{D^b}(\hat{A})$$

with kernel the Poincaré bundle $\mathcal{P}$ on $A \times \hat{A}$ (which is an equivalence) and get an induced isomorphism:

$$M_H(v) \cong M_{\hat{H}}(w)$$

for a Mukai vector w on $\hat{A}$ with rank $r \geqslant 2$ and the induced canonical polarization $\hat{H}$ on $\hat{A}$. The moduli space $M_{\hat{H}}(w)$ is fine and classifies only vector bundles slope stable with respect to $\hat{H}$. We thus have an isomorphism:

$$A^{[n+1]} \times A \cong M_{\hat{H}}(w).$$

Checking that the Albanese morphisms for both spaces are compatible we get an isomorphism of generalized Kummer varieties:

$$\mathrm{Kum}_n(A) \cong K_{\hat{H}}(w).$$

Restricting the universal family on $\hat{A} \times M_{\hat{H}}(w)$ to $\hat{A} \times K_{\hat{H}}(w)$ gives a universal family $\mathcal{U}$ for $K_{\hat{H}}(w)$.

Using the ideas from the first section we prove that there is an ample class D on $K_{\hat{H}}(w) \cong \mathrm{Kum}_n(A)$ such that all wrong-way fibers $\mathcal{U}_{\hat{a}}$ are slope stable with respect to D. We prove that the wrong-way fibers are images of the integral functor

$$\Phi_{\mathcal{I}_{\mathcal{Z}}} : \mathrm{D^b}(A) \to \mathrm{D^b}(\mathrm{Kum}_n(A))$$

with kernel the universal ideal sheaf $\mathcal{I}_{\mathcal{Z}}$ on $A \times \mathrm{Kum}_n(A)$. By the results of Meachan, see Remark 2.9, this integral functor is a $\mathbb{P}^{n-1}$-functor. Hence we can reduce cohomological computations from $\mathrm{Kum}_n(A)$ to A. These computations show that $\hat{A}$ embeds as a smooth connected component into some moduli space $\mathcal{M}$ of stable vector bundles on $\mathrm{Kum}_n(A)$. Providing another example of a connected component of a moduli spaces of stable vector bundles on higher dimensional hyperkähler varieties, that is not hyperkähler.

The second main result can be summarized as

Theorem. *All subquestions in Question 1.9 have a positive answer in the following example:*

$(\hat{A}, \hat{H})$ is the dual of a given polarized abelian surface (A, H), $M = \mathrm{Kum}_n(A)$ and $\mathcal{U}$ is the universal family on $\hat{A} \times \mathrm{Kum}_n(A)$, where $\mathrm{Kum}_n(A)$ is isomorphic to a generalized Kummer variety $K_{\hat{H}}(w)$ in some moduli space of stable vector bundle $M_{\hat{H}}(w)$ on $\hat{A}$.

4. **Smooth components on special iterated Hilbert schemes.**

The main result of [R4] also answers all subquestions in Question 1.9 positively, albeit for a family of subschemes instead of sheaves. More exactly we have:

Theorem. *Assume S is a smooth projective surface with $p_g = q = 0$ and let $S^{[n]}$ be the Hilbert scheme of length n subschemes of S. Then the universal family $\mathcal{Z}$ in $S \times S^{[n]}$ can be understood as a family of codimension two subschemes in $S^{[n]}$ with common Hilbert polynomial $p(t)$ classified by S such that the classifying morphism identifies S with a smooth connected component of the Hilbert scheme $\mathrm{Hilb}^{p(t)}(S^{[n]})$.*

The proof is the same as in the case of the Hilbert scheme $X^{[n]}$ on a K3 surface X in [R1]. The flatness of the universal family $\mathcal{Z}$ over S follows from [29, Theorem 2.1] and to reduce the cohomological computatiosn from $S^{[n]}$ to S we use Krug and Sosna's result saying that the integral functor $\Phi_{\mathcal{I}_{\mathcal{Z}}}$ is fully faithful.

5. **The Fourier-Mukai transform of a universal family of stable vector bundles.**

We study Question 2.14 in article [R5] and answer it in the negative by giving a counterexample on $(\mathbb{P}^2, h)$.

First we show that the moduli space $M_{\mathbb{P}^2}(4,1,3)$ of h-slope stable torsion free sheaves E with numerical invariants

$$\mathrm{rk}(E) = 4,\ c_1(E) = h \text{ and } c_2(E) = 3$$

is fine and an irreducible smooth projective variety of dimension six, which classifies only vector bundles. The last fact implies that the universal family $\mathcal{U}$ on $\mathbb{P}^2 \times M_{\mathbb{P}^2}(4,1,3)$ is locally free and hence flat over $\mathbb{P}^2$.

The main result of this article is:

Theorem. *The integral functor*

$$\Phi_{\mathcal{U}} : \mathrm{D^b}(\mathbb{P}^2) \to \mathrm{D^b}(M_{\mathbb{P}^2}(4,1,3))$$

with kernel the universal family $\mathcal{U}$ is not fully faithful.

There is a close connection between $M_{\mathbb{P}^2}(4,1,3)$ and the Hilbert scheme of three points $\mathbb{P}^{2[3]}$, as every vector bundle $[E] \in M_{\mathbb{P}^2}(4,1,3)$ can be described by an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \longrightarrow E \longrightarrow I_Z(1) \longrightarrow 0. \tag{3}$$

for some $[Z] \in \mathbb{P}^{2[3]}$.

We construct an explicit family of stable vector bundles $\mathcal{E}$ on $\mathbb{P}^2 \times \mathbb{P}^{2[3]}$ using the exact sequence (3) and the theory of universal extensions. The correct fibers of this family satisfy $[\mathcal{E}_Z] \in M_{\mathbb{P}^2}(4,1,3)$ for every $[Z] \in \mathbb{P}^{2[3]}$. This implies again that $\mathcal{E}$ is flat over $\mathbb{P}^2$, so it makes sense to also study the wrong-way fibers of this family.

We go on and prove that given $[E] \in M_{\mathbb{P}^2}(4,1,3)$ the corresponding subscheme $[Z]$ with $E \cong \mathcal{E}_Z$ is unique if $\mathrm{H}^0(\mathbb{P}^2, E) \cong \mathbb{C}^3$, which occurs if and only if $Z \subset \mathbb{P}^2$ is not collinear. Otherwise $\mathrm{H}^0(\mathbb{P}^2, E) \cong \mathbb{C}^4$ and there is a line $\ell \subset \mathbb{P}^2$ such that $\mathcal{E}_Z \cong \mathcal{E}_{Z'}$ for all $[Z], [Z'] \in \mathbb{P}^{2[3]}$ with $[Z], [Z'] \in \ell^{[3]}$. These facts show that the fiber of the classifying morphism of this family

$$\varphi : \mathbb{P}^{2[3]} \to M_{\mathbb{P}^2}(4,1,3),\ [Z] \mapsto [\mathcal{E}_Z]$$

is either a point or $\ell^{[3]}$ for some line $\ell \subset \mathbb{P}^2$ so φ is a birational morphism, contracting the collinear locus in $\mathbb{P}^{2[3]}$ to the Brill-Noether locus S in $M_{\mathbb{P}^2}(4,1,3)$. Recall that the collinear locus in $\mathbb{P}^{2[3]}$ is a $\mathbb{P}^3$-bundle over the dual projective plane $(\mathbb{P}^2)^*$ of lines $\ell \subset \mathbb{P}^2$ and $S \cong \mathbb{P}^2$. Thus the classifying morphism of the family $\mathcal{E}$ realizes the classical isomorphism

$$\mathbb{P}^{2[3]} \cong \mathrm{Bl}_S(M_{\mathbb{P}^2}(4,1,3)),$$

see [57, Example 5.3] or more classically in [15, Théorème 4].

Using these results one can see that there are isomorphisms of $\mathbb{C}$-vector spaces:

$$\mathrm{Ext}^i_{M_{\mathbb{P}^2}(4,1,3)}(\mathcal{U}_p, \mathcal{U}_q) \cong \mathrm{Ext}^i_{\mathbb{P}^{2[3]}}(\mathcal{E}_p, \mathcal{E}_q) \text{ for } i \geqslant 0.$$

To understand the cohomology of the wrong-way fibers $\mathcal{U}_p$ of the universal family $\mathcal{U}$, it is therefore enough to understand the cohomology of the wrong-way fibers of the family $\mathcal{E}$. Given the explicit description of $\mathcal{E}$, we get an explicit description of the wrong-way fibers $\mathcal{E}_p$ for $p \in \mathbb{P}^2$. Using the fully faithfulness result of Krug and Sosna, we are able to show

$$\dim \operatorname{Ext}^1_{\mathbb{P}^{2[3]}}(\mathcal{E}_p, \mathcal{E}_q) \geqslant 1 \text{ for } p, q \in \mathbb{P}^2 \text{ with } p \neq q.$$

This shows that the Bondal-Orlov criterion is not met for $\Phi_{\mathcal{U}}$, as

$$\operatorname{Ext}^1_{M_{\mathbb{P}^2}(4,1,3)}(\Phi_{\mathcal{U}}(\mathcal{O}_p), \Phi_{\mathcal{U}}(\mathcal{O}_q)) \cong \operatorname{Ext}^1_{M_{\mathbb{P}^2}(4,1,3)}(\mathcal{U}_p, \mathcal{U}_q) \neq 0.$$

Hence this functor is not fully faithful.

Remark. After the publication of this paper I was informed by Dmitrii Pedchenko that this result can be explained by the fact that $\operatorname{Pic}(M_{\mathbb{P}^2}(4,1,3)) \cong \mathbb{Z}$, that is $M_{\mathbb{P}^2}(4,1,3)$ is a so-called moduli space of height zero. More generally $\Phi_{\mathcal{U}}$ is never fully faithful for a fine moduli space $M_{\mathbb{P}^2}(r, c_1, c_2)$ of height zero. Indeed, by [32, Theorem 18.2.4] there is an associated exceptional vector bundle E on $\mathbb{P}^2$, which is h-slope stable and satisfies

$$-3 < \mu + \mu(E) \leqslant 0 \text{ with } \mu = \frac{c_1 h}{r} \text{ and } \chi(V \otimes E) = 0 \text{ for all } [V] \in M_{\mathbb{P}^2}(r, c_1, c_2).$$

These conditions imply $\mathrm{H}^i(\mathbb{P}^2, V \otimes E) = 0$ for $i = 0, 1, 2$ and thus we have

$$\Phi_{\mathcal{U}}(E) = 0,$$

which would be impossible for a fully faithful integral functor.

6. **The symplectic structure on a moduli space on a noncommutative surface.**

In the article [R6], we study Question 3.17 i). We start with a K3 surface X together with an Azumaya algebra $\mathcal{A}$ and consider the noncommutative surface $(X, \mathcal{A})$. First, we introduce the notion of an $\mathcal{A}$-Mukai vector by

$$v_{\mathcal{A}}(E) := ch(E) ch(\mathcal{A})^{-\frac{1}{2}} \sqrt{td(X)}$$

and study the moduli space $M_{\mathcal{A}/X}(v_{\mathcal{A}})$ of generically simple torsion free $\mathcal{A}$-modules with fixed Mukai vector $v_{\mathcal{A}}$. The main result can be stated as

Theorem. *Assume $(X, \mathcal{A})$ is a noncommutative K3 surface with an Azumaya algebra $\mathcal{A}$ on X. Then the moduli space $M_{\mathcal{A}/X}(v_{\mathcal{A}})$ is a hyperkähler variety, deformation equivalent to $X^{[n]}$ with $n = \frac{1}{2}\left(v_{\mathcal{A}}^2 + 2\right)$*

The main idea of the proof is to use the Brauer-Severi variety of $\mathcal{A}$, which classifies certain left ideals in $\mathcal{A}$. More precisely the functor

$$\mathcal{BS}(\mathcal{A}) : \mathrm{Sch}_X^{op} \to, \mathrm{Sets}$$

which maps an X-scheme $Y \to X$ to the set of left ideals $\mathcal{I} \subset \mathcal{A}_Y$, such that $\mathcal{A}_Y / \mathcal{I}$ is a locally free $\mathcal{O}_Y$-module of rank $r(r-1)$, where $\operatorname{rk}(\mathcal{A}) = r^2$, is representable by an X-scheme $\pi : \mathrm{BS}(\mathcal{A}) \to X$. This X-scheme is an étale $\mathbb{P}^{r-1}$-bundle and it is called the Brauer-Severi variety of $\mathcal{A}$. The pullback of $\mathcal{A}$ to $\mathrm{BS}(\mathcal{A})$ splits, which means that

$$\pi^* \mathcal{A} \cong \mathcal{E}nd_{\mathrm{BS}(\mathcal{A})}(G)^{op} \cong \mathcal{E}nd_{\mathrm{BS}(\mathcal{A})}(G^*)$$

for some vector bundle G on $\mathrm{BS}(\mathcal{A})$. Here the last isomorphism is given by the transpose. Using Morita equivalence and ideas of Yoshioka, see [58, Definition 1.3, Lemma 1.5], we study the following category:

$$\mathrm{Coh}(\mathrm{BS}(\mathcal{A}), X) := \left\{ E \in \mathrm{Coh}(\mathrm{BS}(\mathcal{A})) \,|\, \pi^* \pi_*(E \otimes G^*) \xrightarrow{\cong} E \otimes G^* \right\},$$

where the morphism is the canonical morphism coming from the adjunction (π^*, π_*). If we denote the category of coherent left $\mathcal{A}$-modules by $\mathrm{Coh}_l(X, \mathcal{A})$ then we prove that there is an equivalence of categories

$$\mathrm{Coh}_l(X, \mathcal{A}) \xrightarrow{\cong} \mathrm{Coh}(\mathrm{BS}(\mathcal{A}), X)$$

which maps generically simple $\mathcal{A}$-modules to so-called G-twisted stable torsion free sheaves on $\mathrm{BS}(\mathcal{A})$. One checks that the equivalence extends to families of the respective moduli problem, that is it maps a family of generically simple $\mathcal{A}$-modules to a family of G-twisted stable torsion free sheaves. This gives rise to an isomorphism of moduli spaces:

$$M_{\mathcal{A},X}(v_{\mathcal{A}}) \xrightarrow{\cong} M_H^{\mathrm{BS}(\mathcal{A}),G}(v)$$

where the latter space classifies G-twisted torsion free stable sheaves on $\mathrm{BS}(\mathcal{A})$ with Mukai vector v.

By a result of Yoshioka, see [58, Theorem 3.16], the moduli space $M_H^{\mathrm{BS}(\mathcal{A}),G}(v)$ is a hyperkähler variety, deformation equivalent to $X^{[n]}$ with $n = \frac{1}{2}(v^2+2)$, thus so is $M_{\mathcal{A}/X}(v_{\mathcal{A}})$.

7. **Torsion-free rank one sheaves over del Pezzo orders.**

We study Question 3.17 ii) in [R7] for so-called terminal del Pezzo orders $\mathcal{A}$ on $\mathbb{P}^2$. These give rise to noncommutative del Pezzo surfaces $(\mathbb{P}^2, \mathcal{A})$.

We say $\mathcal{A}$ is an order on a smooth projective surface X if it is a sheaf of associative $\mathcal{O}_X$-algebras such that the generic stalk $\mathcal{A}_\eta$ is a central division algebra over the function field $\mathbb{C}(X)$ of X. A maximal order is a maximal element with respect to inclusion of orders in $\mathcal{A}_\eta$. It is a well known fact that a maximal order is a vector bundle and that there is a largest open subset $U \subset X$ such that $\mathcal{A}_{|U}$ is an Azumaya algebra. The complement $X \setminus U$ is called the ramification locus of $\mathcal{A}$ and it consists of finitely many curves $\{C_1, \ldots, C_n\}$. Each curve comes with a natural number $e \geqslant 2$, the so-called ramification index of $\mathcal{A}$ at C. We the restrict to the so-called terminal maximal orders. These orders have global dimension 2, which is the same as for $\mathcal{O}_X$. For a maximal order $\mathcal{A}$ one can define the canonical divisor by:

$$K_{\mathcal{A}} := K_X + \sum_{i=1}^{n}(1 - \frac{1}{e_i})C_i$$

Similar to the classical case, a terminal order $\mathcal{A}$ is called del Pezzo if $-K_{\mathcal{A}}$ is ample.

The main results of this paper are:

Theorem. *Let $(\mathbb{P}^2, \mathcal{A})$ be a noncommutative surface given by a terminal del Pezzo order $\mathcal{A} \neq \mathcal{O}_{\mathbb{P}^2}$ on $\mathbb{P}^2$, then*

i) The moduli space $M_{\mathcal{A}/\mathbb{P}^2}$ is smooth

ii) Every generically simple $\mathcal{A}$-module can be deformed into a locally projective $\mathcal{A}$-module, in other words the locus $M^{lp}_{\mathcal{A}/\mathbb{P}^2}$ of locally projective $\mathcal{A}$-modules is dense in $M_{\mathcal{A}/\mathbb{P}^2}$.

To prove both facts one needs to have good control over certain $\mathrm{Ext}^i_{\mathcal{A}}$-groups. For example the obstruction to smoothness of $M_{\mathcal{A}/\mathbb{P}^2;c_1,c_2}$ at a point $[E]$ is encoded in $\mathrm{Ext}^2_{\mathcal{A}}(E,E)$ and if this group vanishes then all obstructions vanish, so the moduli space is smooth. We actually show that one has

$$\mathrm{Ext}^2_{\mathcal{A}}(E,F) = 0$$

for two generically simple $\mathcal{A}$-modules with $c_1(E) = c_1(F)$. Then we adapt the proof of the deformation argument in [21]. For this we need to prove two facts:

a) the connecting homomorphism $\delta : \mathrm{Ext}^1_{\mathcal{A}}(E,E) \to \mathrm{Ext}^2_{\mathcal{A}}(T,E)$ is surjective,

b) the induced map $\iota_* : \mathrm{Ext}^2_{\mathcal{A}}(T_{E,p},E) \to \mathrm{Ext}^2_{\mathcal{A}}(T_{E,p},E^{**})$ is non zero for all $p \in \mathrm{supp}(T_E)$.

Part a) follows from $Ext^2_{\mathcal{A}}(E^{**},E) = 0$ which in turn follows from the more general fact mentioned above. The proof of b) constitutes the biggest part of the article. We first study the local deformation theory, meaning we study deformations of ideals of finite colength in the completion $\widehat{\mathcal{A}}_p$ at a closed point $p \in \mathbb{P}^2$. Using these results, we can show that we can always deform a sheaf E to E' such that the induced map $(\iota')_*$ is not zero. Then using the smoothness of $M_{\mathcal{A}/\mathbb{P}^2;c_1,c_2}$ and the two facts a) and b), the rest of the argument works as in [21].

8. **Rank one sheaves over quaternion algebras on Enriques surfaces.**

In [R8] we study Question 3.17 ii) for noncommutative Enriques surfaces $(X, \mathcal{A})$ with an Azumaya algebra $\mathcal{A}$ on an Enriques surface X.

Cossec and Dolgachev showed that for an Enriques surface we have

$$\mathrm{Br}(X) = \mathbb{Z}/2\mathbb{Z}.$$

This implies that there is one nontrivial element in the Brauer group. As a first result we prove that this class can in fact be realized by an Azumaya algebra of rank four, that is a quaternion algebra.

As is well known the universal covering $\pi : \overline{X} \to X$ is a double cover and $\overline{X}$ is a K3 surface. Beauville proved that if X is very general in the moduli space of Enriques surfaces, then the pullback $\pi^* : \mathrm{Br}(X) \to \mathrm{Br}(\overline{X})$ is injective, hence $\overline{\mathcal{A}}$ is a nontrivial Azumaya algebra and we have an associated noncommutative K3 surface $(\overline{X}, \overline{\mathcal{A}})$. More general for every coherent sheaf E on X, we define $\overline{E} := \pi^* E$ on $\overline{X}$.

The main result of this article is

Theorem. *Let X be an Enriques surface and let $\mathcal{A}$ be the quaternion algebra on X representing the nontrivial element in the Brauer group* $\mathrm{Br}(X)$*. If X is very general then*

i) The moduli space $M_{\mathcal{A}/X}$ of all generically simple $\mathcal{A}$-modules is smooth.

ii) Every generically simple torsion free $\mathcal{A}$-module can be deformed into a locally projective $\mathcal{A}$-module, in other words the locus $M^{lp}_{\mathcal{A}/X}$ of locally projective $\mathcal{A}$-modules is dense in $M_{\mathcal{A}/X}$.

Let $\overline{X}$ be the universal covering K3 surface of X and denote the pullback of the quaternion algebra to $\overline{X}$ by $\overline{\mathcal{A}}$, then $M_{\overline{\mathcal{A}}/\overline{X};c_1,c_2}$ has a symplectic structure by Theorem 3.15. We have

iii) $M_{\mathcal{A}/X;c_1,c_2}$ is an étale double cover of a Lagrangian subscheme $\mathcal{L} \subset M_{\overline{\mathcal{A}}/\overline{X};\overline{c_1},\overline{c_2}}$.

Again to prove the smoothness, it is enough to prove the vanishing of $\mathrm{Ext}^2_{\mathcal{A}}(E, E)$ for every $[E] \in M_{\mathcal{A}/X;c_1,c_2}$. This time we prove this by first using Serre duality

$$\mathrm{Ext}^2_{\mathcal{A}}(E, E) \cong \mathrm{Hom}_{\mathcal{A}}(E, E \otimes \omega_X)^*$$

and then use the projection formula for the double cover $\pi : \overline{X} \to X$ to see

$$\mathrm{Hom}_{\overline{\mathcal{A}}}(\overline{E}, \overline{F}) \cong \mathrm{Hom}_{\mathcal{A}}(E, F) \oplus \mathrm{Hom}_{\mathcal{A}}(E, F \otimes \omega_X).$$

If E is a generically simple $\mathcal{A}$-module, then $\overline{E}$ is a generically simple $\overline{\mathcal{A}}$-module, hence it is simple, we conclude

$$\mathbb{C} \cong \mathrm{End}_{\overline{\mathcal{A}}}(\overline{E}) \cong \mathrm{End}_{\mathcal{A}}(E) \oplus \mathrm{Hom}_{\mathcal{A}}(E, E \otimes \omega_X)$$

which implies $\mathrm{Hom}_{\mathcal{A}}(E, E \otimes \omega_X) = 0$ as $\mathrm{End}_{\mathcal{A}}(E) \cong \mathbb{C}$.

The proof of the deformation arguments works similar to [21]. We only need to adapt the proof of surjectivity of the connecting homomorphism

$$\delta : \mathrm{Ext}^1_{\mathcal{A}}(E, E) \to \mathrm{Ext}^2_{\mathcal{A}}(T_E, E)$$

This follows from the vanishing of $\mathrm{Ext}^2_{\mathcal{A}}(E^{**}, E)$ with a similar argument as for the vanishing of $\mathrm{Ext}^2_{\mathcal{A}}(E, E)$.

To prove iii) we recall that the relative automorphism group $\mathrm{Aut}(\overline{X}/X) \cong \mathbb{Z}/2\mathbb{Z}$ is generated by an involution $\iota : \overline{X} \to \overline{X}$. This involution induces an involution

$$\iota^* : M_{\overline{\mathcal{A}}/\overline{X};\overline{c_1},\overline{c_2}} \to M_{\overline{\mathcal{A}}/\overline{X};\overline{c_1},\overline{c_2}}$$

We prove that ι^* is antisymplectic, hence $\mathcal{L} := \mathrm{Fix}(\iota^*)$ is a smooth projective Lagrangian subscheme. We then generalize a standard descent result to the noncommutative situation, namely:

Theorem. *Assume F is a simple $\overline{\mathcal{A}}$-module with an isomorphism $F \cong \iota^* F$ of $\overline{\mathcal{A}}$-modules, then there is an $\mathcal{A}$-module E and an isomorphism of $\overline{\mathcal{A}}$-modules $F \cong \overline{E}$.*

It follows that the image of $\pi^* : M_{\mathcal{A}/X;c_1,c_2} \to M_{\overline{\mathcal{A}}/\overline{X};\overline{c_1},\overline{c_2}}$ is exactly $\mathcal{L} = \mathrm{Fix}(\iota^*)$ and thus gives rise to a surjective morphism

$$\varphi : M_{\mathcal{A}/X;c_1,c_2} \to \mathcal{L}$$

We show that each fiber of φ contains exactly two points, showing that φ is an étale double cover.

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EXAMPLES OF SMOOTH COMPONENTS OF MODULI SPACES OF STABLE SHEAVES

FABIAN REEDE AND ZIYU ZHANG

Abstract. Let M be a projective fine moduli space of stable sheaves on a smooth projective variety X with a universal family $\mathcal{E}$. We prove that in four examples, $\mathcal{E}$ can be realized as a complete flat family of stable sheaves on M parametrized by X, which identifies X with a smooth connected component of some moduli space of stable sheaves on M.

Introduction

Background. The starting point of the article is a classical result on the moduli space of stable vector bundles on curves. Let C be a smooth complex projective curve of genus $g \geqslant 2$. We denote the moduli space of stable vector bundles on C of rank n with a fixed determinant line bundle L_d of degree d by M.

If n and d are coprime, then it is known by [MN68, Tju70] that M is a fine moduli space, namely, there exist a universal vector bundle $\mathcal{E}$ on $C \times M$ with the property that the fiber $\mathcal{E}|_{C\times\{m\}}$ over a closed point $m = [E] \in M$ is isomorphic to the bundle E itself. But one can also take a closed point $c \in C$ and consider the fiber

$$\mathcal{E}_c := \mathcal{E}|_{\{c\}\times M},$$

which is a vector bundle on M. In [NR75] the authors proved that $\mathcal{E}_c$ is a simple bundle for every closed point $c \in C$ and that the infinitesimal deformation map

$$T_cC \longrightarrow \mathrm{Ext}^1_M(\mathcal{E}_c, \mathcal{E}_c)$$

is bijective. In fact, for all closed points $c \in C$, the bundles $\mathcal{E}_c$ are stable and pairwise non-isomorphic by [BBPN97, LN05].

Thus if we define $\mathcal{M}$ to be the moduli space of stable vector bundles on M with the same Hilbert polynomial as $\mathcal{E}_c$, then the classifying morphism

$$f : C \longrightarrow \mathcal{M}, \quad c \longmapsto [\mathcal{E}_c]$$

identifies C with a smooth connected component of $\mathcal{M}$, as explained in [LN05].

Other examples in a similar spirit appear in the pioneering work of Mukai [Muk81, Muk99] on abelian varieties and K3 surfaces. In the case of K3 surfaces, Mukai considered a general polarized K3 surface S of a certain degree, along with a 2-dimensional fine moduli space M of stable vector bundles of rank at least 2 on S, admitting a universal family $\mathcal{E}$ on $S \times M$.

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It turns out that M is also a K3 surface, and $\mathcal{E}$ can also be realized as a family of stable bundles on M parametrized by S.

As in the previous example, we can define $\mathcal{M}$ to be the moduli space of stable sheaves on M with the same Hilbert polynomial as $\mathcal{E}|_{\{s\}\times M}$ for any closed point $s \in S$. Mukai proved that the classifying morphism

$$f : S \longrightarrow \mathcal{M}, \quad s \longmapsto [\mathcal{E}|_{\{s\}\times M}]$$

is in fact an isomorphism. In other words, S can be identified with the entire moduli space of stable sheaves on M with some fixed Chern classes.

Main result. Motivated by the above examples, one can formulate the following question under a more general setting:

Question 0.1. Let X be a smooth projective variety and M a projective fine moduli space of stable sheaves on X with universal family $\mathcal{E}$ on $X \times M$. Then

- Is $\mathcal{E}$ also a flat family of stable sheaves on M parametrized by X?
- If so, does the classifying map embed X as a smooth connected component of some moduli space of stable sheaves on M?

A positive answer to the above question, especially when X is of low dimension and M is of higher dimension, would be interesting from two perspectives. First of all, examples of stable sheaves on higher dimensional varieties (in particular on higher dimensional irreducible holomorphic symplectic manifolds) are in general difficult to construct. One important class of examples are the *tautological bundles* on Hilbert schemes, which were studied in [Sch10, Wan14, Wan16, Sta16]. Question 0.1 provides another natural approach for finding new examples. Secondly, moduli spaces of stable sheaves on higher dimensional varieties are in general badly behaved. A positive answer to Question 0.1 would allow us to identify some nicely behaved components of such moduli spaces, and at the same time give an explicit description of a complete family of stable sheaves over these components.

In this article, we consider Question 0.1 in some of the first cases:

Theorem 0.2 (Theorems 1.7, 2.4, 3.3, 4.3)**.** *Question 0.1 has a positive answer in the following cases:*

- *X is a smooth projective variety of dimension $d \geqslant 2$ and we use $M = \mathrm{Hilb}^2(X)$, the Hilbert scheme of 2 points on X;*
- *X is K3 surface and $M = \mathrm{Hilb}^n(X)$ is the Hilbert scheme of n points on X;*
- *X is an abelian surface and $M = \mathrm{Kum}_n(X)$ is the generalized Kummer variety of dimension $2n$ associated to X for any $n \geqslant 2$;*
- *X is a K3 surface of Picard rank 1 and M is some fine moduli space of stable torsion sheaves of pure dimension 1 on X.*

Our proof in the first of the above cases will be completely elementary. In all other cases, the moduli space M is in fact an irreducible holomorphic symplectic manifold, and our proof will be divided into two steps: we first establish the flatness of $\mathcal{E}$ over X and the stability of the fibers $\mathcal{E}_p$ over any closed point $p \in X$, then apply some very convenient results about $\mathbb{P}^n$-functors (see [Add16]) to conclude that X is in fact a component of some moduli space of stable sheaves on M.

It would be much more interesting to study Question 0.1 in more general settings, especially when X and M have trivial canonical classes and $\mathcal{E}$ is torsion free (or even locally free) of higher rank. However, it could be then much more difficult to prove the stability of $\mathcal{E}_p$ for any closed point $p \in X$. Moreover, the corresponding results about $\mathbb{P}^n$-functors are not yet known to us (see [Add16, Conjecture, p.2] and [ADM16, Conjecture 2.1]).

This article consists of four sections, which are devoted to the four cases in Theorem 0.2 respectively. The notion of $\mathbb{P}^n$-functors will be briefly recalled in the beginning of §2, followed immediately by a list of $\mathbb{P}^n$-functors relevant to our discussion. All schemes are defined over the field of complex numbers $\mathbb{C}$.

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1. Hilbert squares of smooth projective varieties

Let X be a smooth projective variety of dimension d, and $M = \mathrm{Hilb}^2(X)$. We denote by $\mathcal{Z} \subseteq X \times M$ the universal closed subscheme and $\mathcal{I}_{\mathcal{Z}}$ the universal ideal sheaf on $X \times M$. Then we have a commutative diagram

$$
\begin{tikzcd}
\mathcal{Z} \arrow[dr, hook] \arrow[drr, bend left, "\pi"] \arrow[ddr, bend right, "\tau"'] & & \\
& X \times M \arrow[r] \arrow[d] & M \\
& X &
\end{tikzcd}
\tag{1}
$$

where π is a flat morphism.

By [FGI+05, Remark 7.2.2.], we have $\mathcal{Z} = \mathrm{Bl}_\Delta(X \times X)$, the blow-up of $X \times X$ along the diagonal Δ. The projection τ can be interpreted as a composition

$$
\tau : \mathcal{Z} = \mathrm{Bl}_\Delta(X \times X) \xrightarrow{b} X \times X \xrightarrow{q_1} X \tag{2}
$$

of the blow-up b and the projection q_1 to the first factor. Moreover, the group $\Sigma_2 = \mathbb{Z}/2\mathbb{Z}$ acts on $\mathcal{Z}$ by switching the two factors, with a fixed-locus given by the exceptional divisor. By [FGI+05, Example 7.3.1(3)], π is the quotient of $\mathcal{Z}$ by Σ_2.

For any closed point $p \in X$, we write

$$
F_p := \tau^{-1}(p) \subseteq \mathcal{Z} \quad \text{and} \quad S_p := \pi(F_p) \subseteq M.
$$

Then we have the following results regarding the fibers of τ:

Lemma 1.1. *We have $S_p \cong F_p \cong \mathrm{Bl}_p(X)$, and the morphism τ is flat.*

Proof. The morphism $\pi|_{F_p}$ can be factored into a composition

$$\pi|_{F_p} : F_p \hookrightarrow \{p\} \times M \xrightarrow{\cong} M,$$

hence π induces an isomorphism from F_p to its image S_p. The canonical isomorphism $F_p \cong \mathrm{Bl}_p(X)$ is well known. Finally, since $\mathcal{Z}$ and X are both smooth and the fibers F_p of τ are irreducible of dimension d for all closed points $p \in X$, we deduce from [Mat86, Theorem 23.1, Corollary] that τ is flat. □

By the description of F_p as a blow-up in Lemma 1.1, we denote the exceptional divisor by $E_p \overset{\alpha}{\hookrightarrow} F_p$, then $E_p \cong \mathbb{P}^{d-1}$. This allows us to state the following result:

Lemma 1.2. *$\pi^{-1}(S_p)$ has simple normal crossing singularities with two irreducible components*

$$\pi^{-1}(S_p) = F_p \cup \sigma(F_p) \quad \textit{such that} \quad F_p \cap \sigma(F_p) = E_p$$

where σ is the non-trivial element of Σ_2.

Proof. This property can be verified analytically locally. Without loss of generality we assume that $X = \mathbb{A}^n$, and $p = (0, \cdots, 0) \in X$. Then we have $X \times X = \mathbb{A}^n \times \mathbb{A}^n$ with coordinates $(x_1, \cdots, x_n, y_1, \cdots, y_n)$. We perform an affine change of coordinates: for each $1 \leqslant i \leqslant n$, we write $s_i = x_i + y_i$ and $d_i = x_i - y_i$. Then the diagonal Δ is given by

$$\Delta = \{(s_1, \cdots, s_n, d_1, \cdots, d_n) \mid d_1 = \cdots = d_n = 0\}.$$

By (2) we have $\mathcal{Z} = \mathrm{Bl}_\Delta(X \times X)$, which is given by a mixture of affine and projective coordinates

$$\mathrm{Bl}_\Delta(X \times X) = \{(s_1, \cdots, s_n, d_1, \cdots, d_n, [u_1 : \cdots : u_n]) \mid [d_1 : \cdots : d_n] = [u_1 : \cdots : u_n]\}.$$

It is covered by n affine pieces, among which the first affine piece $\mathrm{Bl}_\Delta(X \times X)^1$ is given by $u_1 = 1$; in other words

$$\begin{aligned} \mathrm{Bl}_\Delta(X \times X)^1 &= \{(s_1, \cdots, s_n, d_1, \cdots, d_n, u_2, \cdots u_n) \mid d_i = u_i d_1 \text{ for } 2 \leqslant i \leqslant n\} \\ &= \{(s_1, \cdots, s_n, d_1, u_2, \cdots, u_n)\}. \end{aligned}$$

Then we have

$$\begin{aligned} q_1^{-1}(p) &= \{(x_1, \cdots, x_n, y_1, \cdots, y_n) \mid x_1 = \cdots = x_n = 0\} \\ &= \{(s_1, \cdots, s_n, d_1, \cdots, d_n) \mid s_i + d_i = 0 \text{ for } 1 \leqslant i \leqslant n\}. \end{aligned}$$

We write $F_p^1 = F_p \cap \mathrm{Bl}_\Delta(X \times X)^1$, then

$$\begin{aligned} F_p^1 &= \left\{ (s_1, \cdots, s_n, d_1, u_2, \cdots, u_n) \;\middle|\; \begin{array}{c} s_1 + d_1 = 0 \\ s_i + u_i d_1 = 0 \text{ for } 2 \leqslant i \leqslant n \end{array} \right\} \\ &= \left\{ (s_1, \cdots, s_n, d_1, u_2, \cdots, u_n) \;\middle|\; \begin{array}{c} s_1 + d_1 = 0 \\ s_i = u_i s_1 \text{ for } 2 \leqslant i \leqslant n \end{array} \right\}. \end{aligned}$$

Notice that $\mathrm{Bl}_\Delta(X \times X)^1$ is σ_2-invariant. The action of the non-trivial element $\sigma \in \Sigma_2$ is given by

$$g : (s_1, \cdots, s_n, d_1, u_2, \cdots, u_n) \longmapsto (s_1, \cdots, s_n, -d_1, u_2, \cdots, u_n).$$

Therefore we have

$$g(F_p^1) = \left\{ (s_1, \cdots, s_n, d_1, u_2, \cdots, u_n) \;\middle|\; \begin{array}{c} s_1 - d_1 = 0 \\ s_i = u_i s_1 \text{ for } 2 \leqslant i \leqslant n \end{array} \right\}$$

and the quotient $\mathrm{Bl}_\Delta(X \times X)^1/\Sigma_2$ is given by coordinates

$$\mathrm{Bl}_\Delta(X \times X)^1/\Sigma_2 = \{(s_1, \cdots, s_n, e_1, u_2, \cdots, u_n)\}$$

where $e_1 = d_1^2$. We write the image of F_p^1 under the quotient map by

$$S_p^1 := S_p \cap \mathrm{Bl}_\Delta(X \times X)^1/\Sigma_2,$$

then it follows that

$$S_p^1 = \left\{ (s_1, \cdots, s_n, e_1, u_2, \cdots, u_n) \;\middle|\; \begin{array}{c} s_1^2 = e_1 \\ s_i = u_i s_1 \text{ for } 2 \leqslant i \leqslant n \end{array} \right\}.$$

It is now clear that

$$\begin{aligned} \pi^{-1}(S_p^1) &= \left\{ (s_1, \cdots, s_n, d_1, u_2, \cdots, u_n) \;\middle|\; \begin{array}{c} s_1 + d_1 = 0 \\ s_i = u_i s_1 \text{ for } 2 \leqslant i \leqslant n \end{array} \right\} \\ &\cup \left\{ (s_1, \cdots, s_n, d_1, u_2, \cdots, u_n) \;\middle|\; \begin{array}{c} s_1 - d_1 = 0 \\ s_i = u_i s_1 \text{ for } 2 \leqslant i \leqslant n \end{array} \right\} \\ &= F_p^1 \cup \sigma(F_p^1). \end{aligned}$$

Therefore the intersection of the two components is transverse, and given by

$$F_p^1 \cap \sigma(F_p^1) = \{(s_1, \cdots, s_n, d_1, u_2, \cdots, u_n) \mid s_1 = \cdots = s_n = d_1 = 0\}$$

which gives precisely the exceptional divisor E_p in the first affine chart, namely, $E_p \cap \mathrm{Bl}_\Delta(X \times X)^1$. The same argument also applies to all other affine charts of $\mathrm{Bl}_\Delta(X \times X)$, which finishes the proof. □

In the following discussion, for any closed embedding $U \hookrightarrow V$, we denote the corresponding ideal sheaf, conormal sheaf and normal sheaf by $\mathcal{I}_{U/V}$, $\mathcal{C}_{U/V}$ and $\mathcal{N}_{U/V}$ respectively. Now we consider two smooth closed subvarieties Y and Z of a smooth variety, which fit in the following commutative diagram of closed embeddings:

$$\begin{tikzcd} Y \cap Z \arrow[r, hook, "\alpha"] \arrow[d, hook, "i"'] & Z \arrow[d, hook, "j"] \\ Y \arrow[r, hook, "\delta"] & Y \cup Z \end{tikzcd} \tag{3}$$

where the intersection and the union are scheme theoretic. The following lemma will be required in our next result:

Lemma 1.3. *In the situation of* (3), *we have* $\mathcal{C}_{Z/(Y \cup Z)} \cong \alpha_* \mathcal{C}_{(Y \cap Z)/Y}$.

Proof. We obtain by the second and the third isomorphism theorems that

$$\begin{aligned} \mathcal{I}_{Z/Y \cup Z} &\cong (\mathcal{I}_{Y/Y \cup Z} + \mathcal{I}_{Z/Y \cup Z})/(\mathcal{I}_{Y/Y \cup Z}) \\ &= (\mathcal{I}_{Y \cap Z/Y \cup Z})/(\mathcal{I}_{Y/Y \cup Z}) \\ &\cong \delta_* \mathcal{I}_{Y \cap Z/Y}. \end{aligned}$$

Therefore we obtain

$$\begin{aligned}\mathcal{C}_{Z/(Y\cup Z)} &= j^*\mathcal{I}_{Z/(Y\cup Z)}\\ &\cong j^*\delta_*\mathcal{I}_{(Y\cap Z)/Y}\\ &\cong \alpha_* i^*\mathcal{I}_{(Y\cap Z)/Y} = \alpha_*\mathcal{C}_{(Y\cap Z)/Y}\end{aligned}$$

as required, where the second isomorphism uses [Sta18, Tag 02KG]. □

In our situation we pick subvarieties $Y = \sigma(F_p)$ and $Z = F_p$ of $\mathcal{Z}$ in (3), then the morphism α becomes $E_p \xhookrightarrow{\alpha} F_p$. Lemma 1.3 immediately yields

Corollary 1.4. *We have* $\mathcal{C}_{F_p/\pi^{-1}(S_p)} \cong \alpha_*\mathcal{O}_{E_p}(1)$. □

The following result is the key to the main theorem of this section:

Lemma 1.5. *If* $d \geqslant 2$*, then we have* $\dim H^0(S_p, \mathcal{N}_{S_p/M}) = d$.

Proof. We divide the proof in two steps.

STEP 1. We claim that $\mathcal{N}_{S_p/M}$ fits into the exact sequence

$$0 \longrightarrow \mathcal{O}_{F_p}^{\oplus d} \longrightarrow (\pi|_{F_p})^*\mathcal{N}_{S_p/M} \longrightarrow \mathcal{E}xt^1_{F_p}(\alpha_*\mathcal{O}_{E_p}(1), \mathcal{O}_{F_p}) \longrightarrow 0. \tag{4}$$

We consider the chain of closed embeddings

$$F_p \xhookrightarrow{\iota} \pi^{-1}(S_p) \hookrightarrow \mathcal{Z}.$$

By [Gro67, Proposition 16.2.7], we get the exact sequence of conormal sheaves

$$\iota^*\mathcal{C}_{\pi^{-1}(S_p)/\mathcal{Z}} \longrightarrow \mathcal{C}_{F_p/\mathcal{Z}} \longrightarrow \mathcal{C}_{F_p/\pi^{-1}(S_p)} \longrightarrow 0. \tag{5}$$

By Lemma 1.1, $\tau : \mathcal{Z} \to X$ is flat, thus by [Gro67, Proposition 16.2.2 (iii)] we get

$$\mathcal{C}_{F_p/\mathcal{Z}} = (\tau|_{F_p})^*\mathcal{C}_{\{p\}/X} = (\tau|_{F_p})^*\mathcal{O}_{\{p\}}^{\oplus d} = \mathcal{O}_{F_p}^{\oplus d}. \tag{6}$$

Furthermore since $S_p \hookrightarrow M$ is a regular embedding of codimension d, the sheaf $\mathcal{C}_{S_p/M}$ is locally free of rank d. It follows by the flatness of $\pi : \mathcal{Z} \to M$ that

$$\iota^*\mathcal{C}_{\pi^{-1}(S_p)/\mathcal{Z}} = \iota^*(\pi|_{\pi^{-1}(S_p)})^*\mathcal{C}_{S_p/M} = (\pi|_{F_p})^*\mathcal{C}_{S_p/M} \tag{7}$$

is also locally free of rank d. Therefore the first two terms in (5) are locally free sheaves of rank d and the third one is by Corollary 1.4 torsion with support E_p. It follows that the first arrow in (5) is injective. By dualizing (5) we obtain

$$0 \longrightarrow \mathcal{N}_{F_p/\mathcal{Z}} \longrightarrow (\pi|_{F_p})^*\mathcal{N}_{S_p/M} \longrightarrow \mathcal{E}xt^1_{F_p}(\mathcal{C}_{F_p/\pi^{-1}(S_p)}, \mathcal{O}_{F_p}) \longrightarrow 0.$$

Together with (6), (7) and Corollary 1.4 we obtain the claim (4).

STEP 2. We claim that

$$H^0(F_p, \mathcal{E}xt^1_{F_p}(\alpha_*\mathcal{O}_{E_p}(1), \mathcal{O}_{F_p})) = 0. \tag{8}$$

Indeed, the sheaf $\alpha_*\mathcal{O}_{E_p}(1) = \alpha_*\mathcal{O}_{E_p}(-E_p)$ admits the following resolution

$$0 \longrightarrow \mathcal{O}_{F_p}(-2E_p) \longrightarrow \mathcal{O}_{F_p}(-E_p) \longrightarrow \alpha_*\mathcal{O}_{E_p}(-E_p) \longrightarrow 0$$

Dualizing this exact sequence shows

$$\mathcal{E}xt^1_{F_p}(\alpha_*\mathcal{O}_{E_p}(-E_p), \mathcal{O}_{F_p}) = \alpha_*\mathcal{O}_{E_p}(2E_p) = \alpha_*\mathcal{O}_{E_p}(-2).$$

Using $E_p \cong \mathbb{P}^{d-1}$ and $d \geqslant 2$, we finally get:

$$\begin{aligned} H^0(F_p, \mathcal{E}xt^1_{F_p}(\alpha_*\mathcal{O}_{E_p}(1), \mathcal{O}_{F_p})) &= H^0(F_p, \alpha_*\mathcal{O}_{E_p}(-2)) \\ &= H^0(E_p, \mathcal{O}_{E_p}(-2)) = 0. \end{aligned}$$

We conclude the proof by combining the long exact sequence in cohomology associated to (4) and the vanishing result (8). □

The following lemma is the main source for finding components of moduli spaces. The proof follows literally from [BBPN97, Theorem 3.6].

Lemma 1.6. *Let X be a smooth projective variety of dimension d and Y a projective scheme. Assume that a morphism $f : X \to Y$ is injective on closed points, and $\dim T_y Y = d$ for each closed point $y \in f(X)$. Then f is an isomorphism from X to a connected component of Y.*

Proof. Since X is complete, $f(X)$ is a closed subvariety of Y of dimension d. Since $\dim T_y Y = d$ for each closed point $y \in f(X)$, it follows that Y is smooth of dimension d at each closed point $y \in f(X)$ by [GW10, Theorem 6.28], hence $f(X)$ must be a smooth irreducible component of Y, which is also a connected component of Y. Finally, since $f : X \to f(X)$ is a morphism between smooth projective varieties and bijective on closed points, it is an isomorphism by Zariski's Main Theorem. □

Combining the above results, we can now give our first main result:

Theorem 1.7. *Any smooth projective variety X of dimension $d \geqslant 2$ is isomorphic to a smooth connected component of a moduli space of stable sheaves with trivial determinants on* $\mathrm{Hilb}^2(X)$*, by viewing $\mathcal{I}_{\mathcal{Z}}$ as a family of coherent sheaves on* $\mathrm{Hilb}^2(X)$ *parametrized by X.*

Proof. By Lemma 1.1, $\mathcal{Z}$ is flat over X hence $\mathcal{I}_{\mathcal{Z}}$ can be viewed as a flat family of sheaves on $\mathrm{Hilb}^2(X)$ parametrized by X. For each closed point $p \in X$, let $(\mathcal{I}_{\mathcal{Z}})_p$ be the restriction of $\mathcal{I}_{\mathcal{Z}}$ on the fiber $\{p\} \times \mathrm{Hilb}^2(X)$. Then $(\mathcal{I}_{\mathcal{Z}})_p$ is the ideal sheaf $\mathcal{I}_{S_p}$ of the closed embedding of S_p into $\mathrm{Hilb}^2(X)$, hence is a stable sheaf of rank 1. Therefore we obtain an induced classifying morphism

$$f : X \longrightarrow \mathcal{M}, \quad p \longmapsto [\mathcal{I}_{S_p}] \tag{9}$$

where $\mathcal{M}$ denotes the moduli space of stable sheaves on $\mathrm{Hilb}^2(X)$ of the class of $\mathcal{I}_{S_p}$ with trivial determinants. By [KPS18, Lemma B.5.6], $\mathcal{M}$ is isomorphic to the Hilbert scheme of subschemes of $\mathrm{Hilb}^2(X)$ which have the same Hilbert polynomials as S_p since $d \geqslant 2$. It is easy to see that f is injective on closed points. Indeed, for two different closed points $p, q \in X$, S_p and S_q are different subschemes of $\mathrm{Hilb}^2(X)$ of codimension $d \geqslant 2$, hence $\mathcal{I}_{S_p}$ and $\mathcal{I}_{S_q}$ are non-isomorphic ideal sheaves. On the other hand, for any closed point $p \in X$, we have

$$T_{[\mathcal{I}_{S_p}]}\mathcal{M} \cong \mathrm{Hom}_{\mathrm{Hilb}^2(X)}(\mathcal{I}_{S_p}, \mathcal{O}_{S_p}) \cong H^0(S_p, \mathcal{N}_{S_p/\,\mathrm{Hilb}^2(X)}).$$

Hence by Lemma 1.5, we have

$$\dim T_{[\mathcal{I}_{S_p}]}\mathcal{M} = d.$$

Therefore we conclude by Lemma 1.6 that the morphism (9) embeds X as a smooth connected component of $\mathcal{M}$. □

2. Hilbert schemes of points on K3 surfaces

What is particular interesting to us is the case of K3 surfaces. The technique of $\mathbb{P}^n$-functors allows us to obtain similar results for their Hilbert schemes of 0-dimension subschemes of arbitrary length. We first recall the following notion of $\mathbb{P}^n$-functors and its implications.

Definition 2.1. [Add16, Definition 4.1] A functor $F : \mathcal{A} \to \mathcal{B}$ between triangulated categories with adjoints L and R is called a $\mathbb{P}^n$-functor if:

(a) There is an autoequivalence H of $\mathcal{A}$ such that

$$RF \cong \mathrm{id} \oplus H \oplus H^2 \oplus \ldots \oplus H^n$$

(b) The map

$$HRF \hookrightarrow RFRF \xrightarrow{R\epsilon F} RF$$

written in components

$$H \oplus H^2 \oplus \ldots \oplus H^{n+1} \to \mathrm{id} \oplus H \oplus \ldots \oplus H^n$$

is of the form

$$\begin{pmatrix} * & * & \cdots & * & * \\ 1 & * & \cdots & * & * \\ 0 & 1 & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & * \end{pmatrix}$$

(c) We have $R \cong H^n L$. (If $\mathcal{A}$ and $\mathcal{B}$ have Serre functors, this is equivalent to $\mathcal{S}_{\mathcal{B}} F H^n \cong F \mathcal{S}_{\mathcal{A}}$.)

More about $\mathbb{P}^n$-functors and examples can be found in [Add16, §4].

We will focus on the case where $\mathcal{A} = D^b(X)$ and $\mathcal{B} = D^b(Y)$ for two smooth projective varieties X and Y such that $F = \Phi_{\mathcal{F}}$ is an integral functor with kernel $\mathcal{F} \in D^b(X \times Y)$. In fact, we are mostly interested in the case where $\mathcal{F}$ is actually a sheaf on $X \times Y$ and the autoequivalence $H = [-2]$. In this case condition (a) can be stated as

$$RF \cong \mathrm{id} \otimes H^*(\mathbb{P}^n, \mathbb{C}).$$

We will use the following simple consequence under this setting

Proposition 2.2. [ADM16, §2.1] *Assume X and Y are smooth projective varieties and $\mathcal{F}$ is a coherent sheaf on $X \times Y$, flat over X, such that the integral functor $F = \Phi_{\mathcal{F}}$ with kernel $\mathcal{F}$ is a $\mathbb{P}^n$-functor with associated autoequivalence $H = [-2]$. Then for any closed points $x, y \in X$ there is an isomorphism:*

$$\mathrm{Ext}^*_Y(\mathcal{F}_x, \mathcal{F}_y) \cong \mathrm{Ext}^*_X(\mathcal{O}_x, \mathcal{O}_y) \otimes H^*(\mathbb{P}^n, \mathbb{C}),$$

where $\mathcal{F}_x$ and $\mathcal{F}_y$ are fibers of $\mathcal{F}$ over the closed points x and y respectively. □

The following list of $\mathbb{P}^n$-functors will be of interest to us:

i) For a K3 surface S, $\mathrm{Hilb}^n(S)$ is a fine moduli space with universal ideal sheaf $\mathcal{I}_{\mathcal{Z}}$. The integral functor $\Phi_{\mathcal{I}_{\mathcal{Z}}} : D^b(S) \to D^b(\mathrm{Hilb}^n(S))$ is a $\mathbb{P}^{n-1}$-functor with associated autoequivalence $H = [-2]$; see [Add16, Theorem 3.1].
ii) Let $\mathrm{Kum}_n(A)$ be the generalized Kummer variety of an abelian surface A with universal ideal sheaf $\mathcal{I}_{\mathcal{Z}}$. For any $n \geqslant 2$, the integral functor $\Phi_{\mathcal{I}_{\mathcal{Z}}} : D^b(A) \to D^b(\mathrm{Kum}_n(A))$ is a $\mathbb{P}^{n-1}$-functor with associated autoequivalence $H = [-2]$; see [Mea15, Theorem 4.1].
iii) Let S be a K3 surface with $\mathrm{Pic}(S) = \mathbb{Z}[H]$ where H is an ample generator of degree $2g-2$. Assume M is the fine moduli space of stable sheaves on S of Mukai vector $(0, H, d+1-g)$ for some d and $\mathcal{U}$ is the universal sheaf over $S \times M$. Then the integral functor $\Phi_{\mathcal{U}} : D^b(S) \to D^b(M)$ is a $\mathbb{P}^{g-1}$-functor with associated autoequivalence $H = [-2]$; see [ADM16, Theorem A].

