Multiplicity-free Kronecker Products of Characters of the Alternating Groups

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Abstract

In this work we classify the multiplicity-free Kronecker products of irreducible characters of the alternating groups where we provide formulas for the decomposition of the products.

Furthermore, we investigate inner and outer tensor products of irreducible characters of the symmetric groups which only contain constituents with multiplicity 1 and 2. For the outer tensor products we classify the products of irreducible characters which only contain constituents with multiplicity 1 and 2. Additionally, we provide a list of skew characters which we conjecture to be the only ones just containing constituents with multiplicity 1 and 2. We prove that all the characters not listed contain a constituent with multiplicity 3 or higher.

For the inner tensor product we classify all the products of irreducible characters which just contain constituents with multiplicity 1 and 2. We also provide formulas for the decomposition of these products. Together with this we obtain a classification of the inner tensor products of skew characters of the symmetric groups which just contain constituents with multiplicity 1 and 2.

Kurzzusammenfassung

In dieser Arbeit werden die multiplizitätenfreien Kroneckerprodukte von irreduziblen Charakteren der alternierenden Gruppen klassifiziert. Dabei geben wir die Zerlegung der Produkte explizit an.

Darüber hinaus werden die inneren und äußeren Tensorprodukte von irreduziblen Charakteren der symmetrischen Gruppen untersucht. Für die äußeren Tensorprodukte klassifizieren wir die Produke von irreduziblen Charakteren, welche nur Konstituenten mit Multiplizität 1 und 2 enthalten. Darüber hinaus wird eine Vermutung aufgestellt, welche Schiefcharaktere nur Konstituenten mit Multiplizität 1 und 2 enthalten und gezeigt, dass alle anderen Schiefcharaktere einen Konstituenten mit einer Multiplizität von mindestens 3 enthalten.

Für das innere Tensorprodukt klassifizieren wir alle Produkte von irreduziblen Charakteren, welche nur Konstituenten mit Multiplizität 1 oder 2 enthalten. Wir geben Formeln für die Zerlegung dieser Produkte an. Dabei erhalten wir auch eine Klassifikation der inneren Tensorprodukte von Schiefcharakteren, welche nur Konstituenten mit Multiplizität 1 und 2 enthalten.

Keywords: Symmetric Groups, Alternating Groups, Representation theory

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Eidesstattliche Erklärung

Hiermit versichere ich, dass ich die folgende Arbeit selbstständig verfasst und keine anderen als die angegebenen Quellen als Hilfsmittel benutzt sowie Zitate kenntlich gemacht habe.

Hannover, den 13.12.2021

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Contents

CHAPTER 1

Introduction

Kronecker coefficients were introduced by Murnaghan in 1938 [Mur38]. Even though 'one of the main problems in the combinatorial representation theory of the symmetric group is to obtain a combinatorial interpretation for the Kronecker coefficients' (Stanley [Sta99]), 'frustratingly little is known about them' (Bürgisser [Bür09]). Besides their importance in combinatorial representation theory, Kronecker coefficients got further attention from geometric complexity theory [Mul07] and quantum information theory [CHM07].

Over time there has been a lot of progress for coefficients which are labeled by hooks or two-line partitions [Rem89, Rem92, RW94, Ros01, BO06, BWZ10, Liu17, Bla18] and the homogeneous products [BK99] have been classified but apart from these (and some other) special cases Kronecker coefficients still seem mysterious and only very little is known for the general case. Recently Bessenrodt and Bowman classified the multiplicity-free Kronecker products of irreducible characters of the symmetric groups [BB17]. The aim of this thesis is to carry this result over to the alternating groups and classify their multiplicity-free character products.

1. Outline

First, we introduce the basic concepts of representation theory of symmetric groups together with the notation and definitions which will be used throughout the whole thesis. The rest of this thesis is divided into three parts. In the first part, outer tensor products of irreducible characters of the symmetric group which contain only constituents with multiplicity 1 and 2 are classified. Furthermore, skew characters which only contain constituents with multiplicity 1 and 2 are investigated. The results obtained here are used in the second part which classifies the inner tensor products of irreducible characters of the symmetric groups. This is by far the longest part of this thesis but the results will enable us to classify the multiplicity-free character products of the alternating groups in the third part.

Part I: Skew characters and outer tensor products of characters of the symmetric groups.
In this part, we investigate the outer tensor products of irreducible characters of the symmetric groups and skew characters. After a short chapter about basic facts about Littlewood-Richardson coefficients and skew characters, we first look at the outer tensor products of irreducible characters of the symmetric groups. Here we are able to classify all outer tensor products which only contain constituents with multiplicity 1 and 2. In the next chapter we present a conjecture which skew characters only contain constituents with multiplicity 1 and 2. We are not able to show that all the listed products contain no constituents with multiplicity 3 or higher, but we show that all other products contain one. This is sufficient for the results on Kronecker products that we prove in the next part.

Part II: Kronecker products of characters of the symmetric groups.
We use the methods from [BB17] to classify the Kronecker products of characters of the symmetric groups which just contain constituents with multiplicity 1 and 2.

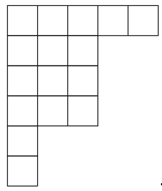
Additionly, we prove for some products that they contain more than one constituent with multiplicity 3 or higher or that they contain a non-symmetric constituent with multiplicity 3 or higher. This will be necessary in order to be able to use these results for the next part. We start with formulas for the decomposition of the products which contain only constituents with multiplicity 1 and 2. These all involve a factor which is labeled by a hook or a two-line partition (with a small exception). Together with the decompositions we prove that these are the only products of two irreducible characters where one is labeled by a hook or a two-line partition such that the product just contains constituents with multiplicity 1 and 2. For most of the other products we prove by induction that these contain a constituent with multiplicity 3 or higher. To do so we need a classification for products of skew characters containing only constituents with multiplicity 1 and 2. At this point, we will just be able to prove this under the assumption that the classification that we stated for the irreducible characters is correct. So after the chapter about skew characters, we can show with induction that most of the products which neither involve a hook nor a two-line contain a constituent with multiplicity 3 or higher. The last four chapters of this part will be spent finding such constituents in the remaining products.

Part III: Multiplicity-free Kronecker products of characters of the alternating groups.
In the last part of this thesis we prove the classification of the multiplicity-free Kronecker products of characters of the alternating groups. For this we use the classification of Kronecker products of characters of the symmetric groups which only contain constituents with multiplicity 1 and 2. Most of the constituents of the multiplicity-free products can be derived directly from the formulas for the multiplicity-free Kronecker products of the symmetric groups ([BB17]). For the products where it is not obvious we provide the decomposition. After a chapter about the preliminaries we start by looking at the products of two characters labeled by non-symmetric partitions. The results of this part can be deduced from [BB17]. In the next chapter we look at products which involve a symmetric partition. For these we need the results of the second part. We finish this part by deducing the multiplicity-free products of three (or more) A_n-characters from the classification of multiplicity-free character products of two characters of the alternating groups.

2. Preliminaries: Representation theory of symmetric groups

Partitions and Young diagrams.
We denote the symmetric group on n letters by S_n. The representation theory of the symmetric and alternating groups is based on partitions and their combinatorics. A *partition* λ of n is a weakly decreasing finite sequence of non-negative integers $\lambda = (\lambda_1, \dots, \lambda_l)$ such that $|\lambda| := \sum_{i=1}^{l} \lambda_i = n$. A partition λ of n will be denoted by $\lambda \vdash n$. We call the number of non-zero parts of λ the *length* of λ and denote it by $l(\lambda)$. By $w(\lambda) := \lambda_1$ we denote the *width* of λ and by $d(\lambda)$ the *Durfee size* of λ. This is the number of boxes on the diagonal, i.e., $d(\lambda) = \max\{i | \lambda_i \geq i\}$. Often we use a more compact notation for a partition. If the part λ_i is repeated r times we indicate that by a superscript r instead of repeating the part λ_i r times. For example we write $(5, 3^3, 1^2)$ instead of $(5, 3, 3, 3, 1, 1)$. If we use this notation, we generally assume that all the parts of same length are grouped together. So if $\lambda = (\lambda_1^{r_1}, \dots, \lambda_l^{r_l})$, we assume that $\lambda_i > \lambda_{i+1}$.

Partitions can be visualized by Young diagrams. Let $\lambda \vdash n$. The *Young diagram* of λ is an array of n boxes arranged in $l(\lambda)$ rows which start in the same column on the left and the ith row (we count, like in matrices, the rows from top to bottom and the columns from left to right) has λ_i boxes for $1 \leq i \leq l(\lambda)$. The box in the ith row and jth column has the coordinate (i, j). For example the Young diagram of the partition $(5, 3^3, 1^2)$ is given by



We do not distinguish strictly between a partition and the associated Young diagram, so by abuse of notation we denote the Young diagram of the partition λ by λ, too.

For $\lambda \vdash n$, the *transposed* or *conjugated* partition λ' is defined to be the partition which we obtain by reflecting the Young diagram of λ on the diagonal (so it coincides with transposition for matrices), for example $(5, 3^3, 1^2)' = (6, 4^2, 1^2)$. If a partition is invariant under conjugation, i.e., $\lambda = \lambda'$, we call it *symmetric*. In some cases λ and λ' behave similarly so we use the notation $\lambda^{(')}$ for λ or λ'.

Let A be a box from the Young diagram of $\lambda \vdash n$. If λ without A is a Young diagram of a partition of $n-1$, we call A *removable* and denote the corresponding partition by λ_A. If there is a partition $\mu \vdash n+1$ and a removable box B of μ such that $\mu_B = \lambda$, we call B an *addable* box of λ and write λ^B for μ. By $\mathrm{Rem}(\lambda)$ resp. $\mathrm{Add}(\lambda)$ we denote the set of removable resp. addable boxes from λ. Further, we denote with $\mathrm{rem}(\lambda) = |\,\mathrm{Rem}(\lambda)|$ and $\mathrm{add}(\lambda) = |\,\mathrm{Add}(\lambda)|$ the number of removable resp. addable boxes of λ.

A lot of names for families of partitions derive from the shape of their Young diagrams. We use the following names:

- $\lambda \vdash n$ is called *linear* if $\lambda^{(')} = (n)$.
- We call a partition $\lambda \vdash n$ a *hook* if $\lambda_2 \leq 1$ and a *proper hook* if λ is a hook and λ is not linear.
- A partition λ is called a *two-line partition* if $\lambda^{(')} = (a, b)$ for suitable a, b. If $\lambda = (a, b)$, we call it a *two-row partition* and if $\lambda' = (a, b)$, we call it a *two-column partition*.
- If $d(\lambda) \leq 2$, we call λ a *double-hook*. If λ is a double-hook but not a hook or a two-line partition, we call it a *proper double-hook*.
- If λ has one removable node, we call λ a *rectangle* and we call λ a *proper rectangle* if λ is a rectangle and $l(\lambda), w(\lambda) \geq 3$. Note that in this definition a two-line rectangle is not a proper rectangle.
- If λ has at most 2 removable nodes, we call it a *fat hook* and a *proper fat hook* if it is a fat hook but not a hook, rectangle nor a two-line partition.

Over the complex numbers the irreducible S_n characters $[\lambda]$ are indexed by partitions of n. This (natural) bijection carries the combinatorics of partitions and tableaux over to modules and characters. Via this bijection we use the attributes that we initially defined for partitions for the corresponding characters, too.

Kronecker coefficients.

Let $\lambda, \mu, \nu \vdash n$. We define the Kronecker coefficient $g(\lambda, \mu, \nu)$ as the multiplicity of $[\nu]$ in the product $[\lambda][\mu]$, i.e.,

$$[\lambda][\mu] = \sum_{\nu \vdash n} g(\lambda, \mu, \nu)[\nu].$$

Thus, the Kronecker coefficient is a more compact notation for special S_n scalar products

$$g(\lambda, \mu, \nu) = \langle [\lambda][\mu], [\nu] \rangle_{S_n}.$$

Moreover, we define for $\lambda, \mu \vdash n$

$$g(\lambda, \mu) := \max\{g(\lambda, \mu, \nu) | \nu \vdash n\}.$$

If $[\lambda][\mu]$ contains r or more constituents with multiplicity l or higher, we denote this by $g(\lambda, \mu)_r \geq l$.

For $\lambda, \mu \vdash n$, we know that $[\lambda][1^n] = [\lambda']$, so the $g(\lambda^{(')}, \mu^{(')})$ all have the same value and we usually just investigate the products up to conjugation. To make the notation a bit more compact we will sometimes write $[\lambda']$ as $[\lambda]'$.

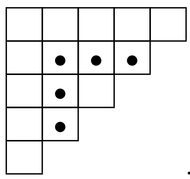
Character values.
Not only the irreducible characters, but also the conjugacy classes of the symmetric group are indexed by partitions. For $\lambda \vdash n$, we write σ_λ for an element of the S_n conjugacy class which contains exactly the elements of cycle type λ.

For $\lambda, \mu \vdash n$, we can compute the character value $[\lambda](\sigma_\mu)$ recursively with the Murnaghan–Nakayama rule. To state the formula we need the definition of a rim hook.

Let λ be a partition of n and (i, j) a box of λ.

(1) The (i, j)-*hook* $H_{(i,j)}$ of λ consists of the box (i, j) together with all the boxes of λ to the right of (i, j) in the row i and the boxes below (i, j) in the column j.

The *hook length* $h_{(i,j)}$ of the hook $H_{(i,j)}$ counts the number of boxes in $H_{(i,j)}$. The *leg length* $L(H_{(i,j)})$ of the hook $H_{(i,j)}$ is the number of boxes below the box (i, j) in the diagram λ. In an example for the partition $(5, 4, 3, 2, 1)$ the $(2, 2)$-hook is given by the dotted boxes. The hook length is 5 and the leg length is 2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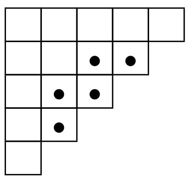
(2) The (i, j)-*rim hook* $R_{(i,j)}$ of λ consists of the rightmost box in the row i, the last box of the column j and their connection along the rim of the diagram of λ.

The $(2, 2)$-rim hook in the diagram $(5, 4, 3, 2, 1)$ is given by the dotted boxes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We have already introduced the irreducible characters of the symmetric groups. Now, with the definition of the rim hooks we can recall the Murnaghan-Nakayama rule which gives us a possibility to evaluate the characters on an element of the symmetric group.

Theorem 1.1 (Murnaghan–Nakayama rule [Mur37, Nak41])**.** *Let λ, μ be partitions of n and $\sigma \in S_n$ an element of cycle type λ. Let σ contain an l-cycle and let*

$\tilde{\sigma} \in S_{n-l}$ be an element of the same cycle type as σ, but with one l-cycle less. We have the following recursive formula for the character value:

$$[\mu](\sigma) = \sum_{\substack{i,j \text{ such that} \\ h_{(i,j)}=l}} (-1)^{L(H_{(i,j)})}[\mu \setminus R_{(i,j)}](\tilde{\sigma}),$$

where $\mu \setminus R_{(i,j)}$ denotes the partition of the diagram μ without the rim hook $R_{(i,j)}$.

Part 1

Skew characters and outer tensor products of characters of the symmetric groups

CHAPTER 2

Preliminaries: The outer tensor product, the Littlewood-Richardson rule and skew characters

1. Skew characters and the Littlewood-Richardson Rule

A *skew partition* of n is a pair of partitions (λ, μ) such that the diagram of μ is contained in the diagram of λ and $|\lambda| - |\mu| = n$. It is denoted by λ/μ. The *skew diagram* corresponding to the skew partition λ/μ is the set-theoretic difference of the diagrams of λ and μ (i.e., the set of boxes that belong to λ but not to μ). A λ/μ *skew tableau* T is a filling of the skew diagram λ/μ with positive integers.

Below, a skew tableau for the skew partition $(6,5,2,1)/(3,2,1)$ is shown

			1	1	1
		1	2	2	
	3				
3					

.

We can assign a word to a (skew) tableau by reading its entries from right to left and from top to bottom. Such a word w is called a *lattice word* if in every initial part of the sequence any number i occurs at least as often as the number $i+1$.

Further, a (skew) tableau T is called *semistandard* if the numbers in T are not decreasing along the rows (from left to right) and are increasing along the columns (from top to bottom).

Let μ be a partition of n. A (skew) tableau T is *of type* μ if for every $i \geq 1$ the entry i occurs μ_i times in T.

Definition 2.1 (Littlewood-Richardson tableaux)**.** A skew tableau of shape λ/μ is called a *Littlewood-Richardson tableau of shape* λ/μ if it is a semistandard tableau such that the word obtained by reading the entries of the rows from right to left and the rows from top to bottom is a lattice word.

Theorem 2.2 (Littlewood-Richardson rule)**.** *Let λ be a partition of n and μ be a partition of m. For $\nu \vdash n+m$ the multiplicity of $[\nu]$ as a composition factor of $[\lambda] \boxtimes [\mu] := ([\lambda] \times [\mu]) \uparrow_{S_n \times S_m}^{S_{n+m}}$ equals the number of Littlewood-Richardson tableaux of shape ν/λ and type ν which we denote by $c(\nu; \lambda, \mu)$, i.e.,*

$$[\lambda] \boxtimes [\mu] = \sum_{\nu \vdash n+m} c(\nu; \lambda, \mu)[\nu].$$

Although the Littlewood-Richardson rule was first stated by these two authors in [LR34] it took some time until Thomas [Tho78, Tho74] and Schützenberger [Sch77] published the first complete proofs.

Whenever we talk about products of characters in this part of the thesis, we mean the outer tensor product which was defined in the previous theorem.

The skew tableau we have already seen is a $(6,5,2,1)/(3,2,1)$-Littlewood-Richardson tableau of type $(4,2,2)$. The word we obtain by reading its rows from right to left is 11122133.

Let λ/μ be a skew partition of n. We define the *skew character* $[\lambda/\mu]$ as

$$\sum_{\nu\vdash n} c(\lambda;\mu,\nu)[\nu].$$

Obviously, we can remove empty rows and columns from a skew diagram without changing the corresponding skew character. We call a skew diagram without empty rows and columns *basic* and usually we restrict ourselves to these skew diagrams. A *proper skew partition* is a skew partition which is neither a partition nor a rotated partition. By [BK99, Lemma 4.4.] the skew character labeled by a proper skew partition has at two distinct irreducible constituents. We refer to such a skew character as a *proper skew character*.

In [Ste01] John Stembrige classified the multiplicity-free products of Schur functions, which is equivalent to the classification of the multiplicity-free outer tensor products of irreducible characters of the symmetric groups. Using this, Christian Gutschwager classified the multiplicity-free skew characters in [Gut10]. Around the same time, Hugh Thomas and Alexander Yong classified the multiplicity-free products of Schubert classes in [TY10], which is equivalent to Gutschwager's result. The classifications are given by the following theorem. The first part is equivalent to the classification of the multiplicity-free outer tensor products of irreducible characters of the symmetric groups. A near-rectangle is a partition which is a rectangle if we remove either one row or one column.

Theorem 2.3. *[Ste01, Gut10, TY10] Let* $\lambda = (\lambda_1^{k_1}, \lambda_2^{k_2}, \dots \lambda_i^{k_i}), \alpha = (\alpha_1^{l_1}, \dots, \alpha_j^{l_j})$ *be partitions and* $k = \sum_{a=1}^{i} k_a, l = \sum_{a=1}^{j} l_a$.

Then $[\lambda/\alpha]$ *is multiplicity-free if and only if one of the following conditions holds:*

(1) λ/α *decomposes into two disconnected skew diagrams* μ *and* ν *for which up to rotation by* 180° *and/or exchanging* μ *with* ν *one of the following conditions holds:*
  - (a) μ *is a one-line rectangle and* ν *is a partition;*
  - (b) μ *is a two-line rectangle and* ν *is a fat hook;*
  - (c) μ *is a rectangle and* ν *a near-rectangle;*
  - (d) μ *and* ν *are rectangles.*

(2) λ/α *is a connected basic skew diagram and one of the following conditions holds:*
  - (a) $i = 1$;
  - (b) $j = 1$ *and one of the following conditions holds:*
    - (i) $\alpha_1 = 1$ *or* $l_1 = 1$;
    - (ii) $\lambda_1 = 1 + \alpha_1$ *or* $k = 1 + l$;
    - (iii) $i = 2$;
    - (iv) $i = 3$ *and one of the following conditions holds:*
      - (A) $\alpha_1 = 2$ *or* $l_1 = 2$;
      - (B) $k_1 = 1$ *or* $\lambda_3 = 1$;
      - (C) $k_2 = 1$ *or* $\lambda_2 = 1 + \lambda_3$;
      - (D) $k_3 = 1$ *or* $\lambda_1 = 1 + \lambda_2$;
      - (E) $k = 2 + l$ *or* $\lambda_1 = 2 + \alpha_1$.
  - (c) $i = 2$ *and one of the following conditions holds:*
    - (i) $\lambda_1 = 1 + \lambda_2$ *or* $k_2 = 1$;
    - (ii) $\lambda_2 = 1$ *or* $k_1 = 1$.
  - (d) $i = 2$ *and* $j = 2$ *and one of the following conditions holds:*
    - (i) $\lambda_1 = 1 + \alpha_1$ *or* $k = 1 + l$;
    - (ii) $\lambda_1 = 2 + \lambda_2$ *or* $k_2 = 2$;

(iii) $\lambda_2 = 2$ *or* $k_1 = 2$*;*
(iv) $\alpha_1 = 1 + \alpha_2$ *or* $l_2 = 1$*;*
(v) $\alpha_2 = 1$ *or* $l_1 = 1$*.*

In the following we want to classify outer-tensor products which just contain constituents with multiplicity 1 and 2.

Theorem 2.4. *A complete list (up to conjugation of all partitions or interchange of the partitions) of outer tensor products of irreducible characters of the symmetric groups which contain only constituents with multiplicity less or equal to 2 is given by:*

(1) *The multiplicity-free outer tensor products (classified in [Ste01]);*
(2) *products of two irreducible characters* $[\lambda] \boxtimes [\mu]$ *if* λ *and* μ *satisfy one of the following conditions:*
  (a) *one of the partitions is a hook and the other one is a fat hook;*
  (b) λ *has two rows and one of the following holds:*
    (i) $\lambda = (2,2)$ *or* $\lambda = (2,1)$ *and* μ *is arbitrary;*
    (ii) $\mu = (m-1,1)$ *or* $m = 2k+1$ *is odd and* $\mu = (k+1,k)$*;*
    (iii) $\mu = (a^b,(a-1)^c)$ *or* $\mu = (a^b,1^c)$*;*
    (iv) $n = 2k+1$ *is odd* $\lambda = (k+1,k)$ *and* μ *is a fat hook;*
    (v) $n = 2k$ *is even,* $\lambda = (k,k)$ *and* $\mu = (a^b,c^d,e^f)$ *with* $a-c=1$ *or* $c-e=1$ *or* $e=1$*;*
    (vi) μ *has three removable nodes and* $\lambda = (3,3)$*.*
  (c) $\lambda = (a^b)$ *is a rectangle and one of the following holds:*
    (i) μ *is a fat hook and* $a = 3$ *and* $b \leq 5$*;*
    (ii) $\mu = (c^2, 2^d)$ *for some* $c,d \geq 2$*;*
    (iii) $b = 3$ *and* $\mu = (c^d,e^f)$ *with* $e \leq 3$ *or* $c-e \leq 3$*;*
    (iv) $\mu = (c^2,d^2)$ *with* $c-d \leq 3$ *or* $d \leq 3$ *or* $a \leq 5$*;*
    (v) $\mu = ((c+2)^2,c^d)$ *or* $\mu = (c^d,(c-2)^2)$*;*
    (vi) $\mu = (c,d,1^e)$*;*
    (vii) $\mu = (c+1,c^d,1)$*;*
    (viii) $a = 3$ *and* $l(\mu) \leq 3$*;*
    (ix) $\mu = (c,d,e)$ *with* $c-1=d$ *or* $d-1=e$*;*
    (x) $\mu = (c^d,c-1,(c-2)^e)$ *with* $d=1$ *or* $e=1$*.*
  (d) λ *is a fat hook and one of the following holds:*
    (i) *all 4 of the possibilities* $\lambda^{(')} = (a^b,1)$ *and* $\mu^{(')} = (c^d,1)$*;*
    (ii) $\lambda = (a^b,a-1)$ *and* $\mu = (c^d,c-1)$*.*
(3) *products of three characters which satisfy one of the following conditions:*
  (a) *[one-row]* $\boxtimes$ *[rectangle]* $\boxtimes$ *[one-column];*
  (b) *[rectangle]* $\boxtimes$ *[rectangle]* $\boxtimes$ $[1]$*;*
  (c) $[1] \boxtimes [1] \boxtimes$ *[anything].*

This is equivalent to characterization of skew characters that decompose into different connected components which are all partitions.

Furthermore, we would like to classify skew characters which contain no constituents with multiplicity 3 or higher, but we are not able to show for all of the products that they only contain constituents with multiplicity 1 and 2. However, we are able to prove that the products which are not in the lists contain a constituent with multiplicity 3 or higher. Since there are a lot more cases, we split our result into several theorems (Theorem 2.5 to 2.7).

For a skew character $[\lambda/\mu]$, we will show that a necessary condition for having only constituents with multiplicity 1 and 2 is that μ, possibly after rotation, has at most 2 removable boxes. If λ/μ is connected and μ has one removable box, the following theorem tells us if the skew character contains a constituent with multiplicity 3 or higher. Since the skew character of a fat hook skewed by a rectangle is multiplicity-free, we can assume that $\mathrm{rem}(\lambda) \geq 3$.

Theorem 2.5. *Let $\lambda = (\lambda_1^{k_1}, \lambda_2^{k_2}, \dots, \lambda_l^{k_l})$ with $l \geq 3$ and $\mu = (a^b)$ be partitions such that $\mu \subset \lambda$, λ/μ is a connected basic skew diagram and $[\lambda/\mu]$ is not multiplicity-free. If up to conjugation none of the following conditions hold, $[\lambda/\mu]$ has a constituent with multiplicity 3 or higher:*

(1) $\mu = (2^2)$;
(2) $a + 2 = \lambda_1$ *and* $b = 2$;
(3) $l = 3$ *and one of the following conditions holds:*
  (a) $\mu = (3^3)$, (4^3), (5^3);
  (b) *there are* $i, j \in \{1, 2, 3\}$ *such that* $\lambda_i - \lambda_{i+1} = 2$ *and* $k_j = 2$;
  (c) *there are* $i \neq j \in \{1, 2, 3\}$ *such that* $\lambda_i - \lambda_{i+1} = 2$, $\lambda_j - \lambda_{j+1} = 2$ *and one of the following conditions holds:*
    (i) $b \leq 5$;
    (ii) *there is an* $r \in \{1, 2, 3\}$ *such that* $k_r \leq 3$.
  (d) $a + 5 \geq \lambda_1$ *and one of the following conditions holds:*
    (i) $b = 3$;
    (ii) $k_2 = 2$ *and* $k_3 = 2$.
  (e) $a + 4 \geq \lambda_1$ *and* $k_1 = 2$;
  (f) $a + 3 \geq \lambda_1$ *and one of the following conditions holds:*
    (i) $b \leq 5$;
    (ii) *there is an* i *such that* $k_i \leq 3$.
  (g) $\lambda_2 = a + 1$ *and one of the following conditions holds:*
    (i) $k_1 = 2$;
    (ii) $b = k_1 + k_2 - 1$ *and one of the numbers* $\lambda_1 - \lambda_2$, λ_3, k_3 *equals 2.*
  (h) $b = 3$ *and there is an* i *such that* $\lambda_i - \lambda_{i+1} \leq 3$.
(4) $l = 4$ *and one of the following conditions holds:*
  (a) $\mu^{(')} = (2^3)$;
  (b) $b = 2$ *and one of the following conditions holds:*
    (i) $a + 3 \geq \lambda_1$;
    (ii) *there is an* i *such that* $\lambda_i - \lambda_{i+1} = 1$.
  (c) $k_1 = 1$ *and* $\lambda_2 = a + 1$;
  (d) *there is an* i *such that* $\lambda_i - \lambda_{i+1} = 1$ *and one of the following holds:*
    (i) $\lambda_1 - \lambda_2 = 1$ *and* $k_3 = k_4 = 1$;
    (ii) $\lambda_2 - \lambda_3 = 1$ *and* $k_1 = k_4 = 1$;
    (iii) $\lambda_3 - \lambda_4 = 1$ *and* $k_1 = k_2 = 1$;
    (iv) $\lambda_4 = 1$ *and* $k_1 = k_2 = 1$ *or* $k_1 = k_3 = 1$ *or* $k_2 = k_3 = 1$.
  (e) *there are* $i \neq j$ *and* $m < n$ *such that* $\lambda_i - \lambda_{i+1}$, $\lambda_j - \lambda_{j+1} = 1$, $k_m = 1$, $k_n = 1$ *and one of the following holds:*
    (i) $\lambda_1 - 2 = \lambda_2 - 1 = \lambda_3$ *and* $(m, n) \neq (1, 3)$;
    (ii) $\lambda_1 - \lambda_2 = 1$ *and* $\lambda_3 - \lambda_4 = 1$ *and* $(m, n) \neq (1, 4), (2, 3)$;
    (iii) $\lambda_1 - 1 = \lambda_2$ *and* $\lambda_4 = 1$ *and* $(m, n) \neq (2, 4)$;
    (iv) $\lambda_2 - 2 = \lambda_3 - 1 = \lambda_4$ *and* $(m, n) \neq (2, 4)$;
    (v) $\lambda_2 - 1 = \lambda_3$ *and* $\lambda_4 = 1$ *and* $(m, n) \neq (3, 4)$;
    (vi) $\lambda_3 = 2$, $\lambda_4 = 1$.
  (f) *for at most one* $i \in \{1, 2, 3, 4\}$ *is* $\lambda_i - \lambda_{i+1} > 1$ *and one of the following conditions holds:*
    (i) $b = 3$;
    (ii) *there is a* $j \in \{1, 2, 3, 4\}$ *such that* $k_j = 1$.
  (g) $\lambda_1 = a + 2$ *and one of the following conditions holds:*
    (i) $b = 3$;

(ii) *there is an* $i \in \{2,3,4\}$ *such that* $k_i = 1$.

In the next theorem we look at the case where λ/μ is connected and μ has two removable boxes.

Theorem 2.6. *Let* $\lambda = (\lambda_1^{k_1}, \lambda_2^{k_2}, ..., \lambda_l^{k_l})$ *with* $l \geq 3$ *and* $\mu = (\mu_1^{r_1}, \mu_2^{r_2})$ *such that* $\mu \subset \lambda$ *and* λ/μ *is a connected basic skew diagram. If up to conjugation none of the following conditions hold,* $[\lambda/\mu]$ *has a constituent with multiplicity 3 or higher:*

(1) $\mu = (2,1)$;
(2) $\mu_1 + 1 = \lambda_1$, $l(\lambda) - 1 = l(\mu)$ *and* $r_1 = \mu_2 = 1$;
(3) $r_1 = r_2 = 1$, $\mu_1 - 1 = \mu_2$ *and* $\lambda_1 = \mu_1 + 1$;
(4) $l = 3$ *and one of the following holds:*
  (a) μ *is a hook;*
  (b) $\lambda_1 - \mu_1 = 1$ *and one of the following holds:*
    (i) $l(\lambda) - l(\mu) = 1$;
    (ii) $r_2 = 1$;
    (iii) $r_1 = 1$ *and* $\mu_1 - \mu_2 = 1$;
    (iv) $k_2 = k_3 = 1$;
    (v) *there is an* i *such that* $k_i = 1$ *and* $\lambda_i - \lambda_{i+1} = 1$ *and* $r_1 = 1$;
    (vi) *there are* i, j *such that* $k_i = 1$ *and* $\mu_j - \mu_{j+1} = 1$.
  (c) *there is an* i *such that* $\lambda_i - \lambda_{i+1} = 1$ *and* $k_{i+1} = 1$ *and one of the following holds:*
    (i) $r_2 = 1$ *and* $\mu_2 = 1$;
    (ii) $r_1 = 1$ *and* $\mu_1 - \mu_2 = 1$.
  (d) *there is an* i *such that* $\lambda_i - \lambda_{i+1} = 1$ *and* $k_i = 1$ *and in addition,* $r_2 = 1$ *and* $\mu_1 - \mu_2 = 1$;
  (e) $\lambda_1 - \lambda_2 = 1$ *and* $k_3 = 1$;
  (f) $\lambda_3 = 1$ *and* $k_1 = 1$ *or* $k_2 = 1$;
  (g) $r_1 = r_2 = 1$ *and one of the following holds:*
    (i) $\mu_1 - \mu_2 = 1$;
    (ii) *there is an* i *such that* $\lambda_i - \lambda_{i+1} = 1$.
  (h) *there are* $i \neq j$ *such that* $\lambda_i - \lambda_{i+1} = \lambda_j - \lambda_{j+1}$ *and* l *such that* $k_l = 1$ *or* $r_l = 1$.

In contrast to the multiplicity-free case, we have outer tensor products of an irreducible character and a skew character which contain only constituents with multiplicity less or equal to 2 (even though we do not show this here). This is equivalent to a skew partition which decomposes into two connected components, one a proper skew partition, the other one a partition.

Theorem 2.7. *Let* ν *be a partition and* λ/μ *be a basic and connected skew partition. If up to rotation of* λ/μ *by* $180°$ *and/or conjugation of* λ/μ *and* ν *they are not from the following list, the outer tensor product* $[\lambda/\mu] \boxtimes [\nu]$ *contains a constituent with multiplicity 3 or higher:*

(1) *If* μ *is a rectangle,* $\lambda = (\lambda_1^{k_1}, \lambda_2^{k_2})$ *and one of the following holds:*
  (a) $\nu = (1)$;
  (b) $\lambda_1 - \lambda_2 = 1$ *or* $\lambda_2 = 1$ *and* ν *has one row;*
  (c) $w(\mu) = \lambda_1 - 1$ *and* ν *has one row.*
(2) *if* ν *is a rectangle and one of the following holds:*
  (a) $\lambda^{(')} = (\lambda_1, \lambda_1 - 1)$;
  (b) λ *is a two-line partition and* $\mu = (1)$;
  (c) $\lambda^{(')} = (\lambda_1^{k_1}, \lambda_1 - 1)$ *and* $\mu = (1)$;
  (d) $\lambda = (\lambda_1^{k_1}, 1)$ *and* $\mu = (\lambda_1 - 1)$ *or both conjugated.*
(3) *if* μ *and* ν *are both one-line partitions and one of the following holds:*

(a) $\lambda_1 - 1 = \mu_1$ *and* $\nu = (1)$*;*
(b) $\lambda = (\lambda_1^{k_1}, \lambda_2^{k_2})$ *and one of the following holds:*
  (i) $l(\mu) = 1$ *and* $w(\nu) = 1$*;*
  (ii) $\lambda_1 - \lambda_2 = 1$ *or* $\lambda_2 = 1$ *and* $l(\mu) = l(\nu) = 1$*;*
  (iii) $l(\lambda) - l(\mu) = 1$ *and* $w(\nu) = 1$.

Conjecture 2.8. *All outer tensor products and skew characters listed in the previous three theorems contain only constituents with multiplicity 1 and 2.*

2. Equalities and inequalities for Littlewood-Richardson coefficients

In this section we collect some known equalities and inequalities which reduce the number of cases we have to investigate. First, we need an operation on skew diagrams. If λ/μ is a skew diagram, we denote with $(\lambda/\mu)^{\mathrm{rot}}$ the skew diagram rotated by 180°. If we talk about rotating a (skew) diagram, we always mean rotating by 180°. There are a lot of symmetries known for Littlewood-Richardson coefficients but we just use the following well known ones:

Theorem 2.9. *Let λ, μ, ν be partitions. Then the following equalities hold:*
(1) $c(\lambda; \mu, \nu) = c(\lambda; \nu, \mu)$*;*
(2) *the conjugation symmetry* $c(\lambda; \mu, \nu) = c(\lambda'; \mu', \nu')$*;*
(3) *the rotation symmetry* $[\lambda/\mu] = [(\lambda/\mu)^{\mathrm{rot}}]$.

These symmetries allow us to reduce the cases we have to look at. For two partitions λ, μ, we define the sum $\lambda + \mu$ by $(\lambda + \mu)_i = \lambda_i + \mu_i$ for all i. We say we add μ to λ (as columns). We also define $\lambda \cup \mu := (\lambda' + \mu')'$. Here we say that we add μ to λ as rows.

Lemma 2.10. *[Gut10, Lemma 3.4.] Let $\lambda = (\lambda_1^{k_1}, ..., \lambda_j^{k_j})$, $\mu = (\mu_1, ..., \mu_l)$, ν be partitions.*
(1) *If $l \leq k_i$ for some $0 \leq i \leq j$, then for all $n \geq 0$,*
$$c(\lambda; \mu, \nu) = c(\lambda \cup (\lambda_i^n); \mu, \nu \cup (\lambda_i^n));$$
(2) *if $\mu_1 \leq \lambda_i - \lambda_i + 1$ (as usual $\lambda_{j+1} = 0$) for some $0 \leq i \leq j$, then let $r_i = \sum_{a=1}^{i} k_a$ and for all $n \geq 0$,*
$$c(\lambda; \mu, \nu) = c(\lambda + (n^{r_i}); \mu, \nu + (n^{r_i})).$$

Lemma 2.11. *[Gut10, Theorem 3.1.] Let λ, μ, ν be partitions and $a \geq b \geq 0$ integers. Then*
$$c(\lambda; \mu, \nu) \leq c(\lambda + (1^a); \mu + (1^b), \nu + (1^{a-b})),$$
as well as
$$c(\lambda; \mu, \nu) \leq c(\lambda \cup (a); \mu \cup (b), \nu \cup (a - b)).$$

The previous lemma generalizes in an obvious way to adding not only rows or columns but suitable triples of partitions as rows or columns. If λ can be obtained from $\tilde{\lambda}$ by adding rows and columns, we write $\lambda \succeq \tilde{\lambda}$.

3. First examples

Lemma 2.12. *If $\lambda \vdash n$ and $\mu \vdash m$ with $\mathrm{rem}(\lambda), \mathrm{rem}(\mu) \geq 3$, the product $[\lambda] \boxtimes [\mu]$ contains a constituent of multiplicity greater or equal to 4.*

Proof: This follows since λ, μ can be obtained from $(3,2,1)$ by adding rows and columns and $[3,2,1] \boxtimes [3,2,1]$ has several constituents (for example $[4,3,2^2,1]$) of multiplicity 3 and 4. We show how to obtain λ from $(3,2,1)$, μ follows in the same way.

Let $\lambda = (\lambda_1^{k_1}, \lambda_2^{k_2}, \ldots, \lambda_l^{k_l})$ with $l \geq 3$. We know that

$$c((4,3,2^2,1);(3,2,1),(3,2,1)) = 4.$$

We obtain λ from $(3,2,1)$ by first adding $(\lambda_1 - 3, \lambda_2 - 2, \lambda_3 - 1)$. Note that this is a partition since we assume that $\lambda_i > \lambda_{i+1}$. Then we add

$$\tilde{\lambda} = (\lambda_1^{k_1-1}, \lambda_2^{k_2-1}, \lambda_3^{k_3-1}, \lambda_4^{k_4}, \ldots, \lambda_l^{k_l})$$

as rows. Thanks to Lemma 2.11 we know

$$c(((4,3,2^2,1) + (\lambda_1 - 3, \lambda_2 - 2, \lambda_3 - 1)) \cup \tilde{\lambda}; \lambda, (3,2,1)) \geq c((4,3,2^2,1);(3,2,1),(3,2,1)) = 4.$$

If we do the same for μ, we obtain $c(\nu; \lambda, \mu) \geq 4$ for some partition ν. □

To find constituents of multiplicity 3 or higher, we often use a small product and Lemma 2.11 to find a constituent for the product we are investigating, like in the previous lemma. However, usually we do not spell out how to obtain λ and μ from the partitions of the small product since this will be quite obvious.

Lemma 2.13. *The outer tensor product of 4 or more characters contains a constituent with multiplicity 3 or higher.*

Proof: This follows since $[1]\boxtimes[1]\boxtimes[1]\boxtimes[1]$ contains $[3,1]$ with multiplicity 3. □

To show that a given product only contains constituents with multiplicity 1 and 2 we have different methods. Sometimes, even though it is an infinite family, we can reduce it to a finite number of cases which we check with the computer. We see this in the following lemma.

Lemma 2.14.
1. *Let $\lambda \vdash n$ be a partition. Then the products $[\lambda] \boxtimes [2,1]$ and $[\lambda] \boxtimes [2,2]$ have only constituents of multiplicity less or equal to 2.*
2. *Let $\lambda \vdash n$ consist of three different parts, i.e., $\lambda = (\lambda_1^{k_1}, \lambda_2^{k_2}, \lambda_3^{k_3})$. The product $[\lambda] \boxtimes [3^2]$ only contains constituents with multiplicity 1 or 2.*

Proof: We start by proving the first statement. Let λ be a partition and μ be either $(2,1)$ or $(2,2)$. If $[\nu]$ is a constituent of $[\lambda] \boxtimes [\mu]$, we know that

$$c(\nu; \lambda, \mu) = \langle [\lambda] \boxtimes [\mu], [\nu] \rangle = \langle [\mu], [\nu/\lambda] \rangle.$$

Without loss of generality we can assume that ν/λ is a basic skew diagram of size $|\mu|$. But it is easy to check with Sage (or even by hand) that all skew characters of size 3 only contain the constituent $[2,1]$ with multiplicity 0, 1 or 2 and that all skew characters of size 4 only contain the constituent $[2,2]$ with multiplicity 0, 1 or 2.

The second part works in the same way. We check with Sage that there is no skew diagram α/β such that $\mathrm{rem}(\alpha) \leq 3$, $|\alpha| - |\beta| = 6$ and $\langle [\alpha/\beta], [3,3] \rangle \geq 3$. □

As we have seen in the proof of the previous lemma, it is sometimes very helpful to switch between skew characters and the outer tensor product of two irreducible characters. So in the following lemma we illustrate some of the methods we have for skew characters with a simple example.

Lemma 2.15. *Let $\lambda = (\lambda_1^{k_1}, \lambda_2^{k_2}, \lambda_3^{k_3}, \lambda_4^{k_4})$ and $\mu = (\mu_1^{r_1})$ be partitions such that λ/μ is a basic skew diagram.*

(1) *If* $\mu^{(\prime)} = (3^2)$*, the skew character* $[\lambda/\mu]$ *only contains constituents with multiplicity* 1 *and 2.*

(2) *If* $r_1 = 2$ *and* $\lambda_1 - \mu_1 \leq 3$*, the skew character* $[\lambda/\mu]$ *only contains constituents with multiplicity* 1 *and 2.*

Proof: We start by proving the first part. Due to the conjugation symmetry (Theorem 2.9) we can assume that $\mu = (2^3)$. Moreover, by Lemma 2.11, we can assume that $\lambda_i - \lambda_{i+1} \geq 2$ and $k_i \geq 3$ for all $1 \leq i \leq 4$. With Lemma 2.10 we can assume that $\lambda_i - \lambda_{i+1} \leq 2$ and $k_i \leq 3$ for all $1 \leq i \leq 4$. Together this shows that it is sufficient to check $[(8^3, 6^3, 4^3, 2^3)/(2^3)]$.

For the second part we know that $r_1 = 2$. With Lemma 2.10 we can assume that $k_1 = 2$. If we add $(3^2)/(3^2)$, Lemma 2.11 tells us that the multiplicities can only grow. But the resulting diagram decomposes into $((\lambda_1 - \mu_1)^2)$ and $(\lambda_2^{k_2}, \lambda_3^{k_3}, \lambda_4^{k_4})$. If $\lambda_1 - \mu_1 = 2$, 3, the previous lemma tells us that this product only contains constituents with multiplicity less or equal to 2. If $\lambda_1 - \mu_1 = 1$, the product is even multiplicity-free (see Theorem 2.3) □

In the last lemma we saw that not only the skew characters help us with the outer tensor products, but it also works in the other direction. In the next lemma we look at a skew character that we will need later for the outer tensor products.

Lemma 2.16. *Let* λ *be a partition and* $\mu = (\lambda_1 - 1, 1^{l(\lambda)-2})$*. The skew character* $[\lambda/\mu]$ *only contains constituents with multiplicity 1 and 2.*

Proof: If μ is a hook with $\mu_1 + 1 = \lambda_1$ and $l(\mu) + 1 = l(\lambda)$, we first add $(1)/(1)$ to λ/μ and in the next step we add $(1)/(1)$ as a row. The resulting skew diagram is no longer connected. Instead, it decomposes into 3 connected components. Above, on the right and at the bottom, on the left there is a component of the shape (1). The component in between these is a partition, too (namely the one that we obtain from λ/μ by deleting the first row and the first column). For a partition $\nu \vdash n$, $[1] \boxtimes [\nu]$ is the same character as $[\nu] \uparrow_{S_n}^{S_{n+1}} = \sum_{A \in \mathrm{Add}(\nu)} [\nu^A]$. From this point of view it is obvious why products of the form $[\nu] \boxtimes [1] \boxtimes [1]$ and therefore, also $[\lambda/\mu]$ contain only constituents with multiplicity less or equal to 2. □

CHAPTER 3

Outer tensor products

In this chapter we want to prove Theorem 2.4 (which classifies the outer tensor products which only contain constituents with multiplicity 1 and 2). Because of Lemma 2.13, we can focus on products of two or three irreducible characters. In the first section we look at products of two partitions, in the second we look at products of three partitions.

1. Products of two irreducible characters

Due to Lemma 2.12 we know that one of the partitions is a rectangle or a fat hook. We look at the different possibilities for this partition.

One of the partitions is a hook.
The following lemma tells us whether a product of a hook with another partition only contains constituents with multiplicity less or equal to 2. Due to the lucidity of hook partitions we can actually construct the possible two Littlewood-Richardson tableaux in this case (which is something that gets quite messy for other cases). Therefore, we give two different proofs for one direction of the following lemma. One is actually looking at the possibilities for the Littlewood-Richardson tableaux, the other one is a bit more structural and generalizes to other cases.

Lemma 3.1. *Let $\lambda \vdash n \geq 4$ be a proper hook and $\mu \vdash m$ a partition such that $[\lambda] \boxtimes [\mu]$ is not multiplicity-free. The product $[\lambda] \boxtimes [\mu]$ contains only constituents with multiplicity less or equal to 2 if and only if μ is a fat hook.*

Proof: There are two directions we have to show. If μ is not a fat hook, μ has at least three different parts, this means $\mu \succeq (3,2,1)$. Because λ is a hook and not linear, we know that $\lambda \succeq (3,1)^{(')}$. Since $[3,2,1] \boxtimes [3,1]^{(')}$ has $[4,3,2,1]$ as constituent with multiplicity 3, we conclude that $[\lambda] \boxtimes [\mu]$ has a constituent of multiplicity at least 3.

Now let us assume that μ is a fat hook. We want to show that all the constituents have multiplicity less or equal to two. Since we already know that $[\lambda] \boxtimes [\mu]$ is multiplicity-free if μ is a rectangle, we can assume that $\mu = (a^b, c^d)$ has two removable nodes. Thanks to Lemma 2.11 we can assume that $a-c$, c, b, d are greater than n. Let $[\nu]$ be a constituent of $[\lambda] \boxtimes [\mu]$. We know that ν/μ decomposes into at most three hooks. Actually, up to two of the three parts could be empty but since the maximal multiplicity of the constituents stays the same or decreases if there are less parts we only elaborate the case that all parts are not empty. We call them α^1, α^2, and α^3. Then we know that

$$c(\nu;\lambda,\mu) = \langle [\nu/\mu], [\lambda] \rangle = \langle [\alpha^1] \boxtimes [\alpha^2] \boxtimes [\alpha^3], [\lambda] \rangle = \langle [\alpha^1] \boxtimes [\alpha^2], [\lambda/\alpha^3] \rangle.$$

But λ/α^3 decomposes into a one-column partition and a one-row partition and therefore, $[\lambda/\alpha^3]$ consists of two hooks. But since α^1 and α^2 are both hooks, we know that all the constituents of $[\alpha^1] \boxtimes [\alpha^2]$ which are hooks only occur once. If $[\beta]$ is a hook and a constituent of $[\alpha^1] \boxtimes [\alpha^2]$, then β/α^1 decomposes into a one-row and a one-column partition. □

Remark 3.2. That products of the form [hook] ⊠ [fat hook] only contain constituents with multiplicity with 1 and 2 can actually be shown with Littlewood-Richardson tableaux in the following way:

We assume that λ is a hook and $\mu = (a^b, c^d)$. Let ν be a partition of $n+m$ such that $\nu \supseteq \mu$ and that there is a Littlewood-Richardson tableau T of shape ν/μ and content λ. Since λ is a hook, we know that $\nu \subseteq (a+n, (a+1)^b, (c+1)^d, 1^n)$. There can be two boxes of T for which the content is not already determined by the shape of ν/μ and the content of the boxes lying in rows above. If such boxes exist, they are:

- the last box of ν/μ in the $(b+1)$-th row if it is not in the $(a+1)$-th column and μ/ν has a box in a row above and
- the last box of ν/μ in the $(b+d+1)$-th row if it is not in the $(c+1)$-th column and μ/ν has a box in a row above.

We conclude that the only partitions which have multiplicity 2 are the ones for which the content of both of these boxes is not already determined and λ is of such a form that one of the boxes has to be filled with a 1 and the other box has to be filled with an entry different from 1.

One of the partitions is a two-row partition.

Lemma 3.3. *Let $\lambda \vdash n \geq 4$ be a two-row partition (neither $(n-1,1)$ nor $(2,2)$) and $\mu \vdash m \geq 4$ not a hook such that λ, μ is not from the list of multiplicity-free outer tensor products. Then all constituents of $[\lambda] \boxtimes [\mu]$ have multiplicity less or equal to 2 if and only if one of the following cases occurs:*

(1) *$\mu = (a^b, (a-1)^c)$ or $\mu = (a^b, 1^c)$ for suitable a, b, c;*
(2) *$n = 2k+1$ is odd, $\lambda = (k+1, k)$ and μ is a fat hook;*
(3) *$n = 2k$ is even, $\lambda = (k, k)$ and μ is of the form (a^b, c^d, e^f) for suitable a, b, c, d, e, f with $a-c=1$ or $c-e=1$ or $e=1$;*
(4) *μ has three removable boxes and $\lambda = (3,3)$.*

Proof: We start with showing that products which are not listed contain constituents with multiplicity 3 or higher. First, let us assume that $\lambda_1 - \lambda_2 \geq 2$ and $\mu \vdash m$ such that λ, μ is not a pair from the list. If $\text{rem}(\mu) \geq 3$, we know that $\mu \succeq (3,2,1)$, so in this case it is sufficient to check $[4,2] \boxtimes [3,2,1]$. If μ is a fat hook of the form (a^b, c^d), we know that $c > 1$ and $a-c > 1$ since the pair λ, μ is not listed in this lemma. Therefore, it is sufficient to check the product $[4,2] \boxtimes [4,2]$. If μ is a rectangle, the product is multiplicity-free, so now we look at the case $\lambda_1 - \lambda_2 \leq 1$.

Case 1: $\lambda_1 - \lambda_2 = 1$. Let us assume that $n = 2k+1 \geq 5$ is odd and that $\lambda = (k+1, k)$. Since we assume the pair λ, μ is not listed in (2), we know that $\text{rem}(\mu) \geq 3$. It is sufficient to check $[3,2] \boxtimes [3,2,1]$.

Case 2: $\lambda_1 = \lambda_2$. Let $n = 2k \geq 6$ be even and $\lambda = (k,k)$. If $\text{rem}(\mu) \geq 4$, we know that $\mu \succeq (4,3,2,1)$ and therefore, we can reduce it to the product $[3,3] \boxtimes [4,3,2,1]$. If $\text{rem}(\mu) = 3$, we know that $n \geq 8$ and that $\mu \succeq (6,4,2)$ so it is sufficient to check $[6,4,2] \boxtimes [4,4]$. Now we verify that the listed products only have constituents with multiplicity less or equal to 2.

(1): We can assume that $a-1$, b and c are greater than n because of Lemma 2.11. From now on we will not always mention Lemma 2.11 if we are using a partition which is bigger than the original one. If $[\nu]$ is a constituent of $[\lambda] \boxtimes [\mu]$, we know that ν/μ decomposes into at most three parts. Two of them are two-row partitions, we call them α^1 and α^2, and the third one which we call α^3 is either (1), (1^2) or empty. So we know that

$$c(\nu; \lambda, \mu) = \langle [\alpha^1] \boxtimes [\alpha^2] \boxtimes [\alpha^3], [\lambda] \rangle = \langle [\alpha^2] \boxtimes [\alpha^3], [\lambda/\alpha^1] \rangle.$$

We know that $[\lambda/\alpha^1]$ is multiplicity-free and all constituents are two-row partitions. If $\alpha^3 = (1^2)$, $[\alpha^2] \boxtimes [\alpha^3]$ only contains one two-row partition. Therefore, $c(\nu; \lambda, \mu) \leq 1$. If $\alpha^3 = (1)$, $[\alpha^2] \boxtimes [\alpha^3]$ contains at most two two-row partitions. Therefore, $c(\nu; \lambda, \mu) \leq 2$.

(2): Let $\lambda = (k+1, k)$ and $\mu = (a^b, c^d)$ be a fat hook. We can assume that $a-c, c, b, d$ are greater than n. If $[\nu]$ is a constituent of $[\lambda] \boxtimes [\mu]$, we know that ν/μ decomposes into at most three parts, which are two-row or one-row partitions. We call them α^1, α^2 and α^3. We know that

$$c(\nu; \lambda, \mu) = \langle [\alpha^1] \boxtimes [\alpha^2] \boxtimes [\alpha^3], [\lambda] \rangle = \langle [\alpha^2] \boxtimes [\alpha^3], [\lambda/\alpha^1] \rangle.$$

Since $\lambda = (k+1, k)$, we know that $(\lambda/\alpha^1)^{\mathrm{rot}}$ is a two-row partition skewed by (1). This means that $[\lambda/\alpha^1]$ decomposes into the sum of at most 2 two-row partitions. It is easy to see that in the product of characters labeled by two two-row partitions every two-row partition has at most multiplicity 1 and therefore, $c(\nu; \lambda, \mu) \leq 2$.

(3): Let λ be a two-row rectangle and $\mu = (a^b, c^d, e^f)$, for which we know that one of the numbers $a-c$, $c-e$, e equals 1. We can assume that the others are greater than n and that b, d, $f > 2$. If $[\nu]$ is a constituent of $[\lambda] \boxtimes [\mu]$, we know that ν/μ decomposes into at most four partitions with up to two rows. We call them α^1, α^2, α^3 and α^4, where α^4 is either (1) or (1^2). With this we get

$$c(\nu; \lambda, \mu) = \langle [\alpha^1] \boxtimes [\alpha^2] \boxtimes [\alpha^3] \boxtimes [\alpha^4], [\lambda] \rangle = \langle [\alpha^2] \boxtimes [\alpha^3] \boxtimes [\alpha^4], [\lambda/\alpha^1] \rangle.$$

But since λ is a two-row rectangle, we know that $[\lambda/\alpha^1] = [\beta]$ for some two-row partition β. Hence,

$$\begin{aligned} c(\nu; \lambda, \mu) &= \langle [\alpha^2] \boxtimes [\alpha^3] \boxtimes [\alpha^4], [\lambda/\alpha^1] \rangle = \langle [\alpha^2] \boxtimes [\alpha^3] \boxtimes [\alpha^4], [\beta] \rangle \\ &= \langle [\alpha^2] \boxtimes [\alpha^3], [\beta/\alpha^4] \rangle. \end{aligned}$$

On the other hand $[\beta/\alpha^4]$ decomposes into at most two two-row partitions and all the constituents of $[\alpha^2] \boxtimes [\alpha^3]$ which are two-row partitions have multiplicity 1. Therefore, $c(\nu; \lambda, \mu) \leq 2$.

(4): This has been proven in Lemma 2.14. □

By the conjugation symmetry for Littlewood-Richardson coefficients this also solves the case in which one of the partitions is a two-column partition. So from now on we assume that neither λ nor μ is a hook or a two-line partition. For the last lemmas we only needed very few small products with multiplicity 3. Sadly, it seems that in the next two lemmas more of these small products are needed. It also gets a bit more complex to see which to which of the small products we can reduce $[\lambda] \boxtimes [\mu]$. There are different ways to organize the proofs. Even though it might look redundant to state some of these small products several times we think that it might be a bit confusing if we mention all the different cases in which the it can be used the first time it comes up.

One of the partitions is a rectangle.

Lemma 3.4. *Let $\lambda = (a^b)$ be a proper rectangle (with $a, b > 2$) and $\mu \vdash n$ neither a hook nor a two-line partition such that λ, μ is not from the list of multiplicity-free outer tensor products. Then the product $[\lambda] \boxtimes [\mu]$ has only constituents with multiplicity less or equal to 2 if and only if one of the following cases occur (up to conjugation of both partitions):*

(1) *μ is a fat hook and $a = 3$ and $b \leq 5$;*
(2) *$\mu = (c^2, 2^d)$ for some $c, d \geq 2$;*
(3) *$b = 3$ and $\mu = (c^d, e^f)$ with $e \leq 3$ or $c - e \leq 3$;*
(4) *$\mu = (c^2, d^2)$ with $c - d \leq 3$ or $d \leq 3$ or $a \leq 5$;*
(5) *$\mu = ((c+2)^2, c^d)$ or $\mu = (c^d, (c-2)^2)$;*

(6) $\mu = (c, d, 1^e)$;
(7) $\mu = (c+1, c^d, 1)$;
(8) $a = 3$ *and* $l(\mu) \leq 3$;
(9) $\mu = (c, d, e)$ *with* $c - 1 = d$ *or* $d - 1 = e$;
(10) $\mu = (c^d, c-1, (c-2)^e)$ *with* $d = 1$ *or* $e = 1$.

Proof: We start by proving that products which are not listed have a constituent with multiplicity 3 or higher. If $\text{rem}(\mu) \geq 4$, we reduce the product $[\lambda] \boxtimes [\mu]$ to $[3^3] \boxtimes [4, 3, 2, 1]$. In the next step we look at μ with $\text{rem}(\mu) = 3$ such that the pair λ, μ is not from the list. If every part of μ occurs with multiplicity 1, we know that $l(\mu) = 3$ and therefore, $a > 3$ and $\mu_1 - \mu_2$, $\mu_2 - \mu_3$, $\mu_3 \geq 2$. Otherwise, the product would be listed in (6), (8) or (9). This implies that $\lambda \succeq (4^3)$ and $\mu \succeq (6, 4, 2)$. So for this case it is sufficient to check that $[6, 4, 2] \boxtimes [4^3]$ has a constituent with multiplicity 3.

Now we assume that exactly one part has multiplicity 2 or higher and the other ones have multiplicity 1. If it is the first part, i.e., $\mu = (c^d, e, f)$ with $d > 1$, we know that $c - e$ or $e - f$ is greater than 1 and that $e \neq 2$. Otherwise, the product would be listed in (10) maybe after conjugation. Therefore, we know that $\mu \succeq (4^2, 3, 1)$ or $\mu \succeq (5^2, 3, 2)$ and we need to check that $[4^2, 3, 1] \boxtimes [3^3]$ and $[5^2, 3, 2] \boxtimes [3^3]$ contain a constituent with multiplicity 3. If the second part occurs multiple times, i.e., $\mu = (c, d^e, f)$ with $e > 1$, we know that $d \neq 2$ and if $f = 1$, we know that $c - d > 1$. Otherwise, it would be listed after conjugation in (6) or in (7). Therefore, if $f \neq 1$, $\mu \succeq (4, 3^2, 2)$ and if $f = 1$, $\mu \succeq (5, 3^2, 1)$ and we check that $[4, 3^2, 2] \boxtimes [3^3]$ and $[5, 3^2, 1] \boxtimes [3^3]$ contain a constituent of multiplicity 3. If it is the third part which is contained multiple times, i.e., $\mu = (c, d, e^f)$, we know that $e \neq 1$ and that $c - d$ or $d - e$ is greater than 1. Otherwise, it would be listed in (6) or (10). Therefore, $\mu \succeq (5, 4, 2^2)$ or $\mu \succeq (5, 3, 2^2)$ and it is sufficient to check that $[5, 4, 2^2] \boxtimes [3^3]$ and $[5, 3, 2^2] \boxtimes [3^3]$ contain a constituent with multiplicity 3.

In the next step we look at the case for μ in which exactly one of the parts has multiplicity 1 and the other two have multiplicity 2 or higher. If the last part has multiplicity 1, i.e., $\mu = (c^d, e^f, g)$, we know that $c > 3$. Otherwise, the product would be listed in (9) after conjugation. Hence, we conclude that $\mu \succeq (4, 3^2, 2)$, $\mu \succeq (4^2, 3, 1)$ or $\mu \succeq (4^2, 2^2, 1)$ and therefore we have to check that $[4^2, 2^2, 1] \boxtimes [3^3]$ contains a constituent with multiplicity 3. We have already checked the other two products. If the second part has multiplicity 1, i.e., $\mu = (c^d, e, f^g)$, we know as before that $c > 3$. Otherwise, the product would be listed in (9) after conjugation. Therefore, we obtain μ from $(4^2, 3, 2^2)$, $(4^2, 3, 1)$ or $(4^2, 2, 1^2)$. We have to check the products $[4^2, 3, 2^2] \boxtimes [3^3]$ and $[4^2, 2, 1^2] \boxtimes [3^3]$. We have already checked the third product. If the first part has multiplicity 1, i.e., $\mu = (c, d^e, f^g)$, we know that $d > 2$. Otherwise, the product would be listed in (6) after conjugation. Therefore, $\mu \succeq (4, 3^2, 2)$ or $\mu \succeq (4, 3^2, 1^2)$. So for this part we have to check that the product $[4, 3^2, 1^2] \boxtimes [3^3]$ contains a constituent with multiplicity 3. Again, we have already checked the other product.

If all of the parts have at least multiplicity 2, we distinguish between two cases. If $b > 3$, $\mu \succeq (3^2, 2^2, 1^2)$ and we obtain the result since $[3^2, 2^2, 1^1] \boxtimes [3^4]$ contains a constituent with multiplicity 3. If $b = 3$, we know that $w(\mu) > 3$. Otherwise, it would be listed in (8) after conjugation. Therefore, we obtain μ from $(4^2, 2^2, 1)$, $(4^2, 3, 1)$ or $(4, 3^2, 2)$, but we have already checked these products.

In the next step we look at μ with $\text{rem}(\mu) = 2$, i.e., $\mu = (c^d, e^f)$. If μ is a rectangle or near-rectangle, the product is multiplicity-free, consequently we assume that $c - e$, d, e, $f > 1$. We start with the case $a = 3$ or $c = 4$. Here we know that $b \geq 6$, d, $f \geq 4$ and $c - e$, $e \geq 2$. Otherwise, the product would be listed, possibly after conjugation, in (1), (3) or (4) or it would be multiplicity-free. Therefore,

$\mu \succeq (4^4, 2^4)$ and we need to check $[3^6] \boxtimes [4^4, 2^4]$. By conjugation symmetry we can assume from now on that a, $b \geq 4$ and c, $d + f \geq 5$. Further, we see that at most one of the numbers $c - e$, d, e, f equals 2; otherwise, the product would be from (2) or (5) or $c = 4$ or $d + f = 4$. By conjugation symmetry we can assume that it is either d or f. If $f \geq 3$, we know that $(6^2, 3^3) \preceq \mu$ and therefore, it is sufficient to check the product $[6^2, 3^3] \boxtimes [4^4]$. If $d \geq 3$, we know that $(6^3, 3^2) \preceq \mu$ and therefore, it is sufficient to check the product $[6^3, 3^2] \boxtimes [4^4]$. In the next step we prove that all the products which are listed in the lemma only contain constituents with multiplicity 1 and 2.

(1): Let $[\nu]$ be a constituent of $[\lambda] \boxtimes [\mu]$. We can assume by Lemma 2.11 that $\mu = (c^d, e^f)$ with $c - e$, e, d, f larger than ab. We know that ν/μ decomposes into at most three parts. We call these parts α^1, α^2 and α^3. We can assume that $|\alpha^1| \geq |\alpha^2| \geq |\alpha^3|$. It follows that

$$\begin{aligned} c(\nu; \lambda, \mu) &= \langle [\lambda] \boxtimes [\mu], [\nu] \rangle = \langle [\lambda], [\nu/\mu] \rangle = \langle [\lambda], [\alpha^1] \boxtimes [\alpha^2] \boxtimes [\alpha^3] \rangle \\ &= \langle [\lambda/\alpha^1], [\alpha^2] \boxtimes [\alpha^3] \rangle, \end{aligned}$$

where $(\lambda/\alpha^1)^{\mathrm{rot}}$ is the diagram of a partition with width smaller or equal to 3 and length smaller or equal to 5. Since $|\alpha^2| \leq 7$, we check with Sage that there are no partitions $\alpha^2 \vdash m_2$, $\alpha^3 \vdash m_3$ such that $15 - m_2 - m_3 \geq m_2 \geq m_3$ and $[\alpha^2] \boxtimes [\alpha^3]$ containing a constituent with multiplicity greater or equal to 3 which has width less or equal to 3.

(2): We can assume that $c - 2$ and d are greater than ab. This means that if we have ν such that $[\nu/\mu]$ contains $[\lambda]$, then ν/μ decomposes into three parts and all three parts are partitions. Let us say that ν/μ decomposes into

- α^1, which is the upper right part and therefore, an at most two-row partition,
- α^2, which is the middle part that we do not know anything about, and
- α^3, which is the part at the lower left end of μ and therefore, a partition with at most two columns.

We know that

$$c(\nu; \lambda, \mu) = \langle [\nu/\mu], [\lambda] \rangle = \langle [\alpha^1] \boxtimes [\alpha^2] \boxtimes [\alpha^3], [\lambda] \rangle = \langle [\alpha^1] \boxtimes [\alpha^3], [\lambda/\alpha^2] \rangle.$$

Since λ is a rectangle, we know that $[\lambda/\alpha^3]$ is an irreducible character (rotation symmetry of Theorem 2.9). Therefore, (2) follows from Lemma 3.3.

(3): We can assume that d and f are greater than 3. We know that for any constituent $[\nu]$ of $[\lambda] \boxtimes [\mu]$ the skew partition ν/μ decomposes into three different parts. We label them in the same way as before. We know that $l(\alpha^i) \leq 3$ for all $i \in \{1, 2, 3\}$ and there is a $j \in \{2, 3\}$ such that $w(\alpha^j) \leq 3$. Let $\alpha^1 = (\alpha^1_1, \alpha^1_2, \alpha^1_2)$. Since $b = 3$, there is an obvious bijection of the Littlewood-Richardson tableaux for $\langle [\alpha^1] \boxtimes [\alpha^2] \boxtimes [\alpha^3], [a^3] \rangle$ and the ones which count $\langle [\alpha^1_1 - \alpha^1_3, \alpha^1_2 - \alpha^1_3] \boxtimes [\alpha^2] \boxtimes [\alpha^3], [(a - \alpha^1_3)^3] \rangle$. Hence, we can assume that $\alpha^1_3 = 0$. Let $k \in \{2, 3\}$ and $j \neq k$. We know

$$\langle [\nu/\mu], [\lambda] \rangle = \langle [\alpha^1] \boxtimes [\alpha^2] \boxtimes [\alpha^3], [\lambda] \rangle = \langle [\alpha^1] \boxtimes [\alpha^2] \boxtimes [\alpha^3], [\lambda] \rangle = \langle [\alpha^1] \boxtimes [\alpha^j], [\lambda/\alpha^k] \rangle.$$

If $\alpha^j \neq (3, 2, 1)$, we know that $[\alpha^j] \boxtimes [\alpha^1]$ only contains constituents with multiplicity less or equal to 2. If α^j is linear or a rectangle, the product is multiplicity-free, if $\alpha^j = (2, 1)$, this follows from Lemma 2.14, if α^j is a hook, this follows from Lemma 3.1, otherwise from Lemma 3.3 since α^1 is a two-row partition. Since λ is a three-row rectangle, we know that $[\lambda/\alpha^k]$ is an irreducible character which corresponds to a partition of length less or equal to 3. For $\alpha^j \neq (3, 2, 1)$ it follows that $[\lambda] \boxtimes [\mu]$ only contains constituents with multiplicity less or equal to 2. If $\alpha^j = (3, 2, 1)$, we show that the constituents of length less or equal to 3 only occur with multiplicity 1 or 2. Since $w((3,2,1)) = 3$, we know by Lemma 2.10 that we only have to check the products of $[3,2,1]$ with the two-row partitions $\beta = (\beta_1, \beta_2)$, where $\beta_1, \beta_1 - \beta_2 \leq 3$, which was done with Sage.

(4): We start with $d = 3$ or $c - d = 3$. In this case we can assume that a and b are greater than $4c$. If $[\nu]$ is a constituent of $[\lambda] \boxtimes [\mu]$, we know that the skew diagram ν/λ decomposes into two diagrams of partitions. If we call them α^1 and α^2, we conclude that

$$\langle [\nu/\lambda], [\mu] \rangle = \langle [\alpha^1] \boxtimes [\alpha^2], [\mu] \rangle = \langle [\alpha^1], [\mu/\alpha^2] \rangle.$$

Thus, we only need to know that $[\mu/\alpha^2]$ only contains constituents with multiplicity less or equal to 2. Since $d = 3$ or $c - d = 3$, this follows from Lemma 2.15 after using the rotation symmetry. If $a \leq 5$, we can assume by Lemma 2.11 that d, $c - d \geq 5 = a$ and that $b \geq 4 = l(\mu)$. Then with Lemma 2.10 we can assume that d, $c - d = 5$ and that $b = 4$. But the product $[5^4] \boxtimes [10^2, 5^2]$ was checked with Sage.

(5): We have two possibilities for μ. We start with $\mu = ((c+2)^2, c^d)$. We can assume that c and d are greater than ab. This time, ν/μ decomposes into $\alpha^1/(2,2)$ and α^2, where α^1 and α^2 are partitions. By Lemma 2.14 $\alpha^1/(2,2)$ only contains constituents with multiplicity less or equal to 2. Additionally, due to rotation symmetry $[\lambda/\alpha^2]$ is an irreducible character. Thus, we know that $[\lambda/\mu]$ only contains constituents with multiplicity 1 and 2. If $\mu = (c^d, (c-2)^2)$, we can assume that d and $c - 2$ are greater than ab. It follows that ν/μ decomposes into at most 3 proper partitions and the middle part is contained in the missing (2^2) square. The statement follows since the product of an irreducible character with any irreducible character which is contained in (2^2) only contains constituents with multiplicity 1 or 2. It is either multiplicity-free or we have seen it in Lemma 2.14.

(6): We assume that $c - d$, $d - 1$, e are greater than ab. Then ν/μ decomposes into at most 4 parts, of which two are one-row partitions and one is a one-column partition. In Lemma 3.6 we will see that their product only contains constituents with multiplicity less or equal to 2. With this the statement follows in the same way as before.

(7): We assume that a and b are greater than $c(d+1)+2$. If $[\nu]$ is a constituent of $[\lambda] \boxtimes [\mu]$, we know that ν/λ decomposes into two partitions. We call them α^1 and α^2. We know that

$$\langle [\nu/\lambda], [\mu] \rangle = \langle [\alpha^1] \boxtimes [\alpha^2], [\mu] \rangle = \langle [\alpha^1], [\mu/\alpha^2] \rangle.$$

Now μ/α^2 is not a partition. If it was the product would be multiplicity-free. But by rotation symmetry and Lemma 2.16 we see that $[\mu/\alpha^2]$ only contains constituents with multiplicity 1 or 2.

(8): We assume that b is greater than 3. If $[\nu]$ is a constituent of $[\lambda] \boxtimes [\mu]$, the skew diagram ν/λ decomposes into two diagrams of partitions. We call them α^1 and α^2. If α^1 is the upper right diagram, we know that $\mu_i - \alpha^1_i \leq 3$ for all $i \in \{1,2,3\}$ and that $l(\alpha^2), w(\alpha^2) \leq 3$. So we know that

$$\langle [\nu/\lambda], [\mu] \rangle = \langle [\alpha^1] \boxtimes [\alpha^2], [\mu] \rangle = \langle [\alpha^2], [\mu/\alpha^1] \rangle.$$

All we have to do is to check all skew characters with length at most 3 and at most 3 boxes in each row for a constituent with multiplicity greater or equal to 3 which fits into (3^3). This was done with Sage.

(9): We know that $c - d$ or $d - e$ equal 1. We can assume that the other one and e are greater than ab. Let $[\nu]$ be a constituent of $[\lambda] \boxtimes [\mu]$. Then ν/μ decomposes into 3 different parts: A one-row partition, a two-row partition skewed by (1) and an arbitrary partition. Since the skew character decomposes into two two-row partitions and the product of a two-line partition with a linear partition is multiplicity-free, part (9) follows.

(10): We know that one of the exponents d or e equals 1. Starting with the case $d = 1$, we can assume that e and $c-2$ are greater than ab. The skew diagram of ν/μ decomposes into a partition and another partition skewed by $(2,1)$. Since a partition skewed by $(2,1)$ only contains constituents with multiplicity less or equal to 2, as we have seen in Lemma 2.14, the statement follows. If $e = 1$, we can assume that d and $c-2$ are greater than ab. Then our skew diagram decomposes into three or four diagrams of partitions or rotated partitions. One of the partition is $(2,1)^{\mathrm{rot}}$ if it decomposes into three parts, or two partitions are (1), in which case we need 4 parts. In the first case the proof follows from Lemma 2.14. In the later case it will follow when we prove Lemma 3.7. □

One of the partitions is a proper fat hook.

Lemma 3.5. *Let $\lambda \vdash n$ be a proper fat hook with $w(\lambda), l(\lambda) > 2$ and $\mu \vdash m$ with $\mathrm{rem}(\mu) \geq 2$ and $w(\mu), l(\mu) > 2$, not a hook. All the constituents of the product $[\lambda] \boxtimes [\mu]$ have multiplicity less or equal to 2 if and only if one of the following cases occurs:*

(1) *$\lambda = (a^b, a-1)$ and $\mu = (c^d, c-1)$ for suitable a, b, c, d;*
(2) *all 4 of the possibilities $\lambda^{(')} = (a^b, 1)$ and $\mu^{(')} = (c^d, 1)$ for suitable a, b, c, d.*

Proof: First we show that products which are not listed contain a constituent with multiplicity at least 3. We know that λ can be obtained from one of the partitions $(3,3,2)$ or $(3,2,2)^{(')}$. If μ has at least three different parts, we can therefore reduce this to either $[3,2,2]^{(')} \boxtimes [3,2,1]$ or $[3,3,2] \boxtimes [3,2,1]$. These products contain constituents with multiplicity 3. So from now on let μ be a fat hook, let us say $\mu = (a^b, c^d)$ and $\lambda = (e^f, g^h)$. If $a - c = 1$ and $e - g = 1$, this can be reduced to $[3^2, 2] \boxtimes [3, 2^2]$. So from now on we can assume that $a - c > 1$ or $e - g > 1$. Without loss of generality we can assume $a - c > 1$. If $c = g = 1$, $d \geq 2$ or $h \geq 2$ and therefore, this can be reduced to $[3^2, 1] \boxtimes [3^2, 1^2]$. If $c = 1$ and $g \neq 1$, we reduce this to $[3^2, 1] \boxtimes [3^2, 2]$ if $f \geq 2$, $[3^2, 1] \boxtimes [4, 2^2]$ if $f = 1$ and $e - g \geq 2$, and $[3, 2^2] \boxtimes [3^2, 1^2]$ if $f = 1$ and $e - g = 1$. So now we know that $a - c, c \geq 2$ and therefore, $\lambda \succeq (4,2,2)$ or $\lambda \succeq (4,4,2)$ in the remaining cases. For μ we know that μ can be obtained from one of the partitions $(3,2,2)^{(')}$ or $(3,3,2)$. We check with Sage that all six possible combinations contain a constituent with multiplicity 3. In the next step we want to prove the other direction.

(1): We can assume that a and b are greater than $c(d+1)$. If $[\nu]$ is a constituent of $[\lambda] \boxtimes [\mu]$, we know that ν/λ decomposes into at most three parts. If it is connected or decomposes into only two parts, the multiplicity of $[\nu]$ is 1. It gets more interesting if it decomposes into three parts. All three parts are partitions. We call them $\alpha^1, \alpha^2, \alpha^3$, with $\alpha^2 = (1)$. Then we know that

$$c(\nu; \lambda, \mu) = \langle [\alpha^1] \boxtimes [\alpha^2] \boxtimes [\alpha^3], [\mu] \rangle = \langle [\alpha^1] \boxtimes [\alpha^2], [\mu/\alpha^3] \rangle.$$

Since $\mu^{\mathrm{rot}} = (w(\mu)^{l(\mu)})/(1)$ and for a partition β the skew character $[\beta/(1)]$ decomposes as $[\beta/(1)] = \sum_{B \in \mathrm{Rem}(\beta)} [\beta_B]$ we obtain

$$\langle [\alpha^1] \boxtimes [\alpha^2], [\mu/\alpha^3] \rangle = \left\langle \sum_{A \in \mathrm{Add}(\alpha^1)} [(\alpha^1)^A], \sum_{B \in \mathrm{Rem}(\beta)} [\beta_B] \right\rangle,$$

where $\beta = ((w(\mu)^{l(\mu)}) \setminus \alpha^3)^{\mathrm{rot}}$. Since $|\alpha^1| + |\alpha^2| + |\alpha^3| = |\mu| = |(w(\mu)^{l(\mu)})| - 1$, we know that $|\alpha^1| + 2 = |\beta|$. With this it is easy to see that if $|\alpha^1 \cap \beta| < |\alpha^1|$, $c(\nu; \lambda, \mu) = 0$. For $|\alpha^1 \cap \beta| = |\alpha^1|$ we have two cases: If both boxes from the set $\beta \setminus \alpha^1$ are addable boxes in α^1, $c(\nu; \lambda, \mu) = 2$. This is equivalent to the fact that both boxes are removable boxes in β. It happens if β/α^1 is not connected. If only one of the boxes in the set $\beta \setminus \alpha^1$ is an addable box in α^1 and the other one is a removable box in β, this happens if β/α^1 is connected, $c(\nu; \lambda, \mu) = 1$.

(2): We do this for $\lambda' = (a^b, 1)$ and $\mu' = (c^d, 1)$. The other case can be solved with the same argument. Like before we assume that $a-1$ and b are greater than $cd+1$. If $[\nu]$ is a constituent of $[\lambda] \boxtimes [\mu]$, we know that ν/λ decomposes into at most two parts. One has the form $\alpha^1/(1)$ for a suitable partition α^1, the other one is a partition we call α^2. We know that

$$c(\nu; \lambda, \mu) = \langle [\alpha^1/(1)] \boxtimes [\alpha^2], [\mu] \rangle = \langle [\alpha^1/(1)], [\mu/\alpha^2] \rangle.$$

We know $[\alpha^1/(1)]$ decomposes as $\sum_{A \in \mathrm{Rem}(\alpha^1)} [(\alpha^1)_A]$, so we have to take a closer look at $[\mu/\alpha^2]$. By rotation symmetry we know that there is a partition β such that $[\mu/\alpha^2] = [\beta/(\beta_1 - 1)]$. If $\beta/(\beta_1 - 1)$ is not connected, it decomposes into $\widehat{\beta}$ and (1), where $\widehat{\beta}$ is β without the first row. If it is connected, we look at the possible Littlewood-Richardson tableaux. We see that $[\beta/(\beta_1 - 1)]$ decomposes as $\sum_{B \in Add(\widehat{\beta})} [\widehat{\beta}^B]$ without $[\widehat{\beta}^{(1,\beta_2+1)}]$ which is the character where we add the box in the first row of $\widehat{\beta}$. Now this part follows with the same argument as we used to prove (1). □

2. Products of three irreducible characters

Lemma 3.6. *Let λ, μ, ν be partitions all different from (1). Then the product $[\lambda] \boxtimes [\mu] \boxtimes [\nu]$ contains only constituents with multiplicity less or equal to 2 if and only if one of the partitions has only one row and another partition has only one column and the third one is a rectangle.*

Proof: First we want to show that products which are not of the form

$$[\text{one-row}] \boxtimes [\text{one-column}] \boxtimes [\text{rectangle}]$$

contain a constituent with multiplicity 3 or higher. By conjugation symmetry we can assume that at least two of the factors have length greater or equal to 2. If all three have length at least 2, $[1^2] \boxtimes [1^2] \boxtimes [1^2]$ shows that the product contains a constituent with multiplicity at least 3. So we know that one of the partitions is of the form (m) for some $m \geq 2$. If one of the partitions has two removable nodes, we know that it has to be one of the partitions of length greater or equal to 2 so this reduces to the case $[2,1] \boxtimes [2] \boxtimes [1^2]$. This contains a constituent with multiplicity greater or equal to 3. We have one one-row partition and the other two partitions can only be rectangles. But if both of them are proper rectangles or two-line rectangle, this reduces to $[2,2] \boxtimes [2,2] \boxtimes [2]$, which contains a constituent with multiplicity 3. All that is left is that one of the partitions is a rectangle and the other ones are a one-column and a one-row partition.

For the other direction look at the product $[m] \boxtimes [1^l] \boxtimes [a^b]$. We know that $[m] \boxtimes [1^l]$ decomposes as $[m, 1^l] + [m+1, 1^{l-1}]$ the sum of two character labeled by hooks. But from the classification of the multiplicity-free outer tensor products of irreducible characters of the symmetric groups we know that the product of a hook and a rectangle is multiplicity-free and therefore, our product only contains constituents with multiplicity 1 and 2. □

Lemma 3.7. *Let λ, μ be partitions. The product $[1] \boxtimes [\mu] \boxtimes [\lambda]$ only contains constituents with multiplicity less or equal to 2 if and only if*

(1) *λ and μ are both rectangles or*
(2) *$\lambda = (1)$ or $\mu = (1)$.*

Proof: One direction follows since one of the partitions has at least two parts, therefore it is $\succeq (2,1)$ and the other partition has at least two boxes so it is $\succeq (2)^{(')}$. Hence, it is sufficient to check that $[1]\boxtimes[2,1]\boxtimes[2]^{(')}$ contains $[3,2,1]$ with multiplicity 3.

Let us now prove the other direction. For a partition $\nu \vdash n$, $[1] \boxtimes [\nu]$ is the same character as $[\nu]\uparrow_{S_n}^{S_{n+1}} = \sum_{A \in Add(\nu)} [\nu^A]$. From this point of view it is obvious why the products from (2) contain only constituents with multiplicity less or equal to 2. For the first part this tells us that $[\lambda] \boxtimes [1]$ consists of two irreducible characters which are both near-rectangles. But since the product of a near-rectangle with a rectangle is multiplicity-free, we know that the product of a sum of two near-rectangles with a rectangle only contains constituents with multiplicity 1 or 2. $\square$

CHAPTER 4

Skew characters

In this chapter we want to prove that all connected skew characters $[\lambda/\mu]$ which are not listed in Theorem 2.5 or 2.6 contain a constituent with multiplicity at least 3. We start by showing that we can assume that λ has more removable boxes than μ. In the next step we show Proposition 4.1. This tells us that if the number of removable boxes of λ is greater or equal to 4 and the number of removable boxes of μ is greater or equal to 3, $[\lambda/\mu]$ contains a constituent with multiplicity greater or equal to 3. Thus, we can focus on the case that μ has one or two removable boxes.

In contrast to the multiplicity-free case we have outer tensor products of an irreducible character and a skew character which only contain constituents with multiplicity less or equal to 2. This is equivalent to a skew partition which decomposes into a proper skew partition and a partition. We deal with this in the next chapter. In this chapter we generally assume that the skew diagram λ/μ is connected.

1. Idea of the proof

We use the symmetry properties for Littlewood-Richardson coefficients from Theorem 2.9 to reduce the number of cases we have to handle. Obviously we can restrict ourselves to basic skew diagrams.

Further, if we have two basic skew diagrams λ/μ and α/β with $(\lambda/\mu)^{\text{rot}} = \alpha/\beta$, we know that $\text{rem}(\lambda)-1 = \text{rem}(\beta)$ and $\text{rem}(\mu)+1 = \text{rem}(\alpha)$. This and the rotation symmetry of skew characters tell us that it is sufficient to consider skew characters $[\lambda/\mu]$ such that $\text{rem}(\lambda) > \text{rem}(\mu)$.

As a first result we show that if $\text{rem}(\lambda) \geq 4$ and $\text{rem}(\mu) \geq 3$, the skew character contains a constituent with multiplicity greater or equal to 3. This together with the previous thought allows us to restrict ourselves to the cases $\text{rem}(\mu) = 1$ or $\text{rem}(\mu) = 2$.

We will show this by reducing the skew partition λ/μ with Lemma 2.11 to a smaller skew partition $\tilde{\lambda}/\tilde{\mu}$ for which we can calculate that $[\tilde{\lambda}/\tilde{\mu}]$ contains a constituent with multiplicity 3 or higher. Such a small skew partition $\tilde{\lambda}/\tilde{\mu}$ we will call a *seed* (for λ/μ). Later we will use slight variations of this in different contexts. But the general concept will always be the same. A seed is a small partition, skew partition, pair of partitions or character from which we can grow the bigger partition, skew partition, pair of partitions or character we are interested in. To keep the sentences a bit shorter we will sometimes talk about constituents of a seed. By this we mean the constituents of the skew character corresponding to that seed.

Proposition 4.1. *Let* $\lambda = (\lambda_1, \ldots, \lambda_l)$ *with* $\text{rem}(\lambda) \geq 4$ *and* $\mu = (\mu_1, \ldots, \mu_m)$ *with* $\text{rem}(\mu) \geq 3$ *such that* λ/μ *is a basic skew diagram. Then* $[\lambda/\mu]$ *has a constituent with multiplicity 3 or higher.*

Proof: If we have the skew partition λ/μ, we create a word consisting of $(0,0), (1,0), (0,1), (1,1)$ as follows: The word starts with $(1,1)$ and is generated according to the following rule. Let $(\lambda/\mu)(i)$ for $2 \leq i \leq l(\lambda)$ be the i-th letter of the word. We define

$$(\lambda/\mu)(i) = \begin{cases} (0,0), & \text{if } \lambda_{i-1} = \lambda_i \text{ and } \mu_{i-1} = \mu_i \\ (1,0), & \text{if } \lambda_{i-1} = \lambda_i \text{ and } \mu_{i-1} > \mu_i \\ (0,1), & \text{if } \lambda_{i-1} > \lambda_i \text{ and } \mu_{i-1} = \mu_i \\ (1,1), & \text{if } \lambda_{i-1} > \lambda_i \text{ and } \mu_{i-1} > \mu_i. \end{cases}$$

We obtain a second word by repeating this process for λ'/μ'. Not every tuple of two such words corresponds to a skew partition, but it contains enough information to regain the skew partition. This means that two different skew partitions do not have the same tuple of words.

We want to use these two words as a kind of book keeping. Lemma 2.11 tells us that if we remove columns and/or rows from λ/μ, the multiplicity of the constituents does not increase and therefore, it is sufficient to prove the proposition for some small seeds. With the words mentioned above we want to see what we need to remove from λ/μ to obtain a small seed for which we can actually calculate the constituents and their multiplicities.

First, we remove skew rows and skew columns from λ/μ in such a way that we obtain λ^1/μ^1 with $\mathrm{rem}(\lambda^1) = 4$, $\mathrm{rem}(\mu^1) = 3$ and λ^1/μ^1 basic. There are several ways to do this. One is the following: If $\mathrm{rem}(\lambda) > 4$ and $\mathrm{rem}(\mu) > 3$, we remove the last row of λ/μ and then we remove empty columns that might appear now. We repeat this until $\mathrm{rem}(\lambda) = 4$ or $\mathrm{rem}(\mu) = 3$. By rotation symmetry we can assume that $\mathrm{rem}(\mu) = 3$. For a λ/μ with $\mathrm{rem}(\lambda) > 4$ and $\mathrm{rem}(\mu) = 3$ it is obvious that we can reduce this to a skew partition λ^1/μ^1 where after removing the empty rows and columns $\mathrm{rem}(\lambda^1) = 4$ and $\mathrm{rem}(\mu^1) = 3$. Note that neither λ/μ nor λ^1/μ^1 have to be connected.

In the next step, we remove all rows λ^1_i/μ^1_i with $(\lambda^1/\mu^1)(i) = (0,0)$ to obtain λ^2/μ^2 and all columns $(\lambda^2)'_i/(\mu^2)'_i$ with $((\lambda^2)'/(\mu^2)')(i) = (0,0)$ to obtain λ^3/μ^3. What we are doing here is removing identical rows and columns to make the skew partition smaller.

If $(\lambda^3/\mu^3)(i) = (0,1)$ and $(\lambda^3/\mu^3)(i+1) = (1,0)$ or $(\lambda^3/\mu^3)(i) = (1,0)$ and $(\lambda^3/\mu^3)(i+1) = (0,1)$, we remove the i-th row and obtain λ^4/μ^4 with $(\lambda^4/\mu^4)(i) = (1,1)$ and $(\lambda^4/\mu^4)(j) = (\lambda^3/\mu^3)(j+1)$ for $j > i$. We do this until none of the two words contains the sequence $(0,1)(1,0)$ or $(1,0)(0,1)$. Then we remove empty rows and columns to make our skew diagram basic. The diagram may contain empty columns if we remove a sequence $(1,0)(0,1)$ from the row-word (and empty rows for the column-word), but removing these empty columns/rows does not reduce the number of removable nodes. We call the resulting skew partition λ^5/μ^5.

Note that $\sum_{i=1}^{l(\lambda^5)} (\lambda^5/\mu^5)(i) = (\mathrm{rem}(\lambda^5), \mathrm{rem}(\mu^5) + 1) = (4,4)$ and further, both words start with $(1,1)$ and do not contain the sequence $(0,1)(1,0)$ or $(1,0)(0,1)$, but there are still 9 possibilities for each word. Even though not every of the 81 combinations of two words corresponds to a skew partition, we do not want to check all of them so we proceed as follows to reduce the number of possibilities.

If $(\lambda^5/\mu^5)(i) = (1,0)$ and $(\lambda^5/\mu^5)(i+1) = (1,1)$ and $(\lambda^5/\mu^5)(i+2) = (0,1)$ and $\lambda^5_{i+1} > \mu^5_i$, we remove $(\lambda^5_i)/(\mu^5_{i+1})$ as rows. If $\lambda^5_{i+1} < \mu^5_i$, the skew partition would not be basic, therefore we can assume that $\lambda^5_{i+1} = \mu^5_i$. But in this case we know that $\lambda^5_i - \lambda^5_{i+1} \geq 2$ and $\mu^5_i - \mu^5_{i+1} \geq 2$. This means we can remove $(1^i)/(1^i)$, maybe multiple times, obtaining a skew partition λ^6/μ^6 with $\lambda^6_{i+1} > \mu^6_i$. Let λ^7/μ^7 be the skew partition we obtain if we eliminate all sequences of the form $(1,0)(1,1)(0,1)$ in both words.

If $(\lambda^7/\mu^7)(i) = (0,1)$ and $(\lambda^7/\mu^7)(i+1) = (1,1)$ and $(\lambda^7/\mu^7)(i+2) = (1,0)$, we remove $(\lambda^7_{i+1})/(\mu^7_i)$ (we can still assume that λ^7/μ^7 is basic). Let λ^8/μ^8 be the partition that results if we eliminate all sequences of the form $(0,1)(1,1)(1,0)$ from both words.

Now the only possible words are

$$(1,1)(1,1)(1,1)(1,1),\ (1,1)(1,0)(1,1)(1,1)(0,1) \text{ and } (1,1)(0,1)(1,1)(1,1)(0,1).$$

Further, we know if λ^8/μ^8 corresponds to the tuple (w_1, w_2), $(\lambda^8/\mu^8)'$ corresponds to the tuple (w_2, w_1). So we have the following 6 tuples to check:

(1) $w_1 = w_2 = (1,1)(1,1)(1,1)(1,1)$ corresponds to the seed $(4,3,2,1)/(3,2,1)$;
(2) $w_1 = (1,1)(1,1)(1,1)(1,1)$ and $w_2 = (1,1)(1,0)(1,1)(1,1)(0,1)$ correspond to the seed $(5,4,3,2)/(3,2,1)$;
(3) $w_1 = (1,1)(1,1)(1,1)(1,1)$ and $w_2 = (1,1)(0,1)(1,1)(1,1)(0,1)$ correspond to $(5,3,2,1)/(4,3,2)$ but this is not a basic skew diagram so this is not possible (it would be reduced to the seed $(4,3,2,1)/(3,2,1)$);
(4) $w_1 = w_2 = (1,1)(1,0)(1,1)(1,1)(0,1)$ corresponds to $(5^2,4,3,2)/(3,2,1)$, which is a seed;
(5) $w_1 = (1,1)(1,0)(1,1)(1,1)(0,1)$ and $w_2 = (1,1)(0,1)(1,1)(1,1)(0,1)$ correspond to the seed $(5,4,3,2^2)/(3^2,2,1)$;
(6) $w_1 = w_2 = (1,1)(0,1)(1,1)(1,1)(0,1)$ corresponds to $(5,3,2,1^2)/(4^2,3,2)$ which is not even a skew partition so this case is not possible. □

Since in the previous proposition we do not make the assumption that λ/μ is connected, we obtain another proof for Lemma 2.13. In addition, we obtain the following useful corollary.

Corollary 4.2. *Let λ/μ be a proper and basic skew diagram. Then $[\lambda/\mu]$ contains a constituent with multiplicity 3 or higher if one of the following holds:*

(1) *λ/μ has at least 4 connected components;*
(2) *λ/μ has 3 connected components and at least one of them is a proper skew diagram;*
(3) *λ/μ has two connected components and one of them is a proper skew diagram α/β with $\operatorname{rem}(\beta) \geq 2$;*
(4) *λ/μ has two connected components which are both proper skew diagrams.*

So now we know that for connected skew diagrams we can focus on the case that μ is a rectangle or a fat hook. We start with the first.

2. μ is a rectangle

In this section we want to prove Theorem 2.5. If $\operatorname{rem}(\lambda) = 2$, the skew character is multiplicity-free so we start with the case $\operatorname{rem}(\lambda) = 3$.

λ has three removable nodes.
We repeat the parts of Theorem 2.5 in which a skew partitions λ/μ with $\operatorname{rem}(\lambda) = 3$ and $\operatorname{rem}(\mu) = 1$ can be listed since we will refer to this quite often in the following proofs. We hope that this makes it easier for the reader to find the references. Therefore, we keep the numbering as in the original theorem, even though the condition of (3) is automatically fulfilled.

The part of Theorem 2.5 for $\operatorname{rem}(\lambda) = 3$. *Let $\lambda = (\lambda_1^{k_1}, \lambda_2^{k_2}, \lambda_3^{k_3})$ and $\mu = (a^b)$ be partitions such that $\mu \subset \lambda$ and λ/μ is a connected basic skew diagram and $[\lambda/\mu]$ is not multiplicity-free. If (up to conjugation) none of the following conditions hold, $[\lambda/\mu]$ has a constituent with multiplicity 3 or higher:*

(1) *$\mu = (2^2)$;*
(2) *$a + 2 = \lambda_1$ and $b = 2$;*
(3) *one of the following conditions holds:*
   (a) *$\mu = (3^3)$, (4^3), (5^3);*

(b) *there are* $i, j \in \{1,2,3\}$ *such that* $\lambda_i - \lambda_{i+1} = 2$ *and* $k_j = 2$*;*
(c) *there are* $i \neq j \in \{1,2,3\}$ *such that* $\lambda_i - \lambda_{i+1} = 2$*,* $\lambda_j - \lambda_{j+1} = 2$ *and one of the following conditions holds:*
  (i) $b \leq 5$*;*
  (ii) *there is an* $r \in \{1,2,3\}$ *such that* $k_r \leq 3$*.*
(d) $a + 5 \geq \lambda_1$ *and one of the following conditions holds:*
  (i) $b = 3$*;*
  (ii) $k_2 = 2$ *and* $k_3 = 2$*.*
(e) $a + 4 \geq \lambda_1$ *and* $k_1 = 2$*;*
(f) $a + 3 \geq \lambda_1$ *and one of the following conditions holds:*
  (i) $b \leq 5$*;*
  (ii) *there is an* i *such that* $k_i \leq 3$*.*
(g) $\lambda_2 = a + 1$ *and one of the following conditions holds:*
  (i) $k_1 = 2$*;*
  (ii) $b = k_1 + k_2 - 1$ *and one of the numbers* $\lambda_1 - \lambda_2$*,* λ_3*,* k_3 *equals 2.*
(h) $b = 3$ *and there is an* i *such that* $\lambda_i - \lambda_{i+1} \leq 3$*.*

If $l(\mu) < 3$, we know that $[\lambda/\mu]$ is multiplicity-free, so we assume that $l(\mu) \geq 3$. In Lemma 4.3 and 4.4 we deal with special cases before we look at the general case. We begin with $l(\mu) = 3$.

Lemma 4.3. *Let* $\lambda = (\lambda_1^{k_1}, \lambda_2^{k_2}, \lambda_3^{k_3})$ *and* $\mu = (a^3)$ *such that* λ/μ *is a basic, connected skew diagram and neither listed in Theorem 2.5 nor multiplicity-free. Then* $[\lambda/\mu]$ *contains a constituent with multiplicity at least 3.*

Proof: We know that $k_i \geq 2$ for $i \in \{1,2,3\}$. Otherwise, $[\lambda/\mu]$ would be multiplicity-free. Further, Theorem 2.5 (3)(h) tells us that $\lambda_i - \lambda_{i+1} \geq 4$ and therefore, $\lambda \succeq (12^2, 8^2, 4^2)$. By Theorem 2.5 (3)(d)(i) we know that $a + 6 \leq \lambda_1$ and from part (3)(a) of the same theorem we conclude that $a \geq 6$. Therefore, λ/μ can be obtained by successively adding skew rows and columns to $(12^2, 8^2, 4^2)/(6^3)$ if $\lambda_2 \neq a + 1$. Hence, with Lemma 2.11 we can conclude that it is sufficient to show that $[(12^2, 8^2, 4^2)/(6^3)]$ contains a constituent with multiplicity 3. If $\lambda_2 = a + 1$, we know by Theorem 2.5 (3)(g) that $k_1 > 2$ and therefore, we need to check $[(13^3, 8^2, 4^2)/(7^3)]$.

Now we explain how to obtain λ/μ from these seeds:

- If $\lambda_2 = a + 1$, we take $(13^3, 8^2, 4^2)/(7^3)$ and add
$$((\lambda_1 - 13)^3, (\lambda_2 - 8)^2, (\lambda_3 - 4)^2)/((a - 7)^3).$$
Next, we add $(\lambda_1^{k_1 - 3}, \lambda_2^{k_2 - 2}, \lambda_3^{k_3 - 2})$ as rows.
- If $\lambda_2 > a + 1$, we take $(12^2, 8^2, 4^2)/(6^3)$ and start by adding
$$((\lambda_1 - 12)^2, (\lambda_2 - 8)^2, (\lambda_3 - 4)^2)/((a - 6)^3).$$
In the second step we add $(\lambda_1^{k_1 - 2}, \lambda_2^{k_2 - 2}, \lambda_3^{k_3 - 2})$ as rows.
- If $\lambda_2 < a + 1$, we take $(12^2, 8^2, 4^2)/(6^3)$ and do it the other way around. We start by adding $(12^{k_1 - 2}, 8^{k_2 - 2}, 4^{k_3 - 2})$ as rows and in the next step we add
$$((\lambda_1 - 12)^{k_1}, (\lambda_2 - 8)^{k_2}, (\lambda_3 - 4)^{k_3})/((a - 6)^3).$$
Since $a + 6 \leq \lambda_1$ and $k_1 > 3$, this is a skew partition. □

By conjugation this is equivalent to the case $w(\mu) = 3$. From now on we assume that $w(\mu)$, $l(\mu) > 3$. In the next lemma we deal with the case that $w(\lambda) - w(\mu) = 3$.

Lemma 4.4. *Let* $\lambda = (\lambda_1^{k_1}, \lambda_2^{k_2}, \lambda_3^{k_3})$ *and* $\mu = (a^b)$ *such that* λ/μ *is a basic and connected skew diagram with* $\lambda_1 - a = 3$ *and it is neither listed in Theorem 2.5 nor multiplicity-free. Then* $[\lambda/\mu]$ *contains a constituent with multiplicity at least 3.*

Proof: Since λ/μ is not listed in Theorem 2.5 (3)(f), we know that $b \geq 6$ and k_1, k_2, $k_3 \geq 4$. Further, we know $\lambda_i - \lambda_{i+1} \geq 2$. Otherwise, the skew character would be multiplicity-free. Finally, if $b+1 = k_1 + k_2$, $[\lambda/\mu]$ is listed in Theorem 2.5 (3)(g). Because if $\lambda_1 - a = 3$ and $\lambda_1 - \lambda_2 \geq 2$, we know that $\lambda_2 \leq a+1$, so if $b \geq k_1$, we know that $\lambda_2 = a+1$ and $\lambda_1 - \lambda_2 = 2$. We consider two different cases.

The first one is $\lambda_2 \geq 5$. We have to check the skew partitions

$$(7^4, 5^4, 3^4)/(4^6),\ (7^4, 5^4, 2^4)/(4^6).$$

The second case is $\lambda_2 = 4$. We know that $k_1 \geq 7$ and we have to check the skew partition

$$(7^7, 4^4, 2^4)/(4^6).$$

How to obtain λ/μ from one of these seeds:

- If $\lambda_2 \geq 5$, we take the seed $(7^4, 5^4, 2^4)/(4^6)$ if $\lambda_3 = 2$, and $(7^4, 5^4, 3^4)/(4^6)$ otherwise. We start with the seed $(7^4, 5^4, c^4)/(4^6)$ and add

  $$(7^{k_1-4}, 5^{k_2-4}, c^{k_3-4})/(4^{b-6})$$

  as rows. Since $b+1 < k_1 + k_2$, this is a skew partition. In the next step we add

  $$((\lambda_1 - 7)^{k_1}, (\lambda_2 - 5)^{k_2}, (\lambda_3 - c)^{k_3})/((a-4)^b).$$

- If $\lambda_2 = 4$, we take the seed $(7^7, 4^4, 2^4), (4^6)$. We start with adding

  $$(7^{k_1-7}, 4^{k_2-4}, 2^{k_3-4})/(4^{b-6})$$

  as rows. In the next step we add $((\lambda_1 - 7)^{k_1})/((a-4)^b)$. □

By conjugation this is equivalent to the case $l(\lambda) - l(\mu) = 3$, so from now on we assume that $w(\mu)$, $l(\mu)$, $w(\lambda) - w(\mu)$, $l(\lambda) - l(\mu) > 3$.

Proposition 4.5. *If* $\lambda = (\lambda_1^{k_1}, \lambda_2^{k_2}, \lambda_3^{k_3})$ *and* $\mu = (a^b)$ *such that* λ/μ *is a connected, basic skew diagram and it is neither listed in Theorem 2.5 nor multiplicity-free,* $[\lambda/\mu]$ *contains a constituent with multiplicity at least 3.*

Proof: We can assume that $a, b, l(\lambda) - l(\mu), w(\lambda) - w(\mu) > 3$. If there are i, j such that $\lambda_i - \lambda_{i+1} = 2$ and $k_j = 2$, λ/μ is listed in Theorem 2.5 (3)(b). By conjugation we can assume that $\lambda_i - \lambda_{i+1} \geq 3$ for all i. If all $k_i = 2$, we know that $l(\lambda) = 6$, but since $l(\mu) > 3$, the skew character would be multiplicity-free. So we look at two different cases. First we assume that there is at most one $i \in \{1, 2, 3\}$ such that $k_i = 2$. Then we look at the case that there are $i \neq j$ such that $k_i = k_j = 2$. If for any i $k_i = 1$, the skew character is multiplicity-free.

1st case: There is at most one $i \in \{1, 2, 3\}$ such that $k_i = 2$. If $\lambda_1 \geq a+5$ and $\lambda_2 \neq a+1$, we obtain λ/μ from one of the seeds

$$(9^2, 6^3, 3^3)/(4^4),\ (9^3, 6^2, 3^3)/(4^4),\ (9^3, 6^3, 3^2)/(4^4).$$

If $\lambda_1 = a+4$ or $\lambda_2 = a+1$, then $k_1 \geq 3$. Additionally, by Theorem 2.5 (3)(e) and (g) we know that if $k_1 + k_2 = b+1$, then $k_3 > 2$. Therefore, we only have to look at the characters which are corresponding to

$$(9^3, 6^2, 3^3)/(5^4),\ (9^3, 6^3, 3^2)/(5^4).$$

How to obtain λ/μ from these seeds:

- If $\lambda_1 \geq a+5$ and $\lambda_2 \neq a+1$, we take $(9^2, 6^3, 3^3)/(4^4)$ if $k_2, k_3 \geq 3$, $(9^3, 6^2, 3^3)/(4^4)$ if $k_2 = 2$, and $(9^3, 6^3, 3^2)/(4^4)$ if $k_3 = 2$. There are two different ways we need to construct the skew partition from the seeds. If $a \geq \lambda_2$, we take the seed $(9^c, 6^d, 3^e)/(4^4)$ and add

  $$(9^{k_1-c}, 6^{k_2-d}, 3^{k_3-e})/(4^{b-4})$$

  as rows. since $a \geq \lambda_2$, we know that $k_1 > b$. Therefore, this is a skew partition. In the next step we add

  $$((\lambda_1 - 9)^{k_1}, (\lambda_2 - 6)^{k_2}, (\lambda_2 - 3)^{k_3})/((a-4)^b).$$

  If $a+1 < \lambda_2$, we take the seed $(9^c, 6^d, 3^e)/(4^4)$ and add

  $$((\lambda_1 - 9)^c, (\lambda_2 - 6)^d, (\lambda_3 - 3)^e)/((a-4)^4).$$

  In the next step we add $(\lambda_1^{k_1-c}, \lambda_2^{k_2-d}, \lambda_3^{k_3-e})/(a^{b-4})$ as rows. If $a \leq \lambda_3$, this is a skew partition since $l(\lambda)-4 \geq l(\mu)$. If $\lambda_3 < a$, we know $k_1+k_2 > b$, so if $c+d=5$, this is a skew partition. If $c+d=6$, we know that $k_3 = 2$ and therefore, that $k_1+k_2+1 > b$ (since $b+3 < k_1+k_2+k_3$). Hence this is a skew partition, too.
- If $\lambda_1 = a+4$ or $\lambda_2 = a+1$, we take the seed $(9^3, 6^2, 3^3)/(5^4)$ if $k_3 \geq 3$, and $(9^3, 6^3, 3^2)/(5^4)$ if $k_3 = 2$. We start with the seed $(9^3, 6^c, 3^d)/(5^4)$ and add

  $$(9^{k_1-3}, 6^{k_2-c}, 3^{k_3-d})/(5^{b-4})$$

  as rows. This is a skew partition since we know that $k_3 > 2$ if $k_1 + k_2 = b+1$. Then we add

  $$((\lambda_1 - 9)^{k_1}, (\lambda_2 - 6)^{k_2}, (\lambda_3 - 3)^{k_3})/((a-5)^b)$$

  to obtain λ/μ.

2nd case: There are $i \neq j \in \{1,2,3\}$ such that $k_i = k_j = 2$. By Theorem 2.5 (3)(c) we know that $a \geq 6$ and $\lambda_r - \lambda_{r+1} \geq 4$. If $k_1 = 2$, we know due to Theorem 2.5 (3)(g)(i) that $\lambda_1 - a \geq 6$ since $\lambda_1 - \lambda_2 \geq 4$. If $k_2 = k_3 = 2$, we know because of Theorem 2.5 (3)(d)(ii) that $\lambda_1 - a \geq 6$. Therefore, we obtain λ/μ from $(12^3, 8^2, 4^2)/(6^4)$ if $k_1 > 2$, $(12^2, 8^3, 4^2)/(6^4)$ if $k_2 > 2$, and $(12^2, 8^2, 4^3)/(6^3)$ if $k_3 > 2$. How to obtain λ/μ from one of these seeds:

- If $a+1 < \lambda_2$, we start with the seed $(12^c, 8^d, 4^e)/(6^f)$ and add

  $$((\lambda_1 - 12)^c, (\lambda_2 - 8)^d, (\lambda_3 - 4)^e)/((a-6)^f).$$

  In the next step we add

  $$(\lambda_1^{k_1-c}, \lambda_2^{k_2-d}, \lambda_3^{k_3-e})/(a^{b-f})$$

  as rows. This is a skew partition since $k_1 + k_2 + k_3 \geq b+4$.
- If $a+1 \geq \lambda_2$, we start again with the seed $(12^c, 8^d, 4^e)/(6^f)$ but this time we first add

  $$(12^{k_1-c}, 8^{k_2-d}, 4^{k_3-e})/(6^{b-f})$$

  as rows and in the next step we add

  $$((\lambda_1 - 12)^{k_1}, (\lambda_2 - 8)^{k_2}, (\lambda_3 - 4)^{k_3})/((a-6)^b).$$

  If $a+1 = \lambda_2$, we know by Theorem 2.5 (3)(g)(ii) that $k_2 + k_3 = 4$. Since we assume that $l(\lambda) - b \geq 4$, we obtain $b \leq k_1$. Therefore, this is a skew partition. □

After solving the case $\mathrm{rem}(\lambda) = 3$ we look at $\mathrm{rem}(\lambda) = 4$.

λ **has four removable nodes.**

We state the part of Theorem 2.5 which is relevant for this subsection since we will refer to this often. Again we preserve the numbering of the original theorem but since we do not need it we skip part (3):

The part of Theorem 2.5 for $\mathrm{rem}(\lambda) = 4$. *Let* $\lambda = (\lambda_1^{k_1}, \lambda_2^{k_2}, \lambda_3^{k_3}, \lambda_3^{k_4})$ *and* $\mu = (a^b)$ *be partitions such that* $\mu \subset \lambda$, λ/μ *is a connected basic skew diagram and* $[\lambda/\mu]$ *is not multiplicity-free. If up to conjugation none of the following conditions hold,* $[\lambda/\mu]$ *has a constituent with multiplicity 3 or higher:*

(1) $\mu = (2^2)$;
(2) $a + 2 = \lambda_1$ *and* $b = 2$;
(4) *one of the following conditions holds:*
  (a) $\mu^{(')} = (2^3)$;
  (b) $b = 2$ *and one of the following conditions holds:*
    (i) $a + 3 \geq \lambda_1$;
    (ii) *there is an* i *such that* $\lambda_i - \lambda_{i+1} = 1$.
  (c) $k_1 = 1$ *and* $\lambda_2 = a + 1$;
  (d) *there is an* i *such that* $\lambda_i - \lambda_{i+1} = 1$ *and one of the following holds:*
    (i) $\lambda_1 - \lambda_2 = 1$ *and* $k_3 = k_4 = 1$;
    (ii) $\lambda_2 - \lambda_3 = 1$ *and* $k_1 = k_4 = 1$;
    (iii) $\lambda_3 - \lambda_4 = 1$ *and* $k_1 = k_2 = 1$;
    (iv) $\lambda_4 = 1$ *and* $k_1 = k_2 = 1$ *or* $k_1 = k_3 = 1$ *or* $k_2 = k_3 = 1$.
  (e) *there are* $i \neq j$ *and* $m < n$ *such that* $\lambda_i - \lambda_{i+1}$, $\lambda_j - \lambda_{j+1} = 1$, $k_m = 1$, $k_n = 1$ *and one of the following holds:*
    (i) $\lambda_1 - 2 = \lambda_2 - 1 = \lambda_3$ *and* $(m, n) \neq (1, 3)$;
    (ii) $\lambda_1 - \lambda_2 = 1$ *and* $\lambda_3 - \lambda_4 = 1$ *and* $(m, n) \neq (1, 4), (2, 3)$;
    (iii) $\lambda_1 - 1 = \lambda_2$ *and* $\lambda_4 = 1$ *and* $(m, n) \neq (2, 4)$;
    (iv) $\lambda_2 - 2 = \lambda_3 - 1 = \lambda_4$ *and* $(m, n) \neq (2, 4)$;
    (v) $\lambda_2 - 1 = \lambda_3$ *and* $\lambda_4 = 1$ *and* $(m, n) \neq (3, 4)$;
    (vi) $\lambda_3 = 2$, $\lambda_4 = 1$.
  (f) *for at most one* $i \in \{1, 2, 3, 4\}$ *is* $\lambda_i - \lambda_{i+1} > 1$ *and one of the following conditions holds:*
    (i) $b = 3$;
    (ii) *there is a* $j \in \{1, 2, 3, 4\}$ *such that* $k_j = 1$.
  (g) $\lambda_1 = a + 2$ *and one of the following conditions holds:*
    (i) $b = 3$;
    (ii) *there is an* $i \in \{2, 3, 4\}$ *such that* $k_i = 1$.

We start by proving some special cases. If λ has four removable nodes, $[\lambda/\mu]$ is multiplicity-free if and only if μ is linear, $l(\mu) + 1 = l(\lambda)$ or $w(\mu) + 1 = w(\lambda)$. Since we assume that $[\lambda/\mu]$ is not multiplicity-free, we start with the case that the length of μ is 2.

Lemma 4.6. *Let* $\lambda = (\lambda_1^{k_1}, \lambda_2^{k_2}, \lambda_2^{k_3}, \lambda_4^{k_4})$ *and* $\mu = (a^2)$ *such that* λ/μ *is a basic and connected skew diagram and the corresponding character is neither multiplicity-free nor listed in Theorem 2.5. Then* $[\lambda/\mu]$ *contains a constituent with multiplicity at least 3.*

Proof: If $a = 3$, $[\lambda/\mu]$ only contains constituents with multiplicity less or equal to 2, see Lemma 2.15, so we can assume that $a > 3$. From Theorem 2.5 (4)(b) we know not only that $\lambda_1 \geq a + 4$, but also that $\lambda_i - \lambda_{i+1} \geq 2$ for all i. We need to check that $[(8, 6, 4, 2)/(4, 4)]$ contains a constituent with multiplicity 3.

We obtain λ/μ from the seed $(8,6,4,2)/(4,4)$ by first adding

$$(8^{k_1-1}, 6^{k_2-1}, 4^{k_3-1}, 2^{k_4-1})$$

as rows and in the next step adding

$$((\lambda_1-8)^{k_1}, (\lambda_2-6)^{k_2}, (\lambda_3-4)^{k_3}, (\lambda_4-2)^{k_4})/((a-4)^2).$$

From Theorem 2.5 (4)(b)(i) follows that $\lambda_1 \geq a+4$. Therefore, this is a skew partition if $k_1 \geq 2$. If $k_1 = 1$, we know that $\lambda_2 \geq a+2$ because of Theorem 2.5 (4)(c). Hence, this is a skew partition. □

By conjugation the case $l(\mu) = 2$ is equivalent to the case $w(\mu) = 2$. From now on we assume that $l(\mu), w(\mu) > 2$. In the next lemma we want to deal with the case $w(\mu)+2 = w(\lambda)$, since the skew character is multiplicity-free, if $w(\mu)+1 = w(\lambda)$.

Lemma 4.7. *Let $\lambda = (\lambda_1^{k_1}, \lambda_2^{k_2}, \lambda_2^{k_3}, \lambda_4^{k_4})$ and $\mu = (a^b)$ with $a, b \geq 3$ and $a+2 = \lambda_1$ such that λ/μ is a basic and connected skew diagram and the corresponding character is neither multiplicity-free nor listed in Theorem 2.5. Then $[\lambda/\mu]$ contains a constituent with multiplicity at least 3.*

Proof: We know that $k_i \geq 2$ for all $i \in \{1,2,3,4\}$ and that $b \geq 4$ ((4)(g) of Theorem 2.5). We have three different cases:

1st case: $\lambda_1 - \lambda_2 \geq 2$. We know $k_1 \geq b+1$ and therefore, we only need to check the seed $(5^5, 3^2, 2^2, 1^2)/(3^4)$.

2nd case: $\lambda_1 - \lambda_2 = 1$ and $k_1 \geq 3$. We use seed $(5^3, 4^2, 3^2, 2^2), (3^4)$ if $\lambda_4 \geq 2$, $(5^3, 4^2, 3^2, 1^2), (3^4)$ if $\lambda_4 = 1$ and $\lambda_3 \geq 3$, and $(5^3, 4^2, 2^2, 1^2), (3^4)$ if $\lambda_4 = 1$ and $\lambda_3 = 2$.

3rd case: $\lambda_1 - \lambda_2 = 1$ and $k_1 = 2$. We know that $k_1 + k_2 > b$ (since $\lambda_3 \leq a$). Hence, one of the multiplicities k_1, k_2 is greater or equal to 3 and $l(\lambda) \geq b+5$. If $k_1 = 2$, we know $k_2 \geq 3$. Therefore, we use the seed $(5^2, 4^3, 3^2, 2^2)/(3^4)$ if $\lambda_4 \geq 2$, $(5^2, 4^3, 3^2, 1^2)/(3^4)$ if $\lambda_4 = 1$ and $\lambda_3 \geq 3$, and $(5^2, 4^3, 2^2, 1^2)/(3^4)$ if $\lambda_4 = 1$ and $\lambda_3 = 2$.

In all cases we obtain λ/μ from the corresponding seed $(5^{c_1}, \nu_2^{c_2}, \nu_3^2, \nu_4^2)/(3^4)$ be by first adding

$$((\lambda_1-5)^{c_1}, (\lambda_2-\nu_2)^{c_2}, (\lambda_3-\nu_3)^2, (\lambda_4-\nu_4)^2)/((a-3)^4).$$

If $c_1 \geq 4$, this is a skew partition since $\lambda_1 - a = 2$. If $c_1 < 4$, we know that $c_1 + c_2 \geq 4$, $\lambda_2 - a = 1$ and $\nu_2 = 4$. Therefore, this is a skew partition. In the next step we add

$$(\lambda_1^{k_1-c_1}, \lambda_2^{k_2-c_2}, \lambda_3^{k_3-2}, \lambda_4^{k_4-2})/(a^{b-4})$$

as rows. If $k_1 > b$, this is a skew partition since $c_1 \leq 5$. If $k_1 \leq b$, we know that $\lambda_1 - \lambda_2 = 1$ and therefore, $\lambda_2 > a$ and $k_1 + k_2 > b$ and $c_1 + c_2 = 5$. Hence this is a skew partition. □

By conjugation this is equivalent to the case $l(\lambda) - l(\mu) = 2$ so from now on we assume that $l(\lambda) - l(\mu)$, $w(\lambda) - w(\mu)$, $l(\mu)$, $w(\mu) > 2$.

If $l(\mu) \geq 3$ and every part of λ occurs with multiplicity 1 (this means $l(\lambda) = 4$), we know that the corresponding skew character is multiplicity-free. So we can assume that one of the parts of λ has at least multiplicity 2. Moreover, we know that if at least three of the k_i's equal 1 that $\lambda_j - \lambda_{j+1} \geq 2$ for all j (Theorem 2.5 (4)(f)(ii)). In the next lemma we look at the case that exactly 3 of the k_i's equal 1. Then we look at the case exactly 2 of the k_i's equal 1 and in the last lemma we deal with the case that at most one k_i equals 1.

Lemma 4.8. *Let $\lambda = (\lambda_1^{k_1}, \lambda_2^{k_2}, \lambda_2^{k_3}, \lambda_4^{k_4})$, where $k_i = 1$ for all but one $i \in \{1,2,3,4\}$ and $\mu = (a^b)$ with $a, b \geq 3$ and $a+2 < \lambda_1$, $b+2 < l(\lambda)$ such that λ/μ is a basic, connected skew diagram. If the corresponding character is neither multiplicity-free nor listed in Theorem 2.5, $[\lambda/\mu]$ contains a constituent with multiplicity at least 3.*

Proof: From Theorem 2.5 (4)(f) we conclude that $\lambda_i - \lambda_{i+1} \geq 2$ for all $1 \leq i \leq 4$ and $a > 3$. Since we assume that $b+2 < l(\lambda)$ we know the k_i which is not 1 is greater or equal to 3. We have 6 different seeds:

(1) If $k_1 > 1$ and $a = 4$, we use the seed $(8^3, 6, 4, 2)/(4^3)$;
(2) if $k_1 > 1$ and $a > 4$, we use the seed $(8^3, 6, 4, 2)/(5^3)$;
(3) if $k_2 > 1$, we use the seed $(8, 6^3, 4, 2)/(4^3)$;
(4) if $k_3 > 1$ and $\lambda_4 = 2$, we use the seed $(9, 7, 5^3, 2)/(4^3)$;
(5) if $k_3 > 1$ and $\lambda_4 > 2$, we use the seed $(9, 7, 5^3, 3)/(4^3)$;
(6) if $k_4 > 1$, we use the seed $(11, 9, 7, 5^3)/(4^3)$.

Even though we have a lot of different seeds, obtaining λ/μ is straightforward. We start with the seed $(\nu_1^{c_1}, \nu_2^{c_2}, \nu_3^{c_3}, \nu_4^{c_4})/(d^3)$ and add

$$(\nu_1^{k_1-c_1}, \nu_2^{k_2-c_2}, \nu_3^{k_3-c_3}, \nu_4^{k_4-c_4})/(d^{b-3})$$

as rows. Since all but at most one of the exponents $k_i - c_i$ equal 0, we see that this is a skew partition. In the next step we add

$$((\lambda_1 - \nu_1)^{k_1}, (\lambda_2 - \nu_2)^{k_2}, (\lambda_3 - \nu_3)^{k_3}, (\lambda_4 - \nu_4)^{k_4})/((a-d)^b).$$

To see that this is a skew partition, we look at every case individually. In (1) $a = d$ so there is nothing to show. In (2) we know that $k_1 \geq b$ and $a+3 \leq \lambda_1$ so this is a skew partition because $l(\lambda) \geq b+3$ and $w(\lambda) \geq a+3$. In (3) we know that $b < k_1 + k_2$ and since $k_1 = 1$, we know that $a+2 \leq \lambda_2$ by Theorem 2.5 (4)(c). In (4) and (5) we know that $b < k_1 + k_2 + k_3$ and that $a < \lambda_3$. In the last case (6) we know that $a < \lambda_4$. □

By conjugation we now know that at least two of the k_i's and $\lambda_i - \lambda_{i+1}$ are greater or equal to 2.

Lemma 4.9. *Let $\lambda = (\lambda_1^{k_1}, \lambda_2^{k_2}, \lambda_2^{k_3}, \lambda_4^{k_4})$, where $k_i = 1$ for exactly two $1 \leq i \leq 4$, $\lambda_i - \lambda_{i+1} = 1$ for exactly two $1 \leq i \leq 4$ and $\mu = (a^b)$ with $a, b \geq 3$, $a+2 < \lambda_1$ and $b+2 < l(\lambda)$ such that λ/μ is a basic, connected skew diagram and the corresponding character is neither multiplicity-free nor listed in Theorem 2.5. Then $[\lambda/\mu]$ contains a constituent with multiplicity at least 3.*

Proof: We look at the different cases for which i, j the difference $\lambda_i - \lambda_{i+1} = 1$ and $\lambda_j - \lambda_{j+1} = 1$ individually.

1st case: $\lambda_1 - 2 = \lambda_2 - 1 = \lambda_3$. We know that $k_1 = k_3 = 1$, otherwise λ/μ is listed Theorem 2.5 (4)(e)(i). Therefore, we check $[(6, 5^2, 4, 2^2)/(3^3)]$. We obtain λ/μ from $(6, 5^2, 4, 2^2)/(3^3)$ by first adding

$$(\lambda_1 - 6, (\lambda_2 - 5)^2, \lambda_3 - 4, (\lambda_4 - 2)^2)/((a-3)^3).$$

This is a skew partition since $\lambda_2 + 1 > a$ because of Theorem 2.5 (4)(c). Then we add

$$(\lambda_2^{k_2-2}, \lambda_4^{k_4-2})/(a^{b-3})$$

as rows and obtain λ/μ. This is a skew partition since $k_2 + k_4 > b$ and if $\lambda_4 < a$, we know that $k_2 + 2 > b$.

2nd case: $\lambda_1 - 1 = \lambda_2$. We know that $k_1 = k_4 = 1$ or $k_2 = k_3 = 1$ are the two parts with multiplicity 1. Otherwise, λ/μ is listed in Theorem 2.5 (4)(e)(ii).

For $k_1 = k_4 = 1$ we need to check the character $[(6, 5^2, 3^2, 2)/(3^3)]$. We obtain λ/μ by first adding $(5^{k_2-2}, 3^{k_3-2})/(3^{b-3})$ as rows to $(6, 5^2, 3^2, 2)/(3^3)$ and then adding

$$(\lambda_1 - 6, (\lambda_2 - 5)^{k_2}, (\lambda_3 - 3)^{k_3}, \lambda_4 - 2)/((a-3)^b).$$

This is a skew partition since $\lambda_2 - a \geq 2$ (because $\lambda_1 - \lambda_2 = 1$ and $\lambda_1 - a \geq 3$).

For $k_2 = k_3 = 1$ we need to check the character $[(6^2, 5, 3, 2^2)/(3^3)]$. We start with adding

$$((\lambda_1 - 6)^2, \lambda_2 - 5, \lambda_3 - 3, (\lambda_4 - 2)^2)/((a-3)^b)$$

to $(6^2, 5, 3, 2^2)/(3^3)$. Again $\lambda_2 - a \geq 2$ therefore, this is a skew partition. In the next step we add $(\lambda_1^{k_1-2}, \lambda_4^{k_4-2})/(a^{b-3})$ as rows to obtain λ/μ. On the one hand we know that $k_1 + k_4 > b$ and on the other hand we know that if $a > k_4$, $b \leq k_1$. Therefore, this is a skew partition.

<u>3rd case: $\lambda_1 - 1 = \lambda_2$ and $\lambda_4 = 1$.</u> We know that $k_2 = k_4 = 1$ are the two parts with multiplicity 1. Otherwise, λ/μ is listed in Theorem 2.5 (4)(e)(iii). Therefore, we need to check the character $[(6^2, 5, 3^2, 1)/(3^3)]$. We obtain λ/μ from $(6^2, 5, 3^2, 1)/(3^3)$ by first adding

$$((\lambda_1 - 6)^2, \lambda_2 - 5, (\lambda_3 - 3)^2)/((a-3)^3).$$

This is a skew partition since $\lambda_1 - a \geq 3$ and therefore, $\lambda_2 - a \geq 2$. In the next step we add $(\lambda_1^{k_1-2}, \lambda_3^{k_3-2})/(a^{b-3})$ as rows. We know that $k_1 + k_2 > b$. Further, if $a > \lambda_3$, we know that $k_1 \geq b$, so this is a skew partition.

<u>4th case: $\lambda_2 - 2 = \lambda_3 - 1 = \lambda_4$.</u> We know that $k_2 = k_4 = 1$ are the two parts with multiplicity 1. Otherwise, λ/μ is listed in Theorem 2.5 (4)(e)(iv). By conjugation this is equivalent to the case $\lambda_1 - \lambda_2 = \lambda_3 - \lambda_4 = 1$ and $k_2 = k_3 = 1$ with which we already dealt.

<u>5th case: $\lambda_2 - 1 = \lambda_3$ and $\lambda_4 = 1$.</u> We know that $k_3 = k_4 = 1$ are the two parts with multiplicity 1. Otherwise, λ/μ is listed in Theorem 2.5 (4)(e)(v). By conjugation this is equivalent to the case $\lambda_1 - \lambda_2 = \lambda_2 - \lambda_3 = 1$ and $k_1 = k_3 = 1$ with which we already dealt.

<u>6th case: $\lambda_3 = 2$ and $\lambda_4 = 1$.</u> Here, for any $r \neq l$ with $k_r = k_l = 1$ the skew partition is listed in Theorem 2.5 (4)(e)(vi). □

Lemma 4.10. *Let $\lambda = (\lambda_1^{k_1}, \lambda_2^{k_2}, \lambda_2^{k_3}, \lambda_4^{k_4})$, where there are at most two i such that $k_i = 1$ and $\mu = (a^b)$ with $a, b \geq 3$, $a + 2 < \lambda_1$ and $b + 2 < l(\lambda)$ such that λ/μ is a basic, connected skew diagram and the corresponding character is neither multiplicity-free nor listed in Theorem 2.5. Then $[\lambda/\mu]$ contains a constituent with multiplicity at least 3.*

Proof: Because of the previous lemma we can assume that $\lambda_i - \lambda_{i+1} = 1$ for at most one i.

<u>1st case: $\lambda_i - \lambda_{i+1} > 1$ for all $i \in \{2, 3, 4\}$ and $k_3 \neq 1$ or $k_4 \neq 1$.</u> Further, we assume that $\lambda_2 - a > 1$ if $k_1 < b$. Note that if $k_3 = k_4 = \lambda_1 - \lambda_2 = 1$, the skew partition λ/μ is listed in Theorem 2.5 (4)(d)(i) and if $k_1 = 1$ and $\lambda_2 - a = 1$ it is listed in Theorem 2.5 (4)(c). Since $\lambda_i - \lambda_{i+1} > 1$ for $i \in \{2, 3, 4\}$, we know that $\lambda \succ (7, 6, 4, 2)$. We look at the different possibilities for which j the multiplicity of λ_j equals 1, i.e., $k_j = 1$.

- If $k_1 = k_2 = 1$, we obtain λ/μ by adding skew rows and columns to $(7, 6, 4^2, 2^2)/(3^3)$. We start with adding

  $$(\lambda_1 - 7, \lambda_2 - 6, (\lambda_3 - 4)^2, (\lambda_4 - 2)^2)/((a-3)^3).$$

  Since $k_1 = k_2 = 1$, we know that $\lambda_3 > a$. Therefore, this is a skew partition. In the next step we add $(\lambda_3^{k_3-2}, \lambda_4^{k_4-2})/(a^{b-3})$ as rows. Since $b < k_3 + k_4$ and if $a > \lambda_4$, $k_3 + 2 > b$, we know that this is a skew partition.

- If $k_1 = 1$ and $k_2, k_4 \geq 2$, we obtain λ/μ by adding skew rows and columns to $(7, 6^2, 4, 2^2)/(3^3)$ if $\lambda_2 - a > 2$, and $(7, 6^2, 4, 2^2)/(4^3)$ if $\lambda_2 - a = 2$. Note that $\lambda_2 - a > 1$ because of Theorem 2.5 (4)(c). We start with the seed $(7, 6^2, 4, 2^2)/(c^3)$ and add

$$(\lambda_1 - 7, (\lambda_2 - 6)^2, \lambda_3 - 4, (\lambda_4 - 2)^2)/((a - c)^3).$$

  In the next step we add $(\lambda_2^{k_2-2}, \lambda_3^{k_3-1}, \lambda_4^{k_4-2})/(a^{b-3})$ as rows. We know that $k_2 + k_3 + k_4 > b + 1$. If $a > \lambda_4$, $k_2 + k_3 + 1 > b$ and if $a > \lambda_3$, $k_2 + 1 > b$. Therefore, this is a skew partition.
- If $k_1 = k_4 = 1$, we obtain λ/μ from $(7, 6^2, 4^2, 2)/(3^3)$ if $\lambda_2 - a > 2$, and from $(7, 6^2, 4^2, 2)/(4^3)$ if $\lambda_2 - a = 2$ (again we have $\lambda_2 - a > 1$). We start with the seed $(7, 6^2, 4^2, 2)/(c^3)$ and add

$$(\lambda_1 - 7, (\lambda_2 - 6)^2, (\lambda_3 - 4)^2, \lambda_4 - 2)/((a - c)^3).$$

  In the next step we add $(\lambda_2^{k_2-2}, \lambda_3^{k_3-2})/(a^{b-3})$ as rows. We know that $k_2 + k_3 > b$ and if $a > \lambda_3$, $k_2 + 1 > b$. Therefore, this is a skew partition.
- If $k_1, k_4 \geq 2$, we obtain λ/μ from $(7^2, 6, 4, 2^2)/(c^3)$, where $c = 3$ if $a = 3$, and $c = 4$ if $a > 3$. If $\lambda_2 - a \leq 1$, we start by first adding $(7^{k_1-2}, 6^{k_2-1}, 4^{k_3-1}, 2^{k_4-2})/(4^{b-3})$ as rows. Then we add

$$((\lambda_1 - 7)^{k_1}, (\lambda_2 - 6)^{k_2}, (\lambda_3 - 4)^{k_3}, (\lambda_4 - 2)^{k_4})/((a - 4)^b).$$

  This is a skew partition since we assume that $\lambda_2 - a > 1$ if $k_1 < b$ and therefore, $k_1 \geq b$. If $\lambda_2 - a \geq 2$, we start by adding

$$((\lambda_1 - 7)^2, \lambda_2 - 6, \lambda_3 - 4, (\lambda_4 - 2)^2)/((a - c)^3).$$

  In the next step we add $(\lambda_1^{k_1-2}, \lambda_2^{k_2-1}, \lambda_3^{k_3-1}, \lambda_4^{k_4-2})/(a^{b-3})$ as rows.
- If $k_1, k_3 \geq 2$ and $k_4 = 1$, we obtain λ/μ from $(7^2, 6, 4^2, 2)/(3^3)$ if $a = 3$ and $(7^2, 6, 4^2, 2)/(4^3)$ if $a > 3$. We start with the seed $(7^2, 6, 4^2, 2)/(c^3)$ and add $(7^{k_1-2}, 6^{k_2-1}, 4^{k_3-2})/(c^{b-3})$ as rows. In the next step we add

$$((\lambda_1 - 7)^{k_1}, \lambda_2 - 6, (\lambda_3 - 4)^{k_3}, \lambda_4 - 2)/((a - c)^b).$$

As we have seen, the arguments for the different cases are very similar, so for the following cases we use a more compact notation, but we use the same two ways to obtain λ/μ from the seed we have seen now.

2nd case: $\lambda_i - \lambda_{i+1} > 1$ for all $i \in \{1, 3, 4\}$ and $k_1 \neq 1$ or $k_4 \neq 1$. Note that if $\lambda_2 - \lambda_3 = 1$ and λ/μ is not listed in Theorem 2.5, we know that $k_1 \neq 1$ or $k_4 \neq 1$. We have to check the following cases:

	Seeds
$k_1 = 1$ and $k_2 = 1$	$(7, 5, 4^2, 2^2)/(3^3)$
$k_1 = 1$ and $k_2, k_4 \geq 2$	$(7, 5^2, 4, 2^2)/(3^3)$
$k_1 \geq 2$ and $k_4 \geq 2$	$(7^2, 5, 4, 2^2)/(3^3)$ if $a = 3$, $(7^2, 5, 4, 2^2)/(4^3)$ if $a > 3$
$k_2 = 1$ and $k_4 = 1$	$(7^2, 5, 4^2, 2)/(3^3)$ if $a = 3$, $(7^2, 5, 4^2, 2)/(4^3)$ if $a > 3$
$k_1, k_2 \geq 2$ and $k_4 = 1$	$(7^2, 5^2, 4, 2)/(3^3)$ if $a = 3$, $(7^2, 5^2, 4, 2)/(4^3)$ if $a > 3$.

As before, we need two different ways.

- If $\lambda_2 > a$ we start with the corresponding seed $(7^{r_1}, 5^{r_2}, 4^{r_3}, 2^{r_4})/(c^3)$ and add

$$((\lambda_1 - 7)^{r_1}, (\lambda_2 - 5)^{r_2}, (\lambda_3 - 4)^{r_3}, (\lambda_4 - 2)^{r_4})/((a - c)^3).$$

  For the cases where $a = 3$ this is even a partition. If $a > 3$, our assumption $\lambda_2 > a$ tells us that $\lambda_2 - 5 \geq a - 4$ (in the second case even that $\lambda_2 - 5 \geq a - 3$, see Theorem 2.5 (4)(c)) and in all but the first case, where $k_1 = k_2 = 1$, $r_1 + r_2 \geq 3$. Hence, this is a skew partition. In the first case we know that $k_1 = k_2 = 1$ and therefore, that $\lambda_3 > a$, so this is a skew partition, too. In the next step we add

$$(\lambda_1^{k_1-r_1}, \lambda_2^{k_2-r_2}, \lambda_3^{k_3-r_3}, \lambda_4^{k_4-r_4})/(a^{b-3})$$

as rows. By assumption $\lambda_2 > a$. If $\lambda_3 < a$, we know that $k_1 + k_2 > b$, but since $r_1 + r_2 \leq 4$, this is a skew partition. If $\lambda_3 \geq a > \lambda_4$, we know that $k_1 + k_2 + k_3 > b$ and if $r_4 = 2$, we know that $r_1 + r_2 + r_3 = 4$. Therefore, it works if $r_4 = 2$. If $r_4 = 1$, this implies $k_4 = 1$. All parts we add are greater than a so there is no problem. If $\lambda_4 \geq a$, we know that $l(\lambda) - 3 \geq b$ and therefore, this is a skew partition.

- If $\lambda_2 \leq a$, we proceed the other way around. We start by adding

$$(7^{k_1-r_1}, 5^{k_2-r_2}, 4^{k_3-r_3}, 2^{k_4-r_4})/(4^{b-3})$$

as rows. Since $\lambda_2 \leq a$, we know that $k_1 > b$. Therefore, this is a skew partition. In the next step we add

$$((\lambda_1 - 7)^{k_1}, (\lambda_2 - 5)^{k}_{2}, (\lambda_3 - 4)^{k_3}, (\lambda_4 - 2)^{k_4})/((a - 4)^b).$$

Since $k_1 > b$ and $\lambda_1 - a \geq 3$, this is a skew partition.

Further, we notice that the previous two cases show that all λ/μ with $\lambda_i - \lambda_{i+1} > 1$ for all $i \in \{1, 2, 3, 4\}$ which are not listed in Theorem 2.5 contain a constituent with multiplicity 3 or higher. So from now on we assume that $\lambda_i - \lambda_{i+1}$ actually equals 1.

3rd case: $\lambda_3 - \lambda_4 = 1$. We know that $k_1 \neq 1$ or $k_2 \neq 1$. We have to check the following cases:

	Partitions
$k_1 = 1$ and $k_2, k_4 \geq 2$	$(7, 5^2, 3, 2^2)/(3^3)$
$k_1 = 1$ and $k_4 = 1$	$(7, 5^2, 3^2, 2)/(3^3)$
$k_2 = 1$ and $k_3 = 1$	$(7^2, 5, 3, 2^2)/(3^3)$ if $\lambda_1 - a > 3$ and $\lambda_2 - 1 > a$, $(7^3, 5, 3, 2^2)/(4^3)$ if $\lambda_1 - a = 3$ or $\lambda_2 - 1 = a$
$k_1 \geq 2$ and $k_3 \geq 2$	$(6^2, 4, 3^2, 2)/(3^3)$
$k_1 \geq 2$ and $k_2 \geq 2$	$(7^2, 5^2, 3, 2)/(3^3)$ if $a = 3$, $(7^2, 5^2, 3, 2)/(4^3)$ if $a > 3$.

Again, we need two different ways to obtain λ/μ. Let $(\nu_1^{r_1}, \nu_2^{r_2}, 3^{r_3}, 2^{r_4})/(c^3)$ be the corresponding seed.

- If $\lambda_2 > a$, we start by adding

$$((\lambda_1 - \nu_1)^{r_1}, (\lambda_2 - \nu_2)^{r_2}, (\lambda_3 - 3)^{r_3}, (\lambda_4 - 2)^{r_4})/((a - c)^3).$$

If $k_1 = 1$, we know by Theorem 2.5 (4)(c) that $\lambda_2 > a + 1$. Therefore, this is a skew partition in the first two cases. For the other cases we check directly that $\lambda_2 - \nu_2 \geq a - c$. In the next step we add

$$(\lambda_1^{k_1-r_1}, \lambda_2^{k_2-r_2}, \lambda_3^{k_3-r_3}, \lambda_4^{k_4-r_4})/(a^{b-3})$$

as rows. Again we have $\sum_{i=1}^{4} k_1 - r_1 \geq b - 3$ since $\sum_{i=1}^{4} r_1 = 6$ and $l(\lambda) - b \geq 3$. If $a > \lambda_3$, we know that $k_1 + k_2 > b$ but since $r_1 + r_2 \leq 4$, we know that $k_1 + k_2 - r_1 - r_2 \geq b - 3$. If $\lambda_3 \geq a > \lambda_4$, we know that $a = \lambda_3$ since $\lambda_3 - 1 = \lambda_4$ therefore, $k_1 + k_2 > b$. So in all cases we are adding a skew partition.

- If $\lambda_2 \leq a$, we know that $k_1 \geq b + 1$. We start by adding

$$(\nu_1^{k_1-r_1}, \nu_2^{k_2-r_2}, 3^{k_3-r_3}, 2^{k_4-r_4})/(c^{b-3})$$

as rows. In the next step we add

$$((\lambda_1 - \nu_1)^{k_1}, (\lambda_2 - \nu_2)^{k_2}, (\lambda_3 - \nu_3)^{k_3}, (\lambda_4 - \nu_4)^{k_4})/((a - c)^b).$$

Both times it is easy to see that we are, indeed, adding skew partitions.

4th case: $\lambda_4 = 1$. If $k_i > 1$ for all i or there is only one i such that $k_i = 1$ and $i \neq 1$, this is by conjugation already part of the previous cases. Therefore, we just need the following seeds:

	Seed
$k_1 = 1$ and $k_2, k_3 \geq 2$	$(7, 5^2, 3^2, 1)/(3^3)$
$k_2 = 1$ and $k_4 = 1$	$(7^2, 5, 3^2, 1)/(3^3)$ if $\lambda_1 - a > 3$ and $\lambda_2 - a > 1$, $(7^3, 5, 3^2, 1)/(4^3)$ if $\lambda_1 - a = 3$ or $\lambda_2 - a = 1$
$k_3 = 1$ and $k_4 = 1$	$(6^2, 4^2, 3, 1)/(3^3)$.

We obtain λ/μ in the same two ways as before. Let $(\nu_1^{r_1}, \nu_2^{r_2}, 3^{r_3}, 1)/(c^3)$ be the corresponding seed.

- If $a < \lambda_2$, we start by adding

  $$((\lambda_1 - \nu_1)^{r_1}, (\lambda_2 - \nu_2)^{r_2}, (\lambda_3 - 3)^{r_3}/((a - c)^3)).$$

  If $k_1 = 1$, we know that $\lambda_2 + 1 > a$ and since $\lambda_4 = 1$, we know that $k_1 + k_2 + k_3 > b + 1$ (both is part (4)(c) of Theorem 2.5), so this is a skew partition in the first case. For the other cases it is obvious. In the next step we add

  $$(\lambda_1^{k_1 - r_1}, \lambda_2^{k_2 - r_2}, \lambda_3^{k_3 - r_3}, 1^{k_4 - 1})/(a^{b-3})$$

  as rows.
- If $\lambda_2 \leq a$, we do it the other way around and start by adding

  $$(\nu_1^{k_1 - r_1}, \nu_2^{k_2 - r_2}, 3^{k_3 - r_3})/(c^{b-3})$$

  as rows and then add

  $$((\lambda_1 - \nu_1)^{k_1}, (\lambda_2 - \nu_2)^{k_2}, (\lambda_3 - \nu_3)^{k_3})/((a - c)^b)$$

  to obtain λ/μ. □

λ has more than four removable nodes.

Lemma 4.11. *If λ/μ is a connected and basic skew partition with $\mathrm{rem}(\lambda) \geq 5$ and $\mu = (a^b)$ is a rectangle such that λ/μ is not listed in Theorem 2.5 and not multiplicity-free, $[\lambda/\mu]$ contains a constituent with multiplicity 3 or higher.*

Proof: By conjugation and Theorem 2.5 (1) we can assume that $a \geq 3$ and that $b < \sum_{i=1}^{l-2} k_i$. We start with the case $\lambda_1 - a \geq 3$. We use one of the following seeds:

	Seed
$\lambda_1 - \lambda_2 \geq 2$	$(6, 4, 3, 2, 1)/(3^2)$
$\lambda_2 - \lambda_3 \geq 2$	$(6, 5, 3, 2, 1)/(3^2)$
$\lambda_3 - \lambda_4 \geq 2$	$(6, 5, 4, 2, 1)/(3^2)$
$\lambda_4 - \lambda_5 \geq 2$	$(6, 5, 4, 3, 1)/(3^2)$
$\lambda_5 \geq 2$	$(6, 5, 4, 3, 2)/(3^2)$

Let $(6, \nu_2, \ldots, \nu_5)/(3^2)$ be the seed. First we add

$$(6^{k_1 - 1}, \nu_2^{k_2 - 1}, \nu_3^{k_3 - 1}, \nu_4^{k_{l-1} - 1}, \nu_5^{k_l - 1})/(3^{c-2})$$

as rows, where $c = \min(k_1 + k_2 + k_3, b)$. Since $\nu_3 \geq 3$, this is a skew partition. In the next step we add

$$((\lambda_1 - 6)^{k_1}, (\lambda_2 - \nu_2)^{k_2 - 1}, (\lambda_3 - \nu_3)^{k_3}, (\lambda_{l-1} - \nu_4)^{k_{l-1}}, (\lambda_l - \nu_5)^{k_l})/((a - 3)^c).$$

If $\lambda_2 + 1 = a$, we know that $\lambda_1 - \lambda_2 \geq 2$ since $\lambda_1 - a \geq 3$ and therefore, that $\nu_2 = 4$. Hence, this is a skew partition. In the last step we add $(\lambda_4^{k_4}, \ldots, \lambda_{l-3}^{k_{l-3}})/(a^{b-c})$ as rows.

The case $\lambda_1 - a = 2$ is still missing but this can be solved with the seeds $(5^2, 4, 3, 2, 1)/(3^3)$ and $(5, 4^2, 3, 2, 1)/(3^3)$. In these last cases obtaining λ/μ from the seed is straightforward so we omit it here. □

This concludes the proof of Theorem 2.5.

3. μ is a fat hook

In the previous section we looked at the case $\mathrm{rem}(\mu) = 1$. Now we focus on $\mathrm{rem}(\mu) = 2$. The aim of this section is to prove Theorem 2.6. By rotation symmetry the case where $\mathrm{rem}(\lambda) = \mathrm{rem}(\mu) = 2$ is equivalent to the case where $\mathrm{rem}(\lambda) = 3$ and $\mathrm{rem}(\mu) = 1$. Therefore, we can assume that $\mathrm{rem}(\lambda) \geq 3$. We start with the case $\mathrm{rem}(\lambda) = 3$ in a first series of lemmas and deal with the case $\mathrm{rem}(\lambda) \geq 4$ in a second one. We will often refer to the different parts of Theorem 2.6. Therefore, as in the previous section we repeat the theorem so that the references can be found easier. But in contrast to the previous section we will not split Theorem 2.6.

Theorem 2.6. *Let $\lambda = (\lambda_1^{k_1}, \lambda_2^{k_2}, ..., \lambda_l^{k_l})$ with $l \geq 3$ and $\mu = (\mu_1^{r_1}, \mu_2^{r_2})$ such that $\mu \subset \lambda$ and λ/μ is a connected basic skew diagram. If (up to conjugation) none of the following conditions hold, $[\lambda/\mu]$ has a constituent with multiplicity 3 or higher:*

(1) $\mu = (2, 1)$;
(2) $\mu_1 + 1 = \lambda_1$, $l(\lambda) - 1 = l(\mu)$ *and* $r_1 = \mu_2 = 1$;
(3) $r_1 = r_2 = 1$, $\mu_1 - 1 = \mu_2$ *and* $\lambda_1 = \mu_1 + 1$;
(4) $l = 3$ *and one of the following holds:*
  (a) μ *is a hook;*
  (b) $\lambda_1 - \mu_1 = 1$ *and one of the following holds:*
    (i) $l(\lambda) - l(\mu) = 1$;
    (ii) $r_2 = 1$;
    (iii) $r_1 = 1$ *and* $\mu_1 - \mu_2 = 1$;
    (iv) $k_2 = k_3 = 1$;
    (v) *there is an* i *such that* $k_i = 1$ *and* $\lambda_i - \lambda_{i+1} = 1$ *and* $r_1 = 1$;
    (vi) *there are* i, j *such that* $k_i = 1$ *and* $\mu_j - \mu_{j+1} = 1$.
  (c) *there is an* i *such that* $\lambda_i - \lambda_{i+1} = 1$ *and* $k_{i+1} = 1$ *and one of the following holds:*
    (i) $r_2 = 1$ *and* $\mu_2 = 1$;
    (ii) $r_1 = 1$ *and* $\mu_1 - \mu_2 = 1$.
  (d) *there is an* i *such that* $\lambda_i - \lambda_{i+1} = 1$ *and* $k_i = 1$ *further* $r_2 = 1$ *and* $\mu_1 - \mu_2 = 1$;
  (e) $\lambda_1 - \lambda_2 = 1$ *and* $k_3 = 1$;
  (f) $\lambda_3 = 1$ *and* $k_1 = 1$ *or* $k_2 = 1$;
  (g) $r_1 = r_2 = 1$ *and one of the following holds:*
    (i) $\mu_1 - \mu_2 = 1$;
    (ii) *there is an* i *such that* $\lambda_i - \lambda_{i+1} = 1$.
  (h) *there are* $i \neq j$ *such that* $\lambda_i - \lambda_{i+1} = \lambda_j - \lambda_{j+1}$ *and* l *such that* $k_l = 1$ *or* $r_l = 1$.

λ has three parts.

For this whole subsection we write $\lambda = (\lambda_1^{k_1}, \lambda_2^{k_2}, \lambda_3^{k_3})$ and $\mu = (\mu_1^{r_1}, \mu_2^{r_2})$. We assume that λ/μ is a skew diagram which is connected, basic and neither is the corresponding skew character multiplicity-free nor listed in Theorem 2.6. We do not mention these assumptions again in the following lemmas.

Lemma 4.12. *If $k_i \geq 2$ for all $i \in \{1, 2, 3\}$ and $r_i \geq 2$ for all $i \in \{1, 2\}$, the skew character $[\lambda/\mu]$ contains a constituent with multiplicity greater or equal to 3.*

Proof: If $l(\lambda) \geq l(\mu)+2$, we take the seed $(3^2, 2^2, 1^2)/(2^2, 1^2)$. If $l(\lambda) = l(\mu)+1$, $\lambda_3 \geq 2$ and r_1 or r_2 is greater or equal to 3 and by Theorem 2.6 (4)(b)(i) that $\lambda_1 \geq \mu_1 + 2$. Therefore, we can use one of the seeds

$$(4^2, 3^2, 2^2)/(2^3, 1^2), \ (4^2, 3^2, 2^2)/(2^2, 1^3).$$

We show how to obtain λ/μ from the seed $(3^2, 2^2, 1^2)/(2^2, 1^2)$ if $l(\lambda) \leq l(\mu)+2$, the other ones work similarly but are a bit easier. First, we handle two special cases before we deal with the generic one.

1st case: $\mu_2 \geq \lambda_2$. We start with the seed $(3^2, 2^2, 1^2)/(2^2, 1^2)$ and add (3^2) as rows. Then we add

$$((\lambda_1 - 3)^4, (\lambda_2 - 2)^2, (\lambda_3 - 1)^2)/((\mu_1 - 2)^2, (\mu_2 - 1)^2).$$

Finally, we add

$$(\lambda_1^{k_1-4}, \lambda_2^{k_2-2}, \lambda_3^{k_3-2})/(\mu_1^{r_1-2}, \mu_2^{r_2-2})$$

as rows. Since $\mu_2 \geq \lambda_2$, we know that $r_1 + r_2 < k_1$ and this is a skew partition.

2nd case: $r_1 + 1 \geq k_1 + k_2$. We know $\mu_1 < \lambda_2$ and $\mu_2 < \lambda_3$. We start with the seed $(3^2, 2^2, 1^2)/(2^2, 1^2)$ and add (1^6). In the next step we add

$$(4^{k_1-2}, 3^{k_2-2}, 2^{k_3-2})/(2^{r_1-2}, 1^{r_2-2})$$

as rows. Since we assume that $l(\lambda) \geq l(\mu) + 2$, this is a skew partition. In the last step we add

$$((\lambda_1 - 4)^{k_1}, (\lambda_2 - 3)^{k_2}, (\lambda_3 - 2)^{k_3})/((\mu_1 - 2)^{r_1}, (\mu_2 - 1)^{r_2}).$$

Since $\lambda_2 > \mu_1$ and $\lambda_3 > \mu_2$, this is a skew partition.

3rd case: $\mu_2 < \lambda_2$ and $r_1 + 1 < k_1 + k_2$. As in the cases before we start with the seed $(3^2, 2^2, 1^2)/(2^2, 1^2)$ and add

$$((\lambda_1 - 3)^2, (\lambda_2 - 2)^2, (\lambda_3 - 1)^2)/((\mu_1 - 2)^2, (\mu_2 - 1)^2).$$

This is a skew partition since $\lambda_1 > \mu_1$ and $\lambda_2 > \mu_2$. Then we add

$$(\lambda_1^{k_1-2}, \lambda_2^{k_2-2}, \lambda_3^{k_3-2})/(\mu_1^{r_1-2}, \mu_2^{r_2-2})$$

as rows. If $\mu_1 \leq \lambda_3$, this is a skew partition because we assume that $l(\lambda) \geq l(\mu) + 2$. If $\mu_1 > \lambda_3$, we know $r_1 - 2 \leq k_1 + k_2 - 4$ because of $r_1 + 1 < k_1 + k_2$ that. Thus, this is a skew partition. □

Note that if $\lambda_i - \lambda_{i+1} = 1$ for two is it follows from Theorem 2.6 (4)(h) that all $k_i, r_i \geq 2$. This is covered in the previous lemma. Further, if $\mu_1 = 2$, Theorem 2.6 (4)(g) and (a) tells us that all $k_i \geq 2$ and both $r_i \geq 2$.

With the previous lemma and the conjugation symmetry we can assume that exactly one of the numbers $\lambda_1 - \lambda_2$, $\lambda_2 - \lambda_3$, λ_3 and/or exactly one of the numbers $\mu_1 - \mu_2$, μ_2 equals 1 and exactly one of the numbers k_1, k_2, k_3 and/or exactly one of the numbers r_1, r_2 equals 1.

Lemma 4.13. *Let $\lambda_3 = 1$. If λ/μ is not listed in Theorem 2.6, $[\lambda/\mu]$ contains a constituent with multiplicity 3 or higher.*

Proof: Since we assume that $\lambda_3 = 1$, we know that $\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, k_1, k_2 > 1$. This follows from the previous lemma and Theorem 2.6 (4)(f). We check two cases. The first one is $k_3 = 1$, where one of the r_is could equal 1 as well. The second one is that $k_3 > 1$ and either $r_1 = 1$ or $r_2 = 1$. In both cases we know since $\lambda_3 = 1$ that $l(\lambda) - l(\mu) \geq 2$.

1st case: $k_3 = 1$. If $\lambda_1 - \mu_1 = 1$, we know due to Theorem 2.6 (4)(b)(ii), (v) and (vi) that $r_1, r_2, \mu_1 - \mu_2, \mu_2 \geq 2$. Since $\mu_1 > \lambda_2$, it follows that $k_1 > r_1 \geq 2$ so this can be obtained from the seed $(5^3, 3^2, 1)/(4^2, 2^2)$.

If $\mu_2 = 1$ or $r_2 = 1$, then $r_1, \mu_1 - \mu_2 > 1$. This follows from Theorem 2.6 (4)(a), (d) and (g)(ii). In this case, λ/μ can be obtained from the seed $(5^2, 3^2, 1)/(3^2, 1)$. If $\mu_2, r_2 \geq 2$, λ/μ can be obtained from the seed $(5^2, 3^2, 1)/(3, 2^2)$.

2nd case: $k_3 \geq 2$. If $r_1 = 1$, we inspect two cases for which we know from the previous lemma and Theorem 2.6 (4)(a), (4)(g)(ii) that $r_2, \mu_2 \geq 2$. First we assume that $\lambda_1 - \mu_1 = 1$. Thus, we know that $\mu_1 - \mu_2 \geq 2$ because of Theorem 2.6 (4)(b)(iii) and we use the seed $(5^2, 3^2, 1^2)/(4, 2^2)$. If $\lambda_1 - \mu_1 > 1$, we use the seed $(5^2, 3^2, 1)/(3, 2^2)$. If $r_2 = 1$, we know from Theorem 2.6 (4)(b)(ii) and (4)(g)(ii) that $\lambda_1 - \mu_1, r_1 > 1$ so we use the seed $(5^2, 3^2, 1)/(3^2, 1)$ if $\mu_2 = 1$ and $(5^2, 3^2, 1^2)/(3^2, 2)$ if $\mu_2 \geq 2$.

In all cases we obtain λ/μ from the seed $(5^{l_1}, 3^2, 1^{l_3})/(\pi_1^{m_1}, \pi_2^{m_2})$ in the same way. We start by adding

$$(5^{k_1 - l_1}, 3^{k_2 - 2}, 1^{k_3 - l_3})/(\pi_1^{r_1 - m_1}, \pi_2^{r_2 - m_2})$$

as rows. We know that $l_1 + 2 = m_1 + m_2 + 1$ and $k_1 + k_2 > r_1 + r_2$ so if $\pi_1 = 3$, this is a skew partition. If $\pi_1 = 4$, we know that $\lambda_1 - \mu_1 = 1$ and therefore, $k_1 > r_1$. It follows that in the remaining cases this is a skew partition, too. In the next step we add

$$((\lambda_1 - 5)^{k_1}, (\lambda_2 - 3)^{k_2})/((\mu_1 - \pi_1)^{r_1}, (\mu_2 - \pi_2)^{r_2}).$$

Because of the way we chose the seeds we know that $\lambda_1 - 5 \geq \mu_1 - \pi_1$. Additionally, since $\pi_1 - \pi_2 \leq \mu_1 - \mu_2$, we know that $\lambda_1 - 5 > \mu_2 - \pi_2$. If $r_1 > k_1$, we know that $\pi_1 = 3$ and $\mu_1 < \lambda_2$. If $r_1 + r_2 > k_1$, we know that if $\pi_1 = 3$, $\lambda_2 - \mu_2 \geq 2$ so this is a skew partition. □

By conjugation the previous lemma is equivalent to the case $k_1 = 1$. So from now on we can assume that $k_1 > 1$. The following two lemmas are structured in the same way as the previous one.

Lemma 4.14. *Let $\lambda_1 - \lambda_2 = 1$. If λ/μ is not listed in Theorem 2.6, $[\lambda/\mu]$ contains a constituent with multiplicity 3 or higher.*

Proof: By Theorem 2.6 (4)(e) we know that $\lambda_2 - \lambda_3, \lambda_3, k_1, k_3 \geq 2$.

1st case: $k_2 = 1$.

If $r_1 = 1$ or $\lambda_1 - \mu_1 = 1$, we know that:

- $r_2 \geq 2$, this follows from Theorem 2.6 (4)(b)(ii) and (4)(g)(ii).
- $\mu_1 - \mu_2$, $\mu_2 \geq 2$, this follows from Theorem 2.6 (4)(a), (4)(b)(vi), (4)(c)(ii).
- $l(\lambda) - l(\mu) \geq 2$, this follows from Theorem 2.6 (4)(b)(i) and (4)(b)(vi).

Therefore, we can use the seed $(5^2, 4, 2^2)/(4, 2^2)$.

If $l(\lambda) - l(\mu) = 1$, we know from Theorem 2.6(4)(b)(i), (ii) and (vi) that $\lambda_1 - \mu_1$, $\mu_1 - \mu_2$, $r_1, r_2 \geq 2$. The same holds if $\mu_2 = 1$. Here because of Theorem 2.6 (4)(b)(vi), (g)(ii), (a) and (c)(i). In both cases we obtain λ/μ from the seed $(5^2, 4, 2^2)/(3^2, 1^2)$.

If $\lambda_1 - \mu_1, l(\lambda) - l(\mu), r_1, \mu_2 \geq 2$, we use the seed $(5^2, 4, 2^2)/(3^2, 2)$. From now on we assume that $k_2 > 1$

2nd case: $r_1 = 1$ We know from Theorem 2.6 (4)(a), (b)(vi) and (g)(ii) that μ_2, $l(\lambda) - l(\mu)$, $r_2 \geq 2$. If $\lambda_1 - \mu_1 \geq 2$, we use the seed $(5, 4^2, 2^2)/(3, 2^2)$. If $\lambda_1 - \mu_1 = 1$, we know by Theorem 2.6 (4)(b)(iii) that $\mu_1 - \mu_2 \geq 2$. Again, we use the seed $(5^2, 4, 2^2)/(4, 2^2)$.

3rd case: If $r_2 = 1$. Then r_1, $\lambda_1 - \mu_1$, $l(\lambda) - l(\mu) \geq 2$ follows from Theorem 2.6 (4)(g)(ii), (b)(ii) and (b)(vi). Therefore, we can use the seed $(5^2, 4, 2^2)/(3^2, 2)$ if $\mu_2 \geq 2$, and $(5, 4^2, 2^2)/(3^2, 1)$ if $\mu_2 = 1$.

We obtain λ/μ from the seed $(5^{l_1}, 4^{l_2}, 2^2)/(\pi_1^{m_1}, \pi_2^{m_2})$ by first adding

$$((\lambda_1 - 5)^{l_1}, (\lambda_2 - 4)^{l_2}, (\lambda_3 - 2)^2)/((\mu_1 - \pi_1)^{m_1}, (\mu_2 - \pi_2)^{m_2}).$$

The seeds are chosen such that $\lambda_1 - 5 \geq \mu_1 - \pi_1$. Further, $\lambda_2 = \lambda_1 - 1$ so if $\pi_1 - 1 = \pi_2$, we know that $\mu_2 - \pi_2 \leq \lambda_2 - 4$ and if $\pi_1 - 2 = \pi_2$, we know that $\mu_1 - 2 \geq \mu_1$. In all cases this is a skew partition. In the next step we add

$$(\lambda_1^{k_1-l_1}, \lambda_2^{k_2-l_2}, \lambda_3^{k_3-2})/(\mu_1^{r_1-m_1}, \mu_2^{r_2-m_2})$$

as rows to obtain λ/μ. □

By conjugation we can from now on assume that $k_3 \geq 2$. In the following two lemmas we obtain λ/μ in the same two ways from the seed as before, first adding the rows vs. first adding the columns. Since this is done in the same way as before and we have seen that a lot already, we just state the seeds for the following two lemma.

Lemma 4.15. *Let $\lambda_2 - \lambda_3 = 1$. If λ/μ is not listed in Theorem 2.6, $[\lambda/\mu]$ contains a constituent with multiplicity 3 or higher.*

Proof: By Lemma 4.13 and 4.14 we know that $\lambda_1 - \lambda_2, \lambda_3, k_1, k_3 \geq 2$. Additionally, we know that at least one of the numbers k_2, r_1 or r_2 equals 1.

1st case: $k_2 = 1$. If $\lambda_1 - \mu_1 = 1$, we know from Theorem 2.6 (4)(b)(i), (ii), (v) and (vi) that $l(\lambda) - l(\mu)$, r_2, r_1, $\mu_1 - \mu_2$, $\mu_2 \geq 2$. We obtain λ/μ from the seed $(5^3, 3, 2^2)/(4^2, 2^2)$. By conjugation this solves the case $l(\lambda) - l(\mu) = 1$, too, since $k_2 = 1$ and $\lambda_2 - \lambda_3 = 1$ correspond by conjugation. If $r_2 = 1$ or $\mu_2 = 1$, it follows from Theorem 2.6 (4)(a), (d) and (g)(ii) that $r_1, \mu_1 - \mu_2 \geq 2$. By conjugation we can assume that $r_2, \mu_2 \geq 2$. Therefore, we obtain λ/μ from the seed $(5^2, 3, 2^2)/(3, 2^2)$. Now we assume that $k_2 > 1$.

2nd case: $r_1 = 1$. Because of Theorem 2.6 (4)(a), (b)(vi) and (g)(ii) we know that $r_2, \mu_2, l(\lambda) - l(\mu) \geq 2$. Further, if $\lambda_1 - \mu_1 = 1$, Theorem 2.6 (4)(b)(iii) tells us that $\mu_1 - \mu_2 \geq 2$. We can use the seed $(5^2, 3^3, 2)/(4, 2^2)$ if $\lambda_1 - \mu_1 = 1$, and $(5^2, 3, 2^2)/(3, 2^2)$, otherwise.

3rd case: $r_2 = 1$. By Theorem 2.6 (4)(g)(ii), (b)(ii) and (vi) we know that $r_1, \lambda_1 - \mu_1, l(\lambda) - l(\mu) \geq 2$. Thus, we obtain λ/μ from $(5^2, 3, 2^2)/(3^2, 1)$ if $\mu_2 = 1$, and $(5^2, 3^2, 2)/(3^2, 2)$, otherwise. □

From now on we assume that $\lambda_i - \lambda_{i+1} \geq 2$ and $k_i \geq 2$.

Lemma 4.16. *Let $\mu_1 - \mu_2 = 1$. If λ/μ is not listed in Theorem 2.6, $[\lambda/\mu]$ contains a constituent with multiplicity 3 or higher.*

Proof: Because of the previous lemmas we can assume that either r_1 or r_2 equals 1.

1st case: $r_2 = 1$. By Theorem 2.6 (4)(b)(ii) we see that $\lambda_1 - \mu_1$, $l(\lambda) - l(\mu) \geq 2$. We use the seed $(5^2, 3^3, 2)/(3^2, 2)$.

2nd case: $r_1 = 1$. Again, we know that $\lambda_1 - \mu_1$, $l(\lambda) - l(\mu) \geq 2$. This time by Theorem 2.6 (4)(b)(ii) and (iii). Here we use the seed $(5^2, 3, 2^2)/(3, 2^2)$. □

By conjugation the case $\mu_1 - \mu_2 = 1$ is equivalent to the case $r_2 = 1$. What is missing is the case $\mu_2 = 1$ and $r_1 = 1$, but these are the skew characters listed in Theorem 2.6 (4)(a).

From now on we focus on the case that λ has more then three removable nodes.

λ has 4 or more parts.

Lemma 4.17. *Let $\lambda = (\lambda_1^{k_1}, \lambda_2^{k_2}, \ldots, \lambda_l^{k_l})$ with $l \geq 4$ and $\mu = (\mu_1^{r_1}, \mu_2^{r_2})$ such that λ/μ is a basic, connected skew diagram and the corresponding character it is neither multiplicity-free nor listed in Theorem 2.6. Then $[\lambda/\mu]$ contains a constituent with multiplicity at least 3.*

Proof: 1st case: $w(\lambda) = w(\mu) + 1$ and $l(\lambda) = l(\mu) + 1$. By Theorem 2.6 (2) we can, possibly after conjugation, assume that $r_1 > 1$. Since the diagram is connected, we know $k_1 > r_1$ and $\lambda_l > \mu_2$. Further, we know that $\mu_1 - \mu_2, r_2 \geq 3$ because $\text{rem}(\lambda) \geq 4$. Therefore, we can use the seed $(5^3, 4, 3, 2)/(4^2, 1^3)$.

To obtain λ/μ from this seed we start with adding

$$((\lambda_1 - 5)^3, \lambda_2 - 4, \lambda_3 - 3, \lambda_4 - 2)/((\mu_1 - 4^2, (\mu_2 - 1)^3))$$

and then we add

$$(\lambda_1^{k_1-3}, \lambda_2^{k_2-1}, \lambda_3^{k_3-1}, \lambda_4^{k_4-1}, \lambda_5^{k_5} \dots, \lambda_l^{k_l})/(\mu_1^{r_1-2}, \mu_2^{r_2-3})$$

as rows. From now on we assume by conjugation and Theorem 2.6 (1) that $w(\mu) \geq 3$ and $l(\lambda) - l(\mu) > 1$.

2nd case: $\mu_1 - \mu_2 > 1$. We use the seed $(4, 3, 2, 1)/(3, 1)$ if $r_1 < \sum_{i=1}^{l-1} k_i - 1$, and $(5, 4, 3, 2^2)/(3^2, 1)$ if $r_1 \geq \sum_{i=1}^{l-1} k_i - 1$. How to obtain λ/μ from the seed:

- If $\lambda_2 - 1 > \mu_2$ and $r_1 < \sum_{i=1}^{l-1} k_i - 1$ we start with adding
$$(\lambda_1 - 4, \lambda_2 - 3, \lambda_{l-1} - 2, \lambda_l - 1)/(\mu_1 - 3, \mu_2 - 1)$$
to the seed $(4, 3, 2, 1)/(3, 1)$. This obviously is a skew partition. In the next step we add
$$(\lambda_1^{k_1-1}, \lambda_2^{k_2-1}, \lambda_3^{k_3} \dots, \lambda_{l-2}^{k_{l-2}}, \lambda_{l-1}^{k_{l-1}-1}, \lambda_l^{k_l-1})/(\mu_1^{r_1-1}, \mu_2^{r_2-1}).$$
First we want to see why $\mu_1^{r_1-1}$ fits into the outer partition. If there is a minimal a such that $\lambda_a < \mu_1$, then $\sum_{i=1}^{a-1} k_i > r_1$. So if $a < l$, we add $\sum_{i=1}^{a-1} k_i - 2 \geq r_1 - 1$ parts that are bigger or equal to μ_1. If $a = l$, we assume that $r_1 < \sum_{i=1}^{l-1} k_i - 1$, so here we add $\sum_{i=1}^{a-1} k_i - 3 \geq r_1 - 1$ parts which are greater or equal to μ_1. If all parts of λ are greater or equal to μ_1, we know that $l(\lambda) - l(\mu) > 1$. In the next step we show why the corner of $\mu_2^{r_2-1}$ fits into the partition. If all parts of λ are greater than μ_2, it follows since $l(\lambda) - l(\mu) > 1$. If there is an a such that $\lambda_a < \mu_2$, we know that $\sum_{i=1}^{a-1} k_i > r_1 + r_2$ so we add at least $\sum_{i=1}^{a-1} k_i - 3 \geq r_1 + r_2 - 2$ parts which are greater or equal to μ_2.
- If $\lambda_2 - 1 \leq \mu_2$, we know that $r_1 < k_1$ and $k_1 + k_2 > r_1 + r_2$, otherwise, the skew diagram would not be connected. This implies $k_1 > 2$ and $l(\lambda) - l(\mu) > 2$. We start with $(4, 3, 2, 1)/(3, 1)$ and add (4) as a row. In the next step we add
$$((\lambda_1 - 4)^2, \lambda_2 - 3, \lambda_3 - 2, \lambda_4 - 1)/(\mu_1 - 3, \mu_2 - 1).$$
This is obviously a skew partition. In the next step we add
$$(\lambda_1^{k_1-2}, \lambda_2^{k_2-1}, \lambda_3^{k_3-1}, \lambda_4^{k_4-1}, \lambda_5^{k_5}, \dots)/(\mu_1^{r_1-1}, \mu_2^{r_2-1})$$
as rows. We know that $r_1 < k_1$ and $r_1 + r_2 < k_1 + k_2$, so this is a skew partition, too.
- If $r_1 \geq \sum_{i=1}^{l-1} k_i - 1$, we start with the seed $(5, 4, 3, 2^2)/(3^2, 1)$ and add
$$(\lambda_1 - 5, \lambda_2 - 4, \lambda_{l-1} - 3, (\lambda_l - 2)^2)/((\mu_1 - 3)^2, \mu_2 - 1)$$

and since $\mu_1, \mu_2 + 2 \leq \lambda_3$, this is a skew partition. In the next step we add

$$(\lambda_1^{k_1-1}, \lambda_2^{k_2-1}, \lambda_3^{k_3}, \dots, \lambda_{l-2}^{k_{l-2}}, \lambda_{l-1}^{k_{l-1}-1}, \lambda_l^{k_l-2})/(\mu_1^{r_1-2}, \mu_2^{r_2-1})$$

as rows to obtain λ/μ.

3rd case: $\mu_1 - \mu_2, \lambda_1 - \mu_1 = 1$. If $= 1$, we know by Theorem 2.6 (3) that $r_1 > 1$ or $r_2 > 1$ and $k_1 > r_1$. So we take one of the seeds $(4^2, 3, 2, 1)/(3^2, 2)$, $(4^2, 3, 2, 1)/(3, 2^2)$. We obtain λ/μ from the seed $(4^2, 3, 2, 1)/(3^a, 2^b)$ by first adding

$$(4^{k_1-2}, 3^{k_2-1}, 2^{k_3-1}, 1^{k_4-1})/(3^{r_1-a}, 2^{r_2-b})$$

as rows. Since $k_1 + k_2 > r_1 + r_2$, this is a skew partition. In the next step we add

$$((\lambda_1 - 4)^{k_1}, \dots, (\lambda_4 - 1)^{k_4}, \lambda_k^{k_5}, \dots, \lambda_l^{k_l})/((\mu_1 - 3)^{r_1}, (\mu_2 - 2)^{r_2}).$$

4th case: $\mu_1 - \mu_2 = 1$ and $\mu_1 + 1 < \lambda_1$. We know that $3 \leq \mu_1 \leq \lambda_1 - 2$ so we know that at least for one $i \in \{1, 2, 3\}$ $\lambda_i - \lambda_{i+1} > 1$ or $\lambda_4 > 1$. We have the following cases:

Condition	Seed
$\lambda_4 > 1$	$(5, 4, 3, 2)/(3, 2)$
$\lambda_3 - \lambda_4 > 1$	$(5, 4, 3, 1)/(3, 2)$
$\lambda_2 - \lambda_3 > 1$	$(5, 4, 2, 1)/(3, 2)$
$\lambda_1 - \lambda_2 > 1$	$(5, 3, 2, 1)/(3, 2)$.