We give a first application of $\mathbb{P}^n$-functors to our problem: let S be a K3 surface and $M = \mathrm{Hilb}^n(S)$ for some positive integer n. Then M is a fine moduli space and the ideal sheaf $\mathcal{I}_{\mathcal{Z}}$ of the universal family $\mathcal{Z}$ is the universal sheaf on $S \times M$. It is well-known that M is an irreducible holomorphic symplectic manifold. The flatness of $\mathcal{I}_{\mathcal{Z}}$ over S follows immediately from the following result:

Lemma 2.3. [KR18, Theorem 2.1] *For every smooth variety X and every positive integer n, the universal family $\mathcal{Z} \subset X \times M$ is flat over X.* □

The above result allows us to obtain a smooth component of the moduli space of stable sheaves on $\mathrm{Hilb}^n(S)$ as follows:

Theorem 2.4. *For any positive integer n, the K3 surface S is isomorphic to a smooth connected component of a moduli space of stable sheaves on $\mathrm{Hilb}^n(S)$, by viewing $\mathcal{I}_{\mathcal{Z}}$ as a family of coherent sheaves on $\mathrm{Hilb}^n(S)$ parametrized by S.*

Proof. By Lemma 2.3, $\mathcal{I}_{\mathcal{Z}}$ can be viewed as a flat family of sheaves on $\mathrm{Hilb}^n(S)$ parametrized by S. For each closed point $s \in S$, let $(\mathcal{I}_{\mathcal{Z}})_s$ be the restriction of $\mathcal{I}_{\mathcal{Z}}$ on the fiber $\{s\} \times \mathrm{Hilb}^n(S)$. Then $(\mathcal{I}_{\mathcal{Z}})_s$ is the ideal sheaf of the closed embedding of $\mathcal{Z} \cap (\{s\} \times \mathrm{Hilb}^n(S))$ into $\mathrm{Hilb}^n(S)$, hence is a stable sheaf of rank 1. Therefore we obtain an induced classifying morphism

$$f : S \longrightarrow \mathcal{M}, \quad s \longmapsto [(\mathcal{I}_{\mathcal{Z}})_s] \tag{10}$$

where $\mathcal{M}$ denotes the moduli space of all stable sheaves on $\mathrm{Hilb}^n(S)$ of the class of $(\mathcal{I}_{\mathcal{Z}})_s$. For any pair of closed points $s_0, s_1 \in S$, we obtain by [Add16, Theorem 3.1] and Proposition 2.2 that

$$\mathrm{Ext}^*_{\mathrm{Hilb}^n(S)}\big((\mathcal{I}_{\mathcal{Z}})_{s_0}, (\mathcal{I}_{\mathcal{Z}})_{s_1}\big) \cong \mathrm{Ext}^*_S(\mathcal{O}_{s_0}, \mathcal{O}_{s_1}) \otimes H^*(\mathbb{P}^{n-1}, \mathbb{C}). \tag{11}$$

In particular, when $s_0 \neq s_1$, it follows from (11) that

$$\mathrm{Hom}_{\mathrm{Hilb}^n(S)}\big((\mathcal{I}_{\mathcal{Z}})_{s_0}, (\mathcal{I}_{\mathcal{Z}})_{s_1}\big) \cong \mathrm{Hom}_S(\mathcal{O}_{s_0}, \mathcal{O}_{s_1}) = 0,$$

which implies that (10) is injective on closed points; when $s_0 = s_1 = s$, it follows from (11) that

$$\mathrm{Ext}^1_{\mathrm{Hilb}^n(S)}\big((\mathcal{I}_{\mathcal{Z}})_s, (\mathcal{I}_{\mathcal{Z}})_s\big) \cong \mathrm{Ext}^1_S(\mathcal{O}_s, \mathcal{O}_s),$$

which implies that

$$\dim T_{[(\mathcal{I}_{\mathcal{Z}})_s]}\mathcal{M} = \dim T_s S = 2.$$

Therefore we conclude by Lemma 1.6 that the morphism (10) embeds S as a smooth connected component of $\mathcal{M}$, as desired. □

3. Generalized Kummer varieties

In this section we apply the technique of $\mathbb{P}^n$-functors to study a component of the moduli space of stable sheaves on generalized Kummer varieties.

Let A be an abelian surface and $\mathrm{Hilb}^{n+1}(A)$ the Hilbert scheme parametrizing closed subschemes of A of length $n+1$. Let the morphism Σ be the composition of the Hilbert-Chow morphism and the summation morphism with respect to the group law on A, namely

$$\Sigma : \mathrm{Hilb}^{n+1}(A) \longrightarrow \mathrm{Sym}^{n+1}(A) \longrightarrow A,$$

then the *generalized Kummer variety* is defined to be its zero fiber, namely

$$\mathrm{Kum}_n(A) := \Sigma^{-1}(0),$$

which is an irreducible holomorphic symplectic manifold. If we denote the restriction of the universal subscheme over $\mathrm{Hilb}^{n+1}(A)$ to $\mathrm{Kum}_n(A)$ by $\mathcal{Z}$, then we have a commutative diagram

$$\begin{tikzcd}
\mathcal{Z} \arrow[dr, hook] \arrow[drr, bend left, "\psi"] \arrow[ddr, bend right, "\varphi"'] & & \\
& A \times \mathrm{Kum}_n(A) \arrow[r, "p_2"'] \arrow[d, "p_1"] & \mathrm{Kum}_n(A) \\
& A &
\end{tikzcd}$$

where φ and ψ are the compositions of the embedding and the projections. We denote the ideal sheaf of $\mathcal{Z}$ in $A \times \mathrm{Kum}_n(A)$ by $\mathcal{I}_{\mathcal{Z}}$. It is clear that $\mathcal{I}_{\mathcal{Z}}$ is flat over $\mathrm{Kum}_n(A)$ since ψ is flat. In fact, $\mathcal{I}_{\mathcal{Z}}$ is also flat over the other factor A.

Lemma 3.1. *The universal ideal sheaf $\mathcal{I}_{\mathcal{Z}}$ is flat over A for any $n \geqslant 2$.*

Proof. It suffices to show that the morphism $\varphi : \mathcal{Z} \to A$ is flat. First of all, we claim that the dimension of the fiber $\varphi^{-1}(a_0)$ is $2n-2$ for any closed point $a_0 \in A$.

On the one hand, since A is smooth, the closed point $a_0 \in A$ is locally defined by two equations. Therefore locally near any point $x \in \varphi^{-1}(a_0)$, the fiber $\varphi^{-1}(a_0)$ is also defined by two equations, hence is of codimension at most 2 by Krull's height theorem; see [Mat80, §12.I, Theorem 18]. In other words, we have

$$\dim \varphi^{-1}(a_0) \geqslant 2n-2. \tag{12}$$

On the other hand, we have

$$\begin{aligned}
\varphi^{-1}(a_0) &= \{(a_0,\xi) \in A \times \mathrm{Kum}_n(A) \mid a_0 \in \mathrm{Supp}(\xi)\} \\
&\cong \{\xi \in \mathrm{Kum}_n(A) \mid a_0 \in \mathrm{Supp}(\xi)\}.
\end{aligned}$$

For any such ξ, we can write the associated 0-cycle $[\xi]$ as

$$[\xi] = \sum_{i=0}^{k} n_i a_i,$$

where $a_0, a_1, \cdots, a_k$ are pairwise distinct closed points, and $n_0, n_1, \cdots, n_k$ are the multiplicities. We further require $n_1 \geqslant \cdots \geqslant n_k > 0$ if $k > 0$. It is clear that

$$\sum_{i=0}^{k} n_i = n + 1 \tag{13}$$

which in particular implies $k \leqslant n$, and

$$\sum_{i=0}^{k} n_i a_i = 0 \in A \tag{14}$$

which utilizes the group law on A. We call the partition of n

$$\vec{n} = (n_0, n_1, \cdots, n_k)$$

the type of ξ. Let $\varphi^{-1}(a_0, \vec{n})$ be the set of all closed points $\xi \in \varphi^{-1}(a_0)$ of type $\vec{n}$, then we have a decomposition

$$\varphi^{-1}(a_0) = \bigsqcup_{\vec{n}} \varphi^{-1}(a_0, \vec{n}). \tag{15}$$

We then compute the dimension of $\varphi^{-1}(a_0, \vec{n})$ for each $\vec{n}$.

When $k = 0$, we have $\vec{n} = (n+1)$, and for any $\xi \in \varphi^{-1}(a_0, \vec{n})$ we have $[\xi] = (n+1)a_0$. It is clear that such $\varphi^{-1}(a_0, \vec{n})$ is non-empty if and only if $a_0 \in A$ is an $(n+1)$-torsion point. When non-empty, $\varphi^{-1}(a_0, \vec{n})$ is the punctual Hilbert scheme $\mathrm{Hilb}^{n+1}_{a_0}(A)$ which parametrizes length $(n+1)$ subschemes of A having support at only one point a_0. By [Iar72, Corollary 1], we have

$$\dim \varphi^{-1}(a_0, \vec{n}) = n \leqslant 2n - 2 \tag{16}$$

for each $(n+1)$-torsion point a_0 and integer $n \geqslant 2$.

When $k \geqslant 1$, every $\xi \in \varphi^{-1}(a_0, \vec{n})$ corresponds to a configuration of pairwise distinct points $\{a_1, \cdots, a_k\}$ satisfying (14). We can choose the first $(k-1)$ points freely, then a_k is uniquely determined up to n_k-torsion. Hence there is a $2(k-1)$-dimensional family of configurations $\{a_1, \cdots, a_k\}$. For any fixed configuration, the possible scheme structures on ξ is classified by the product of punctual Hilbert schemes $\mathrm{Hilb}^{n_0}_{a_0}(A) \times \cdots \times \mathrm{Hilb}^{n_k}_{a_k}(A)$. By [Iar72, Corollary 1] and (13), we obtain

$$\begin{aligned}\dim \varphi^{-1}(a_0, \vec{n}) &= 2(k-1) + \sum_{i=0}^{k} (n_i - 1) \\ &= 2(k-1) + (n+1) - (k+1) \\ &= n + k - 2 \leqslant 2n - 2.\end{aligned}$$

Combining the two cases, we have by (15) that

$$\dim \varphi^{-1}(a_0) \leqslant 2n - 2. \tag{17}$$

It then follows from (12) and (17) that all fibers $\varphi^{-1}(a_0)$ are equidimensional of dimension $2n-2$.

Moreover, since ψ is a surjective flat morphism and $\mathrm{Kum}_n(A)$ is smooth of dimension $2n$, we know $\mathcal{Z}$ is Cohen-Macaulay of dimension $2n$ by [Eis95, Corollary 18.17]. Since A is smooth, we conclude that $\varphi : \mathcal{Z} \to A$ is flat by [Mat86, Theorem 23.1, Corollary], which implies that its ideal sheaf $\mathcal{I}_{\mathcal{Z}}$ is flat over A, as desired. □

Remark 3.2. It is easy to see that the statement of Lemma 3.1 fails for $n = 1$, due to the failure of (16). In fact, in such a case, $\varphi^{-1}(a_0)$ is either a smooth rational curve or a single point, depending on whether a_0 is a 2-torsion point of A.

The above result allows us to obtain a smooth component of the moduli space of stable sheaves on $\mathrm{Kum}_n(A)$ as follows:

Theorem 3.3. *For any $n \geqslant 2$, the abelian surface A is isomorphic to a smooth connected component of a moduli space of stable sheaves on* $\mathrm{Kum}_n(A)$*, by viewing $\mathcal{I}_{\mathcal{Z}}$ as a family of coherent sheaves on* $\mathrm{Kum}_n(A)$ *parametrized by A.*

Proof. By Lemma 3.1, $\mathcal{I}_{\mathcal{Z}}$ can be viewed as a flat family of sheaves on $\mathrm{Kum}_n(A)$ parametrized by A. For each closed point $a_0 \in A$, let $(\mathcal{I}_{\mathcal{Z}})_{a_0}$ be the restriction of $\mathcal{I}_{\mathcal{Z}}$ on the fiber $\{a_0\} \times \mathrm{Kum}_n(A)$. Then $(\mathcal{I}_{\mathcal{Z}})_{a_0}$ is the ideal sheaf of the closed embedding of $\mathcal{Z} \cap (\{a_0\} \times \mathrm{Kum}_n(A))$ into $\mathrm{Kum}_n(A)$, hence is a stable sheaf of rank 1. Therefore we obtain an induced classifying morphism

$$f : A \longrightarrow \mathcal{M}, \quad a_0 \longmapsto [(\mathcal{I}_{\mathcal{Z}})_{a_0}] \tag{18}$$

where $\mathcal{M}$ denotes the moduli space of all stable sheaves on $\mathrm{Kum}_n(A)$ of the class of $(\mathcal{I}_{\mathcal{Z}})_{a_0}$. For any pair of closed points $a_0, a_1 \in A$, we obtain by [Mea15, Theorem 4.1] and Proposition 2.2 that

$$\mathrm{Ext}^*_{\mathrm{Kum}_n(A)}\left((\mathcal{I}_{\mathcal{Z}})_{a_0}, (\mathcal{I}_{\mathcal{Z}})_{a_1}\right) \cong \mathrm{Ext}^*_A(\mathcal{O}_{a_0}, \mathcal{O}_{a_1}) \otimes H^*(\mathbb{P}^{n-1}, \mathbb{C}).$$

From here, a similar argument as in Theorem 2.4 shows that the morphism (18) embeds A as a smooth connected component of $\mathcal{M}$. □

4. Moduli spaces of pure sheaves on K3 surfaces

In this section we extend our discussion to the fine moduli spaces of stable sheaves of pure dimension 1 on a K3 surface of Picard number 1.

Let S be a K3 surface with $\mathrm{Pic}(S) = \mathbb{Z}H$ where H is an ample line bundle of degree $2g-2$. Let $\mathbb{P}(V) \cong \mathbb{P}^g$ be the complete linear system of H where $V = H^0(S, H)$. Since S has Picard number 1, every curve C in the linear system $\mathbb{P}(V)$ is reduced and irreducible of genus g with planar singularities, hence its compactified Jacobian $\overline{\mathrm{Jac}}^d(C)$ is reduced and irreducible of dimension g by [AIK77, Theorem (9)]. We denote by $\mathcal{C}$ the universal curve of the linear system $\mathbb{P}(V)$. Therefore $\mathcal{C}$ is a closed subscheme of $S \times \mathbb{P}(V)$ and admits projections to S and $\mathbb{P}(V)$. All fibers of the first projection $\tau : \mathcal{C} \to S$ are linear subsystems of $\mathbb{P}(V)$ of codimension 1.

Let M be the moduli space of stable sheaves on S with Mukai vector

$$v = (0, H, d + 1 - g).$$

We assume $\gcd(2g-2, d+1-g) = 1$, then M is a smooth fine moduli space of stable torsion sheaves of pure dimension 1, hence admits a universal family $\mathcal{U}$. In fact, M is an irreducible holomorphic symplectic manifold. The corresponding support morphism

$$\eta : M \longrightarrow \mathbb{P}(V)$$

sends a stable sheaf to its support curve.

Alternatively, M can also be interpreted as the relative compactified Jacobian $\overline{\mathrm{Jac}}^d(\mathcal{C}/\mathbb{P}(V))$ of the family $\mathcal{C} \to \mathbb{P}(V)$. Hence the support of the universal family $\mathcal{U}$ is given by

$$T := \mathrm{Supp}(\mathcal{U}) = \mathcal{C} \times_{\mathbb{P}(V)} M.$$

It is more convenient to consider the universal family as a sheaf on T, so we define

$$\mathcal{E} := \iota^* \mathcal{U}$$

where $\iota : T \hookrightarrow S \times M$ is the closed embedding. Then we have $\mathcal{U} \cong \iota_* \mathcal{E}$ by [GW10, Remark 7.35].

The relation among the various spaces and morphisms introduced above can be summarised in the following commutative diagram

$$\begin{tikzcd}
T \arrow[r, "\varphi"'] \arrow[hook, d, "\iota"'] \arrow[rr, bend left, "\psi"] \arrow[dd, bend right=60, "\pi"'] & \mathcal{C} \arrow[r, "\tau"'] \arrow[hook, d] & S \arrow[d, equal] \\
S \times M \arrow[r] \arrow[d] & S \times \mathbb{P}(V) \arrow[r] \arrow[d] & S \\
M \arrow[r, "\eta"] & \mathbb{P}(V) &
\end{tikzcd} \tag{19}$$

where both squares on the left are cartesian.

Moreover, for any closed point $s \in S$, we denote the fiber $\psi^{-1}(s)$ by T_s, with the corresponding closed embedding $i_s : T_s \hookrightarrow T$. We also denote the pullback of $\mathcal{E}$ to the fiber T_s by $\mathcal{E}_s$, and the pullback of $\mathcal{U}$ to the fiber $\{s\} \times M$ by $\mathcal{U}_s$.

The following properties will be used later:

Lemma 4.1. *Both T and T_s (for each closed point $s \in S$) are integral and Gorenstein.*

Proof. We first note that $\mathcal{C}$, being a $\mathbb{P}^{g-1}$-bundle bundle over S, is smooth and irreducible of dimension $g+1$. Consequently $\mathcal{C}$ is integral. Moreover, since both M and $\mathbb{P}(V)$ are smooth, and all closed fibers of η are compactified Jacobians, which are integral of dimension g, the morphism η is flat by [Mat86, Theorem 23.1, Corollary]. It follows that φ is also flat, and every closed fiber of φ is integral. Thus [GW10, Theorem 14.44] implies that the generic fiber of φ is also integral. Therefore T is integral of dimension $2g+1$ by [Sta18, Lemma 0BCM]. This means T is a hypersurface in the smooth variety $S \times M$, hence T is Gorenstein by [Eis95, Corollary 21.19].

For any closed point $s \in S$, the restriction of φ to the fibers over s is given by

$$\varphi_s : T_s \longrightarrow \mathbb{P}^{g-1}.$$

The above properties of φ imply that φ_s is also flat, and that every closed fiber of φ_s is integral. It follows for the same reason as above that T_s is integral of dimension $2g-1$, hence is a hypersurface in the smooth variety M, which implies that T_s is also Gorenstein. □

Now we turn to properties of the universal sheaf:

Lemma 4.2. *The sheaf $\mathcal{E}$ on T is flat over S, and the sheaf $\mathcal{E}_s$ on T_s is stable for each closed point $s \in S$.*

Proof. We observe that the morphism $\mathcal{C} \to \mathbb{P}(V)$ (the composition of the morphisms in the middle column of (19)) is projective, flat and Gorenstein of pure dimension 1. After the base change along η, the induced morphism $\pi : T \to M$ (the composition of the morphisms in the left column of (19)) is also projective, flat and Gorenstein of pure dimension 1. Furthermore $\mathcal{E}$ is flat over M, and for any point $m \in M$, the restriction of $\mathcal{E}$ to the fiber $\pi^{-1}(m)$ is torsion free. It follows by [BK06, Corollary 2.2] that

$$\mathcal{E}xt^i_T(\mathcal{E}, \mathcal{O}_T) = 0$$

for every $i > 0$. Since T is irreducible and Gorenstein, this implies that $\mathcal{E}$ is a maximal Cohen-Macaulay sheaf on T.

We have seen that φ and τ are both flat morphisms, hence ψ is also a flat morphism. The closed embedding $\{s\} \hookrightarrow S$ is a morphism of finite Tor dimension. After a flat base change along ψ, we see that $i_s : T_s \hookrightarrow T$ is also of finite Tor dimension. Since T is irreducible and Gorenstein by Lemma 4.1, [Ari13, Lemma 2.3 (1)] implies

$$Li_s^*\mathcal{E} = i_s^*\mathcal{E}$$

for every closed point $s \in S$, where Li_s^* is the derived pullback functor. It follows by [Huy06, Lemma 3.31] that $\mathcal{E}$ is flat over S.

By Lemma 4.1 we also know T_s is Gorenstein, hence is in particular Cohen-Macaulay. By [Ari13, Lemma 2.3 (2)], $\mathcal{E}_s$ is also maximal Cohen-Macaulay, which by [HK71, Satz 6.1, a) $\Rightarrow$ d)] implies that $\mathcal{E}_s$ is reflexive, and hence in particular torsion free on T_s. Therefore $\mathcal{E}_s$ is stable since it is of rank 1. □

The above result allows us to obtain again a smooth component of the moduli space of stable sheaves on M as follows:

Theorem 4.3. *Under the assumptions in the present section, the K3 surface S is isomorphic to a smooth connected component of a moduli space of stable sheaves on M, by viewing $\mathcal{U}$ as a family of coherent sheaves on M parametrized by S.*

Proof. By Lemma 4.2, we know that the sheaf $\mathcal{U} = \iota_*\mathcal{E}$ is also flat over S, and the fiber $\mathcal{U}_s$ is a stable sheaf on M of pure dimension $2g-1$ for each closed point $s \in S$. Therefore $\mathcal{U}$ is a flat family of stable sheaves on M parametrized by S, with an induced classifying morphism given by

$$f : S \longrightarrow \mathcal{M}, \quad s \longmapsto [\mathcal{U}_s] \tag{20}$$

where $\mathcal{M}$ is the moduli space of all stable sheaves on M of the class of $\mathcal{U}_s$. For any pair of closed points $s_0, s_1 \in S$, we obtain by [ADM16, Theorem A] and Proposition 2.2 that

$$\mathrm{Ext}^*_M(\mathcal{U}_{s_0}, \mathcal{U}_{s_1}) \cong \mathrm{Ext}^*_S(\mathcal{O}_{s_0}, \mathcal{O}_{s_1}) \otimes H^*(\mathbb{P}^{g-1}, \mathbb{C}).$$

From here, a similar argument as in Theorem 2.4 shows that the morphism (20) embeds S as a smooth component of $\mathcal{M}$. $\square$

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STABILITY OF SOME VECTOR BUNDLES ON HILBERT SCHEMES OF POINTS ON K3 SURFACES

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Abstract. Let X be a projective K3 surfaces. In two examples where there exists a fine moduli space M of stable vector bundles on X, isomorphic to a Hilbert scheme of points, we prove that the universal family $\mathcal{E}$ on $X \times M$ can be understood as a complete flat family of stable vector bundles on M parametrized by X, which identifies X with a smooth connected component of some moduli space of stable sheaves on M.

Introduction

Let X be a projective K3 surface, and M a moduli space of semistable sheaves on X. By Mukai's seminal work [15], when M is smooth, it is an example of the so-called irreducible holomorphic symplectic manifolds, which are an important class of building blocks in the classification of compact Kähler manifolds with trivial first Chern class. It is then an interesting question to understand whether the moduli spaces $\mathcal{M}$ of semistable sheaves on M inherit any good properties from M. This paper grew out of an attempt to study this question. When $\dim M > 2$, we cannot expect $\mathcal{M}$ to carry a holomorphic symplectic structure in general, because the Serre duality does not induce a non-degenerate anti-symmetric pairing on the tangent space of $\mathcal{M}$ any more, as opposed to the case of K3 surfaces; however, some components of $\mathcal{M}$ may nevertheless be holomorphic symplectic.

In order to study this question, we need to classify all semistable sheaves on M with fixed Chern classes, which seems difficult in general when we have $\dim M > 2$; it is even a challenging question to construct any non-trivial examples of semistable sheaves on M, due to the fact that stability is difficult to check on higher dimensional varieties in general. When M is a Hilbert scheme of points on the K3 surface X, a natural family of vector bundles on M for considering stability are the so-called tautological bundles, which were proven to be stable with respect to a suitable choice of an ample line bundle on M by Schlickewei [18], Wandel [21] and Stapleton [20]. In fact, Wandel proved that, under some mild assumptions, the connected component of the moduli space containing the tautological bundles is isomorphic to some moduli space of vector bundles on the underlying K3 surface X.

There is another way to construct examples of stable sheaves on M. Assuming that M is a fine moduli space of stable sheaves on X with a universal

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family $\mathcal{E}$ on $X \times M$, and denoting the "wrong-way fiber" $\mathcal{E}|_{\{x\}\times M}$ by E_x for each closed point $x \in X$, we can ask the following questions:

- Is $\mathcal{E}$ also a flat family of coherent sheaves on M parametrized by X?
- If so, are the "wrong-way" fibers E_x stable sheaves on M with respect to some suitable choice of an ample line bundle for every closed point $x \in X$?
- If so, can we identify X with a connected component of the corresponding moduli space of stable sheaves on M?

This idea has also been explored in the literature. In [17], the authors studied some families of ideal sheaves and torsion sheaves of pure dimension 1, and obtained an affirmative answer to the above questions in these cases. A systematic study of the above questions in the case of locally free sheaves was carried out in the very interesting and inspiring thesis of Wray [22]. In order to get around the difficulty of proving stability directly, he invoked the very deep and powerful technique of Hitchin-Kobayashi correspondence to translate the stability problem to the existence of some Hermitian-Einstein metrics, which was then solved by analytic methods to give affirmative answers to the above questions.

The present paper is devoted to study the above questions, in particular the stability of wrong-way fibers E_x with respect to a polarization near the boundary of the ample cone of M, in the very classical way by showing that every proper subsheaf of E_x of a smaller rank has a smaller slope. We will focus on two special cases, namely a projective K3 surface X along with a Mukai vector v such that either

- $\mathrm{NS}(X) = \mathbb{Z}h$ with $h^2 = 4k$ and $v = (k+1, -h, 1)$ for any $k \geqslant 1$; or
- $\mathrm{NS}(X) = \mathbb{Z}e \oplus \mathbb{Z}f$ with intersection matrix $\begin{pmatrix} -2k & 2k+1 \\ 2k+1 & 0 \end{pmatrix}$ for any $k \geqslant 2$ as well as $v = (2k-1, e + (2k-1)f, 2k)$.

We summarize our main results in the following theorem:

Theorem 0.1. *For any projective K3 surface X satisfying either of the above conditions,*

(1) we can explicitly construct a fine moduli space M of stable vector bundles of Mukai vector v on X, isomorphic to the Hilbert scheme of k points on X, along with a universal family $\mathcal{E}$ (see Theorem 2.3 and Theorem 3.7);

(2) there exists an ample divisor H on M such that $\mathcal{E}$ can be regarded as a flat family of μ_H-stable vector bundles on M parametrized by X (see Theorem 2.8 and Theorem 3.15);

(3) the classifying morphism induced by the family $\mathcal{E}$ identifies X with a smooth connected component of a moduli space of μ_H-stable sheaves on M (see Theorem 2.10 and Theorem 3.16).

Let us briefly explain how we achieved the above results. Our choices of the K3 surfaces and the Mukai vectors, as well as the explicit constructions of the moduli space M and the universal family $\mathcal{E}$ in the above two cases, are motivated by [10, Example 5.3.7] and [16, Theorem 1.2] respectively. In fact, in both cases, the stable sheaves on X are given by the spherical twist (or its inverse) of the ideal sheaves of k points on X around $\mathcal{O}_X$, hence their corresponding moduli spaces M are isomorphic to the Hilbert scheme $X^{[k]}$ of k points on X. To show the slope stability of the wrong-way fibers E_x with respect to some ample divisor H on M, we apply the technique developed by Stapleton [20]; namely, we first prove the slope stability of E_x with respect to a natural nef divisor on M by passing to the k-fold product of X, then use the openness of stability to perturb the nef divisor to a nearby ample divisor. In fact, since the perturbation argument in [20] works only for individual sheaves, we need to generalize it so as to find an ample divisor H with respect to which all E_x's are simultaneously stable. Finally, to identify X as a smooth connected component of some moduli space of stable sheaves on M, we interpret E_x's as images of some sheaves or derived objects on X under the integral functor Φ induced by the universal ideal sheaf for $X^{[k]}$. By the fundamental result of Addington [1] that Φ is a $\mathbb{P}^{k-1}$-functor, we can obtain, by computing the relevant cohomology groups, that the E_x's are distinct and the tangent space of deformations of each E_x is of dimension 2, which leads immediately to the conclusion.

The text is organized in three sections. The first section gives background on integral functors, while the other two deal with the two cases mentioned above respectively. All objects in this text are defined over the field of complex numbers $\mathbb{C}$.

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1. Background on spherical twists and $\mathbb{P}^n$-functors

Let X denote a smooth projective variety with $\dim(X) = d$. As we will need them later, we quickly recall some facts about spherical twists and $\mathbb{P}^n$-functors in this section.

Definition 1.1. An object $\mathcal{S} \in \mathrm{D^b}(X)$ is called spherical if

i) $\mathcal{S} \otimes \omega_X \cong \mathcal{S}$

ii) $\mathrm{Ext}^i(\mathcal{S}, \mathcal{S}) = \begin{cases} \mathbb{C} & \text{if } i = 0, d \\ 0 & \text{otherwise} \end{cases}$

Remark 1.2. We note the fact that if X is a K3 surface, then any line bundle $L \in \mathrm{Pic}(X)$ is spherical.

Using spherical objects one can construct autoequivalences of $\mathrm{D^b}(X)$ in the following way: to any object $\mathcal{F} \in \mathrm{D^b}(X)$ one can associate the following object in $\mathrm{D^b}(X \times X)$:

$$\mathcal{P}_{\mathcal{F}} := \mathrm{Cone}(\mathcal{F}^\vee \boxtimes \mathcal{F} \longrightarrow \mathcal{O}_\Delta).$$

We refer to [8, §8] for an exact description of the map $\mathcal{F}^\vee \boxtimes \mathcal{F} \to \mathcal{O}_\Delta$ and more information.

Definition 1.3. The spherical twist

$$T_{\mathcal{S}} \coloneqq \Phi_{\mathcal{P}_{\mathcal{S}}} : \mathrm{D}^{\mathrm{b}}(X) \longrightarrow \mathrm{D}^{\mathrm{b}}(X)$$

associated to a spherical object $\mathcal{S} \in \mathrm{D}^{\mathrm{b}}(X)$ is the Fourier-Mukai transform with kernel $\mathcal{P}_{\mathcal{S}}$.

The most important fact about the spherical twist is

Proposition 1.4. *Let $\mathcal{S}$ be a spherical object in $\mathrm{D}^{\mathrm{b}}(X)$. Then the induced spherical twist*

$$T_{\mathcal{S}} : \mathrm{D}^{\mathrm{b}}(X) \longrightarrow \mathrm{D}^{\mathrm{b}}(X)$$

is an autoequivalence.

The first proof of this proposition was given by Seidel and Thomas, see [19, Theorem 1.2].

Remark 1.5. By [8, Exercise 8.5] the effect of the spherical twist $T_{\mathcal{S}}$ on an object $\mathcal{G} \in \mathrm{D}^{\mathrm{b}}(X)$ can be described by the following distinguished triangle:

$$T_{\mathcal{S}}(\mathcal{G})[-1] \longrightarrow R\operatorname{Hom}(\mathcal{S}, \mathcal{G}) \otimes \mathcal{S} \longrightarrow \mathcal{G} \longrightarrow T_{\mathcal{S}}(\mathcal{G}).$$

As the spherical twist $T_{\mathcal{S}}$ is an autoequivalence one can also study the inverse $T_{\mathcal{S}}^{-1}$. For any object $\mathcal{G} \in \mathrm{D}^{\mathrm{b}}(X)$ there exists the following distinguished triangle, see [8, Remark 8.11]:

$$T_{\mathcal{S}}^{-1}(\mathcal{G}) \longrightarrow \mathcal{G} \longrightarrow R\operatorname{Hom}(\mathcal{S}, \mathcal{G}) \otimes \mathcal{S}[d] \longrightarrow T_{\mathcal{S}}^{-1}(\mathcal{G})[1].$$

We are also interested in another class of integral functors, the so-called $\mathbb{P}^n$-functors, which were introduced by Addington in a very general setting in [1, §4]. We will only need the following special example:

Example 1.6. Let X be a K3 surface, then the integral functor

$$\Phi\colon\ \mathrm{D}^{\mathrm{b}}(X) \longrightarrow \mathrm{D}^{\mathrm{b}}(X^{[k]})$$

whose kernel is the universal ideal sheaf $\mathcal{I}_{\mathcal{Z}}$ on $X \times X^{[k]}$ is a $\mathbb{P}^{k-1}$-functor with corresponding autoequivalence $H = [-2]$ by [1, Theorem 3.1, Example 4.2(2)].

Remark 1.7. The fact that the above integral functor Φ is a $\mathbb{P}^{k-1}$-functor with the corresponding autoequivalence $H = [-2]$ has the following useful consequence, see [2, §2.1]: for any $E, F \in \mathrm{D}^{\mathrm{b}}(X)$ we have an isomorphism of graded vector spaces

$$\operatorname{Ext}^*_{X^{[k]}}(\Phi(E), \Phi(F)) \cong \operatorname{Ext}^*_X(E, F) \otimes H^*(\mathbb{P}^{k-1}, \mathbb{C}).$$

2. K3 surfaces with Picard number one

Throughout this section we assume X is a polarized K3 surface such that $\mathrm{NS}(X) = \mathbb{Z}h$, where h is an ample class with $h^2 = 4k$. We denote the line bundle associated to h by $\mathcal{O}_X(1)$ and the Hilbert scheme of length k subschemes of X by $X^{[k]}$.

2.1. **Explicit construction of a universal family.** In this subsection we generalize [10, Example 5.3.7] to give an explicit construction of a universal family of stable vector bundles on X parametrized by the Hilbert scheme $X^{[k]}$ for $k \geqslant 1$. Let h be the ample generator of $\mathrm{NS}(X)$ and pick the Mukai vector $v = (k+1, -h, 1) \in H^*_{\mathrm{alg}}(X, \mathbb{Z})$. We have the following facts:

Lemma 2.1. *The moduli space $M_h(v)$ of μ_h-stable sheaves on X with Mukai vector v is a smooth projective variety of dimension $2k$ and a fine moduli space. Furthermore every point $[E] \in M_h(v)$ represents a locally free sheaf.*

Proof. We note that every μ_h-semistable sheaf E with $v(E) = v$ is μ_h-stable as $\rho(X) = 1$. Thus $M_h(v)$ is a smooth projective variety. We compute:

$$\dim(M_h(v)) = v^2 + 2 = 4k - 2(k+1) + 2 = 2k.$$

Furthermore $v' = (k+1, -h, a)$ with $a \geqslant 2$ satisfies

$$v'^2 + 2 = 4k - 2a(k+1) + 2 \leqslant 4k - 4(k+1) + 2 = -2 < 0,$$

and thus the second Chern class is minimal (here $c_2(E) = 3k$). This minimality implies that every point $[E]$ in $M_h(v)$ is given by a locally free sheaf E. The condition $\gcd(k+1, 1) = 1$ implies that $M_h(v)$ is a fine moduli space by [10, Remark 4.6.8]. □

The following lemma produces examples of elements in this moduli space:

Lemma 2.2. *For any $[Z] \in X^{[k]}$ the sheaf $I_Z(1)$ is globally generated, i.e. the evaluation morphism*

$$\mathrm{ev} : H^0(I_Z(1)) \otimes \mathcal{O}_X \to I_Z(1)$$

is surjective. Furthermore $E_Z := \ker(\mathrm{ev})$ is a μ_h-stable locally free sheaf with Mukai vector given by $v(E_Z) = (k+1, -h, 1)$.

Proof. The standard exact sequence

$$0 \longrightarrow I_Z(1) \longrightarrow \mathcal{O}_X(1) \longrightarrow \mathcal{O}_Z(1) \longrightarrow 0 \tag{1}$$

shows

$$\chi(I_Z(1)) = \chi(\mathcal{O}_X(1)) - \chi(\mathcal{O}_Z(1)) = (2k+2) - k = k+2.$$

Since Z has codimension two in X, using Serre duality gives

$$H^2(I_Z(1)) \cong \mathrm{Hom}(I_Z(1), \mathcal{O}_X)^\vee \cong H^0(\mathcal{O}_X(-1))^\vee = 0.$$

By [5, Proposition 3.7], the line bundle $\mathcal{O}_X(1)$ is k-very ample which implies that the exact sequence of global sections attached to (1)

$$0 \longrightarrow H^0(I_Z(1)) \longrightarrow H^0(\mathcal{O}_X(1)) \longrightarrow H^0(\mathcal{O}_Z(1)) \longrightarrow 0$$

is still exact. This implies $H^1(I_Z(1)) \cong H^1(\mathcal{O}_X(1)) = 0$ and thus

$$\dim(H^0(I_Z(1))) = \chi(I_Z(1)) = k+2.$$

Now if the evaluation map is not surjective, let $Q := \mathrm{coker}(\mathrm{ev})$ and pick $x \in \mathrm{supp}(Q)$. Then we have an exact sequence

$$0 \longrightarrow I_{Z'}(1) \longrightarrow I_Z(1) \longrightarrow \mathcal{O}_x \longrightarrow 0$$

for a length $k+1$ subscheme Z' containing Z.

Since $I_Z(1)$ is not globally generated at x the last exact sequence gives isomorphisms

$$H^0(I_{Z'}(1)) \cong H^0(I_Z(1)) \text{ and } H^1(I_{Z'}(1)) \cong H^0(\mathcal{O}_x) \neq 0.$$

But $\mathcal{O}_X(1)$ is k-very ample so by definition

$$0 \longrightarrow H^0(I_{Z'}(1)) \longrightarrow H^0(\mathcal{O}_X(1)) \longrightarrow H^0(\mathcal{O}_{Z'}(1)) \longrightarrow 0$$

is still exact, which implies $H^1(I_{Z'}(1)) = 0$, a contradiction. So ev is indeed surjective and we have an exact sequence:

$$0 \longrightarrow E_Z \longrightarrow H^0(I_Z(1)) \otimes \mathcal{O}_X \longrightarrow I_Z(1) \longrightarrow 0. \tag{2}$$

Computing invariants shows $\mathrm{rk}(E_Z) = k+1$, $c_1(E_Z) = -h$ and $c_2(E_Z) = 3k$, hence indeed $v(E_Z) = (k+1, -h, 1)$. The sheaf E_Z is locally free as it is the kernel of a morphism between a locally free and a torsion free sheaf on a smooth surface. The stability of E_Z follows from [23, Lemma 2.1 (2-2)]. □

We can globalize the construction in Lemma 2.2: let $\mathcal{Z} \subset X \times X^{[k]}$ denote the universal length k subscheme, $\mathcal{I}_{\mathcal{Z}}$ its ideal sheaf. There are projections $p : X \times X^{[k]} \to X^{[k]}$ as well as $q : X \times X^{[k]} \to X$. Define a sheaf $\mathcal{E}$ on $X \times X^{[k]}$ by the exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow p^*(p_*(\mathcal{I}_{\mathcal{Z}} \otimes q^* \mathcal{O}_X(1))) \longrightarrow \mathcal{I}_{\mathcal{Z}} \otimes q^* \mathcal{O}_X(1) \longrightarrow 0. \tag{3}$$

Then $\mathcal{E}$ is p-flat and $\mathcal{E}_{|p^{-1}(Z)} \cong E_Z$, which implies that $\mathcal{E}$ is locally free on $X \times X^{[k]}$ by [10, Lemma 2.1.7]. Thus $\mathcal{E}$ defines a classifying morphism

$$\varphi : X^{[k]} \to M_h(v),\ [Z] \mapsto [E_Z].$$

In fact we have:

Theorem 2.3. *The classifying morphism $\varphi : X^{[k]} \to M_h(v)$ is an isomorphism.*

Proof. Looking at Remark 1.5 we see that the sheaf E_Z defined by the exact seqeunce (2) is nothing but the shifted spherical twist of $I_Z(1)$ around $\mathcal{O}_X$, more exactly we have

$$E_Z = T_{\mathcal{O}_X}(I_Z(1))[1],$$

similar to [9, Example 10.3.6]. By Proposition 1.4 the spherical twist $T_{\mathcal{O}_X}$ is an autoequivalence of $\mathrm{D}^{\mathrm{b}}(X)$ likewise is the shift [1]. But then the classifying morphism

$$\varphi : X^{[k]} \to M_h(v),\ \ [Z] \mapsto [E_Z] = [T_{\mathcal{O}_X}(I_Z(1))[1]]$$

is a composition of autoequivalences and thus maps non-isomorphic objects to non-isomorphic objects, hence φ is injective on closed points. Since both $X^{[k]}$ and $M_h(v)$ are smooth of dimension $2k$ the morphism φ is an open embedding and thus an isomorphism as both spaces are irreducible. □

2.2. **Stability of wrong-way fibers.** In the above section, we explicitly constructed a universal family $\mathcal{E}$, which is a locally free sheaf on $X \times X^{[k]}$. In this section we take the alternative point of view and consider $\mathcal{E}$ as a family of vector bundles on $X^{[k]}$ parametrized by X. A "wrong-way fiber" of $\mathcal{E}$ is just the restriction of $\mathcal{E}$ over a point $x \in X$ which gives a locally free sheaf on $X^{[k]}$.

More precisely, we first note that by standard cohomology and base change arguments

$$p_*(\mathcal{I}_{\mathcal{Z}} \otimes q^* \mathcal{O}_X(1)) \otimes \mathcal{O}_{[Z]} \to H^0(I_Z(1))$$

is an isomorphism. Hence

$$K := p_*(\mathcal{I}_{\mathcal{Z}} \otimes q^* \mathcal{O}_X(1)) \tag{4}$$

is a locally free sheaf of rank $k+2$ on $X^{[k]}$. This implies that $\mathcal{E}$ is not only p-flat, but also q-flat since $\mathcal{I}_{\mathcal{Z}} \otimes q^* \mathcal{O}_X(1)$ is both p- and q-flat by [14, Theorem 2.1]. Thus we can restrict the exact sequence (3) to the fiber over a point $x \in X$ and get the following description of the fiber $E_x := \mathcal{E}_{|q^{-1}(x)}$:

$$0 \longrightarrow E_x \longrightarrow K \longrightarrow I_{S_x} \longrightarrow 0, \tag{5}$$

where $S_x := \left\{[Z] \in X^{[k]} \,|\, x \in \operatorname{supp}(Z)\right\}$ is a codimension 2 subscheme of $X^{[k]}$. Hence E_x is a locally free sheaf of rank $k+1$ on $X^{[k]}$.

Before proving the stability of E_x, we recall that for any coherent sheaf F on X there is the associated coherent *tautological sheaf* $F^{[k]}$ on $X^{[k]}$ defined by

$$F^{[k]} := p_*\left(q^*F \otimes \mathcal{O}_{\mathcal{Z}}\right). \tag{6}$$

If F is locally free of rank r then $F^{[k]}$ is locally free of rank kr.

Also recall the well-known fact that $\mathrm{NS}(X^{[k]}) = \mathrm{NS}(X)_k \oplus \mathbb{Z}\delta$. Here d_k is the divisor class on $X^{[k]}$ induced by the divisor class d on X and δ is a divisor class on $X^{[k]}$ such that $2\delta = [E]$ where E is the exceptional divisor of the Hilbert-Chow morphism $X^{[k]} \to X^{(k)}$. In our case this reads

$$\mathrm{NS}(X^{[k]}) = \mathbb{Z}h_k \oplus \mathbb{Z}\delta.$$

Lemma 2.4. *We have* $c_1(E_x) = -h_k + \delta$.

Proof. There is the exact sequence:

$$0 \longrightarrow p_*(\mathcal{I}_{\mathcal{Z}} \otimes q^* \mathcal{O}_X(1)) \longrightarrow p_*q^* \mathcal{O}_X(1) \longrightarrow p_*(\mathcal{O}_{\mathcal{Z}} \otimes q^* \mathcal{O}_X(1)) \longrightarrow 0$$

as $R^1p_*(\mathcal{I}_{\mathcal{Z}} \otimes q^* \mathcal{O}_X(1)) = 0$ since $H^1(I_Z(1)) = 0$ for all $[Z] \in X^{[k]}$.

We also have

$$p_*q^* \mathcal{O}_X(1) \cong H^0(\mathcal{O}_X(1)) \otimes \mathcal{O}_{X^{[k]}}$$

and the sheaf $p_*(\mathcal{O}_{\mathcal{Z}} \otimes q^* \mathcal{O}_X(1))$ is nothing but the tautological sheaf $\mathcal{O}_X(1)^{[k]}$ associated to $\mathcal{O}_X(1)$ on $X^{[k]}$. By [11, Remark 3.20.] we also have

$$H^0(\mathcal{O}_X(1)^{[k]}) = H^0(\mathcal{O}_X(1)).$$

Thus, the above exact sequence can be rewritten as

$$0 \longrightarrow K \longrightarrow H^0(\mathcal{O}_X(1)^{[k]}) \otimes \mathcal{O}_{X^{[k]}} \longrightarrow \mathcal{O}_X(1)^{[k]} \longrightarrow 0. \tag{7}$$

Using [21, Lemma 1.5] we get

$$c_1(K) = -c_1(\mathcal{O}_X(1)^{[k]}) = -h_k + \delta.$$

Now exact sequence (5) gives $c_1(E_x) = c_1(K) = -h_k + \delta$. □

To compute slopes on $X^{[k]}$ we need the following intersection numbers, which can, for example, be found in [21, Lemma 1.10]:

Lemma 2.5. *For the classes h_k and δ from $\mathrm{NS}(X^{[k]})$ we have:*

- $h_k^{2k} = \frac{(2k-1)!}{(k-1)!2^{k-1}}(h^2)^k = \frac{(2k-1)!2^{k+1}}{(k-1)!}k^k > 0$
- $h_k^{2k-1}\delta = 0.$

We also recall the notations introduced in [20, §1]. The ample divisor h on X naturally induces an ample divisor

$$h_{X^k} = \bigoplus_{i=1}^{k} q_i^* h$$

on X^k, where q_i denotes the projection from X^k to the i-th factor, as well as a semi-ample divisor h_k on $X^{[k]}$.

Moreover, we write $X_\circ^k$, $S^k X_\circ$ and $X_\circ^{[k]}$ for the loci of the relevant spaces parametrizing distinct points. Then the natural map

$$\overline{\sigma}_\circ : X_\circ^k \to X_\circ^{[k]}$$

is an étale cover and $j : X_\circ^k \to X^k$ is an open embedding. For any coherent sheaf F on $X^{[k]}$, we denote by $F_\circ$ the restriction of F to $X_\circ^{[k]}$, and define

$$(F)_{X^k} = j_*(\overline{\sigma}_\circ^*(F_\circ))$$

which is a torsion free coherent sheaf if F is.

Proposition 2.6. *The vector bundle K defined in* (4) *is slope stable with respect to h_k.*

Proof. We follow the idea in the proof of [20, Theorem 1.4].

Since $(-)_\circ$ and $\overline{\sigma}_\circ^*(-)$ are exact, and $j_*(-)$ is left exact, by applying these functors to (7) we obtain an exact sequence of $\mathfrak{S}_k$-invariant reflexive sheaves on X^k as follows

$$0 \longrightarrow (K)_{X^k} \longrightarrow (H^0(\mathcal{O}_X(1)) \otimes \mathcal{O}_{X^{[k]}})_{X^k} \xrightarrow{\varphi} (\mathcal{O}_X(1)^{[k]})_{X^k}$$

where φ is not necessarily surjective. It is clear that

$$(H^0(\mathcal{O}_X(1)) \otimes \mathcal{O}_{X^{[k]}})_{X^k} = H^0(\mathcal{O}_X(1)) \otimes \mathcal{O}_{X^k},$$

and we also have

$$(\mathcal{O}_X(1)^{[k]})_{X^k} = \bigoplus_{i=1}^{k} q_i^* \, \mathcal{O}_X(1)$$

by [20, Lemma 1.1]. Hence the above sequence becomes

$$0 \longrightarrow (K)_{X^k} \longrightarrow H^0(\mathcal{O}_X(1)) \otimes \mathcal{O}_{X^k} \xrightarrow{\varphi} \bigoplus_{i=1}^{k} q_i^* \, \mathcal{O}_X(1) \tag{8}$$

where φ is the evaluation map on $X_\circ^k$.

More precisely, for any set of closed points $(x_1,\dots,x_n)\in X^k$ with $x_i\neq x_j$, the morphism of fibers can be identified as

$$\begin{aligned}\varphi_{(x_1,\dots,x_k)}: H^0(\mathcal{O}_X(1)) &\longrightarrow \bigoplus_{i=1}^k \mathcal{O}_X(1)_{x_i}\\ s &\longmapsto (s(x_1),\dots,s(x_k))\end{aligned}$$

Since for any non-trivial $s\in H^0(\mathcal{O}_X(1))$, there are always (many choices of) distinct points $(x_1,\dots x_k)\in X^k$ such that $(s(x_1),\dots,s(x_k))\neq(0,\dots,0)$, we conclude that the map of global sections

$$H^0(\varphi): H^0(\mathcal{O}_X(1))\longrightarrow H^0(\bigoplus_{i=1}^k q_i^*\,\mathcal{O}_X(1))$$

is injective. It follows by exact sequence (8) that $(K)_{X^k}$ has no global sections, that is

$$H^0((K)_{X^k})=0. \tag{9}$$

Note that φ is surjective on $X^k_\circ$, hence $\operatorname{coker}(\varphi)$ is supported on the big diagonal of X^k which is of codimension 2. It follows that

$$c_1((K)_{X^k})=-\sum_{i=1}^k q_i^*h.$$

We claim that $(K)_{X^k}$ has no $\mathfrak{S}_k$-invariant subsheaf which is destabilizing with respect to h_{X^k}. Indeed, assume F is an $\mathfrak{S}_k$-invariant subsheaf of $(K)_{X^k}$, then for some $a\in\mathbb{Z}$:

$$c_1(F)=a(\sum_{i=1}^k q_i^*h).$$

If $a\leqslant -1$, then

$$c_1(F)h_{X^k}^{2k-1}\leqslant c_1((K)_{X^k})h_{X^k}^{2k-1}<0$$

Since $1\leqslant \operatorname{rk}(F)<\operatorname{rk}((K)_{X^k})$, it follows that $\mu_{h_{X^k}}(F)<\mu_{h_{X^k}}((K)_{X^k})$, hence F is not destabilizing.

If $a=0$, we choose a (not necessarily $\mathfrak{S}_k$-invariant) non-zero stable subsheaf $F'\subseteq F$ which has maximal slope with respect to h_{X^k} (e.g. one can take a stable factor in the first Harder-Narasimhan factor of F). Without loss of generality, we can assume F and F' are both reflexive. Since F' is also a subsheaf of $H^0(\mathcal{O}_X(1))\otimes\mathcal{O}_{X^k}$, there must be a projection from $H^0(\mathcal{O}_X(1))\otimes\mathcal{O}_{X^k}$ to a certain direct summand of it, such that the composition of the embedding and projection $F'\to H^0(\mathcal{O}_X(1))\otimes\mathcal{O}_{X^k}\to\mathcal{O}_{X^k}$ is non-zero. Since $\mu_{X^k}(F')\geqslant\mu_{X^k}(F)=0=\mu_{X^k}(\mathcal{O}_{X^k})$, and $\mathcal{O}_{X^k}$ is also stable with respect to h_{X^k}, the map $F'\to\mathcal{O}_{X^k}$ must be injective, and its cokernel is supported on a locus of codimension at least 2. Since both are reflexive, we must have $F'=\mathcal{O}_{X^k}$. Therefore F, and consequently $(K)_{X^k}$, have non-trivial global sections. This contradicts (9).