How to obtain λ/μ from the corresponding seed $(5, a, b, c)/(3, 2)$:

- If $\lambda_2 > \mu_2 + 1$, we start with adding

  $$(\lambda_1 - 5, \lambda_2 - a, \lambda_{l-1} - b, \lambda_l - c)/(\mu_1 - 3, \mu_2 - 2).$$

  This is a skew partition since $\lambda_1 - \mu_1, \lambda_2 - \mu_2 \geq 2$. In the next step we add

  $$(\lambda_1^{k_1-1}, \lambda_2^{k_2-1}, \lambda_3^{k_3}, \dots, \lambda_{l-2}^{k_{l-2}}, \lambda_{l-1}^{k_{l-1}-1}, \lambda_l^{k_l-1})/(\mu_1^{r_1-1}, \mu_2^{r_2-1})$$

  as rows. If $\mu_1 \leq \lambda_l$, this is a skew partition because we know that $l(\lambda) - l(\mu) \geq 2$. If $\mu_1 > \lambda_s$ for some $s \in \{1, \dots, l\}$, we know $\hat{k}_{s-1} > r_1$ where $\hat{k}_{s-1} = \sum_{i=1}^{s-1} k_i$. If $s \leq l - 1$, we get $r_1 < \hat{k}_{s-1}$ so we know $r_1 - 1 \leq \hat{k}_{s-1} - 2$. If $s = l$, we know that $r_1 + r_2 < \hat{k}_{l-1}$ (this follows since $\mu_1 - \mu_2 = 1$ and $\mu_1 > \lambda_l$) which implies $r_1 - 1 \leq \hat{k}_{l-1} - 2$. So the $\mu_1^{r_1-1}$ fits into the outer partition. We can do the same to see that the $\mu_2^{r_2-1}$ also fits inside the outer partition.
- If $\mu_2 + 1 \geq \lambda_2$, we know that $k_1 > r_1$. In particular, $k_1 > 1$ so we start with adding (5) as a row. In the next step we add

  $$((\lambda_1 - 5)^2, \lambda_2 - a, \lambda_3 - b, \lambda_4 - c)/(\mu_1 - 3, \mu_2 - 2).$$

  Since we assume that $\lambda_1 - \mu_1 > 1$, this is a skew partition. In the last step we add

  $$(\lambda_1^{k_1-2}, \lambda_2^{k_2-1}, \lambda_3^{k_2-1}, \lambda_4^{k_4-1}, \lambda_5^{k_5}, \dots)/(\mu_1^{r_1-1}, \mu_2^{r_2-1})$$

  as rows to obtain λ/μ. Since $k_1 > r_1$ and $r_1 + r_2 < k_1 + k_2$ this is a skew partition, too. □

This concludes the proof of Theorem 2.6. So fare we mostly looked at connected skew diagrams. In the next section we look at the outer tensor product of a skew character and a irreducible character. This is equivalent to a skew character where the corresponding diagram decomposes into the diagram of a partition and a skew diagram.

4. Product of a skew character with an irreducible character

In this section we investigate products of a skew character $[\lambda/\mu]$ and an irreducible character $[\nu]$. For these, we prove Theorem 2.7. As in the previous sections we recall the theorem that we want to prove:

Theorem 2.7 *Let ν be a partition and λ/μ be a basic and connected skew partition. If (up to rotation of λ/μ and/or conjugation of λ/μ and ν) they are not from the following list, the outer tensor product $[\lambda/\mu] \boxtimes [\nu]$ contains a constituent with multiplicity 3 or higher:*

(1) *If μ is a rectangle, $\lambda = (\lambda_1^{k_1}, \lambda_2^{k_2})$ and one of the following holds:*
  (a) $\nu = (1)$;
  (b) $\lambda_1 - \lambda_2 = 1$ *or* $\lambda_2 = 1$ *and ν has one row;*
  (c) $w(\mu) = \lambda_1 - 1$ *and ν has one row.*
(2) *if ν is a rectangle and one of the following holds:*
  (a) $\lambda^{(')} = (\lambda_1, \lambda_1 - 1)$;
  (b) *λ is a two-line partition and* $\mu = (1)$;
  (c) $\lambda^{(')} = (\lambda_1^{k_1}, \lambda_1 - 1)$ *and* $\mu = (1)$;
  (d) $\lambda = (\lambda_1^{k_1}, 1)$ *and* $\mu = (\lambda_1 - 1)$ *or both conjugated.*
(3) *if μ and ν are both one-line partitions and one of the following holds:*
  (a) $\lambda_1 - 1 = \mu_1$ *and* $\nu = (1)$;
  (b) $\lambda = (\lambda_1^{k_1}, \lambda_2^{k_2})$ *and one of the following holds:*
    (i) $l(\mu) = 1$ *and* $w(\nu) = 1$;
    (ii) $\lambda_1 - \lambda_2 = 1$ *or* $\lambda_2 = 1$ *and* $l(\mu) = l(\nu) = 1$;
    (iii) $l(\lambda) - l(\mu) = 1$ *and* $w(\nu) = 1$.

Like before, we assume that $\mathrm{rem}(\lambda) > \mathrm{rem}(\mu)$. This follows from the rotation symmetry. In this chapter we call a small product $[\tilde{\lambda}/\tilde{\mu}] \boxtimes [\tilde{\nu}]$, which contains a constituent with multiplicity 3 or higher, a seed for $[\lambda/\mu] \boxtimes [\nu]$ if we can obtain λ/μ from $\tilde{\lambda}/\tilde{\mu}$ and $\tilde{\nu}$ from ν by Lemma 2.11. We will just show how to obtain λ/μ from $\tilde{\lambda}/\tilde{\mu}$ since for ν and $\tilde{\nu}$ it will be obvious. In Corollary 4.2 we have seen that we can restrict ourselves to the case that μ is a rectangle. The only part of Theorem 2.7 where $\mathrm{rem}(\lambda) \geq 3$ is (3)(a). This leads to the following lemma for $\mathrm{rem}(\lambda) \geq 3$.

Lemma 4.18. *Let λ/μ be a basic and connected skew partition with* $\mathrm{rem}(\lambda) \geq 3$, $\mathrm{rem}(\mu) = 1$. *Let ν be a partition. The product $[\lambda/\mu] \boxtimes [\nu]$ contains a constituent with multiplicity 3 or higher if one of the following holds:*

(1) $l(\lambda) - l(\mu) \neq 1$ *and* $w(\lambda) - w(\mu) \neq 1$;
(2) *μ is not linear;*
(3) $\nu \neq (1)$.

Proof: As before, let $\lambda = (\lambda_1^{k_1}, \ldots, \lambda_l^{k_l})$. We split the proof in three parts.

<u>1st case: $l(\lambda) - l(\mu)$, $w(\lambda) - w(\mu) > 1$.</u> Here we use the seed $[(3, 2, 1)/(1)] \boxtimes [1]$. We obtain λ/μ from $(3, 2, 1)/(1)$ by first adding

$$(\lambda_1 - 3, \lambda_{l-1} - 2, \lambda_l - 1)/(\mu_1 - 1).$$

In the next step we add

$$(\lambda_1^{k_1 - 1}, \lambda_2^{k_2}, \ldots, \lambda_{l-2}^{k_{l-2}}, \lambda_{l-1}^{k_{l-1} - 1}, \lambda_l^{k_l - 1})/\hat{\mu}$$

as rows, where $\hat{\mu} = (\mu_2, \mu_3, \ldots)$. This is a skew partition since λ/μ is connected and basic.

<u>2nd case: $w(\lambda) - w(\mu) = 1$ and $l(\mu) \geq 2$.</u> We know that $w(\mu) \geq 2$ and $l(\mu) < k_1$. Therefore, we can use the seed $[(3^3, 2, 1)/(2^2)] \boxtimes [1]$. We first add

$$((\lambda_1 - 3)^3, \lambda_2 - 2, \lambda_3 - 1)/((\mu_1 - 2)^2)$$

and then add

$$(\lambda_1^{k_1-3}, \lambda_2^{k_2-1}, \lambda_3^{k_3-1}, \lambda_4^{k_4}, \ldots, \lambda_l^{k_l})/(\mu_1^{l(\mu)-2})$$

as rows.

3rd case: $w(\lambda) - w(\mu) = 1$ and $l(\mu) = 1$. We know that $\nu \neq (1)$, therefore, we can use one of the seeds $[(3^2, 2, 1)/(2)] \boxtimes [1^2]^{(')}$. We obtain λ/μ from $((3^2, 2, 1)/(2))$ by adding $((\lambda_1 - 3)^2, \lambda_2 - 2, \lambda_3 - 1)/(\mu_1 - 2)$ in the first step and then adding $(\lambda_1^{k_1-2}, \lambda_2^{k_2-1}, \lambda_3^{k_3-1}, \lambda_4^{k_4}, \ldots, \lambda_l^{k_l})$ as rows. □

We will henceforth restrict ourselves to skew partitions where λ has two removable nodes. From now on we assume that $\lambda = (\lambda_1^{k_1}, \lambda_2^{k_2})$. We will use this notation in the following proofs.

Lemma 4.19. *Let λ/μ be a proper and connected skew diagram and ν be a partition with at least two removable nodes. Then $[\lambda/\mu] \boxtimes [\nu]$ has a constituent with multiplicity greater or equal to 3.*

Proof: For every connected and proper skew diagram where λ has two removable nodes, there is either $k_1 \geq 2$ or $\lambda_2 \geq 2$. We assume that μ is a rectangle, as a consequence we know that $l(\lambda) - l(\mu)$ or $w(\lambda) - w(\mu)$ is greater than 1. By conjugation we can assume that $\lambda_2 \geq 2$ and $\lambda_1 - \mu_1 \geq 2$. Therefore, we can use the seed $[(3, 2)/(1)] \boxtimes [2, 1]$. We obtain λ/μ from $(3, 2)/(1)$ by first adding $(\lambda_1 - 3, \lambda_2 - 2)/(\mu_1 - 1)$ and then adding $(\lambda_1^{k_1-1}, \lambda_2^{k_2-1})/(\mu_1^{l(\mu)-1})$ as rows. □

Now we know that μ and ν have to be rectangles but we can refine this even a little bit more. In the next lemma we show that at least one of the partitions μ, ν has to be linear. After that we just look at the three remaining cases.

Lemma 4.20. *Let λ/μ be a basic and connected skew partition with $(2^2) \subseteq \mu$ and $(2^2) \subseteq \nu$ be a partition. Then $[\lambda/\mu] \boxtimes [\nu]$ contains a constituent with multiplicity 3 or higher.*

Proof: Again, we can assume $\lambda = (\lambda_1^{k_1}, \lambda_2^{k_2})$ and μ, ν are rectangles. By conjugation we can assume that $l(\lambda) - l(\mu) > 1$ and $k_1 \geq 3$. If $\lambda_2 \geq 2$, we use the seed $[(3^3, 2)/(2^2)] \boxtimes [2^2]$. If $\lambda_2 = 1$, we use the seed $[(3^3, 1)/(2^2)] \boxtimes [2^2]$. We obtain λ/μ from $(3^3, a)/(2^2)$ by first adding $((\lambda_1 - 3)^3, \lambda_2 - a)/((\mu_1 - 2)^2)$ and then adding $(\lambda_1^{k_1-3}, \lambda_2^{k_2-1})/(\mu_1^{l(\mu)-2})$ as rows. □

ν is a rectangle.

Lemma 4.21. *Let $(2^2) \subseteq \nu$ be a partition, λ/μ a skew partition and $[\lambda/\mu] \boxtimes [\nu]$ is not listed in Theorem 2.7. Then the product contains a constituent with multiplicity 3 or higher.*

Proof: By the previous lemma we can assume that μ is a one-line partition. By conjugation symmetry we can even assume that it is a one-row partition. Then we know that $w(\lambda) > 2$, otherwise, the product would be listed in Theorem 2.7 (2)(b). We divide the proof into different cases. First, we present the seeds. Then we show that these seeds, indeed, cover all cases.

(1) Let $l(\lambda) = 2$. Since we assume $[\lambda/\mu] \boxtimes [\nu]$ is not listed in Theorem 2.7 (2)(a), (b), we know that $\mu_1 \geq 2$ and $\lambda_1 - \lambda_2 \geq 2$. Further, we assume that λ/μ is connected so we know that $\lambda_2 > \mu_1$. This implies $\lambda_1 - \mu_1 \geq 2$. Therefore, we can add $(\lambda_1 - 5, \lambda_2 - 3)/(\mu_1 - 2)$ to $(5, 3)/(2)$ to obtain λ/μ. Hence, $[(5, 3)/(2)] \boxtimes [2^2]$ is a seed for $[\lambda/\mu] \boxtimes [\nu]$.

(2) If k_1, k_2, $\lambda_1 - \lambda_2$, $\mu_1 \geq 2$, we use the seed $[(3^2, 1^2)/(2)] \boxtimes [2^2]$. Note that since $l(\mu) = 1$, we know that $l(\lambda) - l(\mu) \geq 3$. We obtain λ/μ by adding $((\lambda_1 - 3)^2, (\lambda_2 - 1)^2)/(\mu_1 - 2)$ and then adding $(\lambda_1^{k_1-2}, \lambda_2^{k_2-2})$ as rows.

(3) If k_1, λ_2, $\mu_1 \geq 2$, we use the seed $[(3^2,2)/(2)] \boxtimes [2^2]$. We obtain λ/μ from $(3^2,2)/(2)$ by first adding $(3^{k_1-2}, 2^{k_2-1})$ as rows and then adding $((\lambda_1-3)^{k_1}, (\lambda_2-2)^{k_2})/(\mu_1-2)$.

(4) If k_1, $\lambda_1-\mu_1$, $\lambda_1-\lambda_2 \geq 2$, we use the seed $[(3^2,1)/(1)] \boxtimes [2^2]$. We obtain λ/μ from $(3^2,1)/(1)$ by first adding $((\lambda_1-3)^2, \lambda_2-1)/(\mu_1-1)$ and then adding $(\lambda_1^{k_1-2}, \lambda_2^{k_2-1})$ as rows.

(5) If k_2, λ_2, $\lambda_1-\mu_1 \geq 2$, we use the seed $[(3,2^2)/(1)] \boxtimes [2^2]$. We obtain λ/μ from $(3,2^2)/(1)$ by adding $(\lambda_1-3, (\lambda_2-2)^2)/(\mu_1-1)$ and then adding $(\lambda_1^{k_1-1}, \lambda_2^{k_2-2})$ as rows.

We now have a lot of different seeds for the different cases but it is not obvious why these cover all cases, so that is what remains to be shown. If $l(\lambda)=2$, we refer to (1). From now on we assume that $l(\lambda)$, $w(\lambda) \geq 3$. If k_1, λ_2, $\mu_1 \geq 2$, we use (3). If $k_1=1$, we know that $k_2 \geq 2$ since we assume that $l(\lambda) \geq 3$. Moreover λ_2, $\lambda_1-\mu_1 \geq 2$, otherwise, the skew partition would not be connected. Therefore, this is covered by (5).

If $\lambda_2=1$ and $\lambda_1-\mu_1=1$, we know that $k_1 \geq 2$ otherwise, the skew diagram would not be connected. Theorem 2.7 (2)(d) gives us $k_2 \geq 2$. Because we assume $w(\lambda) \geq 3$ we have $\lambda_1-\lambda_2$, $\mu_1 \geq 2$. All together this shows that we can use (2). If $\lambda_2=1$ and $\lambda_1-\mu_1 \geq 2$, we still know that $k_1 \geq 2$ otherwise, the skew diagram would not be connected. Moreover, $\lambda_1-\lambda_2 \geq 2$ since $w(\lambda) \geq 3$. Thus we are in the setting of (4).

If $\mu_1=1$, $k_2 \geq 2$ and $\lambda_2 \geq 2$, we know that $\lambda_1-\mu_1 \geq 2$. In this case we use (5). If $\mu_1=1$ and $k_2=1$ or $\lambda_2=1$, we know that $\lambda_1-\lambda_2$, k_1, $\lambda_1-\mu_1 \geq 2$. This follows because we assume $l(\lambda)$, $w(\lambda) \geq 3$, λ/μ is connected and from Theorem 2.7 (2)(c). Consequently, we can use (4). □

μ is a rectangle.

Lemma 4.22. *Let λ/μ be a connected and proper skew partition such that $(2^2) \subseteq \mu$ and ν be a partition such that $[\lambda/\mu] \boxtimes [\nu]$ is not listed in Theorem 2.7. Then the product contains a constituent with multiplicity greater or equal to 3.*

Proof: By conjugation we can assume that $\lambda_1-\mu_1 \leq l(\lambda)-l(\mu)$. We know that ν has either one row or one column. We start with the latter. By Theorem 2.7 (1)(a), (b) we know that k_1, k_2, $l(\nu) \geq 2$. Additionally, from the assumption $\lambda_1-\mu_1 \leq l(\lambda)-l(\mu)$ follows that $l(\lambda)-l(\mu) \geq 2$. If $\lambda_2 \geq 2$, we use the seed $[(3^2,2^2)/(2^2)] \boxtimes [1^2]$. We obtain λ/μ from $(3^2,2^2)/(2^2)$ by first adding $((\lambda_1-3)^2, (\lambda_2-2)^2)/((\mu_1-2)^2)$ and then adding $(\lambda_1^{k_1-2}, \lambda_2^{k_2-2})/(\mu_1^{l(\mu)-2})$ as rows.

If $\lambda_2=1$, we use the seed $[(3^3,1^2)/(2^2)] \boxtimes [1^2]$. To obtain λ/μ from $(3^3,1^2)/(2^2)$, we by first adding $((\lambda_1-3)^3)/((\mu_1-2)^2)$ and then adding $(\lambda_1^{k_1-3}, 1^{k_2-2})/(\mu_1^{l(\mu)-2})$ as rows.

From now on we assume that ν is a one-row partition. By Theorem 2.7 (1)(a)-(c) we know that $\lambda_1-\lambda_2$, λ_2, $w(\nu)$, $\lambda_1-\mu_1 \geq 2$. The assumption $\lambda_1-\mu_1 \leq l(\lambda)-l(\mu)$ implies $l(\lambda)-l(\mu) \geq 2$. If $k_1 \geq 2$, we use the seed $[(4^2,2)/(2^2)] \boxtimes [2]$. We obtain λ/μ from $(4^2,2)/(2^2)$ by adding $((\lambda_1-4)^2, \lambda_2-2)/((\mu_1-2)^2)$ and then adding $(\lambda_1^{k_1-2}, \lambda_2^{k_2-1})/(\mu_1^{l(\mu)-2})$ as rows. If $k_1=1$, we use the seed $[(5,3^3)/(2^2)] \boxtimes [2]$. We obtain λ/μ from $(5,3^3)/(2^2)$ by adding $(\lambda_1-5, (\lambda_2-3)^3)/((\mu_1-2)^2)$ and then adding $(\lambda_2^{k_2-3})/(\mu_1^{l(\mu)-2})$ as rows. □

μ and ν are one-line partitions.

In the following we want to deal with the case that μ and ν are one-line partitions. Because of Lemma 4.18 we can assume that $\lambda = (\lambda_1^{k_1}, \lambda_2^{k_2})$. If one of the partitions

μ, ν has one row and the other one has one column, the product is listed in Theorem 2.7 (3)(b)(i). By conjugation we can focus on the case where both have one row.

Lemma 4.23. *Let* $\lambda = (\lambda_1^{k_1}, \lambda_2^{k_2})$ *and* μ, ν *are one-row partitions such that* $\mu_1 < \lambda_1$ *and* $[\lambda/\mu] \boxtimes [\nu]$ *is not listed in Theorem 2.7. Then* $[\lambda/\mu] \boxtimes [\nu]$ *contains a constituent with multiplicity greater or equal to 3.*

Proof: By Theorem 2.7 (3)(b)(ii), (iii) we assume that $\lambda_1 - \lambda_2,\ \lambda_2,\ \lambda_1 - \mu_1 \geq 2$. We use the seed $[(4,2)/(2)] \boxtimes [2]$. We add $(\lambda_1 - 2, \lambda_2 - 2)/(2)$ to $(4,2)/(2)$ and then add $(\lambda_1^{k_1-1}, \lambda_2^{k_2-1})$ as rows to obtain λ/μ. □

Part 2

Kronecker products of characters of the symmetric groups

CHAPTER 5

Preliminaries: Kronecker coefficients and adding partitions

In the first part of the thesis we investigated outer tensor products of characters of the symmetric groups. Henceforth, whenever we mention the product of characters, we always mean the Kronecker product of these characters. To prove the classification of the multiplicity-free Kronecker products of characters of the alternating groups in the third part we need a classification of the Kronecker products of the symmetric groups which just contain constituents with multiplicity 1 and 2. This is given by the following theorem:

Theorem 5.1. *Let $\lambda, \mu \vdash n$ be partitions of n. The product $[\lambda][\mu]$ only contains constituents with multiplicity 1 and 2 if and only if (up to conjugating one or both partitions and exchanging λ and μ) one of the following holds:*

(1) *The product is multiplicity-free (classified in [BB17]);*

(2) *λ is a hook and one of the following holds:*
   - (a) *Both partitions are hooks;*
   - (b) *μ is one of the following partitions*
$$\left(\frac{n}{2},\frac{n}{2}\right),\left(\frac{n+1}{2},\frac{n-1}{2}\right),(n-2,2)\,;$$
   - (c) *$l(\lambda) \leq 3$ or $\lambda_1 \leq 3$ and μ is a two-line partition;*
   - (d) *$l(\lambda) \leq 4$ or $\lambda_1 \leq 4$ and μ is a rectangle;*
   - (e) *$\lambda = (n-1,1)$ and $\mathrm{rem}(\mu) \leq 3$;*
   - (f) *$n = ab-1$, $\lambda = (n-2,1^2)$ and $\mu = (a^{b-1}, b-1)$;*
   - (g) *$\lambda = (5,1^4)$ and $\mu = (3^3)$.*

(3) *λ is a two-row partition and one of the following holds:*
   - (a) *$\mu = (n-2,1^2)$;*
   - (b) *$\lambda = (n-2,2)$ and μ is a two-line partition or a hook;*
   - (c) *$n = 2k+1$, $\lambda = (n-2,2)$ and $\mu = (k^2,1)$;*
   - (d) *$n = ab-1$, $\lambda = (n-2,2)$ and $\mu^{(')} = (a^{b-1}, a-1)$;*
   - (e) *$\lambda = (n-3,3)$ and μ is a rectangle;*
   - (f) *$n = 2k+1$, $\lambda = (k+1,k)$ and μ is a hook or $\mu = (\mu_1,\mu_2)$ with $\mu_2 \leq 3$ or $\mu_2 = k, k-1$;*
   - (g) *$n = 2k$, $\lambda = (k,k)$ and μ is $\mu = (n-3,2,1)$ or $\mu = (\mu_1,\mu_2)$ with $\mu_2 \leq 7$ or $\mu_1 - \mu_2 \leq 6$;*
   - (h) *the exceptional cases where one is a two-row partition for $n \leq 18$:*
     - (i) *$\lambda = \mu = (5,3)$;*
     - (ii) *$\lambda = \mu = (6,4)$;*
     - (iii) *$\lambda = (4,3)$ and $\mu = (3,2^2)$;*
     - (iv) *$\lambda = (4^2)$ and $\mu = (3,2^2,1)$, $(3^2,1^2)$, $(3^2,2)$;*
     - (v) *$\lambda = (5,3)$ and $\mu = (3^2,2)$;*
     - (vi) *$\lambda = (5,4)$ and $\mu = (3,2^3)$, $(4^2,1)$;*
     - (vii) *$\lambda = (5^2)$ and $\mu = (4,3^2)$, $(4^2,2)$;*
     - (viii) *$\lambda = (8,4)$, $(7,5)$, $(6,6)$ and $\mu = (4^3)$;*

(ix) $\lambda = (6,3)$, $(5,4)$ *and* $\mu = (3^3)$*;*
(x) $\lambda = (8,7)$ *and* $\mu = (5^3)$*;*
(xi) $\lambda = (9,9)$ *and* $\mu = (6^3)$*.*

(4) *The exceptional cases where no partition is a two-line partition or a hook are:* $n = 9$*,* $\lambda = \mu = (3^3)$ *or* $\lambda = (3^3)$ *and* $\mu = (4^2, 1)$*.*

But this theorem on its own is not enough. To deduce the results for the alternating groups we need to put particular emphasis on the symmetric partitions which we do with the following theorem:

Theorem 5.2. *Let* $\lambda, \mu \vdash n$ *be partitions with* $g(\lambda, \mu) > 2$*.*

(1) *Let* $\lambda = \lambda'$*. There is a* $\nu \neq \lambda$ *such that* $g(\lambda, \mu, \nu) > 2$ *if and only if* λ *and* μ *are not from the following list:*
  (a) $\mu(') = (n-1, 1)$*;*
  (b) $n = 6$*,* $\lambda = (3,2,1)$ *and* $\mu(') = (4,2)$ *or* $\mu(') = (4,1^2)$*;*
  (c) $n = 8$*,* $\lambda = (4,2,1^2)$ *and* $\mu(') = (4^2)$*.*

(2) *If* $\lambda = \lambda'$ *and* $\mu = \mu'$*, there is a* $\nu \neq \nu' \vdash n$ *such that* $g(\lambda, \mu, \nu) > 2$*.*

To prove the previous two theorems we need skew characters. If a skew partition decomposes into several connected components, their order is not important for the skew character. For partitions α and β we use the notation $\alpha * \beta$ for all skew partitions which decompose into two parts and these parts are α and β. The investigation of Kronecker products of skew characters leads to the following results:

Theorem 5.3. *The only product* $\chi\psi$ *of two proper skew characters which only contains constituents with multiplicity 1 and 2 is:*

$$\chi = \psi = [(2,1)/(1)] = [2] + [1^2].$$

Theorem 5.4. *Let* λ/μ *be a basic, proper skew partition of* n *and* $\nu \vdash n$*. The product* $[\lambda/\mu][\nu]$ *only contains constituents with multiplicity 1 and 2 if one of the following holds (up to conjugating the partition and/or the skew partition and/or rotating the skew partition):*

(1) ν *is linear and* λ/μ *only contains constituents with multiplicity 1 and 2;*
(2) $\nu = (n-1, 1)$ *and* λ/μ *is from the following list:*
  (a) $\lambda = (\lambda_1, \lambda_2)^{(')}$ *is a two-line partition and* $\mu = (1)$ *or* $\lambda_1 - \lambda_2 = 1$*;*
  (b) $\lambda = (\lambda_1^{k_1}, \lambda_1 - 1)$ *and* $\mu = (1)$*;*
  (c) $\lambda = (\lambda_1^{k_1}, 1)$ *and* $\mu = (\lambda_1 - 1)$*;*
  (d) $\lambda/\mu = (n-l) * (l)$*;*
  (e) $n = ab + 1$ *and* $\lambda/\mu = (a^b) * (1)$*.*
(3) ν *is a fat hook and* $\lambda/\mu = (n-1) * (1)$*;*
(4) ν *is a rectangle and* λ/μ *is from the following list:*
  (a) λ/μ *equals* $(n-2) * (2)$ *or* $(n-2) * (1^2)$*;*
  (b) $\lambda/\mu = (n-1, 2)/(1)$*;*
  (c) $\lambda/\mu = (n-2, n-2, 1)/(n-3)$*;*
  (d) *the exceptional pairs* $\nu = (3^3)$ *and* λ/μ *equals* $(7,3)/(1)$ *or* $(6,4)/(1)$*.*
(5) $n = 2k$*,* $\nu = (k,k)$ *and* λ/μ *is from the following list:*
  (a) $\lambda/\mu = (\lambda_1, \lambda_2)/(1)$ *with* $\lambda_1 - \lambda_2 \leq 3$ *or* $\lambda_2 \leq 3$*;*
  (b) $\lambda/\mu = (n-2, n-2, 1)/(n-3)$*;*
  (c) $\lambda/\mu = (n-l) * (1^l)$*;*
  (d) *one of the exceptional cases where* λ/μ *is one of the following:*

$$(k+2,k)/(2) \text{ for } k \leq 5,\ (k^2,1)/(1) \text{ for } k \leq 4,\ (2,1)*(1),\ (3)*(3).$$

(6) *the exceptional case for* $n = 5$ *where* $\lambda/\mu = (2^2) * (1)$ *and* $\nu = (3,2)$*.*

1. Results on Kronecker coefficients

In this section we collect some known results on Kronecker coefficients which will be useful later. The following theorem tells us what the maximal length and width of constituents of the product $[\lambda][\mu]$ are.

Theorem 5.5. *[Dvi93, Theorem 1.6],[CM93, 2.1(d)] Let $\lambda, \mu \vdash n$, then*

$$|\lambda \cap \mu| = \max\{w(\nu) \mid \nu \vdash n \text{ with } g(\lambda, \mu, \nu) > 0\} \text{ and}$$
$$|\lambda \cap \mu'| = \max\{l(\nu) \mid \nu \vdash n \text{ with } g(\lambda, \mu, \nu) > 0\}.$$

The next theorem shows that if λ, μ are not linear partitions, the constituents of maximal length and width are different ones.

Theorem 5.6. *[BK99, 3.3] Let $\lambda, \mu, \nu \vdash n$, where neither λ nor μ is a linear partition. If $g(\lambda, \mu, \nu) > 0$,*

$$w(\nu) + l(\nu) < |\lambda \cap \mu| + |\lambda \cap \mu'|.$$

Corollary 5.7. *Let λ and μ be non-linear partitions of n. Further, let $[\alpha]$ resp. $[\beta]$ be a constituent of maximal length resp. width in the product $[\lambda][\mu]$. If $|\lambda \cap \mu| \geq |\lambda \cap \mu'|$, β is not symmetric and if $|\lambda \cap \mu| \leq |\lambda \cap \mu'|$, α is not symmetric. In particular, one of the partitions α and β is always non-symmetric.*

Proof: Let $[\beta]$ be a constituent of maximal width. We can assume that $|\lambda \cap \mu| \geq |\lambda \cap \mu'|$. We show that β is not symmetric. The second statement follows by conjugation. Due to Theorem 5.5 we know $w(\beta) = |\lambda \cap \mu|$. If β was symmetric, this would imply $w(\beta) = l(\beta)$. So $w(\beta) + l(\beta) = 2|\lambda \cap \mu| \geq |\lambda \cap \mu| + |\lambda \cap \mu'|$. But this is a contradiction to Theorem 5.6, which concludes the proof. □

There are two different techniques we use to estimate the value of a Kronecker coefficient.

Manivel's semigroup property for Kronecker coefficients.
One of our main tactics when investigating Kronecker coefficients is to reduce a product to a smaller one with Manivel's semigroup property [Man11].

Theorem 5.8. *Let $\alpha, \beta, \gamma \vdash m$ and $\lambda, \mu, \nu \vdash n$ with $g(\alpha, \beta, \gamma)$, $g(\lambda, \mu, \nu) > 0$, then*

$$g(\lambda + \alpha, \mu + \beta, \nu + \gamma) \geq \max\{g(\alpha, \beta, \gamma), g(\lambda, \mu, \nu)\}.$$

In particular,

$$g(\lambda + \alpha, \mu + \beta) \geq \max\{g(\lambda, \mu), g(\alpha, \beta)\}.$$

Dvir's recursion.
The second main tool will be Dvir's recursion. To state this recursive formula for the Kronecker coefficients we need the following notation.

Notation 5.9. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l) \vdash n$. We write $\widehat{\lambda} = (\lambda_2, \lambda_3, \dots, \lambda_l)$ for the partition which is obtained by removing the first row of λ.

Additionally, we define

$$Y(\lambda) = \{\eta = (\eta_1, \dots) \vdash n \mid \eta_i \geq \lambda_{i+1} \geq \eta_{i+1} \text{ for all } i \geq 1\}.$$

We use this to state the following theorem:

Theorem 5.10. *[Dvi93, Theorems 2.3 and 2.4],[CM93, 2.1(d)]. Let λ, μ and $\nu = (\nu_1, \nu_2, \dots)$ be partitions of n.*

Then

$$g(\lambda, \mu, \nu) = \sum_{\substack{\alpha \vdash \nu_1 \\ \alpha \subseteq \lambda \cap \mu}} \langle [\lambda/\alpha][\mu/\alpha], [\hat{\nu}] \rangle - \sum_{\substack{\eta \in Y(\nu) \\ \eta \neq \nu \\ \eta_1 \leq |\lambda \cap \mu|}} g(\lambda, \mu, \eta) .$$

In particular, if $\nu_1 = |\lambda \cap \mu|$,

$$g(\lambda, \mu, \nu) = \langle [\lambda/\beta][\mu/\beta], [\hat{\nu}] \rangle,$$

where $\beta = \mu \cap \lambda$.

This last part allows us to use induction to prove Theorem 5.1. It implies that $[\lambda][\mu]$ contains a constituent with multiplicity 3 or higher if $[\lambda/(\lambda \cap \mu)][\mu/(\lambda \cap \mu)]$ does.

Formulas for hooks.
For some cases explicit formulas for Kronecker coefficients are known. The following ones are helpful in case we deal with two hooks or a hook and a two-row partition. We use the following notation: If E is an equality or inequality

$$\chi_{(E)} = \begin{cases} 1, & \text{if } E \text{ is true;} \\ 0, & \text{if } E \text{ is false.} \end{cases}$$

The following theorem contained a small mistake which we corrected:

Theorem 5.11. *[Rem89, Theorem 2.1] Let* $g_\lambda := g((n-k, 1^k), (n-i, 1^i), \lambda)$, *where* $k \geq i$ *and* $n - 1 \geq k + i$ *and* λ *is a partition of* n. *Then*

(1) $g_\lambda = 0$ *if* λ *is not a hook or a double-hook;*
(2) $g_\lambda = \chi_{(k-i \leq r \leq k+i)}$ *if* λ *is the hook* $(n-r, 1^r)$;
(3) *if* $\lambda = (p, q, 2^{d_2}, 1^{d_1})$ *is a double-hook, where* $d_1, d_2 \geq 0$ *and* $2 \leq p \leq q$, *we let* $u = \max(q, n-k-i)$, $v_0 = \min(p, n-k, d_2-1)$, $v_1 = \min(p, n-k-d_2)$, $w = 2n - k - i - d_1 - d_2$, *and* $x = \left\lfloor \frac{w}{2} \right\rfloor$. *Then*

$$g_\lambda = \begin{cases} 0, & \text{if } q + d_2 > n - k; \\ \chi_{(u \leq x-1 \leq v_0)} + \chi_{(u \leq x \leq v_1)}, & \text{if } q + d_2 \leq n - k \text{ \& } w \text{ is even;} \\ \chi_{(u \leq x \leq v_0)} + \chi_{(u \leq x \leq v_1)}, & \text{if } q + d_2 \leq n - k \text{ \& } w \text{ is odd.} \end{cases}$$

Theorem 5.12. *[Rem92, Theorem 2.2.] Let* $g_\lambda := g([n-k, k], [n-i, 1^i], \lambda)$, *where* $n \geq 4, \lfloor \frac{n}{2} \rfloor \geq k \geq 2$ *and* $n - 2 \geq i \geq 1$. *Then*

(1) $g_\lambda = 0$ *if* λ *is not a hook or a double-hook;*
(2) *if* $\lambda = (n-j, 1^j)$ *is a hook, then:*
  (a) $g_\lambda = 0$ *if* $j \notin \{i-1, i, i+1\}$;
  (b) $g_\lambda = \chi_{(k \leq \min(n-i, i+1))}$ *if* $\lambda = (n-i-1, 1^{i+1})$;
  (c) $g_\lambda = \chi_{(k \leq \min(n-i, i+1))} + \chi_{(k \leq \min(n-i, i))}$ *if* $\lambda = (n-i, 1^i)$;
  (d) $g_\lambda = \chi_{(k \leq \min(n-i, i))}$ *if* $\lambda = (n-i+1, 1^{i-1})$.
(3) *if* $\lambda = (n_4, n_3, 2^{d_2}, 1^{d_1}) \vdash n$ *is a double-hook, where* $2 \leq n_3 \leq n_4$, *then:*
  (a) $g_\lambda = 0$, *if* $n - i \notin \{d_1 + 2d_1, d_1 + 2d_1 + 1, d_1 + 2d_1 + 2, d_1 + 2d_1 + 3\}$;
  (b) $g_\lambda = \chi_{(n_3 \leq j - d_2 \leq \min(n_4, n_3+d_1))}$, *if* $i = d_1 + 2d_2$;
  (c) *if* $i = d_1 + 2d_2 + 1$, *then*

$$\begin{aligned} g_\lambda =& \chi_{(n_3+1 \leq k-d_2 \leq \min(n_3+d_1, n_4+1))} \\ &+ \chi_{(n_3 \leq k-d_2 \leq \min(n_3+d_1+1, n_4))} \\ &\cdot [1 - \chi_{(k-d_2=n_4=n_3+d_1)}] \\ &+ \chi_{(n_3-1 \leq k-d_2 \leq \min(n_3+d_1-1, n_4-1))}; \end{aligned}$$

  (d) *if* $i = d_1 + 2d_2 + 2$, *then*

$$\begin{aligned} g_\lambda =& \chi_{(n_3+1 \leq k-d_2 \leq \min(n_3+d_1+1, n_4+1))} \\ &+ \chi_{(n_3 \leq k-d_2 \leq \min(n_3+d_1-1, n_4))} \\ &+ \chi_{(n_3-1 \leq k-d_2 \leq \min(n_3+d_1, n_4-1))}; \end{aligned}$$

  (e) $g_\lambda = \chi_{(n_3 \leq k-d_2 \leq \min(n_4, n_3+d_1,))}$ *if* $i = d_1 + 2d_2 + 3$.

For both cases there are formulas from Rosas in [Ros01]. They work for our purpose, too. That we are working with Remmel's formulas is just a fashion choice.

The product with $[n-1,1]$ has a very easy and explicit description which we will need from time to time.

Lemma 5.13. *[BK99, Lemma 4.1] Let $n \geq 3$, and let μ be a partition of n. Then*

$$[\mu][n-1,1] = \left(\sum_{A \in \mathrm{Rem}(\mu)} \sum_{B \in \mathrm{Add}(\mu_A)} [(\mu_A)^B] \right) - [\mu].$$

Constituents in Kronecker squares.
For Kronecker squares for some constituents explicit multiplicities are known. We collect them in the following proposition. The results are from Saxel [Sax87], Zisser [Zis92], and Vallejo [Val97, Val14] in this form it appeared in [BB17, Proposition 4.2.]

Proposition 5.14. *Let $\lambda \vdash n$, $\lambda \neq (n), (1^n)$. Let $h_k = \#\{k\text{-hooks in } \lambda\}$ for $k = 1, 2, 3$ and $h_{21} = \#\{\text{non-linear 3-hooks } H \text{ in } \lambda\}$. Then*

$$\begin{aligned} [\lambda]^2 &= [n] + a_1[n-1,1] + a_2[n-2,2] + b_2[n-2,1^2] + a_3[n-3,3] \\ &\quad + b_3[n-3,1^3] + c_3[n-3,2,1] + \textit{constituents of greater depth} \end{aligned}$$

with $a_1 = h_1 - 1,\ b_2 = (h_1-1)^2,\ a_2 = h_2 + h_1(h_1-2),$ *for* $n \geq 4$,
$a_3 = h_1(h_1-1)(h_1-3) + h_2(2h_1-3) + h_3,$ *for* $n \geq 6$,

$b_3 = h_1(h_1-1)(h_1-3) + (h_1-1)(h_2+1) + h_{21},$ *for* $n \geq 4$,

$c_3 = 2h_1(h_1-1)(h_1-3) + h_2(3h_1-4) + h_1 + h_{21},$ *for* $n \geq 5$.

In particular, for $n \geq 4$ we always have $a_2 > 0$.

Formulas for partitions with small depth.
For products which involve a partition λ of small depth, i.e., $\widehat{\lambda}$ with 2 or 3 boxes, [Val97, Theorem 6.3.] gives us a very powerful tool to compute the decomposition of a product $[\lambda][\mu]$. We will be needing it several times, that is why we state it here despite its length.

For a partition λ and a disjoint union of skew diagrams π, we denote by $r_\lambda(\pi)$ the number of ways we can remove π from λ such that we obtain a partition. For example $r_\lambda(\square) = \mathrm{rem}(\lambda)$, $r_{(4^2,3,2,1^2)}\left(\begin{smallmatrix}\square\\\square\end{smallmatrix}\right) = 2$, $r_{(4^2,3,2,1^2)}(\square\square) = 0$, and $r_{(4^2,3,2,1^2)}\left(\begin{smallmatrix}\square\\\square\end{smallmatrix} \sqcup \square\right) = 6$ (where we use $\sqcup$ for the disjoint union of the skew diagrams, in contrast to $\cup$ which we use for adding as rows).

Theorem 5.15. *[Val97, Theorem 6.3] Let λ, μ be partitions of n.*

(1)

$$g(\lambda, \lambda, (n-2,2)) = r_\lambda(\square\square) + r_\lambda\left(\begin{smallmatrix}\square\\\square\end{smallmatrix}\right) + r_\lambda(\square)\left[r_\lambda(\square) - 2\right].$$

(2)

$$g(\lambda, \mu, (n-2,2)) = \begin{cases} \left(\sum\limits_{\substack{\rho \vdash n-2 \\ \rho \subset \lambda \cap \mu}} c_{(2)}(\lambda, \mu, \rho) \right) - 1, & \textit{if } |\lambda \cap \mu| = n-1; \\ c_{(2)}(\lambda, \mu, \lambda \cap \mu), & \textit{if } |\lambda \cap \mu| = n-2; \\ 0, & \textit{if } |\lambda \cap \mu| < n-2; \end{cases}$$

where $c_{(2)}(\lambda, \mu, \rho)$ is given by the value of $(\lambda/\rho, \mu/\rho)$ in the following table:

	$\square\sqcup\square$	$\square\square$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$
$\square\sqcup\square$	*2*	*1*	*1*
$\square\square$	*1*	*1*	*0*
$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	*1*	*0*	*1.*

(3)

$$g(\lambda,\mu,(n-2,1^2)) = \begin{cases} (r_\lambda(\square)-1)^2, & \text{if } \lambda=\mu; \\ \left(\displaystyle\sum_{\substack{\rho\vdash n-2\\ \rho\subset\lambda\cap\mu}} d_{(1^2)}(\lambda,\mu,\rho)\right)-1, & \text{if } |\lambda\cap\mu| = n-1; \\ d_{(1^2)}(\lambda,\mu,\lambda\cap\mu), & \text{if } |\lambda\cap\mu| = n-2; \\ 0, & \text{if } |\lambda\cap\mu| < n-2; \end{cases}$$

where $d_{(1^2)}(\lambda,\mu,\rho)$ is given by the value of $(\lambda/\rho,\mu/\rho)$ in the following table:

	$\square\sqcup\square$	$\square\square$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$
$\square\sqcup\square$	*2*	*1*	*1*
$\square\square$	*1*	*0*	*1*
$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	*1*	*1*	*0.*

(4)

$$\begin{aligned} g(\lambda,\lambda,(n-3,3)) =& r_\lambda(\square\square\square) + r_\lambda\left(\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}\right) + r_\lambda\left(\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}\right) + r_\lambda\left(\begin{smallmatrix}&\square\\\square&\square\end{smallmatrix}\right) \\ &+ [2r_\lambda(\square)-3]\left[r_\lambda(\square\square) + r_\lambda\left(\begin{smallmatrix}\square\\\square\end{smallmatrix}\right)\right] \\ &+ r_\lambda(\square)\,[r_\lambda(\square)-1][r_\lambda(\square)-3]. \end{aligned}$$

(5)

$$g(\lambda,\mu,(n-3,3)) = \begin{cases} \displaystyle\sum_{\substack{\rho\vdash n-3\\ \rho\subset\lambda\cap\mu}} c_{(3)}(\lambda,\mu,\rho) - \sum_{\substack{\rho\vdash n-2\\ \rho\subset\lambda\cap\mu}} c_{(2)}(\lambda,\mu,\rho), & \text{if } |\lambda\cap\mu| = n-1; \\ \displaystyle\sum_{\substack{\rho\vdash n-3\\ \rho\subset\lambda\cap\mu}} c_{(3)}(\lambda,\mu,\rho) - c_{(2)}(\lambda,\mu,\lambda\cap\mu), & \text{if } |\lambda\cap\mu| = n-2; \\ c_{(3)}(\lambda,\mu,\lambda\cap\mu), & \text{if } |\lambda\cap\mu| = n-3; \\ 0, & \text{if } |\lambda\cap\mu| < n-3; \end{cases}$$

where $c_{(3)}(\lambda,\mu,\rho)$ is given by the value of $(\lambda/\rho,\mu/\rho)$ in the following table:

	$\square\sqcup\square\sqcup\square$	$\square\sqcup\square\square$	$\square\sqcup\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$\square\square\square$	$\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}$	$\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}^{(\mathrm{rot})}$
$\square\sqcup\square\sqcup\square$	*6*	*3*	*3*	*1*	*1*	*2*
$\square\sqcup\square\square$	*3*	*2*	*1*	*1*	*0*	*1*
$\square\sqcup\begin{smallmatrix}\square\\\square\end{smallmatrix}$	*3*	*1*	*2*	*0*	*1*	*1*
$\square\square\square$	*1*	*1*	*0*	*1*	*0*	*0*
$\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}$	*1*	*0*	*1*	*0*	*1*	*0*
$\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}^{(\mathrm{rot})}$	*2*	*1*	*1*	*0*	*0*	*1.*

(6)

$$\begin{aligned} g(\lambda,\lambda,(n-3,2,1)) =& r_\lambda\left(\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}\right) + r_\lambda\left(\begin{smallmatrix}&\square\\\square&\square\end{smallmatrix}\right) + [3r_\lambda(\square)-4]\left[r_\lambda(\square\square) + r_\lambda\left(\begin{smallmatrix}\square\\\square\end{smallmatrix}\right)\right] \\ &+ 2r_\lambda(\square)\,[r_\lambda(\square)-1][r_\lambda(\square)-3] + r_\lambda(\square). \end{aligned}$$

(7)

$$g(\lambda,\mu,(n-3,2,1)) = \begin{cases} \sum\limits_{\substack{\rho\vdash n-3\\ \rho\subset\lambda\cap\mu}} d_{(2,1)}(\lambda,\mu,\rho) \\ \qquad\qquad - \sum\limits_{\substack{\rho\vdash n-2\\ \rho\subset\lambda\cap\mu}} c_{(1^2)}(\lambda,\mu,\rho)+1, & \textit{if } |\lambda\cap\mu| = n-1; \\ \sum\limits_{\substack{\rho\vdash n-3\\ \rho\subset\lambda\cap\mu}} d_{(2,1)}(\lambda,\mu,\rho) - c_{(1^2)}(\lambda,\mu,\lambda\cap\mu), & \textit{if } |\lambda\cap\mu| = n-2; \\ d_{(2,1)}(\lambda,\mu,\lambda\cap\mu), & \textit{if } |\lambda\cap\mu| = n-3; \\ 0, & \textit{if } |\lambda\cap\mu| < n-3; \end{cases}$$

where $d_{(2,1)}(\lambda,\mu,\rho)$ is given by the value of $(\lambda/\rho,\mu/\rho)$ in the following table:

	$\square\sqcup\square\sqcup\square$	$\square\sqcup\square\!\square^{(')}$	$\square\!\square\!\square^{(')}$	$\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}^{(\mathrm{rot})}$
$\square\sqcup\square\sqcup\square$	*12*	*6*	*2*	*4*
$\square\sqcup\square\!\square^{(')}$	*6*	*3*	*1*	*2*
$\square\!\square\!\square^{(')}$	*2*	*1*	*0*	*1*
$\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}^{(\mathrm{rot})}$	*4*	*2*	*1*	*1*

and $c_{(1^2)}(\lambda,\mu,\rho)$ is given by the value of $(\lambda/\rho,\mu/\rho)$ in the following table:

	$\square\sqcup\square$	$\square\!\square$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$
$\square\sqcup\square$	*4*	*2*	*2*
$\square\!\square$	*2*	*1*	*1*
$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	*2*	*1*	*1.*

(8)

$$\begin{aligned} g(\lambda,\lambda,(n-3,1^3)) =& r_\lambda\left(\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}\right) + r_\lambda\left(\begin{smallmatrix}&\square\\\square&\square\end{smallmatrix}\right) + [r_\lambda(\square)-1]\left[r_\lambda(\square\!\square) + r_\lambda\left(\begin{smallmatrix}\square\\\square\end{smallmatrix}\right)+1\right] \\ &+ r_\lambda(\square)\,[r_\lambda(\square)-1][r_\lambda(\square)-3]. \end{aligned}$$

(9)

$$g(\lambda,\mu,(n-3,1^3)) = \begin{cases} \sum\limits_{\substack{\rho\vdash n-3\\ \rho\subset\lambda\cap\mu}} d_{(1^3)}(\lambda,\mu,\rho) \\ \qquad\qquad - \sum\limits_{\substack{\rho\vdash n-2\\ \rho\subset\lambda\cap\mu}} d_{(1^2)}(\lambda,\mu,\rho)+1, & \textit{if } |\lambda\cap\mu| = n-1; \\ \sum\limits_{\substack{\rho\vdash n-3\\ \rho\subset\lambda\cap\mu}} d_{(1^3)}(\lambda,\mu,\rho) - d_{(1^2)}(\lambda,\mu,\lambda\cap\mu), & \textit{if } |\lambda\cap\mu| = n-2; \\ d_{(1^3)}(\lambda,\mu,\lambda\cap\mu), & \textit{if } |\lambda\cap\mu| = n-3; \\ 0, & \textit{if } |\lambda\cap\mu| < n-3; \end{cases}$$

where $d_{(1^3)}(\lambda,\mu,\rho)$ is given by the value of $(\lambda/\rho,\mu/\rho)$ in the following table:

	$\square\sqcup\square\sqcup\square$	$\square\sqcup\square\!\square$	$\square\sqcup\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$\square\!\square\!\square$	$\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}$	$\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}^{(\mathrm{rot})}$
$\square\sqcup\square\sqcup\square$	*6*	*3*	*3*	*1*	*1*	*2*
$\square\sqcup\square\!\square$	*3*	*1*	*2*	*0*	*1*	*1*
$\square\sqcup\begin{smallmatrix}\square\\\square\end{smallmatrix}$	*3*	*2*	*1*	*1*	*0*	*1*
$\square\!\square\!\square$	*1*	*0*	*1*	*0*	*1*	*0*
$\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}$	*1*	*1*	*0*	*1*	*0*	*0*
$\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}^{(\mathrm{rot})}$	*2*	*1*	*1*	*0*	*0*	*1.*

This theorem covers the cases of Proposition 5.14, too. We stated both since we need more than Proposition 5.14, but Proposition 5.14 is a lot more compact and therefore easier to work with in some of the cases.

2. Adding Partitions

We often use the semigroup property of the Kronecker coefficients to conclude that a Kronecker product has a constituent of multiplicity greater or equal to 3. The following results help us to see that a constituent we obtain in this way is not symmetric.

Lemma 5.16. *Let λ and μ be partitions of n such that*

$$w(\lambda) - l(\lambda) \neq w(\mu) - l(\mu)$$

and $\nu \vdash m \in \mathbb{N}$ with $l(\nu) \leq l(\lambda)$ and $l(\nu) \leq l(\mu)$. Then

$$w(\lambda + \nu) - l(\lambda + \nu) \neq w(\mu + \nu) - l(\mu + \nu)$$

In particular, one of the partitions $\lambda + \nu$ and $\mu + \nu$ is non-symmetric.

Proof: For a symmetric partition α we know that $w(\alpha) - l(\alpha) = 0$ but our assumptions on ν imply that $l(\lambda+\nu) = l(\lambda)$, $w(\lambda+\nu) = w(\lambda)+w(\nu)$ and the same for μ. If $\lambda + \nu$ is symmetric this implies

$$0 = l(\lambda + \nu) - w(\lambda + \nu) = l(\lambda) - w(\lambda) - w(\nu)$$

but since $l(\lambda) - w(\lambda) \neq l(\mu) - w(\mu)$, we conclude that $l(\mu + \nu) - w(\mu + \nu) \neq 0$ and therefore, $\mu + \nu$ is not symmetric. □

Lemma 5.17. *Let $\alpha, \beta \vdash n$ and $\lambda \vdash m$, where neither α nor β is a linear partition. Then there are partitions $\mu, \nu \vdash n$ such that $g(\alpha, \beta, \mu) > 0$, $g(\alpha, \beta, \nu) > 0$ and one of the partitions $\lambda + \mu$, $\lambda + \nu$ is not symmetric.*

Proof: From Theorem 5.5 we know that there are partitions $\mu, \nu \vdash n$ such that the corresponding characters are constituents of $[\alpha][\beta]$ with maximal length resp. width. Theorem 5.6 tells us that the length of ν is strictly smaller than the length of μ and the other way around for the width.

We prove that at least one of the partitions $\lambda + \mu$ and $\lambda + \nu$ is not symmetric. We know that

$$\begin{aligned} w(\lambda + \mu) =& w(\lambda) + w(\mu), \;\; l(\lambda + \mu) = \max(l(\lambda), l(\mu)), \\ w(\lambda + \nu) =& w(\lambda) + w(\nu), \;\; l(\lambda + \nu) = \max(l(\lambda), l(\nu)), \end{aligned}$$

and $l(\mu) > l(\nu)$ (because μ is of maximal length),$w(\nu) > w(\mu)$ (because ν is of maximal width). We distinguish between 3 cases:

1st case: $l(\lambda) \geq l(\mu) > l(\nu)$. We know

$$l(\lambda + \mu) = l(\lambda + \nu) = l(\lambda)$$

but

$$w(\lambda + \nu) = w(\lambda) + w(\nu) > w(\lambda) + w(\mu) = w(\lambda + \mu).$$

2nd case: $l(\mu) \geq l(\lambda) \geq l(\nu)$. If we assume that both partitions $\lambda + \mu$ and $\lambda + \nu$ are symmetric, we see that

$$\begin{aligned} l(\lambda + \mu) = l(\mu) = w(\lambda) + w(\mu) \\ l(\lambda + \nu) = l(\lambda) = w(\lambda) + w(\nu) \end{aligned}$$

but this implies that $l(\lambda) > l(\mu)$ which is a contradiction since $w(\nu) > w(\mu)$.

3rd case: $l(\mu) > l(\nu) \geq l(\lambda)$. We know $l(\mu + \lambda) = l(\mu) > l(\nu) = l(\nu + \lambda)$. Since $w(\nu) > w(\mu)$, we conclude

$$w(\mu + \lambda) = w(\mu) + w(\lambda) < w(\nu) + w(\lambda) = w(\nu + \lambda).$$

So not both of the partitions can be symmetric. □

Lemma 5.18. *Let λ and μ be partitions and $l \in \mathbb{N}$. If $|\lambda \cap \mu| = k \in \mathbb{N}$, then $|(\lambda \cup (l)) \cap (\mu \cup (l))| = k + l$.*

Proof: Let i, j be minimal such that $\lambda_i \leq l$ and $\mu_j \leq l$. We can assume that $i \leq j$. In the rows above i and under j the intersection of λ and μ does not change if we add (l) as a row. We see that it is sufficient to consider the partitions $\tilde{\lambda} = (\lambda_i, \ldots, \lambda_{j-1})$ and $\tilde{\mu} = (\mu_i, \ldots, \mu_{j-1})$, where the parts are zero if the index is greater than the length of the partition. We know that $\lambda_i \leq l$ and that $\mu_{j-1} > l$, therefore, $\tilde{\lambda} \subset \tilde{\mu}$ and

$$\tilde{\lambda} \cup (l) = (l, \lambda_i, \ldots, \lambda_{j-1}) \subset (\mu_i, \ldots, \mu_{j-1}, l) = \tilde{\mu} \cup (l).$$

Hence, we know $|\tilde{\lambda} \cap \tilde{\mu}| = \sum_{r=i}^{j-1} \lambda_r$. We see that $|\tilde{\lambda} \cap \tilde{\mu}| + l = |(\tilde{\lambda} \cup (l)) \cap (\tilde{\mu} \cup (l))|$ □

CHAPTER 6

Kronecker products with hooks

Now we want to start proving Theorem 5.1. We begin with the case that one of the partitions λ and μ is a hook. Without loss of generality we can assume this is λ. The products of Theorem 5.1 which involve a hook are:

Proposition 6.1. *Let $\lambda, \mu \vdash n$, where λ is a hook. Then $g(\lambda, \mu) \leq 2$ if and only if one of the following conditions is satisfied (up to conjugation of λ or μ):*

(1) *The product $[\lambda][\mu]$ is multiplicity-free [BB17], i.e., it is one of the following cases:*
   (a) *One of the partitions is linear;*
   (b) *$\lambda = (n-1, 1)$ and μ is a fat hook;*
   (c) *$\lambda = (n-2, 1^2)$ and μ is a rectangle;*
   (d) *$n = 2k$ is even and $\mu = (k, k)$.*

(2) *both partitions are hooks;*

(3) *if μ is one of the following partitions*
$$\left(\frac{n}{2}, \frac{n}{2}\right), \left(\frac{n+1}{2}, \frac{n-1}{2}\right), (n-2, 2);$$

(4) *$l(\lambda) \leq 3$ or $\lambda_1 \leq 3$ and μ is a two-line partition;*

(5) *$l(\lambda) \leq 4$ or $\lambda_1 \leq 4$ and μ is a rectangle;*

(6) *$\lambda = (n-1, 1)$ and $\operatorname{rem}(\mu) \leq 3$;*

(7) *$n = ab - 1$, $\lambda = (n-2, 1^2)$ and $\mu = (a^{b-1}, a-1)$;*

(8) *$\lambda = (5, 1^4)$ and $\mu = (3^3)$.*

Together with this we want to prove another proposition:

Proposition 6.2. *Let $\lambda, \mu \vdash n$ such that λ is a hook and $g(\lambda, \mu) \geq 2$. The product $[\lambda][\mu]$ has two constituents with multiplicity 3 or higher of which at least one is not symmetric unless the pair λ, μ is from the following list:*

(1) *$\lambda^{(')} = (n-1, 1)$;*

(2) *$n = ab + 1$, $\lambda^{(')} = (n-2, 1^2)$ and $\mu^{(')} = (a^b, 1)$;*

(3) *$\lambda^{(')} = (n-3, 1^3)$ and μ is a two-line partition;*

(4) *$\lambda^{(')} = (4, 1^2)$ and $\mu = (3, 2, 1)$.*

1. Formulas for stated products

In this section we prove that all the products stated in Proposition 6.1 only contain constituents with multiplicity 1 and 2. We do this by stating the decompositions of these products. For a lot of these products the decomposition is already known, but still for some of them we need to prove the formulas.

Products with $[n-1, 1]$.
From Lemma 5.13 we obtain the following corollary for products with $[n-1, 1]$:

Corollary 6.3. *For $\mu \vdash n$ the product $[n-1, 1][\mu]$ only contains constituents with multiplicity 1 and 2 if and only if $\operatorname{rem}(\mu) \leq 3$. Further, all the non-symmetric constituents have multiplicity 1 and 2 if and only if $\operatorname{rem}(\mu) \leq 3$ or $\mu = \mu'$.*

Products with $[n-2,1^2]$**.**
For products with $[n-2,1^2]$ we use Theorem 5.15 to calculate the decompositions.

Lemma 6.4. *Let* $\lambda = (\lambda_1, \lambda_2) \vdash n \geq 5$ *be a two-row partition with* $\lambda_1 > \lambda_2 > 1$ *and* $\mu = (n-2, 1^2)$*. The product* $[\lambda][\mu]$ *decomposes as:*

$$\begin{aligned}&[\lambda_1+1, \lambda_2-1] + [\lambda] + 2[\lambda_1, \lambda_2-1, 1] + [\lambda_1-1, \lambda_2-1, 1^2]\\ &+ \chi_{(\lambda_2 \geq 3)}([\lambda_1, \lambda_2-2, 1^2] + [\lambda_1-1, \lambda_2-1, 2] + [\lambda_1+1, \lambda_2-2, 1])\\ &+ \chi_{(\lambda_1-\lambda_2 \geq 2)}([\lambda_1-2, \lambda_2, 1^2] + [\lambda_1-1, \lambda_2+1])\\ &+ \chi_{(\lambda_1-\lambda_2 \geq 3)}[\lambda_1-2, \lambda_2+1, 1] + (1 + \chi_{(\lambda_1-\lambda_2 \geq 2)})[\lambda_1-1, \lambda_2, 1].\end{aligned}$$

Proof: We use part (3) from Theorem 5.15 to prove the formula. Let $[\nu]$ be a constituent of $[\lambda][\mu]$. We know that $|\lambda \cap \nu| \geq n-2$. If $|\lambda \cap \nu| = n-2$, ν is one of the following partitions

$$\begin{aligned}&(\lambda_1-2, \lambda_2, 1^2),\ (\lambda_1-2, \lambda_2, 2),\ (\lambda_1-2, \lambda_2+1, 1),\ (\lambda_1-2, \lambda_2+2),\\ &(\lambda_1-1, \lambda_2-1, 1^2),\ (\lambda_1-1, \lambda_2-1, 2),\ (\lambda_1, \lambda_2-2, 1^2),\ (\lambda_1, \lambda_2-2, 2),\\ &(\lambda_1+1, \lambda_2-2, 1),\ (\lambda_1+2, \lambda_2-2).\end{aligned}$$

For $\lambda_1 - \lambda_2, \lambda_2 < 4$ some of these are not partitions, these do not occur. We order them after the multiset M which contains the basic skew diagrams which correspond to $\{\lambda/(\lambda \cap \nu), \nu/(\lambda \cap \nu)\}$.

- For the following partitions M equals $\left\{\begin{smallmatrix}\square\\\square\end{smallmatrix}, \square\square\right\}$. The corresponding characters occur with multiplicity 1:

  $$(\lambda_1-2, \lambda_2, 1^2),\ (\lambda_1, \lambda_2-2, 1^2),$$

  where the first partition only occurs if $\lambda_1 - 2 \geq \lambda_2$ and the second one only if $\lambda_2 \geq 3$.
- For the following partitions M equals $\{\square\square, \square\square\}$. The corresponding characters do not occur in the decomposition:

  $$(\lambda_1-2, \lambda_2+2),\ (\lambda_1, \lambda_2-2, 2),\ (\lambda_1-2, \lambda_2, 2),\ (\lambda_1+2, \lambda_2-2).$$
- For the following partitions M equals $\{\square \sqcup \square, \square\square\}$. The corresponding characters occur with multiplicity 1:

  $$(\lambda_1-2, \lambda_2+1, 1),\ (\lambda_1-1, \lambda_2-1, 2),\ (\lambda_1+1, \lambda_2-2, 1),$$

  where the first partition only occurs if $\lambda_1 - 3 \geq \lambda_2$ and the other ones only if $\lambda_2 \geq 3$.
- The last partition ν such that the intersection with λ consists of $n-2$ boxes is $(\lambda_1-1, \lambda_2-1, 1^2)$. Here the multiset M is of the form $\left\{\square \sqcup \square, \begin{smallmatrix}\square\\\square\end{smallmatrix}\right\}$. Therefore, the corresponding character occurs with multiplicity 1.

If $|\lambda \cap \mu| = n-1$, we know that ν is one of the partitions

$$(\lambda_1+1, \lambda_2-1),\ (\lambda_1, \lambda_2-1, 1),\ (\lambda_1-1, \lambda_2, 1),\ (\lambda_1-1, \lambda_2+1),$$

where $(\lambda_1-1, \lambda_2+1)$ only occurs if $\lambda \neq (\frac{n+1}{2}, \frac{n-1}{2})$ for an odd n. For these different possibilities for ν we have to look at the partitions $\rho \vdash n-2$ which are contained in $\lambda \cap \nu$ and the multiset $M = \{\lambda/\rho, \nu/\rho\}$.

- For $\nu = (\lambda_1+1, \lambda_2-1)$ the possibilities for ρ are: (λ_1, λ_2-2) and $(\lambda_1-1, \lambda_2-1)$. For both possibilities $M = \{\square\square, \square \sqcup \square\}$. Therefore, $[\lambda_1+1, \lambda_2-1]$ occurs with multiplicity 1.
- For $\nu = (\lambda_1, \lambda_2-1, 1)$ the possibilities for ρ are: (λ_1, λ_2-2) where $M = \{\square\square, \square \sqcup \square\}$ and $(\lambda_1-1, \lambda_2-1)$ where $M = \{\square \sqcup \square, \square \sqcup \square\}$. Therefore, $[\lambda_1, \lambda_2-1, 1]$ occurs with multiplicity 2.

- For $\nu = (\lambda_1 - 1, \lambda_2, 1)$ the possibilities for ρ are: $(\lambda_1 - 2, \lambda_2)$ where $M = \{\square\square, \square \sqcup \square\}$ and $(\lambda_1 - 1, \lambda_2 - 1)$ with $\{\square \sqcup \square, \square \sqcup \square\}$. Therefore, $[\lambda_1 - 1, \lambda_2, 1]$ occurs with multiplicity 2 if $\lambda_1 - 2 \geq \lambda_2$, and multiplicity 1 otherwise.
- For $\nu = (\lambda_1 - 1, \lambda_2 + 1)$ the possibilities for ρ are: $(\lambda_1 - 2, \lambda_2)$ and $(\lambda_1 - 1, \lambda_2 - 1)$ for both $M = \{\square\square, \square \sqcup \square\}$. Therefore, $[\lambda_1 - 1, \lambda_2 + 1]$ occurs with multiplicity 1 if $\lambda_1 - 2 \geq \lambda_2$, and 0 otherwise.

If $|\lambda \cap \nu| = n$, we know that $\nu = \lambda$ and

$$g(\lambda, \lambda, \mu) = (\mathrm{rem}(\lambda) - 1)^2 = (2 - 1)^2 = 1.$$

□

For the rest of this section we will spend finding the decompositions of the products from Proposition 6.1 using Theorem 5.15, like in the last lemma.

Lemma 6.5. (1) *Let $n = ab + a - 1$, where $a \geq 4$ and $b \geq 2$. The product $[n-2, 1^2][a^b, a-1]$ decomposes as follows:*

$$\begin{aligned}
&[a^{b-2}, (a-1)^3, 2] + [a^{b-1}, a-1, a-2, 1^2] + [a^{b-1}, a-1, a-2, 2] \\
&+ 2[a^{b-1}, (a-1)^2, 1] + [a^b, a-3, 1^2] + 2[a^b, a-2, 1] + [a^b, a-1] \\
&+ \chi_{(b>2)}[a+1, a^{b-3}, (a-1)^3, 1] + 2[a+1, a^{b-2}, a-1, a-2, 1] \\
&+ 2[a+1, a^{b-2}, (a-1)^2] + [a+1, a^{b-1}, a-3, 1] + 2[a+1, a^{b-1}, a-2] \\
&+ \chi_{(b>2)}[(a+1)^2, a^{b-3}, a-1, a-2] + [(a+1)^2, a^{b-2}, a-3] \\
&+ \chi_{(b>2)}[a+2, a^{b-3}, (a-1)^3] + [a+2, a^{b-2}, a-1, a-2].
\end{aligned}$$

(2) *Let $n = ab+1$, where $a \geq 3, b \geq 2$. The product $[n-2, 1^2][a^b, 1]$ decomposes as follows:*

$$\begin{aligned}
&[a^{b-2}, (a-1)^2, 2, 1] + \chi_{(a>3)}[a^{b-2}, (a-1)^2, 3] + [a^{b-1}, a-2, 1^3] \\
&+ \chi_{(a>3)}[a^{b-1}, a-2, 2, 1] + 2[a^{b-1}, a-1, 1^2] + 2[a^{b-1}, a-1, 2] + [a^b, 1] \\
&+ \chi_{(b>2)}[a+1, a^{b-3}, (a-1)^2, 1^2] + \chi_{(b>2)}[a+1, a^{b-3}, (a-1)^2, 2] \\
&+ [a+1, a^{b-2}, a-2, 1^2] + \chi_{(a>3)}[a+1, a^{b-2}, a-2, 2] \\
&+ 3[a+1, a^{b-2}, a-1, 1] + [a+1, a^{b-1}] + \chi_{(b>2)}[(a+1)^2, a^{b-3}, a-2, 1] \\
&+ \chi_{(b>2)}[(a+1)^2, a^{b-3}, a-1] + \chi_{(b>2)}[a+2, a^{b-3}, (a-1)^2, 1] \\
&+ [a+2, a^{b-2}, a-1].
\end{aligned}$$

Proof: As for the other proofs in this section we use Theorem 5.15 for both parts.

(1): Let $\lambda = (a^b, a-1)$ and $\mu \vdash n$. Theorem 5.15 (3) provides the multiplicity $g(\lambda, \mu, \nu)$ depending on $\lambda \cap \nu$. If $|\lambda \cap \mu| < n-2$, $[\mu]$ is not a constituent of $[\lambda][n-2, 1^2]$. If $|\lambda \cap \mu| = n-2$, we sort the partitions by the multiset M of the diagrams $\lambda/(\lambda \cap \mu)$ and $\mu/(\lambda \cap \mu)$.

- There is just one partition μ such that $M = \{\square \sqcup \square, \square \sqcup \square\}$ namely
  $$(a+1, a^{b-2}, a-1, a-2, 1).$$
  The corresponding character has multiplicity 2.
- The partitions μ where $M = \{\square\square, \square \sqcup \square\}$ are
  $$(a+2, a^{b-2}, a-1, a-2),\ (a+1, a^{b-1}, a-3, 1),$$
  $$(a^{b-1}, a-1, a-2, 2).$$
  The corresponding characters have multiplicity 1.

- The partitions μ with $M = \left\{\begin{smallmatrix}\square\\ \square\end{smallmatrix}, \square \sqcup \square\right\}$ are

$$((a+1)^2, a^{b-3}, a-1, a-2),\ (a+1, a^{b-3}, (a-1)^3, 1),$$
$$(a^{b-1}, a-1, a-2, 1^2),$$

where $((a+1)^2, a^{b-3}, a-1, a-2)$ and $(a+1, a^{b-3}, (a-1)^3, 1)$ do not occur if $b = 2$. The corresponding characters have multiplicity 1.

- The partitions μ where $M = \left\{\begin{smallmatrix}\square\\ \square\end{smallmatrix}, \square\square\right\}$ are

$$(a+2, a^{b-3}, (a-1)^3),\ ((a+1)^2, a^{b-2}, a-3),$$
$$(a^b, a-3, 1^2),\ \ (a^{b-2}, (a-1)^3, 2),$$

where $(a+2, a^{b-3}, (a-1)^3)$ does not occur if $b = 2$. The corresponding characters have multiplicity 1.

- For the remaining partitions μ where M equals $\{\square\square, \square\square\}$ or $\left\{\begin{smallmatrix}\square\\ \square\end{smallmatrix}, \begin{smallmatrix}\square\\ \square\end{smallmatrix}\right\}$ the corresponding characters have multiplicity 0.

If $|\lambda \cap \mu| = n-1$, we have four possibilities for μ.

- If $\mu = (a+1, a^{b-1}, a-2)$, $(a^b, a-2, 1)$, the possibilities for $\rho \vdash n-2$, with $\rho \subset \lambda \cap \mu$ are $(a^b, a-3)$, $(a^{b-1}, a-1, a-2)$. This means

$$g(\lambda, (n-2, 1^2), (a+1, a^{b-1}, a-2)) = 2 + 1 - 1 = 2,$$
$$g(\lambda, (n-2, 1^2), (a^b, a-2, 1)) = 2 + 1 - 1 = 2.$$

- If $\mu = (a+1, a^{b-2}, (a-1)^2)$, $(a^{b-1}, (a-1)^2, 1)$, the possibilities for $\rho \vdash n-2$, with $\rho \subset \lambda \cap \mu$ are $(a^{b-1}, a-1, a-2)$, $(a^{b-2}, (a-1)^3)$. This means

$$g(\lambda, (n-2, 1^2), (a+1, a^{b-2}, (a-1)^2)) = 2 + 1 - 1 = 2,$$
$$g(\lambda, (n-2, 1^2), (a^{b-1}, (a-1)^2, 1)) = 2 + 1 - 1 = 2.$$

The last case we have to look at is $\mu = \lambda$. Here we know

$$g(\lambda, \lambda, (n-2, 1^2)) = (\mathrm{rem}(\lambda) - 1)^2 = 1.$$

(2): Let $\lambda = (a^b, 1)$ and $\mu \vdash n$. If $|\lambda \cap \mu| < n-2$, $g(\lambda, (n-2, 1^2), \mu) = 0$. If $|\lambda \cap \mu| = n-2$, we sort the partitions by the multiset $M = \{\lambda/(\lambda \cap \mu), \mu/(\lambda \cap \mu)\}$.

- There is no partition $\mu \vdash n$ such that $M = \{\square \sqcup \square, \square \sqcup \square\}$.
- The partitions μ with $M = \{\square\square, \square \sqcup \square\}$ are

$$(a+2, a^{b-2}, a-1),\ (a+1, a^{b-2}, a-2, 2),$$
$$(a+1, a^{b-2}, a-2, 1^2),\ (a^{b-1}, a-2, 2, 1),$$

where the second and the fourth only occur if $a > 3$. The corresponding characters have multiplicity 1.

- The partitions μ with $M = \left\{\begin{smallmatrix}\square\\ \square\end{smallmatrix}, \square \sqcup \square\right\}$ are

$$((a+1)^2, a^{b-3}, a-1),\ (a+1, a^{b-3}, (a-1)^2, 2),$$
$$(a+1, a^{b-3}, (a-1)^2, 1^2),\ (a^{b-2}, (a-1)^2, 2, 1),$$

where the first three only occur if $b > 2$. The corresponding characters have multiplicity 1.

- The partitions μ with $M = \left\{\begin{smallmatrix}\square\\ \square\end{smallmatrix}, \square\square\right\}$ are

$$(a+2, a^{b-3}, (a-1)^2, 1),\ ((a+1)^2, a^{b-3}, a-2, 1),$$
$$(a^{b-1}, a-2, 1^3),\ (a^{b-2}, (a-1)^2, 3),$$

where the first two only occur if $b > 2$ and the last one only if $a > 3$. The corresponding characters have multiplicity 1.

- The remaining partitions μ where M equals $\{\square\square, \square\square\}$ or $\left\{\begin{smallmatrix}\square\\\square\end{smallmatrix}, \begin{smallmatrix}\square\\\square\end{smallmatrix}\right\}$ have multiplicity 0.

If $|\lambda \cap \mu| = n - 1$, we have four possibilities for μ.

- If
$$\mu = (a + 1, a^{b-2}, a - 1, 1),\ (a^{b-1}, a - 1, 2),\ (a^{b-1}, a - 1, 1^2),$$
the possibilities for $\rho \vdash n - 2,\ \rho \subset \lambda \cap \mu$ are
$$(a^{b-1}, a - 1),\ (a^{b-1}, a - 2, 1),\ (a^{b-2}, (a - 1)^2, 1)$$
and we get
$$g(\lambda, (n - 2, 1^2), (a + 1, a^{b-2}, a - 1, 1)) = 2 + 1 + 1 - 1 = 3,$$
$$g(\lambda, (n - 2, 1^2), (a^{b-1}, a - 1, 2)) = 1 + 1 + 1 - 1 = 2,$$
$$g(\lambda, (n - 2, 1^2), (a^{b-1}, a - 1, 1^2)) = 1 + 1 + 1 - 1 = 2.$$

- If $\mu = (a + 1, a^{b-1})$, the only possibility for $\rho \vdash n - 2,\ \rho \subset \lambda \cap \mu$ is $(a^{b-1}, a - 1)$. With this we obtain that $g(\lambda, (n - 2, 1^2), (a + 1, a^{b-1})) = 2 - 1 = 1$.

The last case we again look at is $\mu = \lambda$. We know
$$g(\lambda, \lambda, (n - 2, 1^2)) = (\mathrm{rem}(\lambda) - 1)^2 = 1.$$

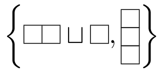
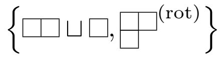
□

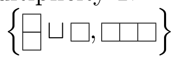
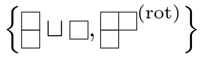
Product of $[n - 3, 1^3]$ with a rectangle.

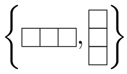
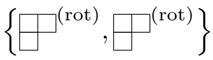
Lemma 6.6. *Let $n = ab$, where $a \geq 4, b \geq 3$. Then $[n - 3, 1^3][a^b]$ is given by*
$$\begin{aligned}
&[a^{b-3}, (a - 1)^3, 3] + [a^{b-2}, a - 1, a - 2, 2, 1] + [a^{b-2}, (a - 1)^2, 1^2] \\
&+ [a^{b-2}, (a - 1)^2, 2] + [a^{b-1}, a - 3, 1^3] + [a^{b-1}, a - 2, 1^2] + [a^{b-1}, a - 2, 2] \\
&+ [a^{b-1}, a - 1, 1] + [a^b] + \chi_{(b>3)}[a + 1, a^{b-4}, (a - 1)^3, 2] \\
&+ [a + 1, a^{b-3}, a - 1, a - 2, 1^2] + [a + 1, a^{b-3}, a - 1, a - 2, 2] \\
&+ 2[a + 1, a^{b-3}, (a - 1)^2, 1] + [a + 1, a^{b-2}, a - 3, 1^2] + 2[a + 1, a^{b-2}, a - 2, 1] \\
&+ [a + 1, a^{b-2}, a - 1] + \chi_{(b>3)}[(a + 1)^2, a^{b-4}, a - 1, a - 2, 1] \\
&+ \chi_{(b>3)}[(a + 1)^2, a^{b-4}, (a - 1)^2] + [(a + 1)^2, a^{b-3}, a - 3, 1] \\
&+ [(a + 1)^2, a^{b-3}, a - 2] + \chi_{(b>3)}[(a + 1)^3, a^{b-4}, a - 3] \\
&+ \chi_{(b>3)}[a + 2, a^{b-4}, (a - 1)^3, 1] + [a + 2, a^{b-3}, a - 1, a - 2, 1] \\
&+ [a + 2, a^{b-3}, (a - 1)^2] + [a + 2, a^{b-2}, a - 2] \\
&+ \chi_{(b>3)}[a + 2, a + 1, a^{b-4}, a - 1, a - 2] + \chi_{(b>3)}[a + 3, a^{b-4}, (a - 1)^3].
\end{aligned}$$

Proof: As before, we use Theorem 5.15 to prove this lemma. This time we need parts (8) and (9) of the theorem. Let $\lambda = (a^b)$ and $\nu \vdash n = ab$. If $|\lambda \cap \nu| < n - 3$,
$$g(\lambda, (n - 3, 1^3), \nu) = 0.$$

If $|\lambda \cap \nu| = n - 3$, we again differ by the multiset $M := \{\lambda/(\lambda \cap \nu), \nu/(\lambda \cap \nu)\}$. There is no partition ν such that $\square \sqcup \square \sqcup \square$ is an element of M. Neither is there a partition ν such that M is given by $\{\square\square^{(\prime)} \sqcup \square, \square\square^{(\prime)} \sqcup \square\}$ (for all four possible choices). Both these statements follow from the fact that (a^b) has one removable and two addable nodes. Thus, we know that $\lambda/(\lambda \cap \nu)$ is connected and $\nu/(\lambda \cap \nu)$ has at most two connected components.

- The possibilities for ν such that M is given by  are
$$(a+2, a^{b-4}, (a-1)^3, 1),\ (a+1, a^{b-4}, (a-1)^3, 2).$$
Both only occur if $b > 3$. The corresponding characters have multiplicity 1.
- The possibilities for ν such that $M =$ 

are
$$(a+2, a^{b-3}, a-1, a-2, 1),\ (a+1, a^{b-3}, a-1, a-2, 2).$$
The corresponding characters have multiplicity 1.
- The possibilities for ν such that $M =$  are
$$((a+1)^2, a^{b-3}, a-3, 1),\ (a+1, a^{b-2}, a-3, 1^2).$$
The corresponding characters have multiplicity 1.
- The possibilities for ν such that M is given by 

are
$$((a+1)^2, a^{b-4}, a-1, a-2, 1),\ (a+1, a^{b-3}, a-1, a-2, 1^2),$$
where the first one only occurs if $b > 3$. The corresponding characters have multiplicity 1.
- The possibilities for ν such that M is given by  are
$$(a+3, a^{b-4}, (a-1)^3),\ ((a+1)^3, a^{b-4}, a-3),\ (a^{b-1}, a-3, 1^3),$$
$$(a^{b-3}, (a-1)^3, 3),$$
where the first two only occur if $b > 3$. The corresponding characters have multiplicity 1.
- The possibilities for ν such that M is given by 

are
$$(a+2, a+1, a^{b-4}, a-1, a-2),\ (a^{b-2}, a-1, a-2, 2, 1),$$
where the first one only occurs if $b > 3$. The corresponding characters have multiplicity 1.
- For partitions $\nu \vdash n$ with $|\lambda \cap \nu| = n-3$ with a different M we have $g(\lambda, (n-3, 1^3), \nu) = 0$.

In the next step we look at the partitions $\nu \vdash n$ such that $|\lambda \cap \nu| = n-2$.

- If ν is one of the partitions
$$(a+2, a^{b-3}, (a-1)^2),\ ((a+1)^2, a^{b-4}, (a-1)^2),$$
$$(a+1, a^{b-3}, (a-1)^2, 1),\ (a^{b-2}, (a-1)^2, 2),$$
$$(a^{b-2}, (a-1)^2, 1^2),$$
where $((a+1)^2, a^{b-4}, (a-1)^2)$ only occurs if $b > 3$, $\lambda \cap \nu = (a^{b-2}, (a-1)^2)$ and $\{\rho \vdash n-3 \,|\, \rho \subset \lambda \cap \nu\} = \{(a^{b-2}, a-1, a-2), (a^{b-3}, (a-1)^3)\}$. This tells us that
$$g(\lambda, (n-3, 1^3), (a+2, a^{b-3}, (a-1)^2)) = 1 + 1 - 1 = 1,$$
$$g(\lambda, (n-3, 1^3), ((a+1)^2, a^{b-4}, (a-1)^2)) = 1 + 0 - 0 = 1,$$
$$g(\lambda, (n-3, 1^3), (a+1, a^{b-3}, (a-1)^2, 1)) = 2 + 1 - 1 = 2,$$
$$g(\lambda, (n-3, 1^3), (a^{b-2}, (a-1)^2, 2)) = 1 + 1 - 1 = 1,$$
$$g(\lambda, (n-3, 1^3), (a^{b-2}, (a-1)^2, 1^2)) = 1 + 0 - 0 = 1,$$
where $((a+1)^2, a^{b-4}, (a-1)^2)$ only occurs if $b > 3$.

- The remaining partitions $\nu \vdash n$ such that $|\lambda \cap \nu| = n-2$ are $(a+2, a^{b-2}, a-2)$, $((a+1)^2, a^{b-3}, a-2)$, $(a+1, a^{b-2}, a-2, 1))$, $(a^{b-1}, a-2, 2)$, $(a^{b-1}, a-2, 1^2)$.

  For these partitions $\lambda \cap \nu = (a^{b-1}, a-2)$ and
  $$\{\rho \vdash n-3 \,|\, \rho \subset \lambda \cap \nu\} = \{(a^{b-1}, a-3), (a^{b-2}, a-1, a-2)\}.$$
  This tells us that
  $$g(\lambda, (n-3, 1^3), (a+2, a^{b-2}, a-2)) = 0 + 1 - 0 = 1,$$
  $$g(\lambda, (n-3, 1^3), ((a+1)^2, a^{b-3}, a-2) = 1 + 1 - 1 = 1,$$
  $$g(\lambda, (n-3, 1^3), (a+1, a^{b-2}, a-2, 1))) = 1 + 2 - 1 = 2,$$
  $$g(\lambda, (n-3, 1^3), (a^{b-1}, a-2, 2)) = 0 + 1 - 0 = 1,$$
  $$g(\lambda, (n-3, 1^3), (a^{b-1}, a-2, 1^2)) = 1 + 1 - 1 = 1.$$

In the next step we look at the partitions $\nu \vdash n$ such that $|\lambda \cap \nu| = n-1$. There are two such partitions, $(a+1, a^{b-2}, a-1)$ and $(a^{b-1}, a-1, 1)$. For both partitions $\lambda \cap \nu = (a^{b-1}, a-1)$ and $\{\rho \vdash n-2 \,|\, \rho \subset \lambda \cap \nu\} = \{(a^{b-1}, a-2), (a^{b-2}, (a-1)^2)\}$. Further,
$$\{\rho \vdash n-3 \,|\, \rho \subset \lambda \cap \nu\} = \{(a^{b-1}, a-3), (a^{b-2}, a-2, a-1), (a^{b-3}, (a-1)^3)\}.$$
As a consequence
$$g(\lambda, (n-3, 1^3), (a+1, a^{b-2}, a-1)) = 0 + 2 + 0 - 1 - 1 + 1 = 1,$$
$$g(\lambda, (n-3, 1^3), (a^{b-1}, a-1, 1)) = 0 + 2 + 0 - 1 - 1 + 1 = 1.$$
Finally, we consider
$$g(\lambda, (n-3, 1^3), \lambda) = 0 + 1 + (1-1)(1+1+1) + 1(1-1)(1-3) = 1.$$
This concludes the proof. □

With these decompositions we now have more or less explicit formulas for all the products from the first half of the Proposition 6.1.

One direction of the proof of Proposition 6.1.

Lemma 6.7. *If $\lambda, \mu \vdash n$ are listed in Proposition 6.1, $g(\lambda, \mu) \leq 2$.*

Proof: We just have to collect the formulas we already have. The multiplicity-free products have been classified in [BB17, Theorem 1.1.]. Proposition 6.1 (2) follows directly from Theorem 5.11.

The products of the form $\left[\frac{n}{2}, \frac{n}{2}\right]$ [*hook*] appear in the list of multiplicity-free Kronecker products of [BB17, Theorem 1.1.]. For the other products of Proposition 6.1 (3) and (4) we take the formula from Theorem 5.12. The theorem tells us that all the constituents are hooks or double-hooks and that hooks have at most multiplicity 2, so we only have to look at the double-hooks. For the double-hooks only Theorem 5.12 (3)(c) and (d) can provide constituents with multiplicity higher than 2. Let $n = 2k+1$, $\lambda \vdash n$ be a hook and $\mu = (k+1, k)$. Further, let $\nu = (n_4, n_3, 2^{d_2}, 1^{d_1})$ be a constituent with multiplicity 3 of $[\lambda][\mu]$ (so it is from Theorem 5.12 (3)(c) or (d)). In both parts we see that $k - d_2 \leq n_3 + d_1 - 1$ and $k - d_2 \leq n_4 - 1$. But if we add both inequalities, we obtain $2k - 2d_2 \leq n_4 + n_3 + d_1 - 2$, which is equivalent to $n - 1 \leq n - 2$, which is obviously a contradiction. If λ is a hook and $\mu = (n-2, 2)$, we see that the first summand from Theorem 5.12 (3)(c) and (d) is 0. If $\lambda = (n-2, 1^2)$ and $\mu = (\mu_1, \mu_2)$ is a two-row partition, we see that $d_1 = 1$ and $d_2 = 0$ for Theorem 5.12 (3)(c) resp. $d_1 = d_2 = 0$ for (d). Therefore, the first summand from Theorem 5.12 (3)(c) and (d) equals 0 if $\mu_2 \neq n_3+1$. But in this case, the third summand equals 0. This proves that the products from Proposition 6.1 (3) and (4), which are not multiplicity-free, only contain constituents with multiplicity 1 and 2.

We proved Proposition 6.1 (5) in Lemma 6.6. Proposition 6.1 (6) follows from Corollary 6.3. Proposition 6.1 (7) we is given by in Lemma 6.5 and to prove Proposition 6.1 (8) we calculate the decomposition with Sage. Therefore, we know that $g((5,1^4),(3^3)) = 2$. □

In the next step we want to collect the information we already have to prove one direction of Proposition 6.2.

Lemma 6.8. *All the products listed in Proposition 6.2 only have one constituent with multiplicity greater or equal to 3 or just symmetric constituents with multiplicity greater or equal to 3.*

Proof: Proposition 6.2 (1) follows from Lemma 5.13. Proposition 6.2 (2) follows from Lemma 6.5. We check Proposition 6.2 (3) with Theorem 5.12. Let $\lambda = (n-3,1^3)$, $\mu = (\mu_1,\mu_2)$ be a two-row partition and $\nu = (n_4, n_3, 2^{d_2}, 1^{d_1})$ be a constituent of their product. Only from Theorem 5.12 (3)(c) we obtain a constituent with multiplicity 3 and with the same argument as for $\lambda = (n-2,1^2)$ in the previous lemma we see that $d_1 = 2$, $d_2 = 0$. Additionally, we see from Theorem 5.12 (3)(c) that we obtain a constituent with multiplicity greater or equal to 3 if and only if $\mu_1 - 2 \geq \mu_2$, $\mu_2 \geq 3$, $\mu_1 - 1 = n_4$ and $\mu_2 - 1 = n_3$. Which proves that the product has at most one constituent with multiplicity 3, namely $\nu = (\mu_1-1, \mu_2-1, 1^2)$, and none with a higher multiplicity. Further, we see that if $n \neq 8$, ν is not symmetric. Proposition 6.2 (4) was checked with Sage. □

2. Other products with hooks contain a constituent with multiplicity 3 or higher

Now we want to prove the other direction of Proposition 6.1 and 6.2. We split the proof into four lemmas. We distinguish the cases according to the diagonal length of the partitions. For that we recall the following notation: If λ is a partition, $d(\lambda)$ is the number of boxes on the main diagonal of λ, called Durfee size of the partition λ. We use the semigroup property of the Kronecker coefficients to reduce the pair (λ,μ) to a pair $(\tilde{\lambda},\tilde{\mu})$ for which we know that $g(\tilde{\lambda},\tilde{\mu}) \geq 3$ and find $\tilde{\nu}$ such that $g(\tilde{\lambda},\tilde{\mu},\tilde{\nu}) \geq 3$ and that one of the partitions that we obtain from $\tilde{\nu}$ with the semigroup property is not symmetric. We will call such a pair $(\tilde{\lambda},\tilde{\mu})$ a *seed* (for λ, μ). Depending on the statement we want to prove we will sometimes need seeds $(\tilde{\lambda},\tilde{\mu})$ such that the corresponding product $[\tilde{\lambda}][\tilde{\mu}]$ has at least 2 constituents with multiplicity 3 or higher. For these seeds we will say that they have 2 constituents with multiplicity 3 or higher even though we mean that the corresponding product $[\tilde{\lambda}][\tilde{\mu}]$ has two constituents with multiplicity 3 or higher. The lemmas we want to prove have all been checked with Sage up to $n = 22$, so in the following proofs we do not have to look at the smaller cases.

Durfee size of 2 or 3.

Lemma 6.9. *Let $\lambda,\mu \vdash n$, where λ is a hook and $d(\mu) \leq 3$.*

(1) *If λ, μ are not listed in Proposition 6.1, $g(\lambda,\mu) \geq 3$.*
(2) *If λ, μ are neither listed in Proposition 6.1 nor in Proposition 6.2, $[\lambda][\mu]$ has two constituents with multiplicity greater or equal to 3 of which one is not symmetric.*

Proof: We verified the lemma up to $n = 22$ so we can assume that $n \geq 23$. If $d(\mu) = 1$ we know that $g(\lambda, \mu) \leq 2$ and λ, μ are listed in Proposition 6.1. Therefore, we can assume that $d(\mu) \geq 2$. Further, by Corollary 6.3 we can assume that $w(\lambda)$, $l(\lambda) \geq 3$.

1st case: $\lambda = (n-i, 1^i)$ and $\mu = (n-j, j)$. We assume that $w(\lambda) \geq l(\lambda)$. We know that $i \geq 3$ and that $n-j \neq j, j+1$ and $j \neq 1, 2$ (otherwise, the product would be listed in Proposition 6.1).

We distinguish between two cases. First let $i \geq 4$. We can choose $a, b, c, d \in \mathbb{N}_0$ in such a way that $\mu = \tilde{\mu} + (a, b) + (c, d)$, where $\tilde{\mu} = (7, 3)$, $(6, 4)$, $(7, 5)$ as well as $a + b = n - i - 6$, $n - i - 7$ and $c + d = i - 4$, $i - 5$. First we cut off (a, b) from μ and $(a + b)$ from λ, then we conjugate the hook and cut off $(c + d)$ from the hook and (c, d) from the two-row partition. We obtain one of the seeds

$$((7, 3), (6, 1^4)),\ ((6, 4), (6, 1^4)),\ ((7, 5), (7, 1^5)).$$

For all three seeds we check that the product of their corresponding characters has two constituents which have multiplicity 3, length and width greater than 2 and the differences of length and width are different, therefore, we can apply Lemma 5.16, which yields the result in this case.

If $i = 3$, we have seen in the previous lemma that $(n - j - 1, j - 1, 1^2)$ is a constituent with multiplicity 3 if $n - 2j \geq 2$ and $j \geq 3$. Alternatively, we could use the semigroup property to reduce it to one of the seeds $((7, 3), (7, 1^3))$ or $((6, 4), (7, 1^3))$. By conjugation symmetry we can assume that μ is not a two-line partition.

2nd case: $\lambda = (n - i, 1^i)$ and $\mu \vdash n$ is a double-hook with $l(\mu)$, $w(\mu) \geq 3$. We prove this by induction. For $n = 21$, 22 we check with Sage, that all products which are neither from Proposition 6.1 nor from Proposition 6.2 contain two constituents $[\alpha]$ and $[\beta]$ with multiplicity 3 or higher such that $w(\alpha), l(\alpha), w(\beta), l(\beta) \geq 2$ and $w(\alpha) - l(\alpha) \neq w(\beta) - l(\beta)$. Since $n \geq 23$ we can assume that $w(\mu) \geq 7$ as well as $w(\lambda) \geq 12$. By Lemma 6.5 we can exclude the case that n is odd, $\lambda = (n - 2, 1^2)$ and $\mu = (\frac{n-1}{2}, \frac{n-1}{2}, 1)$. We remove the second column from the right from μ and the same number of columns of length 1 from λ to obtain $\tilde{\lambda}$ and $\tilde{\mu}$. We remove just a column of length 1 or 2 so the result will follow from Lemma 5.16 if we show that $\tilde{\lambda}$ and $\tilde{\mu}$ are neither from Proposition 6.1 nor from Proposition 6.2. Since $l(\tilde{\mu}) = l(\mu) \geq 3$, $w(\tilde{\mu}) = w(\mu) - 1 \geq 6$ and $d(\tilde{\mu}) = d(\mu) = 2$, know that $\tilde{\mu}$ is not a rectangle nor a two-line partition nor $(a, a, a - 1)$ for some a. Additionally, since we removed the second column from the right from μ, we know that $\tilde{\mu} = (a^2, 1)$ for some a if and only if $\mu = ((a + 1)^2, 1)$. Therefore we see that $\tilde{\lambda}$, $\tilde{\mu}$ are neither from Proposition 6.1 nor from Proposition 6.2 if λ, μ are not listed in these propositions.

3rd case: $d(\mu) = 3$. We can prove this case by induction like the previous case. For $n = 20$, 21, 22 we check with Sage, that all products which are neither from Proposition 6.1 nor from Proposition 6.2 contain two constituents $[\alpha]$ and $[\beta]$ with multiplicity 3 or higher, $w(\alpha)$, $l(\alpha)$, $w(\beta)$, $l(\beta) \geq 3$ and $w(\alpha) - l(\alpha) \neq w(\beta) - l(\beta)$. Since $n \geq 23$ we can assume by conjugation that $w(\mu) \geq 6$ as well as $w(\lambda) \geq 12$. By Lemma 6.5 we can exclude the case that $n = 3a + 1$, $\lambda = (n - 2, 1^2)$ and $\mu = (a^3, 1)$. We remove the fourth column from the left from μ and the same number of columns of length 1 from λ to obtain $\tilde{\lambda}$ and $\tilde{\mu}$. We remove just a column of length 1, 2 or 3 so the result will follow from Lemma 5.16 if we show that $\tilde{\lambda}$ and $\tilde{\mu}$ are neither from Proposition 6.1 nor from Proposition 6.2. We know that $\tilde{\mu} = (a^3)$ is a rectangle if and only if $\mu = ((a + 1)^3)$. Further, $d(\tilde{\mu}) = d(\mu) = 3$. Since $w(\tilde{\mu}) \geq 5$, we know that $\tilde{\mu} \neq (a^3, a - 1)$ for some a. We know that if $l(\mu) = 3$, $w(\mu) \geq 8$. Since we removed the fourth column from the left, we also know that $\tilde{\mu} = (a^2, a - 1)$ for some a if and only if $\tilde{\mu} = ((a + 1)^2, a)$. We still know that $\tilde{\mu} = (a^3, 1)$ for some a if and only if

$\mu = ((a+1)^3, 1)$. Therefore we see that $\tilde{\lambda}$, $\tilde{\mu}$ neither from Proposition 6.1 nor from Proposition 6.2 if λ, μ are not listed in these propositions. □

Squares.

Lemma 6.10. *Let $n = a^2 \geq 16$, $\lambda \vdash n$ be a hook with $l(\lambda), w(\lambda) \geq 5$ and $\mu = (a^a)$. There $\alpha, \beta \vdash n$ with* $\text{rem}(\alpha)$, $\text{rem}(\beta) \geq 2$, *$l(\alpha)$, $w(\alpha) \geq a+1$, $l(\beta) \geq a$, $w(\beta) \geq a+2$ and $l(\alpha) - w(\alpha) \neq l(\beta) - w(\beta)$ such that $g(\lambda, (a^a), \alpha) \geq 3$ and $g(\lambda, (a^a), \beta) \geq 3$.*

Proof: We prove this by induction on a. For $a = 4$ we obtain the result with Sage. For $a \geq 5$ we use the semigroup property. We add a column of length a and a row of length $a+1$. There is a hook $\tilde{\lambda} \vdash a^2$ such that $\lambda \vdash (a+1)^2$ can be obtained from $\tilde{\lambda}$ by adding first (a), then maybe transposing the partition and then adding $(a+1)$. If $l(\lambda) \leq a+5$, $w(\lambda) \geq a^2 + a - 6 \geq 2a + 6$. Thus, $\tilde{\lambda}$ can be chosen in such a way that $w(\tilde{\lambda})$ and $l(\tilde{\lambda}) \geq 5$. By induction we find constituents $[\tilde{\alpha}]$ and $[\tilde{\beta}]$ of $[\tilde{\lambda}][a^a]$ with multiplicity greater or equal to 3, such that $l(\tilde{\alpha}), w(\tilde{\alpha}) \geq a+1$, $\text{rem}(\alpha)$, $\text{rem}(\beta) \geq 2$, $l(\tilde{\beta}) \geq a, w(\tilde{\beta}) \geq a+2$ and $l(\tilde{\alpha}) - w(\tilde{\alpha}) \neq l(\tilde{\beta}) - w(\tilde{\beta})$. We distinguish between the case where we transpose and the case where we do not transpose.

1st case: We do not transpose $\tilde{\lambda} + (a)$ before adding $(a+1)$. In this case the partitions $\alpha := (\tilde{\alpha} + (1^a))' + (1^{a+1})$ and $\beta := (\tilde{\beta}' + (1^a))' + (1^{a+1})$ have the desired properties, since $l(\alpha) = l(\tilde{\alpha})+1$, $w(\alpha) = w(\tilde{\alpha})+1$ and the same holds for β. Thanks to Theorem 5.8 we know that $g(\lambda, ((a+1)^{a+1}), \alpha),\ g(\lambda, ((a+1)^{a+1}), \beta) \geq 3$.

2nd case: We transpose $\tilde{\lambda} + (a)$ before adding $(a+1)$. In this case we have the partitions $\beta := \tilde{\alpha}' + (1^a) + (1^{a+1})$ and $\alpha := \tilde{\beta}' + (1^a) + (1^{a+1})$ with the desired properties, where we can exchange $\tilde{\alpha}$ and $\tilde{\beta}$ by $\tilde{\alpha}'$ and $\tilde{\beta}'$ since (a^a) is symmetric. Here $w(\alpha) = l(\tilde{\beta}) + 2$, $l(\alpha) = w(\tilde{\beta})$, $w(\beta) = l(\tilde{\alpha}) + 2$ and $l(\beta) = w(\tilde{\alpha})$. Thanks to Theorem 5.8 we know that $g(\lambda, ((a+1)^{a+1}), \alpha),\ g(\lambda, ((a+1)^{a+1}), \beta) \geq 3$. □

Durfee size greater than 3.

Lemma 6.11. *Let $\lambda = (n-i, 1^i)$, where $n-i, i+1 \geq 5$ and $\mu \vdash n$ with $d(\mu) \geq 4$. Then $[\lambda][\mu]$ has two constituents with multiplicity greater or equal to 3 of which at least one is not symmetric.*

Proof: If we can reduce λ, μ to a pair $\tilde{\lambda}$, $(d(\mu)^{d(\mu)})$, where $\tilde{\lambda}$ is a hook with $w(\tilde{\lambda}), l(\tilde{\lambda}) \geq 5$, we get the result by Lemma 5.16 and 6.10. It is obvious that we can reduce λ, μ to a pair $\tilde{\lambda}$, $(d(\mu)^{d(\mu)})$, where $\tilde{\lambda}$ is a hook. Now we want to show that we can remove the columns and rows in such a way that $l(\tilde{\lambda})$, $w(\tilde{\lambda}) \geq 5$. We remove all columns and rows individually. We know that all columns and rows that we remove have less or equal to $d(\mu)$ boxes. If $l(\lambda) < d(\mu) + 5$, we know that $w(\lambda) > d(\mu) + 5$ since $l(\lambda) + w(\lambda) - 1 = n \geq d(\mu)^2$ for $d(\mu) \geq 4$. If we repeat this for all rows underneath and all columns to the right of $(d(\mu)^{d(\mu)})$, we obtain the result. □

Lemma 6.12. *Let $\lambda = (n-i, 1^i)$, where $i = 2, 3$ and $\mu \vdash n$ with $d(\mu) \geq 4$ such that λ, μ is not from one of the lists in Proposition 6.1 or Proposition 6.2. Then there are two constituents with multiplicity 3 or higher of which one is not symmetric.*

Proof: 1st case: $i = 3$. We know that μ is not a rectangle, therefore, μ has two rows of different lengths. If $\mu_1 > \mu_2 > \mu_3$ we can remove $(\mu_1 - \mu_2)$ from λ and μ without reducing μ to a rectangle. Thus, we can assume that μ has two rows of the same length which are longer than 3. Hence, we reduce μ to either $(4, 3, 3)$ or $(4, 4, 3)$ if the rows of different length have length ≥ 3, to $(4, 4, 3)'$ if the smaller row has length 2, and to $(4, 3, 3)'$ if the smaller row has length 1. Further, from

λ we only remove boxes from the arm. For all four seeds $((4,3,3)^{(\prime)},(7,1^3))$ and $((4,4,3)^{(\prime)},(8,1^3))$ we have two constituents $[\alpha],[\beta]$ of multiplicity greater or equal to 3 with $l(\alpha)=l(\beta)$, $w(\alpha)\neq w(\beta)$ and $l(\alpha)$, $w(\alpha)$, $l(\beta)$, $w(\beta)\geq 4$. Since we do not conjugate the hook, we can apply Lemma 5.16 if needed.

2nd case: $i=2$ If $\mathrm{rem}(\mu)\geq 3$, this can easily be reduced to one of the seeds $((4,2,1),(5,1^2))$, $((4,3,1),(6,1^2))$ or $((4,3,2),(7,1^2))$. More interesting is the case $\mathrm{rem}(\mu)=2$. Let $\mu=(\mu_1^{r_1},\mu_2^{r_2})$. If $\mu_1-\mu_2\geq 2$, $r_1\geq 3$ and $r_2\geq 2$, we reduce λ and μ to the seed $((9,1^2),(3^3,1^2))$. If $\mu_1-\mu_2=1$, we know that $r_1,r_2\geq 2$ and therefore, this can be reduced to the seed $((12,1^2),(4^2,3^2))$. The case $r_2=1$ is equivalent to $\mu_1-\mu_2=1$ by conjugation. If $r_1\leq 2$, we know that $\mu_2\geq 4$ since $d(\mu)\geq 4$. Therefore, we reduce this to the seed $((9,1^2),(5,3^2))$ if $r_1=1$, and to the seed $((12,1^2),(4^2,3^2))$ if $r_1=2$. All seeds have a pair of constituents which fulfill the requirements of Lemma 5.16. □

CHAPTER 7

Two-line partitions

In the previous chapter we have seen Kronecker products where one of the characters is labeled by a hook. In this chapter we do the same but for a two-line partition. The aim is to prove the following result:

Proposition 7.1. *Let $\lambda, \mu \vdash n$, where $\lambda \neq (n-1, 1)$ is a two-row partition. Then $g(\lambda, \mu) \leq 2$ if and only if one of the following holds (up to conjugation or interchanging of λ and μ if both are two-line partitions):*

(1) *The product $[\lambda][\mu]$ is multiplicity-free [BB17]. This is the case if one of the following holds:*
   (a) *One of the partitions is linear;*
   (b) *$\lambda = (n-2, 2)$ and μ is a rectangle;*
   (c) *$n = 2k+1$ and $\lambda = \mu = (k+1, k)$;*
   (d) *$n = 2k$, $\lambda = (k, k)$ and μ is (k, k), $(k+1, k-1)$, $(n-2, 2)$, $(n-3, 3)$ or a hook;*
   (e) *(λ, μ) is one of the exceptional pairs:*
$$((6,3),(3^3)),\ ((5,4),(3^3)),\ ((6^2),(4^3)).$$
(2) *$\mu = (n-2, 1^2)$;*
(3) *$\lambda = (n-2, 2)$ and μ is a two-line partition or a hook;*
(4) *$n = 2k+1$ and $\lambda = (n-2, 2)$ and $\mu = (k^2, 1)$;*
(5) *$n = ab-1$ and $\lambda = (n-2, 2)$ and $\mu = (a^{b-1}, a-1)$;*
(6) *$\lambda = (n-3, 3)$ and μ is a rectangle;*
(7) *$n = 2k+1$ and $\lambda = (k+1, k)$ and μ is a hook or $\mu = (\mu_1, \mu_2)$ with $\mu_2 \leq 3$ or $\mu_2 = k, k-1$;*
(8) *$n = 2k$, $\lambda = (k, k)$ and $\mu = (n-3, 2, 1)$ or $\mu = (\mu_1, \mu_2)$ with $\mu_2 \leq 7$ or $\mu_1 - \mu_2 \leq 6$;*
(9) *one of the exceptional cases for $n \leq 18$:*
   (a) *$\lambda = (4,3)$ and $\mu = (3, 2^2)$;*
   (b) *$\lambda = (4,4)$ and $\mu = (3, 2^2, 1)$, $(3^2, 1^2)$, $(3^2, 2)$;*
   (c) *$\lambda = \mu = (5,3)$;*
   (d) *$\lambda = (5,3)$ and $\mu = (3^2, 2)$;*
   (e) *$\lambda = (5,4)$ and $\mu = (4^2, 1)$;*
   (f) *$\lambda = (6,3)$, $(5,4)$ and $\mu = (3^3)$;*
   (g) *$\lambda = (5,5)$ and $\mu = (4, 3^2)$, $(4^2, 2)$;*
   (h) *$\lambda = \mu = (6,4)$;*
   (i) *$\lambda = (6,6)$, $(7,5)$, $(8,4)$ and $\mu = (4^3)$;*
   (j) *$\lambda = (8,7)$ and $\mu = (5^3)$;*
   (k) *$\lambda = (8,8)$ and $\mu = (4^4)$;*
   (l) *$\lambda = (9,9)$ and $\mu = (6^3)$.*

Additionally, if $g(\lambda, \mu) > 2$ and μ is symmetric, there is a $\nu \vdash n$, $\nu \neq \mu$ with $g(\lambda, \mu, \nu) > 3$.

Like in the previous chapter we start by finding the decompositions for the listed products.

1. Formulas for the stated products

Products with $[n-2,2]$.

Lemma 7.2. *Let $\lambda = (\lambda_1, \lambda_2) \vdash n$ with $\lambda_1 > \lambda_2 > 1$ and $\mu = (n-2, 2)$. The product $[\lambda][\mu]$ decomposes as follows:*

$$\begin{aligned}
&(1+\chi_{(\lambda_1-\lambda_2>1)})([\lambda]+[\lambda_1-1,\lambda_2,1]) + (1+\chi_{(\lambda_2>2)})[\lambda_1,\lambda_2-1,1]\\
&+[\lambda_1+1,\lambda_2-1] + \chi_{(\lambda_1-\lambda_2>2)}[\lambda_1-1,\lambda_2+1]\\
&+\chi_{(\lambda_1-\lambda_2>1)}[\lambda_1-2,\lambda_2,2] + \chi_{(\lambda_1-\lambda_2>3)}[\lambda_1-2,\lambda_2+2]\\
&+\chi_{(\lambda_2>3)}[\lambda_1,\lambda_2-2,2] + [\lambda_1+2,\lambda_2-2]\\
&+\chi_{(\lambda_1-\lambda_2>2)}[\lambda_1-2,\lambda_2+1,1] + \chi_{(\lambda_2>2)}[\lambda_1-1,\lambda_2-1,2]\\
&+\chi_{(\lambda_2>2)}[\lambda_1+1,\lambda_2-2,1] + [\lambda_1-1,\lambda_2-1,1^2].
\end{aligned}$$

Proof: We use Theorem 5.15 (2) to prove the formula. Let $[\nu]$ be a constituent of $[\lambda][\mu]$. We know that $|\lambda \cap \nu| \geq n-2$. We look at the multiset M of the basic skew partitions corresponding to $\lambda/(\lambda\cap\nu)$ and $\nu/(\lambda\cap\nu)$.

- The partitions for which $M = \{\square\square, \square\square\}$ are:

  $(\lambda_1-2, \lambda_2, 2)$, $(\lambda_1-2, \lambda_1+2)$, $(\lambda_1, \lambda_2-2, 2)$, $(\lambda_1+2, \lambda_2-2)$,

  where the first one only occurs if $\lambda_1-\lambda_2 > 1$, the second one only if $\lambda_1-\lambda_2 > 3$, and the third one only if $\lambda_2 > 3$. The corresponding characters occur with multiplicity 1 in $[\lambda][\mu]$.
- For the following partitions M equals $\{\square\square, \square\sqcup\square\}$:

  $(\lambda_1-2, \lambda_2+1, 1)$, $(\lambda_1-1, \lambda_2-1, 2)$, $(\lambda_1+1, \lambda_2-2, 1)$,

  where the first one only occurs if $\lambda_1-\lambda_2 > 2$, the second and the third one only if $\lambda_2 > 2$. The corresponding characters occur with multiplicity 1 in $[\lambda][\mu]$.
- For $(\lambda_1-1, \lambda_2-1, 1^2)$ M equals $\left\{\begin{smallmatrix}\square\\\square\end{smallmatrix}, \square\sqcup\square\right\}$, therefore, the multiplicity of $[\lambda_1-1, \lambda_2-1, 1^2]$ as a constituent of $[\lambda][\mu]$ equals 1.
- For $\nu \vdash n$ with $|\lambda\cap\nu| = n-2$ with a different M, $g(\lambda,\mu,\nu) = 0$.

If $|\lambda\cap\nu| = n-1$, we know that ν is one of the following partitions:

$$(\lambda_1-1, \lambda_2, 1),\ (\lambda_1-1, \lambda_2+1),\ (\lambda_1, \lambda_2-1, 1),\ (\lambda_1, \lambda_2-1),$$

where $(\lambda_1-1, \lambda_2+1)$ only occurs if $\lambda_1-\lambda_2 > 1$. For these partitions we have to look at the $\rho \vdash n-2$ which are contained in $\lambda\cap\nu$ and at the multiset $M = \{\lambda/\rho, \nu/\rho\}$.

- For $\nu = (\lambda_1-1, \lambda_2, 1)$ the possibilities for ρ are: (λ_1-2, λ_2) which only occurs if $\lambda_1-2 \geq \lambda_2$, where $M = \{\square\square, \square\sqcup\square\}$ and $(\lambda_1-1, \lambda_2-1)$, where $M = \{\square\sqcup\square, \square\sqcup\square\}$. Therefore, $g(\lambda,\mu,\nu) = \chi_{(\lambda_1-\lambda_2>1)} + 1$ since (λ_1-2, λ_2) only occurs if $\lambda_1-2 \geq \lambda_2$.
- For $\nu = (\lambda_1-1, \lambda_2+1)$ the possibilities for ρ are: (λ_1-2, λ_2) which only occurs if $\lambda_1-\lambda_2 > 1$, where $M = \{\square\square, \square\sqcup\square\}$ if $\lambda_1-2 > \lambda_2$, and $M = \left\{\square\square, \begin{smallmatrix}\square\\\square\end{smallmatrix}\right\}$ if $\lambda_1-2 = \lambda_2$. And the second possibility for ρ is $(\lambda_1-1, \lambda_2-1)$, where $M = \{\square\square, \square\sqcup\square\}$. Therefore, we obtain $g(\lambda,\mu,\nu) = \chi_{(\lambda_1-\lambda_2>2)} + 1$.
- For $\nu = (\lambda_1, \lambda_2-1, 1)$ the possibilities for ρ are: (λ_1, λ_2-2), where $M = \{\square\square, \square\sqcup\square\}$ if $\lambda_2 > 2$, and $M = \left\{\square\square, \begin{smallmatrix}\square\\\square\end{smallmatrix}\right\}$ if $\lambda_2 = 2$. The second possibility for ρ is $(\lambda_1-1, \lambda_2-1)$, where $M = \{\square\sqcup\square, \square\sqcup\square\}$. Therefore, $g(\lambda,\mu,\nu) = 1 + \chi_{(\lambda_2>2)}$.
- For $\nu = (\lambda_1+1, \lambda_2-1)$ the possibilities for ρ are: (λ_1, λ_2-2) and $(\lambda_1-1, \lambda_2-1)$. For both $M = \{\square\square, \square\sqcup\square\}$. Therefore, $g(\lambda,\mu,\nu) = 1$.

If $|\lambda \cap \nu| = n$, we know that $\lambda = \nu$. Therefore, $g(\lambda, \lambda, \mu) = 1 + \chi_{(\lambda_1 - \lambda_2 > 1)}$. $\square$

Lemma 7.3.

(1) *Let $n = ab - 1$, for $a \geq 5$ and $b \geq 3$. The product $[n-2, 2][a^{b-1}, a-1]$ decomposes as follows:*

$$\begin{aligned}
&[a^{b-3}, (a-1)^3, 1^2] + [a^{b-2}, a-1, a-2, 1^2] + [a^{b-2}, a-1, a-2, 2] \\
&+ 2[a^{b-2}, (a-1)^2, 1] + [a^{b-1}, a-3, 2] + 2[a^{b-1}, a-2, 1] + 2[a^{b-1}, a-1] \\
&+ \chi_{(b>3)}[a+1, a^{b-4}, (a-1)^3, 1] + 2[a+1, a^{b-3}a-1, a-2, 1] \\
&+ (1 + \chi_{(b>3)})[a+1, a^{b-3}, (a-1)^2] + [a+1, a^{b-2}a-3, 1] \\
&+ 2[a+1, a^{b-2}, a-2] + \chi_{(b>4)}[(a+1)^2, a^{b-5}, (a-1)^3] \\
&+ \chi_{(b>3)}[(a+1)^2, a^{b-4}, a-1, a-2] + [a+2, a^{b-3}, a-1, a-2] \\
&+ [a+2, a^{b-2}, a-3].
\end{aligned}$$

(2) *Let $n = ab + 1$ for $a \geq 3$, $b \geq 2$. The product $[a^b, 1][n-2, 2]$ decomposes as follows:*

$$\begin{aligned}
&[a^{b-2}, (a-1)^2, 1^3] + [a^{b-2}, (a-1)^2, 2, 1] + \chi_{(a>3)}[a^{b-1}, a-2, 2, 1] \\
&+ \chi_{(a>4)}[a^{b-1}, a-2, 3] + 2[a^{b-1}, a-1, 1^2] \\
&+ (1 + \chi_{(a>3)})[a^{b-1}, a-1, 2] + 2[a^b, 1] + \chi_{(b>2)}[a+1, a^{b-3}, (a-1)^2, 1^2] \\
&+ \chi_{(b>2)}[a+1, a^{b-3}, (a-1)^2, 2] + [a+1, a^{b-2}, a-2, 1^2] \\
&+ \chi_{(a>3)}[a+1, a^{b-2}, a-2, 2] + (2 + \chi_{(b>1)})[a+1, a^{b-2}, a-1, 1] \\
&+ [a+1, a^{b-1}] + \chi_{(b>3)}[(a+1)^2, a^{b-4}, (a-1)^2, 1] \\
&+ \chi_{(b>2)}[(a+1)^2, a^{b-3}, a-1] + [a+2, a^{b-2}, a-2, 1] + [a+2, a^{b-2}, a-1].
\end{aligned}$$

Proof: We use (2) of Theorem 5.15 for the proofs of both parts.

(1): Let us assume that $n = ab-1$, where $a \geq 5$ and $b \geq 3$, $\lambda = (a^{b-1}, a-1)$ and $\mu = (n-2, 2)$. If $\nu \vdash n$ with $|\lambda \cap \nu| < n-2$, then $g(\lambda, \mu, \nu) = 0$. If $|\lambda \cap \nu| = n-2$, we look at the different possibilities for the multiset $M := \{\lambda/(\lambda \cap \nu), \nu/(\lambda \cap \nu)\}$. First let us look at the partitions labeling constituents with multiplicity 2. The partition $\nu = (a+1, a^{b-3}, a-1, a-2, 1)$ is the only one for which $M = \{\square \sqcup \square, \square \sqcup \square\}$. This tells us $g(\lambda, \mu, \nu) = 2$.

In the next step we look at the partitions labeling constituents with multiplicity 1. These are the possibilities for ν such that:

- $M = \{\square \sqcup \square, \square\square\}$, which are

  $(a+2, a^{b-3}, a-1, a-2)$, $(a^{b-2}, a-1, a-2, 2)$, $(a+1, a^{b-2}, a-3, 1)$.

- $M = \left\{\square \sqcup \square, \begin{smallmatrix}\square\\\square\end{smallmatrix}\right\}$, which are

  $(a+1, a^{b-4}, (a-1)^3, 1)$, $(a^{b-2}, a-1, a-2, 1^2)$

  $((a+1)^2, a^{b-4}, a-1, a-2)$,

  where the first and the last only occur if $b > 3$.

- $M = \{\square\square, \square\square\}$, which are

  $(a+2, a^{b-2}, a-3)$ and $(a^{b-1}, a-3, 2)$.

- $M = \left\{\begin{smallmatrix}\square\\\square\end{smallmatrix}, \begin{smallmatrix}\square\\\square\end{smallmatrix}\right\}$, which are

  $((a+1)^2, a^{b-5}, (a-1)^3)$ and $(a^{b-3}, (a-1)^3, 1^2)$,

  where the first one only occurs if $b \geq 5$.

The characters labeled by these partitions are constituents of $[\lambda][\mu]$ with multiplicity 1. For the remaining partitions $\nu \vdash n$ with $|\lambda\cap\nu| = n-2$ the multiset $M = \left\{\begin{smallmatrix}\square\\\square\end{smallmatrix}, \square\!\square\right\}$. Therefore, the corresponding characters do not occur in the decomposition.

In the next step we look at the partitions $\nu \vdash n$ such that $|\lambda \cap \nu| = n - 1$.

- For the partitions $(a+1, a^{b-3}, (a-1)^2)$ and $(a^{b-2}, (a-1)^2, 1)$ the intersection with λ is given by $(a^{b-1}, (a-1)^2)$. Hence, the partitions $\rho \vdash n-2$ such that $\rho \subset \lambda \cap \nu$ are $(a^{b-3}, (a-1)^3)$ and $(a^{b-2}, a-1, a-2)$. For both partitions we have
$$g(\lambda, \mu, \nu) = 1 + 2 - 1 = 2.$$
- If ν is given by $(a+1, a^{b-2}, a-2)$ or $(a^{b-1}, a-2, 1)$, the intersection with λ is given by $\lambda \cap \nu = (a^b, a-2)$. Hence, the partitions $\rho \vdash n-2$ such that $\rho \subset \lambda \cap \nu$ are $(a^b, a-3)$ and $(a^{b-1}, a-1, a-2)$. Again for both cases we have
$$g(\lambda, \mu, \nu) = 1 + 2 - 1 = 2.$$

For the multiplicity of $[\lambda]$ in $[\lambda][\mu]$ we obtain
$$g(\lambda, \mu, \lambda) = 1 + 1 + 2(2 - 2) = 2.$$

(2): Let $n = ab + 1$ with $b \geq 2$, $a \geq 3$, $\lambda = (a^b, 1)$ and $\mu = (n-2, 2)$. For $\nu \vdash n$ with $|\lambda\cap\nu| < n-2$, we know that $[\nu]$ is not a constituent of $[\lambda][n-2, 2]$. We divide the partitions $\nu \vdash n$ with $|\lambda\cap\nu| = n-2$ by the multiset $M := \{\lambda/(\lambda\cap\nu), \nu/(\lambda\cap\nu)\}$. There is no ν such that $M = \{\square \sqcup \square, \square \sqcup \square\}$, such a ν would label a constituent with multiplicity 2.

The partitions $\nu \vdash n$ with $|\lambda \cap \nu| = n - 2$ and $g(\lambda, \mu, \nu) = 1$ are:

- The ν such that $M = \{\square \sqcup \square, \square\!\square\}$:
$(a+1, a^{b-2}, a-2, 2)$, $(a+1, a^{b-2}, a-2, 1^2)$, $(a^{b-1}, a-2, 2, 1)$, and $(a+2, a^{b-2}, a-1)$,
where the first and the third one only occur if $a > 3$.
- The ν such that $M = \left\{\square \sqcup \square, \begin{smallmatrix}\square\\\square\end{smallmatrix}\right\}$:
$$(a+1, a^{b-3}, (a-1)^2, 2),\ (a+1, a^{b-3}, (a-1)^2, 1^2),$$
$$(a^{b-2}, (a-1)^2, 2, 1), \text{and } ((a+1)^2, a^{b-3}, a-1),$$
where the first, second and fourth one only occur if $b > 2$.
- The ν such that $M = \{\square\!\square, \square\!\square\}$
$$(a+2, a^{b-2}, a-2, 1) \text{ and } (a^{b-1}, a-2, 3),$$
where the second partition only occurs if $a > 4$.
- The ν such that $M = \left\{\begin{smallmatrix}\square\\\square\end{smallmatrix}, \begin{smallmatrix}\square\\\square\end{smallmatrix}\right\}$
$$((a+1)^2, a^{b-4}, (a-1)^2, 1) \text{ and } (a^{b-2}, (a-1)^2, 1^3),$$
where the first one only appears if $b > 3$.

For the remaining partitions we have $M = \left\{\square\!\square, \begin{smallmatrix}\square\\\square\end{smallmatrix}\right\}$. Thus, the corresponding characters are no constituents of $[\lambda][n-2, 2]$.

In the next step we check the partitions $\nu \vdash n$ with $|\lambda \cap \nu| = n - 1$.

- If $\nu = (a+1, a^{b-2}, a-1, 1)$, $(a^{b-1}, a-1, 2)$ or $(a^{b-1}, a-1, 1^2)$, the intersection $\lambda \cap \nu = (a^{b-1}, a-1, 1)$, so the partitions $\rho \vdash n-2$ such that $\rho \subset \lambda \cap \nu$ are
$$(a^{b-2}, (a-1)^2, 1),\ (a^{b-1}, a-2, 1) \text{ and } (a^{b-1}, a-1).$$

With this we easily see

$$g(\lambda,(n-2,2),(a+1,a^{b-2},a-1,1)) = \begin{cases} 1+1+2-1=3, & \text{if } b>2; \\ 0+1+2-1=2, & \text{if } b=2; \end{cases}$$

$$g(\lambda,(n-2,2),(a^{b-1},a-1,2)) = \begin{cases} 1+1+1-1=2, & \text{if } a>3; \\ 1+0+1-1=1, & \text{if } a=3; \end{cases}$$

$$g(\lambda,(n-2,2),(a^{b-1},a-1,1^2)) = 1+1+1-1=2.$$

- The remaining partition $\nu \vdash n$ with $|\lambda \cap \nu| = n-1$ is $(a+1, a^{b-1})$. The intersection $\lambda \cap \nu = (a^b)$ and the only partition $\rho \vdash n-2$ with $\rho \subset \lambda \cap \nu$ is $(a^{b-1}, a-1)$. Therefore,

$$g(\lambda,(n-2,2),(a+1,a^{b-1})) = 2-1 = 1.$$

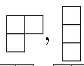
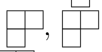
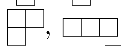
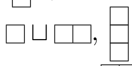
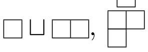
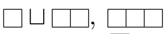
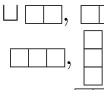
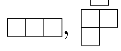
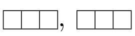
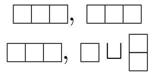
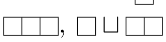
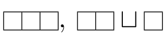
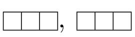
In the last step we calculate $g(\lambda,(n-2,2),\lambda) = 1+1+2(2-2) = 2$. □

Product with $[n-3,3]$**.**
Besides some exceptional products Theorem 5.1 lists two products which involve $(n-3,3)$ where the other factor is not a hook. These are the products of $(n-3,3)$ with $(k+1,k)$ and with (a^b). For these products we provide the decomposition in the following two lemmas.

Lemma 7.4. *Let* $n = 2k+1 \geq 13$*,* $\lambda = (k+1,k)$ *and* $\mu = (n-3,3)$*. The product* $[\lambda][\mu]$ *decomposes as*

$$\begin{aligned} &[\lambda] + [k^2,1] + 2[k+1,k-1,1] + [k+2,k-1] + [k,k-1,1^2] + 2[k,k-1,2] \\ &+ [k+1,k-2,1^2] + 2[k+1,k-2,2] + 2[k+2,k-2,1] + [k+3,k-2] \\ &+ [(k-1)^2,2,1] + [k,k-2,2,1] + [k,k-2,3] + [k+1,k-3,3] \\ &+ [k+2,k-3,2] + [k+3,k-3,1] + [k+4,k-3]. \end{aligned}$$

Proof: We use Theorem 5.15 (4) and (5) for this proof. Let $[\nu]$ be a constituent of $[\lambda][\mu]$. We know that $|\nu \cap \lambda| \geq n-3$. We start by investigating the $\nu \vdash n$ with $|\lambda \cap \nu| = n-3$. We look again at $M = \{\lambda/(\lambda\cap\nu), \nu/(\lambda\cap\nu)\}$. In contrast to the previous lemma not many partitions have the same multiset M. Therefore we just list all partitions $\nu \vdash n$ with $|\nu \cap \lambda| = n-3$. These are:

ν	M	$g(\lambda,\mu,\nu)$
$((k-1)^2,1^3)$		0
$((k-1)^2,2,1)$		1
$((k-1)^2,3)$		0
$(k,k-2,1^3)$		0
$(k,k-2,2,1)$		1
$(k,k-2,3)$		1
$(k+1,k-3,1^3)$		0
$(k+1,k-3,2,1)$		0
$(k+1,k-3,3)$		1
$(k+2,k-3,1^2)$		0
$(k+2,k-3,2)$		1
$(k+3,k-3,1)$		1
$(k+4,k-3)$		1.

Since we assume that $k \geq 6$, all these partitions exist.

If $|\lambda \cap \nu| = n-2$, we know that ν is one of the following partitions:

$$(k, k-1, 1^2),\ (k, k-1, 2),\ (k+1, k-2, 1^2),\ (k+1, k-2, 2),$$
$$(k+2, k-2, 1),\ (k+3, k-2).$$

For the partitions $\rho \vdash n-3$ with $\rho \subset \lambda \cap \nu$ we have to examine the multisets $M = \{\lambda/\rho, \nu/\rho\}$ and $N = \{\lambda/(\lambda \cap \nu), \nu/(\lambda \cap \nu)\}$ for the above listed possibilities for ν.

- If $\nu = (k, k-1, 1^2)$, the possibilities for ρ are
$$((k-1)^2),\ \text{where } M = \left\{\yng(2,1), \square \sqcup \yng(1,1)\right\} \text{ and}$$
$$(k, k-2),\ \text{where } M = \left\{\square \sqcup \yng(2), \square \sqcup \yng(1,1)\right\}.$$
Further, $\lambda \cap \nu = (k, k-1)$ and $N = \left\{\square \sqcup \square, \yng(1,1)\right\}$. This tells us that
$$g(\lambda, \mu, \nu) = 1 + 1 - 1 = 1.$$
- If $\nu = (k, k-1, 2)$, the possibilities for ρ are
$$((k-1)^2),\ \text{where } M = \left\{\yng(2,1), \square \sqcup \yng(2)\right\} \text{ and}$$
$$(k, k-2),\ \text{where } M = \{\square \sqcup \yng(2), \square \sqcup \yng(2)\}.$$
Further, $\lambda \cap \nu = (k, k-1)$ and $N = \{\square \sqcup \square, \yng(2)\}$. This tells us that
$$g(\lambda, \mu, \nu) = 1 + 2 - 1 = 2.$$
- If $\nu = (k+1, k-2, 1^2)$, the possibilities for ρ are
$$(k, k-2),\ \text{where } M = \left\{\square \sqcup \yng(2), \square \sqcup \yng(1,1)\right\} \text{ and}$$
$$(k+1, k-3),\ \text{where } M = \left\{\yng(3), \square \sqcup \yng(1,1)\right\}.$$
Further, $\lambda \cap \nu = (k+1, k-2)$ and $N = \left\{\yng(2), \yng(1,1)\right\}$. This tells us that
$$g(\lambda, \mu, \nu) = 1 + 0 - 0 = 1.$$
- If $\nu = (k+1, k-2, 2)$, the possibilities for ρ are
$$(k, k-2),\ \text{where } M = \{\square \sqcup \yng(2), \square \sqcup \yng(2)\} \text{ and}$$
$$(k+1, k-3),\ \text{where } M = \{\yng(3), \square \sqcup \yng(2)\}.$$
Further, $\lambda \cap \nu = (k+1, k-2)$ and $N = \{\yng(2), \yng(2)\}$. This tells us that
$$g(\lambda, \mu, \nu) = 2 + 1 - 1 = 2.$$
- If $\nu = (k+2, k-2, 1)$, the possibilities for ρ are
$$(k, k-2),\ \text{where } M = \{\square \sqcup \yng(2), \square \sqcup \yng(2)\} \text{ and}$$
$$(k+1, k-3),\ \text{where } M = \{\yng(3), \square \sqcup \square \sqcup \square\}.$$
Further, $\lambda \cap \nu = (k+1, k-2)$ and $N = \{\yng(2), \square \sqcup \square\}$. This tells us that
$$g(\lambda, \mu, \nu) = 2 + 1 - 1 = 2.$$
- If $\nu = (k+3, k-2)$, the possibilities for ρ are
$$(k, k-2),\ \text{where } M = \{\square \sqcup \yng(2), \yng(3)\} \text{ and}$$
$$(k+1, k-3),\ \text{where } M = \{\yng(3), \yng(2) \sqcup \square\}.$$
Further, $\lambda \cap \nu = (k+1, k-2)$ and $N = \{\yng(2), \yng(2)\}$. This tells us that
$$g(\lambda, \mu, \nu) = 1 + 1 - 1 = 1.$$

If $|\lambda \cap \nu| = n-1$, ν is one of the following partitions:

$$(k^2, 1), \ (k+1, k-1, 1), \ (k+2, k-1).$$

For these ν we have to find the partitions $\rho \vdash n-3$ and $\sigma \vdash n-2$ such that $\rho, \sigma \subset \lambda \cap \nu$ and examine the basic skew diagrams which correspond to the multisets $M = \{\lambda/\rho, \nu/\rho\}$ and $N = \{\lambda/\sigma, \nu/\sigma\}$.

- If $\nu = (k^2, 1)$, the possibilities for ρ are:

  $$(k, k-2), \ \text{where } M = \{\square \sqcup \square\square, \square \sqcup \square\square\} \text{ and}$$
  $$((k-1)^2), \ \text{where } M = \left\{\begin{smallmatrix}\square\square\\ \square\end{smallmatrix}, \begin{smallmatrix}\square\\ \square\end{smallmatrix} \sqcup \square\right\}.$$

  The only possibility for σ is $(k, k-1)$, where both skew diagrams in N are of the form $\square \sqcup \square$. This tells us that $g(\lambda, \mu, \nu) = 1$.
- If $\nu = (k+1, k-1, 1)$, the possibilities for ρ are:

  $$(k, k-2), \ \text{where } M = \{\square \sqcup \square\square, \square \sqcup \square \sqcup \square\};$$
  $$((k-1)^2), \ \text{where } M = \left\{\begin{smallmatrix}\square\square\\ \square\end{smallmatrix}, \square\square \sqcup \square\right\} \text{ and}$$
  $$(k+1, k-3), \ \text{where } M = \{\square\square\square, \square\square \sqcup \square\}.$$

  For σ we have the following possibilities:

  $$(k, k-1), \ \text{where } N = \{\square \sqcup \square, \square \sqcup \square\} \text{ and}$$
  $$(k+1, k-2), \ \text{where } N = \{\square\square, \square \sqcup \square\}.$$

  This tells us that $g(\lambda, \mu, \nu) = 2$.
- If $\nu = (k+2, k-1)$, the possibilities for ρ are:

  $$(k, k-2), \ \text{where } M = \{\square \sqcup \square\square, \square \sqcup \square\square\};$$
  $$((k-1)^2), \ \text{where } M = \left\{\begin{smallmatrix}\square\square\\ \square\end{smallmatrix}, \square\square\square\right\} \text{ and}$$
  $$(k+1, k-3), \ \text{where } M = \{\square \sqcup \square\square, \square\square\square\}.$$

  For σ we have the possibilities $(k, k-1)$ and $(k+1, k-2)$, where for both possibilities $N = \{\square\square, \square \sqcup \square\}$. This tells us that $g(\lambda, \mu, \nu) = 1$.

If $\lambda = \nu$, we calculate with Theorem 5.15 (4) that $g(\lambda, \lambda, \mu) = 1$. □

Lemma 7.5. *Let $n = ab$, where $a \geq 6, b \geq 3$. The product $[a^b][n-3, 3]$ decomposes as follows:*

$$\begin{aligned}
&[a^{b-3}, (a-1)^3, 1^3] + [a^{b-2}, a-1, a-2, 2, 1] + [a^{b-2}, (a-1)^2, 1^2] \\
&+ [a^{b-2}, (a-1)^2, 2] + [a^{b-1}, a-3, 3] + [a^{b-1}, a-2, 1^2] + [a^{b-1}, a-2, 2] \\
&+ 2[a^{b-1}, a-1, 1] + [a^b] + \chi_{(b>3)}[a+1, a^{b-4}, (a-1)^3, 1^2] \\
&+ [a+1, a^{b-3}, a-1, a-2, 1^2] + [a+1, a^{b-3}, a-1, a-2, 2] \\
&+ (1 + \chi_{(b>3)})[a+1, a^{b-3}, (a-1)^2, 1] + [a+1, a^{b-2}, a-3, 2] \\
&+ 2[a+1, a^{b-2}, a-2, 1] + (1 + \chi_{(b>3)})[a+1, a^{b-2}, a-1] \\
&+ \chi_{(b>4)}[(a+1)^2, a^{b-5}, (a-1)^3, 1] + \chi_{(b>3)}[(a+1)^2, a^{b-4}, a-1, a-2, 1] \\
&+ \chi_{(b>4)}[(a+1)^2, a^{b-4}, (a-1)^2] + [(a+1)^2, a^{b-3}, a-2] \\
&+ \chi_{(b>5)}[(a+1)^3, a^{b-6}, (a-1)^3] + [a+2, a^{b-3}, a-1, a-2, 1] \\
&+ [a+2, a^{b-3}, (a-1)^2] + [a+2, a^{b-2}, a-3, 1] + [a+2, a^{b-2}, a-2] \\
&+ \chi_{(b>3)}[a+2, a+1, a^{b-4}, a-1, a-2] + [a+3, a^{b-2}, a-3].
\end{aligned}$$

Proof: As for the previous formula we use Theorem 5.15 (4) and (5). Let $\lambda = (a^b) \vdash n$, $\mu = (n-3, 3)$ and $\nu \vdash n$.

If $|\lambda \cap \nu| < n-3$, $g(\lambda, \mu, \nu) = 0$. If $|\lambda \cap \nu| = n-3$, we look at the different possibilities for the multiset $M := \{\lambda/(\lambda \cap \nu), \nu/(\lambda \cap \nu)\}$. There is no partition $\nu \vdash n$ with $|\lambda \cap \nu| = n-3$ such that M contains $\square \sqcup \square \sqcup \square$ or that M is one of the following multisets

$$\{\square \sqcup \square\square, \square \sqcup \square\square\}, \left\{\square \sqcup \square\square, \square \sqcup \begin{smallmatrix}\square\\\square\end{smallmatrix}\right\}, \left\{\square \sqcup \begin{smallmatrix}\square\\\square\end{smallmatrix}, \square \sqcup \begin{smallmatrix}\square\\\square\end{smallmatrix}\right\}$$

(it is easy to see that $\lambda/(\lambda \cap \nu)$ is connected). So all constituents $[\nu]$ for $\nu \vdash n$ of $[\lambda][\mu]$ with $|\lambda \cap \nu| = n-3$ have multiplicity 1. They are labeled by the ν such that:

- $M = \{\square \sqcup \square\square, \square\square\square\}$, which are
$$(a+2, a^{b-2}, a-3, 1),\ (a+1, a^{b-2}, a-3, 2).$$
- $M = \left\{\square \sqcup \square\square, \begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}^{(\mathrm{rot})}\right\}$, which are
$$(a+2, a^{b-3}, a-1, a-2, 1),\ (a+1, a^{b-3}, a-1, a-2, 2).$$
- $M = \left\{\square \sqcup \begin{smallmatrix}\square\\\square\end{smallmatrix}, \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}\right\}$, which are
$$((a+1)^2, a^{b-5}, (a-1)^3, 1),\ (a+1, a^{b-4}, (a-1)^3, 1^2),$$
where the first one only occurs if $b \geq 5$ and the second one only if $b \geq 4$.
- $M = \left\{\square \sqcup \begin{smallmatrix}\square\\\square\end{smallmatrix}, \begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}^{(\mathrm{rot})}\right\}$, which are
$$((a+1)^2, a^{b-4}, a-1, a-2, 1),\ (a+1, a^{b-3}, a-1, a-2, 1^2),$$
where the first one only occurs if $b \geq 4$.
- $M = \{\square\square\square, \square\square\square\}$, which are
$$(a+3, a^{b-2}, a-3), (a^{b-1}, a-3, 3).$$
- $M = \left\{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}, \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}\right\}$, which are
$$((a+1)^3, a^{b-6}, (a-1)^3),\ (a^{b-3}, (a-1)^3, 1^3),$$
where the first one only occurs if $b \geq 6$.
- $M = \left\{\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}^{(\mathrm{rot})}, \begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}^{(\mathrm{rot})}\right\}$, which are are
$$(a+2, a+1, a^{b-4}, a-1, a-2),\ (a^{b-2}, a-1, a-2, 2, 1),$$
where the first one only occurs if $b \geq 4$.

If ν is one of these partitions, we know

$$g(\lambda, \mu, \nu) = 1.$$

The remaining partitions $\nu \vdash n$ with $|\lambda \cap \nu| = n-3$ have a different M and therefore, we know that $g(\lambda, \mu, \nu) = 0$.

Now let us investigate $\nu \vdash n$ with $|\lambda \cap \nu| = n-2$. For the partitions

$$(a+2, a^{b-3}, (a-1)^2), ((a+1)^2, a^{b-4}, (a-1)^2), (a+1, a^{b-3}, (a-1)^2, 1),$$
$$(a^{b-2}, (a-1)^2, 2), (a^{b-2}, (a-1)^2, 1^2)$$

the intersection $\lambda \cap \nu = (a^{b-2}, (a-1)^2)$. The possibilities for $\rho \vdash n-3$ such that $\rho \subset \lambda \cap \nu$ are $(a^{b-3}, (a-1)^3)$ and $(a^{b-2}, a-1, a-2)$. With this we obtain

$$
\begin{aligned}
&g(\lambda, \mu, (a+2, a^{b-3}, (a-1)^2)) = 0 + 1 - 0 = 1,\\
&g(\lambda, \mu, ((a+1)^2, a^{b-4}, (a-1)^2)) = \begin{cases} 1+1-1=1, & \text{if } b>4 \\ 0+1-1=0, & \text{if } b=4 \end{cases},\\
&g(\lambda, \mu, (a+1, a^{b-3}, (a-1)^2, 1)) = \begin{cases} 1+2-1=2, & \text{if } b>3 \\ 0+2-1=1, & \text{if } b=3 \end{cases},\\
&g(\lambda, \mu, (a^{b-2}, (a-1)^2, 2)) = 0 + 1 - 0 = 1,\\
&g(\lambda, \mu, (a^{b-2}, (a-1)^2, 1^2)) = 1 + 1 - 1 = 1.
\end{aligned}
$$

The different cases occur because for small b sometimes the elements of M have a different shape (less connected components) than for the generic case. This happens for example for $(a+2, a^{b-3}, (a-1)^2)$, too, but in that case it has no influence on the result. The remaining possibilities for ν such that $|\lambda \cap \nu| = n-2$ are:

$$
\begin{aligned}
&(a+2, a^{b-2}, a-2),\ ((a+1)^2, a^{b-3}, a-2),\ (a+1, a^{b-2}, a-2, 1),\\
&(a^{b-1}, a-2, 2),\ (a^{b-1}, a-2, 1^2).
\end{aligned}
$$

If ν is one of these partitions, then $\lambda \cap \nu = (a^{b-1}, a-2)$. In this case the possibilities for $\rho \vdash n-3$ such that $\rho \subset \lambda \cap \nu$ are $(a^{b-2}, a-1, a-2)$ and $(a^{b-1}, a-3)$. From this we see that

$$
\begin{aligned}
&g(\lambda, \mu, (a+2, a^{b-2}, a-2)) = 1 + 1 - 1 = 1,\\
&g(\lambda, \mu, ((a+1)^2, a^{b-3}, a-2)) = 1 + 0 - 0 = 1,\\
&g(\lambda, \mu, (a+1, a^{b-2}, a-2, 1)) = 2 + 1 - 1 = 2,\\
&g(\lambda, \mu, (a^{b-1}, a-2, 2)) = 1 + 1 - 1 = 1,\\
&g(\lambda, \mu, (a^{b-1}, a-2, 1^2)) = 1 + 0 - 0 = 1.
\end{aligned}
$$

We could obtain this also from the case $\lambda \cap \nu = (a^{b-2}, (a-1)^2)$ since the tables of Theorem 5.15 are symmetric. However, since we assume that $a \geq 6$, the cases for the small b have no counterpart but we would have to argue why there occur no special cases for $3 \leq b \leq 5$.

In the next step we look at the partitions $\nu \vdash n$ such that $|\lambda \cap \nu| = n-1$. These are $(a^{b-1}, a-1, 1)$ and $(a+1, a^{b-2}, a-1)$. In both cases $\lambda \cap \nu = (a^{b-1}, a-1)$. The partitions $\rho \vdash n-3$ such that $\rho \subset \lambda \cap \nu$ are $(a^{b-3}, (a-1)^3)$, $(a^{b-2}, a-2, a-1)$ and $(a^{b-1}, a-3)$. The partitions $\rho \vdash n-2$ such that $\rho \subset \lambda \cap \nu$ are $(a^{b-2}, (a-1)^2)$ and $(a^{b-1}, a-2)$. This tells us that

$$
\begin{aligned}
&g(\lambda, \mu, (a^{b-1}, a-1, 1)) = 1 + 2 + 1 - 1 - 1 = 2,\\
&g(\lambda, \mu, (a+1, a^{b-2}, a-1)) = \begin{cases} 1+2+1-1-1=2, & \text{if } b>3 \\ 0+2+1-1-1=1, & \text{if } b=3 \end{cases}.
\end{aligned}
$$

Finally we calculate with Theorem 5.15 (5) that $g(\lambda, \mu, \lambda) = 1$. □

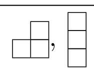
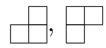
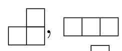
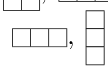
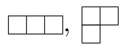
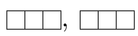
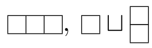
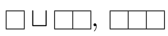
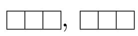
Products with $[k, k]$.

There is one last decomposition which we can prove with Theorem 5.15.

Lemma 7.6. *Let $n = 2k \geq 10$. The product $[k,k][n-3,2,1]$ decomposes as follows:*

$$2[k,k-1,1] + [k+1,k-1] + [(k-1)^2,1^2] + [(k-1)^2,2] + 2[k,k-2,1^2]+$$
$$2[k,k-2,2] + 2[k+1,k-2,1] + [k+2,k-2] + [k-1,k-2,1^3]+$$
$$[k-1,k-2,2,1] + [k-1,k-2,3] + [k,k-3,2,1] + [k+1,k-3,1^2]+$$
$$[k+1,k-3,2] + [k+2,k-3,1]$$

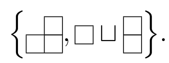
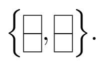
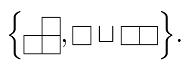
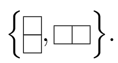
Proof: The result will follow using Theorem 5.15 (6) and (7). Let $\lambda = (k,k)$, $\mu = (n-3,2,1)$ and $[\nu]$ be a constituent of $[\lambda][\mu]$. We know that $|\lambda \cap \nu| \geq n-3$. We start with the partitions $\nu \vdash n$ with $|\lambda \cap \nu| = n-3$. For these ν we have to check the multiset $M = \{\lambda/(\lambda\cap\nu), \nu/(\lambda\cap\nu)\}$. Since there are not many partitions with the same multiset M we list all partitions $\nu \vdash n$ with $|\lambda \cap \nu| = n-3$, like in Lemma 7.4. We get the following results:

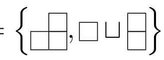
ν	M	$g(\lambda,\mu,\nu)$
$(k-1,k-2,1^3)$		1
$(k-1,k-2,2,1)$		1
$(k-1,k-2,3)$		1
$(k,k-3,1^3)$		0
$(k,k-3,2,1)$		1
$(k,k-3,3)$		0
$(k+1,k-3,1^2)$		1
$(k+1,k-3,2)$		1
$(k+2,k-3,1)$		1
$(k+3,k-3)$		0.

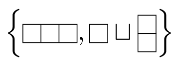
If $|\lambda \cap \nu| = n-2$, we know that ν is one of the following partitions:

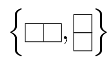
$$((k-1)^2,1^2),\ ((k-1)^2,2),\ (k,k-2,1^2),\ (k,k-2,2),$$
$$(k+1,k-2,1),\ (k+2,k-2).$$

For these possibilities for ν we have to find the partitions $\rho \vdash n-3$, with $\rho \subset \lambda \cap \nu$ and check the multisets $M = \{\lambda/\rho, \nu/\rho\}$ and $N = \{\lambda/(\lambda\cap\nu), \nu/(\lambda\cap\nu)\}$.

- If $\nu = ((k-1)^2,1^2)$, $\rho = (k-1,k-2)$ and $M =$ . Further, $\lambda \cap \nu = ((k-1)^2)$ and $N =$ . Therefore, $g(\lambda,\mu,\nu) = 2-1 = 1$.
- If $\nu = ((k-1)^2,2)$, $\rho = (k-1,k-2)$ and $M =$ . Further, $\lambda \cap \nu = ((k-1)^2)$ and $N =$ . Therefore, $g(\lambda,\mu,\nu) = 2-1 = 1$.
- If $\nu = (k,k-2,1^2)$, we know that:

  ρ is either $(k-1,k-2)$, where $M =$ 

  or $(k,k-3)$, where $M =$ .

  Additionally, we see $\lambda \cap \nu = (k,k-2)$ and $N =$ . Therefore, $g(\lambda,\mu,\nu) = 2+1-1 = 2$.

- If $\nu = (k, k-2, 2)$, we know that ρ is either

$$\begin{aligned} (k-1, k-2),\ \text{where } M &= \left\{\ydiagram{1+1,2}, \square \sqcup \ydiagram{2}\right\} \\ \text{or } (k, k-3),\ \text{where } M &= \{\ydiagram{3}, \square \sqcup \ydiagram{2}\}\,. \end{aligned}$$

Additionally, we see $\lambda \cap \nu = (k, k-2)$ and $N = \{\ydiagram{2}, \ydiagram{2}\}$. Therefore, $g(\lambda, \mu, \nu) = 2 + 1 - 1 = 2$.
- If $\nu = (k+1, k-2, 1)$, we know that ρ is either

$$\begin{aligned} (k-1, k-2),\ \text{where } M &= \left\{\ydiagram{1+1,2}, \ydiagram{2} \sqcup \square\right\} \\ \text{or } (k, k-3),\ \text{where } M &= \{\ydiagram{3}, \square \sqcup \square \sqcup \square\}\,. \end{aligned}$$

Additionally, we see $\lambda \cap \nu = (k, k-2)$ and $N = \{\ydiagram{2}, \square \sqcup \square\}$. Therefore, $g(\lambda, \mu, \nu) = 2 + 2 - 2 = 2$.
- The last possibility for $|\lambda \cap \nu| = n-2$ is $\nu = (k+2, k-2)$. Here ρ is either

$$\begin{aligned} (k-1, k-2),\ \text{where } M &= \left\{\ydiagram{1+1,2}, \ydiagram{3}\right\} \\ \text{or } (k, k-3),\ \text{where } M &= \{\ydiagram{3}, \ydiagram{2} \sqcup \square\}\,. \end{aligned}$$

Additionally, we see $\lambda \cap \nu = (k, k-2)$ and $N = \{\ydiagram{2}, \ydiagram{2}\}$. Therefore, $g(\lambda, \mu, \nu) = 1 + 1 - 1 = 1$.

If $|\lambda \cap \nu| = n-1$, we know that ν is $(k, k-1, 1)$ or $(k+1, k-1)$. For $\rho \vdash n-3$, $\pi \vdash n-2$ with $\rho, \pi \subset \lambda \cap \nu$, we have to look at the multisets $M = \{\lambda/\rho, \nu/\rho\}$ and $N = \{\lambda/\pi, \nu/\pi\}$.

- If $\nu = (k, k-1, 1)$, the possibilities for ρ and the corresponding basic skew diagrams are:

$$\begin{aligned} (k-1, k-2),\ \text{where } M &= \left\{\ydiagram{1+1,2},\ \square \sqcup \square \sqcup \square\right\} \\ \text{and } (k, k-3),\ \text{where } M &= \{\ydiagram{3},\ \ydiagram{2} \sqcup \square\}\,. \end{aligned}$$

The possibilities for π and the corresponding basic skew diagrams are:

$$\begin{aligned} (k-1, k-1),\ \text{where } N &= \left\{\ydiagram{1,1},\ \square \sqcup \square\right\} \\ \text{and } (k, k-2),\ \text{where } N &= \{\ydiagram{2},\ \square \sqcup \square\}\,. \end{aligned}$$

Therefore, $g(\lambda, \mu, \nu) = 4 + 1 - 2 - 2 + 1 = 2$.
- If $\nu = (k+1, k-1)$, the possibilities for ρ and the corresponding basic skew diagrams are:

$$\begin{aligned} (k-1, k-2),\ \text{where } M &= \left\{\ydiagram{1+1,2},\ \ydiagram{2} \sqcup \square\right\} \\ \text{and } (k, k-3),\ \text{where } M &= \{\ydiagram{3},\ \square \sqcup \ydiagram{2}\}\,. \end{aligned}$$

The possibilities for π and the corresponding basic skew diagrams are:

$$\begin{aligned} (k-1, k-1),\ \text{where } N &= \left\{\ydiagram{1,1},\ \ydiagram{2}\right\} \\ \text{and } (k, k-2),\ \text{where } N &= \{\ydiagram{2},\ \square \sqcup \square\}\,. \end{aligned}$$

Therefore, $g(\lambda, \mu, \nu) = 2 + 1 - 1 - 2 + 1 = 1$.

If $\lambda = \nu$, we see directly from (7) of Theorem 5.15 that $g(\lambda, \lambda, \mu) = 0$. □

For the next products with $[k, k]$ we use the combinatorial interpretation of special Kronecker coefficients from [BO06]. For this we use Kronecker tableaux so we need the following two definitions:

Definition 7.7. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_l)$ be a sequence of non-negative integers. A sequence $a_1 a_2 \cdots a_n$ is an *α-lattice permutation* if in any initial factor $a_1 a_2 \cdots a_j$, $1 \le j \le l$, we have for any positive integer i:

$$\text{the number of } i\text{'s} + \alpha_i \ge \text{the number of } (i+1)\text{'s} + \alpha_{i+1}.$$

So lattice words (which we used for the Littlewood-Richardson rule) are exactly the $(0, \dots, 0)$-lattice permutations.

Definition 7.8. A semi-standard Young tableau of shape λ/α and type ν/α whose reverse reading word is an α-lattice permutation is called a *Kronecker tableau* of shape λ/α and type ν/α if

(1) $\alpha_1 = \alpha_2$ or
(2) $\alpha_1 > \alpha_2$ and one of the following two conditions is satisfied:
   (a) The number of 1s in the second row of λ/α is exactly $\alpha_1 - \alpha_2$.
   (b) The number of 2s in the first row of λ/α is exactly $\alpha_1 - \alpha_2$.

Proposition 7.9. *Let $n = 2k$ and $p \in \mathbb{N}$ such that $k \ge 2p$. The product $[n-p, p][k^2]$ is stable. That means that for all $\nu \vdash n$*

$$g((n-p, p), (k, k), \nu) = g((n+2-p, p), (k+1, k+1), \nu + (1^2)).$$

Furthermore, all constituents of $[n+2-p, p][k+1, k+1]$ can be obtained from partitions of n by adding (1^2).

Proof: Let $k \ge 2p$. Then we know from [BO06, Theorem 3.2.] that

$$g((k,k)(n-p,p), \nu) = \sum_{\substack{\alpha \vdash p \\ \alpha \subset (k,k) \cap \nu}} k_{\alpha,\nu}^{(k,k)},$$

where $k_{\alpha,\nu}^{(k,k)}$ equals the number of Kronecker tableaux of shape $(k,k)/\alpha$ and type ν/α.

First, we collect some properties of the constituents of $[n-p, p][k, k]$ from Theorem 5.5. If $g((n-p, p), (k^2), \nu) > 0$, we know that $l(\nu) \le 4$ so $\nu = (k+a, k+b, c, d)$. Since $|\nu \cap (k^2)| \ge n - p$, we know that $-p \le b \le a \le p$, $0 \le d \le c \le p$ and $-p \le a + b$, $c + d \le p$, where $a + b + c + d = 0$. Since we assume that $k \ge 2p$, all these possibilities actually provide partitions of n.

Let us assume we have a Kronecker tableau T of shape $(k, k)/\alpha$ and type ν/α. We know that all the 4s have to be in the second row of T, because there is no box in the first row without a box underneath it in the second row. Since T needs to be semi-standard, we know that the ν_4 rightmost boxes of the second row are filled with 4s. Further, the reverse reading word has to be an α-lattice permutation and $l(\alpha) \le 2$ so there have to be at least λ_4 3s in the first row. Again, they have to be in the λ_4 rightmost boxes. Since every 3 in the first row has to have a 4 in the box underneath it, we know that there are no more 3s in the first row. The remaining 3s are in the second row directly to the left of the 4s. So only for the 1s and 2s there could be some choice. For a Kronecker tableau we have the different possibilities:

(1) $\alpha_1 = \alpha_2$ or
(2) $\alpha_1 > \alpha_2$ and one of the following two conditions is satisfied:
   (a) The number of 1s in the second row of λ/α is exactly $\alpha_1 - \alpha_2$.
   (b) The number of 2s in the first row of λ/α is exactly $\alpha_1 - \alpha_2$.

If T is of type (1), we know that there is no 2 in the first row. Otherwise, the reverse reading word would not be an α-lattice permutation. If T is of type (2), the number of 1s in the second row resp. the number of 2s in the first row is fixed. Therefore, we know that for a given α with $\alpha_1 = \alpha_2$ there is at most one Kronecker tableau of $(k, k)/\alpha$ and type ν/α and for α with $\alpha_1 > \alpha_2$ there are at most two: at most one of type (a) and one of type (b) but we have to be careful, a tableau can be of type (a) and (b) at the same time). Now we look at the different types and show that the existence is independent of k.

1st case: p is even and $\alpha_1 = \alpha_2 = \frac{p}{2}$. The rightmost ν_4 entries of the first row are 3s. This can only lead to an α-lattice permutation if $\nu_4 \leq \alpha_2$. The remaining entries in the first row have to be 1s because $\alpha_1 = \alpha_2$ so there are no 2s in the first row. Since above every box in the second row is one in the first row, there cannot be any 1s in the second row. Therefore, we know that $\nu_1 - \alpha_1 = k - \alpha_1 - \nu_4$. If we substitute ν_1 by $k + a$, we obtain $a = -\nu_4$. This is independent of k. The ν_4 rightmost entries of the second row are the 4s. The next entries are the remaining $\nu_3 - \nu_4$ 3s. Since all the 2s are in the second row, this can only be an α-lattice permutation if $\nu_3 \leq \alpha_2 = \frac{p}{2}$. The remaining $k - \alpha_2 - \nu_3$ entries have to be the 2s which gives $k - \alpha_2 - \nu_3 = \nu_2 - \alpha_2$. Here we substitute ν_2 by $k + b$ to obtain $-\nu_3 = b$ which again is independent of k. But $-\nu_3 = b$ is just equivalent to $a = -\nu_4$ since $a + b + \nu_3 + \nu_4 = 0$. We see that such a tableau exists if and only if $\nu = (k - a, k - b, b, a)$ for some $0 \leq a \leq b \leq \frac{p}{2}$. The other cases are a bit more involved.

2nd case: $\alpha_1 > \alpha_2$ and there are exactly $\alpha_1 - \alpha_2$ 1s in the second row. We know that in the second row from right to left there are ν_4 4s, then $\nu_3 - \nu_4$ 3s, next $k - \nu_3 - \alpha_1$ 2s, and finally $\alpha_1 - \alpha_2$ 1s. The remaining entries have to be in the first row. We fill the boxes in the same order as we read the reverse reading word and check for the conditions when this is a Kronecker tableau. In the first row we only have to check that the reverse reading word is an α-lattice permutation. In the second row we also have to check that it is a semi-standard tableau. We start in the first row at the right with ν_4 3s. So far this is a Kronecker tableau if and only if $\nu_4 \leq \alpha_2$. Then there are the remaining $\nu_2 - \alpha_2 - k + \nu_3 + \alpha_1$ 2s (since we have $\nu_2 - \alpha_2$ 2s in total and $k - \nu_3 - \alpha_1$ 2s are in the second row). Since $\nu_2 = k + b$, this equals $b - \alpha_2 + \nu_3 + \alpha_1$. We obtain $b - \alpha_2 + \nu_3 + \alpha_1 \leq \alpha_1 - \alpha_2$ which is equivalent to $b + \nu_3 \leq 0$. In both rows the rightmost ν_4 entries and the α_1 leftmost boxes are occupied with numbers different from 2 or cut out. This will only be a Kronecker tableau if the number of 2s is at most $k - \alpha_1 - \nu_4$. This leads to the condition $-\alpha_1 - \nu_4 \geq b - \alpha_2$. The leftmost entries in the first row are the $\nu_1 - \alpha_1 - (\alpha_1 + \alpha_2)$ 1s, which are not in the second row. The α-lattice permutation condition is no problem here but there are just $k - \alpha_2 - \nu_4$ boxes which can be filled with 1s. Therefore, $-\alpha_2 - \nu_4 \geq a - \alpha_1$. The second row starts with the ν_4 4s. Above them there are ν_4 3s so there is no problem with them. Next we have $\nu_3 - \nu_4$ 3s. If the reverse reading word is an α-lattice permutation, we know that number of 2s that we have counted so far is $\nu_2 - \alpha_2 - k + \nu_3 + \alpha_1$ plus α_2. This has to be greater or equal to ν_3. We obtain $b + \alpha_1 \geq 0$. Further, under every 2 of the first row there needs to be a 3. Otherwise, it would be a 2 and the tableau would not ne semi-standard. The 2s in the first row and the 3s in the second row start in the same column so $\nu_2 - \alpha_2 - k + \nu_3 + \alpha_1 \leq \nu_3 - \nu_4$ which simplifies to $b + \alpha_1 + \nu_4 \leq \alpha_2$. Since $b + \alpha_1 \geq 0$, this already implies $\nu_4 \leq \alpha_2$, which we got in the first row. The 2s in the second row are no problem for the α-lattice permutation condition and we have already ensured that there are enough 2s so that not two 1s are in the same column by the condition we have on the number of 1s and 2s. In total we have the following conditions, none of them depending on k:

(1) $b + \nu_3 \leq 0$;
(2) $-\alpha_1 - \nu_4 \geq b - \alpha_2$;
(3) $-\alpha_2 - \nu_4 \geq a - \alpha_1$;
(4) $b + \alpha_1 \geq 0$;
(5) $b + \alpha_1 + \nu_4 \leq \alpha_2$.

3rd case: $\alpha_1 > \alpha_2$ and there are exactly $\alpha_1 - \alpha_2$ 2s in the first row. We know that the first row starts with ν_4 3s. The α-lattice permutation condition tells us that $\alpha_2 \geq \nu_4$. Next, there are $\alpha_1 - \alpha_2$ 2s. The last $k - \alpha_1 - \alpha_1 + \alpha_2 - \nu_4$ free boxes in the first row are filled with 1s. In the second row, we can have at most $\alpha_1 - \alpha_2$ additional 1s. Therefore, we obtain $\nu_1 - \alpha_1 \leq k - \alpha_1 - \nu_4$ which simplifies to $a \leq -\nu_4$. But we also need at least $k - \alpha_1 - \alpha_1 + \alpha_2 - \nu_4$ 1s to fill the remaining boxes in the first row. This leads to $a \geq -\alpha_1 + \alpha_2 - \nu_4$. In the second row the first ν_4 entries are the 4s, then follow $\nu_3 - \nu_4$ 3s. These have to be above all the 2s in the first row. Otherwise, the tableau would not be semi-standard. We obtain $\nu_3 - \nu_4 \geq \alpha_1 - \alpha_2$ and ν_3 has to be smaller or equal to the number of 2s we used so far plus α_2. This leads to $\nu_3 \leq \alpha_1$. Next, there are the remaining $\nu_2 - \alpha_2 - \alpha_1 + \alpha_2$ 2s. The number of 2s plus α_2 has to be less or equal to the number of 1s we used so far plus α_1. This leads to $\nu_2 - \alpha_2 + \alpha_2 \leq k - 2\alpha_1 + \alpha_2 - \nu_4 + \alpha_1$, which simplifies to $b \leq -\alpha_1 + \alpha_2 - \nu_4$. In the last step we fill the last boxes of the second row with the remaining 1s. These have to be in some of the $\alpha_1 - \alpha_2$ leftmost boxes of the second row. So we obtain $\nu_1 - \alpha_1 - k + 2\alpha_1 - \alpha_2 + \nu_4 \leq \alpha_1 - \alpha_2$, which simplifies to $a + \nu_4 \leq 0$, but that condition we have already seen. In total we obtain the following conditions:

(1) $\alpha_2 \geq \nu_4$;
(2) $-\alpha_1 + \alpha_2 - \nu_4 \leq a \leq -\nu_4$;
(3) $\nu_3 - \nu_4 \geq \alpha_1 - \alpha_2$;
(4) $\nu_3 \leq \alpha_1$;
(5) $b \leq -\alpha_1 + \alpha_2 - \nu_4$.

4th case: $\alpha_1 > \alpha_2$ and the number of 1s in the second row as well as the number of 2s in the first row equal $\alpha_1 - \alpha_2$. Such a tableau starts with ν_4 3s in the first row. So we obtain the condition $\nu_4 \leq \alpha_2$. Then there are $\alpha_1 - \alpha_2$ 2s. Now we need exactly $k - \alpha_1 - \alpha_1 + \alpha_2 - \nu_4$ 1s since there are exactly $\alpha_1 - \alpha_2$ 1s in the second row. We obtain the equation $\nu_1 - \alpha_1 - \alpha_1 + \alpha_2 = k - \alpha_1 - \alpha_1 + \alpha_2 - \nu_4$ which simplifies to $a = -\nu_4$. The second row starts with the ν_4 4s, then there are $\nu_3 - \nu_4$ 3s. Under every 2 of the first row there needs to be a 3. This leads to $\nu_3 - \nu_4 \geq \alpha_1 - \alpha_2$. The α-lattice permutation condition leads to the inequality $\nu_3 \leq \alpha_1$. Now the remaining boxes are filled with the 2s and 1s. Since from $a = -\nu_4$ follows $b = -\nu_3$, these fit because of the condition we had for the first row. In total we obtain the following conditions:

(1) $\nu_4 \leq \alpha_2$;
(2) $a = -\nu_4$;
(3) $\nu_3 - \nu_4 \geq \alpha_1 - \alpha_2$;
(4) $\nu_3 \leq \alpha_1$.

With these four cases we do not only see that the product is stable for $k \geq 2p$ but we can easily compute the coefficient with a computer. □

With that proposition we can compute $g((k,k),(n-p,p),\nu)$ for $n = 2k \geq 4p$. It would be possible to create a closed formula from the conditions we obtained in the proposition but this is not very compact; nonetheless it can be used to calculate these coefficients with the computer.

For $n \geq 4p$ the Proposition 7.9 provides two ways to compute the multiplicities for the constituents of $[n-p,p][k,k]$. On the one hand we can compute $[3p,p][2p,2p]$. The proposition tells us that the product is stable so we can read off the general formula. On the other hand we can check the conditions for the existence of the Kronecker tableaux easily with a computer and obtain the formula like that. In this way we receive the following decompositions and check the proposition at some examples at the same time.

Lemma 7.10. *Let $n = 2k$. The product $[k,k][n-p,p]$ decomposes for $4 \le p \le 7$ as follows:*

(1) *If $p = 4$ and $n \ge 16$:*

$$[(k-2)^2, 2^2] + [k-1, k-3, 3, 1] + [k-1, k-2, 2, 1] + [k-1, k-2, 3]$$
$$+ [(k-1)^2, 1^2] + [k, k-4, 4] + [k, k-3, 2, 1] + [k, k-3, 3]$$
$$+ 2[k, k-2, 2] + [k, k-1, 1] + [k, k] + [k+1, k-4, 3]$$
$$+ [k+1, k-3, 1^2] + [k+1, k-3, 2] + [k+1, k-2, 1] + [k+2, k-4, 2]$$
$$+ [k+2, k-3, 1] + [k+2, k-2] + [k+3, k-4, 1] + [k+4, k-4].$$

(2) *If $p = 5$ and $n \ge 20$:*

$$[k-2, k-3, 3, 2] + [(k-2)^2, 3, 1] + [k-1, k-4, 4, 1] + [k-1, k-3, 2^2]$$
$$+ [k-1, k-3, 3, 1] + [k-1, k-3, 4] + [k-1, k-2, 2, 1] + [k-1, k-2, 3]$$
$$+ [(k-1)^2, 2] + [k, k-5, 5] + [k, k-4, 3, 1] + [k, k-4, 4] + [k, k-3, 2, 1]$$
$$+ 2[k, k-3, 3] + [k, k-2, 1^2] + [k, k-2, 2] + [k, k-1, 1] + [k+1, k-5, 4]$$
$$+ [k+1, k-4, 2, 1] + [k+1, k-4, 3] + 2[k+1, k-3, 2] + [k+1, k-2, 1]$$
$$+ [k+1, k-1] + [k+2, k-5, 3] + [k+2, k-4, 1^2] + [k+2, k-4, 2]$$
$$+ [k+2, k-3, 1] + [k+3, k-5, 2] + [k+3, k-4, 1] + [k+3, k-3]$$
$$+ [k+4, k-5, 1] + [k+5, k-5].$$

(3) *If $p = 6$ and $n \ge 24$:*

$$[(k-3)^2, 3^2] + [k-2, k-4, 4, 2] + [k-2, k-3, 3, 2] + [k-2, k-3, 4, 1]$$
$$+ [(k-2)^2, 2^2] + [(k-2)^2, 4] + [k-1, k-5, 5, 1] + [k-1, k-4, 3, 2]$$
$$+ [k-1, k-4, 4, 1] + [k-1, k-4, 5] + 2[k-1, k-3, 3, 1]$$
$$+ [k-1, k-3, 4] + [k-1, k-2, 2, 1] + [k-1, k-2, 3] + [(k-1)^2, 1^2]$$
$$+ [k, k-6, 6] + [k, k-5, 4, 1] + [k, k-5, 5] + [k, k-4, 2^2]$$
$$+ [k, k-4, 3, 1] + 2[k, k-4, 4] + [k, k-3, 2, 1] + 2[k, k-3, 3]$$
$$+ 2[k, k-2, 2] + [k, k-1, 1] + [k, k] + [k+1, k-6, 5]$$
$$+ [k+1, k-5, 3, 1] + [k+1, k-5, 4] + [k+1, k-4, 2, 1]$$
$$+ 2[k+1, k-4, 3] + [k+1, k-3, 1^2] + [k+1, k-3, 2] + [k+1, k-2, 1]$$
$$+ [k+2, k-6, 4] + [k+2, k-5, 2, 1] + [k+2, k-5, 3]$$
$$+ 2[k+2, k-4, 2] + [k+2, k-3, 1] + [k+2, k-2] + [k+3, k-6, 3]$$
$$+ [k+3, k-5, 1^2] + [k+3, k-5, 2] + [k+3, k-4, 1] + [k+4, k-6, 2]$$
$$+ [k+4, k-5, 1] + [k+4, k-4] + [k+5, k-6, 1] + [k+6, k-6].$$

(4) *If* $p = 7$ *and* $n \geq 28$*:*

$$[k-3,k-4,4,3]+[(k-3)^2,4,2]+[k-2,k-5,5,2]+[k-2,k-4,3^2]$$
$$+[k-2,k-4,4,2]+[k-2,k-4,5,1]+[k-2,k-3,3,2]$$
$$+[k-2,k-3,4,1]+[k-2,k-3,5]+[(k-2)^2,3,1]$$
$$+[k-1,k-6,6,1]+[k-1,k-5,4,2]+[k-1,k-5,5,1]$$
$$+[k-1,k-5,6]+[k-1,k-4,3,2]+2[k-1,k-4,4,1]$$
$$+[k-1,k-4,5]+[k-1,k-3,2^2]+[k-1,k-3,3,1]$$
$$+2[k-1,k-3,4]+[k-1,k-2,2,1]+[k-1,k-2,3]+[(k-1)^2,2]$$
$$+[k,k-7,7]+[k,k-6,5,1]+[k,k-6,6]+[k,k-5,3,2]$$
$$+[k,k-5,4,1]+2[k,k-5,5]+2[k,k-4,3,1]+2[k,k-4,4]$$
$$+[k,k-3,2,1]+2[k,k-3,3]+[k,k-2,1^2]+[k,k-2,2]+[k,k-1,1]$$
$$+[k+1,k-7,6]+[k+1,k-6,4,1]+[k+1,k-6,5]$$
$$+[k+1,k-5,2^2]+[k+1,k-5,3,1]+2[k+1,k-5,4]$$
$$+[k+1,k-4,2,1]+2[k+1,k-4,3]+2[k+1,k-3,2]$$
$$+[k+1,k-2,1]+[k+1,k-1]+[k+2,k-7,5]+[k+2,k-6,3,1]$$
$$+[k+2,k-6,4]+[k+2,k-5,2,1]+2[k+2,k-5,3]$$
$$+[k+2,k-4,1^2]+[k+2,k-4,2]+[k+2,k-3,1]+[k+3,k-7,4]$$
$$+[k+3,k-6,2,1]+[k+3,k-6,3]+2[k+3,k-5,2]$$
$$+[k+3,k-4,1]+[k+3,k-3]+[k+4,k-7,3]+[k+4,k-6,1^2]$$
$$+[k+4,k-6,2]+[k+4,k-5,1]+[k+5,k-7,2]+[k+5,k-6,1]$$
$$+[k+5,k-5]+[k+6,k-7,1]+[k+7,k-7].$$

Remark 7.11. For $n < 4p$ the formulas for the previous products have less constituents, so at least for these cases the bound from Proposition 7.9 is sharp.

With the computer, we also used Proposition 7.9 to get some numerical evidence for the following claims. For $2k = n \geq 4p$ it seems like

$$g((n-p,p),(k,k)) = \left\lfloor \frac{p}{4} \right\rfloor + 1.$$

Let $N(p,l)$ be the number of constituents with multiplicity at least l in the product $[n-p,p][k,k]$ where still $n \geq 4p$. We suspect that $N(p,l) = N(p+4r, l+r)$ for all $r \geq 1$. We see a small example for this in Lemma 7.10. Here we have $N(4,2) = 1 = N(0,1)$ up to $N(7,2) = 11 = N(3,1)$. It seems like for $n \geq 4p+16$ that

$$g((k,k),(n-p,p),(\lambda_1,\lambda_2,\lambda_3,\lambda_4)) = a$$

if and only if

$$g((k,k),(n-p-4,p+4),(\lambda_1,\lambda_2-2,\lambda_3+2,\lambda_4)) = a+1.$$

In the next step we show that the products $[k,k][k+2,k-2]$ and $[k,k][k+3,k-3]$ only contain constituents with multiplicity 1 and 2. But these products have more constituents as k grows. Therefore, we cannot list them all like in the previous decompositions, instead we use a different presentation. The first product already appeared in [BWZ10] and for the second product we use that paper to prove the decomposition.

We use the notation $Q(n)$ for the partitions of n in at most four parts which are all even or odd. For partitions of length smaller than 4 the 0's count as even parts, e.g. $(4^3) = (4^3, 0) \in Q(12)$ counts as a partition with 4 even parts but $(3^3) \notin Q(9)$. By $R^k(n)$ we denote partitions where every part is repeated at most k times, where we assume that the partition has exactly 4 parts by adding 0 parts, e.g. $(16) = (16, 0^3)$ is not in $R^2(16)$ because the 0 part is repeated three times.

Lemma 7.12. *Let $n = 2k$ be even and $\lambda = (k, k)$.*

(1) *Let $\mu = (k+2, k-2)$. The product $[\lambda][\mu]$ decomposes as*

$$\sum_{\lambda \in Q(n) \cap R^2(n)} [\lambda] + \sum_{\lambda \in R^1,\ l(\lambda) \in \{3,4\}} [\lambda].$$

(2) *Let $\mu = (k+3, k-3)$ and $\nu \vdash n$. $g(\lambda, \mu, \nu) = 0$ if none of the following cases occurs:*
  - (a) *If $l(\nu) = 2$ and $\nu = (\nu_1, \nu_2)$ with $\nu_1 \equiv \nu_2 \equiv 1 \mod 2$ and $\nu_2 > 1$, $g(\lambda, \mu, \nu) = 1$.*
  - (b) *Let $l(\nu) = 3, 4$ and $\nu = (\nu_1, \nu_2, \nu_3, \nu_4)$ (if $l(\nu) = 3$ we have $\nu_4 = 0$). $g(\lambda, \mu, \nu) = 2$ if $\lambda_1 \geq \lambda_2 + 2 \geq \lambda_3 + 4 \geq \lambda_4 + 6$ and not all parts have the same parity. Otherwise, $g(\lambda, \mu, \nu) = 1$ if one of the following holds:*
    - (i) $\nu_1 \equiv \nu_2 \equiv \nu_3 \equiv \nu_4 \mod 2$ *and* $\nu_1 > \nu_2 > \nu_3 > \nu_4$*;*
    - (ii) $\nu_1 \equiv \nu_2 \not\equiv \nu_3 \equiv \nu_4 \mod 2$ *and one of the following holds:*
      - (A) $\nu_2 + 3 \geq \nu_3$*;*
      - (B) $\nu_1 \geq \nu_2 + 2$ *and* $\nu_3 \geq \nu_4 + 2$.
    - (iii) $\nu_1 \not\equiv \nu_2 \not\equiv \nu_3 \not\equiv \nu_4 \mod 2$ *and one of the following holds:*
      - (A) $\nu_2 + 3 \geq \nu_3$*;*
      - (B) $\nu_1 + 3 \geq \nu_2$ *and* $\nu_3 + 3 \geq \nu_4$.
    - (iv) $\nu_1 \not\equiv \nu_2 \equiv \nu_3 \not\equiv \nu_4 \mod 2$ *and one of the following holds:*
      - (A) $\nu_2 \geq \nu_3 + 2$*;*
      - (B) $\nu_1 \geq \nu_2 + 3$ *and* $\nu_3 \geq \nu_4 + 3$.