If $a\geqslant 1$, F would be a subsheaf of the trivial bundle $H^0(\mathcal{O}_X(1))\otimes\mathcal{O}_{X^k}$ of positive slope. Contradiction.

Finally, assume G is a reflexive subsheaf of K. Then $(G)_{X^k}$ is an $\mathfrak{S}_k$-invariant reflexive subsheaf of $(K)_{X^k}$. By the above claim we have

$$\mu_{h_{X^k}}((G)_{X^k}) < \mu_{h_{X^k}}((K)_{X^k})$$

It follows by [20, Lemma 1.2] that $\mu_{h_k}(G) < \mu_{h_k}(K)$. Therefore K is slope stable with respect to h_k, as desired. □

Proposition 2.7. *For any closed point $x \in X$, the bundle E_x is slope stable with respect to h_k.*

Proof. By Lemma 2.4, we have $c_1(E_x) = c_1(K) = -h_k + \delta$. Therefore by Lemma 2.5

$$c_1(E_x)h_k^{2k-1} = c_1(K)h_k^{2k-1} = (-h_k + \delta)h_k^{2k-1} = -h_k^{2k} < 0.$$

Assume F is a destabilizing subsheaf of E_x with $1 \leqslant \mathrm{rk}(F) \leqslant k$ and such that $c_1(F) = ah_k + b\delta$ for some $a, b \in \mathbb{Z}$. Then

$$c_1(F)h_k^{2k-1} = ah_k^{2k}.$$

By the assumption and Proposition 2.6, we have the inequality

$$\mu_{h_k}(E_x) \leqslant \mu_{h_k}(F) < \mu_{h_k}(K),$$

which can be written as

$$\frac{-h_k^{2k}}{k+1} \leqslant \frac{ah_k^{2k}}{\mathrm{rk}(F)} < \frac{-h_k^{2k}}{k+2} \iff -\frac{\mathrm{rk}(F)}{k+1} \leqslant a < -\frac{\mathrm{rk}(F)}{k+2} \text{ as } h_k^{2k} > 0.$$

Such an integer a cannot exist. Contradiction. Hence E_x is stable with respect to h_k. □

2.3. **A smooth connected component.** In this section, we will interpret the universal sheaf $\mathcal{E}$ defined in (3) as a family of stable sheaves on $X^{[k]}$ whose base is a smooth connected component of the corresponding moduli space. We have shown above that each wrong-way fiber E_x of the family $\mathcal{E}$ is μ_{h_k}-stable; however, it would be more preferable to establish the stability with respect to some ample class on $X^{[k]}$. Although the perturbation technique in [20, Proposition 4.8] can be used to achieve this for every single E_x, for our purpose we will have to extend this technique to prove that all sheaves E_x are slope stable with respect to the same ample class near h_k.

Theorem 2.8. *There exists some ample class $H \in \mathrm{NS}(X^{[k]})$ near h_k, such that E_x is μ_H-stable for all $x \in X$ simultaneously.*

Proof. Proposition 2.7 and [4, Theorem 2.3.1] guarantees that the assumptions in [20, Proposition 4.8] are satisfied for each E_x, hence every E_x is slope stable with respect to some ample class near h_k by [20, Proposition 4.8]. In order to find a single ample class H that is independent of the choice of E_x, we can literally use the entire proof of [20, Proposition 4.8] except that we need to reconstruct the non-empty convex open set U so that $\alpha := h_k^{2k-1}$ is in the closure of U, and for every $\gamma \in U$, E_x is stable with respect to γ for all $x \in X$.

We follow the notations in [7, Definition 3.1]. For each $x \in X$, $\mathrm{SStab}(E_x)$ is a convex closed set containing α. Hence the intersection

$$\overline{U} := \cap_{x \in X} \mathrm{SStab}(E_x)$$

is also a convex closed set containing α. We first claim that [7, Theorem 3.4] holds for all E_x simultaneously; namely, we will show that for any $\beta \in \operatorname{Mov}(X^{[k]})^\circ$ (see [7, Definition 2.1] for the notation), there exists a number $e \in \mathbb{Q}^+$, such that $(\alpha + \varepsilon\beta) \in \cap_{x\in X} \operatorname{Stab}(E_x)$ for any real $\varepsilon \in [0, e]$.

To prove the claim, we first note that the slope $c := \mu_\beta(E_x)$ is independent of the choice of $x \in X$. We redefine the set S in the proof of [7, Theorem 3.4] to be

$$S := \{c_1(F) \mid F \subseteq E_x \text{ for some } x \in X \text{ such that } \mu_\beta(F) \geqslant c\}.$$

Since $E_x \subseteq K$ for all $x \in X$ by (5), we obtain that S is a subset of

$$T := \{c_1(F) \mid F \subseteq K \text{ such that } \mu_\beta(F) \geqslant c\},$$

which is finite by [7, Theorem 2.29], hence S is also finite. We can then use the rest of the proof of [7, Theorem 3.4] literally to conclude the claim.

We then claim that $\overline{U}$ is of full dimension $r := \operatorname{rk} N_1(X^{[k]})$. If not, then we have $\alpha \in \overline{U} \subseteq L$ for some hyperplane $L \subset N_1(X^{[k]})_{\mathbb{R}}$. Since $\operatorname{Mov}(X^{[k]})$ is of full dimension, we can choose some $\beta \in \operatorname{Mov}(X^{[k]})^\circ \setminus L$. It follows that $(\alpha + \varepsilon\beta) \in \overline{U} \setminus L$ for some small $\varepsilon > 0$ by the previous claim and the choice of β. Contradiction.

We define U to be the interior of $\overline{U}$ and claim that U is non-empty. Indeed, since $\overline{U}$ is of full dimension r, we can choose $r+1$ points of $\overline{U}$ in general positions, which form an r-simplex. By the convexity of $\overline{U}$, the entire simplex is in $\overline{U}$ hence any interior point of the simplex is also an interior point of $\overline{U}$. The convexity of U follows from the convexity of $\overline{U}$. And it is clear from the construction that $\alpha = h_k^{2k-1}$ is in the closure of U. We finally claim that every $\gamma \in U$ is in $\cap_{x\in X} \operatorname{Stab}(E_x)$. If not, suppose that there exists some $\gamma_0 \in U$ and some $x_0 \in X$, such that $\gamma_0 \in \operatorname{SStab}(E_{x_0}) \setminus \operatorname{Stab}(E_{x_0})$; namely, $\mu_{\gamma_0}(F) = \mu_{\gamma_0}(E_{x_0})$ for some proper subsheaf F of E_{x_0}. Since the slope function is linear with respect to the curve class, and $\mu_\alpha(F) < \mu_\alpha(E_{x_0})$ by Proposition 2.7, one can find a hyperplane in $N^1(X^{[k]})_{\mathbb{R}}$ through γ_0, such that $\mu_\gamma(E_{x_0}) - \mu_\gamma(F)$ takes opposite signs for γ in the two open half-spaces separated by the hyperplane. In particular, F destabilizes E_{x_0} in one of the half-spaces. Since U has non-empty intersection with both half-spaces, this contradicts the condition $U \subseteq \operatorname{SStab}(E_x)$. Therefore we have $U \subseteq \cap_{x\in X} \operatorname{Stab}(E_x)$, as desired. $\square$

We give an alternative description of E_x using the integral functor Φ from Example 1.6:

Lemma 2.9. *For each $x \in X$, let I_x be the ideal sheaf of $x \in X$, then $E_x = \Phi(I_x(1))$.*

Proof. We start with the exact sequence

$$0 \longrightarrow E_x \longrightarrow K \longrightarrow I_{S_x} \longrightarrow 0. \tag{10}$$

We note that $I_{S_x} = \Phi(\mathcal{O}_x)$ as $\mathcal{I}_{\mathcal{Z}}$ is flat over X. Furthermore we have $K = \Phi(\mathcal{O}_X(1))$ since $R^ip_*(\mathcal{I}_{\mathcal{Z}} \otimes q^*\mathcal{O}_X(1)) = 0$ for $i = 1, 2$ as this is true for $H^i(I_Z(1))$ for any $[Z] \in X^{[k]}$. These two facts imply that

$$\operatorname{Hom}_{X^{[k]}}(K, I_{S_x}) = \operatorname{Hom}_{X^{[k]}}(\Phi(\mathcal{O}_X(1)), \Phi(\mathcal{O}_x)) \cong \operatorname{Hom}_X(\mathcal{O}_X(1), \mathcal{O}_x) \cong \mathbb{C}$$

by Remark 1.7. Thus the exact sequence (10) is induced by the exact sequence

$$0 \longrightarrow I_x(1) \longrightarrow \mathcal{O}_X(1) \longrightarrow \mathcal{O}_x \longrightarrow 0.$$

As $K \to I_{S_x}$ is surjective, applying Φ to the last exact sequence shows $E_x = \Phi(I_x(1))$. □

We return to the main result of the section. Let H be an ample class that satisfies Theorem 2.8, and $\mathcal{M}$ the moduli space of μ_H-stable sheaves on $X^{[k]}$ with the same numerical invariants as E_x. Then the universal family $\mathcal{E}$ defines a classifying morphism

$$f\colon X \longrightarrow \mathcal{M}, \quad x \longmapsto [E_x] \tag{11}$$

In fact the morphism f can be described as follows:

Theorem 2.10. *The classifying morphism* (11) *defined by the family $\mathcal{E}$ identifies X with a smooth connected component of $\mathcal{M}$.*

Proof. By [17, Lemma 1.6] we have to prove that f is injective on closed points and that for all $x \in X$ we have $\dim(T_{[E_x]}\mathcal{M}) = 2$.

Now by Lemma 2.9 we know $E_x = \Phi(I_x(1))$, so for $x \neq y$ we find

$$\begin{aligned}\mathrm{Hom}_{X^{[k]}}(E_x, E_y) &= \mathrm{Hom}_{X^{[k]}}(\Phi(I_x(1)), \Phi(I_y(1)))\\ &\cong \mathrm{Hom}_X(I_x(1), I_y(1))\\ &\cong \mathrm{Hom}_X(\mathcal{O}_x, \mathcal{O}_y) = 0\end{aligned}$$

by Remark 1.7 again. This implies f is injective on closed points.

A similar computation shows

$$\begin{aligned}\mathrm{Ext}^1_{X^{[k]}}(E_x, E_x) &= \mathrm{Ext}^1_{X^{[k]}}(\Phi(I_x(1)), \Phi(I_x(1)))\\ &\cong \mathrm{Ext}^1_X(I_x(1), I_x(1))\\ &\cong \mathrm{Ext}^1_X(\mathcal{O}_x, \mathcal{O}_x) \cong T_xX.\end{aligned}$$

Using $T_{[E_x]}\mathcal{M} \cong \mathrm{Ext}^1_{X^{[k]}}(E_x, E_x)$ we thus find $\dim(T_{[E_x]}\mathcal{M}) = 2$ as desired. □

3. K3 surfaces with Picard number two

In this section, we will consider a K3 surface X of Picard number 2, and construct a complete family of stable vector bundles on the Hilbert scheme $X^{[k]}$ for $k \geqslant 2$.

3.1. **The K3 surface.** In this section we assume X is a K3 surface with

$$\mathrm{NS}(X) = \mathbb{Z}e \oplus \mathbb{Z}f$$

such that $e^2 = -2k$, $f^2 = 0$ and $ef = 2k + 1$ for some integer $k \geqslant 2$. The existence of such K3 surfaces is guaranteed by [9, Corollary 14.3.1]. Since $f^2 = 0$, either f or $-f$ is effective. Without loss of generality, we will assume that the divisor class f is effective, after possibly replacing the pair (e, f) by $(-e, -f)$.

In this subsection we collect some helpful properties of X which will be used in the construction of some moduli spaces of stable sheaves in the next section.

Lemma 3.1. *We have $D^2 \geqslant 0$ for all effective divisors on X. Especially there are no smooth curves C on X with $C \cong \mathbb{P}^1$.*

Proof. Any irreducible curve C on S satisfies

$$C^2 = C(C + K_X) = 2p_a(C) - 2 \geqslant -2.$$

So assume $C^2 = -2$ and write $C = me + nf$. Then we have

$$\begin{aligned} C^2 &= (me + nf)^2 = m^2e^2 + 2mnef \\ &= -2km^2 + 2(2k+1)mn \\ &= -2m(km - (2k+1)n). \end{aligned}$$

The equation $C^2 = -2$ translates into $m(km - (2k+1)n) = 1$. This implies $m = \pm 1$ but then one can see that there is no $n \in \mathbb{Z}$ satisfying this equation. □

Lemma 3.2. *The divisor classes $h = e + (2k-1)f$ and $\widehat{h} = (2k)e + (2k-1)f$ are ample.*

Proof. We have

$$\begin{aligned} h^2 &= (e + (2k-1)f)^2 = e^2 + 2(2k-1)ef \\ &= -2k + 2(2k-1)(2k+1) = 8k^2 - 2k - 2. \end{aligned}$$

So $h^2 > 0$ as $k \geqslant 2$. Since also $hf = ef = 2k + 1 > 0$ we see that h is ample by the remark after [9, Corollary 8.1.7].

A similar computation shows $\widehat{h}^2 > 0$ and $\widehat{h}f > 0$. □

Lemma 3.3. *Let m and n be integers. If the class $me + nf$ is effective, then $0 \leqslant m \leqslant \frac{2k+1}{k}n$ and $h(me + nf) \geqslant ((2k-1)(2k+1) - k)m$ (thus in particular $n \geqslant 0$).*

Proof. Let D be an effective divisor with class $me + nf$. Since the claim is additive in m and n, we may assume w.l.o.g. that D is an irreducible curve C.

By Lemma 3.1 we have $C^2 \geqslant 0$. Therefore

$$\begin{aligned} C^2 &= 2m\{-km + (2k+1)n\} \geqslant 0 \\ hC &= (4k^2 - k - 1)m + \{-km + (2k+1)n\} > 0 \end{aligned}$$

which implies $m \geqslant 0$ and $-km + (2k+1)n \geqslant 0$. The last inequality can also be read as

$$(2k+1)n \geqslant km \Leftrightarrow m \leqslant \frac{2k+1}{k}n.$$

Putting everything together shows

$$0 \leqslant m \leqslant \frac{2k+1}{k}n$$

as well as $hC \geqslant ((2k-1)(2k+1) - k)m$. □

Corollary 3.4. *There is a surjective morphism $\pi : X \to \mathbb{P}^1$ such that all fibers are integral curves of arithmetic genus $p_a(C) = 1$, that is X is elliptically fibered.*

Proof. Since $f^2 = 0$ it is known that the linear system $|f|$ induces a surjective map $\pi : X \to \mathbb{P}^1$ with $\pi^* \mathcal{O}_{\mathbb{P}^1}(1) = \mathcal{O}_X(f)$. By the previous lemma the class f cannot be the sum of two effective divisors, hence all fibers C of π are integral and have $p_a(C) = 1$. □

Lemma 3.5. *Let $[Z] \in X^{[k]}$. Assume R is a torsion quotient of $I_Z(e)$ with $c_1(R) = nf$ for some $n \geqslant 0$, then $H^1(R) = 0$.*

Proof. The quotient defines the following exact sequence:

$$0 \longrightarrow K \longrightarrow I_Z(e) \longrightarrow R \longrightarrow 0.$$

Now K is torsion free of rank one, so its double dual K^{**} is locally free of rank one and the natural map $K \to K^{**}$ is injective and the cokernel T has finite support. Especially $c_1(T) = 0$ so

$$c_1(K^{**}) = c_1(K) = c_1(I_Z(e)) - c_1(R) = e - nf$$

and thus $K^{**} \cong \mathcal{O}_X(e - nf)$. The embedding $K \hookrightarrow I_Z(e)$ induces an embedding

$$K^{**} \cong \mathcal{O}_X(e - nf) \hookrightarrow \mathcal{O}_X(e).$$

This embedding is given by a global section of $\mathcal{O}_X(nf)$, that is by an effective divisor $D = \sum_i a_i C_i$ with class nf.

This global section is the pullback along the elliptic fibration π of a global section of $\mathcal{O}_{\mathbb{P}^1}(n)$, with corresponding effective divisor $\sum_i a_i z_i$ on $\mathbb{P}^1$, here $C_i = \pi^{-1}(z_i)$.

Denote by $D \subset X$ also the corresponding closed subscheme (which maybe non-reduced, if $a_i \geqslant 2$ for some i). We get the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & K \arrow[r] \arrow[d] & \mathcal{O}_X(e-nf) \arrow[r] \arrow[d] & T \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & I_Z(e) \arrow[r] \arrow[d] & \mathcal{O}_X(e) \arrow[r] \arrow[d] & \mathcal{O}_Z \arrow[r] & 0 \\
 & R \arrow[r, "\beta"] \arrow[d] & \mathcal{O}_D(e) \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

The snake lemma gives an exact sequence

$$0 \longrightarrow \ker(\alpha) \longrightarrow R \xrightarrow{\beta} \mathcal{O}_D(e) \longrightarrow \operatorname{coker}(\alpha) \longrightarrow 0.$$

Let $R' \subset \mathcal{O}_D(e)$ be the image of β. Since the torsion sheaf $\mathcal{O}_{\sum_i a_i z_i}$ on $\mathbb{P}^1$ has a composition series by skyscraper sheaves $\mathcal{O}_{z_i}$ as composition factors, $\mathcal{O}_D$ has a composition series with composition factors $\mathcal{O}_{C_i}$, thus $\mathcal{O}_D(e)$ has a composition series with composition factors $\mathcal{O}_{C_i}(e)$. The latter is a line bundle of degree

$$e \cdot C_i = ef = 2k + 1$$

on C_i. The quotient $\mathcal{O}_D(e)/R'$ is isomorphic to $\operatorname{coker}(\alpha)$, that is to a quotient Q of $\mathcal{O}_Z$. By intersecting with R' we get a composition series for R' with composition factors which are kernels of a surjection $\mathcal{O}_{C_i}(e) \twoheadrightarrow Q'$ with Q' of length $\leqslant k$. Thus we have exact sequences:

$$0 \longrightarrow L \longrightarrow \mathcal{O}_{C_i}(e) \longrightarrow Q' \longrightarrow 0,$$

with a torsion free sheaf L of rank one on the integral projective curve C_i of arithmetic genus one. Using $\chi(\mathcal{O}_{C_i}) = 0$ and

$$\chi(L) = \chi(\mathcal{O}_{C_i}(e)) - \chi(Q') \geqslant k+1,$$

gives

$$\deg(\mathcal{O}_{C_i}(e)) \geqslant \deg(L) \geqslant k+1.$$

By [6, Proposition 4.6.] all of these composition factors have trivial H^1. By constructing short exact sequences out of the composition series and using the induced exact sequences for H^1, it follows

$$H^1(R') = 0.$$

As $\ker(\beta) = \ker(\alpha) \subseteq T$ has finite support, we also have $H^1(\ker(\beta)) = 0$. Hence

$$H^1(R) = 0.$$ □

3.2. **The construction of a universal family.** In this subsection we want to generalize [16, Theorem 1.2]. Let h be the ample line bundle defined in Lemma 3.2, and $v = (2k-1, h, 2k) \in H^*_{\mathrm{alg}}(X, \mathbb{Z})$ for any integer $k \geqslant 2$. We immediately have the following result:

Lemma 3.6. *The moduli space $M_h(v)$ of μ_h-stable sheaves on X with Mukai vector v is a smooth projective variety of dimension $2k$ and a fine moduli space, and every point $[E] \in M_h(v)$ represents a locally free sheaf.*

Proof. We first observe by [10, Lemma 1.2.7] that every μ_h-semistable sheaf E with $v(E) = v$ is μ_h-stable since $\gcd(2k-1, h^2) = 1$. Thus $M_h(v)$ is a smooth projective variety. We compute:

$$\dim(M_h(v)) = v^2 + 2 = (8k^2 - 2k - 2) - 2(2k-1)(2k) + 2 = 2k.$$

Furthermore $v' = (2k-1, h, a)$ with $a \geqslant 2k+1$ satisfies

$$v'^2 + 2 = h^2 - 2a(2k-1) + 2 \leqslant (8k^2 - 2k - 2) - 2(2k-1)(2k+1) + 2 = 2 - 2k < 0,$$

so again every point $[E]$ in $M_h(v)$ is locally free. $M_h(v)$ is a fine moduli space as $\gcd(2k-1, h^2) = 1$. □

In the following discussion, we will explicitly construct a universal family for the moduli space $M_h(v)$. We first define some integral functors. For any line bundle L on X, we define

$$M_L : \mathrm{D}^{\mathrm{b}}(X) \longrightarrow \mathrm{D}^{\mathrm{b}}(X); \quad (-) \longmapsto (-) \otimes L.$$

Then we consider the composition

$$\Theta := M_{\mathcal{O}_X(f)} \circ T^{-1}_{\mathcal{O}_X} \circ M_{\mathcal{O}_X(e)} : \mathrm{D}^{\mathrm{b}}(X) \longrightarrow \mathrm{D}^{\mathrm{b}}(X), \tag{12}$$

where $T^{-1}_{\mathcal{O}_X}$ is the inverse of the spherical twist induced by $\mathcal{O}_X$. It is clear that Θ is an autoequivalence of $\mathrm{D}^{\mathrm{b}}(X)$ hence a Fourier-Mukai transform.

We denote the corresponding kernel by $\mathcal{P} \in \mathrm{D^b}(X \times X)$. By Remark 1.5, we have an explicit description of $\mathcal{P}$ by the exact triangle

$$\mathcal{P} \longrightarrow \Delta_* \mathcal{O}_X(e+f) \longrightarrow \mathcal{O}_X(e) \boxtimes \mathcal{O}_X(f)[2] \longrightarrow \mathcal{P}[1], \tag{13}$$

where $\Delta : X \hookrightarrow X \times X$ is the diagonal embedding. The kernel $\mathcal{P}$ also defines a Fourier-Mukai transform in the opposite direction, which we denote by

$$\widehat{\Theta} : \mathrm{D^b}(X) \longrightarrow \mathrm{D^b}(X).$$

Since the kernel of each composition factor in (12), viewed as an object in $\mathrm{D^b}(X \times X)$, remains the same under the permutation of the two copies of X, it follows that

$$\widehat{\Theta} = M_{\mathcal{O}_X(e)} \circ T^{-1}_{\mathcal{O}_X} \circ M_{\mathcal{O}_X(f)}. \tag{14}$$

For any $[Z] \in X^{[k]}$, we apply Θ on the ideal sheaf I_Z and define

$$E_Z \coloneqq \Theta(I_Z).$$

A priori E_Z is a derived object on X, but we can show the following:

Theorem 3.7. *E_Z is μ_h-stable locally free sheaf with $v(E_Z) = (2k-1, h, 2k)$.*

Proof. First of all, by (12) and [8, Lemma 8.12], a standard computation of the cohomological Fourier-Mukai transform shows that

$$v(E_Z) = (2k-1, h, 2k).$$

Moreover, by (13) and the fact that $T^{-1}_{\mathcal{O}_X}(\mathcal{O}_X) = \mathcal{O}_X[1]$, we obtain an exact triangle

$$E_Z \longrightarrow I_Z(e+f) \longrightarrow H^*(I_Z(e)) \otimes \mathcal{O}_X(f)[2] \longrightarrow E_Z[1]. \tag{15}$$

In order to compute $H^*(I_Z(e))$, we observe by Lemma 3.3 that

$$h^0(\mathcal{O}_X(e)) = 0 \quad \text{and} \quad h^2(\mathcal{O}_X(e)) = h^0(\mathcal{O}_X(-e)) = 0. \tag{16}$$

It follows by a long exact sequence of cohomology groups that

$$h^0(I_Z(e)) = h^2(I_Z(e)) = 0.$$

Therefore the exact triangle (15) reduces to the short exact sequence

$$0 \longrightarrow H^1(I_Z(e)) \otimes \mathcal{O}_X(f) \longrightarrow E_Z \longrightarrow I_Z(e+f) \longrightarrow 0, \tag{17}$$

where $\dim H^1(I_Z(e)) = \mathrm{rk}(E_Z) - 1 = 2k-2$. For the convenience of analyzing the stability of E_Z, we rewrite the above exact triangle as

$$0 \longrightarrow \mathcal{O}_X^{\oplus(2k-2)} \longrightarrow E_Z(-f) \longrightarrow I_Z(e) \longrightarrow 0.$$

Furthermore, we observe that $\mathcal{O}_X(f) = \Theta(\mathcal{O}_X(-e)[-1])$. Since Θ is an equivalence, we have

$$\mathrm{Hom}(E_Z(-f), \mathcal{O}_X) = \mathrm{Hom}(E_Z, \mathcal{O}_X(f)) = \mathrm{Hom}(I_Z, \mathcal{O}_X(-e)[-1]) = 0.$$

We are now ready to prove that E_Z, or rather $E_Z(-f)$, is μ_h-stable. We first have

$$\mu_h(E_Z(-f)) = \frac{eh}{2k-1} = \frac{-2k + (2k-1)(2k+1)}{2k-1} = 2k+1 - \frac{2k}{2k-1} > 0.$$

Pick a torsion free quotient F of $E_Z(-f)$ with $1 \leqslant \mathrm{rk}(F) \leqslant 2k-2$. We have

$$E_Z(-f) \longrightarrow F \longrightarrow 0$$

with $\mathrm{Hom}(F, \mathcal{O}_X) \hookrightarrow \mathrm{Hom}(E_Z(-f), \mathcal{O}_X) = 0$.

We want to show that we always have $\mu_h(F) > \mu_h(E_Z(-f))$. For this, define the torsion free sheaf F_0 as the image of the composition

$$\mathcal{O}_X^{\oplus(2k-2)} \hookrightarrow E_Z(-f) \longrightarrow F.$$

We get a surjection

$$\mathcal{O}_X^{\oplus(2k-2)} \longrightarrow F_0 \longrightarrow 0.$$

This implies that $c_1(F_0)$ is effective and we have the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K_0 \arrow[r] \arrow[d] & K_1 \arrow[r] \arrow[d] & K_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_X^{\oplus(2k-2)} \arrow[r] \arrow[d] & E_Z(-f) \arrow[r] \arrow[d] & I_Z(e) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F_0 \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & F_1 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{18}$$

Due to the diagram $\mathrm{rk}(F_1) \in \{0, 1\}$.

Case 1: $\mathrm{rk}(F_1) = 1$. Then $\mathrm{rk}(F_0) = \mathrm{rk}(F) - 1$ and $F_1 \cong I_Z(e)$. We conclude

$$c_1(F) = c_1(F_0) + c_1(I_Z(e)) \Rightarrow c_1(F) = c_1(F_0) + e.$$

Using this we find:

$$\mu_h(F) = \frac{c_1(F)h}{\mathrm{rk}(F)} = \underbrace{\frac{c_1(F_0)h}{\mathrm{rk}(F)}}_{\geqslant 0} + \frac{eh}{\mathrm{rk}(F)} > \frac{eh}{2k-1} = \mu_h(E_Z(-f)).$$

So we indeed have $\mu_h(F) > \mu_h(E_x(-f))$.

Case 2: $\mathrm{rk}(F_1) = 0$. Now $\mathrm{rk}(F_0) = \mathrm{rk}(F)$. Write $c_1(F) = me + nf$. Since $c_1(F_0)$ and $c_1(F_1)$ are effective, so is their sum $c_1(F)$, which by Lemma 3.3 implies, that $m \geqslant 0$ as well as

$$\mu_h(F) = \frac{(me+nf)h}{\mathrm{rk}(F)} \geqslant \frac{m((2k-1)(2k+1)-k)}{\mathrm{rk}(F)} \geqslant m(2k+1-\frac{k}{2k-1}).$$

For $m \geqslant 1$ we have

$$\begin{aligned}\mu_h(F) &\geqslant m(2k+1-\frac{k}{2k-1}) \\ &\geqslant 2k+1-\frac{k}{2k-1} \\ &> 2k+1-\frac{2k}{2k-1} = \mu_h(E_Z(-f))\end{aligned}$$

So only the case $m = 0$ remains, i.e. $c_1(F) = nf$. We have

$$\mu_h(F) = \frac{n(2k+1)}{\mathrm{rk}(F)}.$$

If we can prove $n \geqslant \mathrm{rk}(F)$ we are done since then

$$\mu_h(F) \geqslant 2k+1 > 2k+1-\frac{2k}{2k-1} = \mu_h(E_Z(-f)).$$

As $c_1(F) = nf$ is the sum of the two effective divisors $c_1(F_0)$ and $c_1(F_1)$, it follows from Lemma 3.3 that $c_1(F_0) = n_0 f$ and $c_1(F_1) = n_1 f$ with $n_0, n_1 \geqslant 0$ and $n_0 + n_1 = n$.

By Lemma 3.5 we have $H^1(F_1) = 0$ which implies $\mathrm{Ext}^1(F_1, \mathcal{O}_X) = 0$ using Serre duality. So the restriction map

$$\mathrm{Hom}(F, \mathcal{O}_X) \to \mathrm{Hom}(F_0, \mathcal{O}_X)$$

surjective. But we know $\mathrm{Hom}(F, \mathcal{O}_X) = 0$. So

$$\mathrm{Hom}(F_0, \mathcal{O}_X) = 0. \tag{19}$$

Using the elliptic fibration $\pi : X \to \mathbb{P}^1$ we have:

$$h^0(\det(F_0)) = h^0(\mathcal{O}_X(n_0 f)) = n_0 + 1. \tag{20}$$

Now there is a trivial sub-bundle in $\mathcal{O}_X^{\oplus(2k-2)}$ of rank $\mathrm{rk}(F)+1$ such that

$$\mathcal{O}_X^{\oplus(\mathrm{rk}(F)+1)} \xrightarrow{\varphi} F_0$$

is surjective outside a finite subset of X by [3, Lemma 4.60].

Define $R := \mathrm{coker}(\varphi)$. Then there is the exact sequence:

$$0 \longrightarrow F_0' \longrightarrow F_0 \longrightarrow R \longrightarrow 0.$$

As R has finite support, we get:

$$\det(F_0) = \det(F_0') \text{ as well as } H^2(F_0') \cong H^2(F_0).$$

We also have the exact sequence

$$0 \longrightarrow \det(F_0)^{-1} \longrightarrow \mathcal{O}_X^{\oplus(\mathrm{rk}(F)+1)} \longrightarrow F_0' \longrightarrow 0.$$

The end of the induced long cohomology sequence gives:

(21)

$$H^1(F_0') \longrightarrow H^2(\det(F_0)^{-1}) \longrightarrow H^2(\mathcal{O}_X^{\oplus(\mathrm{rk}(F)+1)}) \longrightarrow H^2(F_0') \longrightarrow 0.$$

It follows from (19) by Serre duality that

$$H^2(F_0') \cong H^2(F_0) \cong \operatorname{Hom}(F_0, \mathcal{O}_X)^\vee = 0.$$

Since $H^2(F_0') = 0$, we apply Serre duality again and obtain from (21) that

$$0 \longrightarrow H^0(\mathcal{O}_X^{\oplus(\operatorname{rk}(F)+1)}) \longrightarrow H^0(\det(F_0)).$$

We conclude

$$h^0(\det(F_0)) \geqslant \operatorname{rk}(F) + 1.$$

Using this inequality together with (20) we get:

$$n_0 + 1 = h^0(\det(F_0)) \geqslant \operatorname{rk}(F) + 1 \Rightarrow n_0 \geqslant \operatorname{rk}(F) \Rightarrow n \geqslant \operatorname{rk}(F).$$

We obtain the desired inequality between n and $\operatorname{rk}(F)$, hence $E_Z(-f)$ is stable, and so is E_Z. It then follows by Lemma 3.6 that E_Z is locally free. □

We want to globalize the previous construction. For this we denote the universal closed subscheme of length n by $\mathcal{Z} \subset X \times X^{[k]}$, and the universal ideal sheaf by $\mathcal{I}_\mathcal{Z}$. As a kernel, $\mathcal{I}_\mathcal{Z}$ induces a pair of integral functors (in opposite directions):

$$\Phi : \mathrm{D^b}(X) \longrightarrow \mathrm{D^b}(X^{[k]}) \text{ and } \widehat{\Phi} : \mathrm{D^b}(X^{[k]}) \longrightarrow \mathrm{D^b}(X).$$

Here Φ is a $\mathbb{P}^{k-1}$-functor, see Example 1.6.

The composition of the integral functors

$$\Theta \circ \widehat{\Phi} : \mathrm{D^b}(X^{[k]}) \longrightarrow \mathrm{D^b}(X)$$

is still an integral functor, whose kernel $\mathcal{E} \in \mathrm{D^b}(X^{[k]} \times X)$ can be computed from $\mathcal{P}$ and $\mathcal{I}_\mathcal{Z}$ explicitly. More precisely, let π_{12}, π_{23} and π_{13} be projections from $X^{[k]} \times X \times X$ to each pair of factors, then

$$\mathcal{E} = R\pi_{13*}(\pi_{12}^*\mathcal{I}_\mathcal{Z} \otimes \pi_{23}^*\mathcal{P});$$

see [8, Proposition 5.10]. We have the following property about $\mathcal{E}$:

Proposition 3.8. *$\mathcal{E}$ is a locally free sheaf on $X^{[k]} \times X$ such that for any $[Z] \in X^{[k]}$ we have $\mathcal{E}|_{\{[Z]\}\times X} \cong E_Z$.*

Proof. For any $[Z] \in X^{[k]}$, the derived pullback of $\mathcal{E}$ to the fiber $\{[Z]\} \times X$ can be computed by

$$(\Theta \circ \widehat{\Phi})(\mathcal{O}_{[Z]}) \cong \Theta(I_Z) = E_Z,$$

which is a locally free sheaf by Theorem 3.7. It follows that $\mathcal{E}$ is a sheaf which is flat over $X^{[k]}$ by [8, Lemma 3.31], and locally free by [10, Lemma 2.1.7]. □

In fact, $\mathcal{E}$ is a universal family for the fine moduli space $M_h(v)$. More precisely, we have

Corollary 3.9. *The family $\mathcal{E}$ induces an isomorphism $X^{[k]} \cong M_h(v)$.*

Proof. $\mathcal{E}$ induces a classifying morphism

$$\varphi : X^{[k]} \longrightarrow M_h(v); \quad [Z] \longmapsto [E_Z].$$

Since Θ is an equivalence, we have $E_Z \not\cong E_{Z'}$ for $[Z] \neq [Z']$, hence φ is injective, hence it is an open embedding since $X^{[k]}$ and $M_h(v)$ are both of dimension $2k$. But $X^{[k]}$ is projective, so φ is also closed. Since $X^{[k]}$ and $M_h(v)$ are both irreducible, φ must be an isomorphism. □

Remark 3.10. Although it is not strictly required in our following discussion, the universal family $\mathcal{E}$ can in fact be given in a more explicit form similar to (17). To globalize the construction in Theorem 3.7, we apply the functor $R\pi_{13*}(\pi_{12}^*\mathcal{I}_{\mathcal{Z}} \otimes \pi_{23}^*(-))$ to (13) and obtain

$$\mathcal{E} \longrightarrow R\pi_{13*}(\pi_{12}^*\mathcal{I}_{\mathcal{Z}}\otimes\pi_{23}^*\Delta_*\mathcal{O}_X(e+f)) \longrightarrow R\pi_{13*}(\pi_{12}^*\mathcal{I}_{\mathcal{Z}}\otimes\pi_2^*\mathcal{O}_X(e)\otimes\pi_3^*\mathcal{O}_X(f))[2] \longrightarrow \mathcal{E}[1].$$

We denote the projections from $X^{[k]} \times X$ to the two factors by p and q respectively. Then a simple calculation reduces the above exact triangle to

$$\mathcal{E} \longrightarrow \mathcal{I}_{\mathcal{Z}} \otimes q^*\mathcal{O}_X(e+f) \longrightarrow Rp_*(\mathcal{I}_{\mathcal{Z}} \otimes q^*\mathcal{O}_X(e)) \boxtimes \mathcal{O}_X(f)[2] \longrightarrow \mathcal{E}[1].$$

For the consistency with the following discussion, we denote

$$\mathcal{H} \coloneqq Rp_*(\mathcal{I}_{\mathcal{Z}} \otimes q^*\mathcal{O}_X(e))[1] = \Phi(\mathcal{O}_X(e))[1].$$

We will prove in Lemma 3.11 that $\mathcal{H}$ is in fact a sheaf. Therefore the exact triangle reduces to

$$0 \longrightarrow \mathcal{H} \boxtimes \mathcal{O}_X(f) \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_{\mathcal{Z}} \otimes q^*\mathcal{O}_X(e+f) \longrightarrow 0.$$

3.3. **The wrong-way fibers.** In this subsection we study the wrong-way fibers of $\mathcal{E}$. For any $x \in X$, we define the corresponding wrong-way fiber to be

$$E_x \coloneqq \mathcal{E}|_{X^{[k]}\times\{x\}},$$

which is locally free of rank $2k-1$. As an alternative description, we consider the composition

$$\Phi \circ \widehat{\Theta} : \mathrm{D^b}(X) \longrightarrow \mathrm{D^b}(X^{[k]}),$$

which is also an integral functor with kernel $\mathcal{E}$, in the direction opposite to $\Theta \circ \widehat{\Phi}$. Then we have

$$E_x = (\Phi \circ \widehat{\Theta})(\mathcal{O}_x).$$

The following result gives a concrete description of E_x:

Lemma 3.11. *The locally free sheaf E_x fits in an exact sequence of sheaves*

$$0 \longrightarrow \mathcal{H} \longrightarrow E_x \longrightarrow I_{S_x} \longrightarrow 0, \tag{22}$$

where

$$\mathcal{H} \coloneqq \Phi(\mathcal{O}_X(e))[1]$$

is locally free, and I_{S_x} is the ideal sheaf of

$$S_x \coloneqq \left\{ [Z] \in X^{[k]} \,\middle|\, x \in \mathrm{supp}(Z) \right\} \subset X^{[k]}.$$

Proof. We write $F_x := \widehat{\Theta}(\mathcal{O}_x)$, then $E_x = \Phi(F_x)$. By (14) we have the equality $F_x = T_{\mathcal{O}_X}^{-1}(\mathcal{O}_x) \otimes \mathcal{O}_X(e)$. By applying the inverse spherical functor $T_{\mathcal{O}_X}^{-1}$ to the exact sequence

$$0 \longrightarrow I_x \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_x \longrightarrow 0$$

we obtain an exact triangle

$$T_{\mathcal{O}_X}^{-1}(I_x) \longrightarrow T_{\mathcal{O}_X}^{-1}(\mathcal{O}_X) \longrightarrow T_{\mathcal{O}_X}^{-1}(\mathcal{O}_x) \longrightarrow T_{\mathcal{O}_X}^{-1}(I_x)[1].$$

Since $T_{\mathcal{O}_X}(\mathcal{O}_X) = \mathcal{O}_X[-1]$ and $T_{\mathcal{O}_X}(\mathcal{O}_x) = I_x[1]$, the above triangle becomes

$$\mathcal{O}_x[-1] \longrightarrow \mathcal{O}_X(e)[1] \longrightarrow F_x \longrightarrow \mathcal{O}_x\,.$$

Since $\Phi(\mathcal{O}_x) = I_{S_x}$, we further apply the integral functor Φ to obtain the exact triangle

$$I_{S_x}[-1] \longrightarrow \mathcal{H} \longrightarrow E_x \longrightarrow I_{S_x}, \tag{23}$$

where $\mathcal{H} = \Phi(\mathcal{O}_X(e))[1]$. To compute $\mathcal{H}$, we observe that the short exact sequence of kernels

$$0 \longrightarrow \mathcal{I}_{\mathcal{Z}} \longrightarrow \mathcal{O}_{X^{[k]}\times X} \longrightarrow \mathcal{O}_{\mathcal{Z}} \longrightarrow 0$$

induces an exact triangle

$$\Phi(\mathcal{O}_X(e)) \longrightarrow H^*(\mathcal{O}_X(e)) \otimes \mathcal{O}_{X^{[k]}} \longrightarrow \mathcal{O}_X(e)^{[k]} \longrightarrow \Phi(\mathcal{O}_X(e))[1]. \tag{24}$$

Since $H^i(\mathcal{O}_X(e)) = 0$ for $i \neq 1$ by (16), the exact triangle (24) reduces to the short exact sequence

$$0 \longrightarrow \mathcal{O}_X(e)^{[k]} \longrightarrow \mathcal{H} \longrightarrow H^1(\mathcal{O}_X(e)) \otimes \mathcal{O}_{X^{[k]}} \longrightarrow 0,$$

which in particular implies that $\mathcal{H}$ is a locally free sheaf. It follows that the exact triangle (23) reduces to the short exact sequence (22). $\square$

We will require a technical result in the proof of the stability. For this purpose, we define

$$I^kX := (X^k \times_{S^kX} X^{[k]})_{\mathrm{red}}$$

to be Haiman's isospectral Hilbert scheme, and denote its projections to both factors by p and q respectively. Then the derived McKay correspondence

$$\Psi := (-)^{\mathfrak{S}_k} \circ q_* \circ Lp^* : \mathrm{D^b}(X^k)^{\mathfrak{S}_k} \longrightarrow \mathrm{D^b}(X^{[k]})$$

is an equivalence, and so is $\Psi^{-1} : \mathrm{D^b}(X^{[k]}) \to \mathrm{D^b}(X^k)^{\mathfrak{S}_k}$. We have

Lemma 3.12. *For any coherent sheaf F on $X^{[k]}$, if $\Psi^{-1}(F)$ is a reflexive sheaf, then*

$$\Psi^{-1}(F) = (F)_{X^k}.$$

Proof. We follow the above notation to denote

$$I^kX_\circ := X_\circ^k \times_{S^kX_\circ} X_\circ^{[k]},$$

then we have the commutative diagram

$$\begin{tikzcd} X_\circ^{[k]} \arrow[d, "\alpha"] & I^kX_\circ \arrow[l, "q_\circ"'] \arrow[r, "p_\circ"] \arrow[d, "\beta"] & X_\circ^k \arrow[d, "j"] \\ X^{[k]} & I^kX \arrow[l, "q"'] \arrow[r, "p"] & X^k, \end{tikzcd}$$

where α, β, j and $q_\circ$ are étale morphisms, and $p_\circ$ is an isomorphism. We also have

$$\Psi^{-1} \cong Rp_* \circ q^!,$$
$$(-)_{X^k} = j_* \circ \overline{\sigma}_\circ^* \circ \alpha^*,$$

where the first equation is due to the fact that Ψ^{-1} is the right adjoint of Ψ. It follows that

$$\begin{aligned} j^* \circ \Psi^{-1} &\cong j^* \circ Rp_* \circ q^! \cong p_{\circ *} \circ \beta^* \circ q^! \\ &\cong p_{\circ *} \circ \beta^! \circ q^! \cong p_{\circ *} \circ q_\circ^! \circ \alpha^! \\ &\cong p_{\circ *} \circ q_\circ^* \circ \alpha^* \cong \overline{\sigma}_\circ^* \circ \alpha^*. \end{aligned}$$

Therefore we have

$$j_* \circ j^* \circ \Psi^{-1} \cong (-)_{X^k}.$$

Since $\Delta = X^k \setminus X^k_\circ$ is of codimension 2, if $\Psi^{-1}(F)$ is a reflexive sheaf, then we have

$$\Psi^{-1}(F) \cong j_* \circ j^* \circ \Psi^{-1}(F) \cong (F)_{X^k}$$

as desired. □

Lemma 3.13. *The sheaf $(\mathcal{H})_{X^k}$ fits in an exact sequence of $\mathfrak{S}_k$-invariant locally free sheaves*

$$0 \longrightarrow \bigoplus_{i=1}^k q_i^* \mathcal{O}_X(e) \longrightarrow (\mathcal{H})_{X^k} \longrightarrow H^1(\mathcal{O}_X(e)) \otimes \mathcal{O}_{X^k} \longrightarrow 0. \tag{25}$$

Moreover, $H^0((\mathcal{H})_{X^k})^{\mathfrak{S}_k} = 0$, i.e. every $\mathfrak{S}_k$-invariant global section of $(\mathcal{H})_{X^k}$ vanishes.

Proof. By [12, Theorem 3.6], the composition $\Psi^{-1} \circ \Phi : \mathrm{D^b}(X) \to \mathrm{D^b}(X^n)^{\mathfrak{S}_k}$ agrees with the truncated universal ideal functor defined in [13, Definition 5.1], therefore we have an exact triangle
(26)

$$(\Psi^{-1} \circ \Phi)(\mathcal{O}_X(e)) \longrightarrow H^*(\mathcal{O}_X(e)) \otimes \mathcal{O}_{X^k} \xrightarrow{\delta} \bigoplus_{i=1}^k q_i^* \mathcal{O}_X(e) \longrightarrow (\Psi^{-1} \circ \Phi)(\mathcal{O}_X(e))[1],$$

where each component of δ is an evaluation map. Since $\mathcal{H}$ is a locally free sheaf by Lemma 3.11, it follows by Lemma 3.12 that $\Psi^{-1}(\mathcal{H}) = (\mathcal{H})_{X^k}$. Hence

$$(\Psi^{-1} \circ \Phi)(\mathcal{O}_X(e)) = \Psi^{-1}(\mathcal{H})[-1] = (\mathcal{H})_{X^k}[-1].$$

Together with (16), the exact triangle (26) becomes the short exact sequence (25), which is the universal equivariant extension of $\mathcal{O}_{X^k}$ by $\bigoplus_{i=1}^k q_i^* \mathcal{O}_X(e)$ since δ is a collection of evaluation maps. Therefore its induced connecting map in the long exact sequence of cohomology groups

$$H^0\left(H^1(\mathcal{O}_X(e)) \otimes \mathcal{O}_{X^k}\right)^{\mathfrak{S}_k} \longrightarrow H^1\left(\bigoplus_{i=1}^k q_i^* \mathcal{O}_X(e)\right)^{\mathfrak{S}_k}$$

is naturally an isomorphism, which implies $H^0((\mathcal{H})_{X^k})^{\mathfrak{S}_k} = 0$. □

We will eventually prove the stability of E_x with respect to some ample class $H \in \mathrm{NS}(X^{[k]})$. Similar to the previous section we have

$$\mathrm{NS}(X^{[k]}) = \mathbb{Z}e_k \oplus \mathbb{Z}f_k \oplus \mathbb{Z}\delta.$$

For any $l \in \mathrm{NS}(X)$ and any ample class $h \in \mathrm{NS}(X)$ we have the intersection numbers

$$\begin{aligned} l_k h_k^{2k-1} &= \frac{(2k-1)!}{(k-1)!2^{k-1}}(lh)(h^2)^{k-1}, \\ \delta h_k^{2k-1} &= 0 \end{aligned}$$

by [21, Lemma 1.10]. Moreover, by Lemma 3.11 and [21, Lemma 1.5] we also have

$$c_1(E_x) = c_1(\mathcal{H}) = c_1(\mathcal{O}_X(e)^{[k]}) = e_k - \delta.$$

It follows by the above formulas that for any ample class $h \in \mathrm{NS}(X)$, we have

$$c_1(E_x)h_k^{2k-1} = \frac{(2k-1)!}{(k-1)!2^{k-1}}(eh)(h^2)^{k-1}.$$

However, $\mathcal{O}_X(e)^{[k]}$ is a subsheaf of E_x with the same c_1. For E_x to be μ_{h_k}-stable, it is necessary to have $eh < 0$ since $h^2 > 0$. An easy computation shows that this condition cannot be fulfilled by the class $h = e + (2k-1)f$ from Lemma 3.2, so we cannot hope that E_x is μ-stable with respect to the class h_k induced by this h. However, for the class $\widehat{h} = (2k)e + (2k-1)f$ from Lemma 3.2, we do have

$$\begin{aligned} e\widehat{h} &= (2k)e^2 + (2k-1)ef \\ &= -(4k^2) + (4k^2 - 1) = -1. \end{aligned}$$

Indeed, in the rest of this subsection we will prove that E_x is μ-stable with respect to $\widehat{h}_k$. We use the same notation as in Section 2.2 and also need the following formula: assume F is a coherent sheaf on X^k with $\mathfrak{S}_k$-invariant Chern class

$$c_1(F) = \sum_{i=1}^{k} q_i^* c$$

where $c \in \mathrm{NS}(X)$, then the intersection number

$$c_1(F)\widehat{h}_{X^k}^{2k-1} = a_k(c \cdot \widehat{h})(\widehat{h}^2)^{k-1}$$

where $a_k = \frac{k(2k-1)!}{2^{k-1}}$; see [21, Lemma 1.10]. The main result of this subsection is the following

Proposition 3.14. *E_x is μ-stable with respect to $\widehat{h}_k$.*

Proof. Assume that F is a reflexive subsheaf of E_x of rank $1 \leqslant r \leqslant 2k-2$. We need to show that $\mu_{\widehat{h}_k}(F) < \mu_{\widehat{h}_k}(E_x)$. By [20, Lemma 1.2], it suffices to check that

$$\mu_{\widehat{h}_{X^k}}((F)_{X^k}) < \mu_{\widehat{h}_{X^k}}((E_x)_{X^k}),$$

where $(F)_{X^k}$ is an $\mathfrak{S}_k$-invariant subsheaf of $(E_x)_{X^k}$.

We apply the functor $j_*(\overline{\sigma}^*_{k,\circ}((-)_\circ))$ to (22). Since the functor is left exact, together with [20, Lemma 1.1] we obtain that

$$0 \longrightarrow (\mathcal{H})_{X^k} \longrightarrow (E_x)_{X^k} \longrightarrow (I_{S_x})_{X^k} \longrightarrow Q \longrightarrow 0, \tag{27}$$

such that $\operatorname{supp}(Q) \subseteq \Delta$, where $\Delta = X^k \setminus X^k_\circ$ is the big diagonal. It is also clear that

$$\overline{\sigma}^*_{k,\circ}((I_{S_x})_\circ) = \left.\left(\bigotimes_{i=1}^k q_i^* I_x\right)\right|_{X^k\setminus\Delta}.$$

Since Δ is of codimension 2 in X^k, we have that $c_1((I_{S_x})_{X^k}) = 0$. It follows that

$$c_1((E_x)_{X^k}) = c_1((\mathcal{H})_{X^k}).$$

Moreover, we have by (25) that

$$c_1((\mathcal{H})_{X^k}) = \sum_{i=1}^k q_i^* e.$$

Therefore

$$\begin{aligned} c_1((E_x)_{X^k})\widehat{h}^{2k-1}_{X^k} &= c_1((\mathcal{H})_{X^k})\widehat{h}^{2k-1}_{X^k} \\ &= a_k(e\widehat{h})(\widehat{h}^2)^{k-1} \\ &= a_k(-1)(\widehat{h}^2)^{k-1}. \end{aligned}$$

Since $(F)_{X^k}$ is $\mathfrak{S}_k$-invariant, we have $c_1((F)_{X^k}) = \sum_{i=1}^k q_i^* c$ for some element $c \in \operatorname{NS}(X)$, and

$$c_1((F)_{X^k})\widehat{h}^{2k-1}_{X^k} = a_k(c\cdot\widehat{h})(\widehat{h}^2)^{k-1}.$$

We have the following two cases:

If $c \cdot \widehat{h} \leqslant -1$, then we have

$$c_1((F)_{X^k})\widehat{h}^{2k-1}_{X^k} \leqslant c_1((E_x)_{X^k})\widehat{h}^{2k-1}_{X^k} < 0.$$

Since $\operatorname{rk}((F)_{X^k}) < \operatorname{rk}((E_x)_{X^k})$, it follows that

$$\mu_{\widehat{h}_{X^k}}((F)_{X^k}) < \mu_{\widehat{h}_{X^k}}((E_x)_{X^k}).$$

If $c \cdot \widehat{h} \geqslant 0$, then $c_1((F)_{X^k})\widehat{h}^{2k-1}_{X^k} \geqslant 0$.

We choose a (not necessarily $\mathfrak{S}_k$-invariant) non-zero $\mu_{\widehat{h}_{X^k}}$-stable reflexive subsheaf of maximal slope $F' \subseteq (F)_{X^k}$, then $\mu_{\widehat{h}_{X^k}}(F') \geqslant 0$. However $q_i^*\,\mathcal{O}_X(e)$ is $\mu_{\widehat{h}_{X^k}}$-stable for $i = 1, \dots, k$, and

$$c_1(q_i^*\,\mathcal{O}_X(e))\widehat{h}^{2k-1}_{X^k} = a_k(e\widehat{h})(\widehat{h}^2)^{k-1} = a_k(-1)(\widehat{h}^2)^{k-1} < 0.$$

Hence the only map from F' to $q_i^*\,\mathcal{O}_X(e)$ is zero.

By (27) we obtain a morphism $F' \xrightarrow{\alpha} (I_{S_x})_{X^k}$. It is clear that $(I_{S_x})_{X^k}$ is torsion free, so it is a subsheaf of its double dual $(I_{S_x})^{\vee\vee}_{X^k}$. We also note that the restriction of $(I_{S_x})_{X^k}$ on $X^k \setminus (\Delta \cup q_1^{-1}(\{x\}) \cup \cdots \cup q_k^{-1}(\{x\}))$ is the trivial line bundle, hence

$$(I_{S_x})^{\vee\vee}_{X^k} = \mathcal{O}_{X^k}.$$

Therefore we have

$$F' \xrightarrow{\alpha} (I_{S_x})_{X^k} \hookrightarrow \mathcal{O}_{X^k}\ .$$

If $\alpha \neq 0$, then the composition of both maps is non-zero, hence the stability forces

$$\mu_{\widehat{h}_{X^k}}(F') = 0 = \mu_{\widehat{h}_{X^k}}(\mathcal{O}_{X^k}).$$

Since F' is reflexive, the composition must be the identity map. Since $(I_{S_x})_{X^k} \neq \mathcal{O}_{X^k}$ this is a contradiction. It follows that $\alpha = 0$, which implies by (27) that F' is a subsheaf of $(\mathcal{H})_{X^k}$. By (25) and the above discussion, we can furthermore conclude that F' is isomorphic to a subsheaf of the trivial bundle $H^1(\mathcal{O}_X(e)) \otimes \mathcal{O}_{X^k}$. The stability forces again that

$$\mu_{\widehat{h}_{X^k}}(F') = 0 = \mu_{\widehat{h}_{X^k}}(\mathcal{O}_{X^k})$$

and $F' \cong \mathcal{O}_{X^k}$. Moreover, since all global sections of the trivial bundle $H^1(\mathcal{O}_X(e)) \otimes \mathcal{O}_{X^k}$ in (25) are invariant under the permutation of $\mathfrak{S}_k$, we conclude that F' itself is also $\mathfrak{S}_k$-invariant, which gives a non-trivial $\mathfrak{S}_k$-invariant global section of $\mathcal{H}_{X^k}$. This contradicts Lemma 3.13, therefore $(E_x)_{X^k}$ cannot be destabilized by any $\mathfrak{S}_k$-invariant subsheaf, which concludes that E_x is $\mu_{\widehat{h}_k}$-stable. □

3.4. **A smooth connected component.** In this subsection, we will interpret the universal sheaf $\mathcal{E}$ as a family of stable sheaves on $X^{[n]}$ whose base is a smooth connected component of the corresponding moduli space. We have shown above that all the wrong-way fibers E_x of the family $\mathcal{E}$ are μ-stable with respect to $\widehat{h}_k$. We follow the idea in Theorem 2.8 to show their μ-stability with respect to a certain ample class near $\widehat{h}_k$.

Theorem 3.15. *There exists some ample class $H \in \mathrm{NS}(X^{[k]})$ near $\widehat{h}_k$, such that E_x is μ_H-stable for all $x \in X$ simultaneously.*

Proof. The same as in Theorem 2.8, the value of $c = \mu_\beta(E_x)$ is independent of the choice of $x \in X$. We still define

$$S := \{c_1(F) \mid F \subseteq E_x \text{ for some } x \in X \text{ such that } \mu_\beta(F) \geqslant c\}.$$

The proof of the present result is literally the same as the proof of Theorem 2.8, except that the step which shows that S is a finite set has to be modified.

For this purpose we make a few auxiliary definitions. Let $E'_x = \mathcal{H} \oplus I_{S_x}$ for each $x \in X$. We also define the set

$$S' := \{c_1(F') \mid F' \subseteq E'_x \text{ for some } x \in X \text{ such that } \mu_\beta(F') \geqslant c\}.$$

We claim that $S \subseteq S'$.

Indeed, by (22), every subsheaf $F \subseteq E_x$ is an extension of some subsheaf $F_2 \subseteq I_{S_x}$ by another subsheaf $F_1 \subseteq \mathcal{H}$. It is then clear that $F' = F_1 \oplus F_2$ is a subsheaf of E'_x, and that $c_1(F) = c_1(F')$. If F destabilizes E_x, then F' also destabilizes E'_x, which means that every element of S is also in S', as desired.

It remains to show that S' is finite. In fact, since $E'_x \subseteq (\mathcal{H} \oplus \mathcal{O}_{X^{[k]}})$ for all $x \in X$, we obtain that S' is a subset of

$$T' := \{c_1(F') \mid F' \subseteq (\mathcal{H} \oplus \mathcal{O}_{X^{[k]}}) \text{ such that } \mu_\beta(F') \geqslant c\},$$

which is finite by [7, Theorem 2.29], hence S' is also finite, which further implies the finiteness of S. This concludes the proof. $\square$

Let H be an ample class that satisfies Theorem 3.15, and $\mathcal{M}$ the moduli space of μ_H-stable sheaves on $X^{[k]}$ with the same numerical invariants as E_x. Then the universal family $\mathcal{E}$ defines a classifying morphism

$$f\colon X \longrightarrow \mathcal{M}, \quad x \longmapsto [E_x]. \tag{28}$$

Similar as Theorem 2.10, we obtain

Theorem 3.16. *The classifying morphism* (28) *defined by the family $\mathcal{E}$ identifies X with a smooth connected component of $\mathcal{M}$.*

Proof. For any pair of points $x, y \in X$, since Θ is an equivalence, we have

$$\mathrm{Ext}^*(F_x, F_y) \cong \mathrm{Ext}^*(\mathcal{O}_x, \mathcal{O}_y);$$

moreover by Remark 1.7 we have

$$\mathrm{Ext}^*(E_x, E_y) \cong \mathrm{Ext}^*(F_x, F_y) \otimes H^*(\mathbb{P}^{k-1}, \mathbb{C}).$$

It is clear that

$$\mathrm{Ext}^*_X(\mathcal{O}_x, \mathcal{O}_y) \cong \begin{cases} \Lambda^*(T_{X,x}) & \text{if } x = y \\ 0 & \text{if } x \neq y. \end{cases}$$

Combining the above computations we obtain

$$\begin{aligned} &\mathrm{Hom}_{X^{[k]}}(E_x, E_y) = 0 && \text{for any } x, y \in X \text{ with } x \neq y \\ \text{and} \quad &\mathrm{Ext}^1_{X^{[k]}}(E_x, E_x) \cong T_{X,x} && \text{for any } x \in X. \end{aligned}$$

These imply that f is injective on closed points and that $\dim(T_{[E_x]}\mathcal{M}) = 2$ for all $x \in X$. The claim then follows from an argument similar to the proof of Theorem 2.10. $\square$

Remark 3.17. The stable vector bundles constructed in Theorem 2.8 as well as Theorem 3.15 are not tautological bundles as the rank of a tautological bundle is always divisible by k, but in our cases the ranks are $k+1$ and $2k-1$.

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STABLE VECTOR BUNDLES ON GENERALIZED KUMMER VARIETIES

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Abstract. In this article we explicitly construct two new families of stable vector bundles on the generalized Kummer variety $K_n(A)$ for $n \geqslant 2$ associated to an abelian surface A. The first is the family of tautological bundles associated to stable bundles on A, and the second is the family of the "wrong-way" fibers of a universal family of stable bundles on the dual abelian surface $\widehat{A}$ parametrized by $K_n(A)$. Each family exhibits a smooth connected component in the moduli space of stable bundles on $K_n(A)$, which is holomorphic symplectic but not simply connected, contrary to the case of K3 surfaces.

Introduction

Background. Irreducible holomorphic symplectic manifolds are a type of building blocks in the classification of compact Kähler manifolds with trivial first Chern class. In the very influential paper [3], Beauville constructed two classes of irreducible holomorphic symplectic manifolds, which are the Hilbert schemes $X^{[n]}$ of n-points on K3 surfaces X, and the generalized Kummer varieties $K_n(A)$ associated to abelian surfaces A, obtained as the zero fibers of the summation map $\Sigma : A^{[n+1]} \to A$. The second construction was later generalized by Yoshioka in [27], in which he proved that the fibers $K_H(v)$ of the Albanese morphism $\mathfrak{a}_v : M_H(v) \to A \times \widehat{A}$ for moduli spaces $M_H(v)$ of μ_H-stable sheaves on A with Mukai vector v are deformation equivalent to generalized Kummer varieties.

Main results. The present manuscript is a continuation of the authors' work [22, 23] on the construction of new stable sheaves on irreducible holomorphic symplectic manifolds. The same problem was also studied by various authors, such as in [24, 25, 26]. Recently, Markman [17] and O'Grady [21] also found examples of stable bundles among modular sheaves on K3$^{[n]}$s. The main purpose of this manuscript is to construct new stable vector bundles on generalized Kummer varieties and study some of their properties. We achieved two different constructions.

A natural family of vector bundles on $K_n(A)$ for considering stability are the so-called tautological bundles. In [25] Wandel constructed some examples of tautological bundles on $K_n(A)$ for $n = 1, 2$. Following an idea of Stapleton [24], we generalize Wandel's results by proving that in fact all taulogical bundles on $K_n(A)$ for $n \geqslant 2$ are stable with respect to suitable amples classes. Moreover, under suitable numerical assumptions, we show

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that the tautological bundles form a connected component of the moduli space of stable bundles on $K_n(A)$. This in particular indicates that smooth components of moduli spaces of sheaves on generalized Kummer varieties need not be irreducible holomorphic symplectic manifolds, contrary to the result for K3 surfaces, see [14, Theorem 10.3.10].

For another family of vector bundles on $K_n(A)$, we use the standard Fourier-Mukai transform to construct a fine moduli space $M_{\widehat{H}}(w)$ of stable vector bundles of rank $r \geqslant n+2$ on the dual abelian variety $\widehat{A}$ for some suitable choice of the Mukai vector w, such that $M_{\widehat{H}}(w) \cong A^{[n+1]} \times \widehat{A}$. Then $K_n(A)$ is naturally isomorphic to the zero fiber of the Albanese morphism $\mathfrak{a}_w : M_{\widehat{H}}(w) \to A \times \widehat{A}$. Let $\mathcal{U}$ be the restriction of the universal family on $\widehat{A} \times M_{\widehat{H}}(w)$ to the closed subscheme $\widehat{A} \times K_n(A)$. For each closed point $\widehat{a} \in \widehat{A}$, the further restriction of $\mathcal{U}$ to the slice $\{\widehat{a}\} \times K_n(A)$ gives a vector bundle $\mathcal{U}_{\widehat{a}}$ on $K_n(A)$. Following our approach in [23], we show that each $\mathcal{U}_{\widehat{a}}$ is a stable bundle on $K_n(A)$, hence we obtain a family of stable bundles on $K_n(A)$ parametrized by $\widehat{A}$.

Our main results can be summarized as follows:

Theorem. *Let (A, H) be a polarized abelian surface, and $(\widehat{A}, \widehat{H})$ its dual.*

(1) *(Theorem 1.7) Let E be a μ_H-stable vector bundle of class v on A with $E \neq \mathcal{O}_A$, then the tautological bundle $E^{(n)}$ is a μ_D-stable vector bundle on $K_n(A)$ with respect to some ample divisor D on $K_n(A)$. Moreover, under suitable numerical assumptions on v, the moduli space $M_H(v)$ of μ_H-stable vector bundles of class v on A can be embedded as a connected component of some moduli space of μ_D-stable vector bundles on $K_n(A)$.*

(2) *(Theorem 2.12) Let $\mathcal{U}$ be the restriction of the universal vector bundle on $\widehat{A} \times M_{\widehat{H}}(w)$ to the closed subscheme $\widehat{A} \times K_n(A)$ as described above. Then for each closed point $\widehat{a} \in \widehat{A}$, the fiber $\mathcal{U}_{\widehat{a}}$ is a μ_D-stable bundle on $K_n(A)$ with respect to an ample divisor D. Moreover, $\widehat{A}$ can be embedded as a connected component of a moduli space of μ_D-stable vector bundles on $K_n(A)$.*

Sketch of proof. Let us give a quick overview on how we achieved the above results. Although the setup in both cases looks very different, we will follow a similar chain of ideas to prove the slope stability of the bundles $E^{(n)}$ (resp. $\mathcal{U}_{\widehat{a}}$) with respect to some ample divisor D on $K_n(A)$. The proof consists of the following three major steps.

Step 1. To begin with, let $P_n(A)$ be the codimension 2 subvariety of A^{n+1} parametrizing $(n+1)$-tuples whose components add up to zero under the group law of A. Each bundle $E^{(n)}$ (resp. $\mathcal{U}_{\widehat{a}}$) defines uniquely a reflexive sheaf on $P_n(A)$. We adapt the technique developed by Stapleton in [24] to prove the slope stability of $E^{(n)}$ (resp. $\mathcal{U}_{\widehat{a}}$) with respect to a natural nef divisor H_K on $K_n(A)$ by showing that the corresponding reflexive sheaf cannot be destabilized by any $\mathfrak{S}_{n+1}$-invariant subsheaf on $P_n(A)$. (See Propositions 1.5 and 2.10.)

Step 2. In order to show the slope stability of $E^{(n)}$ (resp. $\mathcal{U}_{\widehat{a}}$) with respect to an ample divisor on $K_n(A)$, we use the openness of stability to perturb H_K to a nearby ample divisor D. This perturbation argument was developed in [9, 24], and generalized in [23]. The main difficulty for our application is to show the existence of the ample divisor D independent of the choice of E in its own moduli $M_H(v)$ (resp. the choice of the fiber $\mathcal{U}_{\widehat{a}}$ in the family $\mathcal{U}$ parametrized by $\widehat{A}$). (See Propositions 1.6 and 2.11.)

STEP 3. Finally, in order to identify $M_H(v)$ (resp. $\widehat{A}$) as a smooth connected component of the moduli space of μ_D-stable sheaves on $K_n(A)$, we interpret $E^{(n)}$ (resp. $\mathcal{U}_{\widehat{a}}$) as the image of E (resp. a line bundle on A) under an integral functor Θ induced by the structure sheaf (resp. the ideal sheaf) of the universal subscheme for $K_n(A)$. By a result of Meachan [18], we can apply the technique of $\mathbb{P}$-functors invented in [1, 2] to compute the relevant cohomology groups, which lead to our conclusion. (See Theorems 1.7 and 2.12.)

Structure of text. The text is organized in two sections, which deal with the two cases mentioned above respectively. All objects are defined over the field of complex numbers $\mathbb{C}$.

1. Tautological Bundles

1.1. **Notations.** For any integer $n \geqslant 2$, let A be an abelian surface, $K_n(A)$ the generalized Kummer variety of dimension $2n$, and $\mathcal{Z} \subset A \times K_n(A)$ the corresponding universal family. The projections from $\mathcal{Z}$ to the two factors A and $K_n(A)$ are denoted by p and q respectively. The following diagram exhibits the relations of some relevant schemes:

$$\begin{tikzcd} P_n(A)_\circ \arrow[r, "\sigma_\circ"] \arrow[d, hook, "j_P"'] & S_n(A)_\circ \arrow[d, hook, "j_S"'] & K_n(A)_\circ \arrow[l, "h_\circ"'] \arrow[d, hook, "j_K"'] \\ P_n(A) \arrow[r, "\sigma"] \arrow[d, hook, "\tau"'] & S_n(A) \arrow[d, hook] & K_n(A) \arrow[l, "h"'] \arrow[d, hook, "\iota"'] \\ A^{n+1} \arrow[r, "\overline{\sigma}"] & A^{(n+1)} & A^{[n+1]} \arrow[l, "\overline{h}"'] \end{tikzcd} \tag{1}$$

Each vertical arrow in the lower half of the diagram is the embedding of a zero fiber of the addition morphism to A; σ and $\overline{\sigma}$ are quotients by the symmetric group $\mathfrak{S}_{n+1}$; h and $\overline{h}$ are Hilbert-Chow morphisms. Moreover, we denote the projections from A^{n+1} to each individual factor by $q_0, q_1, \cdots, q_n$.

Each vertical arrow in the upper half of the diagram is the embedding of an open subscheme parametrizing $n+1$ distinct (ordered or unordered) points in A. It is clear that the complement of each of these embeddings is a closed subscheme of codimension 2. The morphisms $\sigma_\circ$ and $h_\circ$ are restrictions of the morphisms in the second row. Clearly $\sigma_\circ$ is a free $\mathfrak{S}_{n+1}$-quotient and $h_\circ$ is an isomorphism.

Let H be an ample divisor on A. For each $0 \leqslant i \leqslant n$, we define the class $h_i = \tau^* q_i^* H$. Then $H_P = \sum_{i=0}^n h_i$ on $P_n(A)$ is an $\mathfrak{S}_{n+1}$-invariant ample divisor on $P_n(A)$, which descends to an ample divisor H_S on $S_n(A)$, whose pullback $H_K = h^*(H_S)$ is a big and nef divisor on $K_n(A)$. For any $E \in \mathrm{Coh}(A)$, the corresponding *tautological sheaf* $E^{(n)}$ on $K_n(A)$ is defined by $E^{(n)} = q_* p^* E$. Moreover, we write $E_i = \tau^* q_i^* E$ for each $0 \leqslant i \leqslant n$. The goal of this section is to show that if E is a non-trivial μ_H-stable vector bundle on A, then $E^{(n)}$ is slope stable with respect to some ample divisor sufficiently close to H_K. Our approach will mainly follow the idea in [24].

1.2. **Pullback of stable bundles.** We aim to prove Proposition 1.4, which is an analogue of [24, Proposition 4.7] in the Kummer case. We first collect necessary notations and tools required in the course of the proof, following [24, §4].

For any normal projective variety X, let $\gamma \in N_1(X)_{\mathbb{R}}$ be a curve class and $\mathcal{E} \in \mathrm{Coh}(X)$ a torsion-free sheaf. The slope of $\mathcal{E}$ with respect to γ is defined as

$$\mu^{\gamma}(\mathcal{E}) = \frac{c_1(\mathcal{E}) \cdot \gamma}{\operatorname{rk} \mathcal{E}}.$$

It is clear that the slope is linear with respect to γ, and the usual notion of slope $\mu_H(\mathcal{E}) = \mu^{H^{d-1}}(\mathcal{E})$ for any ample class $H \in N^1(X)_{\mathbb{R}}$, where we have $n = \dim X$. The new notion of slope defines a *slope stability* (resp. *semistability*) of $\mathcal{E}$ with respect to a curve class γ, by requiring any torsion-free quotient of $\mathcal{E}$ of a smaller rank to have a smaller (resp. smaller or equal) slope with respect to γ.

The main advantage of studying slope stability with respect to curve classes is the linearity of slopes with respect to the curve parameter. More precisely, we have

Lemma 1.1 ([24, Lemma 4.4]). *Let $\gamma, \delta \in N_1(X)_{\mathbb{R}}$ such that $\mathcal{E}$ is semistable with respect to γ and stable with respect to δ, then $\mathcal{E}$ is stable with respect to $a\gamma + b\delta$ for any $a, b > 0$.* □

Our main tool for determining the slope stability is the following observation

Lemma 1.2 ([24, Corollary 4.6]). *Let $\pi : C_T \to T$ be a family of smooth irreducible closed curves in X with class γ. Suppose that $\mathcal{E}$ is a vector bundle on X such that $\mathcal{E}|_{C_t}$ is stable for all $t \in T$, and that the curves in C_T are dense in X, then $\mathcal{E}$ is stable with respect to the curve class γ. Moreover, the statement also holds if stability is replaced by semistability.* □

The following lemma will be required in the proof of our main result

Lemma 1.3. *Let $S : A^3 \to A$ be the addition with respect to the group law on A, and $q_i : A^3 \to A$ the projection to the i-th factor for $1 \leqslant i \leqslant 3$. Assume H is a sufficiently ample divisor on A. Let $C_i \in |H|$ for $1 \leqslant i \leqslant 3$. Then*

1. *For any fixed point $b \in A$, the scheme theoretic intersection of $S^{-1}(b)$, $q_1^{-1}(C_1)$, $q_2^{-1}(C_2)$, $q_3^{-1}(C_3)$ is a smooth curve C for a generic choice of C_i for $1 \leqslant i \leqslant 3$;*
2. *Each projection $q_i : C \to C_i$ is a finite morphism for $1 \leqslant i \leqslant 3$.*

Proof. For the first statement we use Bertini theorem; see e.g. [7, Proposition 0.5]. We first observe that the addition morphism S is smooth, hence $Y_0 = S^{-1}(b)$ is smooth and irreducible. By assumption the complete linear system $|H|$ has no base point, so does $q_1^{-1}(H)|_{Y_0}$. Hence a generic choice of $C_1 \in |H|$ and $Y_1 = Y_0 \cap q_1^{-1}(C_1)$ are smooth and irreducible by Bertini theorem. For the same reason $q_2^{-1}(H)|_{Y_1}$ has no base point, hence a generic choice of $C_2 \in |H|$ and $Y_2 = Y_1 \cap q_2^{-1}(C_2)$ are smooth and irreducible. Similarly, a generic choice of $C_3 \in |H|$ and $C = Y_2 \cap q_3^{-1}(C_3)$ are smooth and irreducible. A dimension count shows that C is a curve.

For the second statement, since C is smooth irreducible, and the projection q_i is surjective, it follows by [11, Proposition II.6.8] that q_i is a finite morphism. □

It was proven in [24, Proposition 4.7] that the pullback of a slope stable bundle E from A to A^{n+1} via the projection to any factor is stable with respect to a natural $\mathfrak{S}_{n+1}$-invariant ample class. The following proposition shows that a further restriction to $P_n(A)$, the zero fiber of the addition morphism, remains stable.

Proposition 1.4. *Under the above notations, let E be a μ_H-stable bundle on A. Then E_i is a μ_{H_P}-stable bundle on $P_n(A)$ for each $0 \leqslant i \leqslant n$.*

Proof. Without loss of generality, we prove the result for $i = 0$. Moreover, by replacing H with a high tensor power of itself, we can assume that a generic element $C \in |H|$ in the linear system is a smooth curve such that $E|_C$ is slope stable.

Using the notation of slope stability with respect to a real curve class defined in the beginning of the subsection, we need to show that E_0 is slope stable with respect to the curve class H_P^{2n-1}. We expand the product to obtain

$$H_P^{2n-1} = \sum_{\substack{k_0+\cdots+k_n=2n-1 \\ 0\leqslant k_0,\cdots,k_n\leqslant 2}} c_{k_0\cdots k_n} h_0^{k_0}\cdots h_n^{k_n} \tag{2}$$

where each $c_{k_0\cdots k_n}$ is some positive integer. We will analyze the slope stability of E_0 with respect to each term on the right-hand side of (2). Without loss of generality, by permuting the indices $1 \leqslant i \leqslant n$, we need to consider the following 5 cases:

Case 1. $k_0 = 2$, $k_1 = 1$, $k_2 = 0$, and $k_i = 2$ for each $i \geqslant 3$. We consider the family of curves in $P_n(A)$ given by intersecting

$$\{a_0\} \times C_1 \times A \times \{a_3\} \times \cdots \times \{a_n\}$$

with $P_n(A)$ in A^{n+1}, where $a_0, a_3, \cdots, a_n \in A$ and $C_1 \in |H|$. Each curve C in this family lies in the class $h_0^{k_0}\cdots h_n^{k_n}/(H^2)^{n-1}$, and is isomorphic to C_1 via the projection q_1. Therefore a generic choice of C is smooth, and it is clear that $E_0|_C$ is a trivial bundle, hence is slope semistable. We claim that all such curves C cover a dense subset of $P_n(A)$. Indeed, the projection $q' = (q_0, q_1, q_3, \cdots, q_n)$ identifies $P_n(A)$ with A^n, and C with $\{a_0\} \times C_1 \times \{a_3\} \times \cdots \times \{a_n\}$. Since we assume that H is very ample, there is a subset $F \subset |H|$ parametrizing smooth irreducible curves $C_1 \in |H|$, such that the union $U = \bigcup_{C_1\in F} C_1$ is dense in A. By varying $a_0, a_3, \cdots, a_n$, we obtain a family of smooth and irreducible curves in A^n parametrized by $A\times F\times A^{n-2}$, whose union $A\times U\times A^{n-2}$ is dense in A^n. Then the preimages of these curves via q' are smooth irreducible curves in $P_n(A)$, whose union is dense in $P_n(A)$, as desired. Therefore E_0 is slope semistable with respect to the curve class $h_0^{k_0}\cdots h_n^{k_n}$ by Lemma 1.2.

Case 2. $k_0 = 2$, $k_1 = k_2 = k_3 = 1$, and $k_i = 2$ for each $i \geqslant 4$. We consider the family of curves in $P_n(A)$ given by intersecting

$$\{a_0\} \times C_1 \times C_2 \times C_3 \times \{a_4\} \times \cdots \{a_n\}$$

with $P_n(A)$ in A^{n+1}, where $a_0, a_4, \cdots, a_n \in A$ and $C_1, C_2, C_3 \in |H|$. Each curve C in this family lies in the class $h_0^{k_0} \cdots h_n^{k_n}/(H^2)^{n-2}$. By Lemma 1.3, a generic choice of C is smooth, and it is clear that $E_0|_C$ is a trivial bundle, hence is slope semistable. All such curves C cover a dense subset of $P_n(A)$. Therefore E_0 is slope semistable with respect to the curve class $h_0^{k_0} \cdots h_n^{k_n}$ by Lemma 1.2.

Case 3. $k_0 = 1$, $k_1 = 0$, and $k_i = 2$ for each $i \geqslant 2$. We consider the family of curves in $P_n(A)$ given by intersecting $C_0 \times A \times \{a_2\} \times \cdots \{a_n\}$ with $P_n(A)$ in A^{n+1}, where $a_2, \cdots, a_n \in A$ and $C_0 \in |H|$. Each curve C in this family lies in the class $h_0^{k_0} \cdots h_n^{k_n}/(H^2)^{n-1}$, and is isomorphic to C_0 via the projection q_0. Therefore a generic choice of C is smooth, and $E_0|_C$ is isomorphic to $E|_{C_0}$, which by the assumption on H is slope stable for a generic choice of C_0. All such curves C cover a dense subset of $P_n(A)$. Therefore E_0 is slope stable with respect to the curve class $h_0^{k_0} \cdots h_n^{k_n}$ by Lemma 1.2.

Case 4. $k_0 = k_1 = k_2 = 1$, and $k_i = 2$ for each $i \geqslant 3$. We consider the family of curves in $P_n(A)$ given by intersecting $C_0 \times C_1 \times C_2 \times \{a_3\} \times \cdots \times \{a_n\}$ with $P_n(A)$ in A^{n+1}, where $a_3, \cdots, a_n \in A$ and $C_0, C_1, C_2 \in |H|$. Each curve C in this family lies in the class $h_0^{k_0} \cdots h_n^{k_n}/(H^2)^{n-2}$. By Lemma 1.3, a generic choice of C is smooth, and the projection gives a finite morphism $\varphi : C \to C_0$ such that $E_0|_C = \varphi^*(E|_{C_0})$. We know $E|_{C_0}$ is slope stable by the assumption on H for a generic choice of C_0. It follows by [15, Lemma 3.2.3] that $E_0|_C$ is slope semistable. All such curves C cover a dense subset of $P_n(A)$. Therefore E_0 is slope semistable with respect to the curve class $h_0^{k_0} \cdots h_n^{k_n}$ by Lemma 1.2.

Case 5. $k_0 = 0$, $k_1 = 1$, and $k_i = 2$ for each $i \geqslant 2$. We consider the family of curves in $P_n(A)$ given by intersecting $A \times C_1 \times \{a_2\} \times \cdots \{a_n\}$ with $P_n(A)$ in A^{n+1}, where $a_2, \cdots, a_n \in A$ and $C_1 \in |H|$. Each curve C in this family lies in the class $h_0^{k_0} \cdots h_n^{k_n}/(H^2)^{n-1}$, and is isomorphic to C_1 via the projection q_1. Therefore a generic choice of C is smooth. For any fixed choice of $a_2, \cdots, a_n \in A$, let ι be the inverse morphism on A, $b = -(a_2 + \cdots + a_n)$ (with respect to the group law on A), and $t_b : A \to A$ is the corresponding translation. Then $E_0|_C$ is isomorphic to $(t_b^*\iota^*E)|_{C_1}$. Since E is μ_H-stable, $t_b^*\iota^*E$ is also μ_H-stable. Hence $(t_b^*\iota^*E)|_{C_1}$ is slope stable for a generic choice of C_1, and the corresponding curves C cover a dense subset of the intersection of $A \times A \times \{a_2\} \times \cdots \times \{a_n\}$ with $P_n(A)$. When we allow $a_2, \cdots, a_n$ to move in A, it follows that $E_0|_C$ is slope stable for a generic choice of the curve C as described above, and such curves cover a dense subset of $P_n(A)$. Therefore E_0 is slope stable with respect to the curve class $h_0^{k_0} \cdots h_n^{k_n}$ by Lemma 1.2.

Applying Lemma 1.1, the above cases together implies that E_0 is slope stable with respect to the curve class H_P^{2n-1}; in other words, E_0 is μ_{H_P}-stable. $\square$

1.3. **Tautological bundles.** For any torsion free coherent sheaf F on $K_n(A)$, we follow [24, §1] to define an $\mathfrak{S}_{n+1}$-invariant coherent sheaf on $P_n(A)$ by

$$(F)_P = (j_P)_*\sigma_\circ^*(h_\circ^{-1})^* j_K^* F,$$

which is reflexive if F itself is reflexive. Moreover, we observe that an analogue of [24, Lemma 1.2], namely

$$(n+1)! \int_{K_n(A)} c_1(F)\cdot (H_K)^{2n-1} = \int_{P_n(A)} c_1((F)_P)\cdot (H_P)^{2n-1} \tag{3}$$

holds due to the relevant diagonals having codimension 2. The following result is an analogue of [24, Theorem 1.4] in the Kummer case.

Proposition 1.5. *Let E be a μ_H-stable bundle on A not isomorphic to $\mathcal{O}_A$, then $E^{(n)}$ is a μ_{H_K}-stable bundle on $K_n(A)$.*

Proof. It suffices to show that every reflexive subsheaf of $E^{(n)}$ of smaller rank has a smaller slope. Let F be such a subsheaf of $E^{(n)}$, then $(F)_P$ is an $\mathfrak{S}_{n+1}$-invariant reflexive subsheaf of $(E^{(n)})_P$. Using equation (3), it is enough to prove $\mu_{H_P}((F)_P) < \mu_{H_P}((E^{(n)})_P)$. Let G be a non-zero (not necessarily $\mathfrak{S}_{n+1}$-invariant) μ_{H_P}-stable subsheaf of $(F)_P$ of maximal slope; e.g., we can take G to be the first factor in a Jordan-Hölder filtration of $(F)_P$. A similar argument as in [24, Lemma 1.1] shows that $(E^{(n)})_P = E_0 \oplus \cdots \oplus E_n$. (Both sides are reflexive sheaves and isomorphic on $P_n(A)_\circ$ whose complement is of codimension 2.) Therefore, there exists some i such that the composition of the embedding and projection

$$G \hookrightarrow (F)_P \hookrightarrow (E^{(n)})_P \twoheadrightarrow E_i \tag{4}$$

is non-zero. Since E_i is also μ_{H_P}-stable for each $0 \leqslant i \leqslant n$ by Proposition 1.4, we must have $\mu_{H_P}(G) \leqslant \mu_{H_P}(E_i)$.

Case 1. If $\mu_{H_P}(G) < \mu_{H_P}(E_i)$, then

$$\mu_{H_P}((F)_P) \leqslant \mu_{H_P}(G) < \mu_{H_P}(E_i) = \mu_{H_P}((E^{(n)})_P),$$

hence $(F)_P$ does not destabilize $(E^{(n)})_P$.

Case 2. If $\mu_{H_P}(G) = \mu_{H_P}(E_i)$, then the composition map (4) must be an isomorphism. Since $E \ncong \mathcal{O}_A$, we have $E_i \ncong E_j$ for $i \neq j$. (Choose any k different from i and j, then the projection $q'' = (q_0, \cdots, q_{k-1}, q_{k+1}, \cdots, q_n)$ identifies $P_n(A)$ with A^n. The pullback of a non-trivial sheaf via projections to two distinct factors are not isomorphic.) It follows that the composition

$$G \hookrightarrow (F)_P \hookrightarrow (E^{(n)})_P \twoheadrightarrow E_j$$

must be zero for any $j \neq i$. It follows that G is the direct summand E_i of $(E^{(n)})_P$. Since $(F)_P$ is an $\mathfrak{S}_{n+1}$-invariant subsheaf of $(E^{(n)})_P$ containing the direct summand E_i, we obtain $(F)_P = (E^{(n)})_P$, which cannot happen since the left-hand side has a smaller rank than the right-hand side. This concludes the proof. $\square$

In the following we will apply the perturbation argument in [24, §4] to show the slope stability of $E^{(n)}$ with respect to an ample divisor on $K_n(A)$, which stays constant when we deform E in its moduli.

1.4. **The family of stable bundles.** In this subsection we study families of stable tautological bundles. We assume that

$$v = (v_0, v_1, v_2) \in H^0(A, \mathbb{Z}) \oplus \mathrm{NS}(A) \oplus H^4(A, \mathbb{Z})$$

is a Mukai vector satisfying the condition

(†) the projective moduli space $M_H(v)$ of H-semistable sheaves of class v is non-empty and contains only μ_H-stable locally free sheaves.

This condition is easy to achieve: first of all we require $v_0 > 0$; in order for every μ_H-semistable sheaf of class v to be stable, it suffices to require that v_0 and $H \cdot v_1$ are coprime; the nonemptiness can be achieved by requiring

$$\langle v^2 \rangle = v_1^2 - 2v_0v_2 \geqslant 0; \tag{5}$$

finally the local freeness of all μ_H-stable sheaves holds when v_2 takes the largest possible value satisfying (5) for any fixed v_0 and v_1. For instance, if A is an abelian surface with a primitive ample divisor H such that $H^2 = 16$, then the Mukai vector $v = (5, H, 1)$ satisfies the condition (†).

Under the condition (†), we have seen by Proposition 1.5 that the tautological bundle $E^{(n)}$ is μ_{H_K}-stable for each $E \in M_H(v)$. However, H_K lies in the boundary of the ample cone of $K_n(A)$. In order to establish the stability of the tautological bundle with respect to some ample class, we need the following result

Proposition 1.6. *Under condition* (†)*, there exists an ample class* $H' \in \mathrm{NS}(K_n(A))$ *near* H_K*, such that* $E^{(n)}$ *is* $\mu_{H'}$*-stable for all* $[E] \in M_H(v)$.

Proof. By replacing H with a high tensor power of itself if necessary, we assume the complete linear system $|H_K|$ defines (the restriction of) the Hilbert-Chow morphism $h : K_n(A) \to S_n(A)$ as shown in (1). We claim that h is semismall. Indeed, for any partition ξ given by

$$n + 1 = 1 \cdot n_1 + 2 \cdot n_2 + \cdots + r \cdot n_r,$$

we consider the locally closed subscheme $Y_\xi \subset S_n(A)$ parametrizing $n_1 + n_2 + \cdots + n_r$ distinct points, among which are n_i points of multiplicity i for $1 \leqslant i \leqslant r$. We have $\dim Y_\xi = 2(n_1 + n_2 + \cdots + n_r) - 2$ and $\dim h^{-1}(y) = 0 \cdot n_1 + 1 \cdot n_2 + \cdots + (p-1) \cdot n_p$ for any closed point $y \in Y$. It follows that $\dim Y_\xi + 2 \dim h^{-1}(y) = 2n = \dim K_n(A)$ which implies that h is semismall. Therefore H_K is lef by [6, Definition 2.1.3] and satisfies the hard Lefschetz property by [6, Theorem 2.3.1]. It then follows from [24, Proposition 4.8] that each $E^{(n)}$ is $\mu_{H'}$-stable with respect to some ample class H' near H_K. However, in order to find a single H' that works simultaneously for all $E^{(n)}$, we can apply the entire proof of [24, Proposition 4.8] except one step; namely, we need to find a convex open set U such that $\alpha := H_K^{2n-1}$ is in the closure of U, and for each $\gamma \in U$, the tautological bundle $E^{(n)}$ is stable with respect to γ for all $[E] \in M_H(v)$.

We follow the notations in [9, Definition 3.1]. For each $[E] \in M_H(v)$, $\mathrm{SStab}(E^{(n)})$ is a convex closed set containing α. Hence the intersection

$$\overline{U} := \bigcap_{[E] \in M_H(v)} \mathrm{SStab}(E^{(n)})$$

is also a convex closed set containing α.

We claim that [9, Theorem 3.4] holds for all $E^{(n)}$ simultaneously; namely, we will show that for any $\beta \in \mathrm{Mov}(K_n(A))^\circ$ (see [9, Definition 2.1] for the notation), there exists some $e \in \mathbb{Q}^+$, such that $(\alpha+\varepsilon\beta) \in \bigcap_{[E]\in M_H(v)} \mathrm{Stab}(E^{(n)})$ for any real $\varepsilon \in [0, e]$.

To prove the claim, we first note by [8, p.87, Lemma 5(v)] that E is μ_H-stable of class $v = (v_0, v_1, v_2)$ if and only if $E^\vee$ is μ_H-stable of class $v^\vee = (v_0, -v_1, v_2)$. Since μ_H-stable sheaves of class $v^\vee$ are bounded, there exists some positive integer m such that $E^\vee(mH)$ is globally generated and $H^i(A, E^\vee(mH)) = 0$ for all $i > 0$, hence there exists a surjective map $\mathcal{O}_A(-mH)^{\oplus N} \twoheadrightarrow E^\vee$, where m and N are independent of E. Since E is locally free, we can take the dual of the above surjective map to obtain an injective map $E \hookrightarrow \mathcal{O}_A(mH)^{\oplus N}$, and complete it to an exact sequence

$$0 \longrightarrow E \longrightarrow \mathcal{O}_A(mH)^{\oplus N} \longrightarrow Q_E \longrightarrow 0.$$

We apply the functor $q_* \circ p^*$ on the above sequence. It was proven in [22, Lemma 3.1] that the morphism p is flat for $n \geqslant 2$, hence the functor p^* is exact. The morphism q is finite thus q_* is also exact. Therefore we obtain an exact sequence

$$0 \longrightarrow E^{(n)} \longrightarrow q_*p^*\mathcal{O}_A(mH)^{\oplus N} \longrightarrow q_*p^*Q_E \longrightarrow 0. \tag{6}$$

We note that the slope $c := \mu_\beta(E^{(n)})$ is independent of of $[E] \in M_H(v)$, and redefine the set S in the proof of [9, Theorem 3.4] to be

$$S := \{c_1(F) \mid F \subseteq E^{(n)} \text{ for some } [E] \in M_H(v) \text{ such that } \mu_\beta(F) \geqslant c\}.$$

By (6) we see that S is a subset of

$$T := \{c_1(F) \mid F \subseteq q_*p^*\mathcal{O}_A(mH)^{\oplus N} \text{ such that } \mu_\beta(F) \geqslant c\},$$

which is finite by [9, Theorem 2.29]. Thus S is a also a finite set. We can then apply the rest of the proof of [9, Theorem 3.4] literally to conclude the claim.

Now we can show that $\overline{U}$ is of full dimension $r := \mathrm{rk}\, N_1(K_n(A))$. If not, then we have $\alpha \in \overline{U} \subseteq L$ for some hyperplane $L \subset N_1(K_n(A))_\mathbb{R}$. Since $\mathrm{Mov}(K_n(A))$ is of full dimension, we can choose some $\beta \in \mathrm{Mov}(K_n(A))^\circ \setminus L$. It follows that $(\alpha + \varepsilon\beta) \in \overline{U} \setminus L$ for some small $\varepsilon > 0$ by the above claim and the choice of β. Contradiction.

We define U to be the interior of $\overline{U}$ and claim that U is non-empty. Indeed, since $\overline{U}$ is of full dimension r, we can choose $r + 1$ points of $\overline{U}$ in general positions, which form an r-simplex. By the convexity of $\overline{U}$, the entire simplex is in $\overline{U}$ hence any interior point of the simplex is also an interior point of $\overline{U}$. The convexity of U follows from the convexity of $\overline{U}$. And it is clear from the construction that $\alpha = H_K^{2n-1}$ is in the closure of U.

We finally prove that $U \subseteq \bigcap_{[E]\in M_H(v)} \mathrm{Stab}(E^{(n)})$. If not, suppose that there is some $\gamma_0 \in U$ and some $[E] \in M_H(v)$, such that we have in fact $\gamma_0 \in \mathrm{SStab}(E^{(n)}) \setminus \mathrm{Stab}(E^{(n)})$; namely, $\mu_{\gamma_0}(F) = \mu_{\gamma_0}(E^{(n)})$ for some proper subsheaf F of $E^{(n)}$. Since the slope function is linear with respect to the curve class, and $\mu_\alpha(F) < \mu_\alpha(E^{(n)})$ by Proposition 1.4, one can find a hyperplane in $N^1(K_n(A))_\mathbb{R}$ through γ_0, such that $\mu_\gamma(E^{(n)}) - \mu_\gamma(F)$ takes opposite signs for γ in the two open half-spaces separated by the hyperplane. In particular, F destabilizes $\mathcal{U}_{\widehat{a}_0}$ in one of the half-spaces. Since U has non-empty intersection with both half-spaces, this contradicts the condition $U \subseteq \mathrm{SStab}(E^{(n)})$. Therefore we have $U \subseteq \bigcap_{[E]\in M_H(v)} \mathrm{Stab}(E^{(n)})$, as desired. $\square$

1.5. **A component of the moduli space.** In this subsection, we show that under some favorable numerical conditions, $M_H(v)$ is isomorphic to a connected component of a moduli space of stable sheaves on $K_n(A)$.

Indeed, we still assume that v satisfies condition (†); or more precisely, the numerical conditions in the paragraph below (†) that ensure its validity. We further assume that

(‡) for every $[E] \in M_H(v)$, we have $H^i(A, E) = 0$ for $i > 0$.

This condition is also easy to achieve. Since all stable sheaves are bounded, there exists some positive integer m independent of the choice of E, such that $H^i(A, E(mH)) = 0$ for all $i > 0$. By replacing v with $v \cdot \mathrm{ch}(mH)$, we obtain a Mukai vector v satisfying both (†) and (‡).

Under the above assumptions, let H' be the ample line bundle constructed in Proposition 1.6, and $\mathcal{M}$ the moduli space of $\mu_{H'}$-stable sheaves on $K_n(A)$ with the same numerical invariants as $E^{(n)}$. By applying Proposition 1.6, the integral functor $q_* \circ p^*$ induces a morphism

$$f : M_H(v) \longrightarrow \mathcal{M}, \quad [E] \longmapsto [E^{(n)}]. \tag{7}$$

In fact the morphism f can be described as follows:

Theorem 1.7. *Under the assumptions* (†) *and* (‡)*, the classifying morphism* (7) *identifies $M_H(v)$ with a smooth connected component of $\mathcal{M}$.*

Proof. By [22, Lemma 1.6.] we have to prove that f is injective on closed points and that $\dim(T_{[E^{(n)}]}\mathcal{M}) = \dim(T_{[E]}M_H(v))$ for all $[E] \in M_H(v)$.

The main tool for achieving this is [18, Theorem 6.9], which is a formula for computing various extension groups between tautological sheaves. More exactly [18, Theorem 6.9] implies for any $[E_1], [E_2] \in M_H(v)$ that

$$\mathrm{Hom}_{K_n(A)}(E_1^{(n)}, E_2^{(n)}) \cong \mathrm{Hom}_A(E_1, E_2) = \begin{cases} \mathbb{C}, & \text{when } E_1 \cong E_2; \\ 0, & \text{when } E_1 \ncong E_2. \end{cases}$$

In particular, the case of $E_1 \ncong E_2$ implies that f is injective on closed points. Moreover, [18, Theorem 6.9] also implies

$$\begin{aligned} &\mathrm{Ext}^1_{K_n(A)}(E_1^{(n)}, E_2^{(n)}) \\ \cong\ &\mathrm{Ext}^1_A(E_1, E_2) \oplus H^1(A, E_1^\vee) \otimes H^0(A, E_2) \oplus H^0(A, E_1^\vee) \otimes H^1(A, E_2) \\ =\ &\mathrm{Ext}^1_A(E_1, E_2), \end{aligned}$$

where the last equality follows from (‡) and the Serre duality on A. In particular, when $[E_1]$ and $[E_2]$ represent the same closed point $[E] \in M_H(v)$, we obtain that $\dim T_{E^{(n)}}(\mathcal{M}) = \dim T_E(M_H(v))$ as desired. $\square$

2. Universal Bundles

In this section we want to construct a second type of stable bundles on $K_n(A)$. The basic idea is to use the Fourier-Mukai transform to find a fine moduli space of stable sheaves $M_{\widehat{H}}(w)$ on $\widehat{A}$ such that the generalized Kummer $K_{\widehat{H}}(w)$ in $M_{\widehat{H}}(w)$ is isomorphic to $K_n(A)$. We restrict the universal family of $M_{\widehat{H}}(w)$ to $K_{\widehat{H}}(w)$ and study its fibers over a point $\widehat{a} \in \widehat{A}$, which is a sheaf on $K_{\widehat{H}}(w) \cong K_n(A)$.

2.1. **Stable sheaves on abelian surfaces.** Pick $n, r \in \mathbb{N}$ with $n \geqslant 2$ as well as $r \geqslant n+2$ and let A be an abelian surface satisfying

$$\mathrm{NS}(A) = \mathbb{Z}H \text{ such that } H^2 = 2(n+r+1).$$

We denote the dual abelian surface by $\widehat{A}$. We have the Poincare line bundle $\mathcal{P}$ on $A \times \widehat{A}$ which defines the classical Fourier-Mukai transform

$$\Phi : \mathrm{D^b}(A) \to \mathrm{D^b}(\widehat{A}), \;\; E \mapsto Rp_*(\mathcal{P} \otimes q^*(E))$$

where $p : A \times \widehat{A} \to \widehat{A}$ and $q : A \times \widehat{A} \to A$ are the projections.

Using the canonical isomorphism $A \cong \widehat{\widehat{A}}$ (given by the Poincare bundle), we can also understand $\mathcal{P}$ as the Poincare bundle on $\widehat{A} \times A$ up to switching the factors, see [13, p.198, Remark 9.12]. This gives rise to the Fourier-Mukai transform

$$\widehat{\Phi} : \mathrm{D^b}(\widehat{A}) \to \mathrm{D^b}(A), \;\; F \mapsto Rq_*(\mathcal{P} \otimes p^*(F))$$

It is well known that $\det(\Phi(\mathcal{O}_A(H)))^{-1}$ defines the canonical polarization $\widehat{H}$ on $\widehat{A}$ and $\mathrm{NS}(\widehat{A}) = \mathbb{Z}\widehat{H}$, see for example [4].

Now we look at the Mukai vector

$$v = (1, H, r)$$

and denote the moduli space of μ_H-semistable sheaves on A with Mukai vector v by $M_H(v)$. Then there is an isomorphism

$$\epsilon : A^{[n+1]} \times \widehat{A} \xrightarrow{\cong} M_H(v), \;\; (Z, \widehat{a}) \longmapsto I_Z(H) \otimes \mathcal{P}_{\widehat{a}}\,.$$

We compute $\langle v^2 \rangle = H^2 - 2r = 2(n+r+1) - 2r = 2n+2$ and thus

$$\dim(M_H(v)) = 2n+4. \tag{8}$$

Furthermore by the choice of r we have $r > \frac{\langle v^2 \rangle}{2}$, which by [27, Corollary 3.3] implies that every $E \in M_H(v)$ satisfies IT_0 with respect to Φ and that $\Phi(E)$ is a $\mu_{\widehat{H}}$-stable locally free sheaf on $\widehat{A}$ with Mukai vector

$$w = (r, -\widehat{H}, 1).$$

By [27, Prop. 3.2, Cor. 3.3] we get that the Fourier-Mukai transform induces an isomorphism

$$\Phi : M_H(v) \xrightarrow{\cong} M_{\widehat{H}}(w).$$

Remark 2.1. The moduli space $M_{\widehat{H}}(w)$ is fine as $\gcd(r, \widehat{H}^2, 1) = 1$. Furthermore [27, Corollary 3.3] also shows that all sheaves classified by $M_{\widehat{H}}(w)$ are $\mu_{\widehat{H}}$-stable locally free sheaves.

2.2. **Generalized Kummer varieties.** We recall the original construction of the generalized Kummer due to Beauville, see [3, Sect. 7]: the group law on A defines, via the symmetric power and the Hilbert-Chow morphism, a summation morphism:

$$\Sigma : A^{[n+1]} \to A^{(n+1)} \to A.$$

The generalized Kummer variety is then defined by $K_n(A) := \Sigma^{-1}(0_A)$.

This construction was generalized by Yoshioka to moduli spaces of stable sheaves $M_H(v)$ on A, see [27, Theorem 4.1., Definition 4.1.]. We quickly summarize his main results: let v be a primitive Mukai vector with the property $\langle v^2\rangle + 2 \geqslant 6$ and H be a generic polarization, i.e. $\overline{M_H(v)} = M_H(v)$. One finds that the Albanese morphism of $M_H(v)$ is given by

$$\mathfrak{a}_v : M_H(v) \to A \times \widehat{A}$$

with

$$\mathfrak{a}_v(E) = \big(\det(\Phi(E)) \otimes \det(\Phi(E_0))^{-1}, \det(E) \otimes \det(E_0)^{-1}\big)$$

for some fixed $E_0 \in M_H(v)$. Then one can give the following:

Definition 2.2. The generalized Kummer variety $K_H(v)$ in $M_H(v)$ is defined to be the fiber of $\mathfrak{a}_v$ over the point $(0_A, 0_{\widehat{A}})$, i.e. $K_H(v) = \mathfrak{a}_v^{-1}((0_A, 0_{\widehat{A}}))$.

Note that we have $\dim(K_H(v)) = 2n$ by (8). Now assume that v also satisfies all conditions from [27, Corollary 3.3], that is the Fourier-Mukai transform induces an isomorphism $\Phi : M_H(v) \xrightarrow{\cong} M_{\widehat{H}}(w)$. Under these circumstances not only are the moduli spaces are isomorphic, but also the induced generalized Kummer varieties:

Lemma 2.3. *The isomorphism $\Phi : M_H(v) \xrightarrow{\cong} M_{\widehat{H}}(w)$ restricts to an isomorphism between generalized Kummer varieties $K_H(v) \xrightarrow{\cong} K_{\widehat{H}}(w)$.*

Proof. We first note that the Albanese morphism $\mathfrak{a}_w : M_{\widehat{H}}(w) \to \widehat{A} \times \widehat{\widehat{A}}$ can be understood as a morphism $\mathfrak{a}_w : M_{\widehat{H}}(w) \to A \times \widehat{A}$ after identifying $A \cong \widehat{\widehat{A}}$ and switching the factors. It is then given by

$$\mathfrak{a}_w(F) = (\det(F) \otimes \det(F_0)^{-1}, \det(\widehat{\Phi}(F)) \otimes \det(\widehat{\Phi}(F_0))^{-1})$$

with $F_0 = \Phi(E_0) \in M_{\widehat{H}}(w)$.