Proof: The first identity appeared in [BWZ10, Corollary 3.6.]. For the second one we use [BWZ10, Theorem 3.1.], which states that the product $[k, k][k+3, k-3]$ decomposes as

$$\begin{aligned}&[(3,3,0)P] + [(4,3,1)P] + [(5,3,2)P] + [(6,3,3)P]\\ &+ [(5,4,1)P] + [(6,4,2)P] + [(7,4,3)P],\end{aligned}$$

where $[(a, b, c)P]$ is the sum over all $[\alpha]$ for $\alpha \vdash n$ such that $\alpha - (a, b, c)$ is a partition from $Q(n)$. If $l(\nu) = 1$ or $l(\nu) > 4$, we know that $g(\lambda, \mu, \nu) = 0$.

If $l(\nu) = 2$, $[\nu]$ is a constituent of $[\lambda][\mu]$ if and only if $[\nu]$ is a constituent of $[(3, 3, 0)P]$. This proves part (a).

If $l = 3, 4$, either all parts of ν have the same parity or exactly two pairs of two parts have the same parity. For $\nu = (\nu_1, \nu_2, \nu_3, \nu_4)$, $[\nu]$ is a constituent of $[(a, b, c)P]$ if and only if $\nu_1 - a$, $\nu_2 - b$, $\nu_3 - c$ and ν_4 all have the same parity. So we divide

$$\begin{aligned}&[(3,3,0)P] + [(4,3,1)P] + [(5,3,2)P] + [(6,3,3)P]\\ &+ [(5,4,1)P] + [(6,4,2)P] + [(7,4,3)P]\end{aligned}$$

into four disjoint sets according to the parity of the parts. For these we can easily see the different parts of (2)(b). If all parts have the same parity, $[\nu]$ is a constituent of $[\lambda][\mu]$ if and only if $[\nu]$ is a constituent of $[(6, 4, 2)P]$. So from now on we assume that two of the parts of ν are odd and two are even. If $\nu_1 \equiv \nu_2 \mod 2$, $[\nu]$ is a constituent of $[\lambda][\mu]$ if and only if $[\nu]$ is a constituent of $[(3, 3, 0)P] + [(5, 3, 2)P]$. If $\nu_1 \equiv \nu_3 \mod 2$, $[\nu]$ is a constituent of $[\lambda][\mu]$ if and only if $[\nu]$ is a constituent of

$[(5,4,1)P]+[(7,4,3)P]$. The last case is $\nu_1 \equiv \nu_4 \mod 2$. Here $[\nu]$ is a constituent of $[\lambda][\mu]$ if and only if $[\nu]$ is a constituent of $[(4,3,1)P]+[(6,3,3)P]$. □

Products with $[k+1,k]$**.**
The last decomposition we want to prove is the one of $[k+1,k][k+2,k-1]$. To do so we need the following connection between the inner and the outer tensor product of characters. It can be found in [BWZ10, Page 5 (2.3)]:

Lemma 7.13. *Let $\lambda \vdash n$, $\mu \vdash m$ and $\nu, \pi \vdash n+m$. Then*

$$\langle [\mu \boxtimes \lambda], [\nu][\pi] \rangle = \sum_{\substack{\alpha,\beta \vdash |\mu| \\ \alpha \subset \nu, \beta \subset \pi}} g(\alpha,\beta,\mu) g(\lambda, \nu/\alpha, \pi/\beta).$$

Lemma 7.14. *Let $n = 2k+1$. By $R^k(n)$ we denote the $k+1$ regular partitions, like in Lemma 7.12. The product $[k+1,k][k+2,k-1]$ decomposes as*

$$\sum_{\substack{\lambda \vdash n \text{ with} \\ l(\lambda)=2}} [\lambda] + 2 \sum_{\substack{\lambda \in R^1(n) \text{ with} \\ l(\lambda)=3,4}} [\lambda] + \sum_{\substack{\lambda \in R^2(n)\setminus R^1(n) \\ \text{with } l(\lambda)=3,4}} [\lambda].$$

Proof: With Lemma 7.13 we deduce that for a $\lambda \vdash n$

$$\langle [2,1] \boxtimes [\lambda], [k+2,k+2]^2 \rangle = 2g(\lambda,(k+1,k),(k+2,k-1)) + g(\lambda,(k+1,k),(k+1,k)).$$

It follows from Theorem 5.5 that $g((k+1,k),(k+2,k-1),\lambda) = 0$ if $l(\lambda) > 4$. Additionally, we know $g((k+1,k),(k+2,k-1),(n)) = 0$. Thus, we have to look at partitions with two, three or four rows. Thanks to [Gar+12, Theorem 2.3] we know that $[k+2,k+2]^2$ consists of all the partitions of $n+3$ of length four which have only odd parts and all partitions of $n+3$ of length at most four which have only even parts, all with multiplicity 1. If we look at the common constituents with $[2,1] \boxtimes [\lambda]$, we obtain a value for

$$2g(\lambda,(k+1,k),(k+2,k-1)) + g(\lambda,(k+1,k),(k+1,k)).$$

Since $g(\lambda,(k+1,k),(k+1,k)) = \chi_{(l(\lambda)\leq 4)}$ (see [BWZ10, Corollary 5.1.]), we receive $g(\lambda,(k+1,k),(k+2,k-1))$.

1st case: λ is a two-row partition. If $\lambda_2 = 1,2$, it follows from Lemma 5.13 resp. Lemma 7.2 that $g((k+1,k),(k+2,k-1),\lambda) = 1$. If $\lambda = (k+1,k)$, we know the decomposition of $[\lambda]^2$ and $g((k+1,k),(k+2,k-1),(k+1,k)) = 1$ (see [BWZ10, Corollary 5.1.]). From now on we can assume that $\lambda_1 - \lambda_2, \lambda_2 \geq 2$.

If $l(\lambda) = 2$ and $\lambda_1 - \lambda_2, \lambda_2 \geq 2$, $[\lambda] \boxtimes [2,1]$ decomposes as

$$\begin{aligned}&[\lambda_1,\lambda_2,2,1] + [\lambda_1,\lambda_2+1,1^2] + [\lambda_1,\lambda_2+1,2] + [\lambda_1,\lambda_2+2,1] \\ &+ [\lambda_1+1,\lambda_2,1^2] + [\lambda_1+1,\lambda_2,2] + 2[\lambda_1+1,\lambda_2+1,1] \\ &+ [\lambda_1+1,\lambda_2+2] + [\lambda_1+2,\lambda_2,1] + [\lambda_1+2,\lambda_2+1].\end{aligned}$$

In case λ_1 is even which implies λ_2 is odd,

$$[\lambda_1,\lambda_2+1,2],\ [\lambda_1+1,\lambda_2,1^2] \text{ and } [\lambda_1+2,\lambda_2+1]$$

are constituents of $[k+2,k+2]^2$. In case λ_1 is odd,

$$[\lambda_1,\lambda_2+1,1^2],\ [\lambda_1+1,\lambda_2,2] \text{ and } [\lambda_1+1,\lambda_2+2]$$

are constituents of $[k+2,k+2]^2$. So in both cases we obtain

$$3 = 2g(\lambda,(k+1,k),(k+2,k-1)) + g(\lambda,(k+1,k),(k+1,k)).$$

We conclude that $g((k+1,k),(k+2,k-1),\lambda) = 1$.

2nd case: $l(\lambda) = 3$. If $\text{rem}(\lambda) = 1$, we know that

$$[\lambda] \boxtimes [2,1] = [\lambda_1^3,2,1] + [\lambda_1+2,\lambda_1+1,\lambda_1].$$

Therefore, $g((k+1,k),(k+2,k-1),\lambda)=0$.

In the next step we look at λ with $\mathrm{rem}(\lambda)=2$. Let $\lambda_1=\lambda_2>\lambda_3$. The product $[\lambda_1^2,\lambda_3]\boxtimes[2,1]$ decomposes as

$$\begin{aligned}&\chi_{(\lambda_3\geq 2)}[\lambda_1^2,\lambda_3,2,1]+[\lambda_1^2,\lambda_3+1,1^2]+[\lambda_1^2,\lambda_3+1,2]+\chi_{(\lambda_1-\lambda_3\geq 2)}[\lambda_1^2,\lambda_3+2,1]\\&+[\lambda_1+1,\lambda_1,\lambda_3,1^2]+\chi_{(\lambda_3\geq 2)}[\lambda_1+1,\lambda_1,\lambda_3,2]+2[\lambda_1+1,\lambda_1,\lambda_3+1,1]\\&+\chi_{(\lambda_1-\lambda_3\geq 2)}[\lambda_1+1,\lambda_1,\lambda_3+2]+[(\lambda_1+1)^2,\lambda_3,1]+[(\lambda_1+1)^2,\lambda_3+1]\\&+[\lambda_1+2,\lambda_1,\lambda_3,1]+[\lambda_1+2,\lambda_1,\lambda_3+1]+[\lambda_1+2,\lambda_1+1,\lambda_3].\end{aligned}$$

The common constituents of $[\lambda]\boxtimes[2,1]$ and $[(k+2)^2]^2$ are

$$[\lambda_1^2,\lambda_3+1,2],\ [(\lambda_1+1)^2,\lambda_3,1]\text{ and }[\lambda_1+2,\lambda_1,\lambda_3+1]$$

if λ_1 is even and

$$[\lambda_1^2,\lambda_3+2,1],\ [(\lambda_1+1)^2,\lambda_3+1]\text{ and }[\lambda_1+2,\lambda_1,\lambda_3,1]$$

if λ_1 is odd. Note that if λ_1 is odd, λ_3 is odd, too therefore, $\lambda_1-1\neq\lambda_3$. With this we conclude that $g((k+1,k),(k+2,k-1),\lambda)=1$.

If $\mathrm{rem}(\lambda)=2$ and $\lambda_2=1$, we know that λ is a hook and therefore, by Lemma 6.4 $g((k+1,k),(k+2,k-1),\lambda)=1$. So if $\lambda_1>\lambda_2=\lambda_3$, we can assume that $\lambda_2\geq 2$. The product $[\lambda]\boxtimes[2,1]$ decomposes as

$$\begin{aligned}&[\lambda_1,\lambda_2+1,\lambda_2,2]+[\lambda_1,(\lambda_2+1)^2,1]+\chi_{(\lambda_1-\lambda_2\geq 2)}[\lambda_1,\lambda_2+2,\lambda_2,1]\\&+\chi_{(\lambda_1-\lambda_2\geq 2)}[\lambda_1,\lambda_2+2,\lambda_2+1]+[\lambda_1+1,\lambda_2^2,2]+2[\lambda_1+1,\lambda_2+1,\lambda_2,1]\\&+[\lambda_1+1,(\lambda_2+1)^2]+[\lambda_1+1,\lambda_2+2,\lambda_2]+[\lambda_1+2,\lambda_2^2,1]\\&+[\lambda_1+2,\lambda_2+1,\lambda_2]+\text{ constituents of length }>4.\end{aligned}$$

We know that λ_1 is odd. If λ_2 is odd, too, the common constituents are

$$[\lambda_1,\lambda_2+2,\lambda_2,1],\ [\lambda_1+1,(\lambda_2+1)^2]\text{ and }[\lambda_1+2,\lambda_2^2,1].$$

The first one occurs since λ_1 and λ_2 are both odd and therefore, $\lambda_1-\lambda_2\geq 2$. If λ_2 is even, the common constituents are

$$[\lambda_1,(\lambda_2+1)^2,1],\ [\lambda_1+1,\lambda_2^2,2],\text{ and }[\lambda_1+1,\lambda_2+2,\lambda_2].$$

In both cases we obtain that $g((k+1,k),(k+2,k-1),\lambda)=1$.

The last partitions of length 3 we have to check are the ones with three removable nodes. So let $\lambda_1>\lambda_2>\lambda_3\geq 1$. The product $[\lambda]\boxtimes[2,1]$ decomposes as

$$\begin{aligned}&[\lambda_1,\lambda_2,\lambda_3+1,2]+\chi_{(\lambda_2-\lambda_3\geq 2)}[\lambda_1,\lambda_2,\lambda_3+2,1]+\chi_{(\lambda_3\geq 2)}[\lambda_1,\lambda_2+1,\lambda_3,2]\\&+2[\lambda_1,\lambda_2+1,\lambda_3+1,1]+[\lambda_1,\lambda_2+1,\lambda_3+2]+\chi_{(\lambda_1-\lambda_2\geq 2)}[\lambda_1,\lambda_2+2,\lambda_3,1]\\&+\chi_{(\lambda_1-\lambda_2\geq 2)}[\lambda_1,\lambda_2+2,\lambda_3+1]+\chi_{(\lambda_3\geq 2)}[\lambda_1+1,\lambda_2,\lambda_3,2]\\&+2[\lambda_1+1,\lambda_2,\lambda_3+1,1]+\chi_{(\lambda_2-\lambda_3\geq 2)}[\lambda_1+1,\lambda_2,\lambda_3+2]+2[\lambda_1+1,\lambda_2+1,\lambda_3,1]\\&+2[\lambda_1+1,\lambda_2+1,\lambda_3+1]+[\lambda_1+1,\lambda_2+2,\lambda_3]+[\lambda_1+2,\lambda_2,\lambda_3,1]\\&+[\lambda_1+2,\lambda_2,\lambda_3+1]+[\lambda_1+2,\lambda_2+1,\lambda_3]+\text{ constituents of length }>4.\end{aligned}$$

The common constituents with $[k+2,k+2]^2$ are:

- If λ_1 and λ_2 are even:

$$[\lambda_1,\lambda_2,\lambda_3+1,2],\ [\lambda_1,\lambda_2+2,\lambda_3+1],\ [\lambda_1+1,\lambda_2+1,\lambda_3,1],\ [\lambda_1+2,\lambda_2,\lambda_3+1].$$

- If λ_1 and λ_2-1 are even:

$$[\lambda_1,\lambda_2+1,\lambda_3,2],\ [\lambda_1,\lambda_2+1,\lambda_3+2],\ [\lambda_1+1,\lambda_2,\lambda_3+1,1],\ [\lambda_1+2,\lambda_2+1,\lambda_3].$$

- If λ_1 and λ_2-1 are odd:

$$[\lambda_1,\lambda_2+1,\lambda_3+1,1],\ [\lambda_1+1,\lambda_2,\lambda_3,2],\ [\lambda_1+1,\lambda_2,\lambda_3+2],\ [\lambda_1+1,\lambda_2+2,\lambda_3].$$

- If λ_1 and λ_2 are odd:

$$[\lambda_1, \lambda_2, \lambda_3+2, 1],\ [\lambda_1, \lambda_2+2, \lambda_3, 1],\ [\lambda_1+1, \lambda_2+1, \lambda_3+1],\ [\lambda_1+2, \lambda_2, \lambda_3, 1].$$

Hence, in all four cases we obtain

$$5 = 2g(\lambda, (k+1,k), (k+2,k-1)) + g(\lambda, (k+1,k), (k+1,k))$$

and therefore, $g((k+1,k),(k+2,k-1),\lambda) = 2$.

<u>3rd case: $l(\lambda) = 4$.</u> Since n is odd, we know that $\mathrm{rem}(\lambda) \geq 2$. If $\mathrm{rem}(\lambda) = 2$, we know that either $\lambda_1 = \lambda_2 = \lambda_3 > \lambda_4$ or $\lambda_1 > \lambda_2 = \lambda_3 = \lambda_4$. In the first case $[\lambda] \boxtimes [2,1]$ decomposes as

$$\chi_{(\lambda_1-\lambda_4\geq 2)}[\lambda_1+1, \lambda_1^2, \lambda_4+2] + [(\lambda_1+1)^2, \lambda_1, \lambda_4+1] + [\lambda_1+2, \lambda_1^2, \lambda_4+1]$$
$$+ [\lambda_1+2, \lambda_1+1, \lambda_1, \lambda_4] + \text{ constituents of length } > 4.$$

Since $\lambda_1 \not\equiv \lambda_4 \mod 2$, the only common constituent is $[\lambda_1+2, \lambda_1^2, \lambda_4+1]$ and therefore, $g((k+1,k),(k+2,k-1),\lambda) = 0$. If $\lambda_1 > \lambda_2 = \lambda_3 = \lambda_4$, the product $[\lambda] \boxtimes [2,1]$ decomposes as

$$\chi_{(\lambda_1-\lambda_2\geq 2)}[\lambda_1, \lambda_2+2, \lambda_2+1, \lambda_2] + [\lambda_1+1, (\lambda_2+1)^2, \lambda_2] + [\lambda_1+1, \lambda_2+2, \lambda_2^2]$$
$$+ [\lambda_1+2, \lambda_2+1, \lambda_2^2] + \text{ constituents of length } > 4.$$

With the same argument as in the case before the only common constituent is $[\lambda_1+1, \lambda_2+2, \lambda_2^2]$ and therefore, we obtain that $g((k+1,k),(k+2,k-1),\lambda) = 0$, too.

If $\mathrm{rem}(\lambda) = 3$, we look at three different cases. We start with $\lambda_1 = \lambda_2 > \lambda_3 > \lambda_4$. The product $[\lambda] \boxtimes [2,1]$ decomposes as

$$[\lambda_1+2, \lambda_1+1, \lambda_3, \lambda_4] + [\lambda_1+2, \lambda_1, \lambda_3+1, \lambda_4] + [\lambda_1+2, \lambda_1, \lambda_3, \lambda_4+1]$$
$$+ [(\lambda_1+1)^2, \lambda_3+1, \lambda_4] + [(\lambda_1+1)^2, \lambda_3, \lambda_4+1]$$
$$+ \chi_{(\lambda_1-\lambda_3\geq 2)}[\lambda_1+1, \lambda_1, \lambda_3+2, \lambda_4] + 2[\lambda_1+1, \lambda_1, \lambda_3+1, \lambda_4+1]$$
$$+ \chi_{(\lambda_3-\lambda_4\geq 2)}[\lambda_1+1, \lambda_1, \lambda_3, \lambda_4+2] + \chi_{(\lambda_1-\lambda_3\geq 2)}[\lambda_1^2, \lambda_3+2, \lambda_4+1]$$
$$+ [\lambda_1^2, \lambda_3+1, \lambda_4+2] + \text{ constituents of length } > 4.$$

If $\lambda_1 \equiv \lambda_3 \equiv \lambda_4 - 1 \mod 2$, the common constituents are

$$[\lambda_1+2, \lambda_1, \lambda_3, \lambda_4+1],\ [(\lambda_1+1)^2, \lambda_3+1, \lambda_4],\ [\lambda_1^2, \lambda_3+2, \lambda_4+1],$$

where $\lambda_1 \geq \lambda_3 + 2$ since $\lambda_1 > \lambda_3$ and $\lambda_1 \equiv \lambda_3 \mod 2$. If $\lambda_1 \equiv \lambda_3 - 1 \equiv \lambda_4 \mod 2$, the common constituents are

$$[\lambda_1+2, \lambda_1, \lambda_3+1, \lambda_4],\ [(\lambda_1+1)^2, \lambda_3, \lambda_4+1],\ [\lambda_1^2, \lambda_3+1, \lambda_4+2].$$

So we know that $g((k+1,k),(k+2,k-1),\lambda) = 1$.

If $\lambda_1 > \lambda_2 = \lambda_3 > \lambda_4$, $[\lambda] \boxtimes [2,1]$ decomposes as

$$[\lambda_1+2, \lambda_2+1, \lambda_2, \lambda_4] + [\lambda_1+2, \lambda_2^2, \lambda_4+1] + [\lambda_1+1, \lambda_2+2, \lambda_2, \lambda_4]$$
$$+ [\lambda_1+1, (\lambda_2+1)^2, \lambda_4] + 2[\lambda_1+1, \lambda_2+1, \lambda_2, \lambda_4+1]$$
$$+ \chi_{(\lambda_2-\lambda_4\geq 2)}[\lambda_1+1, \lambda_2^2, \lambda_4+2] + \chi_{(\lambda_1-\lambda_2\geq 2)}[\lambda_1, \lambda_2+2, \lambda_2+1, \lambda_4]$$
$$+ \chi_{(\lambda_1-\lambda_2\geq 2)}[\lambda_1, \lambda_2+2, \lambda_2, \lambda_4+1] + [\lambda_1, (\lambda_2+1)^2, \lambda_4+1]$$
$$+ \chi_{(\lambda_2-\lambda_4\geq 2)}[\lambda_1, \lambda_2+1, \lambda_2, \lambda_4+2] + \text{ constituents of length } > 4.$$

If $\lambda_1 \equiv \lambda_2 \equiv \lambda_4 - 1 \mod 2$, the common constituents are:

$$[\lambda_1+2, \lambda_2^2, \lambda_4+1],\ [\lambda_1+1, (\lambda_2+1)^2, \lambda_4],\ [\lambda_1, \lambda_2+2, \lambda_2, \lambda_4+1].$$

If $\lambda_1 \equiv \lambda_2 - 1 \equiv \lambda_4 - 1 \mod 2$, the common constituents are:

$$[\lambda_1+1, \lambda_2+2, \lambda_2, \lambda_4],\ [\lambda_1+1, \lambda_2^2, \lambda_4+2],\ [\lambda_1, (\lambda_2+1)^2, \lambda_4+1].$$

Therefore, in both cases $g((k+1,k),(k+2,k-1),\lambda) = 1$.

If $\lambda_1 > \lambda_2 > \lambda_3 = \lambda_4 \geq 1$, $[\lambda] \boxtimes [2,1]$ decomposes as

$$\begin{aligned}
&[\lambda_1+2, \lambda_2+1, \lambda_3^2] + [\lambda_1+2, \lambda_2, \lambda_3+1, \lambda_3] + [\lambda_1+1, \lambda_2+2, \lambda_3^2] \\
&+ 2[\lambda_1+1, \lambda_2+1, \lambda_3+1, \lambda_3] + \chi_{(\lambda_2-\lambda_3\geq 2)}[\lambda_1+1, \lambda_2, \lambda_3+2, \lambda_3] \\
&+ [\lambda_1+1, \lambda_2, (\lambda_3+1)^2] + \chi_{(\lambda_1-\lambda_2\geq 2)}[\lambda_1, \lambda_2+2, \lambda_3+1, \lambda_3] \\
&+ [\lambda_1, \lambda_2+1, \lambda_3+2, \lambda_3] + [\lambda_1, \lambda_2+1, (\lambda_3+1)^2] \\
&+ \chi_{(\lambda_2-\lambda_3\geq 2)}[\lambda_1, \lambda_2, \lambda_3+2, \lambda_3+1] + \text{ constituents of length } > 4.
\end{aligned}$$

If $\lambda_1 \equiv \lambda_2 - 1 \equiv \lambda_3 \mod 2$, the common constituents are:

$$[\lambda_1+2, \lambda_2+1, \lambda_3^2],\ [\lambda_1+1, \lambda_2, (\lambda_3+1)^2],\ [\lambda_1, \lambda_2+1, \lambda_3+2, \lambda_3].$$

If $\lambda_1 \equiv \lambda_2 - 1 \equiv \lambda_3 - 1 \mod 2$, the common constituents are:

$$[\lambda_1+1, \lambda_2+2, \lambda_3^2],\ [\lambda_1+1, \lambda_2, \lambda_3+2, \lambda_3],\ [\lambda_1, \lambda_2+1, (\lambda_3+1)^2].$$

Therefore, in both cases $g((k+1,k),(k+2,k-1),\lambda) = 1$.

The last case that we have to look at is $l(\lambda) = \text{rem}(\lambda) = 4$. Here, $[\lambda] \boxtimes [2,1]$ decomposes as:

$$\begin{aligned}
&[\lambda_1+2, \lambda_2+1, \lambda_3, \lambda_4] + [\lambda_1+2, \lambda_2, \lambda_3+1, \lambda_4] + [\lambda_1+2, \lambda_2, \lambda_3, \lambda_4+1] \\
&+ [\lambda_1+1, \lambda_2+2, \lambda_3, \lambda_4] + 2[\lambda_1+1, \lambda_2+1, \lambda_3+1, \lambda_4] \\
&+ 2[\lambda_1+1, \lambda_2+1, \lambda_3, \lambda_4+1] + \chi_{(\lambda_2-\lambda_3\geq 2)}[\lambda_1+1, \lambda_2, \lambda_3+2, \lambda_4] \\
&+ 2[\lambda_1+1, \lambda_2, \lambda_3+1, \lambda_4+1] + \chi_{(\lambda_3-\lambda_4\geq 2)}[\lambda_1+1, \lambda_2, \lambda_3, \lambda_4+2] \\
&+ \chi_{(\lambda_1-\lambda_2\geq 2)}[\lambda_1, \lambda_2+2, \lambda_3+1, \lambda_4] + \chi_{(\lambda_1-\lambda_2\geq 2)}[\lambda_1, \lambda_2+2, \lambda_3, \lambda_4+1] \\
&+ [\lambda_1, \lambda_2+1, \lambda_3+2, \lambda_4] + 2[\lambda_1, \lambda_2+1, \lambda_3+1, \lambda_4+1] \\
&+ \chi_{(\lambda_3-\lambda_4\geq 2)}[\lambda_1, \lambda_2+1, \lambda_3, \lambda_4+2] + \chi_{(\lambda_2-\lambda_3\geq 2)}[\lambda_1, \lambda_2, \lambda_3+2, \lambda_4+1] \\
&+ [\lambda_1, \lambda_2, \lambda_3+1, \lambda_4+2] + \text{ constituents of length } > 4.
\end{aligned}$$

We obtain the following common constituents:

- If $\lambda_1 - 1 \equiv \lambda_2 \equiv \lambda_3 \equiv \lambda_4 \mod 2$, they are

$$\begin{aligned}
&[\lambda_1+1, \lambda_2+2, \lambda_3, \lambda_4],\ [\lambda_1+1, \lambda_2, \lambda_3+2, \lambda_4], \\
&[\lambda_1+1, \lambda_2, \lambda_3, \lambda_4+2],\ [\lambda_1, \lambda_2+1, \lambda_3+1, \lambda_4+1].
\end{aligned}$$

- If $\lambda_1 \equiv \lambda_2 - 1 \equiv \lambda_3 \equiv \lambda_4 \mod 2$, they are

$$\begin{aligned}
&[\lambda_1+2, \lambda_2+1, \lambda_3, \lambda_4],\ [\lambda_1+1, \lambda_2, \lambda_3+1, \lambda_4+1], \\
&[\lambda_1, \lambda_2+1, \lambda_3+2, \lambda_4],\ [\lambda_1, \lambda_2+1, \lambda_3, \lambda_4+2].
\end{aligned}$$

- If $\lambda_1 \equiv \lambda_2 \equiv \lambda_3 - 1 \equiv \lambda_4 \mod 2$, they are

$$\begin{aligned}
&[\lambda_1+2, \lambda_2, \lambda_3+1, \lambda_4],\ [\lambda_1+1, \lambda_2+1, \lambda_3, \lambda_4+1], \\
&[\lambda_1, \lambda_2+2, \lambda_3+1, \lambda_4],\ [\lambda_1, \lambda_2, \lambda_3+1, \lambda_4+2].
\end{aligned}$$

- If $\lambda_1 \equiv \lambda_2 \equiv \lambda_3 \equiv \lambda_4 - 1 \mod 2$, they are

$$\begin{aligned}
&[\lambda_1+2, \lambda_2, \lambda_3, \lambda_4+1],\ [\lambda_1+1, \lambda_2+1, \lambda_3+1, \lambda_4], \\
&[\lambda_1, \lambda_2+2, \lambda_3, \lambda_4+1],\ [\lambda_1, \lambda_2, \lambda_3+2, \lambda_4+1].
\end{aligned}$$

So in all four cases $g((k+1,k),(k+2,k-1),\lambda) = 2$. □

One direction of the proof of Proposition 7.1.

Lemma 7.15. *The products in Proposition 7.1 contain only constituents with multiplicity 1 and 2.*

Proof: Proposition 7.1 (1) has been proven in [BB17]. In Lemma 6.4 we have seen Proposition 7.1 (2). The decomposition of the products of Proposition 7.1 (3) is given in Lemma 7.2 if both are two-row partitions, and if one is a hook, we have seen this already in Lemma 6.7. Proposition 7.1 (4) and (5) we have seen in Lemma 7.3. In Lemma 7.5 the formula for Proposition 7.1 (6) is given. In Lemma 7.4 and 7.14 we see the missing parts of Proposition 7.1 (7). Proposition 7.1 (8) has been proven in Lemma 7.6, 7.10 and 7.12. And the exceptional cases of Proposition 7.1 (9) have been checked with Sage. □

2. Other products with a two-row partition contain a constituent with multiplicity 3 or higher

Now we want to prove the other direction of Proposition 7.1. We start with the case that λ and μ are two-row partitions.

Products of two characters labeled by two-row partitions.

Lemma 7.16. *Let $n \in \mathbb{N}, n \geq 6$ and $\lambda = (n-i, i), \mu = (n-j, j)$ with $3 \leq i \leq j \leq \frac{n}{2}$. If none of the following cases occurs:*

(1) $j = \frac{n-1}{2}$ *and* $i = \frac{n-1}{2}$, $i = \frac{n-3}{2}$ *or* $i = 3$*;*
(2) $j = \frac{n}{2}$ *and* $i < 8$ *or* $n - 2i < 8$*;*
(3) $n = 8$ *and* $\lambda = \mu = (5, 3)$ *or* $n = 10$ *and* $\lambda = \mu = (6, 4)$*,*

the product $[\lambda][\mu]$ has a non-symmetric constituent with multiplicity greater or equal to 3. And if, further, none of the following occurs

(a) $i = j = 3$*;*
(b) $i = 4$ *and* $\mu = (k + 1, k)$ *for* $n = 2k + 1$*;*
(c) $i = 8$ *and* $\mu = (k, k)$ *for* $n = 2k$*;*
(d) $n = 12$ *and* $\lambda = \mu = (7, 5)$, $n = 14$ *and* $\lambda = \mu = (8, 6)$ *or* $n = 26$, $\lambda = (18, 8)$, $(17, 9)$ *and* $\mu = (13, 13)$*;*

the product $[\lambda][\mu]$ contains a second constituent with multiplicity 3 or higher.

Proof: We assume that λ and μ are two-row partitions and that they are not from the first list. We check the lemma up to $n = 16$ with Sage. We know that all constituents of $[\lambda][\mu]$ have length at most 4 (see Theorem 5.5). For $n > 16$ we know that width of a constituent has to be larger than 4. Hence, none of the constituents of $[\lambda][\mu]$ is symmetric. We find a seed that contains one constituent with multiplicity 3 if λ and μ are from the second list, and several constituents with multiplicity 3 if they are not from the second list. We start with the case $i = 3$. If $4 \leq j \leq n - j - 2$, we find two constituents with multiplicity 3 or higher with the seed $((7, 3), (6, 4))$. If $j = 3$, this can be reduced to the seed $((6, 3), (6, 3))$. The corresponding product contains one constituent with multiplicity 3. In the other cases we have seen that the product with $[n-3, 3]$ does not contain any constituents with multiplicity 3 or higher.

If $4 \leq i$ and $n - 2j \geq 3$, this can be reduced to the seed $((7, 4), (7, 4))$ which has four constituents with multiplicity 3. Now we have to look at the cases $n - 2j = 2, 1, 0$. For all we can assume that $i > 3$.

If $n - 2j = 2$ and $n - 2i > 2$, we know that $n - 2i \geq 4$ since it is even and we reduce λ and μ to the seed $((8, 4), (7, 5))$. The corresponding product contains four constituents with multiplicity 3. If $n - 2i = 2$, the λ and μ can be obtained from the seed $((9, 7)(9, 7))$. The corresponding product contains three constituent with multiplicity 3 (that $[7,5]^2$ and $[8,6]^2$ contain a constituent with multiplicity 3 was checked with Sage).

If $n-2j=1$, we know that $n-2i \geq 5$. For $i=4$, λ and μ are from the second list. We can reduce them to the seed $((9,4),(7,6))$. The corresponding product contains just one constituent with multiplicity 3. If $i>4$, we reduce λ and μ to $((10,5),(8,7))$. Here it contains three constituents with multiplicity 3.

If $n-2j=0$ and $i=8$, λ and μ can be reduced to the seed $((16,8),(12,12))$. The corresponding product contains just one constituent with multiplicity 3. If $n-2j=0$ and $i>8$, λ and μ can be reduced to the seed $((18,10),(14,14))$ or $((19,9),(14,14))$ which contain several constituents with multiplicity 3. That $[13,13][18,8]$ and $[13,13][17,9]$ contain one constituent with multiplicity 3 was checked with Sage. □

Some special cases.
In the next step we look at some special cases before working on the general case: products with two-row partitions of small depth, $[k,k]$ and $[k+1,k]$. We recall that by $g_2(\lambda,\mu) \geq 3$ we mean that $[\lambda][\mu]$ contains at least 2 constituents with multiplicity 3 or higher.

Lemma 7.17.

(1) *Let $\lambda=(n-2,2)$ and $\mu \vdash n$, where μ is neither hook, a two-row partition, a rectangle, a rectangle where the removable box is removed, nor a rectangle where one of the addable boxes is added. There is a partition ν such that $g(\lambda,\mu,\nu)>2$ and if μ is symmetric, $\nu \neq \mu$.*

(2) *Let $\lambda=(n-3,3)$ and let $\mu \vdash n$ neither be a rectangle nor a hook nor a two-row partition nor (λ,μ) be one of the exceptional cases. Then $g_2(\lambda,\mu) \geq 3$.*

(3) *Let $\lambda=(n-4,4)$ and $\mu \vdash n$ be neither a two-row partition nor a hook nor let (λ,μ) be one of the exceptional pairs. If $\mu \neq (5^3)^{(')}$, $g_2(\lambda,\mu) \geq 3$. The product $[11,4][5^3]$ has only one constituent with multiplicity 3.*

Proof: For all three cases we check the statement with Sage up to $n=21$ so we can always assume that $n \geq 22$. Due to Lemma 7.16 we can assume that $l(\mu) \geq 3$.

(1): Let $\lambda=(n-2,2)$. We can assume that μ has at least two columns of different size, otherwise, the product would be multiplicity-free. First, let us assume μ is a fat hook, i.e., it is of the form (a^b,c^d) for positive integers a, b, c, d.

Let $c=1$. We know that $b>1$, otherwise μ would be a hook, and that $d>1$, otherwise, it would be one of the products from Lemma 7.3. Therefore, we reduce (λ,μ) to the seed $((3^2,1^2),(6,2))$. The seed contains only one constituent with multiplicity 3 but since $c=1$ and μ is a proper fat hook, we know that μ is not symmetric.

Let $c=a-1$. We have seen that $g(\lambda,\mu)=2$ if $d=1$ in Lemma 7.3. Further, $b=1$ is by conjugation equivalent to the previous case. Since $d>1$, we know that μ is not symmetric and we reduce (λ,μ) to the pair $((8,2),(3^2,2^2))$.

If $c \neq 1, a-1$, we can assume that $b>1$, because by conjugation $b=1$ is equivalent to $c=1$. Therefore, we can reduce (λ,μ) to $((8,2),(4^2,2))$. Since $g_2((8,2),(4^2,2)) \geq 3$, we know that $[\lambda][\mu]$ has a constituent with multiplicity 3 or higher which is different from $[\mu]$.

If μ is not a fat hook, we know that $\mu \neq (3,2,1)$ since $n \geq 22$. By conjugation we assume that $w(\mu) \geq 4$. We reduce (λ,μ) to one of the following seeds $((7,2),(4,3,2))$, $((6,2),(4,3,1))$, $((5,2),(4,2,1))$. All three seeds have at least 2 constituents with multiplicity 3 or higher. If we do not mention anything else, all following seeds do, too.

(2): Let $\lambda=(n-3,3)$, $\mu \vdash n$ not be a rectangle and (λ,μ) not be one of the exceptional pairs. By conjugation we can assume that $w(\mu) \geq 4$. Now we get a lot of different cases, but they all work in the same way. We reduce μ to one of the partitions

- $(4,2,1)$, $(4,3,1)$, $(4,3,2)$ if $\text{rem}(\mu) \geq 3$.
- $(4^2,3)$, $(4^2,2)$, $(4^2,1)$ if $\text{rem}(\mu) = 2$ and μ_1 has multiplicity 2 or higher, and $(4,2^2)$, $(4,3^2)$ if μ_1 has multiplicity 1.

λ can be reduced to the two-row partition of the same size with its second row of length 3. All that pairs are seeds which contain at least two constituents with multiplicity 3 of higher.

(3): Let $\lambda = (n-4,4)$ and $\mu \vdash n$ is neither a two-line partition nor a hook. Further, $\mu \neq (4^2,1)^{(')}$, (3^3), $(4^3)^{(')}$. If μ is a rectangle, we have to distinguish two different cases. If $l(\mu) = 3$, we reduce it to the seed $((14,4),(6^3))$. If $l(\mu), w(\mu) \geq 4$, we reduce it to the seed $((12,4),(4^4))$.

Whenever we have $d(\mu) \geq 4$ or $\mu_3 \geq 6$, we can reduce this to the previous cases. Since we assume that $n > 21$ and that $d(\mu) \leq 3$, we get, maybe after conjugation, that $w(\mu) \geq 6$. Let us first look at the case $d(\mu) = 3$. Either $\mu_3 \geq 6$ or (λ,μ) can be reduced to one of the seeds $((7,4),(4^2,3))$, $((6,4),(4,3^2))$.

In the last step we assume that $d(\mu) = 2$. Now we know that $w(\mu) \geq 7$. If $\mu_2 \geq 3$, we reduce μ to one of the following partitions $(5,5,\mu_3),(5,4,\mu_3),(5,3,\mu_3)$, where $\mu_3 \in \{1,2\}$, and λ to the two-row partition $\tilde{\lambda}$ of the same size with $\tilde{\lambda}_2 = 4$. All these seeds contain at least two constituents with multiplicity 3.

If $\mu_2 = 2$, we reduce μ to $(7,2,\mu_3)$, where $\mu_3 \in \{1,2\}$ and λ to the two-row partition $\tilde{\lambda}$ of the same size with $\tilde{\lambda}_2 = 4$. Both seeds have two constituents with multiplicity 3, too. □

Lemma 7.18. (1) *Let $n = 2k \geq 6$, $\lambda = (k,k)$ and $\mu \vdash n$. If μ is neither a hook, nor a two-row partition, nor $(n-3,2,1)$, nor is (λ,μ) one of the exceptional pairs from Proposition 7.1, $g(\lambda,\mu) \geq 3$. If μ is different from $(n-4,2^2)$, $(4,2,1^2)$ for $n = 8$ and $(5,4,1)$ for $n = 10$, $g_2(\lambda,\mu) \geq 3$.*

(2) *Let $n = 2k+1 \geq 5$, $\lambda = (k+1,k)$. If $\mu \vdash n$ is neither a hook nor a two-line partition nor one of the exceptional pairs, $g(\lambda,\mu) \geq 3$. If $\mu^{(')} \neq (4,2,1)$, $g_2(\lambda,\mu) \geq 3$.*

Proof: We checked all the statements up to $n = 25$ so we can assume that $n \geq 26$ and we can assume that $w(\mu) \geq l(\mu)$ so we know that $w(\mu) \geq 6$. Further, can we assume that μ is not a two-row partition.

Both parts of Lemma 7.18 are goning to be proven simultaneously because the proof works in the same way. In both cases we know that there are at least two columns of μ of length greater or equal to 2, otherwise, μ would be a hook. So we know that there are two columns of μ which are congruent modulo 2 and one of them has length greater than 1. In the generic case we remove these two columns from μ to obtain $\tilde{\mu}$ and the fitting number of columns of length 2 from λ to obtain $\tilde{\lambda}$. We obtain the result by induction and Lemma 5.17 if we reduce λ,μ to a pair which contains just one constituent with multiplicity 3 or higher.

But there are some cases where $g(\tilde{\lambda},\tilde{\mu}) \leq 2$. This happens if we reduce μ to a hook, maybe even a one-line partition, a two-row partition or in the case of $\lambda = (k,k)$ to $(\tilde{n},2,1)$ for some $\tilde{n} \in \mathbb{N}$ or to one of the exceptional pairs which are $(4,2^2)$, $(4,3,1)$, $(4,3^2)$, (4^3), (4^4) and (6^3) if $\lambda = (k^2)$ and $(4^2,1)$, (5^3), $(4,2,1)$ if $\lambda = (k+1,k)$. Where we only need to consider exceptional pairs with width ≥ 4. We look at these cases individually. If we reduced μ to (4^4), we would not remove the two columns and have $\mu = (6^4,2^2)$ and $\lambda = (14^2)$ or $\mu = (6^4,2)$ or $(6^4,1^2)$ and $\lambda = (13^2)$. Since $n \geq 26$ and both columns that we remove have the same parity, we know that they are of length 4, 5 or 6. For both cases we check directly that the product contains two constituents with multiplicity 3. In the same way we can solve the other exceptional cases. Note that this is only necessary for (6^3) and (5^3), in the other cases there could have been at most 2 more columns of length 6 but then we still would not have 26 or more boxes.

After the exceptional pairs we look at some special cases. If $\mu = (n-l-2, 2, 1^l)$ with $l > 1$, we reduce this in both cases to the seed $((6,2,1^2),(5,5))$. Again, if we do not mention anything else, all our seeds have at least 2 constituents with multiplicity 3 or higher. If $l = 1$, we know that $\lambda = (k+1, k)$ and this can be reduced to $((5,4),(6,2,1))$. In the case where $\mu^{(')} = (n-l-3, 3, 1^l)$, this can be reduced to $((5,5),(6,3,1))$.

If we reduced μ to a hook, μ has two or three columns larger than 1. We start with the case that μ has three columns larger than 1. Since μ is not of the form $(n-l-3,3,1^l)$, we know that there are at least two columns of length ≥ 3. From μ we remove the smallest column of length ≥ 2 together with one column of length 1 if the length of that column is odd, or two columns of length 1 if the length is even. From λ we just remove columns of length 2. We obtain the result by induction. Now we look at the case that μ has two columns larger than 1. If $\mu = (n-4, 2^2)$, we can reduce this to $((5^2),(6,2^2))$ if $\lambda = (k,k)$. This seed has only one constituent with multiplicity 3 In the case $\mu = (n-4, 2^2)$ and $\lambda = (k+1, 1)$ we can reduce it to $((5,4),(5,2^2))$. This seed contains several constituents with multiplicity 3. Since we assume that $\mu \neq (n-l-2, 2, 1^l)$, we know that the second column is at least of length 3. If the second column is of length 5, we remove (2^2) from μ as rows as well as 2 columns of length 2 from λ. The result follows from Lemma 5.17. If the second column is of length 4, we reduce this to $((4,2^3),(5^2))$ or $((5,2^3,1),(6^2))$, depending on the parity of $l(\mu)$. If the second column is of length 3, we know that $l(\mu) \geq 4$ since we already looked at $\mu = (n-4, 2^2))$. Therefore, we can reduce this to $((4,2^2,1^2),(5^2))$ or $((5,2^2,1),(5^2))$.

If we reduced μ to $(\tilde{n}, 2, 1)$, we know that $\tilde{n} \geq 6$ since $n \geq 26$ and $w(\mu) \geq l(\mu)$. So instead of removing two columns of the same parity, we remove two columns of length 1 and one of length 2 from μ and two columns of length 2 from λ. □

General case.

Lemma 7.19. *Let $\lambda = (n-i, i)$ with $5 \leq i < \frac{n-1}{n}$ and $\mu \vdash n$ is neither a hook nor a two-line partition. Then $g_2(\lambda, \mu) \geq 3$.*

Proof: We prove this with induction on n. As a start for the induction, we verified this up to $n = 25$ so we can assume that $n \geq 26$ and with the assumption $w(\mu) \geq l(\mu)$ also that $w(\mu) \geq 6$.

1st case: $\mu'_3 > 1$. We remove the 3rd and the 6th column from μ and we remove the same number of boxes from λ in such a way that the resulting partition $\tilde{\lambda}$ satisfies $\tilde{\lambda}_2 < \lambda_2$, $\tilde{\lambda}_2 \geq 4$ and $\tilde{\lambda}_1 - \tilde{\lambda}_2 \geq 1$. So induction and Lemma 7.17 if $\lambda_2 = 4$, or Lemma 7.18 if $\lambda_1 - \lambda_2 = 1$ provide the result for $\mu'_3 \geq 2$.

2nd case: $\mu'_3 = 1$. We know that $\mu = (a, 2^b, 1^c)$, where $a \geq 6$, $b \geq 1$, $c \geq 0$ and $b + c \geq 2$. We remove all but the first three rows from μ and the fitting number of columns from λ in such a way that $\tilde{\lambda}$ satisfies $\tilde{\lambda}_2 \geq 4$ and $\tilde{\lambda}_1 - \tilde{\lambda}_2 \geq 1$ like before. Now $\tilde{\mu} = (a, 2, 1)$ or $\tilde{\mu} = (a, 2^2)$. Possibly we get $\tilde{\lambda}_1 - 1 = \tilde{\lambda}_2$. Since we do not want to reduce $\tilde{\lambda}$ to a two-row rectangle, we might only be able to remove an even number of boxes from $\tilde{\lambda}$ and $\tilde{\mu}$. We remove either $(a-6)$ or $(a-7)$ from $\tilde{\mu}$ and the right columns from $\tilde{\lambda}$ such that, again, $\tilde{\lambda}$ satisfies $\tilde{\lambda}_2 \geq 4$ and $\tilde{\lambda}_1 - \tilde{\lambda}_2 \geq 1$. Now we have one of the seeds:

$$((6,2,1),(5,4)),\ ((7,2,1),(6,4)),$$
$$((6,2^2),(5,4)),\ ((7,2^2),(7,4)),\ ((7,2^2),(6,5)).$$

All these seeds contain two constituents with multiplicity greater or equal to 3. □

Proposition 7.20. *Let $\lambda \vdash n$ be a two-row partition and $\mu \vdash n$ such that the pair (λ, μ) is not from Proposition 7.1. Then $[\lambda][\mu]$ contains a constituent with multiplicity 3 or higher which is different from μ if μ is symmetric.*

Proof: In the previous chapter we proved Proposition 6.1 so we can assume that μ is not a hook and that $\lambda \neq (n-1, 1)$. If μ is a two-row partition, we have seen the result in Lemma 7.16. From now on we assume that $l(\mu) > 2$. For $\lambda_2 = 2$, 3, 4, we have seen the result in Lemma 7.17. If $\lambda_1 = \lambda_2$ or $\lambda_2 + 1 = \lambda_1$, the result has been shown in Lemma 7.18. So from now on we can assume that $\lambda_2 > 4$ and $\lambda_2 + 1 < \lambda_1$. But the case that μ is neither a hook nor a two-line partition was proven in Lemma 7.19. □

CHAPTER 8

Kronecker products of skew characters

In this chapter we want to look at the products of a skew character and an irreducible character. If a skew diagram decomposes into different connected components, the order does not matter if we are interested in the corresponding skew character. Therefore, we recall that we write $\alpha^1 * \cdots * \alpha^r$ if the skew diagram decomposes into partitions α^1 up to α^r, no matter in which order they appear. In this section we want to prove that if Theorem 5.1 is true for a fixed $n \in \mathbb{N}$, Theorem 5.4 is also true for that n. This will allow us to use Theorem 5.4 when we prove Theorem 5.1 with induction. Further, the proof of Theorem 5.1 will then finish the proof of Theorem 5.4. We want to recall:

Theorem 5.4. Let λ/μ be a basic, proper skew partition of n and $\nu \vdash n$. The product $[\lambda/\mu][\nu]$ only contains constituents with multiplicity 1 and 2 if one of the following holds (up to conjugating the partition and/or the skew partition and/or rotating the skew partition):

(1) ν is linear and λ/μ only contains constituents with multiplicity 1 and 2;
(2) $\nu = (n-1,1)$ and λ/μ is from the following list:
   (a) $\lambda = (\lambda_1, \lambda_2)^{(\prime)}$ is a two-line partition and $\mu = (1)$ or $\lambda_1 - \lambda_2 = 1$;
   (b) $\lambda = (\lambda_1^{k_1}, \lambda_1 - 1)$ and $\mu = (1)$;
   (c) $\lambda = (\lambda_1^{k_1}, 1)$ and $\mu = (\lambda_1 - 1)$;
   (d) λ/μ decomposes into a one-column and a one-row partition;
   (e) λ/μ decomposes into a rectangle and (1).
(3) ν is a fat hook and $\lambda/\mu = (n-1) * (1)$;
(4) ν is a rectangle and λ/μ is from the following list:
   (a) λ/μ equals $(n-2) * (2)$ or $(n-2) * (1^2)$;
   (b) $\lambda/\mu = (n-1,2)/(1)$;
   (c) $\lambda/\mu = (n-2, n-2, 1)/(n-3)$;
   (d) the exceptional pairs $\nu = (3^3)$ and λ/μ equals $(7,3)/(1)$ or $(6,4)/(1)$.
(5) $n = 2k$, $\nu = (k,k)$ and λ/μ is from the following list:
   (a) $\lambda/\mu = (\lambda_1, \lambda_2)/(1)$ with $\lambda_1 - \lambda_2 \leq 3$ or $\lambda_2 \leq 3$;
   (b) $\lambda/\mu = (n-2, n-2, 1)/(n-3)$;
   (c) $\lambda/\mu = (n-l) * (1^l)$;
   (d) the exceptional cases where λ/μ is one of the following skew partitions:

$$(k+2,k)/(2) \text{ for } k \leq 5,\ (k^2,1)/(1) \text{ for } k \leq 4,\ (2,1) * (1),\ (3) * (3).$$

(6) the exceptional case for $n = 5$ where $\lambda/\mu = (2^2) * (1)$ and $\nu = (3,2)$.

The following result by Gutschwager will be quite useful.

Lemma 8.1. *[Gut11] Any proper skew character of S_n has two neighboring constituents, i.e., constituents $[\lambda]$, $[\mu]$ such that $|\lambda \cap \mu| = n-1$.*

1. Special cases

Before looking at more general cases, we have to deal with some special cases. We look at products of a skew character which contains $[n]^{(\prime)}$ or $[n-1,1]^{(\prime)}$ with an irreducible character and at products of a skew character with $[n-1,1]$, $[n-2,2]$ and $[k,k]$ for $n=2k$ first.

Obviously, the product of a skew character and a linear character only contains constituents with multiplicity less or equal to 2 if and only if the skew character only contains constituents with multiplicity 1 and 2. So now we want to look at the products of a skew character with $[n-1,1]$. For this the following lemma is very helpful:

Products with $[n-1,1]$**.**

Lemma 8.2. *Let χ be a skew character of S_n for $n \geq 3$. If $\chi \boxtimes [1]$ contains a constituent with multiplicity greater or equal to k, $\chi[n-1,1]$ does so, too.*

Proof: Let $\nu \vdash n+1$ such that $[\nu]$ is a constituent of $\chi \boxtimes [1]$ with multiplicity greater or equal to k. This means that there are $\alpha^1, \dots, \alpha^l \vdash n$ all contained in ν with $\langle [\alpha^i], \chi \rangle = k_i$ such that $\sum k_i \geq k$. If $l=1$, χ has a constituent with multiplicity greater or equal to k and therefore, $\chi[n-1,1]$ has one, too. So we can assume that $l \geq 2$. Since $n>2$, we know that one of the α^i is not a rectangle. So without loss of generality we can assume that $\text{rem}(\alpha^1) \geq 2$. Since $\alpha^i \subset \nu$, we know that $|\alpha^i \cap \alpha^1| \geq n-1$. Therefore,

$$\left\langle \chi[n-1,1], [\alpha^1] \right\rangle = (\text{rem}(\alpha^1)-1)k_1 + \sum_{i=2}^{l} k_i \geq \sum_{i=1}^{l} k_i \geq k$$

as required. □

With this we can prove:

Lemma 8.3. *Let λ/μ be a basic and proper skew diagram of n. The product $[\lambda/\mu][n-1,1]$ only contains constituents with multiplicity 1 and 2 if and only if one of the following holds (up to rotation and/or conjugation of λ/μ):*

(1) *$\lambda = (\lambda_1, \lambda_2)^{(\prime)}$ is a two-line partition and $\mu = (1)$ or $\lambda_1 - \lambda_2 = 1$;*
(2) *$\lambda = (\lambda_1^{k_1}, \lambda_1 - 1)$ and $\mu = (1)$;*
(3) *$\lambda = (\lambda_1^{k_1}, 1)$ and $\mu = (\lambda_1 - 1)$;*
(4) *λ/μ decomposes into a one-column and a one-row partition;*
(5) *λ/μ decomposes into a rectangle and (1).*

Proof: First, we want to show that the stated products indeed only contain constituents with multiplicity 1 and 2. First we look how $[\lambda/\mu]$ decomposes:

(1): If $\lambda = (\lambda_1, \lambda_2)$ with $\lambda_1 > \lambda_2 > 1$ and $\mu = (1)$, the skew character $[\lambda/\mu]$ decomposes as $[\lambda_1, \lambda_2 - 1] + [\lambda_1 - 1, \lambda_2]$. By rotation symmetry the case $\lambda_1 - \lambda_2 = 1$ is equivalent to $\mu = (1)$.

(2): If $\lambda = (\lambda_1^{k_1}, \lambda_1 - 1)$ and $\mu = (1)$, then $[\lambda/\mu] = [\lambda_1^{k_1-1}, (\lambda_1-1)^2] + [\lambda_1^{k_1}, \lambda_1 - 2]$.

(3): If $\lambda = (\lambda_1^{k_1}, 1)$ and $\mu = (\lambda_1 - 1)$, the skew character $[\lambda/\mu]$ decomposes into $[\lambda_1^{k_1-1}, 1^2] + [\lambda_1^{k_1-1}, 2]$.

(4): If λ/μ decomposes into a one-row and a one-column partition, $[\lambda/\mu]$ is the sum of two irreducible characters labeled by hooks.

(5): If λ/μ decomposes into a rectangle (a^b) and (1), we know that $[\lambda/\mu]$ decomposes into $[a+1, a^{b-1}] + [a^b, 1]$.

In all five cases the product of $[n-1,1]$ with both irreducible constituents of $[\lambda/\mu]$ is multiplicity-free. Therefore, the sum only contains constituents with multiplicity 1 and 2.

Now we want to prove the other direction. Let λ/μ be a proper skew diagram of n such that it is not listed in this lemma. We start with the case that λ/μ is not connected. Thanks to Lemma 8.2 we know that $[\lambda/\mu] \boxtimes [1]$ only contains constituents with multiplicity 1 and 2. By Lemma 3.6 and 3.7 we know that λ/μ decomposes into two rectangles or any partition and (1). If one of the parts is a rectangle but not linear, we know that the other part has length or width greater or equal to 2. Otherwise, we would be in the fifth case of this lemma. Further, we know by Lemma 8.2 that the other part is a rectangle, too. We know that $[2,2] \boxtimes [2]^{(')}$ contains $[3,2,1] + [2^3]^{(')}$. Since adding further parts cannot reduce the number of removable boxes and neighboring constituents still stay neighboring constituents (see Lemma 5.18), we know that $[\lambda/\mu][n-1,1]$ contains a constituent with multiplicity 3 or higher. If λ/μ decomposes into two linear partitions, we can assume by conjugation that both linear partitions have one row, i.e., $[\lambda/\mu] = [n-a] \boxtimes [a]$ with $2 \leq a \leq n-a$. We know that this contains $[n]+[n-1,1]+[n-2,2]$ and therefore, the multiplicity of $[n-1,1]$ in $[n-1,1][\lambda/\mu]$ is at least 3. If λ/μ decomposes into a partition with two or more removable nodes and (1) this can be reduced to $([2,1] \boxtimes [1])[3,1]$ in the same way as in the first case.

So from now on we can assume that λ/μ is connected. Due to the previous lemma we know that λ/μ has to be from Theorem 2.7. We get two cases: either λ is a fat hook and μ is a rectangle or μ is a one-line partition and $l(\lambda) = l(\mu) + 1$ or $w(\lambda) = w(\mu) + 1$. By conjugation we can assume in the second case that $l(\lambda) = l(\mu) + 1$.

In the most cases we want to find a constituent $[\alpha]$ of $[\lambda/\mu]$ with three or more removable nodes and a neighboring constituent. Then we know that $[\lambda/\mu][n-1,1]$ contains α with multiplicity greater or equal to 3. First, we assume that λ/μ is connected and that λ has two removable nodes.

We start with the case $\lambda = (\lambda_1^{k_1}, \lambda_2^{k_2})$ is a fat hook and μ is a rectangle. Consider the special case that $\mu = (1)$. It is known that $[\lambda/\mu]$ decomposes as $[\lambda_1^{k_1-1}, \lambda_1 - 1, \lambda_2^{k_2}] + [\lambda_1^{k_1}, \lambda_2^{k_2-1}, \lambda_2 - 1]$. If $(\lambda_1^{k_1-1}, \lambda_1 - 1, \lambda_2^{k_2})$ has at most two removable nodes, we know that $k_1 = 1$ or $\lambda_1 - 1 = \lambda_2$. By the assumption that λ/μ is connected and not listed in this lemma each of the conditions $k_1 = 1$ or $\lambda_1 - 1 = \lambda_2$ implies that $k_2 \geq 2$ and $\lambda_2 \geq 2$. Therefore, $(\lambda_1^{k_1}, \lambda_2^{k_2-1}, \lambda_2 - 1)$ has three removable nodes. This tells us that $[\lambda/\mu][n-1,1]$ has a constituent with multiplicity 3 or higher.

From now on we assume that $\mu \neq (1)$. By conjugation we can also assume that $w(\mu) \geq 2$. If $l(\lambda) - l(\mu), w(\lambda) - w(\mu), k_2, \lambda_2 \geq 2$, we obtain λ/μ from the seed $(4,3^2)/(2)$ if $\lambda_2 \geq 3$ or $(4,2^2)/(2)$ if $\lambda_2 = 2$ by adding skew rows and columns in the following way. First, we add $(\lambda_1 - 4, (\lambda_2 - a)^2)/(\mu_1 - 2)$, where a equals 2 or 3 depending on which seed we start with. In the next step we add $(\lambda_1^{k_1-1}, \lambda_2^{k_2-2})/(\mu_1^{r_1-1})$ as rows. Further, neighboring constituents stay neighboring constituents. We know this due to Lemma 5.18. So with Lemma 2.11 it is sufficient to check that $[(4,3^2)/(2)]$ and $[(4,2^2)/(2)]$ contain a constituent with three removable nodes and a neighboring constituent. Now we look at the cases $l(\lambda) - l(\mu) = 1$, $w(\lambda) - w(\mu) = 1$, $k_2 = 1$ or $\lambda_2 = 1$ case by case. Like in this case we will use Lemma 2.11 to reduce λ/μ to a smaller skew partition such that we can check the corresponding character.

If $w(\lambda) - w(\mu) = 1$, we can assume that $k_1 > l(\mu)$. Otherwise, λ/μ would not be connected. Therefore, we can reduce λ/μ to

- $(3^2, 2)/(2)$ if $\lambda_2 \geq 2$;
- $(3^2, 1^2)/(2)$ if $\lambda_2 = 1$ and $l(\mu) = 1$ because this implies $k_2 \geq 2$, otherwise, λ/μ would be listed in (3);
- $(3^3, 1)/(2^2)$ if $\lambda_2 = 1$ and $l(\mu) > 1$.

The corresponding skew characters contain a constituent with three removable nodes and one neighboring constituent. By conjugation this also solves the case $l(\lambda) - l(\mu) = 1$ if λ is not a two-row partition.

Now we look at the case $k_2 = 1$. If $k_1 = 1$, we know that $w(\mu) < \lambda_2 < \lambda_1 - 1$, therefore, we can use the seed $(5,3)/(2)$. The corresponding character decomposes as $[(5,3)/(2)] = [5,1] + [4,2] + [3,3]$. We know that $[\lambda/\mu]$ has a constituent with at least two removable nodes and two neighboring constituents. Therefore, $[\lambda/\mu][n-1,1]$ contains a constituent with multiplicity 3 or higher. This also finishes the case where $l(\lambda) - l(\mu) = 1$. If $k_1 > 1$, we can reduce λ/μ to:

- $(4^2,3)/(2)$ if $\lambda_1 - \lambda_2 = 1$;
- $(4^2,2)/(2)$ if $\lambda_2, \lambda_1 - \lambda_2 \geq 2$;
- and $(4^2,1)/(2)$ if $\lambda_2 = 1$.

All seeds contain a constituent with three removable nodes and a neighboring constituent.

Let $\lambda_2 = 1$. Since $w(\mu) \geq 2$ and $w(\lambda) - w(\mu) \geq 2$, we know that $\lambda_1 > 3$. In addition, we know that $k_1 > 1$, so we take the seed $(4^2,1)/(2)$, again. This tells us that if λ is a fat hook and λ/μ is connected, the classification of the lemma is correct.

The last case is that λ has more than two removable nodes. By conjugation we can assume that $l(\mu) = 1$ and $w(\lambda) - w(\mu) = 1$. We take the seed $(3,2,1)/(2)$. This has a constituent with two removable nodes and two neighboring constituents. □

Before looking at the other special cases for the irreducible character, we first look at what happens if the skew character contains $[n]$ or $[n-1,1]$.

Skew character which contains $[n]$.

Lemma 8.4. *Let $[n] \neq \chi$ be a skew character of S_n such that $\langle \chi, [n] \rangle > 0$, then $\langle \chi, [n] \rangle = 1$. Further, let $\lambda \vdash n$. If $\chi[\lambda]$ only contains constituents with multiplicity 1 and 2, one of the following holds:*

(1) *χ is of the form $[n-a-b] \boxtimes [a] \boxtimes [b]$, where b equals 0 or 1 and $\lambda^{(')} = (n)$;*
(2) *$n \geq 2$, $\chi = [n-1] \boxtimes [1] = [n] + [n-1,1]$ and λ has at most two removable boxes;*
(3) *$n \geq 4$, $\chi = [n-2] \boxtimes [2] = [n] + [n-1,1] + [n-2,2]$ and λ is a rectangle;*
(4) *$n = 6$, $\chi = [3] \boxtimes [3]$ and $\lambda^{(')} = [3,3]$.*

Proof: Let χ be a skew character such that $\langle \chi, [n] \rangle > 0$.

(1): From the Littlewood–Richardson rule it is obvious that $\langle \chi, [n] \rangle = 1$ and that χ is $[a_1] \boxtimes [a_2] \boxtimes \cdots \boxtimes [a_l]$ for some positive integers $a_1, \ldots, a_l$ which sum up to n. So from Lemma 3.6 and Lemma 2.13 we know that $\chi = [n-a-b] \boxtimes [a] \boxtimes [b]$, where b equals 0 or 1.

(2): If $\chi = [n-1] \boxtimes [1] = [n] + [n-1,1]$, we know (see Lemma 5.13)

$$\chi[\lambda] = \mathrm{rem}(\lambda)[\lambda] + \sum_{\substack{\mu \vdash n \\ |\lambda \cap \mu| = n-1}} [\mu].$$

This only contains constituents with multiplicity 1 and 2 if and only if λ has one or two removable nodes.

(3): If $\chi = [n-2] \boxtimes [2] = [n] + [n-1,1] + [n-2,2]$ and λ has more than two removable nodes, we know that the product of $[\lambda]$ with $[n]+[n-1,1]$ already contains $[\lambda]$ three or more times. So we can assume that $\mathrm{rem}(\lambda) = 1, 2$. If $\mathrm{rem}(\lambda) = 2$, we know that $[\lambda]([n]+[n-1,1])$ contains $[\lambda]$ already two times. So if $[\lambda]\chi$ only contains constituents with multiplicity 1 and 2, by Proposition 5.14

$$\langle [\lambda], [\lambda][n-2,2] \rangle = h_2 + h_1(h_1 - 2) = 0,$$

where h_i equals the number of removable i-hooks in λ. But this means that $h_2 = 0$. Since λ has two removable nodes, we obtain $\lambda = (2, 1)$, but this is a contradiction to $n \geq 4$. If $\mathrm{rem}(\lambda) = 1$, we know that $([n] + [n-1, 1])[\lambda]$ and $[n-2, 2][\lambda]$ are multiplicity-free. Therefore, their sum only contains constituents with multiplicity 1 and 2.

(4): The product $([3] \boxtimes [3])[3, 3]$ was checked with Sage.

For the remaining products we show that these contain a constituent with multiplicity 3. Let χ be different from $[n-1] \boxtimes [1]$ and $[n-2] \boxtimes [2]$. Since we assume that χ only contains constituents with multiplicity 1 and 2, we still know that it is of the form as stated in (1). It contains $\chi_0 = [n] + [n-1, 1] + [n-2, 2] + [n-3, 3]$ if $b = 0$ or $\chi_1 = [n] + 2[n-1, 1] + [n-2, 2] + [n-2, 1^2]$ if $b = 1$. Both χ_i contain $[n-2, 2] \boxtimes [2]$ and we have seen in (3) that if λ is not a rectangle, $[n-2, 2] \boxtimes [2][\lambda]$ already contains a constituent with multiplicity 3 or higher. Hence, it is sufficient to show that $\chi_i[c^d]$ contains a constituent with multiplicity 3 or higher for $i = 0, 1$, $c, d > 1$ and $cd = n$. Let us first look at $\chi_0[c^d]$. If $c, d > 2$,

$$\langle [n-3, 3][c^d], [c^d] \rangle = h_1(h_1 - 1)(h_1 - 3) + h_2(2h_1 - 3) + h_3 = 1,$$

where h_i equals the number of removable i-hooks in λ (see Proposition 5.14). Therefore, $\chi_0[\lambda]$ and $\chi[\lambda]$ contain $[c^d]$ with multiplicity 3 or higher. Let us now look at the case $c = 2$ or $d = 2$. By conjugation we can assume that $d = 2$. If $c = 3$, we are in the situation of (4). If $c > 3$, we know the formulas for all involved products. So we see that $\langle [c, c-1, 1], \chi_0[c^2] \rangle = 3$. Hence, in this case $\chi_0[\lambda]$ and $\chi[\lambda]$ contain a constituent with multiplicity 3 or higher. Now let us look at $\chi_1[c^d]$. Here, we easily see that $[c^{d-1}, c-1, 1]$ is contained with multiplicity greater or equal to 3 because for $n \geq 6$ since by [BB17, Proposition 3.6.] all the formulas for the involved products are known. For $n < 6$ this was checked with Sage. □

In the next lemma we look at the skew characters with $[n-1, 1]$ as maximal constituent, i.e., the skew character does not contain $[n]$.

Skew character which contains $[n-1, 1]$.

Lemma 8.5. *Let χ be a skew character of S_n which contains $[n-1, 1]$ as maximal constituent and $\lambda \vdash n$. The product $\chi[\lambda]$ only contains constituents of multiplicity 1 and 2 if and only if one of the following holds (up to conjugation and/or rotation):*

(1) *χ only contains constituents of multiplicity 1 and 2 and $\lambda = (n)$. Then χ is of the form*
  (a) $[(n-1, a+1)/(a)]$ *for* $1 \leq a \leq n-3$;
  (b) $[(n-1-a, 2)/(1)] \boxtimes [a]$ *for* $1 \leq a \leq n-4$;
  (c) $[(n-2-a, a+1)/(a)] \boxtimes [1]$ *for* $1 \leq a \leq n-4$;
  (d) $[n-a-1, 1] \boxtimes [a]$ *for* $1 \leq a \leq n-2$;
  (e) $[n-3, 1] \boxtimes [1] \boxtimes [1]$;
  (f) $[n-2-a] \boxtimes [a] \boxtimes [1^2]$ *for* $1 \leq a \leq n-3$.

(2) *$\lambda = (n-1, 1)$ and χ is $[n-1, 1] + [n-2, 2]$ or $[n-1, 1] + [n-2, 1^2]$;*

(3) *λ is a rectangle and χ is $[n-1, 1] + [n-2, 2]$ or $[n-1, 1] + [n-2, 1^2]$.*

Proof: That all the listed products only contain constituents with multiplicity 1 and 2 is easy to verify. The products of λ with one of the summands from χ in (2) or (3) are multiplicity-free so the sum can just contain constituents with multiplicity 1 and 2. We find (1)(d)-(f) already in Theorem 2.4. (1)(a) is even multiplicity-free and can be found in Theorem 2.3. Since (1)(a) is multiplicity-free and decomposes as $\sum_{i=1}^{a} [n-i, i]$, we see that (1)(c) only contains constituents with multiplicity 1 and 2. For (1)(b) we know that $[(n-1-a, 2)/(1)] = [n-1-a, 1] + [n-2-a, 2]$. The outer tensor product of both summands with $[a]$ is multiplicity-free, which follows from Theorem 2.3, and the sum only contains constituents with multiplicity 1 and 2.

Now we want to prove the other direction. Let χ be a skew character which contains just constituents with multiplicity 1 and 2 and $[n-1,1]$ is its maximal constituent. Since χ does not contain $[n]$, we know that the corresponding skew partition does not consist only of disconnected single-rows. But since it contains $[n-1,1]$, we know that there is one box such that the skew diagram without that box only consists of disconnected single-rows. This tells us that if $\chi = [\nu/\mu]$ and ν/μ is connected, $\nu/\mu = (n-1, a+1)/(a)$ for some $1 \leq a \leq n-3$. If ν/μ decomposes into a proper and connected skew partition and an ordinary partition, we know that they are listed in Theorem 2.7. Further, the ordinary partition can only be $(a-1,1)$ or (a). If the ordinary partition is $(a-1,1)$, we know that the skew character has to contain $[n-a]$, but Lemma 8.4 tells us that this is not possible (see the previous lemma). So the ordinary partition is (a) and the skew character has to contain $[n-a-1,1]$ and does not contain $[n-a]$. Hence, it is of the form (1)(a). Using Theorem 2.7 we find that the possibilities for this are (1)(b) and (1)(c). If ν/μ decomposes into two ordinary partitions, both of them can only be of the form $(a-1,1)$ and (a). We see that χ has to be of the form $[n-a-1,1] \boxtimes [a]$. If ν/μ decomposes into three ordinary partitions, we easily see from Lemma 3.6 and Lemma 3.7 that $\chi = [n-3,1] \boxtimes [1] \boxtimes [1]$ or $\chi = [n-2-a] \boxtimes [a] \boxtimes [1^2]$. Further, we know from Corollary 4.2 that ν/μ cannot decompose into four or more parts or that two of the parts are proper skew partitions. Now we know of which form ν/μ can be.

If we look at products $\chi[\lambda]$ for $\lambda \vdash n$ which is not linear, we still know that the skew character has to be of that form. Thus, we know that χ contains $\chi_0 = [n-1,1] + [n-2,2]$ or $\chi_1 = [n-1,1] + [n-2,1^2]$. If $\text{rem}(\lambda) \geq 4$, $g(\lambda, \lambda, (n-1,1)) \geq 3$. Therefore, $\chi[\lambda]$ contains $[\lambda]$ with multiplicity greater or equal to 3. If $\text{rem}(\lambda) = 3$, we know by Proposition 5.14 that

$$g(\lambda, \lambda, [n-2,2]) = h_2 + h_1(h_1 - 2) \geq h_2 + 3 \geq 3,$$

where h_i is the number of removable i-hooks of λ. Therefore, $\chi_0[\lambda]$ contains $[\lambda]$ with multiplicity at least 3. Further,

$$g(\lambda, \lambda, [n-2,1^2]) = (\text{rem}(\lambda) - 1)^2 = 4$$

and therefore, $\chi_1[\lambda]$ also contains $[\lambda]$ with multiplicity greater than 3. In the next step let $\text{rem}(\lambda) = 2$. We can assume that $\lambda \neq (n-1,1)$ since we have already dealt with this case in Lemma 8.3. We show that for all other λ the products $\chi_0[\lambda]$ and $\chi_1[\lambda]$ contain a constituent with multiplicity 3. Let us begin with χ_0. If $\lambda^{(\prime)} \neq (k+1,k)$ for $n = 2k+1$,

$$g(\lambda, \lambda, [n-2,2]) = h_2 + h_1(h_1 - 2) \geq 2.$$

Therefore, $g(\lambda, \lambda, \chi_0) \geq 3$. If $\lambda = (k+1,k)$, we have the decomposition for this product in Lemma 5.13 and 7.2. Therefore we see that $[\lambda]\chi_0$ contains $(k+1,k-1,1)$ three times. Now we look at the product $[\lambda]\chi_1$. If λ is not a two-row partition or $(a^{b-1}, a-1)$ for $ab-1 = n$, it follows from Proposition 6.1 that the product $[\lambda][n-2,1^2]$ contains a constituent with multiplicity 3 or higher. For the remaining cases we know the decompositions are given in Lemma 5.13, 6.4 and 6.5. We see that $[\lambda]\chi_1$ contains $[a^{b-2}, (a-1)^2, 1]$ three times if $\lambda = (a^{b-1}, a-1)$, and that $[\lambda]\chi_1$ contains $[\lambda_1, \lambda_2 - 1, 1]$ three times if $\lambda = (\lambda_1, \lambda_2)$.

If $\text{rem}(\lambda) = 1$ and $\chi = [n-1,1] + [n-2,1]$ or $\chi = [n-1,1] + [n-2,2]$, we know that the products of both constituents with λ are multiplicity-free. Therefore, the sum only contains constituents with multiplicity 1 and 2. If χ is not one of these characters, we know that it is from (1). Thus, χ contains $\chi_0 = [n-1,1]+[n-2,2]+[n-3,3]$ or $\chi_1 = [n-1,1]+[n-2,2]+[n-2,1^2]$. Let $\lambda = (a^b)$. We can assume that $a \geq b$. The formulas for the products ([BB17, Proposition 3.6.] and Lemma 5.13 and 7.5) tell us that $g(\lambda, \chi_0, (a^{b-1}, a-1, 1)) \geq 3$ and $g(\lambda, \chi_1, (a^{b-1}, a-1, 1)) \geq 3$. □

Now we want to look at the products with $[n-2,2]$ and $[k,k]$ for $n = 2k$ before dealing with the general case.

Product with $[n-2,2]$.

Lemma 8.6. *Let $n \geq 4$, $\lambda = (n-2,2)$ and χ be a skew character of S_n which does not contain $[n]^{(')}$ nor $[n-1,1]^{(')}$ such that $[\lambda]\chi$ only contains constituents with multiplicity 1 and 2. Then $n = 5$ and $\chi = [(3^2,1)/(1^2)]^{(')}$.*

Proof: Thanks to Proposition 7.1 we know that χ can only contain characters that correspond to two-row partitions, hooks, rectangles, $[k^2,1]$ if $n = 2k+1$, and $[a^{b-1}, a-1]$ if $n = ab-1$. Further, by Lemma 8.1 we know that χ contains two neighboring constituents. We check what these constituents could be.

If the neighboring constituents are two hooks $[n-i,1^i]$ and $[n-i-1,1^{i+1}]$, we can assume that $2 \leq i \leq n-4$. With the formula from Theorem 5.12 we know that $g(\lambda,(n-1,1^i),(n-i,1^i)) = 2$ and $g(\lambda,(n-i,1^i),(n-i-1,1^{i+1})) = 1$. Therefore, the multiplicity of $[n-i,1^i]$ in $[\lambda]\chi$ is at least 3.

If the two neighboring constituents are two two-row partitions $[n-i,i]$ and $[n-i-1,i+1]$, we know that $2 \leq i \leq n/2-1$ and

$$g(\lambda,(n-i,i),(n-i,i-1,1)) + g(\lambda,(n-i-1,i+1),(n-i,i-1,1)) \geq 3$$

if $i > 2$ due to Lemma 7.2. In the case of $i = 2$, we get by Lemma 7.2 again that $g(\lambda,\lambda,\lambda) + g(\lambda,(n-3,3),\lambda) \geq 3$. Therefore, $[\lambda]\chi$ contains a constituent with multiplicity 3 or higher.

If the two neighboring constituents are $[n-2,2]$ and $[n-2,1^2]$, we can assume that $n \geq 6$ since we checked the small cases with Sage. We can reduce λ to $(4,2)$ and $(n-2,1^2)$ to $(4,1^2)$. Therefore, we know that $g((n-2,2),\chi,(n-3,2,1)) \geq 4$ since

$$g((4,2),(4,2),(3,2,1)) + g((4,2),(4,1^2),(3,2,1)) = 4$$

and the semigroup property.

The last possibility is $n = 2k+1$ and the neighboring constituents are $[k+1,k]$ and $[k^2,1]$. We check the small cases with Sage. For $n \geq 7$ the decompositions of the involved products are given in Lemma 7.2 and Lemma 7.3. In this case

$$g(\lambda,(k+1,k),(k^2,1)) + g(\lambda,(k^2,1),(k^2,1)) \geq 3.$$

Therefore, $[\lambda]\chi$ contains $[k^2,1]$ with multiplicity 3 or higher. □

To prove the following results, we will proceed in a similar manner. We check the possibilities for the neighboring constituents. Since we assume that χ does not contain $[n]$ nor $[n-1,1]$, there are normally not many possibilities, except for the products with $[k,k]$ which we look at next. This is the most involved one.

Product with $[k,k]$.

Lemma 8.7. *Let λ/μ be a basic and proper skew partition of $n = 2k$ and $\nu = (k,k)$. The product $[\lambda/\mu][\nu]$ only contains constituents with multiplicity 1 or 2 if and only if (up to conjugation and/or rotation) one of the following holds:*

(1) *λ/μ is connected and one of the following holds:*
   (a) *$l(\lambda) = 2$ and $\mu = (1)$ and $\lambda_2 \leq 3$ or $\lambda_1 - \lambda_2 \leq 3$;*

(b) $\lambda = (n-2, n-2, 1)$ *and* $\mu = (n-3)$ *or both conjugated;*

(c) *one of the exceptional cases:*

(i) $\lambda/\mu = (k+2, k)/(2)$ *and* $k \leq 5$*;*

(ii) $\lambda/\mu = (k^2, 1)/(1)$ *and* $k \leq 4$*.*

(2) λ/μ *decomposes into two diagrams of partitions and one of the following holds:*

(a) $\lambda \vdash n+1$ *is a hook and* $\mu = (1)$*;*

(b) $\lambda/\mu^{(')} = (2) * (n-2)$*;*

(c) *one of the exceptional cases:*

(i) *n=4 and* $\lambda/\mu^{(')} = (2,1) * (1)$*;*

(ii) *n=6 and* $\lambda/\mu^{(')} = (3) * (3)$*.*

Proof: We start by showing that the products indeed contain no constituents with multiplicity 3 or higher.

(1): If $l(\lambda) = 2$, then $[\lambda/(1)] = [\lambda_1, \lambda_2 - 1] + [\lambda_1 - 1, \lambda_2]$. We look at the products of $[\lambda_1, \lambda_2 - 1]$ and $[\lambda_1 - 1, \lambda_2]$ with $[k, k]$. If $\lambda_2 \leq 3$, both products are multiplicity-free. Therefore, the sum contains only constituents with multiplicity 1 and 2. If $\lambda = (k+1, k)$, $[\lambda/(1)]$ decomposes as $[k, k] + [k+1, k-1]$. Again, the product of both constituents with $[k, k]$ is multiplicity-free, and therefore, the sum only contains constituents with multiplicity 1 and 2. If $\lambda = (k+2, k-1)$, $[\lambda/(1)]$ decomposes into $[k+1, k-1] + [k+2, k-2]$. From [BWZ10] we know that

$$[k,k][k+1,k-1] = \sum_{\substack{\pi \vdash n, \pi \notin E(n) \\ \ell(\pi) < 4}} [\pi] + \sum_{\substack{\pi \vdash n, \pi \notin O(n) \cup E(n) \\ \ell(\pi) = 4}} [\pi],$$

where $O(n)$ resp. $E(n)$ are the partitions with only odd resp. only even parts. Partitions with less than four parts cannot have only odd parts since $\pi_4 = 0$ counts as even part. Further, we know from Lemma 7.12 that in $[k, k][k+2, k-2]$ only the constituents from $E(n) \cup O(n)$ can appear with multiplicity 2. The other ones appear with multiplicity 1. Therefore, the sum of these products only contains constituents with multiplicity 1 and 2, as well.

If $\lambda = (n-2, n-2, 1)$ and $\mu = (n-3)$, $[\lambda/\mu]$ decomposes as $[n-2, 2] + [n-2, 1^2]$. Again, for both constituents the product with $[k, k]$ is multiplicity-free. Therefore, the sum only contains constituents with multiplicity less or equal to 2.

The exceptional cases have been checked with Sage.

(2): If $\lambda \vdash n+1$ is a hook and $\mu = (1)$, $[\lambda/\mu]$ is the sum of two hooks. The product of each constituent with $[k, k]$ is multiplicity-free. Therefore, the sum just contains constituents with multiplicity 1 and 2.

The product $[(n-2) * (2)][k, k]$ already appeared in Lemma 8.4. The exceptional cases have been checked with Sage.

For the other direction let $[\lambda/\mu][k, k]$ only contain constituents with multiplicity 1 and 2. If $[\nu]$ is a constituent of $[\lambda/\mu]$, $g(\nu, (k, k)) \leq 2$. Since there are a lot of exceptional cases which involve two-line rectangle we check all products of $[k, k]$ with a skew character which contains one of the exceptional factors with Sage. Checking all the skew characters up to $n = 18$ would do the same but that takes a lot longer. After eliminating the exceptionals, according to Proposition 7.1, the possible constituents of $[\lambda/\mu]$ are:

$$(\star) \quad \begin{aligned} &[k,k]^{(')},\ [k+1,k-1]^{(')},\ [k+2,k-2]^{(')},\ [k+3,k-3]^{(')}, \\ &[n-i,i]^{(')} \text{ for } i \leq 7,\ [n-3,2,1]^{(')} \text{ and [hooks].} \end{aligned}$$

If λ/μ decomposes into four or more parts, $[\lambda/\mu]$ already contains a constituent with multiplicity 3 or higher. Obviously, $[\lambda/\mu][k, k]$ does so, too.

We check ($\star$) for possible pairs of neighboring constituents. Since we assume that $[\lambda/\mu]$ does not contain $[n]^{(')}$ nor $[n-1,1]^{(')}$, the possible neighboring constituents are, up to conjugation, $[k,k]$ and $[k+1,k-1]$ or $[k+1,k-1]$ and $[k+2,k-2]$ or $[n-i,1^i]$ and $[n-i-1,1^{i+1}]$ or $[n-i,i]$ and $[n-i-1,i+1]$ for $2 \leq i \leq 6$ or $[n-2,2]$ and $[n-2,1^2]$. If $[\lambda/\mu]$ contains $[n-i,i]$ and $[n-i-1,i+1]$ with $3 \leq i \leq 6$, we see from the formulas in Lemma 7.10 and [BO06, Theorem 4.8] that $[\lambda/\mu][k,k]$ contains $[k,k-2,2]$ at least three times. In the case that $[n-3,2,1]$ as well as one of the characters $[n-2,2]$ or $[n-2,1^2]$ is contained in $[\lambda/\mu]$, the product $[\lambda/\mu][k,k]$ contains $[k,k-1,1]$ at least three times. This follows from Lemma 7.6 and [BB17, Proposition 3.6.]. If $[\lambda/\mu]$ contains $[k+2,k-2]$ and $[k+3,k-3]$, by Lemma 7.12 we see that $[k-1,k-3,3,1]$ is contained at least three times in the product $[\lambda/\mu][k,k]$.

We have the following pairs of possible neighboring constituents, up to conjugation:

- $[k,k]+[k+1,k-1]$;
- $[k+1,k-1]+[k+2,k-2]$;
- $[n-2,2]+[n-3,3]$;
- $[n-2,2]+[n-2,1^2]$;
- $[n-i,1^i]+[n-i-1,1^{i+1}]$.

Now we show that certain combinations of characters cannot occur in the skew character. Let α be a two-row partition from the list ($\star$). Since we know the decompositions for all the products, we see that if α is different from (k,k), $([\alpha]+[n-3,2,1])[k,k]$ contains $[k,k-1,1]$ at least with multiplicity 3. The product $([k,k]+[n-3,2,1])[k,k]$ contains $[k,k-2,1^2]$ 3 times if k is odd and $[k,k-2,2]$ 3 times if k is even. Hence, if $[\lambda/\mu]$ contains a two-row character, it cannot contain $[n-3,2,1]$. Therefore, $[\lambda/\mu]$ only contains characters labeled by two-row partitions and hooks.

In the next step we look at a skew character $[\lambda/\mu]$ which contains one of the two-row characters and a hook. We show that the product $[\lambda/\mu][k,k]$ contains a constituent with multiplicity 3 or higher. If a skew diagram contains a 2×2 square, the corresponding character cannot contain any hooks. If the skew diagram contains 3 boxes in one column, the corresponding character does not contain a character labeled by a two-row partition. This gives quite a few restrictions on the skew diagram. Let us assume that the skew diagram λ/μ is connected. By Proposition 4.1 we know that $\lambda/\mu=(a,b,c)/(b-1,c-1)$ for fitting a,b,c. We can assume that $c>0$, otherwise, the character contains $[n-1,1]$, which we exclude by Lemma 8.5. But with the Littlewood-Richardson rule we see that the skew character $[(a,b,c)/(b-1,c-1)]$ contains $[n-3,2,1]$ if not $a=b$ and $c=1$, but this is case (1)(b). Therefore we can assume that for a connected skew diagram, the corresponding character only consist of characters labeled by two-line partitions or of hooks and $[n-3,2,1]$.

Let the diagram decomposes into a proper skew diagram and a diagram of a regular partition. Due to Corollary 4.2 we conclude that the diagram is of the form $((a,b)/(b-1))*(c,1)$ or $((a,b)/(b-1))*(c)$. In both cases the corresponding character again contains $[n-3,2,1]$. If the diagram decomposes into two diagrams of partitions, these are $(a,1)*(b,1)$, $(a)*(b,1)$ or $(a)*(b)$. The last two cases contain $[n-1,1]$ or $[n]$, so we can exclude them. If the diagram is of the form $(a,1)*(b,1)$, the corresponding character contains $[n-3,2,1]$. If λ/μ decomposes into three parts, all three parts have to be partitions. There are three types of products such that λ/μ decomposes into three parts and only contains constituents with multiplicity 1 and 2. These are:

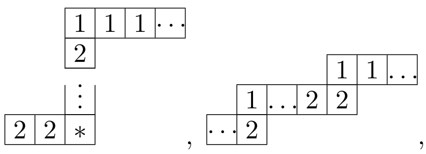
- $(1)*(1)*$(any (rotated) partition)

- (one-row) $*$ (one-column) $*$ (rectangle)
- (rectangle) $*$ (rectangle) $*$ (1)

Since all the rectangles have to be one-line partitions and all the one column partitions have length at most 3, it is easy to see that $[\lambda/\mu]$ contains $[n-3,2,1]$ if it does not contain $[n]$ and $[n-1,1]$. So we know that if the neighboring constituents are two hooks, the skew character only consists of characters labeled by two-line partitions. If the neighboring constituents are two hooks, the skew character only consists of hooks and maybe once $[n-3,2,1]^{(\prime)}$. In particular, this solves the case that the neighboring constituents are $[n-2,2]$ and $[n-1^2]$.

Let us assume $[\lambda/\mu]$ contains two neighboring constituents labeled by two-line partitions. By conjugation we can assume that these are two-row partitions. From our previous thoughts we know that $[\lambda/\mu]$ consists only of characters labeled by two-line partitions. It is easy to see that all of these are actually two-row partitions. First we assume that λ/μ is connected. If $l(\lambda) > 2$, by the Littlewood-Richardson rule $[\lambda/\mu]$ would have a constituent of length >2. So we now that $\lambda/\mu = (\lambda_1, \lambda_2)/(\mu_1)$. By rotation we can assume that $\lambda_1 - \lambda_2 \geq \mu_1$. Now we know that $[\lambda/\mu] = \sum_{i=\lambda_2-\mu_1}^{\lambda_2} [n-i, i]$. It is easy to see that there is a constituent with multiplicity 3 or higher if we are not in the case of (1)(a). If λ/μ is not connected and we have two parts, we know that both parts have to be of length 1. Otherwise, we would have a constituent of length 3 (Littlewood-Richardson rule). But this tells us that $[n]$ is a constituent of $[\lambda/\mu]$ so we are in the case of Lemma 8.4. We see that λ/μ is listed in (2)(a) if one of the parts is just one box, and (2)(b) otherwise.

Now we assume that the neighboring constituents of $[\lambda/\mu]$ are two hooks. First, let us assume that λ/μ is connected. Since $[\lambda/\mu]$ contains hooks, we know that the diagram is one rim hook. Since λ/μ is not a rotated partition, we know that $[\lambda/\mu]$ contains $[w(\lambda/\mu)-1, 2, 1^{l(\lambda/\mu)-2}]$. So we know that $l(\lambda)$ or $w(\lambda)$ is smaller or equal to 3. Otherwise the product with $[k,k]$ would contain a constituent with multiplicity 3 or higher. By conjugation we can assume that $\lambda/\mu = (a,b,c)/(b-1,c-1)$. But the following shapes cannot occur as part of the rim hook because the given filling does not correspond to a hook or $(n-3,2,1)$:



where the first one can be of length 2 or 3. But if $l(\lambda/\mu) = 2$, the corresponding character contains $[n-1,1]$ and that case is already covered in Lemma 8.5. If $l(\lambda/\mu) = 3$, we see that up to rotation $\lambda/\mu = (n-2, 2^2)/(1^2)$.

If λ/μ is not connected, we can have parts of the form $(a)^{(\prime)}$ or $((a,b)/(b-1))^{(\prime)}$. It is easy to see that if this only contains constituents with multiplicity 1 and 2, we are in case (2)(a). $\square$

2. The general case

Now we want to look at the general case. We split this into four lemmas. First we look at the product of a skew character with a hook, then with a two-row partition, then with a rectangle and then with any other partition. In a fifth lemma we look at the product of two skew characters. With the work we have done so far there are only a few cases we have check for each lemma.

Product with a hook.

Lemma 8.8. *Let $\lambda \vdash n$ be a hook, different from $(n-1,1)^{(\prime)}$ and χ be a proper skew character of S_n which contains neither $[n]^{(\prime)}$ nor $[n-1,1]^{(\prime)}$. Then the product $\chi[\lambda]$ contains a constituent with multiplicity greater or equal to 3.*

Proof: We checked this until $n = 10$ so that we do not have to worry about the exceptional case. Thanks to Proposition 6.1 we know that the products which contain just constituents with multiplicity 1 and 2 and have a hook as one of the factors are (up to conjugation):

$$[\text{hook}][\text{hook}],\ [\text{hook}][n-2,2] \text{ and for } n \text{ even } [\text{hook}]\left[\frac{n}{2},\frac{n}{2}\right] \text{ and}$$
$$[n-2,1^2][\text{rectangle}],\ [n-2,1^2][\text{two-line}] \text{ and } [n-3,1^3][\text{rectangle}].$$

We know that χ has two neighboring constituents. If these are labeled by two hooks $[n-i,1^i]$ and $[n-i-1,1^{i+1}]$ and $\lambda = (n-j,1^j)$, then $[n-|i-j|-2,2,1^{|i-j|}]$ has multiplicity 4 or higher in $\chi[\lambda]$. This is because for two hooks $(n-i,1^i)$ and $(n-j,1^j)$ the product $[n-i,1^i][n-j,1^j]$ contains $[n-|i-j|-2,2,1^{|i-j|}]$ and $[n-|i-j|-3,2,1^{|i-j|+1}]$ with multiplicity 2. This easily follows from the fact that $g((3,1^2),(3,1^2),(3,2)^{(\prime)}) = 2$ and the semigroup property but it can also be proven with Theorem 5.11. If the neighboring constituents are $[n-2,1^2]$ and $[n-2,2]$, we obtain with the same argument as before that $[n-i,2,1^{i-2}]$ has at least multiplicity 3 in $\chi[\lambda]$. We obtain that $g((n-2,2),(n-i,1^i),(n-i,2,1^{i-2}) \geq 1$ from the semigroup property since $g((3,2),(3,1^2),(3,2)) = 1$. The last case is $\lambda = (n-2,1^2)$ and the two neighboring constituents of χ correspond to two two-row partitions $[n-a,a]+[n-a-1,a+1]$. We show that $[n-a-1,a,1]$ has at least multiplicity 3 in the product $\chi[\lambda]$. We use Theorem 5.15 (3). This tells us not only that $g((n-2,1^2),(n-a,a),(n-a-1,a,1)) = 2+1-1 = 2$ but also that $g((n-2,1^2),(n-a-1,a+1),(n-a-1,a,1)) = 2+1-1 = 2$. □

Product with a two-row partition.

Lemma 8.9. *Let $\lambda = (\lambda_1,\lambda_2) \vdash n$ be a two-row partition with $2 < \lambda_2 < \lambda_1$ and χ be a proper skew character of S_n which contains neither $[n]^{(\prime)}$ nor $[n-1,1]^{(\prime)}$. Then the product $[\lambda]\chi$ contains a constituent with multiplicity 3 or higher.*

Proof: We check this up to $n = 15$ with Sage so that we do not have to worry about the exceptional cases since the exceptional cases for $n > 15$ are of the form [two-line rectangle][rectangle]. The products of irreducible characters which only contain constituents with multiplicity 1 and 2 and involve a two-row partition $\lambda = (\lambda_1,\lambda_2) \vdash n$ with $2 < \lambda_2 < \lambda_1$ are:

$[n-2,1^2][\text{two-row}]$, $[n-2,2][\text{two-row}]$, $[n-3,3][\text{rectangle}]$,

if $n = 2k+1$, $[k+1,k][\text{hook}]$, $[k+1,k]^2$, $[k+1,k][n-a,a]$ for $a \leq 3$,

if $n = 2k$, $[k,k][n-a,a]$ for $a \leq 7$ and $[k,k][k+a,k-a]$ for $a \leq 3$.

We start with the case $n = 2k+1$ and $\lambda = (k+1,k)$. Here we have four possibilities for pairs of neighboring constituents in χ, namely $[k+1,k]+[k+2,k-1]$, $[n-2,2]+[n-3,3]$, $[n-2,2]+[n-2,1^2]$ and $[n-i,1^i]+[n-i-1,1^{i+1}]$. From the formulas for these products ([BWZ10, Corollary 4.1.] and Lemma 7.14) we see that $[k+1,k]^2$ contains all partitions of n with at most length 4 exactly once and $[k+1,k][k+2,k-1]$ contains all partitions with length 3 or 4, where all parts are different with multiplicity 2. Therefore, these partitions are contained at least three times in the product $[\lambda]\chi$ if χ contains $[k+1,k]+[k+2,k-1]$. If χ contains $[n-2,2]+[n-3,3]$, we use the formulas from Lemma 7.3 and 7.4 to see that

$$g(\lambda,(n-2,2),(k+1,k-1,1)) = g(\lambda,(n-3,3),(k+1,k-1,1)) = 2.$$

Therefore, $[\lambda]\chi$ contains $[k+1,k-1,1]$ at least 4 times if χ contains the neighboring constituents $[n-2,2]+[n-3,3]$. Further, by Lemma 6.4 we know that $g(\lambda,(n-2,1^2),(k+1,k-1,1))=2$ and so $[\lambda]\chi$ contains $[k+1,k-1,1]$ at least 4 times if χ contains $[n-2,2]+[n-2,1^2]$. If χ contains $[n-i,1^i]+[n-i-1,1^{i+1}]$, we can assume that $n-i$ is odd, otherwise, we transpose and interchange both of the summands, and that $n-i\geq 3$ and $i\geq 2$. Then $(n-1,1^i)$ and $(n-i-1,1^{i+1})$ can be obtained by first adding (2) $\frac{n-i-3}{2}$ times to $(3,1^2)$ resp. $(2,1^3)$, then transposing and adding (2) $\frac{i-2}{2}$ times. This tells us that

$$g\left((n-i,1^i),\lambda,\nu\right)\geq g((3,1^2),(3,2),(3,1^2))=2$$

and

$$g\left((n-i-1,1^{i+1}),\lambda,\nu\right)\geq g((3,1^2),(3,2),(2,1^3))=1,$$

where $\nu=\left(\frac{n-i-1}{2},2^{\frac{i-2}{2}},1\right)$ with eventually $2^0=1$. Hence, $[\lambda]\chi$ contains $[\nu]$ at least 3 times.

If $\lambda=(n-a,a)\neq(k+1,k)$ with $a\geq 3$, χ has to contain the neighboring constituents $[n-2,2]+[n-2,1^2]$. We see with Lemma 6.4 and 7.2 that

$$g(\lambda,(n-2,2),(n-a,a-1,1))=g(\lambda,(n-2,1^2),(n-a,a-1,1))=2.$$

Thus, $[\lambda]\chi$ contains $[n-a,a-1,1]$ at least 4 times. □

So far we have looked at products of the form $[\lambda]\chi$ where λ was a two-row partition or a hook. With Proposition 6.1 and 7.1 we knew what the possible constituents of χ were. For the following lemmas we will work under the assumption Theorem 5.1 is true for that n. This will allow us to work in the same way as before. At the moment we will not prove Theorem 5.4 with that. But it allows us to use the classification of Theorem 5.4 when we prove Theorem 5.1 with induction.

Product with a rectangle.

Lemma 8.10. *Assume that Theorem 5.1 holds for a fixed $n\in\mathbb{N}$. Let λ/μ be a basic and proper skew partition of n and ν be a proper rectangle (with $l(\nu),w(\nu)\geq 3$). The product $[\lambda/\mu][\nu]$ only contains constituents with multiplicity 1 or 2 if and only if (up to conjugation and/or rotation) one of the following holds:*

(1) $\lambda/\mu=(n-1,2)/(1)$;
(2) $\lambda/\mu=(n-2,n-2,1)/(n-3)$;
(3) $n=9$, $\nu=(3^3)$ *and λ/μ is one of the following skew partitions:*

$$(7,3)/(1),\ (6,4)/(1),\ (7,6)/(4).$$

Proof: First, we prove that the given products just contain constituents with multiplicity 1 and 2. We have already looked at (1) in Lemma 8.5. For (2) let $\lambda/\mu=(n-2,n-2,1)/(n-3)$. We know that $[\lambda/\mu]=[n-2,2]+[n-2,1^2]$. Both products $[n-2,2][\lambda]$ and $[n-2,1^2][\lambda]$ are multiplicity-free. Therefore, the sum only contains constituents with multiplicity 1 and 2. We checked the exceptional cases with Sage.

For the other direction we check the exceptional cases of Theorem 5.1 with Sage. So we know that $\chi=[\lambda/\mu]$ can only contain the characters

$$[n-2,1^2],\ [n-3,1^3],\ [n-2,2],\ [n-3,3].$$

But we have formulas how the products of these characters with $[\lambda]$ decompose for $n\geq 18$. For $n<18$ we checked it with Sage. If χ contains the neighboring constituents $[n-3,1^3]+[n-2,1^2]$, by [BB17, Proposition 3.6.] and Lemma 6.6 we see that $[\lambda]\chi$ contains $[a+1,a^{b-3},(a-1)^2,1]$ with multiplicity 3 or higher. If χ contains $[n-2,2]+[n-3,3]$, $\chi[\lambda]$ contains $[a+1,a^{b-2},a-2,1]$ with multiplicity 3 or higher ([BB17, Proposition 3.6.] and Lemma 7.5). If χ contains the neighboring constituents $[n-2,2]+[n-2,1^2]$ and nothing else, we are in case (3), so we know that the product only contains constituents with multiplicity 1 and 2. From the previous steps we know that it cannot contain $[n-2,2]+[n-2,1^2]$ and $[n-3,3]$ or $[n-3,1^3]$. The only thing left is that $[n-2,2]$ or $[n-2,1^2]$ could appear with multiplicity higher than 1. But

$$g(\lambda,(n-2,2),(a+1,a^{b-2},a-2,1)) = g(\lambda,(n-2,1^2),(a+1,a^{b-2},a-2,1)) = 1.$$

This tells us that neither $[n-2,2]$ nor $[n-2,1^2]$ can appear more than one time as constituent of χ. □

General case.

Lemma 8.11. *Assume that Theorem 5.1 holds for a fixed $n \in \mathbb{N}$. Let χ be a proper skew character of S_n and $\nu \vdash n$ neither be a rectangle nor a two-line partition nor a hook. Then $\chi[\nu]$ contains a constituent with multiplicity at least 3.*

Proof: We check this up to $n = 12$ so that we do not have to worry about the exceptional cases. Since ν is neither a two-line partition nor a hook nor a rectangle, there are not many possibilities for ν. The non-exceptional products of Theorem 5.1 which involve a character that is neither labeled by a hook nor by a two-line partition are:

$$[a^{b-1},a-1][n-2,1^2],\ [a^{b-1},a-1][n-2,2],\ [k^2,1][n-2,2],\ [n-3,2,1][k,k].$$

The only possible neighboring constituents are $[n-2,2]$ and $[n-2,1^2]$. But for $n \geq 15$ we know that (Lemma 6.5 and 7.3)

$$g((n-2,1^2),(a^{b-1},a-1),(a+1,a^{b-2},a-2)) = 2 \text{ and}$$
$$g((n-2,2),(a^{b-1},a-1),(a+1,a^{b-2},a-2)) = 2.$$

For $n < 15$ it was checked with Sage that $([n-2,1^2]+[n-2,2])[a^{b-1},a-1]$ contains a constituent with multiplicity 3 or higher. Therefore, such a product contains a constituent with multiplicity at least 3. □

Two skew characters.

Lemma 8.12. *Assume that Theorem 5.1 holds for a fixed $n \in \mathbb{N}$. If there is a product of proper skew characters which just contains constituents with multiplicity 1 and 2, $n = 2$ and the product is $[(2,1)/(1)]^2$.*

Proof: It is obvious that $[(2,1)/(1)]^2 = ([2]+[1^2])^2 = 2[2]+2[1^2]$ only contains constituents with multiplicity 2. For the other direction there is not much to do but to collect the results from the previous lemmas. We can assume that $n > 2$. Let χ and ψ be two proper skew characters of S_n such that $\chi\psi$ only contains constituents with multiplicity 1 and 2. We know that χ contains two neighboring constituents $[\lambda]$, $[\mu]$ such that the products $\psi[\lambda]$ and $\psi[\mu]$ are from the previous lemmas. Let us assume that ψ contains $[n]+[n-1,1]$. Since we assume that $n > 2$, we know that not λ and μ are rectangles. Without loss of generality we can assume that λ has two or more removable nodes. Then

$$\langle[\lambda],([n]+[n+1])([\lambda]+[\mu])\rangle = \mathrm{rem}(\lambda)+1 \geq 3.$$

With this we can assume that neither χ nor ψ contains $[n]$, because then it would contain $[n-1,1]$, too, by Lemma 8.4. But for $n > 5$, the smaller cases were checked with Sage again, we look which irreducible characters appear in the previous lemmas, i.e., have a product with a proper skew character that just contains constituents with multiplicity 1 and 2. The possibilities for λ and μ are $(n-1,1)^{(\prime)}$, (a^b) for $n = ab$ and $(k,k)^{(\prime)}$ for $n = 2k$. But there is no possibility such that λ and

μ are two neighboring constituents. Therefore, there can be no products of proper skew characters such that the product only contains constituents with multiplicity 1 and 2 if $n > 5$. □

In this chapter we have proven:

Corollary 8.13. *If Theorem 5.1 is true for a fixed* $n \in \mathbb{N}$*, Theorem 5.3 and 5.4 are also true for that* n*.*

CHAPTER 9

The induction idea and squares

1. The induction

We have proven Theorem 5.1 for products with a hook or a two-line partition. Now we want to look at the remaining cases. We do this by induction. Let $\lambda, \mu \vdash n$ with $\lambda \neq \mu$. We define $\alpha := \lambda/(\lambda \cap \mu)$ and $\beta := \mu/(\lambda \cap \mu)$. The Dvir recursion (Theorem 5.10) tells us that there is a constituent ν with maximal width such that $g(\lambda, \mu, \nu) = \langle [\alpha][\beta], [\widehat{\nu}] \rangle$. Therefore, $[\lambda][\mu]$ has a constituent with multiplicity 3 or higher if $[\alpha][\beta]$ has one. Let $\gamma = \lambda'/(\lambda' \cap \mu)$ and $\delta = \mu/(\lambda' \cap \mu)$. If $[\gamma][\delta]$ also has a constituent with multiplicity 3 or higher, we obtain a constituent $[\tilde{\nu}]$ of $[\lambda][\mu]$ of maximal length with $g(\lambda, \mu, \tilde{\nu}) \geq 3$ with the Dvir recursion. Thanks to [BK99, Theorem 3.3] we know that $w(\nu) \neq l(\tilde{\nu})$. But this means the maximal width is strictly larger than the maximal length or the maximal length is strictly larger than the maximal width. In the first case a constituent of maximal width cannot be symmetric. In the second case a constituent of maximal length cannot be symmetric. In particular, they are always different. Since $|\alpha|, |\beta|, |\gamma|, |\delta| < n$, we can use induction. We assume that Theorem 5.1 is true for all $\tilde{n} < n$ so with Corollary 8.13 we can assume that also Theorem 5.3 and 5.4 are true for all $\tilde{n} < n$. If neither α, β nor γ, δ are listed in Theorem 5.1, 5.3 or 5.4, then $[\lambda][\mu]$ has two constituents with multiplicity 3 or higher of which at least one is not symmetric. This proves Theorem 5.1 and 5.2 for this case. If α, β or γ, μ are listed in Theorem 5.1, 5.3 or 5.4, we can assume by conjugating λ, that α, β are listed in one of these theorems. So now we look at all the different cases listed in these theorems. But before doing that we first have to look at the case $\lambda = \mu$. **For the following chapters we always assume that α and β are from Theorem 5.1 or Theorem 5.4 or $\alpha = \beta = (1) * (1)$. Further, by Chapters 6 and 7 we can assume that neither λ nor μ is a two-line partition or a hook**.

In addition to Theorem 5.1 we want to prove Theorem 5.2. To do so it is sufficient to find a non-symmertric constituent in $[\lambda][\mu]$ if λ and μ are symmetric and two constituents if either λ or μ is symmetric. It is not easy to control if one of the factors is symmetric or not. Therefore, we normally reduce every product to a seed or a known product which contains two constituents with multiplicity 3 or higher. So if we use a seed, we mean that the corresponding product contains two constituents with multiplicity 3 or higher if we do not mention anything else. If λ and μ are symmetric, this implies that α and β are symmetric, too. And here α and β themselves have to be symmetric, not only the corresponding basic skew diagrams. But this is only in a very few cases possible, namely (up to interchanging α and β):

- $\lambda = \mu$;
- $\alpha = (2) * (1^2)$, $\beta = (2^2)$ and β is between the two parts of α;
- $\alpha = \beta = (1) * (1)$ and both parts of β are between the two parts of α.

Only in these cases we have to argue why the constituent that we obtain is not symmetric.

2. Squares

Lemma 9.1. *Let $\lambda \vdash n$. The product $[\lambda]^2$ only contains constituents with multiplicity 1 and 2 if and only if (up to conjugation) one of the following holds:*

(1) *$[\lambda]^2$ is multiplicity-free [BB17, Proposition 4.1.]. This is the case if and only if*
$$\lambda \in \left\{(n),\ (n-1,1),\ \left(\left\lceil\frac{n}{2}\right\rceil, \left\lfloor\frac{n}{2}\right\rfloor\right)\right\}.$$

(2) *If $[\lambda]^2$ is not multiplicity-free, it only contains constituents with multiplicity 1 and 2 if and only if*
$$\lambda \in \{(n-2,2),\ (5,3),\ (3^3),\ (6,4),\ \textit{hooks}\}.$$

Further, $g_2(\lambda,\lambda) \geq 3$ if $g(\lambda,\lambda) \geq 3$ and λ is not from the following list:
$$(n-3,3),\ (7,5),\ (8,6).$$

Proof: That the products listed in (1) are multiplicity-free was proven in [BB17]. We have already seen that $[n-2,2]^2$ and $[\text{hook}]^2$ contain only constituents with multiplicity 1 and 2. The remaining products have been checked with Sage.

If $\text{rem}(\lambda) \geq 3$, we know from Proposition 5.14 that
$$g(\lambda,\lambda,(n-2,1^2)) = (\text{rem}(\lambda)-1)^2 \geq 4 \text{ and}$$
$$g(\lambda,\lambda,(n-2,2)) = h_2 + \text{rem}(\lambda)(\text{rem}(\lambda)-2) \geq 3.$$

If $\lambda = (\lambda_1,\lambda_2)$ is a two-row partition, we know that $3 \leq \lambda_2 \leq \lambda_1 - 2$. If $\lambda_2 = 3$, we reduce it to the seed $((6,3),(6,3))$. Unlike the other seeds we are using this one only contains one constituent with multiplicity 3, but this is sufficient for the claim of the lemma. If $\lambda_2 = \lambda_1 - 2$ and $n \geq 16$, we reduce it to the seed $((9,7),(9,7))$. We checked the smaller products with Sage. If $4 \leq \lambda_2 \leq \lambda_1 - 3$, we reduce this to the seed $((7,4),(7,4))$.

Let $\text{rem}(\lambda) = 2$, but $\lambda = (\lambda_1^{r_1}, \lambda_2^{r_2})$ be not a two-line partition. If $\lambda_2 = 1$, $r_1 \geq 2$, otherwise, λ would be a hook. We use the seed $((3^2,1),(3^2,1))$. If $\lambda_2 \neq 1$, we use one of the seeds $((3^2,2),(3^2,2))$ or $((3,2^2),(3,2^2))$ depending on r_1 and r_2.

For λ a rectangle, we can assume that $w(\lambda) \geq l(\lambda)$ with $w(\lambda) \geq 4$ and $l(\lambda) \geq 3$. Therefore, we can reduce this to the seed $((4^3),(4^3))$. □

CHAPTER 10

α is linear

In the next step we look at the case that at least one of the partitions is linear. We choose λ and μ in such a way that $\alpha := \lambda/(\lambda \cap \mu)$ is linear. If $\beta := \mu/(\lambda \cap \mu)$ is connected and above α, we conjugate λ and μ. Now α and β are conjugated, too. Further, α is above β. So without loss of generality we can assume that α is linear and above β if β is connected. If β is not connected, we can at least assume that there is one part of β which is to the left of α. Normally, we cannot make further assumptions that might need conjugation of λ and μ (as, for example $w(\lambda) \geq l(\lambda)$). But if $\alpha = \beta'$, we can and will do that. We define $m := |\alpha|$.

First we look at the case that $|\lambda \cap \mu| = n - 1$, that means $\alpha = \beta = (1)$.

1. $|\lambda \cap \mu| = n - 1$

Lemma 10.1. *Let $|\lambda \cap \mu| = n - 1$ (i.e., $\alpha = \beta = (1)$), then $g_2(\lambda, \mu) \geq 3$.*

Proof: We checked the lemma up to $n = 25$ with Sage. So we assume that $n \geq 26$. In addition, we can assume $w(\lambda) \geq l(\lambda)$, but this implies since $n > 25$ that $w(\lambda) \geq 6$. Since $\alpha = \beta = (1)$, we know that there are two columns such that λ, μ only differ in these two columns. If we remove these two columns, the resulting partitions are equal. We call that partition γ. Note that $w(\gamma) \geq 4$ since $w(\lambda) \geq 6$. We call the columns which are not equal C_1 and C_2

If γ is not listed in Lemma 9.1, we know that $g_2(\lambda, \mu) \geq 2$. Now we have to look at the cases of Lemma 9.1. If $\gamma = (7, 5)$, $(8, 6)$, we know that $g(\gamma, \gamma) = 3$. Since $n > 25$ and $w(\lambda) \geq l(\lambda)$, we know that $\mu \neq \gamma + (1^l)$ for some $l \in \mathbb{N}$. Therefore, we know because of Lemma 5.17 that $g_2(\lambda, \mu) \geq 3$.

$\underline{\gamma = (\tilde{n}, 3)}$: Because of Lemma 5.17 and since $g(\gamma, \gamma) \geq 3$, we can assume for μ that C_1 or C_2 has no box. We obtain the following possibilities for (λ, μ) :

$$((n-3, 3), (n-4, 4)) \text{ and } ((n-1-l, 4, 1^{l-3}), (n-2-l, 4, 1^{l-2})).$$

However, $((n-3, 3), (n-4, 4))$ are two-row partitions and our assumption for the rest of this part is that neither λ nor μ is a two-line partition or a hook. For the other case we can assume that $l > 3$. But if $l > 3$, we can reduce this to the seeds $((5, 4, 1), (4^2, 1^2))$.

$\underline{\gamma = (\tilde{n})}$: C_1 and C_2 have to be the two leftmost columns. Otherwise, λ and μ would be hooks. We know that without loss of generality $\lambda = (\tilde{n} + 2, 2^a, 1^b)$ and $\mu = (\tilde{n} + 2, 2^{a-1}, 1^{b+2})$, where $a \geq 2$ and $b \geq 0$. Otherwise, one of the partitions would be a hook. But this can easily be reduced to the seed $((3, 2^2), (3, 2, 1^2))$.

$\underline{\gamma = (\tilde{n}, 1)}$: C_1 and C_2 can be the two most left columns. Then we know that $\lambda = (\tilde{n} + 2, 3, 2^a, 1^b)$ and $\mu = (\tilde{n} + 2, 3, 2^{a-1}, 1^{b+2})$ with $\tilde{n} \geq 3, a \geq 1$ and $b \geq 0$. This can be reduced by only removing common rows and columns to the seed $((4, 3, 2), (4, 3, 1^2))$. C_1 and C_2 can be the leftmost column and the third one from the left, a column of length 2. In this case we can assume that $\lambda = (\tilde{n} + 2, 3, 1^a)$ and $\mu = (\tilde{n} + 2, 2, 1^{a+1})$ with $\tilde{n} \geq 3$ and $a \geq 1$. Otherwise, λ would be a two-row partition. By removing some common rows and columns we reduce this to the seed $((4, 3, 1), (4, 2, 1^2))$. It can happen that C_1 and C_2 are the leftmost and the rightmost column. Then $\lambda = (\tilde{n}+2, 2, 1^a)$ and $\mu = (\tilde{n}+1, 2, 1^{a+1})$ with $\tilde{n} \geq 3$ and $a \geq 1$. Here, α and β are removable. By doing so we obtain $[\lambda \cap \mu]^2$, a square which is not from Lemma 9.1. Since we assume that neither λ nor μ is a two-row partition, we know that these are all possible options.

$\underline{\gamma = (k,k)}$: If C_1 and C_2 are the two leftmost columns, $\lambda = (k+2, k+2, 2^a, 1^b)$ and $\mu = (k+2, k+2, 2^{a-1}, 1^{b+2})$ with $a \geq 1$ and $b \geq 0$. This can be reduced to the seed $((4^2, 2), (4^2, 1^2))$. The other possibility is that C_1 and C_2 are the left and the rightmost column. Here, the rightmost columns can be of length 2 and 1 or 1 and 0. In this case $\lambda = (k+2, k+1+a, 1^b)$ and $\mu = (k+1+a, k+1, 1^{b+1})$ with $a \in \{0,1\}$ and $b \geq 1$. By removing common columns and rows we can reduce this to the seed $((4,3,1),(3^2,1^2))$ if $a=0$, and $((4^2,1),(4,3,1^2))$ if $a=1$. Again if the leftmost column would be the same in λ and μ, these would be two-row partitions.

$\underline{\gamma = (k+1,k)}$: If C_1 and C_2 are the two leftmost columns, we know that $\lambda = (k+3, k+2, 2^a, 1^b)$ and $\mu = (k+3, k+2, 2^{a-1}, 1^{b+2})$ with $a \geq 1$ and $b \geq 0$. This can be reduced to the seed $((4,3,2),(4,3,1^2))$. Another possibility is that C_1 and C_2 are the leftmost column and the second column from the right. Then $\lambda = (k+3, k+2, 1^a)$ and $\mu = (k+3, k+1, 1^{a+1})$ with $a \geq 1$. This can be reduced to the seed $((4,3,1),(4,2,1^2))$. The last possibility is that C_1 and C_2 are the leftmost and the rightmost column, but then $\lambda = (k+3, k+1, 1^a)$ and $\mu = (k+2, k+1, 1^{a+1})$ with $a \geq 1$. Here α and β are removable and we obtain $[\lambda \cap \mu]^2$ which is not from Lemma 9.1.

$\underline{\gamma = (\tilde{n},2)}$: If C_1 and C_2 are the two leftmost columns, $\lambda = (\tilde{n}+2, 4, 2^a, 1^b)$ and $\mu = (\tilde{n}+2, 4, 2^{a-1}, 1^{b+2})$ with $a \geq 1$ and $b \geq 0$. This can be reduced to the seed $((5,4,2),(5,4,1^2))$. If C_1 and C_2 are the leftmost column and the fourth one from the left, we can assume that $\lambda = (\tilde{n}, 4, 1^a)$ and $\mu = (\tilde{n}, 3, 1^{a+1})$ with $a \geq 1$. This can be reduced to the seed $((5,4),(5,3,1))$. If C_1 and C_2 are the rightmost and the leftmost column, again, α and β are removable. If we remove them, we obtain $[\lambda \cap \mu]^2$ which is not from Lemma 9.1.

$\underline{\gamma = (5,3)}$: This is not possible since we assume that $n \geq 25$ and $w(\lambda) \geq l(\lambda)$. We know that $w(\lambda) \leq 7$, so we can add at most two columns of length 7, but then there are only 22 boxes.

$\underline{\gamma = (6,4)}$: For the same reason as before the only possibility is $\lambda = (8,6,2^6)$, but this implies $\mu = (8,6,2^5,1^2)$. That product was checked with Sage.

$\underline{\gamma \text{ is a hook}}$: We can assume that $l(\gamma) \geq 3$ and $w(\gamma) \geq 4$. If C_1 and C_2 are the two leftmost columns, we remove all besides the first three rows and all besides the first five columns to obtain $[5,3^2]^2$ which is not from Lemma 9.1. If C_1 and C_2 are the first and the third column, we know that $\lambda = (\tilde{n}, 3^a, 2^b, 1^c)$ and $\mu = (\tilde{n}, 3^{a-1}, 2^{b+1}, 1^{c+1})$ with $a \geq 1, b \geq 0$, $a+b \geq 2$ and $c \geq 0$. We reduce this to the seed $((5,3),(5,2,1))$. If C_1 and C_2 are the leftmost and the rightmost column, again, α and β are removable. So we obtain $[\lambda \cap \mu]^2$ which is not from Lemma 9.1. If C_1 and C_2 are the second and the third column, we know that $\lambda = (\tilde{n}, 3^a, 2^b, 1^c)$ and $\mu = (\tilde{n}, 3^{a-1}, 2^{b+2}, 1^{c-1})$ with $a, c \geq 1$ and $b \geq 0$. We reduce this to the seed $[5,3,1][5,2,2]$. If C_1 and C_2 are the second and the rightmost column, we know that $\lambda = (\tilde{n}, 2^a, 1^b)$ and $\mu = (\tilde{n}+1, 2^{a-1}, 1^{b+1})$, where $a \geq 2$ and $b \geq 0$, otherwise, μ would be a hook. Therefore, we can reduce it to the seed $((5,2,2),(6,2,1))$. □

2. $\alpha = (1^m)$

Now we can assume that $m = |\alpha| = |\beta| > 1$. If $\alpha = (1^m)$, we do not get many restrictions on β. $\beta^{(\mathrm{rot})}$ could be any partition or decompose into different partitions from Theorem 2.4 or be any skew partition from Theorem 2.5 and Theorem 2.6 or decompose into a skew partition and a regular partition from Theorem 2.7. We assume that α is above and to the right of β.

Let us start with the case that $\beta^{(\mathrm{rot})}$ is a partition.

β is a partition up to rotation.

Lemma 10.2. *If $\alpha = (1^m)$ and β or β^{rot} is a partition with $w(\beta) \geq 3$, $g_2(\lambda, \mu) \geq 3$.*

Proof: We remove all the common rows and columns to obtain $\tilde{\lambda}$ and $\tilde{\mu}$. In the next step we remove all the rows which belong to β besides one row of length $w(\beta)$ from $\tilde{\mu}$, the corresponding rows of $\tilde{\lambda}$ and the fitting number of rows where one box belongs to α from $\tilde{\lambda}$ and the corresponding columns of $\tilde{\mu}$ to obtain $\tilde{\lambda} = \tilde{\mu}' \supseteq (4^3)$. The result follows (by conjugation) because of Lemma 9.1. □

It will often happen that in the generic case we start with removing the common rows and columns. The resulting partitions we will call $\tilde{\lambda}$ and $\tilde{\mu}$. We will not always mention that. But $\tilde{\lambda}$ and $\tilde{\mu}$ will always be the partitions that we obtain after we remove rows and columns from λ and μ or if we already removed columns and rows from λ and μ, the partitions that we obtain after removing further columns and rows from $\tilde{\lambda}$ and $\tilde{\mu}$. If we just remove common columns and rows, we call $\tilde{\lambda}/(\tilde{\lambda} \cap \tilde{\mu})$ still α and $\tilde{\mu}/(\tilde{\lambda} \cap \tilde{\mu})$ still β since they differ just by translation. If we remove columns and rows in such a way that $\tilde{\lambda}/(\tilde{\lambda} \cap \tilde{\mu})$ is not just α translated we call it $\tilde{\alpha}$ and if $\tilde{\mu}/(\tilde{\lambda} \cap \tilde{\mu})$ is not just β translated we call it $\tilde{\beta}$. For the rest of this part we will identify α with the part of λ which is outside of $\lambda \cap \mu$. And if we for example say that we remove a row of length l from α, we actually mean that we remove a row (of length $\geq l$) from λ (or $\tilde{\lambda}$ if we already removed rows or columns) where α has l boxes. We do the same for β and μ, too.

Lemma 10.3. *Let $\alpha = (1^m)$ and β with $w(\beta) = 2$ be up to rotation a proper partition. $[\lambda][\mu]$ only contains constituents with multiplicity 1 and 2 if $\lambda = (4^2, 1)$ and $\mu = (3^3)$ or $\lambda = (3^3)$ and $\mu = (3, 2^3)$. Otherwise, $g_2(\lambda, \mu) \geq 3$.*

Proof: For one direction we check with Sage that $[4^2, 1][3^3]$ contains just constituents with multiplicity 1 and 2. The other product follows by conjugating and interchanging λ and μ. Now we show that other products contain two constituents with multiplicity 3 or higher.

1st case: β^{rot} is a partition, but β is not. We start with the exceptional case $\beta^{\mathrm{rot}} = (2, 1)$. If we removed all the common rows and columns, we would obtain $((3^3, 1), (2^5))$. We would have removed too much. But we know that λ has at least three columns, so there needs to be another column. The different possibilities we have to investigate are:

- It is to the left of β. We call such a column C_1.
- It is between α and β. We call such a column C_2.
- It is above α. In that case there also is a row above α which we call R.

If we remove all the common rows and columns but C_1 or C_2 or R, we obtain the seed $((4^3, 2, 1), (3^5))$ if we did not remove C_1, and $((4^3, 1), (3^3, 2^2))$ if we did not remove C_2, and $((3^4, 1), (3, 2^5))$ if we did not remove R. If β^{rot} is different from $(2, 1)$, we remove all the common rows and columns to obtain a two-column rectangle $\tilde{\mu}$ and a proper fat hook $\tilde{\lambda}$ which is different from $(\tilde{n} - 4, 2^2)^{(\prime)}$. The result follows from Lemma 7.18. Alternatively, this case can be reduced to the seeds $((3^4, 1^2), (2^7))$ and $((3^5, 1), (2^8))$.

2nd case: β is a partition. If β is different from (2), $(2, 1)$, $(2, 2)$, $(2, 1^2)$, $(2^2, 1)$, (2^3), which implies $m \geq 5$, we remove all the common rows and columns. Then $\tilde{\mu}$ is a two-column partition, where the second column is at least of length 6 and $\tilde{\lambda}$

is a proper rectangle. The result follows from Lemma 7.18 or 7.19. Alternatively, this case can be reduced to the seeds $((3^8),(2^{12}))$, $((3^7),(2^{10},1))$, $((6^3),(2^8,1^2))$ and $((3^5),(2^6,1^3))$.

Now we turn to the exceptional cases. Again, we know that μ has at least three columns so we know that there is another column. This might be located to the left of β, in which case we call it C_1, it might be between α and β, then we call it C_2, or it might be above α, then there is a common row above α which we call R. If $\beta \neq (2)$, we just remove all the common columns and rows but C_1 or C_2 or R and obtain the following seeds:

β	we do not remove C_1	we do not remove C_2	we do not remove R
$(2,1)$	$((4^3,1^2),(3^4,2))$	$((4^3),(3^3,2,1))$	$((3^4),(3,2^4,1))$
(2^2)	$((4^4,1^2),(3^6))$	$((4^4),(3^4,2^2))$	$((3^5),(3,2^6))$
$(2,1^2)$	$((4^4,1^3),(3^5,2^2))$	$((4^5),(3^5,2^2,1^2))$	$((3^5),(3,2^5,1^2))$
$(2^2,1)$	$((4^5,1^3),(3^7,2))$	$((4^5),(3^5,2^2,1))$	$((3^6),(3,2^7,1))$
(2^3)	$((4^6,1^3),(3^9))$	$((4^6),(3^6,2^3))$	$((3^7),(3,2^9))$

The last case we have to look at is $\beta = (2)$. Again, we know that there needs to be another row R or column C_1 or C_2 like above. But if we removed all the common rows and columns but C_1 or C_2 or R, we would obtain a pair which just contains constituents with multiplicity 1 and 2. We have to remove less. Let us start with the case that there is a common column C_2 between α and β. If there was no common row, λ would be a two-row partition. Therefore, we know there is a row R_1 above α or a row R_2 between α and β or a common column C to the left of β. If we remove all the common columns and rows but C_2 and R_1, we obtain the seed $((4^3),(4,3^3,2))$. If we remove all the common columns and rows but C_2 and C, we obtain the seed $((5^2,1),(4^2,3))$. If we remove all the common columns and rows but C_2 and R_2, we obtain the seed $((4^2,2),(3^2,2^2))$ if C_2 and R_2 are disjoint, and $((4^2,3),(3^3,2))$ if they have a common box. From now on we can assume that there is no common column between α and β and by conjugation and interchanging λ and μ we can also assume that there is no common row between α and β. If there is a column C_1 to the left of β and we removed all common columns and rows but C_1, we would obtain $((4^2,1),(3^3))$ which is listed in the lemma and only contains constituents with multiplicity 1 and 2. We know that there must be more boxes. If C_1 has a box below β, we reduce this to the seed $((4^2,1^2),(3^3,1))$. If there is a common column C_3 to the left of C_1, we remove all common rows and columns but C_3 and C_1 to obtain the seed $((5^2,2),(4^3))$. If there is a common row R above α, we remove all common rows and columns but R and C_1 to obtain the seed $((4^3,1),(4,3^3))$. The case that there is a common row R above α is equivalent to the case that there is a common column C_1 to the left of β by conjugation. □

Remark 10.4. In the part $\alpha' = \beta = (2)$ of the previous proof we maybe conjugate and interchange α and μ but since $\alpha' = \beta$ after conjugating and interchanging λ and μ we still have $\alpha' = \beta = (2)$.

Lemma 10.5. *If $\alpha = \beta = (1^m)$ for some $m > 1$, $g_2(\lambda,\mu) \geq 3$.*

Remark 10.6. By conjugation the previous lemma is equivalent to $\alpha = \beta = (m)$ and will be proven in Lemma 10.11 to 10.13.

β is a skew partition.

Lemma 10.7. *If $\alpha = (1^m)$ and one of the connected parts of β is a proper skew partition and located to the left of α (β can be connected), $g_2(\lambda,\mu) \geq 3$.*

Proof: We remove all the common columns and rows to obtain $\tilde{\lambda}$ and $\tilde{\mu}$. Let β^1 be a connected part of β which is a proper skew partition and located to the left of α. We remove all the rows which belong to β but not to β^1 and the fitting number of rows from α. Now we have $\tilde{\beta} = \beta^1$ and $\tilde{\alpha} = (1^{\tilde{m}})$ for some $\tilde{m} \in \mathbb{N}$, where $\tilde{\alpha} = \tilde{\lambda}/(\tilde{\lambda} \cap \tilde{\mu})$ and $\tilde{\beta} = \tilde{\mu}/(\tilde{\lambda} \cap \tilde{\mu})$. If $w(\tilde{\beta}) = 2$, we remove all rows of $\tilde{\beta}$ except for one of the form $(2)/(1)$, one of the form (2) and one of the form (1) (all theses rows have to exist since $\tilde{\beta}$ is a proper and connected skew partition), the corresponding rows of $\tilde{\lambda}$ and the fitting number of rows from $\tilde{\alpha}$ together with the corresponding rows of $\tilde{\mu}$. We obtain the seed $((3^4, 1), (2^6, 1))$. If $w(\tilde{\beta}) \geq 3$ and $\tilde{\beta}$ has a row of the form $(a)/(b)$ with $a - b \geq 3$ we remove all rows of $\tilde{\beta}$ but this one, the corresponding rows of $\tilde{\lambda}$ and the fitting number of rows from $\tilde{\alpha}$ together with the corresponding rows of $\tilde{\mu}$ to obtain $\tilde{\alpha} = (1^{a-b})$ and $\tilde{\beta} = (a - b)$. After removing all the common rows and columns again, we obtain $((a-b+1)^{a-b}) = \tilde{\lambda} = \tilde{\mu}' \supseteq (4^3)$ and the result follows from Lemma 9.1. If all the rows of $\tilde{\beta}$ only consist of one or two boxes, we know that there have to be at least two rows with two boxes which start in different columns. We remove all the rows from $\tilde{\beta}$ besides these two, the corresponding columns of $\tilde{\lambda}$, the right number of rows from $\tilde{\alpha}$ and the corresponding rows of $\tilde{\mu}$. In the next step we remove all the common rows and columns to obtain one of the seeds $((4^4, 1), (3^5, 2))$ or $((5^4, 2), (4^5, 2))$. The second one can only occur if $w(\tilde{\beta}) \geq 4$ and in this case we can also always obtain the first seed if we choose the rows that we do not remove correctly. □

Lemma 10.8. *If $\alpha = (1^m)$ and β decomposes into two or more parts which are up to rotation all proper partitions, $g_2(\lambda, \mu) \geq 3$.*

Proof: We assume that there is a part of β which is located to the left of α. We distinguish the cases by the number and width of the parts left of α. If β has a row of length 3 to the left of α, the same argument as in Lemma 10.2 tells us that there are two constituents with multiplicity 3 or higher. Now we look at the different possibilities for parts of width at most 2 to the left of α. From Theorem 2.4 we know that β decomposes into at most three parts. We look at the following possibilities:

(1) There are two parts of width 2 to the left of α.
(2) There is a part of width 2 and one of width 1 to the left of α.
(3) There are three parts of width 1 to the left of α.
(4) There are two parts of width 1 to the left of α
(5) There is one part of width 2 to the left of α.
(6) There is one part of width 1 to the left of α.

(1): If two of the connected parts from β which are to the left of α have width 2, this can be reduced to the seed $((5^4, 2), (4^5, 2))$.

(2): If there is exactly one part with width 2 and one with width 1 to the left of α, we reduce this to the seed $((4^3, 1), (3^4, 1))$ or $((4^3, 2), (3^4, 2))$.

(3): If β has three connected parts of width 1 and all are to the left of α, we reduce this to the seed $((4^3, 2, 1), (3^4, 2, 1))$. The next three cases are more complex.

(4): If not both of these parts are (1), this can be reduced to one of the seeds $((3^3, 1^2), (2^5, 1))$ and $((3^3, 1), (2^4, 1^2))$. If both of the parts are (1) and there is no part of β which is to the right of α, we know that μ is not a two-column partition so there is another column. Maybe there is a column C_1 to the left of both parts of β or there is a column C_2 between the two parts of β or there is a column C_3 between α and β or there is a column above α but this implies that there is also a row R above α. If we remove all common columns and rows besides C_1, we obtain the seed $((4^2, 2, 1), (3^3, 2))$. If we remove all common rows and column besides C_2, we obtain the seed $((4^2, 2), (3^3, 1))$. If we remove all common columns and rows besides C_3, we obtain the seed $((4^2, 1), (3^2, 2, 1))$, and if we remove all but R, we obtain the seed $((3^3, 1), (3, 2^3, 1))$. If there are two parts of the form (1) to the left of α and one part of β is to the right of α, we remove all the common rows and columns to obtain $\tilde{\lambda}$ and $\tilde{\mu}$. Now we remove all but one row of every part of β, the corresponding columns of $\tilde{\lambda}$ together with the fitting number of rows from α and the corresponding columns of $\tilde{\mu}$. Now $\tilde{\lambda} = (3^{k+3}, 1)$ and $\mu = (3 + k, 2^{k+3}, 1)$ for some $k \geq 1$. If $k = 1$ we have a seed. If $k > 1$ we remove a column of length 1 from $\tilde{\mu}$ and the row of length 1 from $\tilde{\lambda}$, $\tilde{\beta}$ has exactly one part of width 2 to the left of α and one part to the right of α. This is part of the next case.

(5): If there is exactly one part of β to the left of α and that part has width 2, we start with removing all the common rows and columns to obtain $\tilde{\lambda}$ and $\tilde{\mu}$. If β has two parts to the right of α, we remove one of them, the corresponding rows of $\tilde{\lambda}$ together with the right number of rows from α and the corresponding rows of $\tilde{\mu}$. Then we remove all rows of β except for one of every part, the corresponding rows of $\tilde{\lambda}$ together with the fitting number of rows from α and the corresponding rows of $\tilde{\mu}$. Now $\tilde{\lambda} = (3^{k+3})$ and $\tilde{\mu} = (3 + k, 2^{k+3})$. If $k \leq 3$, we have a seed. If $k > 3$ we remove two rows of length 3 from $\tilde{\lambda}$ and three rows of length 2 from $\tilde{\mu}$, then we remove one row of length 3 from $\tilde{\lambda}$ and (3) from $\tilde{\mu}$. Now $\tilde{\lambda} = (3^k)$ and $\mu = (k, 2^k)$. The diagrams have the same form but k was reduced by 3. We repeat this until we obtain a seed with $w(\mu) \leq 6$.

(6): Now we assume that there is exactly one part of β to the left of α and it has width 1. We start with the case that there are two parts of β to the right of α. We remove all the common rows and columns to obtain $\tilde{\lambda}$ and $\tilde{\mu}$. We remove all but one row from every part of β, the corresponding rows of $\tilde{\lambda}$, the fitting number of rows from α and the corresponding columns of $\tilde{\mu}$. Now $\tilde{\lambda} = (2 + b, 2^{a+b+2})$ and $\tilde{\mu} = (2 + a + b, 2 + b, 1^{a+b+2})$ for some $a, b \geq 1$. We remove $(b - 1)$ from $\tilde{\lambda}$ and $b - 1$ rows of length 1 from $\tilde{\mu}$. Then we remove $b - 1$ rows of length 2 from $\tilde{\lambda}$ and $((b-1)^2)$ from $\tilde{\mu}$ to obtain $\tilde{\lambda} = (3, 2^{a+3})$ and $\tilde{\mu} = (3+a, 3, 1^{a+3})$. If $a \leq 2$, we have a seed which contains two constituents with multiplicity 3 or higher. If $a > 2$, we reduce a similarly to the previous case: we remove a row of length 2 from $\tilde{\lambda}$ and two rows of length 1 from $\tilde{\mu}$, then we remove another row of length 2 from $\tilde{\lambda}$ and (2) from $\tilde{\mu}$. We repeat this until we obtain a seed with $w(\tilde{\mu}) \leq 5$.

From now on we assume that only one part of β is to the right of α. We call the part of β which is to the left of α β^1 and the one which is to the right of α β^2. First we assume that β^2 is neither a one-row diagram nor (1^2). If β^2 has only one column, we know that it has at least length 3. This can easily be reduced to the seed $((2^7), (3^3, 1^5))$

Now we can assume that β^2 does not only consist of rows of length 1. We remove all the common rows and columns, all but one row from β^1 and all but two rows of which one has length at least 2 from β^2 the corresponding rows of $\tilde{\lambda}$, the right number of rows from α and the corresponding rows of $\tilde{\mu}$. Then $\tilde{\lambda} = (2^{a+b+3})$ and $\tilde{\mu} = (2 + a, 2 + b, 1^{a+b+2})$ for some $a \geq b \geq 1$. The result follows from Lemma 7.18.

Let us look at the remaining cases, i.e., β has two connected components, β^1 equals (1^b) and is to the left of α and β^2 equals (1^2) or (a) and is to the right of α. If we removed all the common rows and columns, we would have at most one constituent with multiplicity 3 or higher. We would have removed too much. Since λ is not a two-column partition, we know that there needs to be another column. It can be located to the left of β^1 then we call it C_1, between β^1 and α then we call it C_2, between α and β^2 then we call it C_3 or to the right of β^2. In the last case there is a row above β^2 which we call R. We start with the case $\beta^2 = (1^2)$ (so $m = b+2$). If we remove all the common columns and rows but C_1, we obtain $\tilde{\lambda} = (3^{m+2}, 1^{m-2})$ and $\tilde{\mu} = (4^2, 2^{2m-2})$. This can be reduced to the seed $((3^3, 1), (4, 2^3))$. If we remove all common rows and columns but C_2, $\tilde{\lambda} = (3^{m+2})$ and $\tilde{\mu} = (4^2, 2^m, 1^{m-2})$. This can be reduced to the seed $((3^3), (4, 2^2, 1))$. If we remove all common rows and columns but C_3, $\tilde{\lambda} = (3^2, 2^m)$ and $\tilde{\mu} = (4^2, 1^{2m-2})$. This can be reduced to the seed $((3, 2^2), (4, 1^3))$. If we remove all but R, $\tilde{\lambda} = (3, 2^{m+2})$ and $\tilde{\mu} = (3^3, 1^{2m-2})$. This can be reduced to the seed $((3, 2^3), (3^2, 1^3))$.

If $\beta^2 = (a)$ and $\beta^1 = (1^b)$, where $m = a+b$, it works in a similar way as before, but in the end we have to shrink β^2 to obtain a seed. If we remove all common columns and rows but C_1, we obtain $\tilde{\lambda} = (3^{m+1}, 1^b)$ and $\tilde{\mu} = (3+a, 2^{m+b})$. Now we remove $(3^{b-1}, 1^{b-1})$ from $\tilde{\lambda}$ as rows and (2^{2b-2}) from $\tilde{\mu}$ as rows. If $a \leq 3$ this is a seed. If $a > 3$, we remove (3^2) from $\tilde{\lambda}$ and (2^3) from $\tilde{\mu}$ both as rows. In the next step we remove a row of length 3 from $\tilde{\lambda}$ and (3) from $\tilde{\mu}$. We repeat this until $w(\tilde{\mu}) \leq 6$ to obtain a seed. If we remove all the common rows and columns but C_2, we have $\tilde{\lambda} = (3^{m+1})$ and $\tilde{\mu} = (3+a, 2^m, 1^b)$. First we remove (3^{b-1}) from $\tilde{\lambda}$ as rows and $(2^{b-1}, 1^{b-1})$ from $\tilde{\mu}$ as rows. Then we shrink β^2 exactly like before to obtain one of the seeds $((3^3), (4, 2^2, 1))$, $((3^4), (5, 2^3, 1))$ or $((3^5), (6, 2^4, 1))$. If we remove all common rows and columns but C_3, we have $\tilde{\lambda} = (3, 2^m)$ and $\tilde{\mu} = (3+a, 1^{m+b})$. The result follows from Proposition 6.2 since $\tilde{\mu}$ is a hook. If we remove all common rows and columns but R, $\tilde{\lambda} = (2+a, 2^{m+1})$ and $\tilde{\mu} = ((2+a)^2, 1^{m+b})$. We remove (2^{b-1}) from $\tilde{\lambda}$ as rows and (1^{2b-2}) from $\tilde{\mu}$ as rows. In the next step we remove (2^{a-1}) from $\tilde{\lambda}$ as rows and $((a-1)^2)$ from $\tilde{\mu}$. In a last step we remove $(a-1)$ from $\tilde{\lambda}$ and (1^{a-1}) from $\tilde{\mu}$ as rows to obtain the seed $((3, 2^3), (3^2, 1^3))$. □

We often shrink partitions to seeds like in the last cases of the previous lemma. We will not always spell out every detail since it works always in the same way.

Lemma 10.9. *If $\alpha = (1^m)$ and β decomposes into different parts, $g_2(\lambda, \mu) \geq 3$.*

Proof: In case all of the parts from β are up to rotation proper partitions, the previous lemma provides the result. Therefore, we can assume that one of the parts of β is a proper skew partition. If one of the parts of β which is to the left of α is a proper skew partition, Lemma 10.7 provides the result. From now on we assume that the part of β which is a proper skew partition is to the right of α. Remember that we assume that at least one part of β is to the left of α. Further, from Corollary 4.2 we know that β hast at most two parts if one of them is a proper skew partition.

If the part left of β has width greater or equal to 3, we remove the other part and Lemma 10.2 tells us that $g_2(\lambda, \mu) \geq 3$. For the next step we assume that the part to the left of β has width 2. In the proof of the previous lemma we did not need the existence of additional rows or columns in this case. So if we remove all but one row from the skew part of β which is to the right of α, we obtain the result from the previous lemma.

From now on we assume that β has one part to the left of α which has only one column and one part to the right of α which is a proper skew partition. We call the part of β which is to the left of α β^1 and the other part β^2. We know that $\beta^1 = (1^a)$ and β^2 is a skew partition of b with $a + b = m$. We remove all the common rows and columns to obtain $\tilde{\lambda}$ and $\tilde{\mu}$. Since β^2 is a proper skew partition, we know that there are i, j such that $\beta^2_i = (c)/(d)$ and $\beta^2_j = (e)/(f)$ with $c > e$ and $d > f$. We remove all rows of β^2 but i and j, the corresponding columns of $\tilde{\lambda}$ as well as the right number of rows from α together with the corresponding columns of $\tilde{\mu}$. Further, we remove (2^{a-1}) from $\tilde{\lambda}$ as rows and (1^{2a-2}) from $\tilde{\mu}$ as rows. After removing f common columns of length 2, we obtain $\tilde{\lambda} = (2+d-f, 2^{l+1})$ and $\tilde{\mu} = (2+c-f, 2+e-f, 1^{l+1})$, where $l = c+e-d-f$. If $e \leq d$, the part of $\tilde{\beta}$ which is to the right of $\tilde{\alpha}$ decomposes into two partitions and the result follows because of Lemma 10.8. If $e > d$, we remove $f - d$ rows of length 2 from $\tilde{\lambda}$ and $f - d$ columns of length 2 from $\tilde{\mu}$. Now the part of $\tilde{\beta}$ which is to the right of α decomposes into two diagrams and the result follows from Lemma 10.8. □

Remark 10.10. The cases $\alpha = (1^m)$ and $\alpha = (m)$ are connected by conjugation. We still have to prove the case that $\alpha = \beta = (m)$. But if $\alpha = (m)$ and β decomposes into several parts, we can assume that all of these parts are to the left of α (otherwise it would be covered by the previous lemma).

3. $\alpha = (m)$

β **is a partition.**
If $\alpha = (m)$, β is a partition and we removed all the common rows and columns, we would obtain $\tilde{\lambda} = (w(\beta) + m)$. In this case there is a common row or column which we do not remove and we know that such a row or column exists because λ is not linear. There can be a common row above α or in between α and β or the row can be to the left of β. In the last case there is a common column to the left of β. We look at these three cases in the next three lemmas.

Lemma 10.11. *Let $\alpha = (m)$ and $\beta \vdash m$ is a partition. If there is a common row above α, $g_2(\lambda, \mu) \geq 3$.*

Proof: Let us assume that β is neither a one-column partition nor (2). Then we remove all the common rows and columns besides one row R above α. We obtain $\tilde{\lambda} = ((w(\beta) + m)^2)$ and $\tilde{\mu} = (w(\beta) + m, w(\beta), \beta)$. In this case the statement follows because of Lemma 7.18. Two remaining cases are exceptional. If we removed all the common rows and columns but one above α, we would obtain $\tilde{\lambda} = ((m + 1)^2)$ and $\tilde{\mu} = (m + 1, 1^{m+1})$ if β is a one-column partition and $\tilde{\lambda} = (4^2)$ and $\tilde{\mu} = (4, 2^2)$ if $\beta = (2)$. In both cases $\tilde{\lambda}$ would be a two-row partition so we know that there needs to be another common row.

First we look at the case $\beta = (2)$. Since we assume that λ is not a two-row partition, we know that there is another row R_1 above α or a row R_2 between α and β or a column C to the left of β. We remove all the common rows and columns but R and R_1 or R_2 or C to obtain one of the seeds $((4^3), (4^2, 2^2))$, or $((4^2, 2), (4, 2^3))$, or $((5^2, 1), (5, 3^2))$.

If $\beta = (1^m)$, the situation is similar, but here we have to shrink β to obtain the seed. We know that λ is not a two-row partition, so there has to be another row R_1 above α or a row R_2 between α and β or a column C to the left of β. We start with the case that there is a column C to the left of β. If we remove all the common rows and columns besides R and C, we obtain $\tilde{\lambda} = ((m + 2)^2, 1^m) = \tilde{\mu}'$. The result follows because of Lemma 9.1. If there is another row R_1 above α and we remove all the common rows and columns besides R and R_1, we obtain $\tilde{\lambda} = ((m + 1)^3)$ and $\tilde{\mu} = ((m + 1)^2, 1^{m+1})$. If $m \geq 5$, we do the following: We remove (2^3) from $\tilde{\lambda}$ and (3^2) from $\tilde{\mu}$ and in the next step we remove (3) from $\tilde{\lambda}$ and (1^3) from $\tilde{\mu}$ as rows. We repeat this process until we obtain $\tilde{\lambda}$ with a width smaller or equal to 5. This leads to the seeds $((5^3), (5^2, 1^5))$, $((4^3), (4^2, 1^4))$ and $((3^3), (3^2, 1^3))$. The last case is that there is a row R_2 between α and β. After removing all the common rows and columns but R and R_2 this leads to $\tilde{\lambda} = ((m + 1)^2, 1)$ and $\tilde{\mu} = (m + 1, 1^{m+2})$. If $m > 2$, we have two constituents because of Proposition 6.1 and Proposition 6.2. If $m = 2$, there is only one constituent with multiplicity 3 so we would have removed too much. But we assume that μ is no hook, so we know that there is another common row or column. Because of the previous cases we can assume that there is no other row above α and no column to the left of β. So we know there is a common column C between α and β. We obtain the seed $((4^2, 2), (4, 2, 1^2))$ if R_2 and C have a common box, and $((4^2, 1), (4, 2, 1^3))$ if they do not. □

From now on we can assume that there is no common row above α. In the previous lemma we have seen that removing all common columns and rows but one might be too much in some cases. If it is, this will reduce the cases we have to look at.

Lemma 10.12. *Let $\alpha = (m)$ and $\beta \vdash m$ is a partition. If there is a common row R between α and β, $g_2(\lambda, \mu) \geq 3$.*

Proof: First we assume that $w(\beta) \geq 4$ or $w(\beta) \geq 3$ and $\text{rem}(\beta) \geq 2$. In this case we remove all the common rows and columns but R. We obtain $\tilde{\lambda} = (m + w(\beta), w(\beta))$ and $\tilde{\mu} = (w(\beta)^2, \beta)$. The result follows from Lemma 7.17 if $w(\beta) = 3, 4$, and Lemma 7.19 if $w(\beta) > 4$.

We assume that $\beta = (3^l)$ for some $l \geq 1$. Since λ is not a two-row partition, we know that there is another common row. Because of the previous lemma we can assume that it is not above α, so we know that there is another common row R_1 between α and β or that there is a common column C to the left of β. If there is another row R_1 between α and β, we remove all common rows and columns but R and R_1. In the next step we remove (3^{l-1}) from $\tilde{\mu}$ as rows and $(3(l-1))$ from $\tilde{\lambda}$ to obtain the seed $((6, 3^2), (3^4))$. If there is a common column C to the left of β, we remove all common columns and rows but C and R. If $l = 1$, we obtain the seed $((7, 4, 1), (4^3))$. If $l = 2$, we remove the rightmost column of β, the corresponding column of $\tilde{\lambda}$ together with the right number of boxes from α to obtain the seed $((7, 3, 1^2), (3^4))$. If $l(\beta) > 2$, we remove the two rightmost columns from β, the corresponding columns of $\tilde{\lambda}$ together with the right number of boxes from α. Now $\tilde{\mu}$ is a two-column rectangle and $\tilde{\lambda} = (l(\beta) + 2, 2, 1^{l(\beta)})$. The result follows because of Lemma 7.18.

If $w(\beta) = 2$ and there is a column C to the left of β, this works in the same way. If $l(\beta) = 1, 2$, we remove all the common rows and columns but R and C and obtain the seed $((5, 3, 1), (3^3))$ if $\beta = (2)$, $((6, 3, 1^2), (3^3, 2))$ if $\beta = (2, 1)$ and $((7, 3, 1^2), (3^4))$ if $\beta = (2^2)$. If $l(\beta) > 3$, we remove all the common columns and rows but R and C, then we remove the right column from β and the corresponding column from λ together with the right number of boxes from α and we are in the same situation as in the case that $w(\beta) = 3$ and $l(\beta) > 2$. Even if $w(\beta) = 1$ and $l(\beta) > 2$, this method works.

By conjugation the case $\beta = (1^2)$ and there is a column to the left of β is covered by the previous lemma.

Therefore we can assume for the last missing case that $w(\beta) \leq 2$ and that there is no column to the left of β. We still assume that there is no row above α. In the first step we look at the case $w(\beta) = 2$. We know that λ is not a two-row partition, so there is another row R_1 between α and β. If β is not a rectangle, we remove all the common rows and columns but R and R_1. We obtain $\tilde{\lambda} = (m + 2, 2^2)$ and $\tilde{\mu} = (2^{a+3}, 1^b)$ for some $a, b \in \mathbb{N}$ with $2a + b = m$. Now we remove $(m - 3)$ from $\tilde{\lambda}$ and $(2^{a-1}, 1^{b-1})$ from $\tilde{\mu}$ as rows to obtain the seed $((5, 2^2), (2^4, 1))$. This also follows from the results of Section 2 of Chapter 7. If β was a rectangle, this would maybe only contain one constituent with multiplicity 3. In this case we know that μ is not a two-column partition. Therefore, we know that there is another common column C between α and β. If we remove all common rows and columns but R, R_1 and C, we obtain $\tilde{\lambda} = (3 + m, 3^a, 2^{2-a})$ and $\tilde{\mu} = (3^{1+a}, 2^{2-a+\frac{m}{2}})$, where $a = 0, 1, 2$ is one smaller than the length of C after we removed all the common rows and columns. Now we remove $(m - 2)$ from $\tilde{\lambda}$ and $(2^{\frac{m}{2}-1})$ from $\tilde{\mu}$ as rows to obtain one of the seeds $((5, 3^a, 2^{2-a}), (3^{1+a}, 2^{3-a}))$.

The last case is $w(\beta) = 1$ and there is no column to the left of β nor a row above α. We know that neither λ nor μ is a two-line partition. Since there are no common columns to the left of β and no rows above α, this implies that there are two common columns and another common row between α and β. Since neither λ nor μ is a hook, we can choose them in such a way that the left one of the common columns has length ≥ 2 after removing all the common rows and columns but these 4. After removing all the common rows and columns we remove $(m-1)$ from $\tilde{\lambda}$ and (1^{m-1}) from $\tilde{\mu}$ as rows to obtain one of the following seeds depending on the length of the common rows and columns:

$$((4,2,1),(3,2,1^2)),\ ((4,3,1),(3^2,1^2)),\ ((4,2^2),(3,2^2,1)),$$
$$((4,3,2),(3^2,2,1)),\ ((4,3^2),(3^3,2)).$$

All of these seeds contain several constituents with multiplicity 3 or higher. □

Lemma 10.13. *Let $\alpha = (m)$ and $\beta \vdash m$ is a partition. If there is a common column C to the left of β, $g_2(\lambda,\mu) \geq 3$.*

Proof: Because of the previous two lemmas we can assume that there is no common row above α or between α and β. If β is different from (m), (1^m), $(k,k+1)$ for $m = 2k+1$, (k^2) for $m = 2k$ or (k^3) for $m = 3k$, we remove all the common rows and columns but C. Now $\tilde{\lambda}$ is a hook and the result follows from Proposition 6.2. In the five remaining cases we know that there is another common row or column because λ is not a hook. Since we assume that there is no row above α or between α and β, we know that there is a second column C_1 to the left of β. If $\beta \neq (m)$, we remove all the common rows and columns but C and C_1. If $\beta \neq (1^m)$, we know that $l(\beta) = 2, 3$. We remove all but the leftmost column of β and the corresponding columns of $\tilde{\lambda}$ together with the right number of columns of length 1 from α to obtain the seed $((5,2^2),(3^3))$ if $l(\beta) = 2$ or $((6,2^3),(3^4))$ if $l(\beta) = 3$. If $\beta = (1^m)$ for some $m \geq 2$, we reduce this by 3 as we have done in the previous lemmas to obtain one of the previous seeds or $((7,2^4),(3^5))$. If $\alpha = (m)$, we know that λ and μ are not two-row partitions, so there is a common row R below β which intersects with C or C_1. If we remove all the common rows and columns but C_1, C and R, we obtain $\tilde{\lambda} = (2+2m,2,a)$ and $\tilde{\mu} = ((2+m)^2,a)$, where $a = 1$ if R only intersects with one of the columns C and C_1, and $a = 2$ if it intersects with both. We remove $(2m-2)$ from $\tilde{\lambda}$ and $((m-1)^2)$ from $\tilde{\mu}$ to obtain the seed $((4,2,a),(3^2,a))$. □

Remark 10.14. The previous three lemmas prove the case $\alpha = \beta = (m)$ which is equivalent to the case $\alpha = \beta = (1^m)$. So they prove Lemma 10.5.

β^{rot} is a partition.

Now we want to look at the case that β^{rot} is a partition. Because of the previous three lemmas we can assume that β is not a partition, so β^{rot} has at least two removable nodes.

Lemma 10.15. *If $\alpha = (m)$ and β^{rot} is a partition with three or more removable nodes, $g_2(\lambda,\mu) \geq 3$.*

Proof: We start by removing all the common rows and columns to obtain $\tilde{\lambda}$ and $\tilde{\mu}$. Since β^{rot} has three or more removable nodes, we know β has at least two columns C_1 and C_2 of different length with length strictly smaller than $l(\beta)$. We distinguish two cases. If the length of C_1 and C_2 is not $l(\beta)-1$ and $l(\beta)-2$, we remove all columns from β but C_1 and C_2, and the corresponding columns from $\tilde{\lambda}$ together with the right number of boxes from α. Now $\tilde{\mu}$ is a two-column rectangle and $\tilde{\lambda}$ has three removable boxes and is neither of the form $(l,2,1)^{(\prime)}$ for some $l \in \mathbb{N}$ nor $(5,4,1)$. Lemma 7.18 provides the result. If C_1 has length $l(\beta)-1$ and C_2 has length $l(\beta)-2$, we know that there is a column C_3 of β with length $l(\beta)$. We remove all columns of β but C_1, C_2 and C_3, further, we remove the corresponding columns of $\tilde{\lambda}$ together with the right number of boxes from α. We obtain $\tilde{\lambda} = (l, 2, 1)$ for some $l \in \mathbb{N}$ and $\tilde{\mu} = (3^r)$ for some $r \geq 4$. Then removing $r - 4$ rows of length 3 from $\tilde{\mu}$ and $(3r - 12)$ from $\tilde{\lambda}$ gives us the seed $((9, 2, 1), (3^4))$. □

Lemma 10.16. *If $\alpha = (m)$ and β^{rot} is a partition with two removable nodes, $g_2(\lambda, \mu) \geq 3$.*

Proof: Let $\beta^{\text{rot}} = (a^b, c^d)$. If $a - c \geq 2$ and $d \geq 3$, we remove all the common rows and columns. In the next step we remove all columns from β but two of length b and the corresponding columns of $\tilde{\lambda}$ together with the right number of boxes from α. We obtain $\tilde{\mu}$, a two-column rectangle and $\tilde{\lambda}$, a proper fat hook of the form $(\tilde{n}, 2^3)$ for some $\tilde{n} \in \mathbb{N}$. Therefore, Lemma 7.18 provides the result.

If $a - c \geq 2$ and $d = 2$, we remove all the common columns and rows. Then we remove $c - 1$ columns of length $b + 2$ from β and $a - c - 2$ columns of length b from β, the corresponding columns of $\tilde{\lambda}$ together with the right number of boxes from α. Then we remove (3^{b-1}) from $\tilde{\mu}$ as rows and $(3(b-1))$ from $\tilde{\lambda}$ to obtain the seed $((8, 2^2), (3^4))$.

Now we assume that $d = 1$ and $a - c \geq 4$. After removing all the common rows and columns we have $\tilde{\lambda} = (a + m, a - c)$ and $\tilde{\mu}$ is a proper rectangle, so Lemma 7.19 provides the result.

Now only the exceptional cases are missing. We know that $\beta^{\text{rot}} = (a^b, c^d)$ and $d = 1$ and $a - c = 2, 3$ or $d \geq 1$ and $a - c = 1$. In these cases we do not remove all the common rows and columns. We start with the case $d \geq 1$ and $a - c = 1$. Since we assume that λ is not a hook, we know that there is a common row or column. First we assume that there is a common column C to the left of β. We remove all common rows and columns but C. If $\beta^{\text{rot}} = (2, 1)$, $\beta^{\text{rot}} = (2, 1^2)$ or $\beta^{\text{rot}} = (2^2, 1)$, we already have the seed $((6, 2, 1), (3^3))$, $((7, 2^2, 1), (3^4))$ or $((8, 2, 1^2), (3^4))$. If $l(\beta) = 2, 3$, we remove all but the leftmost and rightmost column of β, the corresponding columns of $\tilde{\lambda}$ and the right number of boxes from α. Then $\tilde{\beta}^{\text{rot}} = (2, 1)$ or $(2, 1^2)$ or $(2^2, 1)$ and we have the same seeds. If $l(\beta) > 3$, we remove all but the leftmost column of β and the corresponding columns of $\tilde{\lambda}$ together with the right number of boxes from α. Now $\tilde{\mu}$ is a two-column rectangle of length ≥ 5 and $\tilde{\lambda}$ has three removable nodes with length and width ≥ 5, so the result follows from Lemma 7.18.

If there is a common row R above α, we remove all common rows and columns but R. In the next step we remove all but the two leftmost columns from β, the corresponding columns of $\tilde{\lambda}$ and the right number of columns from α together with the corresponding columns from $\tilde{\mu}$. Now we know $\tilde{\lambda} = ((\tilde{m} + 2)^2, 1^d)$ and $\tilde{\mu} = (\tilde{m} + 2, 2^{1+b+d})$, where $\tilde{m} = d + 2b$. We start with removing $((b - 1)^2)$ from $\tilde{\lambda}$ and (2^{b-1}) from $\tilde{\mu}$ as rows. In the next step we remove $((b - 1)^2)$ from $\tilde{\lambda}$ and $(2b - 2)$ from $\tilde{\mu}$. Now we remove (1^{d-1}) from $\tilde{\lambda}$ as rows and $(d - 1)$ from $\tilde{\mu}$. In the last step we remove $((d - 1)^2)$ from $\tilde{\lambda}$ and (2^{d-1}) from $\tilde{\mu}$ as rows to obtain the seed $((5^2, 1), (5, 2^3))$.

If there is neither a common row above α nor a common column to the left of β, this implies that there is a common row R between α and β. We start with the case $d > 1$. We remove all common rows and columns but R and if $a > 2$, we remove all but the two leftmost columns of β together with the corresponding columns from $\tilde{\lambda}$ and the right number of columns from α. Now $\tilde{\mu}$ is a two-column rectangle and $\tilde{\lambda} = (\tilde{m} + 2, 2, 1^d)$ has three removable boxes. The result follows from Lemma 7.18 (since $d > 1$ and $l(\tilde{\mu}) \geq 5$). Now we assume that there is a row R between α and β and that $a - c = d = 1$. We start with the case $a \geq 3$. We remove all the common rows and columns but R. In the next step we remove all but the three leftmost columns from β, the corresponding columns from $\tilde{\lambda}$ and the right number of columns (of length 1) from α. If $l(\beta) = 2$ we obtain the seed $((8, 3, 1), (3^4))$.

If $l(\beta) > 2$ we remove $l(\beta) - 2$ rows of length 3 from $\tilde{\beta}$ and $(3(l(\beta) - 2))$ from $\tilde{\lambda}$ to obtain the same seed. Now we assume that $a = 2$, $d = 1$ and that there is a common row R between α and β. If we removed all the common rows and columns but R, we would obtain a pair which is listed in Proposition 7.1. In particular, $\tilde{\mu}$ would be a two-column rectangle. But our assumption is that μ is not a two-line partition. Since we assume that there is not a common row above α nor a common column to the left of β (since we have looked at these cases already), we know there is a common column C between α and β. We remove all the common rows and columns but R and C. The column C can now be of length 1 or 2. We remove $l(\beta) - 2$ rows of length 2 from β and the corresponding number of boxes from α to obtain the seed $((6, 2, 1), (3, 2^3))$ if C has length 1 and $((6, 3, 1), (3^2, 2^2))$ if C has length 2.

The last missing case is $d = 1$ and $a - c = 2, 3$. The case $a - c = 1$ is contained in the previous cases. If we removed all the common rows and columns, $\tilde{\lambda}$ would be a two-row partition. We know that there is a common row R_1 above α or a common row R_2 between α and β or a common column C to the left of β. Let us start with the case that there is a common row R_1 above α. We remove all the common rows and columns but R_1. In the next step we remove all but the rightmost and the leftmost column of β together with the corresponding columns of $\tilde{\lambda}$, the right number of columns from α together with the corresponding columns of $\tilde{\mu}$. If $l(\tilde{\beta}) = 2$, we obtain the seed $((5^2, 1), (5, 2^3))$. If $l(\tilde{\beta}) > 2$, we first remove $(2^{l(\tilde{\beta})-2})$ from $\tilde{\mu}$ as rows and $((l(\tilde{\beta})-2)^2)$ from $\tilde{\lambda}$, then we again remove $((l(\tilde{\beta})-2)^2)$ from $\tilde{\lambda}$ and $(2l(\tilde{\beta}) - 4)$ from $\tilde{\mu}$ to obtain the same seed as for $l(\tilde{\beta}) = 2$.

If there is a common row R_2, we remove all the common rows and columns but R_2. Further, we remove all but the two leftmost and the rightmost column of β, the corresponding columns of $\tilde{\lambda}$ together with the right number of boxes from α. In the last step we remove $l(\beta) - 2$ rows of length 3 from $\tilde{\mu}$ and the same number of boxes from $\tilde{\alpha}$. We obtain the seed $((7, 3, 2), (3^4))$.

The last case is that there is a common column C to to left of β. We remove all the common rows and columns but C. If $l(\beta) = 2, 3$, we remove all but the two leftmost columns of β, the corresponding columns of $\tilde{\lambda}$ and the right number of boxes from α. This gives us one of the seeds $((5, 3, 1), (3^3))$ and $((7, 3, 1), (3^4))$. If $l(\beta) > 3$, we remove all but the leftmost column of β, the corresponding columns of $\tilde{\lambda}$ and the fitting number of boxes from α to obtain $\tilde{\lambda} = (l(\beta) + 1, 2, 1^{l(\beta)-1})$ and $\tilde{\mu} = (2^{l(\beta)+1})$. The result follows from Lemma 7.18. □

β **is a connected skew partition.**

Lemma 10.17. *If $\alpha = (m)$ and $\beta = \beta^1/\beta^2$ is a connected proper skew partition of m, $g_2(\lambda, \mu) \geq 3$.*

Proof: We start with $\mathrm{rem}(\beta^2) \geq 2$. This means $w(\beta) \geq 3$ and that there is a column C_2 such that in this column β^2 is not zero and smaller than $l(\beta^2)$. Let C_1 be the leftmost column of β and C_3 the rightmost column of β. We remove all the common rows and columns. In the next step we remove all columns of β but C_1, C_2 and C_3, the corresponding columns from $\tilde{\lambda}$ and the right number of boxes from α. Note that $\tilde{\beta}$ is not necessarily connected any more. Now we remove the last $l(\tilde{\beta}^1) - l(\tilde{\beta}^2)$ rows from $\tilde{\mu}$ and the corresponding number of boxes from $\tilde{\alpha}$. If there are any common rows, we remove them, too (there is a common column but we do not remove that). If $\tilde{\beta}^2 = (2, 1)$, we have the seed $((6, 2, 1), (3^3))$ or $((5, 2, 1), (3^2, 2))$. If $\tilde{\beta}^2 = (2, 1^2)$, we remove the last row of $\tilde{\mu}$ and the right number of boxes from $\tilde{\lambda}$ to obtain the seed $((4, 2, 1^2), (3^2, 2))$ or $((5, 2, 1^2), (3^3))$. If $\tilde{\beta}^2 = (2^2, 1)$, we have one of the seeds $((5, 2^2, 1), (3^2, 2^2))$, $((6, 2^2, 1), (3^3, 2))$ and $((7, 2^2, 1), (3^4))$. If $\tilde{\beta}^2 = (2^3, 1)$

and $\tilde{\lambda}$ has a row of length 2, we reduce this to the case $\tilde{\beta}^2 = (2^2, 1)$, otherwise, we have the seed $((8, 2^3, 1), (3^5))$. In the other cases we remove the rightmost column of $\tilde{\beta}$, the corresponding column of $\tilde{\lambda}$ and the right number of boxes from $\tilde{\alpha}$. Now $\tilde{\mu}$ is a two-column rectangle and the result follows from Lemma 7.18.

From now on we assume that β^2 is a rectangle. We start with $w(\beta^2) \geq 2$ and $l(\beta^2) \geq 2$. We remove all the common rows and columns. In the next step we remove all the columns of β which are to the right of β^2 besides the leftmost of these columns, further we remove the corresponding columns of $\tilde{\lambda}$ and the right number of boxes from α. For $\tilde{\beta} = \tilde{\beta^1}/\beta^2$ we know that all the columns of $\tilde{\beta^1}$ are longer than $l(\beta^2)$. This means $\tilde{\beta} = ((w(\beta^2)+1)^{l(\beta^2+1)})/\beta^2 \cup \pi$ for some partition π which can be removed. If we remove π together with the fitting number of columns from α, $\tilde{\beta}^{\text{rot}}$ is a partition, so the result follows from the previous lemma from the part $a - c \geq 2$ and $d \geq 2$. In this part of the proof we did not need any additional rows or columns.

Now we look at the cases $l(\beta^2)$ or $w(\beta^2)$ equals 1. We start with the $l(\beta^2) = 1$. Let $\beta^2 = (a)$. If $a \geq 3$, we remove all the common rows and columns. Now $\tilde{\lambda} = (w(\beta) + m, a)$ and $\tilde{\mu}$ is neither a rectangle, a two-line partition, a hook nor can we get one of the exceptional cases. The result follows from Lemma 7.17 if $a = 3, 4$, and Lemma 7.19, otherwise. If $a = 1, 2$, we do not remove all the common rows and columns. Since λ is not a two-row partition, we know that there is a common row R_1 above α or a common row R_2 between α and β or a common column C to the left of β. If there is a common row R_1, we remove all the common rows and columns but R_1. Now $\tilde{\lambda} = ((w(\beta) + m)^2, a)$ and $\tilde{\mu} = (w(\beta) + m, w(\beta), \beta^1)$. Since β is a proper skew partition, we know that $m \geq 4$. We remove the row of length a from $\tilde{\lambda}$ and (a) from $\tilde{\mu}$. Now $\tilde{\lambda}$ is a two-row rectangle and the result follows from Lemma 7.18. If there is a common row R_2, we remove all the common rows and columns but R_2. Now $\tilde{\lambda} = (w(\beta) + m, w(\beta), a)$ and $\tilde{\mu} = (w(\beta)^2, \beta^1)$. If $w(\beta) > 2$, we remove all but the top row of β^1 and the right number of boxes from α. We obtain $\tilde{\lambda} = (2w(\beta) - a, w(\beta), a)$ and $\tilde{\lambda} = (w(\beta)^3)$. This can be reduced to the seed $((6 - a, 3, a), (3^3))$. If $w(\beta) = 2$, we know that the lowest row of β^1 has length 1. We remove all but the highest and the lowest row from β^1 together with the right number of boxes from α to obtain the seed $((4, 2, 1), (2^3, 1))$. If there is a column C to the left of β, we remove all the common rows and columns but C. If $l(\beta) > 2$ we remove all but the leftmost column of β, the corresponding columns of $\tilde{\lambda}$ together with the right number of boxes from α. Now $\tilde{\lambda} = (1 + l(\beta), 2, 1^{l(\beta)-1})$ and $\tilde{\mu} = (2^{l(\beta)+1})$, so the result follows from Lemma 7.18. If $l(\beta) = 2$, we remove all columns of β but the leftmost and the rightmost, the corresponding columns of $\tilde{\lambda}$ together with the right number of boxes from α to obtain the seed $((5, 2, 1), (3^2, 2))$.

From now on we assume that $\beta^2 = (1^a)$ for some $a > 1$. If $a > 3$ or $a = 3$ and $w(\beta^1) > 2$ or $a = 2$ and $w(\beta^1) > 2$ and β^1 is different from $(b^c, 1)$ and $(b^c, c-1)$, we remove all the common rows and columns. Now $\tilde{\lambda}$ is a hook and the result follows from Proposition 6.2. The remaining cases are exceptional. We start with $a = 2$, 3 and β^1 is a two-column partition, but not a two-column rectangle because β is a proper skew partition. We know that μ is not a two-column partition, so there is a common column C_1 to the left of β or a common column C_2 between α and β or a common row R above α. If we remove all the common rows and columns but C_1, we remove the rightmost column of β together with the right number of boxes from α. Now $\tilde{\mu}$ is a two-column rectangle and $\tilde{\lambda}$ has three removable boxes, and further, $l(\tilde{\lambda}) \geq 5$ and $w(\tilde{\lambda}) \geq 4$, so the result follows from Lemma 7.18. If there is a common column C_2, we remove all the common columns and rows except for C_2. Now $\tilde{\lambda}$ is a hook of width and length ≥ 3 and $\tilde{\mu}$ has three columns of different size and is different from $(3, 2, 1)$. The result follows from Proposition 6.2. If there is a common row R, we remove all the common rows and columns but R. In the next step we remove (1^a) from $\tilde{\lambda}$ as rows and (a) from $\tilde{\mu}$. This works since $m > a$. Now $\tilde{\lambda}$ is a two-row rectangle of width ≥ 7 and $\tilde{\mu}$ has three removable nodes, $l(\tilde{\mu}), w(\tilde{\mu}) \geq 4$ (since $m > a+1$) and $\tilde{\mu} \neq (4,2,1^2)$. The result follows from Lemma 7.18. The last case we look at is $a = 2$ and β^1 equals $(b^c, 1)$ or $(b^c, b-1)$ for $b \geq 3$. We know that λ is not a hook, so there is a common row R_1 above α or a common row R_2 between α and β or a common column C to the left of β. Note that informally speaking a common column between α and β only makes λ not a hook if it is of length ≥ 2, but this implies that there is a common row above α or between α and β. If there is a row R_1 above α and we remove all the common rows and columns but R_1, we remove the two rows of length 1 from $\tilde{\lambda}$ and (2) from $\tilde{\mu}$. Now $\tilde{\lambda} = ((w(\beta)+m)^2)$ and $\tilde{\mu} = (w(\beta)+m-2, w(\beta), \beta^1)$ and the result follows from Lemma 7.18. If there is a row R_2 and we remove all the common rows and columns but R_2, we remove all but the two leftmost columns of β, the corresponding columns of $\tilde{\lambda}$ together with the right number of columns from α. In the next step we remove all rows below the fifth one of $\tilde{\mu}$ and the right number of boxes from $\tilde{\alpha}$. Now we obtain the seed $((6,2,1^2),(2^5))$. If there is a common column C, we remove all the common rows and columns but C. If $l(\beta) \geq 4$, we remove all but the leftmost column from β, the corresponding columns of $\tilde{\lambda}$ together with the right number of boxes from α. We obtain $\tilde{\lambda} = (l(\beta), 2^2, 1^{l(\beta)-2})$ and $\tilde{\mu} = (2^{l(\beta)+1})$ and the result follows from Lemma 7.18. If $l(\beta) = 3$, we know that $\beta^1 = (b^2, b-1)$, otherwise, β would not be connected. We remove all but the two leftmost columns of β, the corresponding columns of α together with the right number of boxes from α to obtain the seed $((7,2^2,1),(3^4))$. □

β decomposes into three connected components.

Lemma 10.18. *If $\alpha = (m)$ and β decomposes into three parts, $g_2(\lambda,\mu) \geq 3$.*

Proof: By Corollary 4.2 we know that all three parts of β are partitions or rotated partitions. Further, we assume that all the parts of β are to the left of α (see Remark 10.10). We start with removing all the common rows and columns. Then we remove all but one column from every part of β, the corresponding columns of $\tilde{\lambda}$ together with the fitting number of columns from α. If a part of β is a rotated partition, $\tilde{\lambda}$ and $\tilde{\mu}$ might have common rows now, which we remove. Now $\tilde{\lambda} = (a+b+c+3, 2^a, 1^b)$ and $\tilde{\mu} = (3^{a+1}, 2^b, 1^c)$ for $a,b,c \geq 1$. If $c > 1$, we remove $(c-1)$ from $\tilde{\lambda}$ and (1^{c-1}) from $\tilde{\mu}$ as rows to obtain $\tilde{\lambda} = (4+a+b, 2^a, 1^b)$ and $\tilde{\mu} = (3^{a+1}, 2^b, 1)$. If $b > 2$, we first remove (1^2) from $\tilde{\lambda}$ as rows together with (2) from $\tilde{\mu}$ as row. In the next step we remove (2) from $\tilde{\lambda}$ and (2) from $\tilde{\mu}$ as row. We repeat this process until we obtain $\tilde{\lambda} = (4+a+\tilde{b}, 2^a, 1^{\tilde{b}})$ and $\tilde{\mu} = (3^{a+1}, 2^{\tilde{b}}, 1)$ with $\tilde{b} = 1,2$. With essentially the same procedure for a we can reduce these partitions to one of the seeds $\tilde{\lambda} = (4+\tilde{a}+\tilde{b}, 2^{\tilde{a}}, 1^{\tilde{b}})$ and $\tilde{\mu} = (3^{\tilde{a}+1}, 2^{\tilde{b}}, 1)$, where $\tilde{a} = 1,2,3$ and $\tilde{b} = 1,2$. □

β decomposes into two connected components.