Using the isomorphism $\varphi : A \times \widehat{A} \to A \times \widehat{A}$ given by $\varphi := 1_A \times (-1_A)^*$ we claim that the following diagram commutes:

$$\begin{tikzcd} M_H(v) \arrow[r, "\Phi"] \arrow[d, "\mathfrak{a}_v"'] & M_{\widehat{H}}(w) \arrow[d, "\mathfrak{a}_w"'] \\ A \times \widehat{A} \arrow[r, "\varphi"] & A \times \widehat{A}. \end{tikzcd}$$

To see this we simply note that since every $E \in M_H(v)$ is IT_0 with respect to Φ, we have the following isomorphism by [19, Corollary 2.4.]:

$$\widehat{\Phi}(\Phi(E)) \cong (-1_A)^* E.$$

Using $\varphi((0_A, 0_{\widehat{A}})) = (0_A, 0_{\widehat{A}})$ and the commutativity, we see that Φ restricts to an isomorphism $K_H(v) \cong K_{\widehat{H}}(w)$. $\square$

2.3. **Construction of a universal family.** In this section we want to construct a universal family for the generalized Kummer variety $K_{\widehat{H}}(w)$. For this we first note that $M_H(v)$ is a fine moduli space, that is there is a universal family on the product $A \times M_H(v)$. Denote the restriction of the universal family along the closed immersion $A \times K_H(v) \hookrightarrow A \times M_H(v)$ by $\mathcal{E}$.

Remark 2.4. For the Mukai vector $v = (1, H, r)$ we have the following isomorphism:

$$K_n(A) \xrightarrow{\cong} K_H(v), \quad [Z] \longmapsto I_Z(H).$$

By making a careful choice of the line bundles on A and $\widehat{A}$ representing $\det(E_0)$ and $\det(\Phi(E_0))$ as in [10, §3.1], an explicit computation similar to [10, Lemma 3.2] shows that there is a commutative diagram

$$\begin{tikzcd} A^{[n+1]} \times \widehat{A} \arrow[r, "\epsilon"] \arrow[d, "\Sigma\times 1_{\widehat{A}}"'] & M_H(v) \arrow[d, "\mathfrak{a}_v"] \\ A \times \widehat{A} \arrow[r, "\rho"] & A \times \widehat{A} \end{tikzcd}$$

with the isomorphism

$$\rho : A \times \widehat{A} \xrightarrow{\cong} A \times \widehat{A}, \quad (a, \widehat{a}) \longmapsto (-a + \phi_{\widehat{H}^{-1}}(\widehat{a}), \widehat{a}).$$

Again, as $\rho(0_A, 0_{\widehat{A}}) = (0_A, 0_{\widehat{A}})$, we find that the isomorphism ϵ restricts to an isomorphism between the fibers of $\Sigma \times 1_{\widehat{A}}$ and $\mathfrak{a}_v$ over $(0_A, 0_{\widehat{A}})$. It remains to note that these fibers are $K_n(A)$ and $K_H(v)$ by definition.

Using the last remark we will, from now on, understand the universal family $\mathcal{E}$ on $A \times K_H(v)$ as a family on $A \times K_n(A)$, which is easily seen to be given by

$$\mathcal{E} = \mathcal{I}_{\mathcal{Z}} \otimes \pi_1^* \mathcal{O}_A(H)$$

where $\pi_1 : A \times K_n(A) \to A$ is the projection and $\mathcal{I}_{\mathcal{Z}}$ is the universal ideal sheaf on $A \times K_n(A)$.

We now define a family $\mathcal{U}$ on $\widehat{A} \times K_n(A)$ using the Fourier-Mukai transform relative to $K_n(A)$ following [20, Sect. 1]. For this we introduce some notation:

$$\begin{tikzcd} & & & A \times \widehat{A} \times K_n(A) \arrow[dll, "p_A"'] \arrow[d, "q"] \arrow[dr, "p_{\widehat{A}}"] & \\ A & & A \times K_n(A) \arrow[ll, "\pi_1"'] & A \times \widehat{A} & \widehat{A} \times K_n(A) \end{tikzcd}$$

Then the relative Fourier-Mukai transform is defined by:

$$\Psi : \mathrm{D}^{\mathrm{b}}(A \times K_n(A)) \to \mathrm{D}^{\mathrm{b}}(\widehat{A} \times K_n(A)), \quad \mathcal{F} \mapsto Rp_{\widehat{A}*}(q^* \mathcal{P} \otimes p_A^*(\mathcal{F}))$$

Using this we define the following family on $\widehat{A} \times K_n(A)$:

$$\mathcal{U} := \Psi(\mathcal{E}).$$

The restriction of $\mathcal{E}$ to the fiber over $[Z] \in K_n(A)$ is just $I_Z(H)$ which is IT_0 with respect to Φ, implying that $\mathcal{U}$ is WIT_0 and that $\Psi(\mathcal{E})$ commutes with arbitrary base change $T \to K_n(A)$ by [19, Theorem 1.6.]. By choosing $T = \{[Z]\}$ for some $[Z] \in K_n(A)$ we see that there is an isomorphism

$$\mathcal{U} \otimes \mathcal{O}_{[Z]} = \Psi(\mathcal{E}) \otimes \mathcal{O}_{[Z]} \cong \Phi(\mathcal{E} \otimes \mathcal{O}_{[Z]}) \cong \Phi(I_Z(H)).$$

As $\Phi(I_Z(H))$ is locally free the last equation shows that $\mathcal{U}$ is locally free by [15, Lemma 2.1.7].

Furthermore, since $H^1(A, I_Z(H)) = 0$ for all $[Z] \in K_n(A)$ we see, using standard results from the theory of cohomology and base change, that for every morphism $\alpha : S \to \widehat{A} \times K_n(A)$ we get a diagram

$$\begin{tikzcd} A & A \times S \arrow[l, "t_1"'] \arrow[r, "\beta"] \arrow[d, "t_2"'] & A \times \widehat{A} \times K_n(A) \arrow[r, "p_A"] \arrow[d, "p_{\widehat{A}}"] & A \times K_n(A) \\ & S \arrow[r, "\alpha"] & \widehat{A} \times K_n(A) & \end{tikzcd} \tag{9}$$

together with an isomorphism:

$$\begin{aligned} \alpha^*\mathcal{U} &= \alpha^*(Rp_{\widehat{A}*}(q^*\mathcal{P} \otimes p_A^*\mathcal{E})) \\ &\cong Rt_{2*}\beta^*(q^*\mathcal{P} \otimes p_A^*\mathcal{E}). \end{aligned}$$

We sum up the results from this subsection in the following:

Lemma 2.5. *The family $\mathcal{U} = \Psi(\mathcal{E})$ on $\widehat{A} \times K_n(A)$ is a locally free universal family for $K_{\widehat{H}}(w)$, namely, its classifying morphism $K_n(A) \to M_{\widehat{H}}(w)$ induces the isomorphism*

$$K_n(A) \xrightarrow{\cong} K_{\widehat{H}}(w), \ \ [Z] \longmapsto \Phi(I_Z(H)).$$

2.4. **Stability of the wrong-way fibers.** In this section we want to study the stability of the wrong-way fibers of $\mathcal{U}$, that is the fibers over points $\widehat{a} \in \widehat{A}$. For this we choose in the diagram (9) the base change along the inclusion $j_{\widehat{a}}$ of the fiber over $\widehat{a}$ of the projection $\widehat{A} \times K_n(A) \to \widehat{A}$, that is

$$\begin{tikzcd} A & A \times K_n(A) \arrow[l, "t_1"'] \arrow[r, "i_{\widehat{a}}"] \arrow[d, "t_2"'] & A \times \widehat{A} \times K_n(A) \arrow[r, "p_A"] \arrow[d, "p_{\widehat{A}}"] & A \times K_n(A) \\ & K_n(A) \arrow[r, "j_{\widehat{a}}"] & \widehat{A} \times K_n(A) & \end{tikzcd} \tag{10}$$

where the morphisms $j_{\widehat{a}}$ and $i_{\widehat{a}}$ are given on closed points by

$$\begin{aligned} j_{\widehat{a}} &: K_n(A) \hookrightarrow \widehat{A} \times K_n(A), \quad [Z] \mapsto (\widehat{a}, [Z]) \\ i_{\widehat{a}} &: A \times K_n(A) \hookrightarrow A \times \widehat{A} \times K_n(A), \quad (a, [Z]) \mapsto (a, \widehat{a}, [Z]). \end{aligned}$$

Going through the base change we see that we can describe the wrong-way fibers in the following way:

$$\begin{aligned} \mathcal{U}_{\widehat{a}} = j_{\widehat{a}}^*\mathcal{U} &= j_{\widehat{a}}^*(Rp_{\widehat{A}*}(q^*\mathcal{P} \otimes p_A^*\mathcal{E}))) \\ &\cong Rt_{2*}i_{\widehat{a}}^*(q^*\mathcal{P} \otimes p_A^*(\mathcal{I}_{\mathcal{Z}} \otimes \pi_1^*\mathcal{O}_A(H))) \\ &\cong Rt_{2*}(\mathcal{I}_{\mathcal{Z}} \otimes t_1^*(\mathcal{P}_{\widehat{a}}(H))). \end{aligned}$$

We recall the integral functor

$$\Theta : \mathrm{D^b}(A) \to \mathrm{D^b}(K_n(A)), \quad E \mapsto Rt_{2*}(\mathcal{I}_{\mathcal{Z}} \otimes t_1^* E),$$

which is a $\mathbb{P}^{n-1}$-functor by [18, Theorem 4.1].

We see that the wrong-way fiber is given by

$$\mathcal{U}_{\widehat{a}} = \Theta(\mathcal{P}_{\widehat{a}}(H)) \tag{11}$$

and sits in the exact sequence:

(12)

$$0 \longrightarrow \mathcal{U}_{\widehat{a}} \longrightarrow Rt_{2*}(t_1^*(\mathcal{P}_{\widehat{a}}(H))) \longrightarrow Rt_{2*}(\mathcal{O}_{\mathcal{Z}} \otimes t_1^*(\mathcal{P}_{\widehat{a}}(H))) \longrightarrow 0$$

We also have:

$$Rt_{2*}(t_1^*(\mathcal{P}_{\widehat{a}}(H))) \cong H^0(\mathcal{P}_{\widehat{a}}(H)) \otimes \mathcal{O}_{K_n(A)}$$

by cohomology and base change. Furthermore

$$Rt_{2*}(\mathcal{O}_{\mathcal{Z}} \otimes t_1^*(\mathcal{P}_{\widehat{a}}(H))) = (\mathcal{P}_{\widehat{a}}(H))^{(n)}$$

is the tautological bundle of rank $n+1$ on $K_n(A)$ induced by $\mathcal{P}_{\widehat{a}}(H)$.

A quick diagram chase shows that we have

$$(\mathcal{P}_{\widehat{a}}(H))^{(n)} \cong \iota^*((\mathcal{P}_{\widehat{a}}(H))^{[n+1]})$$

where $\iota : K_n(A) \hookrightarrow A^{[n+1]}$ is the inclusion and $(\mathcal{P}_{\widehat{a}}(H))^{[n+1]}$ is the tautological bundle induced by $\mathcal{P}_{\widehat{a}}(H)$ on $A^{[n+1]}$.

For the next results we recall that we have $\mathrm{NS}(K_n(A)) = \mathrm{NS}(A)_K \oplus \mathbb{Z}\delta$. Here D_K is the divisor class on $K_n(A)$ induced by the divisor class D on A and δ is a divisor class on $K_n(A)$ such that $2\delta = [E]$ where E is the exceptional divisor of the Hilbert-Chow morphism $K_n(A) \to S_n(A)$. In our case this reads

$$\mathrm{NS}(K_n(A)) = \mathbb{Z}H_K \oplus \mathbb{Z}\delta.$$

Remark 2.6. Note that we can also write $\mathrm{NS}(K_n(A)) = \iota^* \mathrm{NS}(A^{[n+1]})$, with

$$\mathrm{NS}(A^{[n+1]}) = \mathrm{NS}(A)_{n+1} \oplus \mathbb{Z}\Delta \oplus \Sigma^* \mathrm{NS}(A).$$

where $\mathrm{NS}(A)_{n+1}$ are the divisor classes on $A^{[n+1]}$ induced from A and Δ is the class such that 2Δ is the class of the exceptional divisor of the morphism $A^{[n+1]} \to A^{(n+1)}$. We have $\iota^* H_{n+1} = H_K$ and $\iota^*\Delta = \delta$.

Lemma 2.7. *We have $c_1(\mathcal{U}_{\widehat{a}}) = -H_K + \delta$.*

Proof. By the exact sequence (12) we get:

$$\begin{aligned} c_1(\mathcal{U}_{\widehat{a}}) &= -c_1((\mathcal{P}_{\widehat{a}})^{(n)}) \\ &= -c_1(\iota^*((\mathcal{P}_{\widehat{a}}(H))^{[n+1]})) \\ &= -\iota^* c_1((\mathcal{P}_{\widehat{a}}(H))^{[n+1]}) \\ &= -\iota^*(H_{n+1} - \Delta) = -H_K + \delta \end{aligned}$$

where we use [25, Lemma 1.5] in the second to last step. □

To compute slopes on $K_n(A)$ we need the following intersection numbers, which can, for example, be found in [5, 1.2., 1.4.]:

Lemma 2.8. *For the classes H_K and δ from $\mathrm{NS}(K_n(A))$ we have:*

- $H_K^{2n} = \frac{(n+1)(2n)!}{(n)!2^n}(H^2)^n > 0$
- $H_K^{2n-1}\delta = 0$.

Lemma 2.9. *There is an isomorphism*

$$\mathrm{NS}(A) \xrightarrow{\cong} \mathrm{NS}(P_n(A))^{\mathfrak{S}_{n+1}}, \ \ H \longmapsto \sum_{i=0}^{n} \tau^* q_i^* H.$$

Proof. We note that $P_n(A)$ is itself an abelian variety (isomorphic to A^n via projection) hence its integral cohomology is torsion free. This implies especially that its Neron-Severi group $\mathrm{NS}(P_n(A))$ is torsion free and hence so is $\mathrm{NS}(P_n(A))^{\mathfrak{S}_{n+1}}$.

Furthermore by [12, Lemma 3] we have an isomorphism

$$\mathrm{NS}(P_n(A))^{\mathfrak{S}_{n+1}} \otimes \mathbb{Q} \cong \left(\mathrm{NS}(P_n(A)) \otimes \mathbb{Q}\right)^{\mathfrak{S}_{n+1}}.$$

and so it is enough to prove the lemma over the field of rational numbers $\mathbb{Q}$.

We start with the morphisms

$$P_n(A) \xhookrightarrow{\tau} A^{n+1} \xrightarrow{S} A$$

where $S = \sum_{i=0}^{n} q_i$ is the summation morphism using the group law on A.

The natural inclusion τ has the following retract:

$$A^{n+1} \to P_n(A), \ \ (a_0, \dots, a_n) \mapsto (a_0, \dots, a_{n-1}, -\sum_{i=0}^{n-1} a_i),$$

which shows that we have a surjection

$$H^2(A^{n+1}, \mathbb{Q}) \xrightarrow{\tau^*} H^2(P_n(A), \mathbb{Q}) \longrightarrow 0$$

As we work over $\mathbb{Q}$ and $\mathfrak{S}_{n+1}$ is finite we get an induced surjection:

$$H^2(A^{n+1}, \mathbb{Q})^{\mathfrak{S}_{n+1}} \xrightarrow{\tau^*} H^2(P_n(A), \mathbb{Q})^{\mathfrak{S}_{n+1}} \longrightarrow 0.$$

It is well known, see for example [16, Theorem 2.15], that:

$$H^2(A^{n+1}, \mathbb{Q})^{\mathfrak{S}_{n+1}} \cong H^2(A, \mathbb{Q}) \oplus \Lambda^2\left(H^1(A, \mathbb{Q})\right)$$

where the maps are given by:

$$H^2(A, \mathbb{Q}) \hookrightarrow H^2(A^{n+1}, \mathbb{Q})^{\mathfrak{S}_{n+1}}, \ \ c \mapsto \sum_{i=0}^{n} q_i^* c$$

as well as (using $\Lambda^2\left(H^1(A, \mathbb{Q})\right) \cong H^2(A, \mathbb{Q})$):

$$\Lambda^2\left(H^1(A, \mathbb{Q})\right) \hookrightarrow H^2(A^{n+1}, \mathbb{Q})^{\mathfrak{S}_{n+1}}, \ \ c \wedge d \mapsto \sum_{i,j}\left(q_i^* c \wedge q_j^* d\right)$$

Now since $S = \sum_{i=0}^{n} q_i$ we get similar to Beauville in [3, Proposition 8.]:

$$\sum_{i,j}\left(q_i^* c \wedge q_j^* d\right) = \left(\sum_{i=0}^{n} q_i^* c\right) \wedge \left(\sum_{j=0}^{n} q_j^* d\right) = S^*(c \wedge d).$$

This implies $\Lambda^2\left(H^1(A,\mathbb{Q})\right) \cong \mathrm{Im}(S^*)$. But then

$$\tau^*(S^*(c \wedge d)) = (S \circ \tau)^*(c \wedge d) = 0$$

which shows that we have

$$H^2(P_n(A),\mathbb{Q})^{\mathfrak{S}_{n+1}} \cong \tau^* H^2(A,\mathbb{Q}). \tag{13}$$

Using the Lefschetz $(1,1)$-theorem gives

$$(\mathrm{NS}(P_n(A)) \otimes \mathbb{Q})^{\mathfrak{S}_{n+1}} \cong \tau^*\left(\mathrm{NS}(A) \otimes \mathbb{Q}\right),$$

which is what we wanted to prove. $\square$

Proposition 2.10. *The vector bundle $\mathcal{U}_{\widehat{a}}$ defined in* (11) *is slope stable with respect to H_K.*

Proof. We follow the idea in the proof of [24, Theorem 1.4].

Since $j_K^*(-)$, $(h_\circ^{-1})^*(-)$ and $\sigma_\circ^*(-)$ are exact, and $(j_P)_*$ is left exact, by applying these functors to (12) we obtain an exact sequence of $\mathfrak{S}_{n+1}$-invariant reflexive sheaves on $P_n(A)$ as follows:

$$0 \longrightarrow (\mathcal{U}_{\widehat{a}})_P \longrightarrow (H^0(\mathcal{P}_{\widehat{a}}(H)) \otimes \mathcal{O}_{K_n(A)})_P \xrightarrow{\varphi} (\mathcal{P}_{\widehat{a}}(H))^{(n)})_P$$

where φ is not necessarily surjective. It is clear that

$$(H^0(\mathcal{P}_{\widehat{a}}(H)) \otimes \mathcal{O}_{K_n(A)})_P = H^0(\mathcal{P}_{\widehat{a}}(H)) \otimes \mathcal{O}_{P_n(A)},$$

and we also have

$$((\mathcal{P}_{\widehat{a}}(H))^{(n)})_P = \bigoplus_{i=0}^{n} \tau^* q_i^*\left(\mathcal{P}_{\widehat{a}}(H)\right)$$

by a similar argument as in [24, Lemma 1.1] (see also Proposition 1.5). Hence the above sequence becomes

$$0 \longrightarrow (\mathcal{U}_{\widehat{a}})_P \longrightarrow H^0(\mathcal{P}_{\widehat{a}}(H)) \otimes \mathcal{O}_{P_n(A)} \xrightarrow{\varphi} \bigoplus_{i=0}^{n} \tau^* q_i^*\left(\mathcal{P}_{\widehat{a}}(H)\right) \tag{14}$$

where φ is the evaluation map on $P_n(A)_\circ$.

More precisely, for any set of closed points $(a_0,\ldots,a_n) \in P_n(A)$ with $a_i \neq a_j$, the morphism of fibers can be identified as

$$\begin{aligned} \varphi_{(a_0,\ldots,a_n)} : H^0(\mathcal{P}_{\widehat{a}}(H)) &\longrightarrow \bigoplus_{i=0}^{n} \left(\mathcal{P}_{\widehat{a}}(H)\right)_{x_i} \\ s &\longmapsto (s(a_0),\ldots,s(a_n)) \end{aligned}$$

Since for any non-trivial $s \in H^0(\mathcal{P}_{\widehat{a}}(H))$, there are always (many choices of) distinct points $(a_0,\ldots a_n) \in P_n(A)$ such that $(s(a_0),\ldots,s(a_n)) \neq (0,\ldots,0)$, we conclude that the map of global sections

$$H^0(\varphi) : H^0(\mathcal{P}_{\widehat{a}}(H)) \longrightarrow H^0(\bigoplus_{i=0}^{n} \tau^* q_i^* \, \mathcal{P}_{\widehat{a}}(H))$$

is injective. It follows by (14) that $H^0((\mathcal{U}_{\widehat{a}})_P) = 0$.

Note that φ is surjective on $P_n(A)_\circ$, hence $\mathrm{coker}(\varphi)$ is supported on the big diagonal of $P_n(A)$ which is of codimension 2. It follows that

$$c_1((\mathcal{U}_{\widehat{a}})_P) = -\sum_{i=0}^{n} \tau^* q_i^* H.$$

We claim that $(\mathcal{U}_{\widehat{a}})_P$ has no $\mathfrak{S}_{n+1}$-invariant subsheaf which is destabilizing with respect to H_P. Indeed, assume F is an $\mathfrak{S}_{n+1}$-invariant subsheaf of $(\mathcal{U}_{\widehat{a}})_P$, then $c_1(F) \in \mathrm{NS}(P_n(A))^{\mathfrak{S}_{n+1}}$ and thus by Lemma 2.9 we have:

$$c_1(F) = a(\sum_{i=0}^{n} \tau^* q_i^* H) \text{ for some } a \in \mathbb{Z}.$$

If $a \leqslant -1$, then

$$c_1(F) \cdot H_P^{2n-1} \leqslant c_1((\mathcal{U}_{\widehat{a}})_P) \cdot H_P^{2n-1} < 0$$

Since $1 \leqslant \mathrm{rk}(F) < \mathrm{rk}((\mathcal{U}_{\widehat{a}})_P)$, it follows that $\mu_{H_P}(F) < \mu_{H_P}((\mathcal{U}_{\widehat{a}})_P)$, hence F is not destabilizing.

If $a = 0$, we choose a (not necessarily $\mathfrak{S}_{n+1}$-invariant) non-zero stable subsheaf $F' \subseteq F$ which has maximal slope with respect to H_P (e.g. one can take a stable factor in the first Harder-Narasimhan factor of F). Without loss of generality, we can assume F and F' are both reflexive. Since F' is also a subsheaf of the trivial bundle $H^0(\mathcal{P}_{\widehat{a}}(H)) \otimes \mathcal{O}_{P_n(A)}$, there must be a projection from $H^0(\mathcal{P}_{\widehat{a}}(H)) \otimes \mathcal{O}_{P_n(A)}$ to a certain direct summand of it, such that the composition of the embedding and projection

$$F' \to H^0(\mathcal{P}_{\widehat{a}}(H)) \otimes \mathcal{O}_{P_n(A)} \to \mathcal{O}_{P_n(A)}$$

is non-zero. Since $\mu_{P_n(A)}(F') \geqslant \mu_{P_n(A)}(F) = 0 = \mu_{P_n(A)}(\mathcal{O}_{P_n(A)})$, and $\mathcal{O}_{P_n(A)}$ is also stable with respect to H_P, the map $F' \to \mathcal{O}_{P_n(A)}$ must be injective, and its cokernel is supported on a locus of codimension at least 2. Since both are reflexive, we must have $F' = \mathcal{O}_{P_n(A)}$. Therefore F, and consequently $(\mathcal{U}_{\widehat{a}})_P$, have non-trivial global sections. Contradiction.

If $a \geqslant 1$, F would be a subsheaf of the trivial bundle $H^0(\mathcal{P}_{\widehat{a}}(H)) \otimes \mathcal{O}_{P_n(A)}$ of positive slope. Contradiction.

Finally, assume G is a reflexive subsheaf of $\mathcal{U}_{\widehat{a}}$. Then $(G)_P$ is an $\mathfrak{S}_{n+1}$-invariant reflexive subsheaf of $(\mathcal{U}_{\widehat{a}})_P$. By the above claim we have

$$\mu_{H_P}((G)_P) < \mu_{H_P}((\mathcal{U}_{\widehat{a}})_P).$$

It follows from equation (3) that $\mu_{H_K}(G) < \mu_{H_K}(\mathcal{U}_{\widehat{a}})$. Therefore $\mathcal{U}_{\widehat{a}}$ is slope stable with respect to H_K, as desired. □

Proposition 2.11. *There exists some ample class $H' \in \mathrm{NS}(K_n(A))$ near H_K, such that $\mathcal{U}_{\widehat{a}}$ is $\mu_{H'}$-stable for all $\widehat{a} \in \widehat{A}$ simultaneously.*

Proof. By Proposition 1.6 the divisor H_K is lef so that Proposition 2.10 and [6, Theorem 2.3.1] guarantee that the assumptions in [24, Proposition 4.8] are satisfied for each $\mathcal{U}_{\widehat{a}}$. Hence every $\mathcal{U}_{\widehat{a}}$ is slope stable with respect to some ample class near H_K. In order to find a single ample class H' that is independent of the choice of $\mathcal{U}_{\widehat{a}}$, we can use the entire proof of [24, Proposition 4.8] except that we need to reconstruct the non-empty convex open set U so that $\alpha := H_K^{2n-1}$ is in the closure of U, and for every $\gamma \in U$, $\mathcal{U}_{\widehat{a}}$ is stable with respect to γ for all $\widehat{a} \in \widehat{A}$.

We follow the notations in [9, Definition 3.1]. For each $\widehat{a} \in \widehat{A}$, $\mathrm{SStab}(\mathcal{U}_{\widehat{a}})$ is a convex closed set containing α. Hence the intersection

$$\overline{U} := \bigcap_{\widehat{a} \in \widehat{A}} \mathrm{SStab}(\mathcal{U}_{\widehat{a}})$$

is also a convex closed set containing α. We first claim that [9, Theorem 3.4] holds for all $\mathcal{U}_{\widehat{a}}$ simultaneously; namely, we will show that for any $\beta \in \mathrm{Mov}(K_n(A))^\circ$ (see [9, Definition 2.1] for the notation), there exists a number $e \in \mathbb{Q}^+$, such that $(\alpha + \varepsilon\beta) \in \cap_{x \in X} \mathrm{Stab}(\mathcal{U}_{\widehat{a}})$ for any real $\varepsilon \in [0, e]$.

To prove the claim, we first note that the slope $c := \mu_\beta(\mathcal{U}_{\widehat{a}})$ is independent of the choice of $\widehat{a} \in \widehat{A}$. We redefine the set S in the proof of [9, Theorem 3.4] to be

$$S := \{c_1(F) \mid F \subseteq \mathcal{U}_{\widehat{a}} \text{ for some } \widehat{a} \in \widehat{A} \text{ such that } \mu_\beta(F) \geqslant c\}.$$

Since $\mathcal{U}_{\widehat{a}} \subseteq H^0(\mathcal{P}_{\widehat{a}}(H)) \otimes \mathcal{O}_{K_n(A)}$ for all $\widehat{a} \in \widehat{A}$, we obtain that S is a subset of

$$T := \{c_1(F) \mid F \subseteq H^0(\mathcal{P}_{\widehat{a}}(H)) \otimes \mathcal{O}_{K_n(A)} \text{ such that } \mu_\beta(F) \geqslant c\},$$

which is finite by [9, Theorem 2.29], hence S is also finite. We can then use the rest of the proof of [9, Theorem 3.4] literally to conclude the claim.

We then claim that $\overline{U}$ is of full dimension $r := \mathrm{rk}\, N_1(K_n(A))$. If not, then we have $\alpha \in \overline{U} \subseteq L$ for some hyperplane $L \subset N_1(K_n(A))_\mathbb{R}$. Since $\mathrm{Mov}(K_n(A))$ is of full dimension, we can choose some $\beta \in \mathrm{Mov}(K_n(A))^\circ \setminus L$. It follows that $(\alpha + \varepsilon\beta) \in \overline{U} \setminus L$ for some small $\varepsilon > 0$ by the previous claim and the choice of β. Contradiction.

We define U to be the interior of $\overline{U}$ and claim that U is non-empty. Indeed, since $\overline{U}$ is of full dimension r, we can choose $r+1$ points of $\overline{U}$ in general positions, which form an r-simplex. By the convexity of $\overline{U}$, the entire simplex is in $\overline{U}$ hence any interior point of the simplex is also an interior point of $\overline{U}$. The convexity of U follows from the convexity of $\overline{U}$. And it is clear from the construction that $\alpha = H_K^{2n-1}$ is in the closure of U. We finally claim that every $\gamma \in U$ is in $\bigcap_{\widehat{a} \in \widehat{A}} \mathrm{Stab}(\mathcal{U}_{\widehat{a}})$. If not, suppose that there exists some class $\gamma_0 \in U$ and some closed point $\widehat{a}_0 \in \widehat{A}$, such that $\gamma_0 \in \mathrm{SStab}(\mathcal{U}_{\widehat{a}_0}) \setminus \mathrm{Stab}(\mathcal{U}_{\widehat{a}_0})$; namely, $\mu_{\gamma_0}(F) = \mu_{\gamma_0}(\mathcal{U}_{\widehat{a}_0})$ for some proper subsheaf F of $\mathcal{U}_{\widehat{a}_0}$. Since the slope function is linear with respect to the curve class, and $\mu_\alpha(F) < \mu_\alpha(\mathcal{U}_{\widehat{a}_0})$ by Proposition 2.10, one can find a hyperplane in $N^1(K_n(A))_\mathbb{R}$ through γ_0, such that $\mu_\gamma(\mathcal{U}_{\widehat{a}_0}) - \mu_\gamma(F)$ takes opposite signs for γ in the two open half-spaces separated by the hyperplane. In particular, F destabilizes $\mathcal{U}_{\widehat{a}_0}$ in one of the half-spaces. Since U has non-empty intersection with both half-spaces, this contradicts the condition $U \subseteq \mathrm{SStab}(\mathcal{U}_{\widehat{a}})$. Therefore we have $U \subseteq \bigcap_{\widehat{a} \in \widehat{A}} \mathrm{Stab}(\mathcal{U}_{\widehat{a}})$, as desired. □

2.5. **A component of the moduli space.** We start this subsection by making a brief digression to consider again the integral functor

$$\Theta\colon \mathrm{D^b}(A) \longrightarrow \mathrm{D^b}(K_n(A))$$

whose kernel is the universal ideal sheaf $\mathcal{I}_\mathcal{Z}$ on $A \times K_n(A)$. Recall that Θ is a $\mathbb{P}^{n-1}$-functor, which implies by [2, §2.1] that for any $E, F \in \mathrm{D^b}(A)$ we have an isomorphism of graded vector spaces

$$\mathrm{Ext}^*_{K_n(A)}(\Theta(E),\Theta(F)) \cong \mathrm{Ext}^*_A(E,F)\otimes H^*(\mathbb{P}^{n-1},\mathbb{C}). \tag{15}$$

We now turn to the main result of the section. Let H' be an ample class that satisfies Proposition 2.11, and $\mathcal{M}$ the moduli space of $\mu_{H'}$-stable sheaves on $K_n(A)$ with the same numerical invariants as $\mathcal{U}_{\widehat{a}}$. Then the universal family $\mathcal{U}$ defines a classifying morphism

$$f\colon \widehat{A} \longrightarrow \mathcal{M}, \quad \widehat{a} \longmapsto [\mathcal{U}_{\widehat{a}}] \tag{16}$$

In fact the morphism f can be described as follows:

Theorem 2.12. *The classifying morphism* (16) *defined by the family $\mathcal{U}$ identifies $\widehat{A}$ with a smooth connected component of $\mathcal{M}$.*

Proof. By [22, Lemma 1.6.] we have to prove that f is injective on closed points and that $\dim(T_{[\mathcal{U}_{\widehat{a}}]}\mathcal{M}) = 2$ for all $\widehat{a}\in\widehat{A}$.

Now we know $\mathcal{U}_{\widehat{a}} = \Theta(\mathcal{P}_{\widehat{a}}(H))$, so for $\widehat{a}_1 \neq \widehat{a}_2$ we find by (15) that

$$\begin{aligned}\mathrm{Hom}_{K_n(A)}(\mathcal{U}_{\widehat{a}_1},\mathcal{U}_{\widehat{a}_2}) &= \mathrm{Hom}_{K_n(A)}(\Theta(\mathcal{P}_{\widehat{a}_1}(H)),\Theta(\mathcal{P}_{\widehat{a}_2}(H)))\\ &\cong \mathrm{Hom}_A(\mathcal{P}_{\widehat{a}_1}(H),\mathcal{P}_{\widehat{a}_2}(H))\\ &\cong H^0(A,\mathcal{P}^\vee_{\widehat{a}_1}\otimes\mathcal{P}_{\widehat{a}_2}) = 0,\end{aligned}$$

where the last step follows from [13, Lemma 9.9]. This implies f is injective on closed points.

A similar computation shows

$$\begin{aligned}\mathrm{Ext}^1_{K_n(A)}(\mathcal{U}_{\widehat{a}},\mathcal{U}_{\widehat{a}}) &= \mathrm{Ext}^1_{K_n(A)}(\Theta(\mathcal{P}_{\widehat{a}}(H)),\Theta(\mathcal{P}_{\widehat{a}}(H)))\\ &\cong \mathrm{Ext}^1_A(\mathcal{P}_{\widehat{a}}(H),\mathcal{P}_{\widehat{a}}(H))\\ &\cong \mathrm{Ext}^1_A(\mathcal{P}_{\widehat{a}},\mathcal{P}_{\widehat{a}})\\ &\cong \mathrm{Ext}^1_{\widehat{A}}(\mathcal{O}_{\widehat{a}},\mathcal{O}_{\widehat{a}}) \cong T_{\widehat{a}}\widehat{A}\end{aligned}$$

where the second to last isomorphism uses $\mathcal{P}_{\widehat{a}} \cong \widehat{\Phi}(\mathcal{O}_{\widehat{a}})$ and the fact that $\widehat{\Phi}$ is an equivalence from $\mathrm{D^b}(\widehat{A})$ to $\mathrm{D^b}(A)$.

Using $T_{[\mathcal{U}_{\widehat{a}}]}\mathcal{M} \cong \mathrm{Ext}^1_{K_n(A)}(\mathcal{U}_{\widehat{a}},\mathcal{U}_{\widehat{a}})$ we find $\dim(T_{[\mathcal{U}_{\widehat{a}}]}\mathcal{M}) = 2$ as desired. □

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SMOOTH COMPONENTS ON SPECIAL ITERATED HILBERT SCHEMES

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Abstract. Let S be a smooth projective surface with $p_g = q = 0$. We show how to use derived categorical methods to study the geometry of certain special iterated Hilbert schemes associated to S by showing that they contain a smooth connected component isomorphic to S.

1. Introduction

Hilbert schemes are ubiquitous in modern algebraic geometry. But even in good situations these schemes can behave badly. This became clear with Mumford's famous example, which shows that there is an irreducible component of the Hilbert scheme of smooth irreducible curves in $\mathbb{P}^3$ of degree 14 and genus 24 that is generically non-reduced, see [17]. More exactly Mumford constructs a 56-dimensional irreducible family of such curves, such that the tangent space at each point of this component has dimension 57 and proves that this family is not contained in any other irreducible family of dimension > 56.

In this note we prove that there are certain iterated Hilbert schemes which contain at least one smooth connected component. More exactly the main result of this note is:

Theorem 1. *Assume S is a smooth projective surface with $p_g = q = 0$ and let $S^{[n]}$ be the Hilbert scheme of length n subschemes of S. Then the universal family $\mathcal{Z}$ in $S \times S^{[n]}$ can be understood as a family of codimension two subschemes in $S^{[n]}$ with common Hilbert polynomial $p(t)$ classified by S such that the classifying morphism identifies S with a smooth connected component of the Hilbert scheme $\mathrm{Hilb}^{p(t)}(S^{[n]})$.*

This theorem has its origin in a result by Lange and Newstead, who proved a similar result for curves and moduli spaces of stable vector bundles on these curves in [15]. More exactly they show that if M is a fine moduli space of stable vector bundles on a smooth projective curve C of genus $g \geqslant 2$ with universal family $\mathcal{U}$ on $C \times M$, then for any $c \in C$ the vector bundle $\mathcal{U}_c$ on M is stable. Furthermore they show that for $c \neq c'$ we have $\mathcal{U}_c \ncong \mathcal{U}_{c'}$. Together with a previous result by Narasimhan and Ramanan these results imply that C embeds as a smooth connected component in a moduli spaces of stable vector bundles o M.

The main input into the proof of the main result of this note is a result by Krug and Sosna which states that the integral functor $\Phi : \mathrm{D^b}(S) \to \mathrm{D^b}(S^{[n]})$

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with kernel the universal ideal sheaf $\mathcal{I}_{\mathcal{Z}}$ is fully faithful. This result allows to reduce the computation of certain Ext-groups on $S^{[n]}$ to the computation of easier Ext-groups on S.

All objects in this note are defined over the field of complex numbers $\mathbb{C}$.

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2. Proof of the Main Theorem

Let S be a smooth projective surface with $p_g = q = 0$, that is we have

$$\mathrm{H}^1(S, \mathcal{O}_S) = \mathrm{H}^2(S, \mathcal{O}_S) = 0.$$

Remark 1. Around the year 1870 Max Noether posed the question if surfaces with $p_g = q = 0$ are necessarily rational, see [2, §3, Question 1]. By now it is known that the answer to this question is negative. In fact besides rational surfaces there is a huge class of surfaces satisfying these conditions which are not rational, most classically Enriques surfaces which have Kodaira dimension zero. But there are also surfaces of general type satisfying these conditions for example Godeaux surfaces, Campedelli surfaces or Beauville surfaces, see [1, 7] for more information and examples.

In the following we denote the Hilbert scheme of length n subschemes of S by $S^{[n]}$, that is we have as sets:

$$S^{[n]} = \left\{[Z] \mid Z \subset S \text{ is a zero-dimensional subscheme with } h^0(Z, \mathcal{O}_Z) = n\right\}.$$

It is well known that $S^{[n]}$ is smooth and that $\dim(S^{[n]}) = 2n$. Using this notation we have the universal subscheme

$$\mathcal{Z} = \left\{(s, [Z]) \in S \times S^{[n]} \mid s \in \operatorname{supp}(Z)\right\} \subset S \times S^{[n]} \tag{1}$$

coming with the corresponding universal ideal sheaf $\mathcal{I}_{\mathcal{Z}} \hookrightarrow \mathcal{O}_{S \times S^{[n]}}$.

Remark 2. Recall that the universal family $\mathcal{Z}$ is flat over $S^{[n]}$. Indeed, using definition (1) one can see that the restriction of $p : S \times S^{[n]} \to S^{[n]}$ to $\mathcal{Z}$ is finite and flat of degree n. But as a matter of fact $\mathcal{Z}$ is also flat over S due to [13, Theorem 2.1].

As we use integral functors in the following, we quickly recall their definition: let X and Y be smooth projective varieties and denote their bounded derived categories of coherent sheaves by $\mathrm{D^b}(X)$ and $\mathrm{D^b}(Y)$ respectively, then the integral functor with kernel $\mathcal{K} \in \mathrm{D^b}(X \times Y)$ is defined by

$$\Phi_{\mathcal{K}} : \mathrm{D^b}(X) \to \mathrm{D^b}(Y), \;\; E \mapsto \mathbf{R}p_*(q^*E \otimes^L \mathcal{K})$$

where p and q are the projections $X \times Y \to Y$ resp. $X \times Y \to X$, see [10, §5].

The description of the image of an integral transform is rather easy for a skyscraper sheaf $\mathcal{O}_x$ of a closed point $x \in X$, which we will collect as

Example 1. [10, Examples 5.4 (vi)] Assume the kernel of the integral functor $\Phi_{\mathcal{K}}$ is in fact a coherent sheaf on $X \times Y$ flat over X, then we have for every closed point $x \in X$

$$\Phi_{\mathcal{K}}(\mathcal{O}_x) \cong \mathcal{K}_x,$$

where the fiber $\mathcal{K}_x := \mathcal{K}_{|\{x\}\times Y}$ is considered as a sheaf on Y via the second projection $\{x\} \times Y \to Y$.

Interpreting the universal ideal sheaf $\mathcal{I}_{\mathcal{Z}}$ as an element in $\mathrm{D^b}(S \times S^{[n]})$ (as a complex concentrated in degree zero), we can look at the integral functor with kernel given by $\mathcal{I}_{\mathcal{Z}}$:

$$\Phi : \mathrm{D^b}(S) \to \mathrm{D^b}(S^{[n]}), \;\; E \mapsto \mathbf{R}p_*(q^*E \otimes \mathcal{I}_{\mathcal{Z}}).$$

In our case this integral functor has some very good properties. The main input into this note is the following very useful fact discovered by Krug and Sosna:

Theorem 2. [12, Theorem 1.2] *Let S be a smooth projective surface which satisfies $p_g = q = 0$, then the integral functor Φ is fully faithful.*

As an application of Theorem 2 we have the following

Corollary 1. *Assume S is a smooth projective surface with $p_g = q = 0$ then for all $E, F \in \mathrm{D^b}(S)$ and $i \geqslant 0$ there is an isomorphism*

$$\mathrm{Ext}^i_{S^{[n]}}(\Phi(E), \Phi(F)) \cong \mathrm{Ext}^i_S(E, F).$$

Remark 3. There are also fully faithfulness results for universal families of moduli spaces of stable bundles of rank two and degree one on smooth projective curves of genus $g \geqslant 2$ by Narasimhan as well as Fonarev and Kuznetsov, see [18], [19] and [9]. Recently these results were generalized to higher rank and degree by Belmans and Mukhopadhyay as well as Lee and Moon, see [5] and [16]. These results can be used to give a proof of the result of Lange and Newstead in the spirit of this note.

Proof of Theorem 1. Remark 2 shows that the family $\mathcal{Z} \subset S \times S^{[n]}$ is flat over S. Denote the fiber over a closed point $s \in S$ by $\mathcal{Z}_s$ and its image in $S^{[n]}$ (via the second projection $\{s\} \times S^{[n]} \cong S^{[n]}$, which is an isomorphism) by F_s. This identification together with Example 1 gives isomorphisms

$$I_{F_s} \cong (\mathcal{I}_{\mathcal{Z}})_s \cong \Phi(\mathcal{O}_s). \tag{2}$$

Remark 4. By the definition of $\mathcal{Z}$ given in (1) we have

$$F_s = \Big\{[Z] \in S^{[n]} \mid s \in \mathrm{supp}(Z)\Big\} \subset S^{[n]}.$$

That is, F_s is the subscheme of $S^{[n]}$ classifying all length n subschemes containing the closed point $s \in S$ in its support. Note that $\dim(F_s) = 2n-2$, that is F_s is a subscheme of codimension two in $S^{[n]}$.

Since S is integral the Hilbert polynomial of F_s does not depend on $s \in S$ by [11, Proposition 2.1.2], call it $p(t)$. We thus have a well defined classifying morphism

$$\varphi : S \to \mathrm{Hilb}^{p(t)}(S^{[n]}), \;\; s \mapsto [F_s]\,.$$

Remark 5. Here we can choose any ample line bundle $L \in \mathrm{Pic}(S^{[n]})$ to define the Hilbert polynomial $p(t)$ of F_s, as there is no distinguished ample line bundle on $S^{[n]}$. The choice of a different ample line bundle $\overline{L}$ would give rise to a different Hilbert polynomial $\overline{p}(t)$, but it would not change the proof of the main theorem.

A more conceptual way would be to choose a n-very ample line bundle M on S, then by [6] there is a closed embedding

$$S^{[n]} \to \mathrm{Gr}(n, \mathrm{H}^0(S, M)^*), \quad [Z] \mapsto \mathrm{H}^0(S, M \otimes \mathcal{O}_Z)^*.$$

Composing this morphism with the Plücker embedding of the Grassmannian, we get a closed embedding $S^{[n]} \to \mathbb{P}^N$ for some N and we can pullback $\mathcal{O}_{\mathbb{P}^N}(1)$ to get an ample line bundle L on $S^{[n]}$.

We claim that the morphism φ identifies S with a smooth connected component of $\mathrm{Hilb}^{p(t)}(S^{[n]})$. To see this we have to show that the morphism is injective on closed points and that for every closed point $s \in S$ we have

$$\dim(T_{[F_s]} \mathrm{Hilb}^{p(t)}(S^{[n]})) = 2.$$

We start by picking two closed points $s_1 \neq s_2 \in S$ and note that using equation (2) as well as Corollary 1, we get:

(3)

$$\mathrm{Hom}_{S^{[n]}}(I_{F_{s_1}}, I_{F_{s_2}}) \cong \mathrm{Hom}_{S^{[n]}}(\Phi(\mathcal{O}_{s_1}), \Phi(\mathcal{O}_{s_2}) \cong \mathrm{Hom}_S(\mathcal{O}_{s_1}, \mathcal{O}_{s_2}) = 0.$$

If $[F_{s_1}] = [F_{s_2}] \in \mathrm{Hilb}^{p(t)}(S^{[n]})$, then we would have an induced isomorphism $\mathcal{O}_{F_{s_1}} \cong \mathcal{O}_{F_{s_2}}$ and the exact sequences (for $i = 1, 2$)

$$0 \longrightarrow I_{F_{s_i}} \longrightarrow \mathcal{O}_{S^{[n]}} \longrightarrow \mathcal{O}_{F_{s_i}} \longrightarrow 0$$

would give rise to a commutative diagram of short exact sequences (with the identity between $\mathcal{O}_{S^{[n]}}$) which shows that there is a nontrivial morphism between $I_{F_{s_1}}$ and $I_{F_{s_2}}$. But this is impossible by 3. So the classifying morphism φ is indeed injective on closed points.

To find $\dim(T_{[F_s]} \mathrm{Hilb}^{p(t)}(S^{[n]}))$ we remark that

$$T_{[F_s]} \mathrm{Hilb}^{p(t)}(S^{[n]}) \cong \mathrm{Hom}_{S^{[n]}}(I_{F_s}, \mathcal{O}_{F_s}),$$

see for example [11, Proposition 2.2.7].

As $q = 0$ we have $\mathrm{Pic}^0(S) = 0$, but by [8, Theorem 5.4.] we also have an isomorphism

$$\mathrm{Pic}^0(S) \xrightarrow{\cong} \mathrm{Pic}^0(S^{[n]})$$

and thus $\mathrm{Pic}^0(S^{[n]}) = 0$. So we can use [14, Lemma B.5.6.] which gives an isomorphism

$$\mathrm{Hom}_{S^{[n]}}(I_{F_s}, \mathcal{O}_{F_s}) \cong \mathrm{Ext}^1_{S^{[n]}}(I_{F_s}, I_{F_s}).$$

We find, using again equation (2) and Corollary 1:

$$\mathrm{Ext}^1_{S^{[n]}}(I_{F_s}, I_{F_s}) \cong \mathrm{Ext}^1_{S^{[n]}}(\Phi(\mathcal{O}_s), \Phi(\mathcal{O}_s)) \cong \mathrm{Ext}^1_S(\mathcal{O}_s, \mathcal{O}_s) \cong T_s S.$$

Putting all results together shows $\dim(T_{[F_s]} \mathrm{Hilb}^{p(t)}(S^{[n]})) = 2$ as desired. $\square$

Remark 6. Theorem 2 can be generalized in the case $n = 2$ to all smooth projective varieties X having the property $\mathrm{H}^i(X, \mathcal{O}_X) = 0$ for $i \geqslant 1$ (that is $\mathcal{O}_X$ is exceptional). The integral functor $\Phi : \mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}^{\mathrm{b}}(X^{[2]})$ with kernel the universal ideal sheaf $\mathcal{I}_{\mathcal{Z}}$ is also fully faithful in these cases by [3, Theorem A]. Thus our proof of the main result is also valid in these cases.

Remark 7. In the case of surfaces, the proof of the main result only works for those surfaces with $p_g = q = 0$, since for $n \geqslant 2$ the integral functor $\Phi : \mathrm{D}^{\mathrm{b}}(S) \to \mathrm{D}^{\mathrm{b}}(S^{[n]})$ is fully faithful if and only if $p_g = q = 0$ by [4, Theorem A.]. But there are similar results for K3 surfaces as well as abelian surfaces, see [20]. In these cases the integral functor Φ is a so-called $\mathbb{P}^n$-functor, which again allows to reduce cohomological computations on $S^{[n]}$ to computations on S.

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THE FOURIER-MUKAI TRANSFORM OF A UNIVERSAL FAMILY OF STABLE VECTOR BUNDLES

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Abstract. In this note we prove that the Fourier-Mukai transform $\Phi_{\mathcal{U}}$ of the universal family of the moduli space $\mathcal{M}_{\mathbb{P}^2}(4,1,3)$ is not fully faithful.

Introduction

To every smooth projective variety X one can associate its bounded derived category of coherent sheaves $\mathrm{D}^{\mathrm{b}}(X)$. The derived category contains a lot of geometric information about X. In some cases one can even recover X from $\mathrm{D}^{\mathrm{b}}(X)$ but there are also examples of different varieties with equivalent derived categories, see [7] for an introduction.

To compare the derived categories of two smooth projective varieties X and Y, one needs to study functors between them. As it turns out, most of the interesting functors are Fourier-Mukai transforms $\Phi_{\mathcal{F}} : \mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}^{\mathrm{b}}(Y)$ for some object $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}(X \times Y)$.

In this note we are interested in fully faithful Fourier-Mukai transforms because they give a semi-orthogonal decomposition of the derived category $\mathrm{D}^{\mathrm{b}}(Y)$ into smaller admissible subcategories. For example Krug and Sosna prove in [10] that the Fourier-Mukai transform $\Phi_{\mathcal{I}_{\mathcal{Z}}} : \mathrm{D}^{\mathrm{b}}(S) \to \mathrm{D}^{\mathrm{b}}(S^{[n]})$ induced by the universal ideal sheaf $\mathcal{I}_{\mathcal{Z}}$ of the Hilbert scheme $S^{[n]}$ is fully faithful for a surface S with $p_g = q = 0$, hence $\mathrm{D}^{\mathrm{b}}(S)$ is an admissible subcategory in $\mathrm{D}^{\mathrm{b}}(S^{[n]})$. This result was generalized for the Hilbert square $X^{[2]}$ to smooth projective varieties X with exceptional structure sheaf and arbitrary dimension $\dim(X) \geq 2$, see [1].

Another example of this behaviour is given by the moduli space $\mathcal{M}_C(2,L)$ of stable rank two vector bundles with fixed determinant L of degree one on a smooth projective curve C of genus $g \geq 2$. This moduli space is fine and thus there is a universal family $\mathcal{U}$ on $C \times \mathcal{M}_C(2,L)$. By work of Narasimhan, see [18] and [19], as well as Fonarev and Kuznetsov, see [6], it is known that the Fourier-Mukai transform $\Phi_{\mathcal{U}} : \mathrm{D}^{\mathrm{b}}(C) \to \mathrm{D}^{\mathrm{b}}(\mathcal{M}_C(2,L))$ is fully faithful. Thus $\mathrm{D}^{\mathrm{b}}(C)$ is an admissible subcategory of $\mathrm{D}^{\mathrm{b}}(\mathcal{M}_C(2,L))$. This also solves the so-called Fano visitor problem for smooth projective curves of genus $g \geq 2$. This result was generalized in [2] to the higher rank case $\mathcal{M}_C(r,L)$ for a line bundle L of degree d such that $\gcd(r,d) = 1$ and curves of genus $g \geq g_0$ for some $g_0 \in \mathbb{N}$.

In light of these examples one can ask if the Fourier-Mukai transform of the universal family $\mathcal{U}$ on a fine moduli space $\mathcal{M}_{\mathbb{P}^2}(r,c_1,c_2)$ of stable sheaves

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on $\mathbb{P}^2$ is also fully faithful. Our main result is, that this is not always the case. We prove:

Theorem. *The Fourier-Mukai transform*

$$\Phi_{\mathcal{U}} : \mathrm{D^b}(\mathbb{P}^2) \to \mathrm{D^b}(\mathcal{M}_{\mathbb{P}^2}(4,1,3))$$

induced by the universal family $\mathcal{U}$ of the moduli space $\mathcal{M}_{\mathbb{P}^2}(4,1,3)$ is not fully faithful.

The structure of this note is as follows: in section 1 we recall some facts about the moduli space we are interested in. We construct an explicit family of stable sheaves for this moduli space in section 2. The computation of some cohomology groups for the family of stable sheaves can be found in section 3. In the final section 4 we prove the main result.

Everything in this note is defined over the field of complex numbers $\mathbb{C}$. The projective plane $\mathbb{P}^2$ is polarized by $H = \mathcal{O}_{\mathbb{P}^2}(1)$, thus μ-stability means μ_H-stability. A cohomology group written in lowercase characters simply denotes its dimension as a $\mathbb{C}$-vector space.

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1. The moduli space

We begin by studying the moduli space $\mathcal{M}_{\mathbb{P}^2}(4,1,3)$ of S-equivalence classes of μ-semistable torsion-free sheaves E on the projective plane $\mathbb{P}^2$ with the following numerical data:

$$\mathrm{rk}(E) = 4, \;\; c_1(E) = 1 \;\; c_2(E) = 3.$$

(Since the first Chern class is just an integer multiple of the polarization H, we simply identify it with this number.)

By this choice of rank r and Chern classes c_1 resp. c_2 we get:

Lemma 1.1. *The moduli space $\mathcal{M}_{\mathbb{P}^2}(4,1,3)$ is fine and there are no proper semistable sheaves.*

Proof. We have

$$\gcd\left(r, c_1.H, \frac{1}{2}c_1.(c_1 - K_{\mathbb{P}^2}) - c_2\right) = \gcd(4,1,-1) = 1.$$

The result now follows from [8, Corollary 4.6.7] and [8, Lemma 1.2.14]. $\square$

Remark 1.2. This lemma shows that the moduli space $\mathcal{M}_{\mathbb{P}^2}(4,1,3)$ has a universal family, that is a sheaf $\mathcal{U}$ on $\mathbb{P}^2 \times \mathcal{M}_{\mathbb{P}^2}(4,1,3)$ flat over $\mathcal{M}_{\mathbb{P}^2}(4,1,3)$ such that for every E with $[E] \in \mathcal{M}_{\mathbb{P}^2}(4,1,3)$ there is an isomorphism

$$\mathcal{U}_{[E]} \cong E,$$

where $\mathcal{U}_{[E]}$ denotes the restriction of $\mathcal{U}$ to the fiber over $[E]$.

The following properties of the moduli space are probably well known:

Lemma 1.3. *The moduli space $\mathcal{M}_{\mathbb{P}^2}(4,1,3)$ is a smooth projective variety of dimension six. Furthermore all sheaves E classified by this moduli space are locally free.*

Proof. The space $\mathcal{M}_{\mathbb{P}^2}(4,1,3)$ is projective by construction. Since every sheaf E is stable, we get by Serre duality

$$\mathrm{Ext}^2(E,E) \cong \mathrm{Hom}(E,E(-3))^\vee = 0$$

hence $\mathcal{M}_{\mathbb{P}^2}(4,1,3)$ is smooth and by [5] it is also irreducible. We also recall

$$\dim(\mathcal{M}_{\mathbb{P}^2}(r,c_1,c_2)) = \Delta - (r^2-1)\chi(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2})$$

where $\Delta = 2rc_2 - (r-1)c_1^2$ is the discriminant. So $\dim(\mathcal{M}_{\mathbb{P}^2}(4,1,3)) = 6$.

The double dual of a μ-stable torsion-free sheaf E is still μ-stable and defines a smooth point in $\mathcal{M}_{\mathbb{P}^2}(4,1,3-\ell)$ with $\ell = \mathrm{length}(E^{\vee\vee}/E)$. If E were not locally free we would have $\ell \geq 1$ and $\mathcal{M}_{\mathbb{P}^2}(4,1,3-\ell)$ would have negative dimension, which is not possible. □

Remark 1.4. Using Lemma 1.3 together with [8, Lemma 2.1.7.] shows that the universal family $\mathcal{U}$ is itself locally free on $\mathbb{P}^2 \times \mathcal{M}_{\mathbb{P}^2}(4,1,3)$. This implies that the sheaves $\mathcal{U}_p$, the restriction to the fiber over $p \in \mathbb{P}^2$, are also locally free on the moduli space.

The sheaves classified by $\mathcal{M}_{\mathbb{P}^2}(4,1,3)$ can be described more explicitly:

Lemma 1.5. *Let E be a locally free sheaf on $\mathbb{P}^2$ with $[E] \in \mathcal{M}_{\mathbb{P}^2}(4,1,3)$, then there is a length three subscheme $Z \subset \mathbb{P}^2$ and an exact sequence*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \longrightarrow E \longrightarrow I_Z(1) \longrightarrow 0.$$

Proof. Hirzebruch-Riemann-Roch shows $\chi(\mathbb{P}^2,E) = 3$. The stability of E implies that we have $h^2(\mathbb{P}^2,E) = 0$ and thus $h^0(\mathbb{P}^2,E) \geq 3$.

Choose a 3-dimensional subspace $U \subset H^0(\mathbb{P}^2,E)$, then by [17, Lemma 1.5] the natural evaluation map $\varphi : U \otimes \mathcal{O}_{\mathbb{P}^2} \to E$ is injective with torsion-free quotient $Q = \mathrm{Coker}(\varphi)$. We get the exact sequence

$$0 \longrightarrow U \otimes \mathcal{O}_{\mathbb{P}^2} \xrightarrow{\varphi} E \longrightarrow Q \longrightarrow 0.$$

By comparing Chern classes we see that we must have $Q \cong I_Z(1)$ for a length three subscheme $Z \subset \mathbb{P}^2$, that is $[Z] \in \mathbb{P}^{2[3]}$. This gives the desired exact sequence. □

Lemma 1.5 shows that there is a close connection between $\mathcal{M}_{\mathbb{P}^2}(4,1,3)$ and $\mathbb{P}^{2[3]}$. This connection will become clearer in the next sections.

2. Construction of a family

In this section we want to construct a $\mathbb{P}^{2[3]}$-family of μ-stable locally free sheaves such that every member of this family is classified by $\mathcal{M}_{\mathbb{P}^2}(4,1,3)$. The construction is based on a construction of Mukai, see [15, Section 3]

The starting point of our construction is the observation that

$$\mathrm{ext}^1(I_Z(1),\mathcal{O}_{\mathbb{P}^2}) = h^1(\mathbb{P}^2, I_Z(-2)) = 3 \tag{1}$$

for every $[Z] \in \mathbb{P}^{2[3]}$.

We define $V := \mathrm{Ext}^1(I_Z(1), \mathcal{O}_{\mathbb{P}^2})$ and observe the isomorphism

$$\mathrm{Ext}^1(I_Z(1), V^\vee \otimes \mathcal{O}_{\mathbb{P}^2}) \cong \mathrm{Ext}^1(I_Z(1), \mathcal{O}_{\mathbb{P}^2}) \otimes V^\vee \cong \mathrm{Hom}(V, V).$$

Hence there is a distinguished extension class $e \in \mathrm{Ext}^1(I_Z(1), V^\vee \otimes \mathcal{O}_{\mathbb{P}^2})$ corresponding to $\mathrm{id}_V \in \mathrm{Hom}(V, V)$, giving rise to:

$$0 \longrightarrow V^\vee \otimes \mathcal{O}_{\mathbb{P}^2} \longrightarrow E_Z \longrightarrow I_Z(1) \longrightarrow 0. \tag{2}$$

Remark 2.1. The sheaf E_Z is called the universal extension of $I_Z(1)$ by $\mathcal{O}_{\mathbb{P}^2}$. By construction we have $\mathrm{Hom}(E_Z, \mathcal{O}_{\mathbb{P}^2}) = 0$.

We want to study some of the properties of the sheaf E_Z. For example we have:

Lemma 2.2. *The sheaf E_Z is a locally free sheaf on $\mathbb{P}^2$.*

Proof. Tensor the exact sequence (2) with $\omega_{\mathbb{P}^2}$:

$$0 \longrightarrow V^\vee \otimes \omega_{\mathbb{P}^2} \longrightarrow E_Z \otimes \omega_{\mathbb{P}^2} \longrightarrow I_Z(-2) \longrightarrow 0 \ .$$

Now for every subscheme $Z' \subsetneq Z$ of length $0 \leq d < 3$ we have

$$h^1(\mathbb{P}^2, I_{Z'}(-2)) < h^1(\mathbb{P}^2, I_Z(-2)),$$

which by [20, Lemma 1.2.] implies that $E_Z \otimes \omega_{\mathbb{P}^2}$ is locally free, hence so is E_Z. $\square$

We also have the following result concerning the stability of E_Z:

Lemma 2.3. *The locally free sheaf E_Z is μ-stable.*

Proof. This follows from a more general result, see [17, Lemma 1.4.]. But in this situation we can also give a direct proof:

Let F be a torsion free quotient of E_Z with $1 \leq \mathrm{rk}(F) \leq 3$, then there is the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \arrow[r] \arrow[d] & E_Z \arrow[r] \arrow[d] & I_Z(1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F_0 \arrow[r] & F \arrow[r] & F_1 \arrow[r] & 0
\end{tikzcd}$$

with $F_0 = \mathrm{Im}(\mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \hookrightarrow E_Z \to F)$. Thus all vertical arrows are surjective. Since F_0 is a quotient of a free sheaf we have $c_1(F_0).H \geq 0$. Furthermore $\mathrm{rk}(F_1) \in \{0, 1\}$ as F_1 is a quotient of a torsion free sheaf of rank 1. We distinguish two cases.

<u>Case $\mathrm{rk}(F_1) = 1$:</u>

In this case $F_1 \cong I_Z(1)$ and hence $c_1(F).H = (c_1(F_0) + c_1(F_1)).H \geq 1$. This implies

$$\mu(F) = \frac{c_1(F).H}{\mathrm{rk}(F)} \geq \frac{1}{3} > \frac{1}{4} = \mu(E_Z).$$

<u>Case $\mathrm{rk}(F_1) = 0$:</u>

In this case we have $c_1(F_1).H \geq 0$ as F_1 is a torsion sheaf. The only critical case is $c_1(F_0) = c_1(F_1) = 0$, since otherwise $c_1(F) = d \geq 1$ and thus $\mu(F) \geq \frac{1}{3} > \frac{1}{4} = \mu(E_Z)$.

So assume $c_1(F_0) = c_1(F_1) = 0$. Then F_0 is free itself, see for example [14, p. 302], and F_1 is supported in finitely many points. This implies

$$\operatorname{Hom}(F, \mathcal{O}_{\mathbb{P}^2}) \cong \mathbb{C}^{\operatorname{rk}(F_0)}.$$

On the other hand $\operatorname{Hom}(F, \mathcal{O}_{\mathbb{P}^2}) \hookrightarrow \operatorname{Hom}(E_Z, \mathcal{O}_{\mathbb{P}^2}) = 0$ by Remark 2.1. This shows $\operatorname{rk}(F_0) = 0$ and hence $\operatorname{rk}(F) = 0$. So for $\operatorname{rk}(F) \geq 1$ the case $c_1(F_0) = c_1(F_1) = 0$ cannot occur and E_Z is stable. □

The last two lemmas show:

Corollary 2.4. *The sheaf E_Z defines a point $[E_Z] \in \mathcal{M}_{\mathbb{P}^2}(4, 1, 3)$ for every $[Z] \in \mathbb{P}^{2[3]}$.*

We want to put the μ-stable locally free sheaves E_Z in a family classified by $\mathbb{P}^{2[3]}$. To do this we need the following maps:

$$\begin{tikzcd} & \mathcal{Z} \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^{2[3]} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathbb{P}^2 & & \mathbb{P}^{2[3]} \end{tikzcd}$$

where $\mathcal{Z}$ is the universal family of length 3 subschemes.

Remark 2.5. Recall that for any coherent sheaf F on $\mathbb{P}^2$ there is the associated coherent *tautological sheaf* $F^{[3]}$ on $\mathbb{P}^{2[3]}$ defined by

$$F^{[3]} := q_* \left(p^* F \otimes \mathcal{O}_{\mathcal{Z}} \right).$$

If F is locally free of rank r then $F^{[3]}$ is locally free of rank $3r$.

To construct the family of stable sheaves, we first put the $\operatorname{Ext}^1(I_Z(1), \mathcal{O}_{\mathbb{P}^2})$ for $[Z] \in \mathbb{P}^{2[3]}$ in a family:

Lemma 2.6. *The first relative* Ext-*sheaf* $\mathcal{V} := \mathcal{E}\mathrm{xt}^1_{\mathrm{q}}(\mathcal{I}_{\mathcal{Z}} \otimes p^* \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^{2[3]}})$ *is a locally free sheaf of rank three on* $\mathbb{P}^{2[3]}$. *It commutes with base change and there is an isomorphism*

$$\mathcal{E}\mathrm{xt}^1_{\mathrm{q}}(\mathcal{I}_{\mathcal{Z}} \otimes p^* \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^{2[3]}})^\vee \cong \mathcal{O}_{\mathbb{P}^2}(-2)^{[3]}. \tag{3}$$

Proof. The morphism q is proper and flat and the map

$$\phi : \mathbb{P}^{2[3]} \to \mathbb{N}, \ \ [Z] \mapsto \operatorname{ext}^1(I_Z(1), \mathcal{O}_{\mathbb{P}^2})$$

is constant due to (1). So by [3, Satz 3.] the first relative Ext-sheaf is locally free of rank three on $\mathbb{P}^{2[3]}$ and commutes with base change, that is for every $[Z] \in \mathbb{P}^{2[3]}$ we have

$$\mathcal{E}\mathrm{xt}^1_{\mathrm{q}}(\mathcal{I}_{\mathcal{Z}} \otimes p^* \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^{2[3]}}) \otimes k(Z) \cong \operatorname{Ext}^1(I_Z(1), \mathcal{O}_{\mathbb{P}^2}).$$

Using relative Serre duality, see [9, Corollary(24)], gives an isomorphism

$$\begin{aligned} \mathcal{E}\mathrm{xt}^1_{\mathrm{q}}(\mathcal{I}_{\mathcal{Z}} \otimes p^* \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^{2[3]}}) &\cong \mathcal{E}\mathrm{xt}^1_{\mathrm{q}}(\mathcal{I}_{\mathcal{Z}} \otimes p^* \mathcal{O}_{\mathbb{P}^2}(-2), \omega_q) \\ &\cong \mathcal{H}om(R^1 q_*(\mathcal{I}_{\mathcal{Z}} \otimes p^* \mathcal{O}_{\mathbb{P}^2}(-2)), \mathcal{O}_{\mathbb{P}^{2[3]}}). \end{aligned}$$

The exact sequence

$$0 \longrightarrow \mathcal{I}_{\mathcal{Z}} \otimes p^* \mathcal{O}_{\mathbb{P}^2}(-2) \longrightarrow p^* \mathcal{O}_{\mathbb{P}^2}(-2) \longrightarrow \mathcal{O}_{\mathcal{Z}} \otimes p^* \mathcal{O}_{\mathbb{P}^2}(-2) \longrightarrow 0$$

and standard cohomology and base change results, see [16, II.5.], show that there is an isomorphism

$$q_*(\mathcal{O}_{\mathcal{Z}} \otimes p^* \mathcal{O}_{\mathbb{P}^2}(-2)) \cong R^1 q_*(\mathcal{I}_{\mathcal{Z}} \otimes p^* \mathcal{O}_{\mathbb{P}^2}(-2)).$$

We see that $R^1 q_*(\mathcal{I}_{\mathcal{Z}} \otimes p^* \mathcal{O}_{\mathbb{P}^2}(-2)) \cong \mathcal{O}_{\mathbb{P}^2}(-2)^{[3]}$ is locally free of rank three and thus we get the desired isomorphism (3). □

As the main result of this section we can now construct the desired family:

Theorem 2.7. *There is a locally free $\mathbb{P}^{2[3]}$-family $\mathcal{E}$ of μ-stable locally free sheaves, given by the exact sequence*

$$0 \longrightarrow q^*\mathcal{V}^\vee \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_{\mathcal{Z}} \otimes p^* \mathcal{O}_{\mathbb{P}^2}(1) \longrightarrow 0,$$

i.e. for every $[Z] \in \mathbb{P}^{2[3]}$ the restriction to the fiber over Z defines a point $[\mathcal{E}_Z] \in \mathcal{M}_{\mathbb{P}^2}(4, 1, 3)$.

Proof. For every $[Z] \in \mathbb{P}^{2[3]}$ we have $\mathrm{Hom}(I_Z(1), \mathcal{O}_{\mathbb{P}^2}) = 0$, so

$$\mathcal{E}xt^0_q(\mathcal{I}_{\mathcal{Z}} \otimes p^* \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^{2[3]}}) = q_*\mathcal{H}om(\mathcal{I}_{\mathcal{Z}} \otimes p^* \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^{2[3]}}) = 0.$$

Using this fact and the projection formula for relative Ext-sheaves [13, Lemma 4.1.], the five term exact sequence of the spectral sequence

$$H^i(\mathbb{P}^{2[3]}, \mathcal{E}xt^j_q(\mathcal{I}_{\mathcal{Z}} \otimes p^* \mathcal{O}_{\mathbb{P}^2}(1), q^*\mathcal{V}^\vee)) \Rightarrow \mathrm{Ext}^{i+j}(\mathcal{I}_{\mathcal{Z}} \otimes p^* \mathcal{O}_{\mathbb{P}^2}(1), q^*\mathcal{V}^\vee)$$

reduces to an isomorphism

$$\begin{aligned} \mathrm{Ext}^1(\mathcal{I}_{\mathcal{Z}} \otimes p^* \mathcal{O}_{\mathbb{P}^2}(1), q^*\mathcal{V}^\vee) &\cong H^0(\mathbb{P}^{2[3]}, \mathcal{E}xt^1_q(\mathcal{I}_{\mathcal{Z}} \otimes p^* \mathcal{O}_{\mathbb{P}^2}(1), q^*\mathcal{V}^\vee)) \\ &\cong H^0(\mathbb{P}^{2[3]}, \mathcal{E}xt^1_q(\mathcal{I}_{\mathcal{Z}} \otimes p^* \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^{2[3]}}) \otimes \mathcal{V}^\vee) \\ &\cong \mathrm{Hom}(\mathcal{V}, \mathcal{V}). \end{aligned}$$

The identity $\mathrm{id}_{\mathcal{V}}$ gives rise to an extension on $\mathbb{P}^2 \times \mathbb{P}^{2[3]}$:

$$0 \longrightarrow q^*\mathcal{V}^\vee \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_{\mathcal{Z}} \otimes p^* \mathcal{O}_{\mathbb{P}^2}(1) \longrightarrow 0 \tag{4}$$

with $\mathcal{E}$ flat over $\mathbb{P}^{2[3]}$, since both other terms are. Restricting to the fiber over a point $[Z] \in \mathbb{P}^{2[3]}$ defines by flatness of $\mathcal{I}_{\mathcal{Z}} \otimes p^* \mathcal{O}_{\mathbb{P}^2}(1)$ a map

$$\mathrm{Ext}^1(\mathcal{I}_{\mathcal{Z}} \otimes p^* \mathcal{O}_{\mathbb{P}^2}(1), q^*\mathcal{V}^\vee) \to \mathrm{Ext}^1(I_Z(1), V^\vee \otimes \mathcal{O}_{\mathbb{P}^2}).$$

By [13, Lemma 2.1.] the extension defined by $\mathrm{id}_{\mathcal{V}}$ restricts to the extension given by id_V on the fiber over $[Z] \in \mathbb{P}^{2[3]}$. Thus the pullback of (4) to the fiber over $[Z] \in \mathbb{P}^{2[3]}$ is exactly the exact sequence (2), hence it defines a locally free sheaf classified by $\mathcal{M}_{\mathbb{P}^2}(4, 1, 3)$. Using [8, Lemma 2.1.7.] again, we see that $\mathcal{E}$ is itself locally free. □

By the universal property of $\mathcal{M}_{\mathbb{P}^2}(4, 1, 3)$ the family $\mathcal{E}$ comes with a classifying morphism

$$f_{\mathcal{E}} : \mathbb{P}^{2[3]} \to \mathcal{M}_{\mathbb{P}^2}(4, 1, 3), \ [Z] \mapsto [\mathcal{E}_Z].$$

Furthermore there is $L \in \mathrm{Pic}(\mathbb{P}^{2[3]})$ and an isomorphism

$$(\mathrm{id}_{\mathbb{P}^2} \times f_{\mathcal{E}})^*\mathcal{U} \otimes q^*L \cong \mathcal{E}. \tag{5}$$

We need to study some properties of the morphism $f_{\mathcal{E}}$. For this we need:

Lemma 2.8. *Assume $[Z] \in \mathbb{P}^{2[3]}$ is not collinear. If there is an isomorphism $\alpha : E_{Z'} \cong E_Z$ for some $[Z'] \in \mathbb{P}^{2[3]}$, then $[Z] = [Z']$.*

Proof. We look at the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \arrow[r, "\iota"] & E_{Z'} \arrow[r] \arrow[d, "\cong"', "\alpha"] & I_{Z'}(1) \arrow[r] & 0 \\ 0 \arrow[r] & \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \arrow[r] & E_Z \arrow[r, "q"] & I_Z(1) \arrow[r] & 0. \end{tikzcd}$$

Since Z is not collinear the composition $q \circ \alpha \circ \iota$ is zero. Consequently the free submodule of $E_{Z'}$ maps injectively to the free submodule of E_Z, which must be an isomorphism then, so we get in fact the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \arrow[r] \arrow[d, "\cong"'] & E_{Z'} \arrow[r] \arrow[d, "\cong"', "\alpha"] & I_{Z'}(1) \arrow[r] \arrow[d, "\cong"'] & 0 \\ 0 \arrow[r] & \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \arrow[r] & E_Z \arrow[r] & I_Z(1) \arrow[r] & 0. \end{tikzcd}$$

Therefore there is an induced isomorphism $I_{Z'}(1) \cong I_Z(1)$ and so $[Z] = [Z']$. □

Thus the non-collinear subschemes in $\mathbb{P}^{2[3]}$ define sheaves E_Z with exactly three global sections. It makes sense to study the Brill-Noether-locus S in $\mathcal{M}_{\mathbb{P}^2}(4,1,3)$:

$$S := \left\{ [E] \in \mathcal{M}_{\mathbb{P}^2}(4,1,3) \,|\, h^0(\mathbb{P}^2, E) = 4 \right\}.$$

Remark 2.9. We can write down the inverse g to $f_{\mathcal{E}}$ on the complement of S:

$$g : \mathcal{M}_{\mathbb{P}^2}(4,1,3) \setminus S \to \mathbb{P}^{2[3]}, \;\; E \mapsto \operatorname{supp}(Q^{\vee\vee}/Q)$$

where Q is the cokernel of the (in this case) canonical evaluation map from Lemma 1.5.

By [20, Corollary, p.14, lines 3-5] we get for the E_Z with collinear subschemes Z:

Lemma 2.10. *Assume $[Z], [Z'] \in \mathbb{P}^{2[3]}$ are collinear with $[Z] \neq [Z']$ such that there is a line $\ell \subset \mathbb{P}^2$ containing both Z and Z', then $E_Z = E_{Z'}$.*

Remark 2.11. This shows that for a sheaf E_Z with a collinear subscheme $Z \subset \mathbb{P}^2$ we have

$$f_{\mathcal{E}}^{-1}([E_Z]) = \ell^{[3]} \cong \mathbb{P}^3,$$

where $\ell \subset \mathbb{P}^2$ is the line containing Z.

The last two lemmas suggest that $f_{\mathcal{E}}$ is the blow up of S in $\mathcal{M}_{\mathbb{P}^2}(4,1,3)$. This is indeed the case since by [21, 5.29, Example 5.3.] we have:

Lemma 2.12. *The Brill-Noether-locus S in $\mathcal{M}_{\mathbb{P}^2}(4,1,3)$ is isomorphic to $\mathbb{P}^2$ and there is an isomorphism $\mathbb{P}^{2[3]} \cong \operatorname{Bl}_S \mathcal{M}_{\mathbb{P}^2}(4,1,3)$ such that $f_{\mathcal{E}}$ can be identified with the blow up of S in $\mathcal{M}_{\mathbb{P}^2}(4,1,3)$.*

Remark 2.13. This description goes back to Drezet who proved this in terms of Kronecker modules in [4, Théorème 4.].

Corollary 2.14. *For every locally free sheaf F on $\mathcal{M}_{\mathbb{P}^2}(4,1,3)$ we have isomorphisms*

$$H^i(\mathcal{M}_{\mathbb{P}^2}(4,1,3),F) \cong H^i(\mathbb{P}^{2[3]}, f_{\mathcal{E}}^* F).$$

Proof. Since $f_{\mathcal{E}}$ is birational by Lemma 2.12, the result follows from the well known formula $R^i\,(f_{\mathcal{E}})_*\,\mathcal{O}_{\mathbb{P}^{2[3]}} = 0$ for $i \geq 1$, the projection formula and the Leray spectral sequence. □

3. Computations

We want to understand the family $\mathcal{E}$ as a $\mathbb{P}^2$-family, that is we want to understand the sheaves $\mathcal{E}_p$ on $\mathbb{P}^{2[3]}$ for $p \in \mathbb{P}^2$. For this we first note that $\mathcal{Z}$ is not just flat over $\mathbb{P}^{2[3]}$ but also over $\mathbb{P}^2$, see [12, Theorem 2.1.], so restricting the exact sequence

$$0 \longrightarrow \mathcal{I}_{\mathcal{Z}} \longrightarrow \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^{2[3]}} \longrightarrow \mathcal{O}_{\mathcal{Z}} \longrightarrow 0$$

to the fiber over point $p \in \mathbb{P}^2$ gives the exact sequence

$$0 \longrightarrow I_{S_p} \longrightarrow \mathcal{O}_{\mathbb{P}^{2[3]}} \longrightarrow \mathcal{O}_{S_p} \longrightarrow 0, \tag{6}$$

where $S_p := \left\{[Z] \in \mathbb{P}^{2[3]} \mid p \in \operatorname{supp}(Z)\right\}$ is a codimension two subscheme in $\mathbb{P}^{2[3]}$.

Since $\mathcal{Z}$ is flat over $\mathbb{P}^2$ we see, using [7, Examples 5.4 vi)], that we have $\mathcal{O}_{S_p} = k(p)^{[3]}$ is the tautological sheaf on $\mathbb{P}^{2[3]}$ associated to the skyscraper sheaf $k(p)$ of the point $p \in \mathbb{P}^2$. This implies we can use [10, Theorem 3.17.,Remark 3.20.] to find the following cohomology groups:

$$\operatorname{ext}^i(\mathcal{O}_{\mathbb{P}^{2[3]}}, \mathcal{O}_{\mathbb{P}^2}(-2)^{[3]}) = 0 \text{ for all } i \text{ and } \operatorname{ext}^i(\mathcal{O}_{S_p}, \mathcal{O}_{\mathbb{P}^2}(-2)^{[3]}) = \begin{cases} 1 & i=2 \\ 0 & i \neq 2 \end{cases} \tag{7}$$

as well as

$$\operatorname{ext}^i(\mathcal{O}_{\mathbb{P}^2}(-2)^{[3]}, \mathcal{O}_{\mathbb{P}^{2[3]}}) = \begin{cases} 6 & i=0 \\ 0 & i \geq 1 \end{cases} \text{ and } \operatorname{ext}^i(\mathcal{O}_{\mathbb{P}^2}(-2)^{[3]}, \mathcal{O}_{S_p}) = \begin{cases} 7 & i=0 \\ 0 & i \geq 1. \end{cases} \tag{8}$$

Using these results we can prove:

Lemma 3.1. *For $p \in \mathbb{P}^2$ we have*

$$\operatorname{ext}^i(I_{S_p}, \mathcal{O}_{\mathbb{P}^2}(-2)^{[3]}) = \begin{cases} 1 & i=1 \\ 0 & i \neq 1 \end{cases} \text{ and } \operatorname{ext}^i(\mathcal{O}_{\mathbb{P}^2}(-2)^{[3]}, I_{S_p}) = \begin{cases} \geq 1 & i=1 \\ 0 & i \geq 2. \end{cases}$$

Proof. Applying $\operatorname{Hom}(-, \mathcal{O}_{\mathbb{P}^2}(-2)^{[3]})$ to exact sequence (6) shows that we have $\operatorname{Ext}^6(I_{S_p}, \mathcal{O}_{\mathbb{P}^2}(-2)^{[3]}) = 0$. Also there are isomorphisms:

$$\operatorname{Ext}^i(I_{S_p}, \mathcal{O}_{\mathbb{P}^2}(-2)^{[3]}) \cong \operatorname{Ext}^{i+1}(\mathcal{O}_{S_p}, \mathcal{O}_{\mathbb{P}^2}(-2)^{[3]}) \text{ for } 1 \leq i \leq 5,$$

since $\operatorname{Ext}^i(\mathcal{O}_{\mathbb{P}^{2[3]}}, \mathcal{O}_{\mathbb{P}^2}(-2)^{[3]}) = 0$ for all i by (7). We furthermore find $\operatorname{Hom}(I_{S_p}, \mathcal{O}_{\mathbb{P}^2}(-2)^{[3]}) = 0$ by further using $\operatorname{Ext}^i(\mathcal{O}_{S_p}, \mathcal{O}_{\mathbb{P}^2}(-2)^{[3]}) = 0$ for $i = 0, 1$. This proves the first claim.

For the second claim we apply $\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^2}(-2)^{[3]}, -)$ to exact sequence (6). As $\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^2}(-2)^{[3]}, \mathcal{O}_{\mathbb{P}^{2[3]}}) = 0$ by (8) the first part of the long exact sequence gives

$$0 \longrightarrow \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^2}(-2)^{[3]}, I_{S_p}) \longrightarrow \mathbb{C}^6 \longrightarrow \mathbb{C}^7 \longrightarrow \mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^2}(-2)^{[3]}, I_{S_p}) \longrightarrow 0$$

which shows that $\mathrm{ext}^1(\mathcal{O}_{\mathbb{P}^2}(-2)^{[3]}, I_{S_p}) \geq 1$. We also get isomorphisms

$$\mathrm{Ext}^i(\mathcal{O}_{\mathbb{P}^2}(-2)^{[3]}, I_{S_p}) \cong \mathrm{Ext}^{i-1}(\mathcal{O}_{\mathbb{P}^2}(-2)^{[3]}, \mathcal{O}_{S_p}) \quad \text{for } 2 \leq i \leq 6.$$

Again using (8) proves the second claim. □

To study the locally free sheaves $\mathcal{E}_p$ on $\mathbb{P}^{2[3]}$ we note that the exact sequence

$$0 \longrightarrow q^*\mathcal{V}^\vee \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_{\mathcal{Z}} \otimes p^*\,\mathcal{O}_{\mathbb{P}^2}(1) \longrightarrow 0$$

restricts to the fiber over $p \in \mathbb{P}^2$ as

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-2)^{[3]} \longrightarrow \mathcal{E}_p \longrightarrow I_{S_p} \longrightarrow 0 \tag{9}$$

by using flatness of $\mathcal{I}_{\mathcal{Z}}$ over $\mathbb{P}^2$ and Lemma 2.6. We can now prove:

Theorem 3.2. *Let $\mathcal{E}$ be the $\mathbb{P}^{2[3]}$-family of μ-stable locally free sheaves, then for any pair of closed points $p, q \in \mathbb{P}^2$ with $p \neq q$ we have*

$$\mathrm{ext}^1(\mathcal{E}_p, \mathcal{E}_q) \geq 1.$$

Proof. By [11, Theorem 1.2] the Fourier-Mukai transform

$$\Phi_{\mathcal{I}_{\mathcal{Z}}} : \mathrm{D^b}(\mathbb{P}^2) \to \mathrm{D^b}(\mathbb{P}^{2[3]})$$

is fully faithful, that is for $p, q \in \mathbb{P}^2$ with $p \neq q$ we have by flatness of $\mathcal{I}_{\mathcal{Z}}$ over $\mathbb{P}^2$:

$$\mathrm{Ext}^i(I_{S_p}, I_{S_q}) \cong \mathrm{Ext}^i(k(p), k(q)) = 0 \text{ for } 0 \leq i \leq 6.$$

So applying $\mathrm{Hom}(I_{S_q}, -)$ with $q \neq p$ to (9) gives isomorphisms

$$\mathrm{Ext}^i(I_{S_q}, \mathcal{E}_p) \cong \mathrm{Ext}^i(I_{S_q}, \mathcal{O}_{\mathbb{P}^2}(-2)^{[3]}) \text{ for } 0 \leq i \leq 6.$$

If we apply $\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^2}(-2)^{[3]}, -)$ and use [10, Theorem 3.17.] again to see

$$\mathrm{ext}^i(\mathcal{O}_{\mathbb{P}^2}(-2)^{[3]}, \mathcal{O}_{\mathbb{P}^2}(-2)^{[3]}) = \begin{cases} 1 & i = 0 \\ 0 & i \geq 1 \end{cases}$$

we get an exact sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^2}(-2)^{[3]}, \mathcal{E}_p) \longrightarrow \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^2}(-2)^{[3]}, I_{S_p}) \longrightarrow 0$$

and isomorphisms

$$\mathrm{Ext}^i(\mathcal{O}_{\mathbb{P}^2}(-2)^{[3]}, \mathcal{E}_p) \cong \mathrm{Ext}^i(\mathcal{O}_{\mathbb{P}^2}(-2)^{[3]}, I_{S_p}) \text{ for } 1 \leq i \leq 6.$$

Finally applying $\mathrm{Hom}(-, \mathcal{E}_q)$ with $q \neq p$ to (9) we get the following relevant part of the induced long exact sequence:

$$\longrightarrow \mathrm{Ext}^1(\mathcal{E}_p, \mathcal{E}_q) \longrightarrow \mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^2}(-2)^{[3]}, \mathcal{E}_q) \longrightarrow \mathrm{Ext}^2(I_{S_p}, \mathcal{E}_q) \longrightarrow$$

With the previous results this sequence gets:

$$\longrightarrow \operatorname{Ext}^1(\mathcal{E}_p,\mathcal{E}_q) \longrightarrow \operatorname{Ext}^1(\mathcal{O}_{\mathbb{P}^2}(-2)^{[3]}, I_{S_q}) \longrightarrow \operatorname{Ext}^2(I_{S_p},\mathcal{O}_{\mathbb{P}^2}(-2)^{[3]}) \longrightarrow$$

Using Lemma 3.1 we have $\operatorname{Ext}^2(I_{S_p},\mathcal{O}_{\mathbb{P}^2}(-2)^{[3]}) = 0$ and thus

$$\operatorname{ext}^1(\mathcal{E}_p,\mathcal{E}_q) \geq \operatorname{ext}^1(\mathcal{O}_{\mathbb{P}^2}(-2)^{[3]}, I_{S_q}) \geq 1. \qquad \square$$

4. Non-full faithfulness of the universal family

We want to study the full faithfulness of the Fourier-Mukai transform

$$\Phi_{\mathcal{U}} : \mathrm{D^b}(\mathbb{P}^2) \to \mathrm{D^b}(\mathcal{M}_{\mathbb{P}^2}(4,1,3))$$

induced by the universal family $\mathcal{U}$ of the moduli space $\mathcal{M}_{\mathbb{P}^2}(4,1,3)$.

We will use the following corollary of the Bondal-Orlov criterion for full faithfulness:

Lemma 4.1. [7, Corollary 7.5] *Let X and Y be two smooth projective varieties and $\mathcal{P}$ a coherent sheaf on $X\times Y$, flat over X. Then the Fourier-Mukai transform*

$$\Phi_{\mathcal{P}} : \mathrm{D^b}(X) \to \mathrm{D^b}(Y)$$

is fully faithful if and only if the following two conditions are satisfied

i) For any closed point $x \in X$ one has $\operatorname{Ext}^i(\mathcal{P}_x,\mathcal{P}_x) = \begin{cases} \mathbb{C} & i=0 \\ 0 & i>\dim(X) \end{cases}$

ii) For any pair of closed points $x,y \in X$ with $x \neq y$ and for all i one has $\operatorname{Ext}^i(\mathcal{P}_x,\mathcal{P}_y) = 0$.

To apply this lemma to $\mathcal{P} = \mathcal{U}$, the universal family of $\mathcal{M}_{\mathbb{P}^2}(4,1,3)$, we need to be able to compute $\operatorname{Ext}^i(\mathcal{U}_p,\mathcal{U}_q)$. The following lemma reduces this problem to computing $\operatorname{Ext}^i(\mathcal{E}_p,\mathcal{E}_q)$:

Lemma 4.2. *Let $\mathcal{U}$ be the universal family of $\mathcal{M}_{\mathbb{P}^2}(4,1,3)$ and $\mathcal{E}$ be the $\mathbb{P}^{2[3]}$-family, then for any two points $p,q \in \mathbb{P}^2$ there are the following isomorphisms for all i:*

$$\operatorname{Ext}^i(\mathcal{U}_p,\mathcal{U}_q) \cong \operatorname{Ext}^i(\mathcal{E}_p,\mathcal{E}_q).$$

Proof. We have the following chain of isomorphisms:

$$\begin{aligned}
\operatorname{Ext}^i(\mathcal{U}_p,\mathcal{U}_q) &\cong H^i(\mathcal{M}_{\mathbb{P}^2}(4,1,3), \mathcal{H}om(\mathcal{U}_p,\mathcal{U}_q)) \\
&\cong H^i(\mathbb{P}^{2[3]}, f_{\mathcal{E}}^*\mathcal{H}om(\mathcal{U}_p,\mathcal{U}_q)) \\
&\cong H^i(\mathbb{P}^{2[3]}, \mathcal{H}om(f_{\mathcal{E}}^*\mathcal{U}_p, f_{\mathcal{E}}^*\mathcal{U}_q)) \\
&\cong \operatorname{Ext}^i(f_{\mathcal{E}}^*\mathcal{U}_p, f_{\mathcal{E}}^*\mathcal{U}_q) \\
&\cong \operatorname{Ext}^i(f_{\mathcal{E}}^*\mathcal{U}_p \otimes L, f_{\mathcal{E}}^*\mathcal{U}_q \otimes L) \\
&\cong \operatorname{Ext}^i(\mathcal{E}_p,\mathcal{E}_q)
\end{aligned}$$

Here the first and third isomorphism use the locally freeness of $\mathcal{U}_p$, see Remark 1.4. The second isomorphism is Corollary 2.14 since $\mathcal{U}_q$ is also locally free. The fourth isomorphism uses locally freeness of $f_{\mathcal{E}}^*\mathcal{U}_p$, while the sixth isomorphism follows from restricting (5) to the fiber over $p \in \mathbb{P}^2$. $\square$

We can now prove the main theorem of this note:

Theorem 4.3. *The Fourier-Mukai transform*

$$\Phi_{\mathcal{U}} : \mathrm{D^b}(\mathbb{P}^2) \to \mathrm{D^b}(\mathcal{M}_{\mathbb{P}^2}(4,1,3))$$

induced by the universal family $\mathcal{U}$ of the moduli space $\mathcal{M}_{\mathbb{P}^2}(4,1,3)$ is not fully faithful.

Proof. For the Fourier-Mukai transform $\Phi_{\mathcal{U}}$ to be fully faithful one needs

$$\mathrm{Ext}^i(\mathcal{U}_p, \mathcal{U}_q) = 0 \tag{10}$$

for any pair of points $p \neq q \in \mathbb{P}^2$ and any i according to Lemma 4.1.

Lemma 4.2 shows that (10) is equivalent to

$$\mathrm{Ext}^i(\mathcal{E}_p, \mathcal{E}_q) = 0.$$

But we have $\mathrm{Ext}^1(\mathcal{E}_p, \mathcal{E}_q) \neq 0$ by Theorem 3.2, so $\Phi_{\mathcal{U}}$ cannot be fully faithful. □

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THE SYMPLECTIC STRUCTURE ON THE MODULI SPACE OF LINE BUNDLES ON A NONCOMMUTATIVE AZUMAYA SURFACE

FABIAN REEDE

ABSTRACT. In this note we prove that the moduli space of torsion-free modules of rank one over an Azumaya algebra on a $K3$ surface is an irreducible symplectic variety deformation equivalent to a Hilbert scheme of points on the $K3$ surface.

INTRODUCTION

Assume X is a smooth projective $K3$ surface over $\mathbb{C}$ and let $\mathcal{A}$ be an Azumaya algebra on X, that is, $\mathcal{A}$ is a twisted form of a matrix algebra on X. The pair $(X, \mathcal{A})$ can be thought of as a noncommutative surface and it is also an example of what is called a Calabi-Yau order.

Locally projective $\mathcal{A}$-modules of rank one can be considered as line bundles on this noncommutative surface. In [HS05a] the authors construct moduli schemes for such line bundles. These schemes can be seen as noncommutative versions of the classical Picard schemes. By allowing torsion-free $\mathcal{A}$-modules and by fixing invariants, for example the Mukai vector, theses moduli schemes $M_{\mathcal{A}/X}(v)$ are shown to be projective schemes over $\mathbb{C}$. Furthermore the authors show that these schemes are smooth and possess a symplectic structure. So one can ask the question: are these moduli schemes irreducible symplectic varieties (hyperkähler manifolds)?

There are 4 known classes of hyperkähler manifolds: the Hilbert schemes of points on a smooth projective $K3$ surface and the generalized Kummer varieties associated to an abelian surface, both of these classes are due to Beauville. Furthermore there is a class of 6-dimensional examples and one class of 10-dimensional examples. Both classes are due to O'Grady, using symplectic desingularization. All other known examples of hyperkähler manifolds are deformation equivalent to one of the four examples mentioned above. The main result of this note is the following

Theorem. *Assume $v = v_{\mathcal{A}}(E)$ is a primitive Mukai vector for some torsion-free $\mathcal{A}$-module E of rank one, then $M_{\mathcal{A}/X}(v)$ is an irreducible symplectic variety deformation equivalent to $Hilb^{\frac{v^2}{2}+1}(X)$.*

The structure of this note is as follows. In section 1 we review basic facts about Azumaya algebras and associated Brauer-Severi varieties. We study

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the relationship between modules on which an Azumaya algebra acts and modules on the associated Brauer-Severi variety. In section 2 we review facts about the moduli schemes in question and relate them to moduli schemes known to be deformation equivalent to Hilbert schemes. These schemes were constructed by Yoshioka in [Yos06].

1. Modules over Azumaya algebras and Brauer-Severi varieties

In this section we assume X is any fixed scheme of finite type over $\mathbb{C}$, that is $X \in Sch_{\mathbb{C}}$. Following [HL10], we denote by $Sch_{\mathbb{C}}$ the category of schemes of finite type over $\mathbb{C}$. For $X \in Sch_{\mathbb{C}}$, we denote the category of schemes of finite type over X by Sch_X.

Definition 1.1. A sheaf of $\mathcal{O}_X$-algebras $\mathcal{A}$ is called Azumaya algebra if $\mathcal{A}$ is locally free of finite rank and for every closed point $x \in X$ the fiber $\mathcal{A}(x) = \mathcal{A} \otimes k(x)$ is a central simple algebra over the residue field $k(x)$.

Remark 1.2. The rank of an Azumaya algebra $\mathcal{A}$ is always a square, that is, $\mathrm{rk}(\mathcal{A}) = r^2$ for some $r \in \mathbb{N}$. In this note we will always assume that $\mathcal{A}$ is nontrivial, so that $r > 1$.

We also note that for $f : T \to X$ the $\mathcal{O}_T$-algebra $\mathcal{A}_T := f^*\mathcal{A}$ is an Azumaya algebra on T.

Definition 1.3. Let $\mathcal{A}$ be an Azumaya algebra on X. The functor

$$\mathcal{BS}(\mathcal{A}) : (Sch_X)^{op} \to Sets$$

which sends an X-scheme $f : T \to X$ to the set of left ideals $\mathcal{I} \subseteq \mathcal{A}_T$, such that $\mathcal{A}_T/\mathcal{I}$ is a locally free $\mathcal{O}_T$-module of rank $r(r-1)$, is called the Brauer-Severi functor associated to the Azumaya algebra $\mathcal{A}$ on X.

We recall some facts about this functor which follow directly from [GW10, 8.13, Exercise 8.14.] and [Gro68, Théorème 8.2, Corollaire 8.3]:

Lemma 1.4. *The functor $\mathcal{BS}(\mathcal{A})$ is representable by an X-scheme*

$$\pi : BS(\mathcal{A}) \to X.$$

The morphism π is faithfully flat and proper and exhibits $BS(\mathcal{A})$ as an étale $\mathbb{P}^{r-1}$-bundle over X.

Remark 1.5. For every X-scheme $f : T \to X$, there is a functorial isomorphism

$$BS(\mathcal{A}_T) \cong BS(\mathcal{A}) \times_X T.$$

The scheme $BS(\mathcal{A})$ is called the Brauer-Severi variety associated to $\mathcal{A}$. In the following we will just write $\pi : Y \to X$ for this X-scheme.

Lemma 1.6. *Assume $\mathcal{A}$ is an Azumaya algebra on X and let $\pi : Y \to X$ be the associated Brauer-Severi variety, then one has for every $E \in Coh(X)$:*

$$R^i\pi_*(\pi^*E) \cong \begin{cases} E & \text{if } i = 0 \\ 0 & \text{if } i \geq 1. \end{cases}$$

Proof. The problem is étale local, so one may assume $\mathcal{A} = \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})$. Using Morita equivalence shows $BS(\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})) \cong \mathbb{P}(\mathcal{E}^\vee)$. Now the result follows from [TT90, 4.5.(f)]. □

Definition 1.7. Assume $\mathcal{A}$ is an Azumaya algebra on X and let $\pi : Y \to X$ be the associated Brauer-Severi variety. Define a locally free sheaf G of rank r on Y by choosing some extension

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow G \longrightarrow \mathcal{T}_{Y/X} \longrightarrow 0.$$

This extension is unique up to scalars, see [Yos06, Lemma 1.1.].

Remark 1.8. The Brauer-Severi variety $\pi : Y \to X$ associated to an Azumaya algebra $\mathcal{A}$ has the following splitting property: there is an isomorphism of $\mathcal{O}_Y$-algebras

$$\pi^*\mathcal{A} \cong \mathcal{E}nd_{\mathcal{O}_Y}(G)^{op},$$

see for example [Qui73, 8.4.]. This shows that we have $\pi^*(\mathcal{A}^{op}) \cong \mathcal{E}nd_{\mathcal{O}_Y}(G)$ and hence

$$\mathcal{A}^{op} \cong \pi_*\pi^*(\mathcal{A}^{op}) \cong \pi_*(\mathcal{E}nd_{\mathcal{O}_Y}(G)).$$

Using ideas of Yoshioka, see [Yos06, Lemma 1.5.], we define the following subcategory of $Coh(Y)$:

Definition 1.9. Assume $\mathcal{A}$ is an Azumaya algebra on X and let $\pi : Y \to X$ be the associated Brauer-Severi variety, then one defines:

$$Coh(Y, X) := \{E \in Coh(Y) | \pi^*\pi_*(E \otimes G^\vee) \xrightarrow{\sim} E \otimes G^\vee\}.$$

Here the morphism is the canonical morphism coming from the adjunction between π^* and π_*.

Furthermore we denote the category of coherent sheaves on X having the structure of a left respectively right $\mathcal{A}$-module by $Coh_l(X, \mathcal{A})$ respectively $Coh_r(X, \mathcal{A})$.

Lemma 1.10. *Assume $\mathcal{A}$ is an Azumaya algebra on X and denote the associated Brauer-Severi variety by $\pi : Y \to X$, then there is an equivalence of categories:*

$$Coh_l(X, \mathcal{A}) \cong Coh(Y, X).$$

Proof. The category $Coh_l(X, \mathcal{A})$ is isomorphic to $Coh_r(X, \mathcal{A}^{op})$, hence it is enough to show that $Coh_r(X, \mathcal{A}^{op})$ and $Coh(Y, X)$ are equivalent.

For this, we define the following functor:

$$F : Coh_r(X, \mathcal{A}^{op}) \to Coh(Y, X), \;\; E \mapsto \pi^*E \otimes_{\pi^*(\mathcal{A}^{op})} G.$$

We need to verify that F is well-defined, that is we have to show that $\pi^*E \otimes_{\pi^*(\mathcal{A}^{op})} G$ belongs to $Coh(Y, X)$. For this we look at the canonical morphism

$$\pi^*\pi_*((\pi^*E \otimes_{\pi^*(\mathcal{A}^{op})} G) \otimes G^\vee) \to (\pi^*E \otimes_{\pi^*(\mathcal{A}^{op})} G) \otimes G^\vee.$$

By definition we have $G \otimes G^\vee \cong \mathcal{E}nd_{\mathcal{O}_Y}(G) \cong \pi^*(\mathcal{A}^{op})$. So we have to see that

$$\pi^*\pi_*\pi^*E \to \pi^*E$$

is an isomorphism. But this follows since $E \to \pi_*\pi^*E$ is an isomorphism by 1.6 and (π^*, π_*) is a pair of adjoint functors.

We note that $G^\vee$ is a right $\mathcal{E}nd_{\mathcal{O}_Y}(G)$-module, hence so is $E \otimes G^\vee$. Then $\pi_*(E \otimes G^\vee)$ is a right $\pi_*(\mathcal{E}nd_{\mathcal{O}_Y}(G)) \cong \mathcal{A}^{op}$-module by 1.8. So it makes sense to define the following functor:

$$H : Coh(Y, X) \to Coh_r(X, \mathcal{A}^{op}), \;\; E \mapsto \pi_*(E \otimes G^\vee).$$

It remains to study the compositions of F and H. For $E \in Coh_r(X, \mathcal{A}^{op})$ we have:

$$H(F(E)) = \pi_*((\pi^* E \otimes_{\pi^*(\mathcal{A}^{op})} G) \otimes G^\vee) \cong \pi_* \pi^* E \cong E$$

again by 1.6. For $E \in Coh(Y, X)$ we get

$$F(H(E)) = \pi^*(\pi_*(E \otimes G^\vee)) \otimes_{\pi^*(\mathcal{A}^{op})} G \cong (E \otimes G^\vee) \otimes_{\pi^*(\mathcal{A}^{op})} G \cong E.$$

The first isomorphism follows from $E \in Coh(Y, X)$ and the second isomorphism follows from $G^\vee \otimes_{\pi^*(\mathcal{A}^{op})} G \cong \mathcal{O}_Y$, see [GW10, Proposition 8.26.].

This shows that these categories are equivalent, therefore $Coh_l(X, \mathcal{A})$ and $Coh(Y, X)$ are also equivalent. □

Remark 1.11. If we denote the Brauer class of $\mathcal{A}$ by α, that is we have $\alpha = [\mathcal{A}] \in Br(X)$, then it can be shown, that the categories $Coh(Y, X)$ and $Coh_l(X, \mathcal{A})$ are also equivalent to the category of so called α-twisted coherent sheaves, denoted by $Coh(X, \alpha)$, see [HS06].

The definition of an α-twisted coherent sheaf involves an appropriate analytic (or étale) cover of X, representing the class α as a Čech 2-cocycle on this cover and then "twisting" the gluing functions of the sheaf by this 2-cocycle. For an exact definition see [Cal00, Definition 1.2.1].

The pair (X, α) is sometimes called a twisted variety. Huybrechts and Stellari have studied properties of twisted $K3$-surfaces in detail, see [HS05b] and [HS06].

All results in this note can be rephrased in terms of α-twisted sheaves. Our approach of using Azumaya algebras has the advantage of avoiding working with open covers and gluing functions. On the other hand, our approach forces the ranks of the sheaves involved to be considerably larger.

2. Moduli spaces of line bundles on a noncommutative Azumaya surface

In this section we work again with a fixed scheme X, which in this case should be a smooth projective $K3$ surface over $\mathbb{C}$. Thus, we have the Mukai pairing $\langle -, - \rangle$ on $H^{2*}(X, \mathbb{Q})$ given by

$$\langle x, y \rangle = -\int_X x^\vee y \qquad x, y \in H^{2*}(X, \mathbb{Q}),$$

see for example [HL10, 2.6.1.5]. To shorten notation we write x^2 for the term $< x, x >$.