Lemma 10.19. *If $\alpha = (m)$ and β decomposes into two parts of which one is a proper skew partition or a rotated partition (but not a rectangle), $g_2(\lambda,\mu) \geq 3$.*

Proof: In all cases we start with removing all the common rows and columns to obtain $\tilde{\lambda}$ and $\tilde{\mu}$ and we assume that all the parts of β are to the left of α (see Remark 10.10). If β has a part β^1 which is a proper skew partition, there are two skew columns which start in different rows and end in different rows. We call these columns C_1 and C_2. Further, we choose a column C_3 from the other connected part of β. Now we remove all columns of β but C_1, C_2, C_3 and the fitting number of columns from α. If there is no row where C_1 and C_2 both have a box, the rest of the skew partition decomposes into two parts. The result follows from the previous lemma. From now on we can assume that after removing all those rows and columns $\tilde{\beta}$ still has only two connected components. We start with the case that the right part of β is the skew partition. Maybe after removing common rows, $\tilde{\lambda} = (3+\tilde{m}, 2^a, 1^{b+c})$ and $\tilde{\mu} = (3^{a+b+1}, 2^c, 1^d)$, where $\tilde{m} = a+2b+c+d$. Now we remove (d) from $\tilde{\lambda}$ and (1^d) from $\tilde{\mu}$ as rows. If $a=b=c=1$, we have the seed $((7,2,1^2),(3^3,2))$. If $a+b+c>3$, we remove $(a+b+1)$ from $\tilde{\lambda}$ and (1^{a+b+1}) from $\tilde{\mu}$. Since $\tilde{\mu}$ is a two-column rectangle of length ≥ 5 and $\tilde{\lambda}$ has three removable nodes and $l(\tilde{\lambda}), w(\tilde{\lambda}) \geq 4$, the result follows from Lemma 7.18. If the skew part is the left one of the connected parts of β, the same method again leads to a rectangular two-column partition $\tilde{\mu}$ of length ≥ 5 and a partition with three removable nodes $\tilde{\lambda}$ with $l(\tilde{\lambda}), w(\tilde{\lambda}) \geq 4$ or to the seed $((8,2,1),(3^2,2^2,1))$.

Now let us assume that β has one part which is a rotated partition. We call the lower left part of β β^2 and the upper right part β^1. If β^1 is a rotated partition, we remove all but one column from β^2, all but the leftmost and the rightmost column from β^1, the corresponding columns of $\tilde{\lambda}$ together with the fitting number of columns of length 1 from α. Then we remove the remaining boxes of β^2 and the fitting number of boxes from α. If $l(\beta^1) = 2,3$ or $\tilde{\beta^1}^{\mathrm{rot}} = (2,1^3)$, we have the seed $((6,2,1),(3^3))$ if $\tilde{\beta^1}^{\mathrm{rot}} = (2,1)$, $((7,2^2,1),(3^4))$ if $\tilde{\beta^1}^{\mathrm{rot}} = (2,1^2)$, $((8,2,1^2),(3^4))$ if $\tilde{\beta^1}^{\mathrm{rot}} = (2^2,1)$, and $((8,2^3,1),(3^5))$ if $\tilde{\beta^1}^{\mathrm{rot}} = (2,1^3)$. If $l(\tilde{\beta}^2) > 3$ and $\tilde{\beta^1}^{\mathrm{rot}} \neq (2,1^3)$, we remove the rightmost column of $\tilde{\beta}^1$, the corresponding column of $\tilde{\lambda}$ together with the right number of boxes from $\tilde{\alpha}$. Now $\tilde{\mu}$ is a two-column rectangle and $\tilde{\lambda}$ has three removable boxes and since $\tilde{\beta^1}^{\mathrm{rot}} \neq (2,1^3)$, we cannot get an exceptional case. The result follows from Lemma 7.18.

If β^2 is a rotated partition, we remove all but the leftmost and the rightmost column from β^2, all but the leftmost column from β^1, the corresponding columns of $\tilde{\lambda}$ together with the right number of columns of length 1 from α. We obtain $\tilde{\lambda} = (3+\tilde{m}, 2^a, 1^b)$ and $\tilde{\mu} = (3^{a+1}, 2^{b+c})$ for some a, b, $c \in \mathbb{N}$, where $\tilde{m} = a+b+2c$. If $c \geq 2$, we remove $(2c-2)$ from $\tilde{\lambda}$ and (2^{c-1}) from $\tilde{\mu}$. If $a=b=1$, we have the seed $((7,2,1),(3^2,2^2))$. If $a+b>2$, we remove $(a+1)$ from $\tilde{\lambda}$ and the column of length $a+1$ from $\tilde{\mu}$. Now $\tilde{\mu}$ is a two-column rectangle of length greater or equal to 5 and $\tilde{\lambda}$ has three removable nodes and $w(\tilde{\lambda})$, $l(\tilde{\lambda}) \geq 4$. So the result follows from Lemma 7.18. □

Lemma 10.20. *If $\alpha = (m)$ and β decomposes into two parts which are both partitions, $g_2(\lambda,\mu) \geq 3$.*

Proof: We call the upper right part of β β^1 and the lower left part β^2. Again, we can assume that both parts of β are to the left of α. We start with the case $l(\beta^1) \geq 2$ and $w(\beta^2) \geq 2$. First we remove all the common rows and columns. We remove all but the leftmost column from β^1 and all but the two leftmost columns from β^2, the corresponding columns of $\tilde{\lambda}$ and the right number of boxes from α. Now $\tilde{\lambda} = (3+\tilde{m}, 2^{l(\beta^1)})$ and $\tilde{\mu} = (3^{l(\beta^1)+1}, 2^b, 1^c)$. In the next step we remove $(2^{b-1}, 1^c)$ from $\tilde{\mu}$ as rows and $(2b+c-2)$ from $\tilde{\lambda}$. If $l(\beta^1) = 2$, we have the seed $((7,2^2),(3^3,2))$. If $l(\beta^1) > 2$, we remove the column of length $l(\beta^1)+1$ from $\tilde{\mu}$ and $(a+1)$ from $\tilde{\lambda}$. Now $\tilde{\mu}$ is a two-column rectangle and $\tilde{\lambda} = (4, 2^{l(\beta^1)})$. The result follows from Lemma 7.18.

Now we look at $w(\beta^2) = 1$. In this case we do not remove all the common rows and columns. We know that λ is not a hook, therefore, there is a common row R_1 above α or a common row R_2 between α and β^1 or a common column C_1 to the left of β^2 or a common column C_2 between β^1 and β^2. If there is a common R_1 above

α, we remove all the common rows and columns but R_1. If $m = 2$, we have the seed $((4^2, 1), (4, 2^2, 1))$. If $m = 3$, we have one of the seeds $((5^2, 1), (5, 2^2, 1^2))$ or $((5^2, 1^2), (5, 2^3, 1))$ or $((6^2, 1), (6, 3^2, 1))$. If $m > 3$, we remove $l(\beta^1)$ rows of length 1 from $\tilde{\lambda}$ and $(l(\beta^1))$ from $\tilde{\mu}$. Now $\tilde{\lambda}$ is a two-row rectangle of width at least 6 and $\tilde{\mu}$ has three removable nodes and the second column has length at least 3. The result follows from Lemma 7.18. If there is a common row R_2 between α and β^1, we remove all the common rows and columns but R_2. If $w(\beta^1) = 1$, $\tilde{\mu}$ is a two-column partition and the second column is of length at least 3, $\tilde{\lambda}$ has three removable nodes and the result follows from Lemma 7.17 if the second column is of length 3 or 4, Lemma 7.18 if $\beta^2 = (1)$, and Lemma 7.19, otherwise. If $w(\beta^1) > 1$, we remove all but the leftmost column of β^1, the corresponding columns of $\tilde{\lambda}$ and the right number of boxes from α. This reduces it to the case $w(\beta^1) = 1$. If there is a common column C_1 to the left of β, we remove all the common columns and rows but C_1. Then we remove all but the leftmost column of β^1, the corresponding columns of $\tilde{\lambda}$ together with the right number of boxes from α. Now $\tilde{\beta^1}$ and $\tilde{\beta^2}$ have width 1. We successively reduce the length of $\tilde{\beta^1}$ by 3 and the length of $\tilde{\beta^2}$ by 2 to obtain the seed $((3 + a + b, 2^a, 1^b), (3^{a+1}, 2^b))$ for $1 \leq a \leq 3$ and $1 \leq b \leq 2$. If there is a common column C_2 between β^1 and β^2 and $l(\beta^1) > 1$, we remove all the common rows and columns but C_2. Then we remove all but the leftmost column of β^1 to obtain $\tilde{\lambda} = (3 + a + b, 2^a)$ and $\tilde{\mu} = (3^{a+1}, 1^b)$, where $a \geq 2$. We remove $(b - 1)$ from $\tilde{\lambda}$ and (1^{b-1}) from $\tilde{\mu}$. If $a \geq 5$, we reduce a successively by 3 to obtain a seed $((4 + a, 2^a), (3^{a+1}, 1))$ for $2 \leq a \leq 4$. If $l(\beta^1) = 1$, $\tilde{\lambda}$ would be a two-row partition if we removed all the common rows and columns but C_2. So we know that there is another common row. But we can assume that this row is not above α nor to the left of β nor between α and β^1 because we have already looked at these cases. Therefore, we assume that there is a common row R between β^1 and β^2. We remove all the common rows and columns but C_2 and R, then we remove all but one column of β^1, the corresponding columns of $\tilde{\lambda}$ together with the right number of boxes from α. Further, we remove all but the topmost row of β^2 and the right number of boxes from $\tilde{\alpha}$. We obtain the seed $((5, 2^2), (3^2, 2, 1))$ if R and C_2 have a common box, and $((5, 2, 1), (3^2, 1^2))$ if they do not.

Now we look at the case $l(\beta^1) = 1$. We can assume that $w(\beta^2) > 1$. If we removed all the common rows and columns, $\tilde{\lambda}$ would be a two-row partition. Since λ is not a two-row partition, we know that there is a common row R_1 above α or a common row R_2 between α and β^1 or a common row R_3 between β^1 and β^2 or a common column C to the left of β. If there is a common row R_1 above α, we remove all the common rows and columns but R_1 and all but one column of β^1 and all but one column of β^2, the corresponding columns of $\tilde{\lambda}$ with the right number of columns from α and the corresponding columns of $\tilde{\mu}$. Now $\tilde{\lambda} = ((3 + a)^2, 1)$ and $\tilde{\mu} = (3 + a, 2^2, 1^a)$. We successively reduce a by 2 until we obtain a seed $(((3 + a)^2, 1), (3 + a, 2^2, 1^a))$ for $1 \leq a \leq 2$. If there is a row R_2 or R_3, we remove all the common rows and columns but R_2 or R_3. Then we remove all but the topmost row of β^2 and the right number of boxes from α. In the last step we remove all columns of $\tilde{\beta}$ but one from β^1 and two from $\tilde{\beta}^2$, the corresponding columns of $\tilde{\lambda}$ together with the right number of boxes from $\tilde{\alpha}$. We obtain the seed $((6, 3, 2), (3^3, 2))$ if we did not remove R_2 and $((6, 2^2), (3^2, 2^2))$ if we did not remove R_3. If there is a common column C to the left of β, we remove all the common rows and columns but C. In the next step we remove all the columns from β but one from β^1 and one from β^2. Now $\tilde{\lambda} = (4 + a, 2, 1^a)$ and $\tilde{\mu} = (3^2, 2^a)$. We successively reduce a by 2 until we obtain a seed $((4 + a, 2, 1^a), (3^2, 2^a))$ for $1 \leq a \leq 2$. □

CHAPTER 11

α is equivalent to $(m-1,1)$

We know that α and β are up to conjugation and/or rotation from Theorem 5.1, 5.3 and 5.4. If α and ν are (skew) partitions of m, we use the notation $\alpha \equiv \nu$ for α equals ν up to rotation and/or conjugation. We say α is equivalent to ν. In this chapter we assume that α or β is equivalent to $(m-1,1)$. Again, we can assume that $\alpha \equiv (m-1,1)$ and that α is above and to the right of one part of β. Since $\alpha \equiv (m-1,1)$ and $[\alpha][\beta]$ just contains constituents with multiplicity 1 and 2, we have the following possibilities for β (again up to equivalence):

(1) β is a partition with $\mathrm{rem}(\beta) \leq 3$;
(2) β is a skew partition and one of the following holds:
  (a) $\lambda = (\lambda_1, \lambda_2)^{(')}$ is a two-line partition and $\mu = (1)$ or $\lambda_1 - \lambda_2 = 1$;
  (b) $\lambda = (\lambda_1^{k_1}, \lambda_1 - 1)$ and $\mu = (1)$;
  (c) $\lambda = (\lambda_1^{k_1}, 1)$ and $\mu = (\lambda_1 - 1)$;
  (d) λ/μ decomposes into a one-column and a one-row partition;
  (e) λ/μ decomposes into a rectangle and (1).

We can assume that β is not linear since we have dealt with that already in the previous chapter.

We consider two different cases: first we look at the case that β is up to rotation a proper partition, then at the case that β is a proper skew partition.

1. Up to rotation β is a partition

Lemma 11.1. *If $\alpha^{(\mathrm{rot})} = (m-1,1)$ and β is up to rotation a proper partition with three or less removable nodes, $g_2(\lambda, \mu) \geq 3$.*

Proof: We start with the case $\alpha = (m-1,1)$ and β is a proper partition. Here we have a lot of exceptional cases with which we deal first. If $\beta = (2, 1^{m-2})$ or $m = 2k$ is even and $\beta = (2^k)$ or $m = 2k+1$ and $\beta = (2^k, 1)$, we have exceptional families. If $\alpha = (5,1)$ and $\beta = (3^2)$, we have an exceptional case. We start with the case $\beta = \alpha' = (2, 1^{m-2})$. We know that λ is not a two-row partition so there is another row R_1 above α or R_2 between α and β. By conjugation the case that there is a column to the left of β is equivalent to the case that there is a row R_1 above α. If we remove all the common rows and columns but R_1, we obtain $\tilde{\lambda} = ((m+1)^2, 3)$ and $\tilde{\mu} = (m+1, 2^3, 1^{m-2})$. We successively reduce m by 2 until we obtain the seed $((4^2, 3), (4, 2^3, 1))$ if m is odd and $((5^2, 3), (5, 2^3, 1^2))$ if m is even. If we remove all the common rows and columns but R_2, we obtain $\tilde{\lambda} = (m+1, 3, 2)$ and $\tilde{\mu} = (2^4, 1^{m-2})$. The result follows from Lemma 7.17. Alternatively, this can be reduced to the seed $((5,3,2), (2^4, 1^2))$.

The two cases $m = 2k$ and $\beta = (2^k)$ and $m = 2k+1$ and $\beta = (2^k, 1)$ work very similarly. We remove the same rows and columns but get different seeds. We show the $\beta = (2^k)$ case explicitly and state the $\beta = (2^k, 1)$ case in brackets. We know that λ is not a two-row partition, so there is another row. If there is a row R_1 above α, we remove all the common rows and columns besides R_1 and obtain $\tilde{\lambda} = ((m+1)^2, 3)$ and $\tilde{\mu} = (m+1, 2^{k+2})$ (for $\beta = (2^k, 1)$ we obtain $\tilde{\lambda} = ((m+1)^2, 3)$ and $\tilde{\mu} =$

$(m+1, 2^{k+2}, 1))$. We remove (1^2) from $\tilde{\lambda}$ and (2) from $\tilde{\mu}$ as row and then (1^2) from $\tilde{\lambda}$ and (2) from $\tilde{\mu}$. We repeat this procedure until we obtain the seed $((5^2, 3), (5, 2^4))$ (resp. $((6^2, 3), (6, 2^4, 1)))$. If there is a row R_2 between α and β, we remove all the common rows and columns but R_2 to obtain $\tilde{\lambda} = (m+1, 3, 2)$ and $\tilde{\mu} = (2^{k+3})$ (resp. $\tilde{\lambda} = (m+1, 3, 2)$ and $\tilde{\mu} = (2^{k+3}, 1))$. The result follows from Lemma 7.18. If the common row is below β, we know that there also is a common column C to the left of β. We remove all the common rows and columns but C to obtain $\tilde{\lambda} = (m+2, 4, 1^k)$ and $\tilde{\mu} = (3^{k+2})$ (resp. $\tilde{\lambda} = (m+2, 4, 1^{k+1})$ and $\tilde{\mu} = (3^{k+2}, 2))$. If $l(\tilde{\lambda}) \geq 7$ (resp. ≥ 8), we remove (6) from $\tilde{\lambda}$ and (3^2) from $\tilde{\mu}$ as rows and (1^3) from $\tilde{\lambda}$ as rows and (3) from $\tilde{\mu}$ as row. We repeat this until $l(\tilde{\lambda}) = 4$, 5, 6 (resp. 5, 6, 7). We obtain one of the seeds $((6, 4, 1^2), (3^4))$, $((8, 4, 1^3), (3^5))$, $((10, 4, 1^4), (3^6))$ (resp. $((7, 4, 1^3), (3^4, 2))$, $((9, 4, 1^4), (3^5, 2))$, $((11, 4, 1^5), (3^6, 2)))$.

In the exceptional case $\alpha = (5, 1)$ and $\beta = (3^2)$ we proceed in the same manner. We know, since λ has more than two rows, that there is a common row. If there is a common row R_1 above α and we remove all the common rows and columns but R_1, we obtain the seed $((8^2, 4), (8, 3^4))$. If there is a common row R_2 between α and β and we remove all the common rows and columns but R_2, we obtain the seed $((8, 4, 3), (3^5))$. If there is a common column C to the left of β and we remove all the common rows and columns but C, we obtain the seed $((9, 5, 1^2), (4^4))$.

If β is a two-column partition, we know that β has two rows of length 2 and two rows of length 1. After removing all the common rows and columns this can easily be reduced to the seed $((7, 3), (2^4, 1^2))$.

If $w(\beta) > 2$, we remove all the common rows and columns. In this case we obtain $\tilde{\lambda} = (m + w(\beta) - 1, w(\beta) + 1)$ and $\tilde{\mu} = (w(\beta)^2, \beta)$. The result follows from Lemma 7.17 if $w(\beta) = 3$, and Lemma 7.19 if $w(\beta) > 3$.

If $\alpha^{\text{rot}} = (m-1, 1)$ and β is a partition, still assuming that β is not linear, we have an exceptional case if $\alpha^{\text{rot}} = (2, 1)$ and $\beta = (2, 1)$. We know that λ is not a two-row partition and therefore, there is another row R_1 above α or a row R_2 in between α and β or a column C to the left of β. If we remove all common rows and columns but R_1, we obtain the seed $((4^3), (4, 3, 2^2, 1))$. If we remove all common rows and columns except of R_2, we obtain the seed $((4^2, 2), (3, 2^3, 1))$. If we remove all common rows and columns except of C, we obtain the seed $((5^2, 1^2), (4, 3^2, 2))$.

In all the other cases we remove all the common rows and columns to obtain $\tilde{\lambda}$ which is a two-row rectangle and $\tilde{\mu}$ which is a proper fat hook or has three or more removable nodes and $w(\tilde{\mu}), l(\tilde{\mu}) \geq 4$. Lemma 7.18 tells us that $g_2(\tilde{\lambda}, \tilde{\mu}) \geq 3$.

If $\alpha = (m-1, 1)$ and β^{rot} is a partition, we can assume $\text{rem}(\beta^{\text{rot}}) \geq 2$ and that β^{rot} is different from $(2, 1)$, which excludes the exceptional case. If $l(\beta) = 2$, we know that $\beta^{\text{rot}} = (a, b)$ with $b \geq 1$ and $a \geq 3$. We remove all the common rows and columns to obtain $\tilde{\lambda} = (a + m - 1, a + 1, a - b)$ and $\tilde{\mu} = (a^4)$. If $b \geq 2$, we remove $(m + a - 8, a - 3, a - b - 1)$ from $\tilde{\lambda}$ and $((a-3)^4)$ from $\tilde{\mu}$ to obtain the seed $((7, 4, 1), (3^4))$. If $b = 1$, we remove $(m + a - 7, a - 3, a - 3)$ and $((a-3)^4)$ to obtain the seed $((6, 4, 2), (3^4))$. If $l(\beta) > 2$, we remove all the common rows and columns to obtain $\tilde{\lambda}$ and $\tilde{\mu}$. Then we remove all but the leftmost and the rightmost column from $\tilde{\mu}$ and the corresponding columns of $\tilde{\lambda}$ together with the right number of columns of length 1 from α. We obtain $\tilde{\mu}$ which is a two-column rectangle and $\tilde{\lambda}$ which has three removable nodes. Since we assume that $l(\beta) > 2$, we know $g_2(\tilde{\lambda}, \tilde{\mu}) \geq 3$ (see Lemma 7.18).

If $\alpha^{\text{rot}} = (m-1, 1)$ and β^{rot} is a partition, we assume, again, that β is not a partition. We remove all the common rows and columns to obtain $\tilde{\lambda}$ and $\tilde{\mu}$. Now we remove all but the rightmost and leftmost column from $\tilde{\mu}$, the corresponding columns from $\tilde{\lambda}$ and the fitting number of columns of length 2 from $\tilde{\lambda}$ and length

1 from $\tilde{\mu}$. If $\tilde{\lambda} \neq \tilde{\mu}'$, we remove (1^2) from $\tilde{\lambda}$ and (2) from $\tilde{\mu}$ as row and again, (1^2) from $\tilde{\lambda}$ and (2) from $\tilde{\mu}$ (this time not as row). We repeat this process until $\tilde{\lambda} = \tilde{\mu}'$. The result now follows from Lemma 9.1. □

Lemma 11.2. *If $\alpha^{(\mathrm{rot})} = (2, 1^{m-2})$ and β is up to rotation a proper partition, $g_2(\lambda, \mu) \geq 3$.*

Proof: The case $m = 3$ is contained in the previous lemma. So we assume that α is above β, $m \geq 4$ and that β is not linear. We start with the case $\alpha = (2, 1^m)$ and β is a partition. We have the exceptional case $\alpha = (2, 1^2)$ and $\beta = (2^2)$. If $\alpha = (2, 1^2)$ and $\beta = (2^2)$, we know that μ is not a two-column partition. Like in the exceptional cases in the previous lemma we get three seeds. If there is a common column C_1 to the left of β and we remove all common columns and rows but C_1, we obtain $((5, 4^2, 1^2), (3^5))$. If there is a common column C_2 between α and β and we remove all common columns and rows but C_2, we obtain $((5, 4^2), (3^3, 2^2))$. If there is a common row R above α and we remove all common columns and rows but R, we obtain $((4^2, 3^2), (4, 2^5))$. Now let us look at the general case. Here we remove all the common rows and columns. If $w(\beta) = 2$, $\tilde{\mu}$ is a two-column partition and $\tilde{\lambda} = (4, 3^{m-2})$. Since we assume that $\beta \neq (2^2)$, $g_2(\tilde{\lambda}, \tilde{\mu}) \geq 3$. This follows from Lemma 7.18 if β is (2^k) or $(2^k, 1)$, and Lemma 7.19, otherwise. If $w(\beta) \geq 3$ and $\beta = \alpha' = (3, 1)$, we obtain the seed $((5, 4^2), (3^4, 1))$. If $\beta \neq (3, 1)$, we know that $m \geq 5$. Further, α and β are removable. If we remove them, we obtain $\tilde{\lambda} = \tilde{\mu}$ which are rectangles and contain (3^4). The result follows from Lemma 9.1.

In the we assume that $\alpha = (2, 1^{m-2})$ and β^{rot} is a partition but β is not a partition. We remove all the common rows and columns. If $w(\beta) = 2$, we can assume that $\beta^{\mathrm{rot}} \neq (2, 1^{m-2})$ because by conjugating and interchanging λ and μ this is contained in the case $\alpha^{\mathrm{rot}} = (m - 1, 1)$. So we reduce this to the seed $((4, 3^3, 1), (2^7))$. If $w(\beta) \geq 3$ and $\beta^{\mathrm{rot}} = (m-1, 1)$, we have the seed $((5, 4^2, 2), (3^5))$ if $m = 4$. If $m > 4$, we remove the two lowest rows of $\tilde{\lambda}$ and the corresponding rows of $\tilde{\mu}$. Now $\tilde{\alpha} = (2, 1^{m-3})$ and $\tilde{\beta} = (m - 1)$ are removable. After removing them, $\tilde{\lambda} = \tilde{\mu}$ are rectangles which contain (4^3). The result follows from Lemma 9.1. If $\beta^{\mathrm{rot}} \neq (m - 1, 1)$, we remove all the rows of $\tilde{\mu}$ which belong to β besides the lowest one and the corresponding rows of $\tilde{\lambda}$ together with all but the lowest $w(\beta)$ rows from α and the corresponding rows from $\tilde{\mu}$. We obtain $\tilde{\lambda} = ((w(\beta) + 1)^{w(\beta)}) = \tilde{\mu}'$. The result follows from Lemma 9.1.

In the next step let us look at the case $\alpha^{\mathrm{rot}} = (2, 1^{m-2})$ and β is a partition. We start with removing all the common rows and columns. We can assume that β is neither $(m - 1, 1)$ nor $(2, 1^{m-2})$. By conjugating and interchanging λ and μ these are contained in the previous case and the case $\alpha = (m - 1, 1)$. If $w(\beta) = 2$, we know that $\beta = (2^a, 1^b)$ with $a \geq 2$. We remove (4^{m-4}) from $\tilde{\lambda}$ as rows and $(3^{m-4}, 2^{a-2}, 1^b)$ from $\tilde{\mu}$ as rows to obtain the seed $((4^3), (3^2, 2^3))$. If $w(\beta) \geq 3$, we know that $m - \beta_1 \geq 2$ since we assume that β is neither linear nor $(m - 1, 1)$. Therefore, we can remove all rows but β_1 of β from $\tilde{\mu}$, all but the top β_1 rows from $\tilde{\lambda}$ and the corresponding rows from $\tilde{\mu}$. We obtain $\tilde{\lambda} = ((\beta_1 + 2)^{\beta_1})$ and $\tilde{\mu} = ((\beta_1 + 1)^{\beta_1}, \beta_1)$. If we remove a column of length β_1 from $\tilde{\lambda}$ and a row of length β_1 from $\tilde{\mu}$, we obtain $\tilde{\lambda} = \tilde{\mu} = ((\beta_1 + 1)^{\beta_1}) \supseteq (4^3)$. The result follows from Lemma 9.1.

The last case is $\alpha^{\mathrm{rot}} = (2, 1^{m-2})$ and β^{rot} is a partition and β is not. We start with removing all the common rows and columns. If $w(\beta) = 2$, we reduce this to the seed $((4^2, 1), (3, 2^3))$. If $w(\beta) \geq 2$ and $\alpha = \beta'$, we have $\tilde{\lambda} = \tilde{\mu}'$. Therefore, the result follows from Lemma 9.1. From now on we know that $\beta^{\mathrm{rot}} \neq (m - 1, 1)$. This tells us that $m - w(\beta^{\mathrm{rot}}) \geq 2$. Now we remove all the rows of $\tilde{\mu}$ which belong to β but the lowest one, the corresponding rows of $\tilde{\lambda}$ together with all but the highest

$w(\beta^{\text{rot}})$ rows of $\tilde{\lambda}$ and the corresponding rows from $\tilde{\mu}$. After removing a common column we obtain $\tilde{\lambda} = \tilde{\mu}' = ((w(\beta)+1)^{w(\beta)}) \supseteq (4^3)$. Therefore, the result follows from Lemma 9.1. □

2. β is a skew partition

We only have to look at the skew partitions $\beta = \beta^1/\beta^2$ such that $[m-1,1][\beta]$ only contains constituents with multiplicity 1 and 2. By Lemma 8.3 we know that the possibilities for β are (up to equivalence):

(1) β^1 is a two-line partition (not a rectangle) and $\beta^2 = (1)$;
(2) $\beta^1 = ((\beta^1_1)^{k_1}, \beta^1_1 - 1)$ and $\beta^2 = (1)$;
(3) $\beta^1 = ((\beta^1_1)^{k_1}, 1)$ and $\mu = (\beta^1_1 - 1)$;
(4) β^1/β^2 decomposes into a one-column and a one-row partition;
(5) β^1/β^2 decomposes into a rectangle and (1).

Lemma 11.3. *If $\alpha^{\text{rot}} = (m-1,1)^{(\prime)}$ and β is (up to equivalence) from Lemma 8.3, $g_2(\lambda,\mu) \geq 3$.*

Proof: We remove all common rows and columns. We look at the five possibilities for β individually. We start with the first one. By rotation and conjugation we have eight cases but we will look at the pairs where β just differs by rotation together. For the two where $\alpha^{\text{rot}} = (m-1,1)$ and β is a two-row skew partition, we remove all columns of β but the rightmost, the leftmost and one column of length 2, the corresponding columns from $\tilde{\lambda}$ and all but two columns of length 1 and one of length 2 from α and the corresponding columns of $\tilde{\mu}$ to obtain the seed $((6^2,1),(5,3^2,2))$. If $\alpha^{\text{rot}} = (2,1^{m-2})$ and β is a two-column skew partition, we do the same as before but for rows instead of columns and obtain the seed $((4^3,1),(3^2,2^3,1))$. If $\alpha^{\text{rot}} = (m-1,1)$ and $\tilde{\mu}$ has two columns, we remove a row of length 1 from $\tilde{\lambda}$ and one of length 1 from $\tilde{\mu}$. Since $\beta^2 = (1)$ or the difference of the two columns of β^2 equals 1, $\tilde{\beta}$ is now a (rotated) partition and the result follows from Lemma 11.1. Since β was a proper skew partition, $m \geq 4$ and we do not have the exceptional case of that lemma. If $\alpha^{\text{rot}} = (2,1^{m-1})$ and $\beta = (a,b)/(c)$ is a two-row skew partition, we remove $\min(a-b,c) = 1$ columns of length m from $\tilde{\lambda}$ and $\tilde{\mu}$. α does not change but $\tilde{\beta}$ is now a (rotated) partition so the result follows from Lemma 11.2.

Let $\alpha^{\text{rot}} = (m-1,1)$, $\beta^1 = (a^b, a-1)$ for $a > 2$ and $b > 1$ and $\beta^2 = (1)$. We remove all but the leftmost and the rightmost column of β, the corresponding columns of $\tilde{\lambda}$, the right number columns of length 1 from α and the corresponding columns of $\tilde{\mu}$. Now $\tilde{\beta} = (2^b,1)/(1)$ is a two-column skew partition, $\alpha^{\text{rot}} = (\tilde{m}-1,1)$ and the result follows from the previous case. If $\alpha^{\text{rot}} = (2,1^{m-2})$, $\beta^1 = (a^b, a-1)$ and $\beta^2 = (1)$, we can do the same but with rows instead of columns to reduce this to the case that β is a two-row skew partition. Further, these cases are invariant under conjugation and rotation of β, so here we just have these ones.

If $\beta^1 = (a^b,1)$ and $\beta^2 = (a-1)$, the case is, again, invariant under rotation, but this time not under conjugation, therefore, we also have to look at the conjugated case for both possibilities of α. If $\alpha^{\text{rot}} = (m-1,1)$, we remove all columns of β but the rightmost and the leftmost, the corresponding columns of $\tilde{\lambda}$, the right number of columns of length 1 from α and the corresponding columns of $\tilde{\mu}$. Now $\tilde{\alpha}^{\text{rot}} = (\tilde{m}-1,1)$ and $\tilde{\beta}$ is a two-column partition skewed by (1), so the result follows from that case. If $\alpha^{\text{rot}} = (2,1^{m-2})$ and $\beta = (a+1,a^b)/(1^b)$, we can do the same but with rows instead of columns to reduce it to the case $\alpha^{\text{rot}} = (2,1^{\tilde{m}-2})$ and β is a two-row partition skewed by (1). If $\alpha^{\text{rot}} = (m-1,1)$ and $\beta = (a+1,a^b)/(1^b)$, we remove all columns of β but the two leftmost columns, the corresponding columns of $\tilde{\lambda}$, the right number of columns of length 1 from α together with the corresponding columns from $\tilde{\mu}$. Now $\tilde{\alpha}^{\mathrm{rot}} = (\tilde{m}-1, 1) = (\tilde{\beta}^{\mathrm{rot}})'$, so the result follows from Lemma 9.1. Again, if α and β are conjugated, we can do the same for rows instead of columns. The result follows again from Lemma 9.1.

If β decomposes into one column and one row, we do not assume that any part of β is to the left of α. Therefore, we can assume that $\alpha^{\mathrm{rot}} = (m-1, 1)$. If the column of β is actually just one box, we remove all but two columns from the row of β, the corresponding columns of $\tilde{\lambda}$, the right number of columns of length 1 from α and the corresponding columns of $\tilde{\mu}$ to obtain one of the following seeds:

Position of β	Seed
Both parts are to the left of α	$((5^2,2),(4,3^2,2))$ or $((5^2,1),(4,3^2,1))$
One part is to the left and one is to the right of α	$((3^3),(5,2,1^2))$ or $((4^3,5,3,2^2),(5,3,2^2))$
Both parts are to the right of α	$((4,2^3),(5,4,1))$ or $((3,2^3),(5,3,1^2))$.

From now on we assume that the column of β has two or more boxes. We remove all but one column from the row of β, the corresponding columns of $\tilde{\lambda}$, the right number of columns of length 1 from α and the corresponding columns of $\tilde{\mu}$. Now we want to shrink the column of β or reduce this to a previous result, but this works differently regarding the position. First we assume that both parts of β are to the left of α. If the column of β is the leftmost part, we have $\tilde{\lambda} = ((\tilde{m}+1)^2, 1)$ and $\tilde{\mu} = (\tilde{m}, 2^2, 1^{\tilde{m}-1})$. If $\tilde{m} = 3$, 4, we check this with Sage. If $\tilde{m} > 4$ we remove a row of length 1 from $\tilde{\lambda}$ and $\tilde{\mu}$. Now $\tilde{\lambda}$ is a two-row rectangle and the result follows from Lemma 7.18. If the column is the right part of β, we have $\tilde{\lambda} = ((\tilde{m}+1)^2, 1^{\tilde{m}-1})$ and $\tilde{\mu} = (\tilde{m}, 2^{\tilde{m}}, 1)$. We remove (1) from $\tilde{\lambda}$ as row and from $\tilde{\mu}$ as row. Now $\tilde{\lambda}' = \tilde{\mu}$ and the result follows from Lemma 9.1. Next we assume that α is in between the two parts of β. If the column is to the left of α, we obtain $\tilde{\lambda} = (\tilde{m}^3)$ and $\tilde{\mu} = (\tilde{m}+1, \tilde{m}-1, 1^{\tilde{m}})$. We check the cases $\tilde{m} = 3$, 4, 5 with Sage. If $\tilde{m} \geq 6$, we remove (2^3) from $\tilde{\lambda}$ and (3^2) from $\tilde{\mu}$. In the next step we remove (1^3) from $\tilde{\lambda}$ and (1^3) from $\tilde{\mu}$ as rows. We repeat this until $w(\tilde{\lambda}) \leq 6$ to obtain one of the pairs we checked for $\tilde{m} = 3$, 4, 5. If the column is to the right of β, we have $\tilde{\lambda} = (\tilde{m}^{\tilde{m}+1})$ and $\tilde{\mu} = ((\tilde{m}+1)^{\tilde{m}-1}, \tilde{m}-1, 1^2)$. If $\tilde{m} = 3$, we check this with Sage. If $\tilde{m} > 3$, we remove a column of length $\tilde{m}+1$ from $\tilde{\lambda}$ and $(\tilde{m}-1, 1^2)$ from $\tilde{\mu}$ as rows. Now $\tilde{\lambda} = \tilde{\mu}' = ((\tilde{m}-1)^{\tilde{m}+1})$ and the result follows from Lemma 9.1. Last we assume that both parts of β are to the right of α. If the column is the left part of β, we have $\tilde{\lambda} = (\tilde{m}, (\tilde{m}-1)^{\tilde{m}+1})$ and $\tilde{\mu} = (\tilde{m}+1, \tilde{m}^{\tilde{m}-1}, \tilde{m}-2)$. If $\tilde{m} = 3$, we check this with Sage. If $\tilde{m} > 3$, we remove $(\tilde{m}, \tilde{m}-1)$ from $\tilde{\lambda}$ as rows and $(\tilde{m}+1, \tilde{m}-2)$ from $\tilde{\mu}$ as rows. Now $\tilde{\lambda}' = \tilde{\mu} = (\tilde{m}^{\tilde{m}-1})$, so the result follows from Lemma 9.1. If the column is the right part of β, we obtain $\tilde{\lambda} = (\tilde{m}^{\tilde{m}-1}, (\tilde{m}-1)^3)$ and $\tilde{\mu} = ((\tilde{m}+1)^{\tilde{m}-1}, \tilde{m}, \tilde{m}-2)$. If $\tilde{m} = 3$, we check this with Sage. If $\tilde{m} > 3$, we remove $((\tilde{m}-1)^2)$ as rows from $\tilde{\lambda}$ and $(\tilde{m}, \tilde{m}-2)$ as rows from $\tilde{\mu}$. In the next step we remove the last row of length $\tilde{m}-1$ from $\tilde{\lambda}$ and a column of length $\tilde{m}-1$ from $\tilde{\mu}$. Now $\tilde{\lambda} = \tilde{\mu}' = (\tilde{m}^{\tilde{m}-1})$.

If β decomposes into a rectangle and a single node, we can assume that it the rectangle has width and length at least 2 (otherwise, we are in the previous case). Again, we assume that $\alpha^{\mathrm{rot}} = (m-1, 1)$, but we cannot make any assumptions on the order of α and β. We remove all but one column of the rectangle and the fitting number of columns of length 1 from α so that we reduce this case to the previous one. □

Lemma 11.4. *If $\alpha = (m-1, 1)^{(\prime)}$ and β is (up to equivalence) from Lemma 8.3, $g_2(\lambda, \mu) \geq 2$.*

Proof: We have the same possibilities for β as in the lemma before. Again, we look at these case by case and start in all cases with removing all the common rows and columns. If it is not mentioned otherwise, we assume that α is to the right of β. We start with the case $\alpha = (m-1,1)$ and $\beta = (a,b)/(c)$ is a two-row skew partition. We remove all columns of β but one of the skew columns of length 1, one column of length 2 and one not-skew column of length 1, the corresponding columns of $\tilde{\lambda}$ and all columns of α but the one of length 2 and two of length 1. We obtain the seed $((6,4,1),(3^3,2))$. If $\alpha = (2,1^{m-2})$ and β is a two-column skew partition, we can do the same but for rows instead of columns to obtain the seed $((4,3^2,1),(2^5,1))$. If $\alpha = (2,1^2)$ and $\beta = (3,2)/(1)$, we check that $g_2((5,4^2,1),(3^4,2)) \geq 3$. If $\alpha = (2,1^{m-2})$ and $\beta = (a,b)/(c)$, we can assume that $m \geq 4$. We remove $\min(a-b,c) = 1$ columns of length m from $\tilde{\lambda}$ and $\tilde{\mu}$. Now $\tilde{\beta}$ is a partition if $c \leq a-b$, or a rotated partition if $c \geq a-b$. The result follows from Lemma 11.2. If $\alpha = (m-1,1)$ and $\beta = (2^a,1^b)/(1^c)$, we have $\tilde{\lambda} = (m+1,3,1^c)$ and $\tilde{\mu} = (2^{2+a},1^b)$. Since $\tilde{\mu}$ is a two-column partition, the result follows from Lemma 7.18 if $b = 1$, and Lemma 7.19, otherwise.

The second case can be reduced to the two-row or two-column case like in the previous lemma.

The third case also works like in the previous lemma (with very tiny adjustments).

Let β decomposes into a column and a row. We assume that $\alpha = (m-1,1)$ but drop the assumption that α is to the right of (a part) of β. If the column is of length 1, we remove all but two boxes from the row of β, the corresponding columns from $\tilde{\lambda}$, all but one column of length 1 and the column of length 2 from α and the corresponding columns of $\tilde{\mu}$ to obtain one of the following seeds:

Position of β	Seed
Both parts are to the left of α	$((5,4,1),(3^3,1))$ or $((5,4,2),(3^3,2))$
One part is to the left and one is to the right of α	$((3^2,2),(5,1^3))$ or $((4^2,3),(5,2^3))$
Both parts are to the right of α	$((4,2^2,1),(5,4))$ or $((3,2^2,1),(5,3))$.

From now on we know that the column of β has at least two boxes. We remove all but one box of the row from β, the corresponding columns from $\tilde{\lambda}$ and the right number of columns of length 1 from α with the corresponding columns of $\tilde{\mu}$. First we assume that both parts of β are to the left of α. If the column is the leftmost part of β, this can easily be reduced to the seed $((4,3,1),(2^3,1^2))$ by removing boxes from the column of β and the arm of α. If the column is the right part of β, the result follows from Lemma 7.18 since $\tilde{\mu} = (2^{\tilde{m}+1},1)$ and $\tilde{\lambda} = (\tilde{m}+1,3,1^{\tilde{m}-1})$ has three removable nodes. Next we assume that α is in between the two parts of β. If the column is to the left of α, $\tilde{\mu}$ is a hook with $w(\tilde{\mu}) \geq 4$ and $l(\tilde{\mu}) \geq 5$ and $\tilde{\lambda} = (\tilde{m}^2,2)$. The result follows from Proposition 6.2. If the column is to the right of β, $\tilde{\lambda} = (\tilde{m}^{\tilde{m}},2)$ and $\tilde{\mu} = ((\tilde{m}+1)^{\tilde{m}-1},1^3)$. If $\tilde{m} = 3$, we check this with Sage. If $\tilde{m} \geq 4$, we remove a row of length 2 from $\tilde{\lambda}$ and two rows of length 1 from $\tilde{\mu}$. Then we remove a common column of length $\tilde{m}$ from $\tilde{\lambda}$ and $\tilde{\mu}$. Now $\tilde{\lambda}' = \tilde{\mu} = (\tilde{m}^{\tilde{m}-1})$. The result follows from Lemma 9.1. In the last step we assume that both parts of β are to the right of α. If the column is the left part of β, we have $\tilde{\lambda} = (\tilde{m},(\tilde{m}-1)^{\tilde{m}},1)$ and $\tilde{\mu} = (\tilde{m}+1,\tilde{m}^{\tilde{m}-1})$. If $\tilde{m} = 3$, we check this with Sage. If $\tilde{m} > 3$, we remove $(\tilde{m},1)$ from $\tilde{\lambda}$ as rows and $(\tilde{m}+1)$ from $\tilde{\mu}$ as row. Now $\tilde{\lambda}' = \tilde{\mu} = (\tilde{m}^{\tilde{m}-1})$ and the result follows from Lemma 9.1. If the column is the right part of β, we have $\tilde{\lambda} = (\tilde{m}^{\tilde{m}-1},(\tilde{m}-1)^2,1)$ and $\tilde{\mu} = ((\tilde{m}+1)^{\tilde{m}-1},\tilde{m})$. Again, if $\tilde{m} = 3$, we check this with Sage. If $\tilde{m} > 3$, we remove $(\tilde{m}-1,1)$ from $\tilde{\lambda}$ as rows and the row of length $\tilde{m}$ from $\tilde{\mu}$. Now we remove a common column of length $\tilde{m}-1$ from both and obtain $\tilde{\lambda}' = \tilde{\mu} = (\tilde{m}^{\tilde{m}-1})$. So the result follows from Lemma 9.1.

Like in the previous lemma the case β decomposes into a rectangle with length and width at least 2 and one box can be reduced to the previous case. $\square$

This concludes the part that $\alpha \equiv (m-1, 1)$. Together with the part $\alpha \equiv (m)$ this was the most complex part because here we did not know much about β. In the remaining chapters we know much more about the possibilities for β.

CHAPTER 12

α is a skew partition

In this chapter we assume that α is a skew partition. Further, we assume that β is not linear or equivalent to $(m-1,1)$. Since we work under the assumption that the classification is correct for m we only have to look at the products from Theorem 5.4.

In contrast to the previous sections, we do not assume that α is to the right of β, because if α decomposes into different parts this might get a bit confusing. Instead, we can choose for either α or β if we just look at α or α' resp. β or β'.

1. The exceptional pairs

Lemma 12.1. *If α, β is an exceptional pair from Theorem 5.4, $g_2(\lambda,\mu) \geq 3$.*

Proof: In all but two cases we remove the common rows and columns to obtain the seed.

If $\alpha^{(\mathrm{rot})} = (2,1) * (1)$ and $\beta = (2,2)$, both are invariant under conjugation. Therefore, we can assume that $(2,1)^{(\mathrm{rot})}$ is to the left of β. We obtain six seeds by removing all the common rows and columns depending on the position of (1). These are:

$\alpha =$	Position of (1)	Seed
$(2,1)*(1)$	To the left of $(2,1)$	$((3^3,2,1),(5^2,1^2))$
$(2,1)*(1)$	Between $(2,1)$ and $(2,2)$	$((3^3,2,1),(5^2,2))$
$(2,1)*(1)$	To the right of $(2,2)$	$((5,2^3,1),(4^3))$
$(2,2)/(1)*(1)$	To the left of $(2,1)$	$((3^4,1),(5^2,2,1))$
$(2,2)/(1)*(1)$	Between $(2,1)$ and $(2,2)$	$((3^3,2^2),(5^2,2,1))$
$(2,2)/(1)*(1)$	To the right of $(2,2)$	$((5,2^4),(4^3,1))$.

If $\alpha^{(')} = (3)*(3)$ and $\beta^{(')} = (3,3)$, both parts of α are of the same shape. By conjugation we can assume that $\beta = (3,3)$. We have the two possibilities, $(3)*(3)$ and $(1^3)*(1^3)$ for α, and three possibilities for the position of β. In total we have six seeds, these are:

$\alpha =$	Position of β	Seed
$(3)*(3)$	β is to the left of α	$((9,6),(6,3^3))$
$(3)*(3)$	β is between the two parts of α	$((9,3^3),(6^3))$
$(3)*(3)$	β is to the right of α	$((6^3,3),(9^2,3))$
$(1^3)*(1^3)$	β is to the left of α	$((5^3,4^3),(4^3,3^5))$
$(1^3)*(1^3)$	β is between the two parts of α	$((5^3,1^5),(4^5))$
$(1^3)*(1^3)$	β is to the right of α	$((2^5,1^3),(5^2,1^3))$.

Let $m=2k$, $3 \leq k \leq 5$, $\alpha^{(')} = (k+2,k)/(2)$ and $\beta^{(')} = (k,k)$. By conjugation we can assume that $\beta = (k,k)$. Further, α is invariant under rotation. So α can be $(k+2,k)/(2)$ or $(2^k,1^2)/(1^2)$ and we have two possibilities for the position of β. So in total we obtain four seeds for each $m \in \{6,8,10\}$. These are:

$\alpha =$	Position of α	Seed
$(k+2,k)/(2)$	α is to the left of β	$(((k+2)^3,k),((2k+2)^2,2))$
$(k+2,k)/(2)$	α is to the right of β	$((2k+2,2k),(k+2,k^3))$
$(2^k,1^2)/(1^2)$	α is to the left of β	$((2^{k+2},1^2),((k+2)^2,1^2))$
$(2^k,1^2)/(1^2)$	α is to the right of β	$(((k+2)^k,(k+1)^2),((k+1)^2,k^{k+2}))$.

All these seeds can be checked directly, but for $k=5$ the last seed $((7^5,6^2),(6^2,5^7))$ takes some time. Instead we can reduce this further to $((7^5),(5^7))$ and the result follows from Lemma 9.1.

In the next step we look at the exceptional cases $m=2k$ for $2\le k\le 4$, $\alpha\equiv(k^2,1)/(1)$ and $\beta^{(')}=(k^2)$. Here we have two cases where we do not obtain seeds if we remove all the common rows and columns. We look at these first. If $\alpha=(2^2,1)/(1)$, $\beta=(2,2)$ and β is to the right of α, we cannot remove all the common rows and columns to obtain a seed. If we removed all the common rows and columns, $\tilde{\lambda}$ would be a two-column partition. Since we assume that λ is not a two-column partition, we know that there is an extra column. If there is an extra column to the left of β, we call it C_1. If it is between α and β, we call it C_2. If there is an extra column above α, there is an extra row above α which we call R. We remove all the common rows and columns but C_1, C_2 or R to obtain one of the following seeds:

We do not remove	Seed
C_1	$((3^4,2),(5^2,2,1^2))$
C_2	$((3^2,2^2,1),(5^2,1))$
R	$((4,2^4,1),(4^3,1))$.

If $\alpha=(3,2)/(1)$ is to the right of $\beta=(2,2)$, we know that λ is not a two-row partition. Therefore, there needs to be another row R_1 above α or R_2 between α and β or a column C to the left of β. If we remove all common rows and columns but R_1, R_2 or C, we obtain one of the following seeds:

We do not remove	Seed
R_1	$((5^2,4),(5,3,2^3))$
R_2	$((5,4,2),(3,2^4))$
C	$((6,5,1^2),(4,3^3))$.

In the remaining cases we obtain the seed by removing all the common rows and columns. Let $m=2k$ for $2\le k\le 4$, $\alpha\equiv(k^2,1)/(1)$ and $\beta^{(')}=(k^2)$. We can, again, assume by conjugation that $\beta=(k^2)$. We have four possibilities for α (α can be conjugated and rotated) and two for the position of α. For every $k>2$ we obtain eight seeds. For $k=2$ two of the cases are covered in the previous paragraph and these are no seeds, further, for $k=2$ α is invariant under rotation. The seeds are:

$\alpha=$	Position of α	Seed
$(k^2,1)/(1)$	α is left of β	$((k^4,1),((2k)^2,1))$
$(k^2,1)/(1)$	α is right of β	$(((2k)^2,k+1),(k+1,k^4))$
$(3,2^{k-1})/(1)$	α is left of β	$((3^3,2^{k-1}),((3+k)^2,1))$
$(3,2^{k-1})/(1)$	α is right of β	$((k+3,(k+2)^{k-1}),(k+1,k^{k+1}))$
$(k^2,k-1)/(k-1)$	α is left of β	$((k^4,k-1),((2k)^2,k-1))$
$(k^2,k-1)/(k-1)$	α is right of β	$(((2k)^2,2k-1),(2k-1,k^4))$
$(3^{k-1},2)/(1^{k-1})$	α is left of β	$((3^{k+1},2),((k+3)^2,1^{k-1}))$
$(3^{k-1},2)/(1^{k-1})$	α is right of β	$(((k+3)^{k-1},k+2),((k+1)^{k-1},k^3))$.

Where the entries in the first and fourth row are seeds for $k=3$, 4 the other ones for $k=2$, 3, 4.

For the last exceptional pairs $\beta = (3^3)$ and α is equivalent to $(7,3)/(1)$ or $(6,4)/(1)$. Since β is invariant under conjugation, we can assume that α is a two-row skew partition. After removing all the common rows and columns we obtain the following seeds:

α=	Position of α	Seed
$(6,4)/(1)$	α is to the left of β	$((6^4,4),(9^3,1))$
$(6,4)/(1)$	α is to the right of β	$((9,7),(4,3^4))$
$(6,5)/(2)$	α is to the left of β	$((6^4,5),(9^3,2))$
$(6,5)/(2)$	α is to the right of β	$((9,8),(5,3^4))$
$(7,3)/(1)$	α is to the left of β	$((7^4,3),(10^3,1))$
$(7,3)/(1)$	α is to the right of β	$((10,6),(4,3^4))$
$(7,6)/(4)$	α is to the left of β	$((7^4,6),(10^3,4))$
$(7,6)/(4)$	α is to the right of β	$((10,9),(7,3^4))$.

□

2. α is connected

Lemma 12.2. *If $m = ab$ for $a, b \geq 2$, $\alpha^{(\text{rot})} = (m-1,2)/(1)$ and $\beta = (a^b)$, $g_2(\lambda,\mu) \geq 3$.*

Proof: We can assume that $m \geq 6$ because $m = 4$ is covered in the exceptional cases. We remove all the common columns and rows. If $\alpha = (m-1,2)/(1)$ is to the right of β, $\tilde{\lambda} = (a+m-1, a+2)$ is a two-row partition and $\tilde{\mu} = (a+1, a^{b+1})$ is a proper fat hook, so the result follows from Lemma 7.17 if $a = 2$, and Lemma 7.19 if $a > 2$. If $\alpha = (m-1,m-2)/(m-3)$ is to the right of β, $\tilde{\lambda} = (a+m-1, a+m-2)$ and $\tilde{\mu} = (a+m-3, a^{b+1})$. The result follows from Lemma 7.18.

If α is to the left of β and $b = 2$, we reduce this to the seed $((3^3,2),(5^2,1))$ (for both possibilities of α). If $b > 2$, we remove a column of length $b+1$ from $\tilde{\lambda}$ and $\tilde{\mu}$. Now $\tilde{\alpha}^{(\text{rot})} = (m-2,2)$ and $\tilde{\beta} = \beta$ is still a rectangle. These cases follow when we prove Lemma 13.7. □

Lemma 12.3. *If $\beta = (k,k)$ and α is a two-line skew partition, $g_2(\lambda,\mu) \geq 3$.*

Proof: We can assume that $m = 2k \geq 6$ because we have seen $n = 4$ in the exceptional cases. We remove all the common rows and columns to obtain $\tilde{\lambda}$ and $\tilde{\mu}$. If α is a two-row skew partition, this can be reduces to $((3^3,2),(5^2,1))$ if α is to the left of β. If α is to the right of β, we can assume that $\alpha \neq (3,2)/(1)$ because we have seen this case in the exceptional cases. Since $m \geq 6$, $\tilde{\lambda}$ is a two-row partition with $w(\tilde{\lambda}) \geq 7$ and $\tilde{\mu}$ is a proper fat hook. The result follows from Lemma 7.18 or from Lemma 7.19.

If $\alpha = (2^a,1^b)/(1^c)$ is to the left of β, we can assume that $\alpha \neq (2^2,1)/(1)$ because we have seen this in the exceptional cases. Therefore, $\tilde{\lambda} = (2^{a+2},1^b)$ is a two-column partition (and not a rectangle) with $l(\tilde{\mu}) \geq 6$ and $\tilde{\mu} = ((k+2)^2,1^c)$ is a proper fat hook. So the result follows from Lemma 7.18 if $b = 1$ or from Lemma 7.19 if $b \geq 2$.

If $\alpha = (2^a,1^b)/(1^c)$ is to the right of β, we obtain $\tilde{\lambda} = ((k+2)^a,(k+1)^b)$ and $\tilde{\mu} = ((k+1)^c, k^{a+b-c+2})$. We remove $\min(b,c)$ rows of length $k+1$ from $\tilde{\lambda}$ and $\tilde{\mu}$. Now $\tilde{\alpha}$ is a two-column partition and the result will follow from Lemma 14.1 and 14.2. □

Lemma 12.4. *If $m = 2k$, $\alpha \equiv ((m-2)^2,1)/(m-3)$ and $\beta = (k,k)$, $g_2(\lambda,\mu) \geq 3$.*

Proof: We can assume that $m \geq 6$ because $m = 4$ is covered in the exceptional cases. Note that α is invariant under rotation. We remove all the common rows and columns. If $\alpha = ((m-2)^2,1)/(m-3)$, we remove all columns of length 1 from

α but two, the corresponding columns of $\tilde{\mu}$ and all but three columns of β together with the corresponding columns from $\tilde{\lambda}$. We obtain the seed $((7^2,4),(6,3^4))$ if α is to the right of β, and $((4^4,1),(7^2,3))$ if α is to the left of β.

If $\alpha=(3,2^{m-3})/(1^{m-3})$ and β is to the left of α, $\tilde{\lambda}=(k+3,(k+2)^{m-3})$ and $\tilde{\mu}=((k+1)^{m-3},k^3)$. If $m=6$, we have the seed $((6,5^3),(4^3,3^3))$. If $m>6$, we remove two columns of length $m-2$ and one of length 1 from $\tilde{\lambda}$, a column of length $m-3$ and one of length m from $\tilde{\mu}$. Now $\tilde{\alpha}=((k-1)^2)$ and $\tilde{\beta}=(1^{m-2})$ are removable and the result follows from Lemma 9.1.

If $\alpha=(3,2^{m-3})/(1^{m-3})$ and β is to the right of α, $\tilde{\lambda}=(3^3,2^{m-3})$ and $\tilde{\mu}=((k+3)^2,1^{m-3})$. If $m=6$, we have the seed $((3^3,2^3),(6^2,1^3))$. If $m>6$, we remove a column of length 3 from $\tilde{\lambda}$ and three rows of length 1 from $\tilde{\mu}$. Now $\tilde{\lambda}=(2^m)$ and $\tilde{\mu}=((k+3)^3,1^{m-6})$ and the result follows from Lemma 7.18. □

Lemma 12.5. *If $m=ab$, for $a,b\geq 3$, $\alpha=((m-2)^2,1)/(m-3)$ and $\beta=(a^b)$, $g_2(\lambda,\mu)\geq 3$.*

Proof: We remove all the common columns and rows. We remove all but two columns of β, the corresponding columns from $\tilde{\lambda}$, all but $2b-4$ columns of length 1 from α and the corresponding columns of $\tilde{\mu}$. Now $\tilde{\beta}$ is a two-column rectangle and $\tilde{\alpha}$ is of the same form as α but with $2b$ instead of m boxes, so the result follows from Lemma 12.4. □

3. α decomposes into two parts

Lemma 12.6. *If $\alpha^{(\prime)}=(m-1)*(1)$ and $\beta\vdash m$ has at most two removable nodes, $g_2(\lambda,\mu)\geq 3$.*

Proof: By conjugation we assume that $\alpha=(m-1)*(1)$ but we have six different possibilities for the ordering of the parts of α and β. Since β is not linear nor $(m-1,1)^{(\prime)}$, we know that $m\geq 4$. We start with the case that (1) is the rightmost part, followed by $(m-1)$ and the leftmost part is β. We remove all common rows and columns to obtain $\tilde{\lambda}=(w(\beta)+m,w(\beta)+m-1)$ and $\tilde{\mu}=(w(\beta)+m-1,w(\beta),\beta)$. The result follows from Lemma 7.18.

Next, let us look at the case that $(m-1)$ is the rightmost part, followed by (1) and the leftmost part is β. Again, we remove all the common rows and columns to obtain $\tilde{\lambda}=(w(\beta)+m,w(\beta)+1)$ and $\tilde{\mu}=(w(\beta)+1,w(\beta),\beta)$. Since $w(\beta)\geq 2$, the result follows from Lemma 7.17 if $w(\beta)=2,3$, and Lemma 7.19 if $w(\beta)\geq 4$.

For the next case we assume that $(m-1)$ is to the right of β and (1) to the left of β. If μ is not a two-row rectangle, we remove all the common rows and columns to obtain $\tilde{\lambda}=(w(\beta)+m,1^{l(\beta)+1})$ and $\tilde{\mu}=(w(\beta)+1,\beta_1+1,\beta_2+1,\dots)$. If $\tilde{\mu}$ is a rectangle, we know that $l(\beta)>2$, therefore, $w(\tilde{\lambda}),l(\tilde{\lambda})\geq 5$ and $l(\tilde{\mu})\geq 4$. The result follows from Proposition 6.2. If β is not a rectangle, $\tilde{\mu}$ is a fat hook and $w(\tilde{\lambda}),l(\tilde{\lambda})\geq 4$. The result follows, again, from Proposition 6.2. If β is a two-row rectangle, we know that μ is not a hook. Therefore, there is another row or column. If we remove all the common rows and columns but a row R_1 above the upper part of α or a row R_2 between the upper part of α and β, we remove all but two columns of β, the corresponding columns of $\tilde{\lambda}$ and all but three columns of the upper part of α together with the corresponding columns of $\tilde{\mu}$. We obtain the seed $((6^2,1^3),(6,3^3))$ if there is a common row R_1, or $((6,3,1^3),(3^4))$ if there is a common row R_2. If there is a common column C_1 to the left of the left part of α or a common column C_2 between the left part of α and β, we can do the same to obtain the seed $((7,2^3),(4^3,1))$ if there is a common column C_1, and $((7,2^2,1),(4^3))$ if there is a common column C_2.

If (1) is to the right of β and $(m-1)$ is to the left of β, we distinguish two cases. In both cases we start by removing all the common rows and columns. If $l(\beta)=2$, we remove all but the two leftmost columns of β (the two remaining columns of β are of length 2 since $\beta \neq (m-1,1)$), the corresponding columns of $\tilde{\lambda}$, all but three columns from the left part of α and the corresponding columns of $\tilde{\mu}$. We obtain the seed $((6,3^3),(5^3))$. If $l(\beta) > 2$, we remove all but the leftmost column of β, the corresponding columns of $\tilde{\lambda}$, the column of length 1 from the right part of α, all but $l(\beta)$ columns from the left part of α and the corresponding columns of $\tilde{\mu}$. In the next step we remove the common row to obtain $\tilde{\lambda} = (l(\beta)^{l(\beta)+1}) = \tilde{\mu}'$. The result follows from Lemma 9.1.

If both parts of α are to the left of β, this works essentially as before. If $l(\beta) = 2$, we obtain the seed $((4^3,3),(6^2,3))$ if (1) is the right part of α, and $((4^3,1),(6^2,1))$ if (1) is the left part of α. If $l(\beta) > 2$, we obtain the result with Lemma 9.1. $\square$

Lemma 12.7. *If* $\alpha^{(')} = (m-1)*(1)$ *and* β^{rot} *has two removable nodes,* $g_2(\lambda,\mu) \geq 3$.

Proof: Again, we assume that $\alpha = (m-1)*(1)$ and look at the six different possibilities that we have for the ordering of α and β. We remove all the common rows and columns to obtain $\tilde{\lambda}$ and $\tilde{\mu}$. If $(m-1)$ is to the left of β and $l(\beta) = 2$, we remove all but the leftmost and the rightmost column of β, the corresponding columns of $\tilde{\lambda}$, all but two columns of $(m-1)$ and the corresponding columns of $\tilde{\mu}$. We obtain one of the seeds $((5,3,2^2),(4^3))$, $((4,3^2,2),(5^2,2))$ and $((4,3^2,1),(5^2,1))$ depending on the position of (1). If $l(\beta) > 2$ we remove all but the rightmost column of β, the corresponding columns of $\tilde{\lambda}$, the column with (1) from α, the right number of columns from $(m-1)$ and the corresponding columns of $\tilde{\mu}$. After removing the common rows and columns again, we obtain $\tilde{\lambda} = (l(\beta)^{l(\beta)+1}) = \tilde{\mu}'$.

If β is to the left of both parts of α, we remove all but the leftmost and the rightmost column of β, the corresponding columns of $\tilde{\lambda}$ and the fitting number of columns from $(m-1)$ together with the corresponding columns of $\tilde{\mu}$. Now $\tilde{\beta} = (2^a)/(1^b)$. If $b > 2$, we remove (1^2) from λ as rows and (2) from μ until we obtain $\tilde{\lambda}$ and $\tilde{\mu}$ with $\tilde{\beta} = (2^{\tilde{a}})/(1^{\tilde{b}})$ with $\tilde{b} \in \{1,2\}$. If $(m-1)$ is the rightmost part, we reduce this to the seed $((5,3,1),(3,2^3))$ if $\tilde{b} = 1$, and $((6,3,1^2),(3,2^4))$ if $\tilde{b} = 2$. If (1) is the rightmost part, $\tilde{\lambda} = (\tilde{m}+2,\tilde{m}+1,1^{\tilde{b}})$ with $\tilde{b} \in \{1,2\}$ and $\tilde{\mu} = (\tilde{m}+1,2^{\tilde{a}+1})$. We remove (1) from $\tilde{\lambda}$ and (1) from $\tilde{\mu}$. Now $\tilde{\alpha}^{\text{rot}} = (\tilde{m}-1,1)$ and $\tilde{\beta}$ is a rotated two-column partition. The result follows from Lemma 11.1.

If (1) is to the left of β and $(m-1)$ is to the right of β, we remove all but the leftmost and the rightmost column from β, the corresponding columns from $\tilde{\lambda}$ together with the right number of boxes from $(m-1)$. Now $\tilde{\lambda} = (m+2,2^a,1^b)$ and $\tilde{\mu} = (3^{a+b})$. We reduce a and b by 3 until we obtain a seed with $a,b \in \{1,2,3\}$. $\square$

In the next step we want to look at the case that $\alpha = (n-2)*((2)^{(')})$ and β is a rectangle. But if $\alpha = (2)*(1^2)$ and $\beta = (2^2)$, it can happen that λ and μ are symmetric. So in this case we do not only want to find two constituents with multiplicity 3 or higher but we want to know that one of them is not symmetric. Therefore, we first look at that case.

Lemma 12.8. *If* $\alpha = (2)*(1^2)$ *and* $\beta = (2^2)$, $g_2(\lambda,\mu) \geq 3$. *If* β *is between the two parts of* α*, the product* $[\lambda][\mu]$ *contains a non-symmetric constituent with multiplicity 3 or higher.*

Proof: We start with the case that both parts of α are on one side of β. By conjugation we can assume that this is the right side. In both cases we remove all the common rows and columns. If (2) is the rightmost part, we obtain the seed $((5,3^2),(3,2^4))$. If (1^2) is the rightmost part, we obtain the seed $((5^2,4),(4^2,2^3))$.

Now we assume that β is between the two parts of α. If (2) is to the left of β, we remove all the common rows and columns to obtain the seed $((5^2,2^3),(4^4))$, which contains a non-symmetric constituent (for example $(3^4, 2^2)$) with multiplicity higher than 3. If there are two or more common rows above β or two or more common columns to the left of β, the non-symmetric constituent in $[\lambda][\mu]$ follows from Lemma 5.17. If there is just one common row above β and/or one common column to the left of β, we see that the constituent that we obtain from $(3^4, 2^2)'$ cannot be symmetric if we compare length and width.

If (1^2) is to the left of β and (2) is to the right of β, we know that λ is not a hook. Therefore, we know that there is a common row R_1 above (2) or a common row R_2 between (2) and β or a common column to the left of (1^2) or a common column between (1^2) and β. By symmetry and conjugation we can assume that there is a common row. If we remove all common rows and columns but R_1, we obtain the seed $((5^2, 1^4), (5, 3^3))$. We obtain the seed $((5, 3, 1^4), (3^4))$ if we remove all the common rows and columns but R_2. Both seeds contain a non-symmetric constituent with multiplicity 3 or higher for example $(3^3, 2, 1^3)$ and $(3^3, 1^3)$. With the same argument as before we obtain that the product $[\lambda][\mu]$ contains a non-symmetric constituent. □

Lemma 12.9. *If $m = ab$ for $a, b \geq 2$, $\alpha^{(')} = (m-2) * (2)$ or $\alpha^{(')} = (m-2) * (1^2)$ and $\beta = (a^b)$, $g_2(\lambda, \mu) \geq 3$.*

Proof: By conjugation we can assume that the part of α which has two boxes is (2). We start with the case that $\alpha = (2) * (1^{m-2})$. Because of the previous lemma we can assume that $m \geq 6$. We remove all the common rows and columns. If both parts of α are to the right of β, we start with the case that m is even and $\beta = (2^{\frac{m}{2}})$. This can be reduced to the seed $((5, 3^2), (3, 2^4))$ if (2) is the right part of α, and $((5^2, 4), (4^2, 2^3))$ if (1^m) is the right part of α. If $a > 2$, we remove all but one row of β, the row of α of length 2, the corresponding row of $\tilde{\mu}$ and the right number of rows of length 1 from α together with the corresponding rows of $\tilde{\mu}$. If (2) is the left part of α, we have two common columns that we remove. Now in both cases $\tilde{\lambda} = ((a+1)^a) = \tilde{\mu}'$, so the result follows from Lemma 9.1.

Now we assume that one of the parts of α is to the right of β and the other one is to the left of β. If (2) is to the right of β, $\tilde{\alpha}$ is a hook and, since $m \geq 6$, the result follows from Proposition 6.2. If (2) is to the left of β, we remove all but one row of β, the corresponding rows of $\tilde{\lambda}$, the row of length 2 of α and the right number of rows of length 1 from α together with the corresponding rows from $\tilde{\mu}$. If $a = 2$, we obtain the seed $((5^2, 2), (4^3))$. If $a > 2$, we remove the two common columns to obtain $\tilde{\lambda} = ((a+1)^a) = \tilde{\mu}'$. The result follows from Lemma 9.1.

In the next step we assume that both parts of α are to the left of β. If (1^{m-2}) is the left part of α, we remove all but two rows from β, the corresponding rows from $\tilde{\lambda}$ and the right number of rows of length 1 from α. In the next step we remove all but two columns of $\tilde{\beta}$ and all but two rows of length 1 from $\tilde{\alpha}$. We obtain the seed $((3^3, 1^2), (5^2, 1))$. If (2) is the left part of α, $\tilde{\lambda} = (3^{m-2+b}, 2)$ and $\tilde{\mu} = ((3+a)^b, 2^{m-2})$. If we remove a column of length $m-2+b$ from both partitions, $\tilde{\lambda}$ is a two-column rectangle and the result follows from Lemma 7.18.

Now we look at the second case $\alpha = (m-2) * (2)$. If both parts are to the right of β, $\tilde{\lambda}$ is a two-row partition and $\tilde{\mu}$ is a proper fat hook. The result follows from Lemma 7.17 if $a = 2$ and (2) is the left part of α, and Lemma 7.19, otherwise.

For the next case we assume that one part of α is to the right of β. If $b = 2$, this can be reduced to the seed $((6, 2^3), (4^3))$. If $b > 2$ and $(m-2)$ is to the left of β, we remove all but one column of β, the corresponding columns of $\tilde{\lambda}$, the two columns of length 1 from the right part of α as well as all but b columns of the left part of α together with the corresponding columns of $\tilde{\mu}$. After removing the common row of length $b+1$, we obtain $\tilde{\lambda} = (b^{b+1}) = \tilde{\mu}'$. The result follows from

Lemma 9.1. If $b > 2$ and (2) is to the left of β, we remove all but one column from β and the fitting number of columns of length 1 from α. Now $\tilde{\beta} = (1^b)$ and the result follows from Lemma 10.8 (from part (5) where we do not need the existence of extra rows or columns).

The last case is that both parts of α are to the left of β. If $b = 2$, this can be reduced to the seed $((4^3, 2), (6^2, 2))$. If $b > 2$, we remove all but one column of β, all columns from α but b columns from $(m-2)$ and the corresponding columns from $\tilde{\mu}$. Maybe after removing a common row, $\tilde{\lambda} = (b^{b+1}) = \tilde{\mu}'$. The result follows from Lemma 9.1. □

Lemma 12.10. *If $m = 2k$, $\alpha = (m-l) * (1^l)$ for $3 \leq l \leq m-3$ and $\beta^{(')} = (k, k)$, $g_2(\lambda, \mu) \geq 3$.*

Proof: By conjugation we can assume that $\beta = (k, k)$. We remove all the common rows and columns. We start with the case that both parts of α are to the right of β. We remove an even number of columns from $(m-l)$ such that one or two boxes are left, the corresponding columns of $\tilde{\mu}$, the right number of columns from $\tilde{\beta}$ and the corresponding columns of $\tilde{\lambda}$. If there are two boxes left, the result follows from Lemma 12.9, if there is just one box left, it follows from Lemma 12.6.

For the next case we assume that β is between the two parts of α. If $(m-l)$ is to the left of β, we can do the same as before. If (1^l) is to the left of β and $(m-l)$ is to the right of β, $\tilde{\lambda}$ is a hook and the result follows from Proposition 6.2.

For the last case we assume that both parts of α are to the left of β. If $(m-l)$ is the left part of α, we can reduce this part so that the result follows from Lemma 12.9 or 12.6 like before. If (1^l) is the left part of α, we remove an even number of boxes from (1^l) and (k, k) until there are one or two boxes of (1^l) left. Then we remove an even number of columns from $(m-l)$, the corresponding columns from $\tilde{\mu}$ and the right number of boxes from $\tilde{\beta}$. We obtain the seed $((4^3, 1), (6^2, 1))$ if l is odd, and $((3^3, 1^2), (5^2, 1))$ if l is even. □

4. α and β are skew partitions

Lemma 12.11. *If $m = 2$, $\alpha = (1)*(1)$ and $\beta = (1)*(1)$, the product $[\lambda][\mu]$ contains two constituents with multiplicity 3 or higher of which one is not symmetric.*

Proof: There are exactly four rows and four columns of λ and μ which are not identical. If we remove all the identical rows and columns, we end up with $\tilde{\lambda}$ and $\tilde{\mu}$ which are partitions of 8 which are inside (4^4) such that α and β are still $(1) * (1)$. Up to exchanging $\tilde{\lambda}$ and $\tilde{\mu}$ we have the following possible pairs:

$$((4, 3, 1), (3, 2^2, 1)),\ ((4, 2^2), (3^2, 1^2)),\ ((4, 2, 1^1), (3, 2^2))$$

In all three cases the corresponding products contain $[4, 1^4]$ and $[4, 2, 1^2]$ with multiplicity 3 or higher. Why is one of the constituents from $[\lambda][\mu]$ which arises from these not symmetric? We can obtain $\tilde{\lambda}$ and $\tilde{\mu}$ from λ and μ by removing parts in such a way that we can apply Lemma 5.16 or Lemma 5.17. □

CHAPTER 13

α is a fat hook, a rectangle or has three removable nodes

Since we assume that α and β are listed (up to equivalence) in Theorem 5.1, we know that $\mathrm{rem}(\alpha^{(\mathrm{rot})})$, $\mathrm{rem}(\beta^{(\mathrm{rot})}) \leq 3$. In almost all the remaining products α or β is equivalent to a hook or a two-line partition. The products from Theorem 5.1 where none of the factors is a hook or a two-line partition are $[3^3]^2$, $[3^3][4^2,1]$ and $[3^3][3,2^3]$. Before we look at the case that α and β are equivalent to a hook or a two-line partition, we start with the cases where one of them is neither a hook nor a two-line partition. Like in the previous chapters we can assume (by conjugating and exchanging λ and μ) that α is neither a hook nor a two-line partition. In contrast to the previous chapter we assume again that α is to the right and above β.

1. α has three removable nodes

If $\mathrm{rem}(\alpha^{(\mathrm{rot})}) = 3$, there are not many possibilities for α and β. Namely there are (up to equivalence):

- $m = 2k$ is even, $\alpha = (m-3, 2, 1)$ and $\beta = (k, k)$;
- $m = 8$, $\beta = (4, 4)$ and $\alpha = (3, 2^2, 1)$.

We assume again that α is to the right of β.

Lemma 13.1. *If $m = 2k$, $\alpha \equiv (m-3, 2, 1)$ and $\beta^{(')} = (k, k)$, $g_2(\lambda, \mu) \geq 3$.*

Proof: We start by removing all the common rows and columns to obtain $\tilde{\lambda}$ and $\tilde{\mu}$. If $\beta = (2^k)$ and α is a partition, $\tilde{\mu}$ is a two-column rectangle and the result follows from Lemma 7.18.

If $\beta = (k, k)$, α is a partition and $m = 6$, we have the seed $((6, 5, 4), (3^5))$. If $m > 6$, we remove α and β and the result follows from Lemma 9.1.

If $\alpha^{\mathrm{rot}} = (m-3, 2, 1)$ and $\beta = (k, k)$, this can easily be reduced to the seed $((6^3), (5, 4, 3^3))$.

If $\alpha^{\mathrm{rot}} = (3, 2, 1^{m-5})$ and $\beta = (k, k)$, we can assume that $m > 6$, since $m = 6$ is covered in the previous case. We remove one row of β, the row of length 3 and $k-3$ rows of length 1 from α and the corresponding rows of $\tilde{\mu}$. Now $\tilde{\beta} = (k)$ and $l(\tilde{\alpha}) \geq 3$, therefore, the result follows from Lemma 10.2.

If $\alpha^{\mathrm{rot}} = (3, 2, 1^{m-5})$ and $\beta = (2^k)$, this can easily be reduced to the seed $((5^3), (4, 3, 2^4))$.

If $\alpha^{\mathrm{rot}} = (m-3, 2, 1)$ and $\beta = (2^k)$, we can again assume that $m > 6$. $\tilde{\lambda} = ((m-1)^3)$ and $\tilde{\mu} = (m-2, m-3, 2^{k+1})$. We remove a row of length $m-1$ from $\tilde{\lambda}$ and we remove a row of length $m-3$ and one of length 2 from $\tilde{\mu}$. Now $\tilde{\lambda}$ is a two-row rectangle and the result follows from Lemma 7.18. □

Lemma 13.2. *If $m = 8$, $\alpha \equiv (3, 2^2, 1)$ and $\beta^{(')} = (4, 4)$, $g_2(\lambda, \mu) \geq 3$.*

Proof: We remove all the common rows and columns to obtain the seed. We have four possibilities for α and two possibilities for β so in total we get eight seeds. There are:

α	β	Seed
$(3, 2^2, 1)$	$(4, 4)$	$((7, 6^2, 5), (4^6))$
$(3, 2^2, 1)$	(2^4)	$((5, 4^2, 3), (2^8))$
$(4, 3, 1)$	$(4, 4)$	$((8, 7, 5), (4^5))$
$(4, 3, 1)$	(2^4)	$((6, 5, 3), ((2^7))$
$(3^4)/(2, 1^2)$	$(4, 4)$	$((7^4), (6, 5^2, 4^3))$
$(3^4)/(2, 1^2)$	(2^4)	$((5^4), (4, 3^2, 2^5))$
$(4^3)/(3, 1)$	$(4, 4)$	$((8^3), (7, 5, 4^3))$
$(4^3)/(3, 1)$	(2^4)	$((5^3), (4, 3^2, 2^5))$.

□

For the exceptional cases from Theorem 5.1 we often just have to remove all the common rows and columns, like in the previous lemma, to obtain a seed.

2. α is a proper fat hook

If α is a proper fat hook, we know that α and β are up to equivalence a pair from the following list (since we can exclude the cases that β is equivalent to (m) or $(m-1, 1)$):

(1) If $m = ab - 1$ and $\alpha = (a^{b-1}, a-1)$ and $\beta = (m-2, 1^2)$ or $\beta = (m-2, 2)$;
(2) if $m = 2k + 1$ is odd and $\alpha = (k^2, 1)$ and $\beta = (n-2, 2)$;
(3) one of the exceptional pairs:

$$((4, 3), (3, 2^2)),\ ((4^2), (3^2, 1^2)),\ ((4^2), (3^2, 2)),\ ((5, 3), (3^2, 2)),$$
$$((5, 4), (3, 2^3)),\ ((5^2), (4, 3^2)),\ ((5^2), (4^2, 2), ((3^3), (4^2, 1)).$$

Lemma 13.3. *If $m = ab - 1$ for $a, b \geq 3$, $\alpha \equiv (a^{b-1}, a-1)$ and $\beta \equiv (m-2, 1^2)$ or $\beta \equiv (m-2, 2)$, $g_2(\lambda, \mu) \geq 3$.*

Proof: We start with removing all common rows and columns. We assume that α is above β. We start with the case that α and β are partitions. If $l(\alpha) \geq 3$ and $w(\beta) \geq 3$ and one of them is strictly bigger, we remove α and β to obtain two rectangles which contain $(4^3)^{(')}$, so the result follows from Lemma 9.1. If $l(\alpha)$ and $w(\beta)$ equals three, we remove all but the one columns resp. row of length 3 from α and β to obtain the seed $((4^3), (3^4))$. This solves the three cases where α is a partition and $\beta = (m-2, 1^2)$, $(m-2, 2)$, $(3, 1^{m-3})$. If α and β are partitions, only the case $\beta = (2^2, 1^{m-4})$ is missing. Here $\tilde{\mu}$ is a two-column partition and the result follows from Lemma 7.19.

For the next case we assume that $\alpha = (a^b)/(1)$ and β is a partition. If $w(\beta) \geq 3$ and b or $w(\beta)$ is strictly greater than 3, we remove (a^b) from $\tilde{\lambda}$ and $(w(\beta) + 1, \hat{\beta})$ from $\tilde{\mu}$ as rows. We obtain two rectangles $\tilde{\lambda} = ((w(\beta))^b) = \tilde{\mu}$. The result follows from Lemma 9.1. If $l(\alpha) = w(\beta) = 3$, we know that $\beta = (3, 1^{m-3})$. We remove columns of length 3 from $\tilde{\lambda}$ and the corresponding number of rows of length 1 from β until $w(\tilde{\lambda}) = 5$ to obtain the seed $((5^3), (4, 3^3, 1^2))$. What is missing is the case $\beta = (2^2, 1^{m-4})$. This can be reduced to the seed $((4^3), (3, 2^4, 1))$.

From now on we can assume that β^{rot} is a partition. If $\alpha^{(\mathrm{rot})} = (a^{b-1}, a-1)$ and $\beta^{\mathrm{rot}} = (m-2, 2)$, we remove a column of length 2 and $b-2$ columns of length 1 from β, the corresponding columns from $\tilde{\lambda}$ and a column of length b from α. Now $\tilde{\beta}^{\mathrm{rot}} = (\tilde{m} - 1, 1)$ and the result follows from Lemma 11.2.

If $\beta^{\mathrm{rot}} = (2^2, 1^{m-4})$ and α is a partition,$\tilde{\lambda} = ((a+2)^{b-1}, a+1, 1^{m-4})$ and $\tilde{\mu} = (2^{m-2+b})$. The result follows from Lemma 7.18.

If $\beta^{\mathrm{rot}} = (2^2, 1^{m-4})$ and α^{rot} is a partition, we remove a row of length 1 from $\tilde{\lambda}$ and a column of length 1 from $\tilde{\mu}$. Now $\tilde{\lambda} = ((a+2)^b)$ and $\tilde{\mu} = (2^{m-2+b})$. The result follows from Lemma 7.18.

If $\beta^{\text{rot}} = (m-2, 1^2)$, we remove all columns of α but one of length b, the corresponding columns of $\tilde{\mu}$ and all columns of β but b columns of length 1 together with the corresponding columns of $\tilde{\lambda}$. After removing two common rows, we obtain $\tilde{\lambda} = (b^{b+1}) = \tilde{\mu}'$ and the result follows from Lemma 9.1.

If $\beta^{\text{rot}} = (3, 1^{m-3})$, we remove all of α but two length a, the corresponding rows of $\tilde{\mu}$, the right number of rows of length 1 from β and the corresponding rows of $\tilde{\lambda}$. Now $\tilde{\lambda} = ((a+3)^2, 2^{2a-3})$ and $\tilde{\mu} = (3^{2a})$, where $\tilde{\alpha} = (a^2)$ and $\tilde{\beta}^{\text{rot}} = (3, 1^{2a-3})$. This will be solved in Lemma 14.3. □

Lemma 13.4. *If $m = 2k+1 \geq 7$ is odd, $\alpha \equiv (k^2, 1)$ and $\beta \equiv (n-2, 2)$, $g_2(\lambda, \mu) \geq 3$.*

Proof: In all cases we start with removing all common rows and columns to obtain $\tilde{\lambda}$ and $\tilde{\mu}$. Let us start with the case that α and β are partitions. If $\beta = (m-2, 2)$ and α is a partition, α and β are removable. After removing them, we obtain two rectangles $\tilde{\lambda} = \tilde{\mu}$ which contain (4^3). The result follows from Lemma 9.1. If $\beta = (2^2, 1^{m-4})$ and $\alpha = (k^2, 1)$, $\tilde{\lambda} = ((k+2)^2, 3)$ and $\tilde{\mu} = (2^5, 1^{m-4})$. We remove $m-7$ rows of length 1 from β and $\frac{m-7}{2}$ columns of length 2 from α to obtain the seed $((5^2, 3), (2^5, 1^3))$. If $\beta = (2^2, 1^{m-4})$ and $\alpha = (3, 2^{k-1})$, we remove $k-3$ rows of length 4 from $\tilde{\lambda}$ and $k-3$ times $(2, 1^2)$ from $\tilde{\mu}$ as rows to obtain the seed $((5, 4^2), (2^5, 1^3))$.

In the next step we look at the case that α^{rot} and β are partitions. If $\beta = (m-2, 2)$ and $\alpha^{\text{rot}} = (3, 2^{k-1})$, $\tilde{\lambda} = ((m+1)^k)$ and $\tilde{\mu} = ((m-1)^{k-1}, (m-2)^2, 2)$. We remove $k+1$ columns of length k from $\tilde{\lambda}$ to obtain $((k+1)^k)$. From $\tilde{\mu}$ we remove the $k-1$ leftmost columns and the rightmost one which are $(k^{k-1}, (k-1)^2, 2)$ to obtain (k^{k+1}). The result follows, again, from Lemma 9.1. If $\beta = (m-2, 2)$ and $\alpha^{\text{rot}} = (k^2, 1)$, $\tilde{\lambda} = ((3k-1)^3)$ and $\tilde{\mu} = (3k-2, (2k-1)^3, 2)$. We remove $((3k-9)^3)$ from $\tilde{\lambda}$ and $(3k-9, (2k-6)^3)$ to obtain the seed $((8^3), (7, 5^3, 2))$. If $\beta = (2^2, 1^{m-4})$ and $\alpha^{\text{rot}} = (k^2, 1)$, $\tilde{\lambda} = ((k+2)^3)$ and $\tilde{\mu} = (k+1, 2^4, 1^{m-4})$. For $k = 3, 4, 5$, this is a seed. If $k > 5$, we remove (1^3) from $\tilde{\lambda}$ and (3) from $\tilde{\mu}$ and in the next step (2^3) from $\tilde{\lambda}$ and six rows of length 1 from $\tilde{\mu}$. We repeat this until we reach one of the seeds. If $\beta = (2^2, 1^{m-4})$ and $\alpha^{\text{rot}} = (3, 2^{k-1})$, $\tilde{\lambda} = (5^k)$ and $\tilde{\mu} = (3^{k-1}, 2^3, 1^{m-4})$. We remove (5^{k-3}) as rows from $\tilde{\lambda}$ and $(3^{k-3}, 1^{m-7})$ from $\tilde{\mu}$ as rows to obtain the seed $((5^3), (3^2, 2^3, 1^3))$.

In the next paragraph we look at the case that α and β^{rot} are partitions. We start with $\beta^{\text{rot}} = (m-2, 2)$ and $\alpha = (k^2, 1)$. We have $\tilde{\lambda} = ((3k-1)^2, 2k, 2k-3)$ and $\tilde{\mu} = ((2k-1)^5)$. We remove $((2k-6)^5)$ from $\tilde{\mu}$ and $((3k-9)^2, (2k-6)^2)$ from $\tilde{\lambda}$ to obtain the seed $((8^2, 6, 3), (5^5))$. If $\beta^{\text{rot}} = (m-2, 2)$ and $\alpha = (3, 2^{k-1})$, $\tilde{\lambda} = (2k+2, (2k+1)^{k-1}, 2k-3)$ and $\tilde{\mu} = ((2k-1)^{k+2})$. We remove $((k-1)^{k+2})$ from $\tilde{\mu}$ and $(k+1, k^{k-1}, k-3)$ from $\tilde{\lambda}$ to obtain $\tilde{\lambda} = ((k+1)^k, k)$ and $\tilde{\mu} = (k^{k+2})$. We remove the common row of length k to obtain two rectangles $\tilde{\lambda} = \tilde{\mu}'$. The result follows from Lemma 9.1. If $\beta^{\text{rot}} = (2^2, 1^{m-4})$ and $\alpha = (k^2, 1)$, we obtain $\tilde{\lambda} = ((k+2)^2, 3, 1^{m-4})$ and a two-column rectangle $\tilde{\mu} = (2^{m+1})$. The result follows from Lemma 7.18. If $\alpha = (3, 2^{k-1})$ and $\beta^{\text{rot}} = (2^2, 1^{m-4})$, we obtain $\tilde{\lambda} = (5, 4^{k-1}, 1^{2k-3})$ and a two-column rectangle $\tilde{\mu} = (2^{3k-1})$. Again, the result follows from Lemma 7.18.