Pick an Azumaya algebra $\mathcal{A}$ of rank r^2 on X and let $\pi : Y \to X$ be the associated Brauer-Severi variety. Given $S \in Sch_{\mathbb{C}}$, we have the following commutative diagram for every $s \in S$:

$$\begin{array}{ccccc} Y_s & \xrightarrow{j_s} & Y \times S & \xrightarrow{q} & Y \\ {\scriptstyle \pi_s}\downarrow & & {\scriptstyle \pi_S}\downarrow & & \downarrow{\scriptstyle \pi} \\ X_s & \xrightarrow{i_s} & X \times S & \xrightarrow{p} & X \end{array}$$

where X_s is the fiber of $X \times S \to S$ over $s \in S$.

The following objects live on the various schemes in this diagram:

- $\mathcal{A}$ on X, $\mathcal{A}_S := p^* \mathcal{A}$ on $X \times S$ and $\mathcal{A}_s := i_s^* p^* \mathcal{A}$ on X_s.

- $\pi^*(\mathcal{A}^{op}) \cong \mathcal{E}nd_{\mathcal{O}_Y}(G)$ on Y, $\pi_S^*(\mathcal{A}_S^{op}) \cong \mathcal{E}nd_{\mathcal{O}_{Y\times S}}(G_S)$ on $Y \times S$ with $G_S = q^*G$ and $\pi_s^*(\mathcal{A}_s^{op}) \cong \mathcal{E}nd_{\mathcal{O}_{Y_s}}(G_s)$ on Y_s with $G_s = j_s^*q^*G$.

Using 1.5, we have $Y \times S \cong BS(\mathcal{A}_S)$ and $Y_s \cong BS(\mathcal{A}_s)$, so that both schemes are also Brauer-Severi varieties.

Remark 2.1. At the generic point $\eta \in X$ the Azumaya algebra $\mathcal{A}$ is given by the central simple algebra $\mathcal{A}_\eta = M_n(D)$ for some division ring D over the function field $\mathbb{C}(X)$. Without loss of generality, we may assume $n = 1$.

This is because we are only interested in the Brauer class $[\mathcal{A}] \in Br(X)$. (Brauer equivalent algebras have equivalent module categories and hence isomorphic moduli schemes.) But we have $[M_n(D)] = [D] \in Br(\mathbb{C}(X))$, so by injectivity of $Br(X) \to Br(\mathbb{C}(X))$, $[\mathcal{A}] \mapsto [\mathcal{A}_\eta]$ all we need to do is find an Azumaya algebra $\mathcal{D}$ on X with $\mathcal{D}_\eta = D$. The existence of such an algebra is due the following theorem, see [CT04, Théorème 2.5.]:

Theorem 2.2. *Assume X is a regular integral scheme of dimension at most two with function field K. If A is a central simple K-algebra of dimension n^2 whose image in $Br(K)$ is unramified at every point of codimension one in X, then there is an Azumaya algebra $\mathcal{A}$ of rank n^2 on X such that $\mathcal{A} \otimes K = A$.*

Using this theorem, we see that a generically simple $\mathcal{A}$-module E, that is, E_η is a simple $\mathcal{A}_\eta$-module, must be generically of rank one over $\mathcal{A}$, hence has rank r^2 over X. We call such modules $\mathcal{A}$-modules of rank one.

We recall the definition of the Mukai vector $v_{\mathcal{A}}$ for a coherent left $\mathcal{A}$-module from [Ree13, Definition 2.7]. For $E \in Coh_l(X, \mathcal{A})$ we have:

$$v_{\mathcal{A}}(E) = \frac{ch(E)}{\sqrt{ch(\mathcal{A})}}\sqrt{td(X)}.$$

Furthermore Yoshioka defines in [Yos06, Definition 3.1.] for $E \in Coh(Y, X)$:

$$v_G(E) = \frac{ch(\pi_*(E \otimes G^\vee))}{\sqrt{ch(\pi_*(G \otimes G^\vee))}}\sqrt{td(X)}.$$

(Actually he defines this vector using the derived direct image, but the sheaf $E \otimes G^\vee$ does not have higher direct images for $E \in Coh(Y, X)$ by 1.6.)

Using the equivalence 1.10, we have $v_G(F(E)) = v_{\mathcal{A}}(E)$ as well as the equality $v_{\mathcal{A}}(H(E)) = v_G(E)$. This is because we have $\mathcal{A}^{op} \cong \pi_*(G \otimes G^\vee)$ by 1.8 and $ch(\mathcal{A}^{op}) = ch(\mathcal{A})$. So these Mukai vectors are the same and in the following we will omit the subscript and just write v for a fixed Mukai vector.

Now we can study the moduli functors of interest. First we define

$$\mathcal{M}_{\mathcal{A}/X}(v) : (Sch_{\mathbb{C}})^{op} \to Sets$$

which sends a $\mathbb{C}$-scheme S to the set of isomorphism classes of families of torsion-free $\mathcal{A}$-modules of rank one with Mukai vector v over S.

There is the following theorem describing the corresponding moduli space, see [HS05a, Theorem 2.4., Theorem 3.6.]:

Theorem 2.3. *The moduli functor $\mathcal{M}_{\mathcal{A}/X}(v_{\mathcal{A}})$ has a coarse moduli scheme $M_{\mathcal{A}/X}(v_{\mathcal{A}})$. Furthermore $M_{\mathcal{A}/X}(v_{\mathcal{A}})$ is a smooth projective scheme with a symplectic form on its tangent bundle.*

Remark 2.4. There is a more general and abstract construction of theses spaces as algebraic stacks, see for example [Lie04] and [Lie07]. Lieblich studies theses spaces also in terms of twisted sheaves.

The other moduli functor of interest is the following:

$$\mathcal{M}_H^{Y,G}(v) : (Sch_{\mathbb{C}})^{op} \to Sets$$

which sends a $\mathbb{C}$-scheme S to the set of isomorphism classes of families of torsion-free G-twisted semistable $\mathcal{O}_Y$-modules of rank r with Mukai vector v over S.

Here H is an ample divisor on X such that G-twisted stability is viewed with respect to H. For the definition of G-twisted stability, see [Yos06, Definition 2.2.].

The following theorem states some facts about the corresponding moduli space, which follow from [Yos06, Theorem 2.1.,Proposition 3.6., Theorem 3.11.].

Theorem 2.5. *The moduli functor $\mathcal{M}_H^{Y,G}(v)$ has a coarse moduli scheme $M_H^{Y,G}(v)$. If v is primitive and H is general with respect to v, then all G-twisted semistable sheaves are G-twisted stable. In particular, $M_H^{Y,G}(v)$ is a smooth projective scheme with a symplectic form on its tangent bundle.*

The following lemma gives a connection between theses moduli spaces:

Lemma 2.6. *Assume v is a Mukai vector with $v = v_{\mathcal{A}}(E)$ for some torsion-free $\mathcal{A}$-module E of rank one, then, using the equivalence $Coh_l(X, \mathcal{A}) \cong Coh(Y, X)$, torsion-free $\mathcal{A}$-modules of rank one with Mukai vector v correspond to torsion-free G-twisted semistable $\mathcal{O}_Y$-modules of rank r with Mukai vector v. In fact, every torsion-free G-twisted semistable $\mathcal{O}_Y$-module of rank r with Mukai vector $v_G = v$ is G-twisted stable.*

Proof. If $E \in Coh_l(X, \mathcal{A})$ is given such that E is a torsion-free $\mathcal{A}$-module of rank one with Mukai vector $v = v_{\mathcal{A}}(E)$, then we have $\mathrm{rk}(E) = r^2$. Now E has no $\mathcal{A}$-submodules $E' \subsetneq E$ with $0 < \mathrm{rk}(E') < r^2$ because any such module must satisfy $r^2 | \mathrm{rk}(E')$ which is impossible. So $F(E)$ is a torsion-free $\mathcal{O}_Y$-module of rank r, has Mukai vector $v_G = v$ and has no nontrivial submodules in $Coh(Y, X)$, since F preserves submodules. $F(E)$ is therefore G-twisted stable.

On the other hand if $E \in Coh(Y, X)$ is given and E is a torsion-free G-twisted semistable $\mathcal{O}_Y$-module of rank r with $v_G(E) = v$, then we have $H(E) \in Coh_l(X, \mathcal{A})$ and $H(E)$ is torsion-free and of rank one, as we have $\mathrm{rk}(H(E)) = r^2$. We can conclude that E must be G-twisted stable, because any nontrivial submodule $E' \subsetneq E$ in $Coh(Y, X)$ with $0 < \mathrm{rk}(E') < r$ would give rise to an $\mathcal{A}$-submodule $H(E') \subsetneq H(E)$ in $Coh_l(X, \mathcal{A})$ with $0 < \mathrm{rk}(H(E')) < r^2$ as H also preserves submodules. But this is impossible since $H(E)$ is a torsion-free $\mathcal{A}$-module of rank one. $\square$

Remark 2.7. The previous lemma 2.6 shows that given a Mukai vector $v = v_{\mathcal{A}}(E)$ for some torsion-free $\mathcal{A}$-module E of rank one, we do not need to worry if the polarization H is general with respect to v, see 2.5. All torsion-free G-twisted semistable $\mathcal{O}_Y$-modules of rank r and with Mukai vector $v_G = v$ are automatically G-twisted stable. So $M_H^{Y,G}(v)$ is a smooth projective scheme without choosing a special polarization H.

Looking at the definition of the moduli functors, we see that we are working with two kinds of families of sheaves:

i) a family of torsion-free $\mathcal{A}$-modules of rank one with Mukai vector v over a $\mathbb{C}$-scheme S is a sheaf $\mathcal{E} \in Coh_l(X \times S, \mathcal{A}_S)$ such that $\mathcal{E}$ is flat over S and $\mathcal{E}_s = i_s^*\mathcal{E}$ is a torsion-free $\mathcal{A}_s$-module of rank one on X_s with Mukai vector v for every closed point $s \in S$, especially $\mathcal{E}_s \in Coh_l(X_s, \mathcal{A}_s)$.

ii) a family of torsion-free G-twisted semistable $\mathcal{O}_Y$-modules of rank r with Mukai vector v over a $\mathbb{C}$-scheme S is a sheaf $\mathcal{F} \in Coh(Y \times S, X \times S)$ such that $\mathcal{F}$ is flat over S and $\mathcal{F}_s = j_s^*\mathcal{F}$ is a torsion-free G-twisted semistable $\mathcal{O}_Y$-module of rank r on Y_s with Mukai vector v for every closed point $s \in S$, especially $\mathcal{F}_s \in Coh(Y_s, X_s)$.

For every $S \in Sch_{\mathbb{C}}$ we have $X \times S \in Sch_{\mathbb{C}}$. Furthermore $\mathcal{A}_S$ is an Azumaya algebra on $X \times S$ with a functorial isomorphism $BS(\mathcal{A}_S) \cong Y \times S$, so lemma 1.10 shows that there is an equivalence

$$Coh_l(X \times S, \mathcal{A}_S) \cong Coh(Y \times S, X \times S)$$

which has the following property:

Lemma 2.8. *The equivalence $Coh_l(X \times S, \mathcal{A}_S) \cong Coh(Y \times S, X \times S)$ maps families of type i) to families of type ii) with semistable replaced by stable and vice versa.*

Proof. Let $\mathcal{E}$ be a family of type i). Define $\mathcal{F} := \pi_S^*\mathcal{E} \otimes_{\pi_S^*(\mathcal{A}_S^{op})} G_S$, then $\mathcal{F}$ is a family of type ii).

We have $\mathcal{F} \in Coh(Y \times S, X \times S)$ and $\mathcal{F}$ is flat over S. To see this, note that π_S is faithfully flat, so $\pi_S^*\mathcal{E}$ is flat over S, see [GD65, 2.2.11 (iii)]. Furthermore G_S is a flat $\pi_S^*(\mathcal{A}_S^{op})$-module, so $\mathcal{F}$ is flat over S. We also see that

$$\mathcal{F}_s = j_s^*(\pi_S^*\mathcal{E} \otimes_{\pi_S^*(\mathcal{A}_S^{op})} G_S) = \pi_s^*\mathcal{E}_s \otimes_{\pi_s^*(\mathcal{A}_s^{op})} G_s.$$

So $\mathcal{F}_s \in Coh(Y_s, X_s)$ and $\mathcal{F}_s$ is a torsion-free G-twisted stable $\mathcal{O}_Y$-module of rank r on Y_s with Mukai vector v by 2.6.

Let $\mathcal{F}$ be a family of type ii). Define $\mathcal{E} := \pi_{S*}(\mathcal{F} \otimes G_S^\vee)$, then $\mathcal{E}$ is a family of type i).

We have $\mathcal{E} \in Coh_l(X \times S, \mathcal{A}_S)$ and $\mathcal{E}$ is flat over S. To see this, we note that this can be tested after pullback with a faithfully flat morphism $f : Z \to X \times S$ by [GD65, 2.2.11 (iii)]. So we use $f = \pi_S$. But then $\pi_S^*\mathcal{E} = \pi_S^*\pi_{S*}(\mathcal{F} \otimes G_S^\vee) \cong \mathcal{F} \otimes G_S^\vee$ and the latter is flat over S since $\mathcal{F}$ and $G_S^\vee$ are. Finally

$$\begin{aligned}\mathcal{E}_s = i_s^*\pi_{S*}(\mathcal{F} \otimes G_S^\vee) &\cong \pi_{s*}\pi_s^*i_s^*\pi_{S*}(\mathcal{F} \otimes G_S^\vee) \\ &= \pi_{s*}j_s^*\pi_S^*\pi_{S*}(\mathcal{F} \otimes G_S^\vee) \cong \pi_{s*}j_s^*(\mathcal{F} \otimes G_S^\vee) = \pi_{s*}(\mathcal{F}_s \otimes G_s^\vee).\end{aligned}$$

So $\mathcal{E}_s \in Coh_l(X_s, \mathcal{A}_s)$ and $\mathcal{E}_s$ is a torsion-free $\mathcal{A}_s$-module of rank one on X_s with Mukai vector v by 2.6. ☐

Lemma 2.9. *Assume $v = v_{\mathcal{A}}(E)$ is a Mukai vector for some torsion-free $\mathcal{A}$-module E of rank one, then the functors $\mathcal{M}_{\mathcal{A}/X}(v)$ and $\mathcal{M}_H^{Y,G}(v)$ are isomorphic.*

Proof. Let $S, T \in Sch_{\mathbb{C}}$ with $f : T \to S$, then we have the following commutative diagram:

$$\begin{array}{ccccc} Y \times T & \xrightarrow{f_Y} & Y \times S & \xrightarrow{q} & Y \\ {\scriptstyle \pi_T}\big\downarrow & & {\scriptstyle \pi_S}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ X \times T & \xrightarrow{f_X} & X \times S & \xrightarrow{p} & X \end{array}$$

with $f_X = id_X \times f$ and $f_Y = id_Y \times f$. This gives

$$\mathcal{M}_{\mathcal{A}/X}(v)(f) : \mathcal{M}_{\mathcal{A}/X}(v)(S) \to \mathcal{M}_{\mathcal{A}/X}(v)(T),\ [\mathcal{E}] \mapsto [f_X^*(\mathcal{E})]$$
$$\mathcal{M}_H^{Y,G}(v)(f) : \mathcal{M}_H^{Y,G}(v)(S) \to \mathcal{M}_H^{Y,G}(v)(T), \quad [\mathcal{F}] \mapsto [f_Y^*(\mathcal{F})].$$

Define a natural transformation $\Psi : \mathcal{M}_H^{Y,G}(v) \to \mathcal{M}_{\mathcal{A}/X}(v)$ by

$$\Psi_S : \mathcal{M}_H^{Y,G}(v)(S) \to \mathcal{M}_{\mathcal{A}/X}(v)(S),\ [\mathcal{F}] \mapsto [\pi_{S*}(\mathcal{F} \otimes G_S^{\vee})].$$

This map is well-defined by 2.8. One computes

$$\begin{aligned} \Psi_T(\mathcal{M}_H^{Y,G}(v)(f)[\mathcal{F}]) &= [\pi_{T*}(f_Y^*\mathcal{F} \otimes G_T^{\vee})] = [\pi_{T*}(f_Y^*(\mathcal{F} \otimes G_S^{\vee}))] \\ &= [\pi_{T*}(f_Y^*(\pi_S^*\pi_{S*}(\mathcal{F} \otimes G_S^{\vee})))] \\ &= [\pi_{T*}\pi_T^*(f_X^*(\pi_{S*}(\mathcal{F} \otimes G_S^{\vee})))] \\ &= [f_X^*(\pi_{S*}(\mathcal{F} \otimes G_S^{\vee}))] = \mathcal{M}_{\mathcal{A}/X}(v)(f)(\Psi_S([\mathcal{F}])) \end{aligned}$$

and hence $\Psi_T \circ \mathcal{M}_H^{Y,G}(v)(f) = \mathcal{M}_{\mathcal{A}/X}(v)(f) \circ \Psi_S$.

We define a another natural transformation $\eta : \mathcal{M}_{\mathcal{A}/X}(v) \to \mathcal{M}_H^{Y,G}(v)$ by

$$\eta_S : \mathcal{M}_{\mathcal{A}/X}(v)(S) \to \mathcal{M}_H^{Y,G}(v)(S),\ [\mathcal{E}] \mapsto [\pi_S^*\mathcal{E} \otimes_{\pi_S^*(\mathcal{A}_S^{op})} G_S].$$

This map is also well-defined by 2.8. η is a natural transformation due to the fact that pullbacks commute with tensor products.

Finally by what we have already seen Ψ_S are η_S are inverse bijections for every S, so Ψ is a natural isomorphism between these moduli functors. □

Corollary 2.10. *Assume $v = v_{\mathcal{A}}(E)$ is a Mukai vector for some torsion-free $\mathcal{A}$-module E of rank one, then the moduli schemes $M_{\mathcal{A}/X}(v)$ and $M_H^{Y,G}(v)$ are isomorphic.*

Proof. Since these schemes are coarse moduli spaces they corepresent their corresponding moduli functors, see [HL10, Definition 2.2.1]. Thus the natural isomorphisms η and Ψ from 2.9 induce natural isomorphisms

$$Hom_{Sch_{\mathbb{C}}}(-, M_{\mathcal{A}/X}(v)) \cong Hom_{Sch_{\mathbb{C}}}(-, M_H^{Y,G}(v))$$

by the universal property. But then Yoneda implies $M_{\mathcal{A}/X}(v) \cong M_H^{Y,G}(v)$. □

Using this corollary and [Yos06, Theorem 3.16.], we finally get our main result:

Theorem 2.11. *Assume $v = v_{\mathcal{A}}(E)$ is a primitive Mukai vector for some torsion-free $\mathcal{A}$-module E of rank one, then $M_{\mathcal{A}/X}(v)$ is an irreducible symplectic variety deformation equivalent to $Hilb^{\frac{v^2}{2}+1}(X)$. Moreover one has:*

- $M_{\mathcal{A}/X}(v) \neq \emptyset$ *if and only if* $v^2 \geq -2$

- *if $v^2 = 0$, then $M_{\mathcal{A}/X}(v)$ is a $K3$ surface.*

Corollary 2.12. *Assume $v = v_{\mathcal{A}}(E)$ is a primitive Mukai vector for some torsion-free $\mathcal{A}$-module E of rank one, then one has:*

- $h^{p,q}(M_{\mathcal{A}/X}(v)) = h^{p,q}(Hilb^{\frac{v^2}{2}+1}(X))$
- $b_i(M_{\mathcal{A}/X}(v)) = b_i(Hilb^{\frac{v^2}{2}+1}(X))$

Here $h^{p,q}$ are the Hodge numbers and b_i are the Betti numbers.

Proof. This follows from 2.11, using the fact that the Betti numbers and the Hodge numbers are invariant with respect to deformation equivalence. $\square$

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TORSION-FREE RANK ONE SHEAVES OVER DEL PEZZO ORDERS

NORBERT HOFFMANN AND FABIAN REEDE

Abstract. Let $\mathcal{A}$ be a del Pezzo order on the projective plane over the field of complex numbers. We prove that every torsion-free $\mathcal{A}$-module of rank one can be deformed into a locally free $\mathcal{A}$-module of rank one.

Introduction

An *order* on an algebraic variety X is a torsion-free coherent sheaf of $\mathcal{O}_X$-algebras whose generic stalk is a central division algebra over the function field of X. A surface together with an order on it can be thought of as a noncommutative surface. In this article we are interested in terminal del Pezzo orders on the projective plane $\mathbb{P}^2$ over the field of complex numbers $\mathbb{C}$. These orders are noncommutative analogues of classical del Pezzo surfaces and have been completely classified by D. Chan and C. Ingalls in the course of their proof of the minimal model program for orders over surfaces, see [CI05].

Let $\mathcal{A}$ be a terminal del Pezzo order on $\mathbb{P}^2$. Left $\mathcal{A}$-modules which are locally free and generically of rank one can be thought of as line bundles on this noncommutative surface. There is a quasi-projective coarse moduli scheme for these line bundles [HS05], a noncommutative analogue of the classical Picard scheme. To compactify this moduli scheme, that is to get a projective moduli scheme, one has to allow torsion-free left $\mathcal{A}$-modules which are generically of rank one.

We study the boundary of this compactification by studying the deformation theory of torsion-free $\mathcal{A}$-modules. The main result of this article is the following

Theorem. *Let $\mathcal{A} \neq \mathcal{O}_{\mathbb{P}^2}$ be a terminal del Pezzo order on $\mathbb{P}^2$ over $\mathbb{C}$. Then every torsion-free $\mathcal{A}$-module E of rank one can be deformed to a locally free $\mathcal{A}$-module E'.*

As a corollary, we obtain that every irreducible component of the compactification of the noncommutative Picard scheme contains a point defined by an $\mathcal{A}$-line bundle.

The structure of this paper is as follows. We review the definition and some basic facts about terminal del Pezzo orders in section 1. In section 2 we study in detail the local deformation theory of $\mathcal{A}$-modules in this setting. We look at the homological algebra of torsion-free $\mathcal{A}$-modules and study

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the compactification of the noncommutative Picard scheme and some of its properties in section 3. In the final section 4 we study the global deformation theory and prove the main result.

1. Noncommutative del Pezzo surfaces

Let X be a smooth projective surface over $\mathbb{C}$.

Definition 1.1. An *order* $\mathcal{A}$ on X is sheaf of associative $\mathcal{O}_X$-algebras such that

- $\mathcal{A}$ is coherent and torsion-free as an $\mathcal{O}_X$-module, and
- the stalk $\mathcal{A}_\eta$ at the generic point $\eta \in X$ is a central division algebra over the function field $\mathbb{C}(X) = \mathcal{O}_{X,\eta}$ of X.

We can now look at all orders in $\mathcal{A}_\eta$ and order them by inclusion. A maximal element will be called a maximal order. These are the algebras we are interested in. Maximal orders have some nice properties, for example they are locally free $\mathcal{O}_X$-modules.

Furthermore, it is well known that there is a largest open subset $U \subset X$ on which $\mathcal{A}$ is even an Azumaya algebra, see for example [Tan81, Proposition 6.2]. The complement $D := X \setminus U$ is called the ramification locus of $\mathcal{A}$. It is the union of finitely many curves $C \subset X$, and contains valuable informations about the order $\mathcal{A}$.

The ramification of a maximal order $\mathcal{A}$ can be seen in the Artin-Mumford sequence:

Theorem 1.2 ([Tan81, Lemma 4.1])**.** *Let X be a smooth projective surface over $\mathbb{C}$. Then there is a canonical exact sequence*

$$0 \longrightarrow \mathrm{Br}(X) \longrightarrow \mathrm{Br}(\mathbb{C}(X)) \longrightarrow \bigoplus_{\substack{C \subset X \\ \textit{irreducible curve}}} \mathrm{H}^1(\mathbb{C}(C), \mathbb{Q}/\mathbb{Z}).$$

Here the Galois cohomology group $\mathrm{H}^1(\mathbb{C}(C), \mathbb{Q}/\mathbb{Z})$ classifies isomorphism classes of cyclic extensions of $\mathbb{C}(C)$. The ramification curves are exactly the curves where the Brauer class of $\mathcal{A}_\eta$ has nontrivial image in $\mathrm{H}^1(\mathbb{C}(C), \mathbb{Q}/\mathbb{Z})$. Thus every ramification curve C comes with a finite cyclic field extension $L/\mathbb{C}(C)$. The degree $e_C := [L : \mathbb{C}(C)]$ is called the ramification index of $\mathcal{A}$ at C.

We are interested in a special class of maximal orders on X, the so called terminal orders. To give a definition of terminal orders, let e, e' and f be positive integers such that e' divides e. We look at the complete local ring $R = \mathbb{C}[[u, v]]$ and define

$$S := R\langle x, y\rangle \text{ with the relations } x^{e'} = u, y^{e'} = v \text{ and } yx = \zeta xy$$

where ζ is a primitive e'-th root of unity. Then S is of finite rank over R, the center of S is R, and the tensor product $S \otimes_R K$ with the field of fractions $K := \mathrm{Quot}(R)$ is a division ring. Define the following R-subalgebra:

$$B := \begin{pmatrix} S & \cdots & \cdots & S \\ xS & S & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ xS & \cdots & xS & S \end{pmatrix} \subset M_{e/e'}(S) \tag{1}$$

Then we define the R-algebra A as a full matrix algebra over B:

$$A := M_f(B). \tag{2}$$

Note that the algebra $A = A_{e,e',f}$ depends on the integers e, e' and f. The following theorem describes some of its properties:

Theorem 1.3 ([CI05, Proposition 2.8]). *Let $A = A_{e,e',f}$ be the R-algebra defined by* (2)*.*

i) *A has global dimension two.*
ii) *If $e = e' = 1$, then A is unramified.*
iii) *If $e > e' = 1$, then A is ramified on $u = 0$, with ramification index e.*
iv) *If $e' > 1$, then A is ramified on $uv = 0$, with ramification index e on $u = 0$, and with ramification index e' on $v = 0$.*

Definition 1.4 ([CI05, Corollary 4.3]). A maximal order $\mathcal{A}$ on a smooth projective surface X over $\mathbb{C}$ is called *terminal* if and only if for every closed point $p \in X$ there is

- an isomorphism of complete local rings $\widehat{\mathcal{O}}_{X,p} \cong \mathbb{C}[[u, v]]$, and
- a $\mathbb{C}[[u, v]]$-algebra isomorphism $\mathcal{A}_p \otimes \widehat{\mathcal{O}}_{X,p} \cong A_{e,e',f}$ for some integers e, e' and f.

Definition 1.5 ([CK03, Lemma 8]). Assume $\mathcal{A}$ is a terminal order on a smooth projective surface X over $\mathbb{C}$, with ramification curves $\{C_i\}$ and ramification indices $\{e_i\}$. Then we define the canonical divisor class $K_{\mathcal{A}}$ of $\mathcal{A}$ by:

$$K_{\mathcal{A}} = K_X + \sum (1 - \frac{1}{e_i})C_i.$$

Lemma 1.6. *If $\mathcal{A}$ is a terminal order on a smooth projective surface X over $\mathbb{C}$, then*

$$K_{\mathcal{A}} = K_X - \frac{2c_1(\mathcal{A})}{\mathrm{rk}(\mathcal{A})}.$$

Proof. Theorem 1.83 in [Ree13] states that $c_1(\mathcal{A}) = -\frac{\mathrm{rk}(\mathcal{A})}{2} \sum (1 - \frac{1}{e_i})C_i$. □

Definition 1.7 ([CK03, Definition 7, Lemma 8]). A terminal order $\mathcal{A}$ on a smooth projective surface X over $\mathbb{C}$ is called a *del Pezzo order* if $-K_{\mathcal{A}}$ is ample.

If $\mathcal{A}$ is a terminal del Pezzo order on $\mathbb{P}^2$, then its ramification is rather limited:

Proposition 1.8 ([CI05, Proposition 3.21]). *Assume $\mathcal{A}$ is a terminal del Pezzo order on $\mathbb{P}^2$ with ramification locus $D = \bigcup C_i$ and ramification indices $\{e_i\}$. Then all ramification indices e_i are equal, and we have the inequalities $3 \le deg(D) \le 5$.*

Furthermore there are more constraints for the common ramification index $e \in \mathbb{N}$ depending on the degree of D, see for example [CI05, Proposition 3.21].

2. Punctual deformations of rank one modules

In this section we study the local situation. That is we replace the surface X over $\mathbb{C}$ by the complete local ring $R = \mathbb{C}[[u, v]]$, and the terminal order $\mathcal{A}$ on X by the R-algebra

$$A = A_{e,e',f}$$

defined in (2). The role of the dualizing sheaf will be played by the A-bimodule

$$A^* := \mathrm{Hom}(A, R).$$

Left ideals $I \subset A$ of R-colength $l < \infty$ are parameterized by the punctual Hilbert scheme

$$\mathrm{Hilb}_A(l),$$

which is a closed subscheme of the punctual Quot-scheme $\mathrm{Quot}_R(A, l)$ and hence projective over $\mathbb{C}$. We say that I can be *deformed* to another left ideal $I' \subset A$ if I' has the same colength $l < \infty$, and lies in the same connected component of $\mathrm{Hilb}_A(l)$.

Equivalently, $I \subset A$ can be deformed to $I' \subset A$ if and only if there is a sheaf of left ideals $\mathcal{I} \subset A_T := A \otimes_{\mathbb{C}} \mathcal{O}_T$ for some connected scheme T over $\mathbb{C}$ such that $A_T/\mathcal{I}$ is flat over $\mathcal{O}_T$, and $\mathcal{I}$ has fibers $\mathcal{I}_t = I$ and $\mathcal{I}_{t'} = I'$ for some points $t, t' \in T(\mathbb{C})$.

We consider three different cases, depending on the ramification of A.

2.1. **No ramification: $e = e' = 1$.** In this case, $A = M_f(R)$ is a full matrix algebra over $R = \mathbb{C}[[u, v]]$. We assume $f > 1$.

Lemma 2.1. *Every proper left ideal $I \subset A$ of finite colength can be deformed to a proper left ideal $I' \subset A$ of finite colength such that $I'A^* \not\subseteq A^*I'$.*

Proof. The left ideal $I \subset A$ is Morita equivalent to an R-submodule $M \subset R^f$ of some colength $l < \infty$. Choose an ideal $J \subset R$ of colength l. Then the R-submodule

$$M' := J \oplus R^{f-1} \subset R^f \tag{3}$$

is Morita equivalent to some left ideal $I' \subset A$. Since the punctual Quot-scheme

$$\mathrm{Quot}_R(R^f, l)$$

is irreducible according to [EL99, Proposition 6], $M \subset R^f$ can be deformed to $M' \subset R^f$. Therefore $I \subset A$ can be deformed to $I' \subset A$. It remains to prove $I'A^* \not\subseteq A^*I'$.

Assume for contradiction that $I'A^* \subseteq A^*I'$. Then $I'A \subseteq AI'$, because $A^* \cong A$ as A-bimodules by means of the trace form $A \otimes_R A \to R$. Hence I' is a two-sided ideal. Consequently, $I' = M_f(J')$ for some ideal $J' \subset R$. Therefore,

$$M' = (J')^f \subset R^f.$$

Since $f > 1$, this contradicts (3). Hence indeed $I'A^* \not\subseteq A^*I'$. $\square$

2.2. **Smooth ramification:** $e > e' = 1$**.** In this case, our algebra is given by $A = A_{e,1,f}$ over $R = \mathbb{C}[[u,v]]$ is ramified over $u = 0$, with ramification index e. Explicitly, we have $A = M_f(B)$ for

$$B = \begin{pmatrix} R & \cdots & \cdots & R \\ uR & R & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ uR & \cdots & uR & R \end{pmatrix} \subset M_e(R). \tag{4}$$

The aim of this subsection is to prove an analogue of Lemma 2.1 in this situation.

We have $A^* = M_f(B^*)$ for the B-bimodule $B^* := \mathrm{Hom}_R(B, R)$. The trace map

$$\mathrm{tr} : B_K := B \otimes_R K = M_e(K) \to K$$

allows us to identify B^* with the set of all $b \in B_K$ for which $\mathrm{tr}(bB) \subseteq R$; explicitly,

$$B^* = \begin{pmatrix} R & u^{-1}R & \cdots & u^{-1}R \\ R & R & \ddots & \vdots \\ \vdots & \vdots & \ddots & u^{-1}R \\ R & R & \cdots & R \end{pmatrix} \subset B_K = M_e(K).$$

In particular, $B^* = b^*B = Bb^*$ and $A^* = b^*A = Ab^*$ for the matrix

$$b^* := \begin{pmatrix} 0 & u^{-1} & 0 & \cdots & 0 \\ \vdots & 0 & u^{-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & 0 & u^{-1} \\ 1 & 0 & \cdots & \cdots & 0 \end{pmatrix} \in B^*, \tag{5}$$

where elements of $A = M_f(B)$ are multiplied componentwise by $b^* \in B^*$.

We see from (4) that B has exactly e two-sided maximal ideals $\mathfrak{m}_i$, given by replacing R by its maximal ideal $\mathfrak{m}$ in the diagonal entry (i,i) respectively. So there are also exactly e non-isomorphic simple B-modules $S_i := B/\mathfrak{m}_i$. We have

$$B^* \otimes_B S_1 \cong S_e \quad \text{and} \quad B^* \otimes_B S_i \cong S_{i-1} \quad \text{for } i \geq 2,$$

because $b^*\mathfrak{m}_1 = \mathfrak{m}_e b^*$ and $b^*\mathfrak{m}_i = \mathfrak{m}_{i-1}b^*$ for $i \geq 2$, as is easily checked. Using Morita equivalence, we see that there are e simple left A-modules, all of R-length f.

Corollary 2.2. *$Hilb_A(l)$ is nonempty if and only if f divides l.*

Lemma 2.3. *Let $I \subset A$ be a left ideal such that $IA^* \subseteq A^*I$. Then I is a two-sided ideal. In particular, $I = M_f(J)$ for some two-sided ideal $J \subset B$ such that $JB^* \subseteq B^*J$.*

Proof. Let $b^* \in B^*$ still be the matrix given by (5). The finitely generated R-modules A/I and $A^*/A^*I = b^*A/b^*I$ are isomorphic, as b^* is invertible in $M_e(K)$. The R-linear map

$$\phi : A/I \to A^*/A^*I, \qquad a + I \mapsto ab^* + A^*I$$

is well-defined since $Ib^* \subseteq A^*I$ by assumption, and surjective since we have $Ab^* = A^*$. Therefore, ϕ is also injective, according to [Mat89, Theorem 2.4]. Since ϕ is by definition A-linear from the left, and $IA^* \subseteq A^*I$ by assumption, we conclude that $IA \subseteq I$. $\square$

Lemma 2.4. *Let $J \subset B$ be a left ideal such that $JB^* \subseteq B^*J$. Then*

$$J = \begin{pmatrix} J_e & J_{e-1} & \cdots & J_1 \\ uJ_1 & J_e & \ddots & \vdots \\ \vdots & \ddots & \ddots & J_{e-1} \\ uJ_{e-1} & \cdots & uJ_1 & J_e \end{pmatrix} \subset M_e(R) \tag{6}$$

for some chain of ideals $R \supseteq J_1 \supseteq J_2 \supseteq \cdots \supseteq J_e$ with $J_e \supseteq uJ_1$.

Proof. Since Lemma 2.3 applies to $J \subset B$, it shows that J is a two-sided ideal in B. We denote the standard basis elements of the free R-module B by

$$b_{i,j} \in M_e(R), \qquad 1 \leq i, j \leq e. \tag{7}$$

In other words, the matrix $b_{i,j}$ has a single nonzero entry in row i and column j, which is 1 for $i \leq j$ and u for $i > j$. Since J is two-sided, we have $b_{i,i}Jb_{j,j} \subseteq J$, and therefore

$$b_{i,i}Jb_{j,j} = J_{i,j}b_{i,j}$$

for some ideals $J_{i,j} \subseteq R$. As $b_{1,1} + b_{2,2} + \cdots + b_{e,e} = 1$ in B, we conclude that

$$J = \begin{pmatrix} J_{1,1} & J_{1,2} & \cdots & J_{1,e} \\ uJ_{2,1} & J_{2,2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & J_{e-1,e} \\ uJ_{e,1} & \cdots & uJ_{e,e-1} & J_{e,e} \end{pmatrix} \subset M_e(R).$$

Using this description, the other assumption $Jb^* \subseteq b^*J$ directly implies

$$J_{i,e} \subseteq J_{i+1,1} \subseteq J_{i+2,2} \subseteq \cdots \subseteq J_{e,e-i} \subseteq J_{1,e-i+1} \subseteq J_{2,e-i+2} \subseteq \cdots \subseteq J_{i,e}$$

for $i = 1, \ldots, e$. Hence these inclusions are all equalities, and (6) holds with $J_i := J_{i,e}$. Using (6), the assumption $J \supseteq b_{1,2}J$ directly implies that we must have $J_1 \supseteq J_2 \supseteq \cdots \supseteq J_e \supseteq uJ_1$. $\square$

Proposition 2.5. *Every proper left ideal $I \subset A$ of finite colength can be deformed to a proper left ideal $I' \subset A$ of finite colength such that $I'A^* \not\subseteq A^*I'$.*

Proof. We may assume $IA^* \subseteq A^*I$, since otherwise there is nothing to prove. Using Lemma 2.3 and Lemma 2.4, we get $I = M_f(J)$ with $J \subset B$ given by (6) for some ideals

$$R \supseteq J_1 \supseteq J_2 \supseteq \cdots \supseteq J_e$$

of finite colength, not all equal to R, such that $J_e \supseteq uJ_1$. It suffices to deform J to a left ideal $J' \subset B$ such that $J'B^* \not\subseteq B^*J'$. Changing J only in the first row, we will take

$$J' = \begin{pmatrix} J'_e & J'_{e-1} & \cdots & J'_1 \\ uJ_1 & J_e & \cdots & J_2 \\ \vdots & \ddots & \ddots & \vdots \\ uJ_{e-1} & \cdots & uJ_1 & J_e \end{pmatrix} \subset M_e(R) \tag{8}$$

for some ideals $J'_1, \dots, J'_e \subseteq R$, chosen as follows.

Suppose that $J_1 = \cdots = J_e$. Since $\mathfrak{m}J_e \neq J_e$ by Nakayama's lemma, the vector space $J_e/\mathfrak{m}J_e$ over $R/\mathfrak{m} = \mathbb{C}$ has a one-dimensional quotient. Hence we can find an ideal

$$J'_e \subseteq J_e \quad \text{with} \quad J_e/J'_e \cong \mathbb{C} \quad \text{as } R\text{-modules.}$$

Since $J_1 \neq R$ by assumption, the R-module R/J_1 of finite length has a simple submodule, which is necessarily isomorphic to $R/\mathfrak{m} = \mathbb{C}$. Hence we can find an ideal

$$J'_1 \supseteq J_1 \quad \text{with} \quad J'_1/J_1 \cong \mathbb{C} \quad \text{as } R\text{-modules.}$$

Finally, we take $J'_i = J_i$ for $i \neq e, 1$ in this case.

Now suppose that $J_1 = \cdots = J_e$ is not true. Choose an index m with $J_m \neq J_{m+1}$. Then the R-module J_m/J_{m+1} of finite length has a simple submodule and a simple quotient, which are both necessarily isomorphic to $R/\mathfrak{m} = \mathbb{C}$. Hence we can find two ideals

$$J_m \supseteq J'_m, J'_{m+1} \supseteq J_{m+1} \quad \text{with} \quad J_m/J'_m \cong \mathbb{C} \cong J'_{m+1}/J_{m+1} \quad \text{as } R\text{-modules.}$$

Finally, we take $J'_i = J_i$ for $i \neq m, m+1$ in this case.

To show that the R-submodule $J' \subseteq B$ defined by (8) is a left ideal, we check that the basis elements $b_{i,j} \in B$ in (7) satisfy $b_{i,j}J' \subseteq J'$. This clearly holds for $i = j = 1$, and also for $i, j \geq 2$ because J is a left ideal. In each of the two cases considered above, the ideals $J'_1, \dots, J'_e \subseteq R$ satisfy by construction

$$J_i \subseteq J'_{i-1} \quad \text{for} \quad i \geq 2, \quad \text{and} \quad uJ_1 \subseteq J'_e.$$

This directly implies $b_{1,2}J' \subseteq J'$. Similarly, $J'_1, \dots, J'_e$ also satisfy by construction

$$J'_i \subseteq J_{i-1} \quad \text{for} \quad i \geq 2, \quad \text{and} \quad uJ'_1 \subseteq J_e.$$

This directly implies $b_{e,1}J' \subseteq J'$. Using $b_{1,i} = b_{1,2}b_{2,i}$ and $b_{i,1} = b_{i,e}b_{e,1}$ for $i \geq 2$, we conclude that that $J' \subseteq B$ is indeed a left ideal.

Since J' is by construction not of the form (6), Lemma 2.4 shows that $J'B^* \not\subseteq B^*J'$. It remains to prove that J can be deformed to J'.

The left B-modules $J/(J \cap J')$ and $J'/(J \cap J')$ are, by construction of J', both isomorphic to the simple module $S_1 = B/\mathfrak{m}_1$. Consequently, the sum $J + J' \subseteq B$ satisfies

$$\frac{J + J'}{J \cap J'} \cong \frac{J}{J \cap J'} \oplus \frac{J'}{J \cap J'} \cong S_1 \oplus S_1 \cong \mathbb{C}^2$$

where all these B-modules are $\mathbb{C}$-vector spaces because B acts on them via $B \twoheadrightarrow B/\mathfrak{m}_1 \cong \mathbb{C}$. We consider the $\mathbb{P}^1$ of lines in this $\mathbb{C}^2$. The universal quotient

$$\mathbb{C}^2 \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^1} \twoheadrightarrow \mathcal{O}_{\mathbb{P}^1}(1)$$

over this $\mathbb{P}^1$ gives rise to a family of B-module quotients

$$(J + J') \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^1} \twoheadrightarrow \mathcal{O}_{\mathbb{P}^1}(1).$$

Its kernel $\mathcal{J} \subset B \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^1}$ restricts to J over $[1 : 0] \in \mathbb{P}^1$, and to J' over $[0 : 1] \in \mathbb{P}^1$. Therefore, $\mathcal{J}$ is the required deformation of J to J'. □

2.3. **Singular ramification:** $e = e' > 1$**.** In this case, our algebra is given by $A = A_{e,e,f}$ over $R = \mathbb{C}[[u, v]]$ is ramified over $u = 0$ and over $v = 0$, with common ramification index e. Explicitly, we have $A = M_f(S)$ for

$$S = R\langle x, y\rangle \text{ with the relations } x^e = u, y^e = v \text{ and } yx = \zeta xy$$

where ζ is a primitive e-th root of unity. The ring S is local in the sense that it has a unique two-sided maximal ideal $\mathfrak{n} \subset S$, which is generated by x and y.

In this situation, the analogue of Lemma 2.1 is no longer true; a counterexample is given by $f = 1$ and $I = \mathfrak{n}$. However, the following fact will suffice for our purposes.

Lemma 2.6. $\mathrm{Hilb}_A(l)$ *is connected if f divides l, and it is empty otherwise.*

Proof. The unique simple S-module $S/\mathfrak{n} \cong \mathbb{C}$ has R-length one. Therefore, $S/\mathfrak{n}$ is Morita equivalent to a unique simple left A-module, whose R-length is f.

Now one can just copy the corresponding part in the proof of [HS05, Theorem 3.6. iii)] and replace the Quot- and the Flag-scheme by the punctual versions. The main point is that induction also works in this case, because A has just one simple left module. □

3. Moduli spaces of rank one sheaves

Let $\mathcal{A}$ be a terminal order on a smooth projective surface X over $\mathbb{C}$.

Definition 3.1 ([CK03, Definition 4])**.** The *canonical bimodule* of $\mathcal{A}$ is

$$\omega_{\mathcal{A}} := \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{A}, \omega_X).$$

Lemma 3.2 ([Ree13, Theorem 1.58])**.** *Let E and F be two $\mathcal{O}_X$-coherent left $\mathcal{A}$-modules. Then there is the following form of Serre duality:*

$$\mathrm{Ext}^i_{\mathcal{A}}(E, F) \cong \mathrm{Ext}^{2-i}_{\mathcal{A}}(F, \omega_{\mathcal{A}} \otimes_{\mathcal{A}} E)^\vee$$

for $i \in \{0, 1, 2\}$. Here $(\_)^\vee$ denotes the $\mathbb{C}$-dual.

Lemma 3.3 ([Ree13, Lemma 1.62])**.** *Let E and T be $\mathcal{O}_X$-coherent left $\mathcal{A}$-modules such that E is locally projective and T is an Artinian module of finite length. Then the map*

$$\mathrm{Ext}^2_{\mathcal{A}}(T, E) \to \mathrm{Ext}^2_{\mathcal{O}_X}(T, E)$$

induced by the forgetful functor $\mathcal{A}$-mod $\to \mathcal{O}_X$-mod is injective.

Definition 3.4. A left $\mathcal{A}$-module E is called a *torsion-free $\mathcal{A}$-module of rank one* if

- E is coherent and torsion-free as an $\mathcal{O}_X$-module, and
- the stalk E_η at the generic point $\eta \in X$ has dimension 1 over the division ring $\mathcal{A}_\eta$.

Lemma 3.5 ([CC15, Proposition 4.2.]). *Let E be a torsion-free $\mathcal{A}$-module of rank one which is a locally free $\mathcal{O}_X$-module, then for every closed point $p \in X$ there is an isomorphism of completions*

$$\widehat{E}_p \cong \widehat{\mathcal{A}}_p.$$

Thus E is locally free over $\mathcal{A}$ if and only if E is locally free over $\mathcal{O}_X$.

Lemma 3.6 ([Ree13, Theorem 1.84]). *If E is a torsion-free $\mathcal{A}$-module of rank one, then*

$$c_1(\mathcal{A}^* \otimes_{\mathcal{A}} E) = c_1(E) - 2c_1(\mathcal{A})$$

where $\mathcal{A}^* := \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{A}, \mathcal{O}_X)$ denotes the dual sheaf of $\mathcal{A}$.

Definition 3.7. A *family of torsion-free $\mathcal{A}$-modules of rank one* over a $\mathbb{C}$-scheme T is a left module $\mathcal{E}$ under the pullback $\mathcal{A}_T$ of $\mathcal{A}$ to $X \times T$ with the following properties:

- $\mathcal{E}$ is coherent over $\mathcal{O}_{X\times T}$ and flat over T;
- for every $t \in T$, the fiber $\mathcal{E}_t$ is a torsion-free $\mathcal{A}_{\mathbb{C}(t)}$-module of rank one.

Here $\mathbb{C}(t)$ is the residue field of T at t, and the fiber is the pullback of $\mathcal{E}$ to $X \times \operatorname{Spec} \mathbb{C}(t)$.

Now one can define the moduli functor

$$\mathcal{M}_{\mathcal{A}/X;P} : \mathrm{Schemes}_{\mathbb{C}} \to \mathrm{Sets}$$

which sends a $\mathbb{C}$-scheme T to the set of isomorphism classes of families $\mathcal{E}$ of torsion-free $\mathcal{A}$-modules of rank one over T with Hilbert polynomial P.

Theorem 3.8 ([HS05, Theorem 2.4]). *There is a coarse moduli scheme $M_{\mathcal{A}/X;P}$ for the functor $\mathcal{M}_{\mathcal{A}/X;P}$. The scheme $M_{\mathcal{A}/X;P}$ is of finite type and projective over $\mathbb{C}$.*

Instead of fixing the Hilbert polynomial, one can also fix the Chern classes of these modules. We will work with the moduli space $M_{\mathcal{A}/X;c_1,c_2}$ of torsion-free $\mathcal{A}$-modules of rank one over X with Chern classes $c_1 \in \mathrm{NS}(X)$ and $c_2 \in \mathbb{Z}$.

Lemma 3.9. *Let $\mathcal{A}$ be a terminal del Pezzo order on $\mathbb{P}^2$ over $\mathbb{C}$. If E and F are torsion-free $\mathcal{A}$-modules of rank one with $c_1(E) = c_1(F)$, then $\mathrm{Ext}^2_{\mathcal{A}}(E, F) = 0$.*

Proof. Assume for contradiction that $\mathrm{Ext}^2_{\mathcal{A}}(E, F) \neq 0$. Then Serre duality for $\mathcal{A}$-modules states that there is a nonzero map

$$\phi : F \to \omega_{\mathcal{A}} \otimes_{\mathcal{A}} E.$$

Since E and F are generically simple and torsion-free, ϕ is generically bijective and therefore injective, and its cokernel is a torsion sheaf. This means that the divisor class

$$c_1(\omega_{\mathcal{A}} \otimes_{\mathcal{A}} E) - c_1(F) \tag{9}$$

is effective. On the other hand, Definition 3.1, Lemma 3.6 and Lemma 1.6 imply that

$$\begin{aligned} c_1(\omega_{\mathcal{A}} \otimes_{\mathcal{A}} E) &= c_1(\mathcal{A}^* \otimes_{\mathcal{A}} E) + \mathrm{rk}(\mathcal{A})c_1(\omega_{\mathbb{P}^2}) \\ &= c_1(E) - 2c_1(\mathcal{A}) + \mathrm{rk}(\mathcal{A})K_{\mathbb{P}^2} \\ &= c_1(E) + \mathrm{rk}(\mathcal{A})K_{\mathcal{A}}. \end{aligned}$$

Hence the class in (9) equals $\mathrm{rk}(\mathcal{A})K_{\mathcal{A}}$. But $\mathcal{A}$ is a del Pezzo order, so $-K_{\mathcal{A}}$ is ample. Since $\mathrm{Pic}(\mathbb{P}^2) = \mathbb{Z} \cdot [\mathcal{O}(1)]$, we conclude that $K_{\mathcal{A}}$ and the class in (9) are negative multiples of $[\mathcal{O}(1)]$, and therefore not effective. This contradiction proves $\mathrm{Ext}^2_{\mathcal{A}}(E, F) = 0$. □

Theorem 3.10. *If $\mathcal{A}$ is a terminal del Pezzo order on $\mathbb{P}^2$ over $\mathbb{C}$, then the moduli space $M_{\mathcal{A}/\mathbb{P}^2;c_1,c_2}$ of torsion-free $\mathcal{A}$-modules of rank one with Chern classes c_1 and c_2 is smooth.*

Proof. Let E be a torsion-free $\mathcal{A}$-module of rank one with Chern classes c_1 and c_2. Then $\mathrm{Ext}^2_{\mathcal{A}}(E, E) = 0$ according to Lemma 3.9. In particular, all obstruction classes in $\mathrm{Ext}^2_{\mathcal{A}}(E, E)$ vanish. This implies that $M_{\mathcal{A}/\mathbb{P}^2;c_1,c_2}$ is smooth at the point $[E]$. □

4. Deformations of torsion-free rank one sheaves

Let $\mathcal{A}$ be a terminal del Pezzo order of rank $n^2 > 1$ on the projective plane $\mathbb{P}^2$ over $\mathbb{C}$. Let $D \subset \mathbb{P}^2$ denote the ramification divisor. Proposition 1.8 states that $\mathcal{A}$ has the same ramification index e at every component of D. We put $f := n/e$.

Proposition 4.1. *Let E be a locally free left $\mathcal{A}$-module of rank one. Let*

$$\pi : E \twoheadrightarrow T \tag{10}$$

be a nonzero quotient of finite length. Then π can be deformed to a nonzero quotient

$$\pi' : E \twoheadrightarrow T' \tag{11}$$

of finite length such that the following induced map is not injective:

$$\pi'_* : \mathrm{Ext}^2_{\mathcal{A}}(T', E) \to \mathrm{Ext}^2_{\mathcal{A}}(T', T') \tag{12}$$

Proof. Choose $p \in \mathbb{P}^2$ in the support of T. As T has finite length, its support is finite, and

$$T = T_p \oplus T_{\neq p}$$

where T_p is supported at p, and $T_{\neq p}$ is supported outside p. We distinguish three cases, depending on the ramification of $\mathcal{A}$ at p.

The first case is that p is a smooth point of the ramification divisor D. Let $A := \widehat{\mathcal{A}}_p$ denote the completion of $\mathcal{A}$ at p, that is we have $A \cong A_{e,1,f}$. Choosing an isomorphism of completions given by Lemma 3.5

$$\widehat{E}_p \cong A, \tag{13}$$

we can identify the quotient T_p of $\widehat{E}_p$ with A/I for some left ideal $I \subset A$ of finite colength. Proposition 2.5 allows us to deform I to a left ideal $I' \subset A$ of finite colength such that

$$I'A^* \not\subseteq A^*I'.$$

Therefore, T_p can be deformed to $T'_p := A/I'$ as a quotient of A, and the given quotient π in (10) can be deformed to the quotient

$$\pi' : E \twoheadrightarrow T' := T'_p \oplus T_{\neq p}.$$

To prove that π'_* in (12) is not injective, we choose an element $a^* \in A^*$ with $I'a^* \not\subseteq A^*I'$. Then the left A-module homomorphism

$$\phi : A \to A^*/A^*I' = A^* \otimes_A T'_p, \qquad a \mapsto aa^* + A^*I',$$

does not vanish on I', and hence does not factor through $A/I' = T'_p$. Therefore, the map

$$\mathrm{Hom}_A(T'_p, A^* \otimes_A T'_p) \to \mathrm{Hom}_A(A, A^* \otimes_A T'_p)$$

induced by the projection $A \twoheadrightarrow T'_p$ is not surjective, as its image does not contain ϕ. Using the identification (13) and the decomposition $T' = T'_p \oplus T_{\neq p}$, we conclude that

$$(\pi')^* : \mathrm{Hom}_{\mathcal{A}}(T', \omega_{\mathcal{A}} \otimes_{\mathcal{A}} T') \to \mathrm{Hom}_{\mathcal{A}}(E, \omega_{\mathcal{A}} \otimes_{\mathcal{A}} T')$$

is not surjective. Hence the map π'_* in (12) is not injective, by Serre duality for $\mathcal{A}$-modules.

The second case is that $\mathcal{A}$ is unramified at p. This case is simpler than the first case. However, the same argument works, using Lemma 2.1 instead of Proposition 2.5.

The third case is that p lies in the singular locus D^{sing} of the ramification divisor D. Let l be the $\mathcal{O}_{\mathbb{P}^2}$-length of T_p. Then $\pi_p : E \twoheadrightarrow T_p$ defines a point in the scheme

$$\mathrm{Quot}_{\mathcal{A}}(E, l)$$

that classifies left $\mathcal{A}$-module quotients of E with $\mathcal{O}_{\mathbb{P}^2}$-length l. This is a closed subscheme of $\mathrm{Quot}_{\mathcal{O}_{\mathbb{P}^2}}(E, l)$, and hence projective over $\mathbb{C}$. It comes with a Hilbert-Chow morphism

$$\mathrm{supp} : \mathrm{Quot}_{\mathcal{A}}(E, l) \to \mathrm{Sym}^l(\mathbb{P}^2), \tag{14}$$

whose fiber over $l \cdot q$ for $q \in \mathbb{P}^2$ is the punctual Hilbert scheme for the completion $\widehat{\mathcal{A}}_q$:

$$\mathrm{supp}^{-1}(l \cdot q) = \mathrm{Hilb}_{\widehat{\mathcal{A}}_q}(l). \tag{15}$$

For $q = p$, this fiber contains the point T_p, and is therefore non-empty. Using Lemma 2.6, we conclude that f divides l. Hence (15) is non-empty for each ramified point $q \in D$ by Corollary 2.2. In other words, the image of the morphism supp in (14) contains the diagonally embedded

$$D \subset \mathbb{P}^2 \hookrightarrow \mathrm{Sym}^l(\mathbb{P}^2).$$

Let $\Delta \subset D$ be the finite set of all points $q \neq p$ in D^{sing} or in the support of T. Choose an irreducible component $C \subseteq D \setminus \Delta$ with $p \in C$. Let Q_i be the connected components of

$$\mathrm{supp}^{-1}(C) \subseteq \mathrm{Quot}_{\mathcal{A}}(E, l).$$

Since the morphism supp in (14) is projective, the image $\mathrm{supp}(Q_i)$ is closed in C. But the union of these images is all of C, which is irreducible. Hence

$$\mathrm{supp}(Q_i) = C$$

for some such connected component Q_i. Since $\operatorname{supp}^{-1}(l \cdot p)$ is connected by Lemma 2.6, and intersects Q_i by construction, it is contained in Q_i. In particular, the point given by

$$\pi_p : E \twoheadrightarrow T_p \tag{16}$$

lies in Q_i. Now choose a point $q \neq p$ in C, and a quotient

$$\pi'_q : E \twoheadrightarrow T'_q \tag{17}$$

corresponding to a point in Q_i over q. The restriction of the universal quotient to

$$Q_i \subset \operatorname{Quot}_{\mathcal{A}}(E, l)$$

provides a deformation of the quotient (16) to the quotient (17). Since $\operatorname{supp}(Q_i) = C \subset \mathbb{P}^2$ does not intersect the support of $T_{\neq p}$, we can take the direct sum with the component

$$\pi_{\neq p} : E \twoheadrightarrow T_{\neq p}$$

of π to obtain a deformation of the given quotient (10) to the quotient

$$\pi'_q \oplus \pi_{\neq p} : E \twoheadrightarrow T'_q \oplus T_{\neq p}.$$

As the support of this quotient contains the point $q \in D \setminus D^{\mathrm{sing}}$, we can apply the first case treated above to deform it further to a quotient (11) with the required property. $\square$

Theorem 4.2. *Let $\mathcal{A} \neq \mathcal{O}_{\mathbb{P}^2}$ be a terminal del Pezzo order on $\mathbb{P}^2$ over $\mathbb{C}$. Then every torsion-free $\mathcal{A}$-module E of rank one can be deformed to a locally free $\mathcal{A}$-module E'.*

Proof. We adapt the proof of [HS05, Theorem 3.6.(iii)] and start with the exact sequence

$$0 \longrightarrow E \xrightarrow{\iota} E^{**} \xrightarrow{\pi} T \longrightarrow 0 \tag{18}$$

induced by E. The functor $\operatorname{Hom}_{\mathcal{A}}(T, \_)$ turns (18) into the long exact sequence

$$\ldots \longrightarrow \operatorname{Ext}^2_{\mathcal{A}}(T, E) \xrightarrow{\iota_*} \operatorname{Ext}^2_{\mathcal{A}}(T, E^{**}) \xrightarrow{\pi_*} \operatorname{Ext}^2_{\mathcal{A}}(T, T) \longrightarrow 0.$$

Applying Proposition 4.1 to the quotient $\pi : E^{**} \twoheadrightarrow T$, and replacing E by the kernel of the resulting deformed quotient $\pi' : E^{**} \twoheadrightarrow T'$, we may assume that π_* is not injective. Then $\iota_* \neq 0$. The functor $\operatorname{Hom}_{\mathcal{A}}(\_, E)$ turns (18) into the long exact sequence

$$\ldots \longrightarrow \operatorname{Ext}^1_{\mathcal{A}}(E, E) \xrightarrow{\partial} \operatorname{Ext}^2_{\mathcal{A}}(T, E) \longrightarrow \operatorname{Ext}^2_{\mathcal{A}}(E^{**}, E) \longrightarrow \ldots$$

whose connecting homomorphism ∂ is surjective by Lemma 3.9. Hence the composition

$$\operatorname{Ext}^1_{\mathcal{A}}(E, E) \xrightarrow{\partial} \operatorname{Ext}^2_{\mathcal{A}}(T, E) \xrightarrow{\iota_*} \operatorname{Ext}^2_{\mathcal{A}}(T, E^{**})$$

is nonzero. We choose a class $\gamma \in \operatorname{Ext}^1_{\mathcal{A}}(E, E)$ whose image in $\operatorname{Ext}^2_{\mathcal{A}}(T, E^{**})$ is nonzero. The infinitesimal deformation of E given by γ can be extended to a deformation $\mathcal{E}$ of E over a smooth connected curve C, since we have $\operatorname{Ext}^2_{\mathcal{A}}(E, E) = 0$ by Lemma 3.9.

Let E' be the fiber of $\mathcal{E}$ over a general point of C. Lemma 3.3 states that the forgetful functor induces an injective map

$$\mathrm{Ext}^2_{\mathcal{A}}(T, E^{**}) \hookrightarrow \mathrm{Ext}^2_{\mathcal{O}_{\mathbb{P}^2}}(T, E^{**}).$$

So the class γ, seen as an element in $\mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}^2}}(E, E)$, has nonzero image in $\mathrm{Ext}^2_{\mathcal{O}_{\mathbb{P}^2}}(T, E^{**})$.

We can thus use a result of Artamkin, which says that the length of $(E')^{**}/E'$ is strictly smaller than the length of E^{**}/E, see [Art91, Corollary 1.3]. Using induction over this length, we may assume that E' can already be deformed to a locally free $\mathcal{A}$-module. ☐

Corollary 4.3. *Every irreducible component of the moduli space $M_{\mathcal{A}/\mathbb{P}^2;c_1,c_2}$ contains a point defined by a locally free $\mathcal{A}$-module.*

Proof. Every connected component of $M_{\mathcal{A}/\mathbb{P}^2;c_1,c_2}$ contains such a point by Theorem 4.2. But these connected components are smooth by Theorem 3.10, and hence irreducible. ☐

Corollary 4.4. *The open locus $M^{\mathrm{lf}}_{\mathcal{A}/\mathbb{P}^2;c_1,c_2}$ of locally free $\mathcal{A}$-modules is dense in $M_{\mathcal{A}/X;c_1,c_2}$.*

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RANK ONE SHEAVES OVER QUATERNION ALGEBRAS ON ENRIQUES SURFACES

FABIAN REEDE

Abstract. Let X be an Enriques surface over the field of complex numbers. We prove that there exists a nontrivial quaternion algebra $\mathcal{A}$ on X. Then we study the moduli scheme of torsion free $\mathcal{A}$-modules of rank one. Finally we prove that this moduli scheme is an étale double cover of a Lagrangian subscheme in the corresponding moduli scheme on the associated covering K3 surface.

Introduction

A *noncommutative variety* is a pair $(X, \mathcal{A})$ consisting of a classical complex algebraic variety X and a sheaf of noncommutative $\mathcal{O}_X$-algebras $\mathcal{A}$ of finite rank as an $\mathcal{O}_X$-module.

The algebras of interest in this article are Azumaya algebras. These are algebras locally isomorphic to a matrix algebra $M_r(\mathcal{O}_X)$ with respect to the étale topology. Especially interesting are the first nontrivial examples for $r = 2$, the so called quaternion algebras, Azumaya algebras of rank four. These are generalizations of the classical quaternions $\mathbb{H}$.

Since the generic stalk of a nontrivial quaternion algebra $\mathcal{A}$ is a central division algebra over the function field of X, locally projective left $\mathcal{A}$-modules which are generically of rank one can be understood as line bundles on $(X, \mathcal{A})$. There is a quasi-projective moduli scheme for these line bundles, a noncommutative Picard scheme, which can be compactified to a projective moduli scheme $\mathrm{M}_{\mathcal{A}/X}$ by adding torsion free $\mathcal{A}$-modules generically of rank one.

We study in detail the situation of Enriques surfaces. We prove that every Enriques surface X gives rise to a noncommutative Enriques surface $(X, \mathcal{A})$ with a quaternion algebra $\mathcal{A}$ on X. The main results of this article can be summarized as follows

Theorem. *Let X be an Enriques surface, then there is a quaternion algebra $\mathcal{A}$ on X representing the nontrivial element in the Brauer group* $\mathrm{Br}(X)$*. If X is very general then*

i) The moduli scheme $\mathrm{M}_{\mathcal{A}/X}$ of torsion free $\mathcal{A}$-modules of rank one is smooth.

ii) Every torsion free $\mathcal{A}$-module of rank one can be deformed into a locally projective $\mathcal{A}$-module, i.e. the locus $\mathrm{M}^{lp}_{\mathcal{A}/X}$ of locally projective $\mathcal{A}$-modules is dense in $\mathrm{M}_{\mathcal{A}/X}$.

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Let $\overline{X}$ be the universal covering K3 surface of X and denote the pullback of the quaternion algebra to $\overline{X}$ by $\overline{\mathcal{A}}$, then $\mathrm{M}_{\overline{\mathcal{A}}/\overline{X}}$ has a symplectic structure. For fixed Chern classes c_1 and c_2 we have

iii) $\mathrm{M}_{\mathcal{A}/X,c_1,c_2}$ *is an étale double cover of a Lagrangian subscheme $\mathcal{L}$ in* $\mathrm{M}_{\overline{\mathcal{A}}/\overline{X},\overline{c_1},\overline{c_2}}$.

The structure of this paper is as follows. We compare properties of modules over an Azumaya algebra on a smooth projective variety W to those of the pullbacks to an étale double cover $\overline{W}$ in section 1. In section 2 we prove that a classical descent result for modules on the double cover is also true in the noncommutative setting. We look at the existence of Azumaya algebras on Enriques surfaces in section 3. In the final section 4 we study moduli schemes of sheaves generically of rank one on a noncommutative Enriques surface, these were constructed in [10]. Many of the results in the last section are noncommutative analogues of results found by Kim in [11]. We work over the field of complex numbers $\mathbb{C}$.

1. Modules over an Azumaya algebra and double coverings

In this section W denotes a smooth projective complex variety of dimension d together with a nontrivial 2-torsion line bundle L. By [3, I.17] there is an étale Galois double cover

$$q : \overline{W} \to W$$

with covering involution $\iota : \overline{W} \to \overline{W}$ such that

$$q_*\mathcal{O}_{\overline{W}} \cong \mathcal{O}_W \oplus L.$$

Remark 1.1. We make the following convention: for every coherent sheaf E on W we write $\overline{E}$ for the pullback to $\overline{W}$ along q, that is $\overline{E} := q^*E$.

Definition 1.2. A sheaf of $\mathcal{O}_W$-algebras $\mathcal{A}$ is called an *Azumaya algebra* if it is locally free of finite rank and for every point $w \in W$ the fiber $\mathcal{A}(w)$ is a central simple algebra over the residue field $\mathbb{C}(w)$. Such a sheaf is called a quaternion algebra if $rk(\mathcal{A}) = 4$. Furthermore a coherent $\mathcal{O}_W$-module E is said to be an Azumaya module or an $\mathcal{A}$-module if E is also a left $\mathcal{A}$-module.

Azumaya algebras on W are classified up to similarity by the Brauer group $\mathrm{Br}(W)$ of W. We say $\mathcal{A}$ is trivial if there is a locally free $\mathcal{O}_W$-module P with $\mathcal{A} \cong \mathcal{E}nd_W(P)$ or equivalently $[\mathcal{A}] = 0 \in \mathrm{Br}(W)$. From now on, if not otherwise stated, by an Azumaya algebra $\mathcal{A}$ we mean a nontrivial Azumaya algebra. Furthermore we assume that there is a nontrivial Azumaya algebra $\mathcal{A}$ on W such that $\overline{\mathcal{A}}$ is nontrivial on $\overline{W}$.

Lemma 1.3. *Assume E and F are $\mathcal{A}$-modules and $f : Z \to W$ is a flat morphism, then*

$$\mathcal{H}om_{f^*\mathcal{A}}(f^*E, f^*F) \cong f^*\mathcal{H}om_{\mathcal{A}}(E, F).$$

Proof. First we note that by [9, 0.4.4.6] there is a natural morphism

$$f^*\mathcal{H}om_{\mathcal{A}}(E, F) \to \mathcal{H}om_{f^*\mathcal{A}}(f^*E, f^*F).$$

So after a faithfully flat étale base change we may assume that $\mathcal{A}$ is trivial. Then Morita equivalence for $\mathcal{A} = \mathcal{E}nd_W(P)$ reduces this problem to the case $\mathcal{A} = \mathcal{O}_W$. Now the lemma follows from [9, 0.6.7.6] since f is flat by assumption. $\square$

Lemma 1.4. *Assume E and F are $\mathcal{A}$-modules, then*

$$\mathrm{Hom}_{\overline{\mathcal{A}}}(\overline{E}, \overline{F}) \cong \mathrm{Hom}_{\mathcal{A}}(E, F) \oplus \mathrm{Hom}_{\mathcal{A}}(E, F \otimes L).$$

Proof. By the previous Lemma 1.3 we have an isomorphism

$$\mathcal{H}om_{\overline{\mathcal{A}}}(\overline{E}, \overline{F}) \cong \overline{\mathcal{H}om_{\mathcal{A}}(E, F)}.$$

This lemma is then a consequence of the following chain of isomorphisms, where the third line uses the projection formula for finite morphisms, [1, Lemma 5.7]:

$$\begin{aligned} q_*\mathcal{H}om_{\overline{\mathcal{A}}}(\overline{E}, \overline{F}) &\cong q_*\overline{\mathcal{H}om_{\mathcal{A}}(E, F)} \\ &= q_*q^*\mathcal{H}om_{\mathcal{A}}(E, F) \\ &\cong \mathcal{H}om_{\mathcal{A}}(E, F) \otimes q_*\mathcal{O}_{\overline{W}} \\ &\cong \mathcal{H}om_{\mathcal{A}}(E, F) \oplus \mathcal{H}om_{\mathcal{A}}(E, F \otimes L). \end{aligned}$$

$\square$

Corollary 1.5. *Assume E is an $\mathcal{A}$-module. If $\overline{E}$ is a simple $\overline{\mathcal{A}}$-module, then E is a simple $\mathcal{A}$-module and* $\mathrm{Hom}_{\mathcal{A}}(E, E \otimes L) = 0$.

Proof. As $\overline{E}$ is a simple $\overline{\mathcal{A}}$-module, we have $\mathrm{End}_{\overline{\mathcal{A}}}(\overline{E}) \cong \mathbb{C}$. Lemma 1.4 gives

$$\mathrm{End}_{\overline{\mathcal{A}}}(\overline{E}) \cong \mathrm{End}_{\mathcal{A}}(E) \oplus \mathrm{Hom}_{\mathcal{A}}(E, E \otimes L)$$

and as $\mathrm{id}_E \in \mathrm{End}_{\mathcal{A}}(E)$ we find $\mathrm{End}_{\mathcal{A}}(E) \cong \mathbb{C}$ and $\mathrm{Hom}_{\mathcal{A}}(E, E \otimes L) = 0$. $\square$

Proposition 1.6. [10, Proposition 3.5.] *Assume E and F are $\mathcal{A}$-modules, then there is the following variant of Serre duality:*

$$\mathrm{Ext}^i_{\mathcal{A}}(E, F) \cong \left(\mathrm{Ext}^{d-i}_{\mathcal{A}}(F, E \otimes \omega_W)\right)^\vee.$$

We assume now furthermore that $\dim W = 2$. Denote the $\mathcal{O}_W$-double dual of E by E^{**}.

Lemma 1.7. *Assume E is an $\mathcal{A}$-module which is torsion free as an $\mathcal{O}_W$-module. If $\overline{E^{**}}$ is a simple $\overline{\mathcal{A}}$-module, then*

$$\mathrm{Hom}_{\mathcal{A}}(E, E^{**} \otimes L) = 0.$$

Proof. We first observe that there is an isomorphism

$$\mathrm{End}_{\mathcal{A}}(E^{**}) \cong \mathrm{Hom}_{\mathcal{A}}(E, E^{**}). \tag{1}$$

To see this, we note that there is an exact sequence of $\mathcal{A}$-modules

$$0 \longrightarrow E \longrightarrow E^{**} \longrightarrow T \longrightarrow 0 \tag{2}$$

with $\dim \mathrm{supp}(T) = 0$ as E is torsion free and $\dim W = 2$. It is known that E^{**} is a locally free $\mathcal{O}_W$-module, hence a locally projective $\mathcal{A}$-module. This immediately implies $\mathrm{Hom}_{\mathcal{A}}(T, E^{**}) = 0$ since T is torsion. Furthermore this also shows $\mathrm{Ext}^1_{\mathcal{A}}(T, E^{**}) = 0$ by using Proposition 1.6, the local-to-global spectral sequence and the fact that T is supported in dimension zero. Applying $\mathrm{Hom}_{\mathcal{A}}(-, E^{**})$ to (2) and using the vanishing results gives the desired isomorphism.

Using the same argument for $\overline{E}$ shows that we also have an isomorphism

$$\mathrm{End}_{\overline{\mathcal{A}}}(\overline{E^{**}}) \cong \mathrm{Hom}_{\overline{\mathcal{A}}}(\overline{E}, \overline{E^{**}}) \tag{3}$$

since $\overline{E^{**}} \cong \overline{E}^{**}$ by [9, 0.6.7.6.].

We can now conclude as follows: as $\overline{E^{**}}$ is simple, the isomorphism (3) gives $\mathrm{Hom}_{\overline{\mathcal{A}}}(\overline{E}, \overline{E^{**}}) \cong \mathbb{C}$. By Corollary 1.5 and isomorphism (1) we also have $\mathrm{Hom}_{\mathcal{A}}(E, E^{**}) \cong \mathbb{C}$. Finally Lemma 1.4 gives an isomorphism

$$\mathrm{Hom}_{\overline{\mathcal{A}}}(\overline{E}, \overline{E^{**}}) \cong \mathrm{Hom}_{\mathcal{A}}(E, E^{**}) \oplus \mathrm{Hom}_{\mathcal{A}}(E, E^{**} \otimes L).$$

The last three isomorphisms show that we have $\mathrm{Hom}_{\mathcal{A}}(E, E^{**} \otimes L) = 0$. □

2. Noncommutative descent

We use the same notation as in the previous section. We have the étale Galois double cover $q : \overline{W} \to W$ with $\mathrm{Aut}(\overline{W}/W)$ generated by the covering involution ι:

$$\begin{tikzcd} \overline{W} \arrow[rr, "\iota"] \arrow[dr, "q"'] & & \overline{W} \arrow[dl, "q"] \\ & W & \end{tikzcd}$$

Definition 2.1. We say a coherent sheaf F of $\mathcal{O}_{\overline{W}}$-modules on $\overline{W}$ *descends* to W, if there is a coherent sheaf E of $\mathcal{O}_W$-modules on W together with an isomorphism $F \cong \overline{E}$.

Since $q : \overline{W} \to W$ is an étale Galois double cover with automorphism group $\mathrm{Aut}(\overline{W}/W) = \langle \iota \rangle \cong \mathbb{Z}/2\mathbb{Z}$, the descent condition for a coherent sheaf F on $\overline{W}$, see [17, Lemma 0D1V], reduces to the existence of an isomorphism $\varphi_\iota : F \to \iota^* F$ such that (using $\varphi_{\iota^2} = \mathrm{id}$):

$$\iota^* \varphi_\iota \circ \varphi_\iota = \mathrm{id}\,. \tag{4}$$

But we have $\iota^* \varphi_\iota \circ \varphi_\iota : F \to \iota^* \iota^* F \cong F$. So, for example, if F is simple, then any isomorphism φ_ι satisfies $\iota^* \varphi_\iota \circ \varphi_\iota \in \mathrm{End}_{\overline{W}}(F) = \mathbb{C} \cdot \mathrm{id}_F$. Hence after multiplication with an appropriate scalar, φ_ι satisfies (4) and F descends. Summarizing:

Proposition 2.2. *Assume F is a simple coherent $\mathcal{O}_{\overline{W}}$-module on $\overline{W}$ together with an isomorphism $F \cong \iota^* F$, then F descends to W.*

In the rest of this section we want to prove a similar result for $\overline{\mathcal{A}}$-modules on $\overline{W}$. For this we need some notation: let $p : Y \to W$ be the Brauer-Severi variety of $\mathcal{A}$, see [13] for more information. By functoriality the Brauer-Severi variety $\overline{p} : \overline{Y} \to \overline{W}$ of $\overline{\mathcal{A}}$ is given by $\overline{Y} = Y \times_W \overline{W}$ and thus $\overline{q} : \overline{Y} \to Y$ is also an étale Galois double cover with covering involution $\overline{\iota}$. All this fits into the following diagram with both squares cartesian:

$$\begin{tikzcd} \overline{Y} \arrow[r, "\overline{\iota}"] \arrow[d, "\overline{p}"'] & \overline{Y} \arrow[r, "\overline{q}"] \arrow[d, "\overline{p}"'] & Y \arrow[d, "p"] \\ \overline{W} \arrow[r, "\iota"] & \overline{W} \arrow[r, "q"] & W \end{tikzcd} \tag{5}$$

The Brauer-Severi variety of $\mathcal{A}$ has the property that $\mathcal{A}_Y := p^*\mathcal{A}$ is split, more exactly we have

$$\mathcal{A}_Y^{op} \cong \mathcal{E}nd_Y(G)$$

for a locally free sheaf G on Y. Note that G is only unique up to scalars, see [13, Remark 1.8], but there is a canonical choice using $R^1p_*\Omega_{Y/W} \cong \mathcal{O}_W$.

In the following we will frequently use, without further mention, the fact that a coherent left $\mathcal{A}$-module is the same as a coherent right $\mathcal{A}^{op}$-module. Denote these isomorphic categories by $\mathrm{Coh}_l(W, \mathcal{A})$ and $\mathrm{Coh}_r(W, \mathcal{A}^{op})$ respectively.

We also define

$$\mathrm{Coh}(Y, W) = \left\{ E \in \mathrm{Coh}(Y) \,|\, p^*p_*(E \otimes G^*) \xrightarrow{\cong} E \otimes G^* \right\}.$$

Then by [13, Lemma 1.10] we have the following equivalences

$$\begin{aligned} \phi &: \mathrm{Coh}_r(W, \mathcal{A}^{op}) \to \mathrm{Coh}(Y, W),\ E \mapsto p^*E \otimes_{\mathcal{A}_Y^{op}} G \\ \psi &: \mathrm{Coh}(Y, W) \to \mathrm{Coh}_r(W, \mathcal{A}^{op}),\ E \mapsto p_*(E \otimes G^*) \end{aligned}$$

We have similar equivalences $\overline{\phi}$ and $\overline{\psi}$ involving $\overline{\mathcal{A}}_{\overline{Y}}^{op} \cong \mathcal{E}nd_{\overline{Y}}(\overline{q}^*G)$, $\overline{Y}$ and $\overline{W}$.

Lemma 2.3. *Assume F is an $\overline{\mathcal{A}}$-module, then*

$$\mathrm{End}_{\overline{\mathcal{A}}}(F) \cong \mathrm{End}_{\overline{Y}}(\overline{\phi}(F)).$$

Proof. Using $\mathcal{E}nd_{\overline{\mathcal{A}}}(F) = \mathcal{E}nd_{\overline{\mathcal{A}}^{op}}(F)$, the following chain of isomorphisms gives the result:

$$\begin{aligned} \mathcal{E}nd_{\overline{\mathcal{A}}^{op}}(F) &\cong \overline{p}_*\overline{p}^*\mathcal{E}nd_{\overline{\mathcal{A}}^{op}}(F) && \text{by [13, Lemma 1.6]} \\ &\cong \overline{p}_*\mathcal{E}nd_{\overline{\mathcal{A}}_{\overline{Y}}^{op}}(\overline{p}^*F) && \text{by Lemma 1.3} \\ &\cong \overline{p}_*\mathcal{E}nd_{\mathcal{O}_{\overline{Y}}}(\overline{p}^*F \otimes_{\overline{\mathcal{A}}_{\overline{Y}}^{op}} \overline{q}^*G) && \text{by Morita equivalence} \\ &= \overline{p}_*\mathcal{E}nd_{\mathcal{O}_{\overline{Y}}}(\overline{\phi}(F)). \end{aligned}$$

□

Lemma 2.4. *Assume F is an $\overline{\mathcal{A}}$-module such that there is an isomorphism $F \cong \iota^*F$ of $\overline{\mathcal{A}}$-modules, then $\overline{\phi}(F) \cong \overline{\iota}^*(\overline{\phi}(F))$ as $\mathcal{O}_{\overline{Y}}$-modules.*

Proof. There are the following isomorphisms:

$$\begin{aligned} \overline{\iota}^*(\overline{\phi}(F)) &= \overline{\iota}^*(\overline{p}^*F \otimes_{\overline{\mathcal{A}}_{\overline{Y}}^{op}} \overline{q}^*G) \\ &\cong \overline{\iota}^*\overline{p}^*F \otimes_{\overline{\iota}^*\overline{\mathcal{A}}_{\overline{Y}}^{op}} \overline{\iota}^*\overline{q}^*G && \text{by [9, 0.4.3.3]} \\ &\cong \overline{p}^*\iota^*F \otimes_{\overline{\mathcal{A}}_{\overline{Y}}^{op}} \overline{q}^*G && \text{by (5)} \\ &\cong \overline{p}^*F \otimes_{\overline{\mathcal{A}}_{\overline{Y}}^{op}} \overline{q}^*G \\ &= \overline{\phi}(F). \end{aligned}$$

□

Lemma 2.5. *Assume F is an $\overline{\mathcal{A}}$-module such that there is $M \in Coh(Y)$ with $\overline{\phi}(F) \cong \overline{q}^*M$, then $M \in \mathrm{Coh}(Y, W)$.*

Proof. We have to prove that the canonical morphism

$$p^*p_*(M \otimes G^*) \to M \otimes G^* \tag{6}$$

is an isomorphism. It is enough to prove this after the faithfully flat base change $\overline{q} : \overline{Y} \to Y$:

$$\begin{array}{lll} & \overline{q}^*(p^*p_*(M \otimes G^*)) & \to \overline{q}^*(M \otimes G^*) \\ \cong & \overline{p}^*q^*p_*(M \otimes G^*) & \to \overline{q}^*M \otimes (\overline{q}^*G)^* \quad \text{by (5) and [9, 0.6.7.6]} \\ \cong & \overline{p}^*\overline{p}_*(\overline{q}^*M \otimes \overline{q}^*G^*)) & \to \overline{q}^*M \otimes (\overline{q}^*G)^* \quad \text{by (5) and [17, Lemma 02KH]} \\ \cong & \overline{p}^*\overline{p}_*(\overline{\phi}(F) \otimes \overline{q}^*G^*)) & \to \overline{\phi}(F) \otimes (\overline{q}^*G)^* \end{array}$$

But $\overline{\phi}(F) \in \mathrm{Coh}(\overline{Y}, \overline{W})$, so the last morphism is an isomorphism, hence so is (6). ☐

We can now prove the main result of this section:

Theorem 2.6. *Assume F is a simple $\overline{\mathcal{A}}$-module together with an isomorphism $F \cong \iota^*F$ of $\overline{\mathcal{A}}$-modules, then there is an $\mathcal{A}$-module E and an isomorphism of $\overline{\mathcal{A}}$-modules $F \cong \overline{E}$.*

Proof. Since F satisfies $F \cong \iota^*F$, by Lemma 2.4 we get an isomorphism $\overline{\phi}(F) \cong \overline{\iota}^*(\overline{\phi}(F))$. Since furthermore the $\mathcal{O}_{\overline{Y}}$-module $\overline{\phi}(F)$ is simple using Lemma 2.3, it descends to Y, so $\overline{\phi}(F) \cong \overline{q}^*M$ for some coherent $\mathcal{O}_Y$-module M. But then $M \in \mathrm{Coh}(Y, W)$ due to Lemma 2.5. Define $E := \psi(M)$ then $E \in Coh_l(W, \mathcal{A})$ and $\overline{E} \cong F$ since:

$$\overline{E} = q^*\psi(M) = q^*p_*(M \otimes G^*) \cong \overline{p}_*(\overline{q}^*M \otimes (\overline{q}^*G)^*) \cong \overline{p}_*(\overline{\phi}(F) \otimes (\overline{q}^*G)^*) \cong F.$$

☐

3. Quaternion algebras on Enriques surfaces

Definition 3.1. A smooth projective surface X is called an *Enriques surface* if:

- $H^1(X, \mathcal{O}_X) = 0$
- ω_X is 2-torsion, i.e. $\omega_X \neq \mathcal{O}_X$ but $\omega_X \otimes \omega_X \cong \mathcal{O}_X$.

The 2-torsion element $\omega_X \in \mathrm{Pic}(X)$ induces an étale Galois double cover

$$\pi : \overline{X} \to X.$$

It is well known that $\overline{X}$ is a K3 surface hence π is a universal cover of X. Denote the associated involution by $\iota : \overline{X} \to \overline{X}$.

By results of Cossec and Dolgachev, see [7, Theorem 1.1.3., Corollary 5.7.1.], we have:

Theorem 3.2. *Assume X is an Enriques surface over $\mathbb{C}$, then*

$$\mathrm{Br}(X) \cong \mathbb{Z}/2\mathbb{Z}.$$

This result shows that there is one nontrivial element b_X in the Brauer group of an Enriques surface X. The first question is if we can find a representative of this element in terms of Azumaya algebras.

Proposition 3.3. *The nontrivial element in the Brauer group of an Enriques surface X can be represented by a quaternion algebra $\mathcal{A}$ on X.*

Proof. The result of Cossec and Dolgachev shows that the nontrivial element $b_X \in \mathrm{Br}(X)$ has order two. As X is smooth by [6, Théorème 2.4.] the restriction to the generic point η gives an injection

$$r_\eta : \mathrm{Br}(X) \hookrightarrow \mathrm{Br}(\mathbb{C}(X)).$$

So the image $r_\eta(b_X)$ has order two in $\mathrm{Br}(\mathbb{C}(X))$.

The field $\mathbb{C}(X)$ has property C_2, see [16, II.4.5.(b)]. By a result of Platonov (simultaneously found by Artin and Harris) the element $r_\eta(b_X)$ can be represented by a quaternion algebra A over $\mathbb{C}(X)$, see [14, Theorem 5.7] ([2, Theorem 6.2.]).

Since the element $[A] = r_\eta(b_X)$ comes from $\mathrm{Br}(X)$ it is unramified at every point of codimension one in X, and thus by [6, Théorème 2.5.] there is a quaternion algebra $\mathcal{A}$ on X with $\mathcal{A}\otimes\mathbb{C}(X) = A$ such that $[\mathcal{A}] = b_X$. □

One natural question to ask then: Is the pullback of the nontrivial element still nontrivial in $\mathrm{Br}(\overline{X})$, i.e. is $\pi^* : \mathrm{Br}(X) \to \mathrm{Br}(\overline{X})$ injective? Beauville gives a complete answer to this question, see [4, Corollary 4.3., Corollary 5.7., Corollary 6.5.]:

Theorem 3.4. *The morphism $\pi^* : \mathrm{Br}(X) \to \mathrm{Br}(\overline{X})$ is trivial if and only if there is $L \in \mathrm{Pic}(\overline{X})$ with $\iota^* L = L^{-1}$ and $c_1(L)^2 \equiv 2 \pmod 4$. The surfaces X with $\pi^* b_X = 0$ form an infinite, countable union of (non-empty) hypersurfaces in the moduli space $\mathcal{M}$ of Enriques surfaces.*

Thus if X is a very general Enriques surface (in the sense of the previous theorem) then the pullback of the quaternion algebra $\mathcal{A}$ constructed in Proposition 3.3 to $\overline{X}$ represents the nontrivial element $\pi^* b_X \in \mathrm{Br}(\overline{X})$.

Remark 3.5. For a description of the (non)triviality of $\pi^* : \mathrm{Br}(X) \to \mathrm{Br}(\overline{X})$ using lattice theory, group cohomology and the Hochschild-Serre spectral sequence, see [12].

4. Moduli schemes of sheaves over quaternion algebras

Assume W is a smooth projective d-dimensional variety and $\mathcal{A}$ is an Azumaya algebra on W, then we can think of the pair $(W, \mathcal{A})$ as a noncommutative version of W. In this section, we want to study moduli schemes of sheaves on such noncommutative pairs.

Definition 4.1. A sheaf E on W is called a generically simple torsion free $\mathcal{A}$-module, if E is a left $\mathcal{A}$-module such that E is coherent and torsion free as a $\mathcal{O}_W$-module and the stalk E_η over the generic point $\eta \in W$ is a simple module over $\mathcal{A}_\eta$. If furthermore $\mathcal{A}_\eta$ is a division ring over $\mathbb{C}(W)$ then such a module is also called a torsion free $\mathcal{A}$-module of rank one.

Remark 4.2. A generically simple torsion free $\mathcal{A}$-module E is simple, see [10].

Apart from being simple, these modules share many properties with classical stable sheaves, for example we have

Lemma 4.3. *Assume E and F are generically simple torsion free $\mathcal{A}$-modules with the same Chern classes, then $\mathrm{Hom}_{\mathcal{A}}(E, F) \neq 0$ implies $E \cong F$.*

Proof. A nontrivial $\mathcal{A}$-morphism ϕ must be generically bijective as E and F are generically simple. As E and F are torsion free this implies that ϕ is injective, so we get an exact sequence with $Q = \mathrm{Coker}(\phi)$:

$$0 \longrightarrow E \xrightarrow{\phi} F \longrightarrow Q \longrightarrow 0$$

But E and F have the same Chern classes, so $Q = 0$ and hence $E \cong F$. $\square$

By fixing the Hilbert polynomial P of such sheaves, Hoffmann and Stuhler showed that these modules are classified by a moduli scheme, see [10, Theorem 2.4. iii), iv)]:

Theorem 4.4. *There is a projective moduli scheme $\mathrm{M}_{\mathcal{A}/W,P}$ classifying generically simple torsion free $\mathcal{A}$-modules with Hilbert polynomial P on W.*

We want to study these moduli schemes for a noncommutative Enriques surfaces $(X, \mathcal{A})$, where X is a very general Enriques surface and $\mathcal{A}$ is a quaternion algebra representing the nontrivial element in $\mathrm{Br}(X)$. Note that the $\mathcal{O}_X$-rank of a torsion free $\mathcal{A}$-module of rank one is four in this case.

We also have an associated noncommutative K3 surface $(\overline{X}, \overline{\mathcal{A}})$. Now we first recall some facts about the moduli schemes for such pairs, see [10, Theorem 3.6.]:

Theorem 4.5. *Let $\overline{X}$ be a K3 surface which is a double cover of a very general Enriques surface X and let $\overline{\mathcal{A}}$ be the quaternion algebra coming from the quaternion algebra on X which represents the nontrivial element in* $\mathrm{Br}(X)$.

i) The moduli scheme $\mathrm{M}_{\overline{\mathcal{A}}/\overline{X}}$ of torsion free $\overline{\mathcal{A}}$-modules of rank one is smooth.

ii) There is a nowhere degenerate alternating 2-form on the tangent bundle of $\mathrm{M}_{\overline{\mathcal{A}}/\overline{X}}$

iii) Every torsion free $\overline{\mathcal{A}}$-module of rank one can be deformed into a locally projective $\overline{\mathcal{A}}$-module, i.e. the locus $\mathrm{M}^{lp}_{\overline{\mathcal{A}}/\overline{X}}$ of locally projective $\mathcal{A}$-modules is dense in $\mathrm{M}_{\overline{\mathcal{A}}/\overline{X}}$.

iv) For fixed Chern classes $\overline{c_1}$ and $\overline{c_2}$ we have

$$\dim \mathrm{M}_{\overline{\mathcal{A}}/\overline{X},\overline{c_1},\overline{c_2}} = \frac{\overline{\Delta}}{4} - c_2(\overline{\mathcal{A}}) - 6$$

where $\overline{\Delta} = 8\overline{c_2} - 3\overline{c_1}^2$ is the discriminant and $\overline{c_i} = \pi^* c_i$.

One can also define the $\overline{\mathcal{A}}$-Mukai vector for an $\overline{\mathcal{A}}$-module E by

$$\overline{v}_{\overline{\mathcal{A}}}(E) = ch(E)\sqrt{\mathrm{td}(\overline{X})}\sqrt{ch(\overline{\mathcal{A}})^{-1}}.$$

As in the case of $\mathcal{O}_X$-modules, it has the property that

$$\left(\overline{v}_{\overline{\mathcal{A}}}(E)\right)^2 + 2 = \dim \mathrm{M}_{\overline{\mathcal{A}}/\overline{X},\overline{c_1},\overline{c_2}}.$$

Then by [13, Theorem 2.11] we also have:

Theorem 4.6. *Let the pair $(\overline{X}, \overline{\mathcal{A}})$ be as in Theorem 4.5. Assume $\overline{v}$ is a fixed primitive $\overline{\mathcal{A}}$-Mukai vector, then $\mathrm{M}_{\overline{\mathcal{A}}/\overline{X},\overline{v}}$ is an irreducible holomorphic symplectic manifold deformation equivalent to $\mathrm{Hilb}^{\frac{\overline{v}^2}{2}+1}(\overline{X})$. Furthermore $\mathrm{M}_{\overline{\mathcal{A}}/\overline{X},\overline{v}} \neq \emptyset$ if and only if $\overline{v}^2 \geq -2$.*

The covering involution $\iota : \overline{X} \to \overline{X}$ induces an involution

$$\iota^* : \mathrm{M}_{\overline{\mathcal{A}}/\overline{X},\overline{c_1},\overline{c_2}} \to \mathrm{M}_{\overline{\mathcal{A}}/\overline{X},\overline{c_1},\overline{c_2}}, \;\; [F] \mapsto [\iota^* F]$$

Lemma 4.7. *The involution ι^* is antisymplectic, that is if we denote the symplectic form on the tangent bundle of $\mathrm{M}_{\overline{\mathcal{A}}/\overline{X}}$ by ω, then we have the equality $\omega(\iota^* f_1, \iota^* f_2) = -\omega(f_1, f_2)$.*

Proof. By [10, Theorem 3.6. ii)], and similar to Mukai's construction, after the identification $T_{[F]} \mathrm{M}_{\overline{\mathcal{A}}/\overline{X}} \cong \mathrm{Ext}^1_{\overline{\mathcal{A}}}(F, F)$ the symplectic form is defined by the Yoneda product

$$\mathrm{Ext}^1_{\overline{\mathcal{A}}}(F, F) \times \mathrm{Ext}^1_{\overline{\mathcal{A}}}(F, F) \to \mathrm{Ext}^2_{\overline{\mathcal{A}}}(F, F).$$

composed with the trace map $\mathrm{tr}_{\overline{\mathcal{A}}} : \mathrm{Ext}^2_{\overline{\mathcal{A}}}(F, F) \to H^2(\overline{X}, \mathcal{O}_{\overline{X}})$.

Using the functoriality of the Yoneda pairing (the cup product) we get the following commutative diagram

$$\begin{tikzcd}
\mathrm{Ext}^1_{\overline{\mathcal{A}}}(F, F) \times \mathrm{Ext}^1_{\overline{\mathcal{A}}}(F, F) \arrow[r] \arrow[d, "\iota^*"] & \mathrm{Ext}^2_{\overline{\mathcal{A}}}(F, F) \arrow[d, "\iota^*"] \\
\mathrm{Ext}^1_{\overline{\mathcal{A}}}(\iota^* F, \iota^* F) \times \mathrm{Ext}^1_{\overline{\mathcal{A}}}(\iota^* F, \iota^* F) \arrow[r] & \mathrm{Ext}^2_{\overline{\mathcal{A}}}(\iota^* F, \iota^* F)
\end{tikzcd}$$

According to the definition in [10] the trace map $\mathrm{tr}_{\overline{\mathcal{A}}}$ is the composition of the forgetful functor from $\overline{\mathcal{A}}$-modules to $\mathcal{O}_{\overline{X}}$-modules and the usual trace map $\mathrm{tr}_{\mathcal{O}_{\overline{X}}}$, so $\mathrm{tr}_{\overline{\mathcal{A}}}$ is also functorial and we get the following commutative diagram

$$\begin{tikzcd}
\mathrm{Ext}^2_{\overline{\mathcal{A}}}(F, F) \arrow[r, "\mathrm{tr}_{\overline{\mathcal{A}}}"] \arrow[d, "\iota^*"] & H^2(\overline{X}, \mathcal{O}_{\overline{X}}) \arrow[d, "\iota^*"] \\
\mathrm{Ext}^2_{\overline{\mathcal{A}}}(\iota^* F, \iota^* F) \arrow[r, "\mathrm{tr}_{\overline{\mathcal{A}}}"] & H^2(\overline{X}, \mathcal{O}_{\overline{X}})
\end{tikzcd}$$

But $\iota^* : H^2(\overline{X}, \mathcal{O}_{\overline{X}}) \to H^2(\overline{X}, \mathcal{O}_{\overline{X}})$ is multiplication by -1. This follows from the identification $H^2(\overline{X}, \mathcal{O}_{\overline{X}}) \cong \mathbb{C}$ by using $H^0(\overline{X}, \omega_{\overline{X}}) = \mathbb{C}\sigma$ with the symplectic form σ on $\overline{X}$ and the fact that ι^* is antisymplectic with respect to σ as $H^0(X, \omega_X) = 0$.

Putting both diagrams together, we see that ι^* is in fact antisymplectic. □

Corollary 4.8. *The locus of fixed points of the involution*

$$\mathrm{Fix}(\iota^*) \subset \mathrm{M}_{\overline{\mathcal{A}}/\overline{X},\overline{c_1},\overline{c_2}}$$

is a smooth projective Lagrangian subscheme.

Proof. $\mathrm{Fix}(\iota^*)$ is smooth and projective by [8, 3.1., 3.4.]. Since ι^* is antisymplectic, it follows from [5, Lemma 1.] that $\mathrm{Fix}(\iota^*)$ is also Lagrangian. □

For the rest of this section we need the following

Remark 4.9. For a torsion free $\mathcal{A}$-module E of rank one on X, the $\mathcal{A}$-modules E^{**} and $E \otimes L$ for $L \in \mathrm{Pic}(X)$ are also torsion free of rank one. In addition $\overline{E}$ is a torsion free $\overline{\mathcal{A}}$-module of rank one on $\overline{X}$ since π is flat.

Theorem 4.10. *Let X be a very general Enriques surface and let $\mathcal{A}$ be a quaternion algebra on X representing the nontrivial element in* $\mathrm{Br}(X)$*.*

i) The moduli scheme $\mathrm{M}_{\mathcal{A}/X}$ of torsion free $\mathcal{A}$-modules of rank one is smooth.

ii) Every torsion free $\mathcal{A}$-module of rank one can be deformed into a locally projective $\mathcal{A}$-module, i.e. the locus $\mathrm{M}^{lp}_{\mathcal{A}/X}$ of locally projective $\mathcal{A}$-modules is dense in $\mathrm{M}_{\mathcal{A}/X}$.

iii) For fixed Chern classes c_1 and c_2 we have

$$\dim \mathrm{M}_{\mathcal{A}/X,c_1,c_2} = \frac{\Delta}{4} - c_2(\mathcal{A}) - 3$$

where $\Delta = 8c_2 - 3c_1^2$ is the discriminant.

Proof. i) We note that, as in the classical case of $\mathcal{O}_X$-modules, there is a deformation theory for $\mathcal{A}$-modules, see [10, Sect. 3]. Thus for a given point $[E] \in M_{\mathcal{A}/X}$ we have to show that all obstruction classes in $\mathrm{Ext}^2_{\mathcal{A}}(E,E)$ vanish. But by Proposition 1.6 we have:

$$\mathrm{Ext}^2_{\mathcal{A}}(E,E) \cong (\mathrm{Hom}_{\mathcal{A}}(E, E \otimes \omega_X))^{\vee}.$$

As $\overline{E}$ is a simple $\overline{\mathcal{A}}$-module, we get $\mathrm{Hom}_{\mathcal{A}}(E, E\otimes\omega_X) = 0$ by Corollary 1.5. Thus all obstructions vanish and $M_{\mathcal{A}/X}$ is smooth at $[E]$.

ii) The proof of [10, Theorem 3.6.iii)] carries over to our situation with one small change: the surjectivity of the connecting homomorphisms δ in the diagram:

$$\begin{tikzcd}
\mathrm{Ext}^1_{\mathcal{A}}(E,E) \arrow[r, "\delta"] & \mathrm{Ext}^2_{\mathcal{A}}(T,E) \arrow[r, "\pi^*"] \arrow[d, "\iota_*"] & \mathrm{Ext}^2_{\mathcal{A}}(E^{**},E) \\
 & \mathrm{Ext}^2_{\mathcal{A}}(T,E^{**}) \arrow[r, equal] & \bigoplus_{i=1}^{l} \mathrm{Ext}^2_{\mathcal{A}}(T_{x_i},E^{**})
\end{tikzcd}$$

follows from the fact that

$$\mathrm{Ext}^2_{\mathcal{A}}(E^{**},E) = 0.$$

This vanishing can be seen as follows: using Proposition 1.6 we have

$$\mathrm{Ext}^2_{\mathcal{A}}(E^{**},E) \cong (\mathrm{Hom}_{\mathcal{A}}(E, E^{**} \otimes \omega_X))^{\vee}.$$

But the last space is zero by Lemma 1.7. The rest of the proof works unaltered.

iii) Using ii) it suffices to compute the dimension of

$$T_{[E]}M_{\mathcal{A}/X} \cong \mathrm{Ext}^1_{\mathcal{A}}(E,E) \cong H^1(X, \mathcal{E}nd_{\mathcal{A}}(E))$$

for a locally projective $\mathcal{A}$-module E of rank one.

Again as in [10, Theorem 3.6.iv)] we have:

$$c_1(\mathcal{E}nd_{\mathcal{A}}(E)) = 0 \ \text{ and } \ c_2(\mathcal{E}nd_{\mathcal{A}}(E))) = \frac{\Delta}{4} - c_2(\mathcal{A})$$

where Δ is the discriminant of E. So by Hirzebruch-Riemann-Roch:

$$\chi(X, \mathcal{E}nd_{\mathcal{A}}(E)) = -\frac{\Delta}{4} + c_2(\mathcal{A}) + 4\chi(X, \mathcal{O}_X)$$

Using $\operatorname{End}_{\mathcal{A}}(E) \cong \mathbb{C}$, $\operatorname{Ext}^2_{\mathcal{A}}(E,E) = 0$ and $\chi(X, \mathcal{O}_X) = 1$ we get our result.

□

Remark 4.11. The proof of i) also implies $E \ncong E \otimes \omega_X$ for all torsion free $\mathcal{A}$-modules of rank one.

Similar to the involution ι, using Remark 4.9, the projection $\pi : \overline{X} \to X$ induces a morphism

$$\pi^* : \mathrm{M}_{\mathcal{A}/X,c_1,c_2} \to \mathrm{M}_{\overline{\mathcal{A}}/\overline{X},\overline{c_1},\overline{c_2}}, \quad [E] \mapsto \left[\overline{E}\right].$$

Our goal is to understand this morphism:

Theorem 4.12. *Let the pair $(X, \mathcal{A})$ be as in Theorem 4.10. The pullback map*

$$\pi^* : \mathrm{M}_{\mathcal{A}/X,c_1,c_2} \to \mathrm{M}_{\overline{\mathcal{A}}/\overline{X},\overline{c_1},\overline{c_2}}$$

realizes $\mathrm{M}_{\mathcal{A}/X,c_1,c_2}$ as an étale double cover of the Lagrangian subscheme $\operatorname{Fix}(\iota^*) \subset \mathrm{M}_{\overline{\mathcal{A}}/\overline{X},\overline{c_1},\overline{c_2}}$.

Proof. We have

$$\iota^*\overline{E} = \iota^*\pi^*E \cong (\pi \circ \iota)^*E = \pi^*E = \overline{E}.$$

So $\operatorname{Im}(\pi^*) \subset \operatorname{Fix}(\iota^*)$ and hence π^* factors through $\operatorname{Fix}(\iota^*)$ giving rise to

$$\varphi : \mathrm{M}_{\mathcal{A}/X,c_1,c_2} \to \operatorname{Fix}(\iota^*).$$

By Theorem 2.6 we also have $\operatorname{Fix}(\iota^*) \subset \operatorname{Im}(\pi^*)$. So $\operatorname{Im}(\pi^*) = \operatorname{Fix}(\iota^*)$ and the morphism φ is surjective.

Assume $\varphi([E]) = \varphi([F])$ that is $\overline{E} \cong \overline{F}$ and $\operatorname{Hom}_{\overline{\mathcal{A}}}(\overline{E}, \overline{F}) \neq 0$. Then Lemma 1.4 says

$$\operatorname{Hom}_{\overline{\mathcal{A}}}(\overline{E}, \overline{F}) \cong \operatorname{Hom}_{\mathcal{A}}(E, F) \oplus \operatorname{Hom}_{\mathcal{A}}(E, F \otimes \omega_X)$$

and so by Lemma 4.3 and Remark 4.9 we have

$$E \cong F \quad \text{or} \quad E \cong F \otimes \omega_X$$

but not both by Remark 4.11. So φ is an unramified 2 : 1-morphism. Moreover the computations also show that φ is a flat morphism by [15, Lemma, p.675], hence φ is étale. □

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