The last case we have to look at is α^{rot} and β^{rot} are partitions. If $\alpha^{\text{rot}} = (k^2, 1)$ and $\beta^{\text{rot}} = (m-2, 2)$, $\tilde{\lambda} = ((3k-1)^3, 2k-3)$ and $\tilde{\mu} = (3k-2, (2k-1)^4)$. We remove $((3k-9)^3, 2k-6)$ from $\tilde{\lambda}$ and $(3k-9, (2k-6)^4)$ from $\tilde{\mu}$ to obtain the seed $((8^3, 3), (7, 5^4))$. If $\alpha^{\text{rot}} = (k^2, 1)$ and $\beta^{\text{rot}} = (2^2, 1^{m-4})$, we obtain the partitions $\tilde{\lambda} = ((k+2)^3, 1^{2k-3})$ and $\tilde{\mu} = (k+1, 2^{2k+1})$. If $k = 3, 4, 5$ these are seeds. If $k > 5$, we remove (2^3) from $\tilde{\lambda}$ and (2^3) from $\tilde{\mu}$ as rows. In the next step we remove (3) from $\tilde{\lambda}$ and (1^3) from $\tilde{\mu}$ and (1^6) from $\tilde{\lambda}$ as rows together with (2^3) from $\tilde{\mu}$ as rows. We repeat this until we obtain one of the seeds $((5^3, 1^3), (4, 2^7))$, $((6^3, 1^5), (5, 2^9))$ and

$((7^3, 1^7), (6, 2^{11}))$. If $\alpha^{\text{rot}} = (3, 2^{k-1})$ and $\beta^{\text{rot}} = (2^2, 1^{m-4})$, we have $\tilde{\lambda} = (5^k, 1^{2k-3})$ and $\tilde{\mu} = (3^{k-1}, 2^{2k})$. We remove $(5^{k-3}, 1^{2k-6})$ from $\tilde{\lambda}$ as rows and $(3^{k-3}, 2^{2k-6})$ from $\tilde{\mu}$ as rows to obtain the seed $((5^3, 1^3), (3^2, 2^6))$. If $\alpha^{\text{rot}} = (3, 2^{k-1})$ and $\beta^{\text{rot}} = (m-2, 2)$, we obtain $\tilde{\lambda} = ((2k+2)^k, 2k-3)$ and $\tilde{\mu} = ((2k)^{k-1}, (2k-1)^3)$. We remove $((k+1)^k, k-3)$ from $\tilde{\lambda}$ and $(k^{k-1}, (k-1)^3)$ from $\tilde{\mu}$. In the next step we remove the common row of length k to obtain $\tilde{\lambda}' = (k^{k+1}) = \tilde{\mu}$. The result follows from Lemma 9.1. □

Lemma 13.5. *If α is equivalent to a proper fat hook and (α, β) is one of the exceptional pairs, $g_2(\lambda, \mu) \geq 3$.*

Proof: In all eight cases we do the same, we just remove all the common rows and columns and obtain a seed which contains two constituents with multiplicity greater or equal to 3. But if β is not a rectangle and α and β are not symmetric, we have sixteen different seeds. So this is easy but tedious work. By conjugation we can assume that α is above β. We get the following seeds:

$((4,3),(3,2^2))$: We have sixteen seeds. These are:

$$((7,6^2),(4^4,3)),\ ((7^2,5),(4^4,3)),\ ((5,4^2),(2^6,1)),\ ((5^2,3),(2^6,1)),$$
$$((7,6^2,1),(4^5)),\ ((7^2,5,1),(4^5)),\ ((5,4^2,1),(2^7)),\ ((5^2,3,1),(2^7)),$$
$$((7^3),(5^2,4^2,3)),\ ((7^3),(6,4^3,3)),\ ((5^3),(3^2,2^4,1)),\ ((5^3),(4,2^5,1)),$$
$$((7^3,1),(5^2,4^3)),\ ((7^3,1),(6,4^4)),\ ((5^3,1),(3^2,2^5)),\ ((5^3,1),(4,2)).$$

$((4^2),(3^2,1^2))$: We have eight seeds. These are:

$$((7^2,5^2),(4^6))\ ((8,6^2),(4^5)),\ ((5^2,3^2),(2^8)),\ ((6,4^2),(2^7)),$$
$$((7^4),(6^2,4^4)),\ ((8^3),(6^2,4^3)),\ ((5^4),(4^2,2^6)),\ ((6^3),(4^2,2^5)).$$

$((4^2),(3^2,2))$: We have four seeds. These are:

$$((7^2),(3^4,2)),\ ((7^2,1),(3^5)),\ ((5^4),(6^3,2)),\ ((5^4,1),(3^7)).$$

$((5,3),(3^2,2))$: We have eight seeds. These are:

$$((8^2,7)(5^4,3)),\ ((5^2,4),(2^6,1^2)),\ ((8^2,7,2),(5^5)),\ ((5^2,4,1^2),(2^8)),$$
$$((8^3),(6,5^3,3)),\ ((5^3),(3,2^5,1^2)),\ ((8^3,2),(6,5^4)),\ ((5^3,1^2),(3,2^7)).$$

$((5,4),(3,2^3))$: We have sixteen seeds. These are:

$$((8,7^3),(5^5,4)),\ ((9^2,6),(5^4,4)),\ ((5,4^3),(2^8,1)),\ ((6^2,3),(2^7,1)),$$
$$((8^4),(6^3,5^2,4)),\ ((9^3),(8,5^3,4)),\ ((5^4),(3^3,2^5,1)),((6^3),(5,2^6,1)),$$
$$((8,7^3,1),(5^6)),\ ((9^2,6,1),(5^5)),\ ((5,4^3,1),(2^9)),\ ((6^2,3,1),(2^8)),$$
$$((8^4,1),(6^3,5^3)),\ ((9^3,1),(8,5^4)),\ ((5^4,1),(3^3,2^6)),\ ((6^3,1),(5,2^7)).$$

$((5,5),(4,3^2))$: We have eight seeds. These are:

$$((9,8^2),(5^5)),\ ((8^3,6),(5^6)),\ ((6,5^2),(2^8)),\ ((5^3,3),(2^9)),$$
$$((9^3),(6^2,5^3)),\ ((8^4),(7,5^5)),\ ((6^3),(3^2,2^6)),\ ((5^4),(4,2^8)).$$

$((5,5),(4^2,2))$: We have eight seeds. Theses are:

$$((9^2,7),(5^5)),\ ((8^2,7^2),(5^6)),\ ((6^2,4),(2^8)),\ ((5^2,4^2),(2^9)),$$
$$((9^3),(7,5^4)),\ ((8^4),(6^2,5^4)),\ ((6^3),(4,2^7)),\ ((5^4),(3^2,2^7)).$$

$((3^3),(4^2,1))$: We have four seeds. These are:

$$((7^2,4),(3^6)),\ ((6,5^3),(3^7)),\ ((7^3),(6,3^5)),\ ((6^4),(4^3,3^4)).$$

□

3. α is a proper rectangle

In this section we assume that α is a proper rectangle. We know that α and β are up to equivalence a pair from the following list (since we can exclude that β is equivalent to (m) and $(m-1,1)$):

(1) β is a hook with $w(\beta) \leq 4$ or $l(\beta) \leq 4$;
(2) $\beta = (m-2,2)$ or $(m-3,3)$;
(3) the exceptional cases:

$((5,1^4),(3^3)),\ ((7,5),(4^3)),\ ((8,4),(4^3)),\ ((6,6),(4^3)),$

$((6,3),(3^3)),\ ((5,4),(3^3)),\ ((8,7),(5^3)),\ ((8,8),(4^4)),\ ((9,9),(6^3)),((3^3),(3^3)).$

We start with the exceptional cases:

Lemma 13.6. *If (α,β) is an exceptional pair from Proposition 7.1 and α is a rectangle, $g_2(\lambda,\mu) \geq 3$.*

Proof: The idea is always the same. We remove all common rows and columns. Sometimes we get a seed like this. Sometimes we still have to remove a bit more because the partitions are still too big. And in two case we have to remove less. This leads up to eight different seeds for one exceptional pair.

$((3^3),(5,1^4))$: If $\beta = (5,1^4)$, we obtain the seed $((8^3),(5^4,1^4))$ after removing all the common rows and columns. If $\beta^{\text{rot}} = (5,1^4)$, we obtain $\tilde{\lambda} = (8^3,4^4)$ and $\tilde{\mu} = (5^8)$. We remove two rows and two columns of length 8 and obtain the seed $((8,4^4),(3^8))$.

$((4^3),(7,5))$: If β or β^{rot} equals $(2^5,1^2)$, we remove all the common rows and columns and obtain one of the seeds:

$$((6^3),(2^8,1^2))\,((5^4),(2^9,1^2)),\,((6^3,1^2),(2^{10})),\,((5^4,1^2),(2^{11})).$$

In the other cases we have to remove more than just the common rows and columns. If $\alpha = (4^3)$ and $\beta = (7,5)$, we obtain $\tilde{\lambda} = (11^3)$ and $\tilde{\mu} = (7^4,5)$ after removing all the common rows and columns. Here we remove five columns of length 3 from $\tilde{\lambda}$ and three columns of length 5 from $\tilde{\mu}$ to obtain the seed $((6^3),(4^4,2))$. If $\alpha = (4^3)$ and $\beta^{\text{rot}} = (7,5)$, we obtain $\tilde{\lambda} = (11^3,2)$ and $\tilde{\mu} = (7^5)$. Here we remove five columns of length 3 from $\tilde{\lambda}$ and three columns of length 5 from $\tilde{\mu}$ to obtain the seed $((6^3,2),(4^5))$. If $\alpha = (3^4)$ and $\beta = (7,5)$, we obtain $\tilde{\lambda} = (10^4)$ and $\tilde{\mu} = (7^5,5)$. Here we remove three columns of length 4 from $\tilde{\lambda}$ and one row of length 7 together with the row of length 2 from $\tilde{\mu}$. We end up with the seed $((7^4),(7^4))$. If $\alpha = (3^4)$ and $\beta^{\text{rot}} = (7,5)$, we obtain $\tilde{\lambda} = (10^4,2)$ and $\tilde{\mu} = (7^6)$. In a first step we remove a row of length 10 together with the one of length 2 from $\tilde{\lambda}$ and two columns of length 6 from $\tilde{\mu}$ to obtain $\tilde{\lambda} = (10^3)$ and $\tilde{\mu} = (5^6)$. In the next step we remove four columns of length 3 from $\tilde{\lambda}$ and two columns of length 6 from $\tilde{\mu}$ to obtain the seed $((6^3),(3^6))$.

$((4^3),(8,4))$: If $\beta = (8,5)$, we remove α and β after removing all the common rows and columns and in both cases obtain two rectangles $\tilde{\lambda} = \tilde{\mu}$ which contain (4^3). The result follows from Lemma 9.1. If β or β^{rot} equals $(2^4,1^2)$, we obtain the seeds directly after removing all the common rows and columns. These are:

$$((6^3),(2^7,1^4)),\,((5^4),(2^8,1^4)),\,((6^3,1^4),(2^{11})),\,((5^4,1^4),(2^{12})).$$

The missing cases are all where $\beta^{\text{rot}} = (8,4)$. If $\alpha = (4^3)$, we obtain $\tilde{\lambda} = (12^3,4)$ and $\tilde{\mu} = (8^5)$ after removing all common rows and columns. Here we remove five columns of length 3 and three columns of length 5 to obtain the seed $((7^3,4),(5^5))$. If $\alpha = (3^4)$, we have $\tilde{\lambda} = (11^4,4)$ and $\tilde{\mu} = (8^6)$ where we remove six columns of length 4 and four columns of length 6 to obtain the seed $((5^4,4),(4^6))$.

$((4^3),(6^2))$: If $\beta=(6^2)$, we start for both choices of α with removing all the common rows and columns. Now α and β are removable and if we remove them, we obtain two rectangles $\tilde{\lambda}=\tilde{\mu}$ which contain (4^3). If $\beta=(2^6)$ and $\alpha=(3^4)$ and we remove all the common rows and columns, we obtain the seed $((5^4),(2^{10}))$. If $\beta=(2^6)$ and $\alpha=(4^3)$, we would obtain $((6^3),(2^9))$ if we removed all the common rows and columns. But this is an exceptional case from Theorem 5.1. Here we would have removed too much. But we know that μ is not a two-column partition, so there is a common column C_1 to the left of β or a common column C_2 between α and β or there is a common column to the right of α but that means there is a common row R above α. If we remove all common rows and columns but C_1, we obtain the seed $((7^3,1^6),(3^9))$. If we remove all common rows an columns but C_2, we obtain the seed $((7^3),(3^3,2^6))$. If we remove all common rows and columns but R we obtain $((6^4),(6,2^9))$.

$((3^3),(6,3))$: Here we just remove all the common rows and columns and obtain one of the following seeds:

$$((9^3),(6^4,3)),\ ((9^3),(6^5)),\ ((5^3),(2^6,1^3)),\ ((5^3,1^3),(2^9)).$$

$((3^3),(5,4))$: If $\beta=(2^4,1)$, we would remove too much if we removed all the common rows and columns. Again, we know that μ is not a two-column partition, so there is a common column C_1 to the left of β or a common column C_2 between α and β or there is a common row R above α. We obtain the seed $((6^3,1^5),(3^7,2))$ if we remove all the common rows and columns but C_1, $((6^3),(3^3,2^4,1))$ if we remove all the common rows and columns but C_2, and $((5^4),(5,2^7,1))$ if we remove all the common rows and columns but R. In the three other cases we obtain the seed by removing all the common rows and columns. These are:

$$((8^3),(5^4)),\ ((8^3,1),(5^5)),\ ((5^3,1),(2^8)).$$

$((5^3),(8,7))$: In all cases we remove all the common rows and columns. If β or β^{rot} equals $(2^7,1)$, we directly obtain one of the following seeds:

$$((7^3),(2^{10},1)),\ \ ((7^3,1),(2^{11})),\ \ ((5^5),(2^{12},1)),\ \ ((5^5,1),(2^{13})).$$

If $\beta=(8,7)$, α and β are removable. After removing them we obtain two rectangles $\tilde{\lambda}=\tilde{\mu}$ which contain (4^3). If $\beta^{\mathrm{rot}}=(8,7)$ and $\alpha=(5^3)$, we obtain $\tilde{\lambda}=(13^3,1)$ and $\tilde{\mu}=(8^5)$. Here we remove five columns of length 3 from $\tilde{\lambda}$ and three columns of length 5 from $\tilde{\mu}$ to obtain the seed $((8^3,1),(5^5))$. If $\beta^{\mathrm{rot}}=(8,7)$ and $\alpha=(3^5)$, we obtain $\tilde{\lambda}=(11^5,1)$ and $\tilde{\mu}=(8^7)$. We remove seven columns of length 5 from $\tilde{\lambda}$ and five columns of length 7 from $\tilde{\mu}$ to obtain the seed $((3^7),(4^5,1))$.

$((4^4),(8^2))$: In both cases we remove all the common rows and columns. If $\beta=(2^8)$, we directly obtain the seed $((6^4),(2^{12}))$. If $\beta=(8^2)$, $\tilde{\lambda}=(12^4)$ and $\tilde{\mu}=(8^6)$. We remove two rows of length 2 from $\tilde{\lambda}$ and four columns of length 6 from $\tilde{\mu}$ to obtain the seed $((12^2),(4^6))$.

$((6^3),(9^2))$: In all cases we remove all the common rows and columns. If $\beta=(9^2)$, we remove α and β to obtain two rectangles $\tilde{\lambda}=\tilde{\mu}$ which contain (4^3). The result follows from Lemma 9.1. If $\beta=(2^9)$, we remove all the common rows and columns and obtain the seed $((8^3),(2^{12}))$ if $\alpha=(6^3)$, and $((5^6),(2^{15}))$ if $\alpha=(3^6)$.

$((3^3),(3^3))$: After removing all common rows and columns we obtain the seed $((6^3),(3^6))$. □

In the next step we look at the case $\beta\equiv(m-2,2)$ or $\beta\equiv(m-3,3)$.

Lemma 13.7. *If $m=ab$ for $a,b\geq 3$, $\alpha=(a^b)$ is a proper rectangle and β is equivalent to $(m-2,2)$ or $(m-3,3)$, $g_2(\lambda,\mu)\geq 3$.*

Proof: In all cases we start with removing all the common rows and columns. If $\beta = (m-2,2)$ or $\beta = (m-3,3)$, α and β are removable. We remove them to obtain two rectangles $\tilde{\lambda} = \tilde{\mu}$ which contain (4^3). The result follows from Lemma 9.1.

If $\beta^{\text{rot}} = (m-2,2)$ or $\beta^{\text{rot}} = (m-3,3)$, we remove all but one column of α, all but the b leftmost columns from β and the corresponding columns from $\tilde{\lambda}$. After removing a common row $\tilde{\lambda} = ((b+1)^b) = \tilde{\mu}'$. The result follows from Lemma 9.1.

If $\beta = (2^2, 1^{m-4})$ or $\beta' = (3^2, 1^{m-6})$, $\tilde{\lambda} = ((a+2)^b)$ and $\tilde{\mu} = (2^{b+2}, 1^{m-4})$ or $\tilde{\mu} = (2^{b+3}, 1^{m-6})$. In both cases $\tilde{\mu}$ is a two-column partition and the result follows from Lemma 7.19.

If $\beta^{\text{rot}} = (2^2, 1^{m-4})$ or $\beta^{\text{rot}} = (2^3, 1^{m-6})$, $\tilde{\lambda} = ((a+2)^b, 1^{m-4})$ and $\tilde{\mu} = (2^{b+m-2})$ or $\tilde{\lambda} = ((a+2)^b, 1^{m-6})$ and $\tilde{\mu} = (2^{b+m-6})$. In both cases $\tilde{\mu}$ is a two-column rectangle and the result follows from Lemma 7.18. □

The last thing we have to look at in this chapter are the products of a hook and a rectangle.

Lemma 13.8. *If α is a proper rectangle and β is equivalent to a hook with $l(\beta) \leq 4$ or $w(\beta) \leq 4$, $g_2(\lambda, \mu) \geq 3$.*

Proof: In all cases we remove all common rows and columns. First we assume that β is a hook (because of Lemma 11.1 we can assume that $l(\beta), w(\beta) \geq 3$). If $w(\beta) = l(\alpha) = 3$, we remove all but one column from α and all rows of length 1 from β and obtain the seed $((4^3), (3^4))$. Now we can assume that $w(\beta)$ or $l(\alpha)$ is greater than 3. We remove α and β to obtain two rectangles $\tilde{\lambda} = \tilde{\mu}$ which contain $(4^3)^{(')}$ and the result follows from Lemma 9.1.

From now on we assume that β^{rot} is a hook with $l(\beta) \leq 4$ or $w(\beta) \leq 4$. We start with the case $l(\beta) \leq 4$ so we know $w(\beta) > b$. We remove all but one column from α and all but the b leftmost columns from β together with the corresponding columns from $\tilde{\lambda}$. After removing the common rows we obtain $\tilde{\lambda} = ((b+1)^b) = \tilde{\mu}'$. The result follows from Lemma 9.1. If $w(\beta) \leq 4$, we remove the topmost $b - w(\beta) + 1$ rows and $(b - w(\beta) + 1)a$ rows where β is of length 1 from both $\tilde{\lambda}$ and $\tilde{\mu}$. In the next step we remove the column which contains more than just one box of $\beta$and the corresponding column of $\tilde{\lambda}$ together with all but one column of α. Now we remove some of the common rows to obtain the seed $((3^2, 2^4), (2^7))$ if $w(\beta) = 3$, and $((4^3, 3^2), (3^6))$ if $w(\beta) = 4$. □

CHAPTER 14

α and β are hooks or two-line partitions

The only products from Theorem 5.1 which are missing are the ones where α and β are equivalent to two-line partitions or hooks. We start with the case that $m = 2k$ is even and $\alpha = (k, k)^{(')}$.

1. $\alpha = (k, k)^{(')}$

Let $\alpha = (k, k)$. Since we assume that our classification is correct for m, we know that $[\alpha][\beta]$ only contains constituents with multiplicity 1 and 2 if and only if β is (up to equivalence) from the following list:

(1) $\beta = (m - a, a)$ with $a \leq 7$;
(2) $\beta = (k + a, k - a)$ with $a \leq 3$;
(3) $\beta = (m - 3, 2, 1)$ (we have seen that in Lemma 13.1);
(4) β is a hook ;
(5) one of the exceptional cases (we have already seen them in the previous sections).

Lemma 14.1. *If $m = 2k$, $\alpha^{(')} = (k, k)$ and $\beta^{(')} = (\beta_1, \beta_2) \vdash m$ is a two-line partition (with $\beta_2 > 1$), $g_2(\lambda, \mu) \geq 3$.*

Proof: By conjugation of λ and μ we can assume that α is above β. We start with some exceptional cases. First the three cases $\alpha = \beta = (2^2)$, (3^2), (4^2). If we removed all the common rows and columns, we would obtain $(((2k)^2), (k^4))$. For $k = 2$, 3, 4 this is not a seed. But we assume that λ is not a two-row partition, so we know there is a third row. This might be located above α or between α and β or to the left of β. If there is a row R_1 is above α and we remove all the common rows and columns besides R_1, we obtain the seed $(((2k)^3), (2k, k^4))$ for $k = 2$, 3, 4. If there is a row to the left of β, we know that there also is a column C_1 to the left of β. If we remove all the common rows and columns besides C_1, we obtain the seed $(((2k+1)^2, 1^2), ((k+1)^4)$. In the last step we assume that there is a common row R_2 between α and β. We can assume that there is neither a row above α nor a column left of β. If we remove all the common rows and columns besides R_2, we obtain $(((2k)^2, k), (k^5))$. For $k = 3, 4$ this is a seed. If $k = 2$, we would have removed too much. But for $k = 2$ we know that μ is not a two-column partition and since we assume that there is no column to the left of β and no row above α, the only possibility is that there is a column C_2 between α and β. If we remove all common rows and columns besides R_2 and C_2, we obtain the seed $((5^2, 2), (3^2, 2^3))$ if R_2 and C_2 have no common box, and $((5^2, 3), (3^3, 2^2))$ if they do.

Now let us look at the case $\alpha = (2^3)$ and $\beta = (2^2, 1^2)$. If we removed all the common rows and columns, we would obtain $((4^3), (2^5, 1^2))$. But we know μ is not a two-column partition. Therefore, there is a column C_1 to the left of β or C_2 between α and β or above (and maybe to the right) of α, then there is a row R above α. We obtain the seed $((5^3, 1^4), (3^5, 2^2))$ if we remove all the common rows and columns but C_1, $((5^3), (3^3, 2^2, 1^2))$ if we remove all the common rows and columns but C_2, and $((4^4), (4, 2^5, 1^2))$ if we remove all the common rows and columns but R.

Let $\alpha = (k,k)$ and $\beta = (2^a, 1^b)$ be a two-column partition. If we removed all the common rows and columns, $\tilde{\lambda}$ would be $((k+2)^2)$ and $\tilde{\mu}$ would be a two-column partition. In a lot of cases we would have removed too much. But we know that λ is not a two-row partition, so we know that there is another row. If that row R_1 is between α and β, we remove all the common rows and columns besides R_1 and then successively remove (1^2) from $\tilde{\lambda}$ and (2) or (1^2) from $\tilde{\mu}$ as rows until we obtain the seed $((5^2,2),(2^6))$ if $b=0$ and $((5^2,2),(2^5,1^2))$ if $b>0$. If there is a row R_2 above α, we remove all the common rows and columns besides R_2. We obtain $\tilde{\lambda} = ((k+2)^3)$ and $\tilde{\mu} = (k+2, 2^{a+2}, 1^b)$. Now we successively remove first (1^3) from $\tilde{\lambda}$ and (3) from $\tilde{\mu}$ and then (2^3) from $\tilde{\lambda}$ and (2^3) or $(2^2,1^2)$ or $(2,1^4)$ or (1^6) from $\tilde{\mu}$ as rows until $w(\tilde{\lambda}) = 4,5,6$. If $w(\tilde{\lambda}) = 4$ (and we removed the correct parts from $\tilde{\mu}$), we obtain the seed $((4^3),(4,2^4))$. If $w(\tilde{\lambda}) = 5$, we obtain the seed $((5^3),(5,2^5))$ if $a \geq 3$, and $((5^3),(5,2^4,1^2))$ if $b \geq 2$. If $w(\tilde{\lambda}) = 6$, we obtain the seed $((6^3),(6,2^6))$ if $a \geq 4$, $((6^3),(6,2^5,1^2))$ if $a \geq 3$ and $b \geq 2$, and $((6^3),(6,2^4,1^4))$ if $b \geq 4$. If the common column C is to the left of β, we remove all the common rows and columns besides C. If $b > 0$, we remove (1^2) from $\tilde{\lambda}$ and (2) from $\tilde{\mu}$ as row. In the next step we remove (1^2) from $\tilde{\lambda}$ as rows and (2) from $\tilde{\mu}$ as row. We do this $\frac{b}{2}$ times. Now $\tilde{\alpha} = \tilde{\beta}' = (a,a)$. If $a > 2$ and we conjugate $\tilde{\lambda}$ and $\tilde{\mu}$, this is the previous case. If $a = 2$ we have the seed $((5^2,1^2),(3^4))$.

We have checked the exceptional cases. For all the other cases we start with removing all the common rows and columns. If $\beta = \alpha = (k,k)$, we know because of the previous cases that $k \geq 5$, so we can reduce this to the seed $((10^2),(5^4))$. If β is a two-row partition with $\beta_1 > \beta_2$, we know that $\beta_2, \beta_1 - \beta_2 \geq 2$ since m is even and $\beta \neq (m-1,1)$. This can be reduced to the seed $((7^2),(4^3,2))$.

If $\alpha = (2^k)$ and β is a two-row partition, we can assume that $k > 2$. For the case $\alpha = (2^3)$ and $\beta = (3^2)$ we check the seed $((5^3),(3^5))$. If $\beta \neq (3^2)$, we know that $w(\beta) > 3$. When we remove α and β, we obtain two rectangles $\tilde{\lambda} = \tilde{\mu}$ which contain (4^3). The result follows from Lemma 9.1.

If $\alpha = (2^k)$ and $\beta = (2^a,1^b)$ is a two-column partition, by conjugation we can assume that $\beta \neq \alpha$. Since we already checked $\beta = (2^2,1^2)$ with the exceptional cases, we know that $k > 3$ and can reduce these to the seed $((4^4),(2^6,1^4))$ if $a = 2$, and $((4^4),(2^7,1^2))$ if $a > 2$. □

Lemma 14.2. *If $m = 2k$ is even, $\alpha^{(')} = (k^2)$ and $\beta^{\text{rot}} \vdash m$ is a two-line partition, $g_2(\lambda,\mu) \geq 3$.*

Proof: We can assume that α is above β. Further, we can assume that β is not a two-line rectangle nor $\beta \equiv (m-1,1)$. In all cases we remove all the common rows and columns. We start with $\alpha = (k^2)$ and $\beta^{\text{rot}} = (a,b)$ is a two-row partition. We remove (2^2) from $\tilde{\lambda}$ and (1^4) from $\tilde{\mu}$ $b-2$ times and $(3^2,2)$ from $\tilde{\lambda}$ and (2^4) from $\tilde{\mu}$ $\frac{a-b}{2} - 1$ times. We obtain the seed $((7^2,2),(4^4))$.

If $\beta^{\text{rot}} = (2^a,1^b)$, $\tilde{\mu}$ is in both cases a two-column rectangle and the result follows from Lemma 7.18.

Let $\alpha = (2^k)$ and β^{rot} be a two-row partition. If $k = 3$, after removing all the common columns and rows we obtain the seed $((6^3,2),(4^5))$. If $k \geq 4$, we can always remove from $\tilde{\mu}$ columns which contain k boxes of β and the corresponding columns of $\tilde{\lambda}$ together with one column of α such that the remaining partition $\tilde{\beta}$ still has width at least 3. The new $\tilde{\alpha} = (1^k)$ and $\tilde{\beta}$ is a one-row or a rotated two-row partition. The result follows from Lemma 10.2. □

Lemma 14.3. *If $m = 2k$, $\alpha^{(')} = (k,k)$ and $\beta \vdash m$ is equivalent to a hook, $g_2(\lambda,\mu) \geq 3$.*

Proof: Again, we assume that α is above β and that $\beta \not\equiv (m-1,1)$. Since the length and width of β is greater or equal to 3, we know that $m \geq 6$. In all cases we remove all the common rows and columns to obtain $\tilde{\lambda}$ and $\tilde{\mu}$. We start with the case $\alpha = (2^k)$ and β is a hook. If $k = 3$, we have the seed $((5^3), (3^4, 1^3))$ if $w(\beta) = 3$, and $((6^3), (4^4, 1^2))$ if $w(\beta) = 4$. If $k > 3$, we remove α and β to obtain two rectangles $\tilde{\lambda} = \tilde{\mu} = ((w(\beta))^k)$ which contain (3^4). The result follows from Lemma 9.1.

If $\alpha = (k^2)$ and β is a hook, $\tilde{\lambda}$ is a two-row rectangle and the result follows from Lemma 7.18.

Let $\alpha = (2^k)$ and β^{rot} be a hook. We know that $w(\beta) \geq 3$. We remove $(w(\beta)+2, (w(\beta)-1)^2)$ from $\tilde{\lambda}$ as rows and $((w(\beta))^3)$ from $\tilde{\mu}$ as rows until one of the following happens:

- If $w(\beta) = 3, 4$, we obtain the seed $((5^2, 2^3), (3^7))$ or $((6^3, 3^2), (4^6))$;
- if $w(\beta) > 4$ is even, we do this until $l(\tilde{\beta}) = 1$. Since $w(\tilde{\beta}) \geq 6$ we know that $l(\tilde{\alpha}) \geq 3$, so the result follows from Lemma 10.2;
- if $w(\beta) \geq 5$ is odd, we do this until $l(\tilde{\beta}) = 2$. Now the result follows from Lemma 11.1.

In the last case we look at $\alpha = (k^2)$ and β^{rot} is a hook. If $w(\beta) = 3$, we know that $l(\beta)$ is even, so we remove a column of length $l(\beta)+2$ (which equals m) from $\tilde{\mu}$ and α from $\tilde{\lambda}$. Now $\tilde{\mu} = (2^m)$ and $\tilde{\lambda} = (3^2, 2^{m-3})$. The result follows from Lemma 7.18. If $w(\beta) \geq 5$ is odd, we remove a column of length $l(\beta)+2$ from $\tilde{\mu}$ and $\frac{l(\beta)+2}{2}$ rows of length 2 from $\tilde{\lambda}$. In the next step we remove a row of length $w(\beta)-1$ from $\tilde{\mu}$ and $\frac{w(\beta)-1}{2}$ rows of length 2 from $\tilde{\lambda}$. Now $\tilde{\lambda} = \tilde{\mu} = ((w(\beta)-1)^{l(\beta)+1})$. The result follows from Lemma 9.1. If $w(\beta) \geq 4$ is even, we do it the other way around, we start with removing a row of length $w(\beta)$ from $\tilde{\mu}$ and $\frac{w(\beta)}{2}$ columns of length 2 from $\tilde{\lambda}$. In the next step we remove a column of length $l(\beta)+1$ from $\tilde{\mu}$ and $\frac{l(\beta)+1}{2}$ columns of length 2 from $\tilde{\lambda}$. Again, $\tilde{\lambda} = \tilde{\mu} = ((w(\beta)-1)^{l(\beta)+1})$ and the result follows from Lemma 9.1. □

2. α and β are two-line partitions

Now we assume that α and β are two-line partitions. Here we can exclude the case that one of them is $(k,k)^{(')}$ because of lemma Lemma 14.1. We solve the remaining cases in the following lemma. Since we assume that $\alpha \not\equiv (m-1,1) \not\equiv \beta$, we know that $m \geq 5$.

Lemma 14.4. *If α and β are equivalent to two-line partitions, $g_2(\lambda, \mu) \geq 3$.*

Proof: Let α be above β. We have four possibilities for α and β, so in total sixteen cases where some are equivalent by conjugation. We order them as follows:

(1) α and β are two-row partitions. By conjugation this equivalent to α and β being two-column partitions;
(2) α is a two-column partition and β is a two-row partition;
(3) α is a two-row and β a two-column partition;
(4) α^{rot} is a two-row partition and β is a two-row partition. By conjugation this is equivalent to α being a two-column partition and β^{rot} being a two-column partition;
(5) α^{rot} is a two-row partition and β is a two-column partition. By conjugation this is equivalent to α being a two-row partition and β^{rot} being a two-column partition;
(6) α is a two-row partition and β^{rot} is a two-row partition. By conjugation this is equivalent to α^{rot} being a two-column partition and β being a two-column partition;

(7) α is a two-column partition and β^{rot} is a two-row partition. By conjugation this is equivalent to α^{rot} being a two-column partition and β being a two-row partition;

(8) α^{rot} and β^{rot} are two-row partitions. By conjugation this is equivalent to α^{rot} and β^{rot} being two-column partitions;

(9) α^{rot} is a two-row partition and β^{rot} a two-column partition;

(10) α^{rot} is a two-column partition and β^{rot} a two-row partition.

We solve these cases one by one.

(1): We remove all common rows and columns to obtain $\tilde{\lambda} = (\beta_1 + \alpha_1, \beta_1 + \alpha_2)$ and $\tilde{\mu} = (\beta_1^3, \beta_2)$. Now the result follows from Lemma 7.18 or Lemma 7.19.

(2): We remove again all common rows and columns. If $\alpha = (2^2, 1)$ and $\beta = (3, 2)$, we have the seed $((5^2, 4), (3^4, 2))$. If $\alpha \neq (2^2, 1)$, we know not only that $l(\alpha), w(\beta) > 3$ but also that α and β are removable. We remove α and β to obtain two rectangles $\tilde{\lambda} = \tilde{\mu} \supseteq (4^4)$. The result follows from Lemma 9.1, again.

(3): This is a bit more complicated. We have some exceptional cases, these are: $\alpha = \beta' = (3, 2)$, $\alpha = \beta' = (4, 2)$, $m = 2k + 1$ is odd and (maybe after conjugation) $\alpha = (k + 1, k)$ and $\beta = (2^k, 1)$, $\beta = (2^{k-1}, 1^3)$ or $\beta = (2^2, 1^{m-4})$. In all of the exceptional cases we proceed in the same way. We know that λ is not a two-row partition. Therefore, there is another row or column in λ which we do not remove. Let us start with the case $\alpha = \beta' = (3, 2), (4, 2)$. If there is a common row R_1 above α and we remove all common rows and columns besides R_1, we obtain the seed $((5^2, 4), (5, 2^4, 1))$ or $((6^2, 4), (6, 2^4, 1^2))$. If there is a row R_2 between α and β and we remove all the common rows and columns besides R_2, we obtain the seed $((5, 4, 2), (2^5, 1))$ or $((6, 4, 2), (2^5, 1^2))$. If there is no common row above β, there is one to the left of β, but this means there is a column C to the left of β. By conjugation this is equivalent to the case that there is a row R_1 above α. In the next step we look at the case $m = 2k + 1$, $\alpha = (k + 1, k)$ and $\beta = (2^k, 1)$, $\beta = (2^{k-1}, 1^3)$ or $\beta = (2^2, 1^{m-4})$. We have the same options for an extra row or column as in the case before. If there is a common row R_2 between α and β, we remove all the common rows and columns except for R_2. We remove (1^2) from $\tilde{\lambda}$. From $\tilde{\mu}$ we remove (2) as row if $\beta \neq (2^2, 1^{m-4})$, and (1^2) as rows if $\beta = (2^2, 1^{m-4})$. We do this until we obtain the seed $((5, 4, 2), (2^5, 1))$ if $\beta = (2^k, 1)$ or $\beta = (2^2, 1^{m-4})$, and $((4, 3, 2), (2^3, 1^3))$ if $\beta = (2^{k-1}, 1^3)$. If there is a common row R_1 above α, we remove all the common rows and columns but R_1. Then we remove (2^3) from $\tilde{\lambda}$ and (2^3) from $\tilde{\mu}$ as rows if $\beta \neq (2^2, 1^{m-4})$ resp. (1^6) as rows if $\beta = (2^2, 1^{m-4})$. In the next step we remove (1^3) from $\tilde{\lambda}$ and (3) from $\tilde{\mu}$. We repeat these two steps ($\lfloor \frac{k-1}{3} \rfloor$ times) until $w(\tilde{\lambda}) \leq 6$ if $\beta \neq (2^2, 1^{m-4})$ resp. $w(\tilde{\lambda}) \leq 7$ if $\beta = (2^2, 1^{m-4})$ to obtain one of the following seeds: If $\beta = (2^k, 1)$, we obtain $((4^2, 3), (4, 2^3, 1), ((5^2, 4), (5, 2^4, 1))$ or $((6^2, 5), (6, 2^5, 1)$. If $\beta = (2^{k-1}, 1^3)$, we obtain $((4^2, 3), (4, 2^2, 1^3))$, $((5^2, 4), (5, 2^3, 1^3))$ or $((6^2, 5), (6, 2^4, 1^3)$. If $\beta = (2^2, 1^{m-4})$, we obtain $((5^2, 4), (5, 2^4, 1))$, $((6^2, 5), (6, 2^4, 1^3))$ or $((7^2, 6), (7, 2^4, 1^5))$. If there is a common column to the left of β and $\beta' = \alpha$, this is equivalent to the previous case by conjugation. The case $\beta = (2^{k-1}, 1^3)$ can be solved with exactly the same procedure as in the previous case (after conjugation). We obtain the three seeds (which are again stated with $\tilde{\alpha}$ above $\tilde{\beta}$) $((5, 4, 1^3), (3^2, 2^3))$, $((6, 5, 1^4), (3^3, 2^3))$ and $((7, 6, 1^5), (3^4, 2^3))$. For the case $\beta = (2^2, 1^{m-4})$ we can reduce k by 1 with a similar idea as before to obtain the seed $((6, 5, 1^3), (3^4, 2))$. In the generic case we remove all common rows and columns and obtain a product of two two-line partitions which contains two constituent with multiplicity 3 or higher, see Lemma 7.16.

(4): We remove all common rows and columns. Now $\tilde{\lambda}$ is a two-row rectangle and the result follows from Lemma 7.18.

(5): If $m > 5$, we remove all common rows and columns. Now $\tilde{\lambda}$ is a two-row rectangle and the result follows from Lemma 7.18. If $m = 5$, we know that $\alpha \equiv \beta \equiv (3,2)$. If we removed all common rows and columns, we would just have one constituent with multiplicity 3. But in this case we know that λ is not a two-row partition. Therefore, there is a common row R_1 above α or R_2 between α and β or a common column C to the left of β. If we remove all the common columns and rows but R_1, we obtain the seed $((5^3),(5,3,2^3,1))$. If we remove all the common rows and columns except for R_2, we obtain the seed $((5^2,2),(3,2^4,1))$. If we remove all all the common rows and columns except for C, we obtain the seed $((6^2,1^3),(4,3^3,2))$.

(6): We remove all common rows and columns. Then we remove all columns which belong to α besides two of length 2 and one of length 1 together with all the columns of $\tilde{\mu}$ which belong to β besides two of length 2 and one of length 1 and the corresponding columns of $\tilde{\lambda}$ to obtain the seed $((6,5,1),(3^4))$.

(7): We remove all the common rows and columns. If $m = 5$, 6, we directly obtain the seed $((5^2,4,1),(3^5))$ or $((6^2,5^2,2),(4^6))$. If $m = 7$ and $\alpha' = (4,3) = \beta^{\mathrm{rot}}$, we obtain the seed $((6^3,5,1),(4^6))$. In all other cases we remove the rightmost column of α and the minimal number of columns of β with the same number of boxes and the corresponding columns from $\tilde{\lambda}$. Now $\tilde{\alpha} = (1^{l(\alpha)})$ and $w(\tilde{\beta}) \geq 3$. The result follows from Lemma 10.2.

(8): This can easily be reduced to the seed $((6^2,1),(4,3^3))$ after removing all the common rows and columns.

(9): We remove all the common rows and columns to obtain $\tilde{\lambda} = (a^2,1^b)$ and $\tilde{\mu} = (c,2^d)$ for some $a \geq 5$, $b \geq 1$, $c,d \geq 3$. If $c = b+2$, the result follows from Lemma 9.1 by conjugating $\tilde{\mu}$. So by conjugating and exchanging $\tilde{\lambda}$ and $\tilde{\mu}$ we can assume that $c < b+2$. After removing $(c-2)$ from $\tilde{\mu}$ and (1^{c-2}) from $\tilde{\lambda}$, $\tilde{\mu}$ is a two-column rectangle and the result follows from Lemma 7.18.

(10): We remove all the common rows and columns. If $m = 5$, we obtain the seed $((5^3,1),(4,3^4))$. If $m = 6$, we obtain the seed $((6^4,2),(5^2,4^4))$. If $m = 7$ and $(\alpha^{\mathrm{rot}})' = \beta^{\mathrm{rot}} = (4,3)$, we obtain the seed $((6^4,1),(5,4^5)$. In all other cases we proceed in the following way: We remove the column of length b from $\tilde{\mu}$ together with the corresponding column of length $a+b$ from $\tilde{\lambda}$ and the minimal number of columns from $\tilde{\lambda}$ and $\tilde{\mu}$ such that the β part of these columns has a boxes. We obtain $\tilde{\alpha}$ which is a one-column partition and $\tilde{\beta}$ with $w(\tilde{\beta}) \geq 3$. The result follows from Lemma 10.2. □

3. α and β are hooks

Lemma 14.5. *If α and β are equivalent to hooks, $g_2(\lambda,\mu) \geq 3$.*

Proof: We assume that neither α nor β is linear or equivalent to $(m-1,1)$. We remove all common columns and rows. We split the proof into three parts.

1st case: α and β are hooks. We can assume that α is above β. If $l(\alpha) > 3$ or $w(\beta) > 3$, we remove α and β to obtain $\tilde{\lambda} = \tilde{\mu} \supseteq (4^3)^{(')}$ two rectangles and the result follows from Lemma 9.1. If $l(\alpha) = w(\beta) = 3$, we remove $(m-3)$ from $\tilde{\lambda}$ and (1^{m-3}) from $\tilde{\mu}$ to obtain the seed $((4^3),(3^4))$.

2nd case: α or β is a hook and the other one is a rotated hook. Without loss of generality we can assume that α is above β and that α^{rot} and β are hooks. We remove rows of length $w(\alpha)+w(\beta)$ from $\tilde{\lambda}$ and $(w(\alpha)+w(\beta)-1,1)$ from $\tilde{\mu}$ as rows until $l(\tilde{\alpha}) = 2$ or $l(\tilde{\beta}) = 2$. The result follows from Lemma 7.18 and 11.2 (we know that $|\tilde{\alpha}| = \tilde{m} \geq 4$ and $w(\tilde{\beta}) \geq 3$, this excludes the exceptional cases from these lemmas which are relevant for this).

3rd case: α^{rot} and β^{rot} are hooks. Here we remove $(w(\alpha)+w(\beta), w(\beta)-1)$ from $\tilde{\lambda}$ as rows and $(w(\alpha)+w(\beta)-1, w(\beta))$ from $\tilde{\mu}$ as rows until $l(\tilde{\alpha}) = 2$ or $l(\tilde{\beta}) = 2$. The result follows from Lemma 11.1 and 11.2. □

4. α is a hook and β is a two-line partition

Lemma 14.6. *If α is equivalent to a hook and β is equivalent to a two-line partition, $g_2(\lambda, \mu) \geq 3$.*

Proof: Without loss of generality we can assume that α is above β, that $\alpha, \beta \not\equiv (m-1, 1)$ and that β is not a two-line rectangle. In all cases we start with removing all the common rows and columns to obtain $\tilde{\lambda}$ and $\tilde{\mu}$.

1st case: α is a hook and β is a two-row partition. We directly obtain the seed $((6, 4^2), (3^4, 2))$ if $m = 5$. If $m > 5$, we know that $w(\beta) > 3$ since $\beta \neq (3, 3)$. We remove α and β to obtain two rectangles which contain (4^3). The result follows from Lemma 9.1.

2nd case: α is a hook and $\beta^{(\mathrm{rot})}$ is a two-column partition. Now $\tilde{\mu}$ is a two-column partition and $\tilde{\lambda}$ is a proper fat hook or has three removable nodes. The result follows from Lemma 7.18 or 7.19.

3rd case: α^{rot} is a hook and β is a two-row partition. If $l(\alpha) = 3, 4$, this can easily be reduced to the seed $((6^3), (5^2, 3^2, 2))$ if $l(\alpha) = 3$, and $((5^4), (4^3, 3^2, 2))$ if $l(\alpha) = 4$. If $l(\alpha) > 4$, we remove all the columns where α has a column of length 1 and a fitting number of columns where β has columns of length 1 or 2 from $\tilde{\lambda}$ and $\tilde{\mu}$. This works since the two rows of β are of different length. Now $\tilde{\alpha}$ and $\tilde{\beta}$ are removable. We remove them to obtain two rectangles $\tilde{\lambda} = \tilde{\mu}$ which contain (3^5). The result follows from Lemma 9.1.

4th case: α^{rot} is a hook and $\beta^{(\mathrm{rot})}$ is a two-column partition. We remove all but one of the rows of length 1 from α, the corresponding rows of $\tilde{\mu}$ and the right number of rows from β in such a way that $w(\tilde{\beta}) = 2$ with the corresponding columns of $\tilde{\lambda}$ if β is a skew partition. Now $\tilde{\alpha}^{\mathrm{rot}} = (\tilde{m}-1, 1)$. If $\tilde{\beta}$ is a two-column partition, $\tilde{\lambda}$ is a two-row rectangle and the result follows from Lemma 7.18. Otherwise, the result follows from Lemma 11.1. Since $\tilde{\alpha}^{\mathrm{rot}} \neq (2, 1)$, we do not get one of the exceptional cases from the lemma.

5th case: $\alpha^{(\mathrm{rot})}$ is a hook and $\beta^{\mathrm{rot}} = (a, b)$ is a two-row partition. If $l(\alpha) = 3$, this can be reduced to the seed $((6, 4^2, 1), (3^5))$ if α is a hook, and $((6^3, 1), (5^2, 3^3))$ if α^{rot} is a hook. If $l(\alpha) = 4$, it can be reduced to the seed $((5, 4^3, 1), (3^6))$ if α is a hook, and $((5^4, 1), (4^3, 3^3))$ if α^{rot} is a hook. If $l(\alpha) \geq 5$, we remove all the columns of length 1 from α, the corresponding columns if α is a rotated hook, together with the minimal number of columns from β with the same number of boxes and the corresponding columns of $\tilde{\alpha}$. Now $\tilde{\alpha} = (1^{l(\alpha)})$ and $w(\tilde{\beta}) \geq 3$. The result follows from Lemma 10.2. □

This concludes the proof of Theorem 5.1 and Theorem 5.2. By Corollary 8.13 we also proved Theorem 5.3 and Theorem 5.4. Equipped with these results we now look at the multiplicity-free Kronecker products of characters of the alternating groups.

Part 3

Multiplicity-free Kronecker products of characters of the alternating groups

CHAPTER 15

Preliminaries: Representation theory of the alternating groups

1. Irreducible A_n characters and conjugacy classes

The background for the representation theory of the alternating groups can be found in [JK81].

Irreducible A_n characters.
We denote the alternating group on n letters by A_n. Let $\lambda \vdash n$. By $[\lambda] \downarrow_{A_n}$ we denote the restriction of $[\lambda]$ to A_n. As before, over the complex numbers.

- If λ is not symmetric, $[\lambda] \downarrow_{A_n} = [\lambda'] \downarrow_{A_n}$ is irreducible. We write $\{\lambda\}$ for this A_n character.
- If λ is symmetric, the character $[\lambda] \downarrow_{A_n}$ decomposes as the sum of two distinct, irreducible A_n characters which we call $\{\lambda\}_+$ and $\{\lambda\}_-$, i.e., $[\lambda] \downarrow_{A_n} = \{\lambda\}_+ + \{\lambda\}_-$.

This is a complete list of the irreducible A_n characters and the listed characters are pairwise distinct.

The characters $\{\lambda\}_+$ and $\{\lambda\}_-$ are conjugated. Therefore, they often behave similarly so we write $\{\lambda\}_\pm$ for $\{\lambda\}_+$ or $\{\lambda\}_-$. If we have a $\lambda \vdash n$, a priori we do not know if λ is symmetric or not so we write $\{\lambda\}_{(\pm)}$ for $\{\lambda\}$ if λ is not symmetric, and $\{\lambda\}_\pm$ if λ is symmetric.

A_n conjugacy classes.
Let $\lambda \vdash n$. The elements of cycle type λ form a conjugacy class of S_n. Let $\operatorname{sgn}(\lambda) = \prod_{i=1}^{l(\lambda)} (-1)^{\lambda_i - 1} = 1$. The elements of cycle type λ are also elements of A_n.

- If λ has a nonzero part with multiplicity 2 or higher or an even nonzero part, also in A_n all elements of cycle type λ are conjugated. We denote that conjugacy class by C_λ and an arbitrary element from that class by σ_λ.
- If all the nonzero parts are pairwise distinct and odd, the elements of cycle type λ form two conjugacy classes in A_n, we denote them by C_λ^+ and C_λ^-. Further, we denote by C_λ the union of these two classes. An arbitrary element from the class C_λ^+ we denote by σ_λ^+ resp. an element from C_λ^- by σ_λ^-.

There is a bijection from the set of symmetric partitions of n to the set of partitions of n with only pairwise different odd parts. For $\lambda \vdash n$ we denote by $h(\lambda)$ the partition with $d(\lambda)$ parts and the ith part has $h_{(i,i)}$ boxes, i.e., $h(\lambda)$ is the partition of n which parts are the length of the hooks on the main diagonal of λ. If λ is symmetric, $h(\lambda)$ consists only of odd and pairwise different parts. This yields a bijection from the set of symmetric partitions of n to the set of partitions of n with only pairwise different odd parts.

This bijection transfers a connection between a symmetric character and a splitting conjugation class.

Character values.

- If $\lambda \vdash n$ is not symmetric, $\{\lambda\}(\sigma) = [\lambda](\sigma)$ for all $\sigma \in A_n$.
- If $\lambda \vdash n$ is symmetric and $\sigma \notin C_{h(\lambda)}$,

$$\{\lambda\}_+(\sigma) = \{\lambda\}_-(\sigma) = \frac{1}{2}[\lambda](\sigma).$$

- If $\lambda \vdash n$ is symmetric and $h(\lambda) = (h_1, \dots h_d)$, then

$$[\lambda](\sigma_{h(\lambda)}) = (-1)^{(n-d)/2} =: e_\lambda.$$

  Further,

$$\{\lambda\}_\pm(\sigma^+_{h(\lambda)}) = \frac{1}{2}\left(e_\lambda \pm \sqrt{e_\lambda \prod_{i=1}^{d} h_i}\right),$$

$$\{\lambda\}_\pm(\sigma^-_{h(\lambda)}) = \frac{1}{2}\left(e_\lambda \mp \sqrt{e_\lambda \prod_{i=1}^{d} h_i}\right).$$

So the classes $C^+_{h(\lambda)}$ and $C^-_{h(\lambda)}$ are the only classes on which the characters $\{\lambda\}_+$ and $\{\lambda\}_-$ differ and $\{\lambda\}_+$ and $\{\lambda\}_-$ are the only irreducible characters which differ on these conjugacy classes. This leads to the idea of critical classes [Bes18].

With that notation we can state the main result.

2. Main Theorem

The goal of this part is to prove the following theorem. For this we use the results we have proven in the previous part.

Theorem 15.1. *Let $\lambda, \mu \vdash n$ be partitions. The product $\{\lambda\}_{(\pm)}\{\mu\}_{(\pm)}$ is multiplicity-free if and only if up to exchanging λ and μ and/or conjugating λ and/or μ one of the following cases occurs:*

1. *One of the characters is the trivial character;*
2. *$\lambda = (n-1,1)$ and μ is symmetric with at most 3 removable nodes or μ is not symmetric, has at most 2 removable nodes and is different from $(k^{k-1}, k-2)^{(\prime)}$, $(k,1^k)^{(\prime)}$, $(3,2)^{(\prime)}$;*
3. *$n = 2k+1 \geq 15$ is odd and $\lambda = \mu = (k+1,k)$;*
4. *$n = 2k$, $\lambda = (k,k)$ and one of the following holds:*
   (a) *$k \geq 7$ or $k \in \{3,5\}$ and $\mu = (k,k)$ or $\mu = (k+1,k-1)$;*
   (b) *$k \neq 4$ and $\mu = (n-2,2)$ or $\mu = (n-3,3)$;*
   (c) *μ is a hook different from $(k,1^k)^{(\prime)}$.*
5. *$n = ab$ for $a,b \geq 3$, $n \neq 12$, $\lambda = (a^b)$ and $\mu = (n-2,2)$ or $\mu = (n-2,1^2)$;*
6. *$n = 2k+1$ is odd and $\lambda = \mu = (k+1,1^k)$*
7. *(λ,μ) is one of the exceptional pairs:*

$$((2,2),(2,2)),\ ((3,1^2),(3,2)),\ ((3,2,1),(3,3)),\ ((4,4),(3^2,2)),$$
$$((5,1^4),(3^3)),\ ((3^3),(3^3)),\ ((3^3),(5,4)),\ ((3^3),(6,3)).$$

To prove this we want to use the results from the previous part. The following lemma makes the connection between S_n products and A_n products more evident.

3. Transition from S_n to A_n products

Lemma 15.2. *Let $n \geq 4$, and let λ, μ, ν be partitions of n.*

(1) *If λ, μ, ν are not symmetric,*
$$\langle \{\lambda\}\{\mu\}, \{\nu\} \rangle_{A_n} = \langle [\lambda][\mu], [\nu] \rangle_{S_n} + \langle [\lambda][\mu], [\nu'] \rangle_{S_n}.$$
(2) *If $\lambda = \lambda'$ is symmetric, while $\mu \neq \mu'$ and $\nu \neq \nu'$ are not,*
$$\langle \{\lambda\}_\pm \{\mu\}, \{\nu\} \rangle_{A_n} = \langle [\lambda][\mu], [\nu] \rangle_{S_n}.$$
(3) *If $\lambda = \lambda'$ and $\nu = \nu'$ are symmetric with $\lambda \neq \nu$, while $\mu \neq \mu'$ is not,*
$$\langle \{\lambda\}_\pm \{\mu\}, \{\nu\}_\pm \rangle_{A_n} = \frac{1}{2} \langle [\lambda][\mu], [\nu] \rangle_{S_n}.$$

Proof: The first part follows by restricting the S_n product $[\lambda][\mu]$ and using the results from Section 1 of this chapter.

Let us look at the second part. Since λ is symmetric (and μ, ν are not), we know that
$$\langle [\lambda][\mu], [\nu] \rangle_{S_n} = \langle [\lambda][\mu], [\nu'] \rangle_{S_n}.$$
So if we restrict this to the alternating group, we get
$$\langle (\{\lambda\}_+ + \{\lambda\}_-)\{\mu\}, \{\nu\} \rangle_{A_n} = 2 \langle [\lambda][\mu], [\nu] \rangle_{S_n}.$$
From [BK99, Lemma 5.3] we know that
$$\langle \{\lambda\}_+ \{\mu\}, \{\nu\} \rangle_{A_n} = \langle \{\lambda\}_- \{\mu\}, \{\nu\} \rangle_{A_n}.$$
This proves part (2). Further, [BK99, Lemma 5.3] tells us that for $\lambda \neq \nu$ both symmetric and μ not symmetric
$$\langle \{\lambda\}_\pm \{\mu\}, \{\nu\}_\pm \rangle_{A_n} = \langle \{\lambda\}_\pm \{\mu\}, \{\nu\} \pm \rangle_{A_n}$$
for all choices of the signs. Therefore, part (3) follows in the same way. ☐

Since $\overline{\{\lambda\}_+}$ stays $\{\lambda\}_+$ (if $e_\lambda = 1$) or becomes $\{\lambda\}_-$ (if $e_\lambda = -1$) the previous lemma leads to the following corollary.

Corollary 15.3. (1) *Let $\lambda, \mu \vdash n \geq 4$ be not symmetric. The product $\{\lambda\}\{\mu\}$ is multiplicity-free if and only if $[\lambda][\mu]$ is multiplicity-free and $g(\lambda, \mu, \nu) + g(\lambda, \mu, \nu') \leq 1$ for all non-symmetric $\nu \vdash n$.*

(2) *Let $\lambda, \mu \vdash n \geq 4$, where $\lambda = \lambda'$ is symmetric and $\mu \neq \mu'$ is not. For $\{\lambda\}_\pm \{\mu\}$ to be multiplicity free it is necessary that $g(\lambda, \mu, \nu) \leq 1$ for all non-symmetric $\nu \vdash n$ and $g(\lambda, \mu, \nu) \leq 2$ for all symmetric $\lambda \neq \nu \vdash n$. In particular, if $g(\lambda, \mu)_2 \geq 3$, $\{\lambda\}_\pm \{\mu\}$ is not multiplicity-free.*

(3) *Let $\lambda, \mu \vdash n \geq 4$, $\lambda \neq \mu$ both be symmetric. If there is a non-symmetric $\nu \vdash n$ such that $g(\lambda, \mu, \nu) \geq 3$, none of the products $\{\lambda\}_\pm \{\mu\}_\pm$ is multiplicity-free.*

With this corollary, the results of [BB17] and Part 2 we will be able prove Theorem 15.1 for the majority of all partitions.

CHAPTER 16

Non-symmetric products

In this chapter we want to investigate which of the multiplicity-free products of non-symmetric characters of the symmetric groups stay multiplicity-free if we restrict them to the alternating group. Since we know the explicit formulas for these products, it is easy to check when a product of the symmetric group contains two conjugated constituents (and therefore, is not multiplicity-free if we restrict it). For these multiplicity-free A_n products the decomposition in irreducible characters can easily be derived from the decomposition of the S_n product. Therefore, it is not stated here.

For the multiplicity-free Kronecker products of characters of the symmetric groups we have the following theorem:

Theorem 16.1. *[BB17, Theorem 1.1.]*

Let λ, μ be partitions of n. The product $[\lambda][\mu]$ is multiplicity-free if and only if the partitions λ, μ satisfy one of the following conditions (up to conjugation of one or both of the partitions):

(1) *One of the partitions is linear, the other one arbitrary;*
(2) *one of the partitions is $(n-1,1)$ the other one is a fat hook;*
(3) *$n = 2k$ and $\lambda = \mu = (k,k)$ or $n = 2k+1$ and $\lambda = \mu = (k+1,k)$;*
(4) *$n = 2k$, one of the partitions is (k,k), the other one is one of $(k+1,k-1)$, $(n-3,3)$ or a hook;*
(5) *one of the partitions is a rectangle, the other one is one of $(n-2,2)$, $(n-2,1^2)$;*
(6) *the partition pair is one of the exceptional pairs $((3^3),(6,3))$, $((3^3),(5,4))$, and $((4^3),(6^2))$.*

We look at all the products from the previous theorem which do not involve symmetric partitions. We do this case by case.

Obviously, products of the form $\{n\}\{\lambda\}$ are still irreducible. From now on we focus on products where none of the factors is the trivial character. Further, there is the exceptional multiplicity-free S_{12} product $[4^3][6^2]$. For this we check with Sage that it contains for example $[6^2]$ and $[2^6]$. Therefore, the restriction to A_{12} is not multiplicity-free.

1. Products with $\{n-1,1\}$

We recall Lemma 5.13: For $\lambda, \nu \vdash n$,

$$g((n-1,1),\lambda,\nu) = \begin{cases} \mathrm{rem}(\lambda)-1, & \text{if } \lambda = \nu; \\ 1, & \text{if } |\lambda \cap \nu| = n-1; \\ 0, & \text{otherwise.} \end{cases}$$

We use this to get the following result for the alternating groups:

Lemma 16.2. *Let $\lambda \vdash n$ be non-symmetric. The product $\{\lambda\}\{n-1,1\}$ is multiplicity-free if and only if λ is a fat hook and up to conjugation of λ none of the following occurs:*

(1) $n = k^2 + 1 \geq 4$ *and* $\lambda = (k^k, 1)$;
(2) $n = k^2 - 2 \geq 4$ *and* $\lambda = (k^{k-1}, k-2)$;
(3) $n = 2k$ *and* $\lambda = (k+1, 1^{k-1})$;
(4) $n = 8$ *and* $\lambda = (4, 2^2)$.

Proof: Let $\lambda \vdash n$ be non-symmetric. If λ is not a fat hook, $[\lambda][n-1,1]$ has a constituent with multiplicity 2 of higher (namely $[\lambda]$) and therefore, by Corollary 15.3 we know that the product $\{\lambda\}\{n-1,1\}$ is not multiplicity-free.

Let λ be from (1)-(3). We know that $|\lambda \cap \lambda'| = n-1$. Hence,

$$g(\lambda, \lambda, (n-1,1)) = g(\lambda, \lambda', (n-1,1)) = 1.$$

By Corollary 15.3 the product $\{\lambda\}\{n-1,1\}$ is not multiplicity-free. Moreover, the exceptional case $\{7,1\}\{4,2^2\}$ has $\{4,3,1\}$ as constituent with multiplicity 2.

From now on let neither λ nor λ' be from (1)-(4). Let $[\nu]$ be a constituent of $[\lambda][n-1,1]$ for some non-symmetric $\nu \vdash n$. We will show that $[\nu']$ is not a constituent of $[\lambda][n-1,1]$. Let us first look at some special cases:

1st case: λ is a hook. If $\lambda = (a, 1^b)$, the product $[n-1,1][a,1^b]$ decomposes as:

$$[a+1, 1^{b-1}] + \chi_{(b>1)}[a, 2, 1^{b-2}] + [a, 1^b] + \chi_{(a>2)}[a-1, 2, 1^{b-1}] + [a-1, 1^{b+1}].$$

Here, $(a+1, 1^{b-1})' = (a-1, 1^{b+1})$ if and only if $a = b+1$ so the hook would be symmetric. The same holds for $(a, 2, 1^{b-2})' = (a-1, 2, 1^{b-1})$. If and only if $a = b$, $(a+1, 1^{b-1})$ and $(a, 1^b)$ are conjugated which is the third case of the lemma (by conjugation). The last case is $(a, 1^b)' = (a-1, 1^{b+1})$ this holds just if $a = b+2$ which is the third case of the lemma.

2nd case: λ rectangle. We assume that $\lambda = (a^b) \vdash n$, where $a > b \geq 2$. There is only one box we can remove and two we can add. So

$$[a^b][n-1,1] = [a+1, a^{b-2}, a-1] + [a^{b-1}, a-1, 1].$$

But $(a+1, a^{b-2}, a-1)' = (a^{b-1}, a-1, 1)$ just if $a = b$ so the square would be symmetric.

3rd case: $\lambda = (a^b, c)$. We know that $[n-1,1][\lambda]$ decomposes as:

$$[a^b, c] + [a^{b-1}, a-1, c, 1] + \chi_{(b>1)}[a+1, a^{b-2}, a-1, c] + \chi_{(a>c+1)}[a^{b-1}, a-1, c+1] + \chi_{(c>1)}[a^b, c-1, 1] + [a+1, a^{b-1}, c-1].$$

The constituents have the following length and width:

ν	length	width
(a^b, c)	$b+1$	a
$(a^{b-1}, a-1, c, 1)$	$b+2$	a
$(a+1, a^{b-2}, a-1, c)$	$b+1$	$a+1$
$(a^{b-1}, a-1, c+1)$	$b+1$	a
$(a^b, c-1, 1)$	$b+2$	a
$(a+1, a^{b-1}, c-1)$	$b+\chi_{(c>1)}$	$a+1$

where $(a+1, a^{b-2}, a-1, c)$ only appears if $b > 1$, $(a^{b-1}, a-1, c+1)$ only appears if $a > c+1$ and $(a^b, c-1, 1)$ only appears if $c > 1$. We see that two of the constituents can only be conjugated if $a \in \{b, b+1, b+2\}$. If $a = b$ and $c > 1$, the only partitions which could be conjugated (from the values of the table) are $(a+1, a^{a-2}, a-1, c)$ and $(a+1, a^{a-1}, c-1)$, but they are not. If $a = b$ and $c = 1$, this is the first case of the lemma.

If $a = b+1$, the partitions (a^{a-1}, c) and $(a^{a-2}, a-1, c+1)$ are conjugated if and only if $c = a-2$. This is the second case of the lemma. Further, from the length and width one of the partitions $(a^{a-2}, a-1, c, 1)$ and $(a^{a-1}, c-1, 1)$ could be the conjugate of $(a+1, a^{a-3}, a-1, c)$ or $(a+1, a^{a-2}, c-1)$, to the second one only if $c > 1$. We see that the only times we got a pair of conjugates is if $c = a-2$, then $(a^{a-2}, a-1, a-2, 1)' = (a+1, a^{a-3}, a-1, a-2)$, this is again the second case of the lemma, or if $c = a-1$ but then λ would be symmetric.

If $a = b+2$, only $(a^{a-3}, a-1, c, 1)$ and $(a^{a-2}, c-1, 1)$ could be conjugated but they are not.

4th case: $\lambda = (a^b, 1^d)$. Here we assume that $b, d > 1$ and $a > 2$. The product $[n-1, 1][\lambda]$ decomposes as

$$[a+1, a^{b-1}, 1^{d-1}] + [a+1, a^{b-2}, a-1, 1^d] + [a^b, 2, 1^{d-2}] + [a^b, 1^d] + [a^{b-1}, a-1, 2, 1^{d-1}] + [a^{b-1}, a-1, 1^{d+1}].$$

In the following table it is easy to see when two constituents are conjugated.

ν	$\nu_1 - \nu_2$	$\nu'_1 - \nu'_2$
$(a+1, a^{b-1}, 1^{d-1})$	1	$d-1$
$(a+1, a^{b-2}, a-1, 1^d)$	$1 + \chi_{(b>2)}$	d
$(a^b, 2, 1^{d-2})$	0	$d-2$
$(a^b, 1^d)$	0	d
$(a^{b-1}, a-1, 2, 1^{d-1})$	1	$d-1$
$(a^{b-1}, a-1, 1^{d+1})$	$\chi_{(b>2)}$	$d+1$

There are no constituents which can be conjugated except for $(a+1, a^{b-1}, 1^{d-1})$ and $(a^{b-1}, a-1, 2, 1^{d-1})$ for $d = 2$, and indeed, for $d = 2$, $a = 3$ and $b = 2$ they are (this is the fourth case of the lemma).

5th case: $\lambda = (a^b, c^d)$. Here (assuming $a-c, c, b, d > 1$) $[n-1, 1][\lambda]$ decomposes as :

$$[a^b, c^d] + [a+1, a^{b-1}, c^{d-1}, c-1] + [a+1, a^{b-2}, a-1, c^d] + [a^b, c+1, c^{d-2}, c-1] + [a^{b-1}, a-1, c+1, c^{d-1}] + [a^b, c^{d-1}, c-1, 1] + [a^{b-1}, a-1, c^d, 1].$$

Here, we combine the previous methods. But first we notice that we can exclude $[a^b, c^d]$ because it is the only constituent with 2 removable nodes.

ν	$l(\nu)$	$w(\nu)$	$\nu_1 - \nu_2$	$\nu'_1 - \nu'_2$
$(a+1, a^{b-1}, c^{d-1}, c-1)$	$b+d$	$a+1$	1	$\chi_{(c>2)}$
$(a+1, a^{b-2}, a-1, c^d)$	$b+d$	$a+1$	$1 + \chi_{(b>2)}$	0
$(a^b, c+1, c^{d-2}, c-1)$	$b+d$	a	0	$\chi_{(c>2)}$
$(a^{b-1}, a-1, c+1, c^{d-1})$	$b+d$	a	$\chi_{(b>2)}$	0
$(a^b, c^{d-1}, c-1, 1)$	$b+d+1$	a	0	$1 + \chi_{(c>2)}$
$(a^{b-1}, a-1, c^d, 1)$	$b+d+1$	a	$\chi_{(b>2)}$	1

By looking at length and width we see that two constituents can only be conjugated if $a \in \{b+d-1, b+d, b+d+1\}$, where the cases $a = b+d-1$ and $a = b+d+1$ are conjugated. For $a = b+d-1$ we see by including the last two columns that there are no partition which can be conjugated. If $a = b+d$ $(a+1, a^{b-1}, c^{d-1}, c-1)$ and $(a^{b-1}, a-1, c^d, 1)$ can be conjugated, and they are if and only if μ is symmetric. The same holds for the pairs $(a+1, a^{b-2}, a-1, c^d)$ with $(a^b, c^{d-1}, c-1, 1)$ and $(a^b, c+1, c^{d-2}, c-1)$ with $(a^{b-1}, a-1, c+1, c^{d-1})$. ☐

2. Products of two-row partitions and hooks

Now we look what happens with the products of two-row partitions. In this case the decomposition of the products is known. For the general case they can be found in [RW94, Ros01] (where the formula in [RW94] contains some errors but can be fixed [Bri06]). Conveniently for the special cases that we need the formulas have appeared in [BWZ10, Corollary 3.5., Corollary 5.1.], [Gar+12, Theorem 2.3] and [Man10, Theorem 1.]. These are a bit more handy than the general ones. In this form they are from [BB17, Proposition 3.3.].

Proposition 16.3. *Let $k \in \mathbb{N}$.*

(1) *For $n = 2k + 1$, we have*

$$[k+1,k]^2 = \sum_{\substack{\lambda \vdash n \\ l(\lambda) \leq 4}} [\lambda].$$

(2) *Let $n = 2k$, we let $E(n)$ and $O(n)$ denote the sets of partitions of n into only even parts and only odd parts, respectively, then*

$$[k,k]^2 = \sum_{\substack{\lambda \in E(n) \\ l(\lambda) \leq 4}} [\lambda] + \sum_{\substack{\lambda \in O(n) \\ l(\lambda) = 4}} [\lambda]\,.$$

(3) *Let $n = 2k$, then*

$$[k,k][k+1,k-1] = \sum_{\substack{\lambda \vdash n, \lambda \notin E(n) \\ l(\lambda) < 4}} [\lambda] + \sum_{\substack{\lambda \vdash n, \lambda \notin O(n) \cup E(n) \\ l(\lambda) = 4}} [\lambda]\,.$$

We derive the following results for the alternating groups.

Corollary 16.4. *Let $2 \leq k \in \mathbb{N}$, then $\{k+1,k\}^2$ is multiplicity-free if and only if $k \geq 7$.*

Lemma 16.5. *Let $n = 2k \geq 6$ be even and $\lambda \vdash n$ non-symmetric. The product $\{k,k\}\{\lambda\}$ multiplicity-free if up to conjugation of λ one of the following holds:*

(1) *$k \neq 4, 6$ and $\lambda = (k,k)$ or $\lambda = (k+1,k-1)$;*
(2) *$k \neq 4$ and $\lambda = (n-2,2)$;*
(3) *$k \neq 4$ and $\lambda = (n-3,3)$;*
(4) *λ is a hook and different from $(k,1^k)$.*

Proof: Let $\lambda \vdash n = 2k \geq 6$ be non-symmetric. Because of Theorem 16.1 and Corollary 15.3 we know that $\{\lambda\}\{k,k\}$ contains a constituent with multiplicity 2 or higher if $\lambda^{(\prime)}$ is neither a hook, $(n-2,2)$, $(n-3,3)$, (k,k) nor $(k+1,k-2)$.

We start with the case that λ is $(n-2,2)$, $(n-3,3)$, (k,k) or $(k+1,k-2)$. We know that all the constituents of $[\lambda][k,k]$ have length less or equal to 4 (Theorem 5.5). Therefore, $\{\lambda\}\{k,k\}$ is multiplicity-free if $n > 16$. For $n \leq 16$ we calculate the products with GAP and see for which n they are multiplicity-free.

From now on we assume that $\lambda = (n-i,1^i)$ is a hook. We know that $\{\lambda\}\{k,k\}$ is multiplicity-free if and only if $g(\lambda,(k,k),\nu) + g(\lambda,(k,k),\nu') \leq 1$ for all non-symmetric $\nu \vdash n$ (Corollary 15.3). Thanks to Theorem 5.12 and [Ros01, Theorem 4] we know that all the constituents of $[k,k][\lambda]$ are hooks or double-hooks, so we only have to look at these cases. Let $\nu \vdash n$ with $g(\lambda,(k,k),\nu) = g(\lambda,(k,k),\nu') = 1$.

First case: $\nu = (n-j,1^j)$ is a hook. By Theorem 5.12 we know that

$$j+1 \in \{i-1,i,i+1\} \text{ and } n-j \in \{i-1,i,i+1\}.$$

This is only possible if $i = k$ or $i = k-1$. But the only partitions of this form are $(k,1^k)$ and its conjugated.

Second case: ν is a proper double-hook. Let $\nu = (n_4,n_3,2^{d_2},1^{d_1}) \vdash n$ with $2 \leq n_3 \leq n_4$. Like in the first case with Theorem 5.12 we obtain that ν and ν' can only be constituents of $[k,k][\lambda]$ if

$$i \in \{d_1+2d_2,\ d_1+2d_2+1,\ d_1+2d_2+2,\ d_1+2d_2+3\} \text{ and}$$
$$i \in \{n_3+n_4-4,\ n_3+n_4-3,\ n_3+n_4-2,\ n_3+n_4-1\}.$$

From this follows $n-4 \leq 2i \leq n+2$ since $n_4+n_3+2d_2+d_1 = n$. Therefore, $i = k-2,\ k-1,\ k,\ k+1$. We have already excluded the cases $i = k-1,\ k$. We assume $i = k-2$ the other case follows by conjugation. We show that

$$g(\lambda,(k,k),\nu) = g(\lambda,(k,k),\nu') = 1$$

implies that $\nu = \nu'$. Theorem 5.12 tells us that $k-2 \leq d_1 + 2d_2 \leq k+1$ (since $g(\lambda, (k,k), \nu) = 1$) and $k-2 \leq n_4 + n_3 - 4 \leq k+1$ (since $g(\lambda, (k,k), \nu') = 1$). From these two inequalities we obtain $d_1 + 2d_2 = k-2$. This means that the first two columns have the same number of boxes as the first two rows, therefore, it is sufficient to prove that $d_2 + 2 = n_3$.

Due to Theorem 5.12 we know that for $d_1 + d_2 = k - 2 = i$

$$g((k,k), \lambda, \nu) = \chi_{(n_3 \leq k - d_2 \leq \min\{n_4, n_3 + d_1\})}.$$

It follows that

$$\begin{aligned} & k - d_2 \leq n_3 + d_1 \\ \Leftrightarrow \quad & k \leq n_3 + d_1 + d_2 \\ \Leftrightarrow \quad & k \leq n_3 + d_1 + d_2 + k - 2 - 2d_2 - d_1 \\ \Leftrightarrow \quad & d_2 + 2 \leq n_3. \end{aligned}$$

We can repeat the same calculation for ν' to obtain $d_2 + 2 \geq n_3$. Together we get $d_2 + 2 = n_3$ which implies that ν is symmetric. □

3. Products involving rectangles

Now we want to look at the multiplicity-free S_n products with a rectangle. Since we have already looked at the cases (k,k), we assume that length and width of the rectangle are greater or equal to 3.

Lemma 16.6. *Let $n = ab$, where $b > a \geq 3$. The products $\{n-2,2\}\{a^b\}$ and $\{n-2,1^2\}\{a^b\}$ are multiplicity-free if and only if $n \neq 12$.*

Proof: The formulas for the S_n products are known (see [BO06, Corollary 4.6.] and [BB17, Proposition 3.6.]) so we check these like in the $(n-1,1)$ case (Lemma 16.2). We start with the product $[n-2,2][a^b]$. The possible constituents are:

Partition	length	width
(a^b)	b	a
$(a^{b-1}, a-1, 1)$	$b+1$	a
$(a^{b-2}, (a-1)^2, 1^2)$	$b+2$	a
$((a+1)^2, a^{b-4}, (a-1)^2)$	b	$a+1$
$(a+1, a^{b-2}, a-1)$	b	$a+1$
$(a+1, a^{b-3}, (a-1)^2, 1)$	$b+1$	$a+1$
$(a+2, a^{b-2}, a-2)$	b	$a+2$
$(a+1, a^{b-2}, a-2, 1)$	$b+1$	$a+1$
$(a^{b-1}, a-2, 2)$	$b+1$	a

where $(a^{b-1}, a-2, 2)$ only occurs if $a > 3$. From the length and width we see that two of the constituents can only be conjugated if $b = a+1$, $a+2$. If $b = a+2$, there is one constituent which could be symmetric (but is not) but not two which could be conjugated.

If $b = a+1$, by looking at the length and width, $((a+1)^2, a^{a-3}, (a-1)^2)$ and $(a+1, a^{a-1}, a-1)$ could be conjugated and $(a+2, a^{a-1}, a-2)$ could be conjugated to one of the partitions $(a+1, a^{a-2}, (a-1)^2, 1)$ or $(a+1, a^{a-1}, a-2, 1)$. However, we see that $((a+1)^2, a^{a-3}, (a-1)^2)$ and $(a+1, a^{a-1}, a-1)$ cannot be conjugated, neither can $(a+2, a^{a-1}, a-2)$ and $(a+1, a^{a-2}, (a-1)^2, 1)$. The last thing we check is that $(a+2, a^{a-1}, a-2)$ and $(a+1, a^{a-1}, a-2, 1)$ are conjugated if and only if $a = 3$ (and $b = 4$) which is the case exactly if $n = 12$.

Now we do the same for the product $[n-2,1^2][a^b]$. The possible constituents are:

Partition	length	width
$(a+2, a^{b-3}, (a-1)^2)$	b	$a+2$
$(a+1, a^{b-2}, a-1)$	b	$a+1$
$(a+1, a^{b-2}, a-2, 1)$	$b+1$	$a+1$
$(a^{b-2}, (a-1)^2, 2)$	$b+1$	a
$(a+1, a^{b-3}, (a-1)^2, 1)$	$b+1$	$a+1$
$((a+1)^2, a^{b-3}, a-2)$	b	$a+1$
$(a^{b-1}, a-2, 1^2)$	$b+2$	a
$(a^{b-1}, a-1, 1)$	$b+1$	a .

Again we only have to look at the case $b = a+1$ (for the same reasons). From looking at the length and width, $(a+1, a^{a-1}, a-1)$ and $((a+1)^2, a^{a-2}, a-2)$ could be conjugated and $(a+2, a^{a-2}, (a-1)^2)$ could be conjugated to $(a+1, a^{a-1}, a-2, 1)$ or $(a+1, a^{a-2}, (a-1)^2, 1)$. Here, $(a+1, a^{a-1}, a-1)$ and $((a+1)^2, a^{a-2}, a-2)$ are conjugated if and only if $a = 3$ but $(a+2, a^{a-2}, (a-1)^2)$ is never conjugated to $(a+1, a^{a-1}, a-2, 1)$ or $(a+1, a^{a-2}, (a-1)^2, 1)$. □

We have checked for all multiplicity-free products of irreducible S_n characters labeled by non-symmetric partitions if they stay irreducible if we restrict them to A_n. From Corollary 15.3 we know that the multiplicity-free A_n products we obtained in that way are all, where both characters are labeled by non-symmetric partitions. In the next chapter we will look what happens if one or both characters are labeled by symmetric partitions.

CHAPTER 17

Symmetric products

In this chapter we want to look at multiplicity-free products which involve irreducible A_n characters which are labeled by symmetric partitions. First, we derive the formulas for the products of Theorem 15.1 where at least one of the factors is labeled by a symmetric partition.

1. Decomposition of multiplicity-free symmetric products

We start with investigating products of the form $\{\lambda\}_\pm\{\mu\}$, where one of the factors is symmetric and the other one is not. Here, the multiplicities of all the constituents but $\{\lambda\}_\pm$ can directly be derived from the known S_n formulas with Lemma 15.2. Thus, we just state these multiplicities instead of the whole decomposition of the product.

We name the multiplicity of $\{\lambda\}_\pm$ in the product $\{\mu\}\{\lambda\}_+$ with $m_\pm$, where the signs have to coincide, i.e., $m_\pm := \langle\{\mu\}\{\lambda\}_+, \{\lambda\}_\pm\rangle$. Then we know that $m_\mp = \langle\{\mu\}\{\lambda\}_-, \{\lambda\}_\pm\rangle$, where the signs have to differ. With this we can state a result which connects the multiplicity of a character $\{\mu\}$ for a non-symmetric partition μ in the products $\{\lambda\}_\pm\{\lambda\}_\pm$ with the Kronecker coefficient $g(\lambda, \lambda, \mu)$ and the character value $[\mu](\sigma_{h(\lambda)})$. Here, we use e_λ, defined in Section 1 of Chapter 15 in the subsection about character values, as an index, where it stands for $+$ or $-$ instead of $+1$ or -1.

Lemma 17.1. *Let $\lambda = \lambda' \vdash n$ be a symmetric partition and $\mu \neq \mu'$ be a non-symmetric partition. Then*

$$\langle(\{\lambda\}_\pm)^2, \{\mu\}\rangle = m_{e_\lambda} \text{ and } \langle\{\lambda\}_+\{\lambda\}_-, \{\mu\}\rangle = m_{-e_\lambda}.$$

Further,

$$m_\pm = \frac{1}{2}(g(\lambda, \lambda, \mu) \pm [\mu](\sigma_{h(\lambda)})).$$

Proof: By definition $m_\pm = \langle\{\mu\}\{\lambda\}_+, \{\lambda\}_\pm\rangle = \langle\{\mu\}, \overline{\{\lambda\}_+}\{\lambda\}_\pm\rangle$. But we know that $\overline{\{\lambda\}_+} = \{\lambda\}_{e_\lambda}$ which shows the first claim. To prove the second one we look at $m_+ + m_-$ and $m_+ - m_-$. We know that

$$\{\mu\}\{\lambda\}_\pm = m_\pm\{\lambda\}_+ + m_\mp\{\lambda\}_- + \text{ other constituents.}$$

But this tell us that

$$g(\lambda, \lambda, \mu) = m_+ + m_-.$$

From [Bes18, Lemma 2.3. (2)] we know that $[\mu](\sigma_{h(\lambda)}) = m_+ - m_-$. These two facts tell us that

$$2m_\pm = g(\lambda, \lambda, \mu) \pm [\mu](\sigma_{h(\lambda)})$$

which proves the second part. □

With the previous lemma we compute the unknown multiplicities for products in which one of the factors is symmetric.

Products involving one symmetric partition.

Lemma 17.2. *Let $\lambda = \lambda'$ be a symmetric partition of $n \geq 4$. The products*

$$\{\lambda\}_{\pm}\{n-1,1\}$$

are multiplicity-free if and only if λ has at most three removable nodes. If λ is a rectangle,

$$\langle \{\lambda\}_{+}\{n-1,1\}, \{\lambda\}_{\pm}\rangle = \langle \{\lambda\}_{-}\{n-1,1\}, \{\lambda\}_{\pm}\rangle = 0.$$

If λ has two removable nodes,

$$\langle \{\lambda\}_{\pm}\{n-1,1\}, \{\lambda\}_{\pm}\rangle = 0 \text{ and } \langle \{\lambda\}_{\mp}\{n-1,1\}, \{\lambda\}_{\pm}\rangle = 1.$$

If λ has three removable nodes,

$$\langle \{\lambda\}_{+}\{n-1,1\}, \{\lambda\}_{\pm}\rangle = \langle \{\lambda\}_{-}\{n-1,1\}, \{\lambda\}_{\pm}\rangle = 1.$$

Proof: Let $\lambda \vdash n$ be symmetric. If $\nu \vdash n$ with $\nu \neq \lambda$ we know from Lemma 5.13 that

$$g(\lambda, (n-1,1), \nu) = \begin{cases} 1, & \text{if } |\lambda \cap \nu| = n-1; \\ 0, & \text{otherwise.} \end{cases}$$

By Lemma 15.2 the multiplicity of $\{\nu\}_{(\pm)}$ as constituent of $\{n-1,1\}\{\lambda\}_{\pm}$ is 1 if $|\lambda \cap \nu| = n-1$, and 0, otherwise. So now we calculate the multiplicities $m_{\pm}$ with the previous lemma. If λ is a square, then $g(\lambda, \lambda, (n-1,1)) = 0$ and therefore, this case is trivial.

If λ is a fat hook, we know that $[n-1,1](\sigma_{h(\lambda)}) = -1$ because λ has no hook of length 1 on the main diagonal. With Theorem 1.1 we see that all but the last rim hook that we remove are in the arm of $(n-1,1)$. Since $g(\lambda, \lambda, (n-1,1)) = 1$ we obtain $m_{+} = 0$ and $m_{-} = 1$. So

$$\langle \{\lambda\}_{\pm}\{\mu\}, \{\lambda\}_{\pm}\rangle = 0 \text{ and } \langle \{\lambda\}_{\pm}\{\mu\}, \{\lambda\}_{\mp}\rangle = 1.$$

If λ has three removable nodes, we know that $\sigma_{h(\lambda)}$ has one fix-point, because $h(\lambda)$ has exactly one part of length 1. Since the character value of $[n-1,1]$ can be interpreted as number of fix-points of $\sigma_{h(\lambda)}$ minus one, $[n-1,1](\sigma_{h(\lambda)}) = 0$. Together with $g(\lambda, \lambda, (n-1,1)) = 2$ this tells us that $m_{+} = m_{-} = 1$ and therefore,

$$\langle \{\lambda\}_{+}\{n-1,1\}, \{\lambda\}_{\pm}\rangle = \langle \{\lambda\}_{-}\{n-1,1\}, \{\lambda\}_{\pm}\rangle = 1.$$

If λ has 4 or more removable nodes, we know that $[n-1,1](h(\lambda)) \in \{0, -1\}$, because if $\mathrm{rem}(\lambda)$ is odd, it has 1 fix-point and if it is even, none. But

$$g(\lambda, \lambda, (n-1,1)) = \mathrm{rem}(\lambda) - 1 > 2$$

and therefore, m_{+} or m_{-} is strictly greater than 1. □

To see how the constituents of $[\lambda][\mu]$ for a symmetric partition λ are divided between $\{\lambda\}_{+}\{\mu\}$ and $\{\lambda\}_{-}\{\mu\}$ we often look at the critical classes $h(\lambda)$.

Lemma 17.3. *If $n = a^2 \geq 9$, the products $\{a^a\}_{\pm}\{n-2,2\}$ and $\{a^a\}_{\pm}\{n-2,1^2\}$ are multiplicity-free, where*

$$\langle \{a^a\}_{\pm}\{n-2,2\}, \{a^a\}_{\pm}\rangle = 0 \text{ and } \langle \{a^a\}_{\pm}\{n-2,2\}, \{a^a\}_{\mp}\rangle = 1$$

and

$$\langle \{a^a\}_{\pm}\{n-2,1^2\}, \{a^a\}_{\pm}\rangle = \langle \{a^a\}_{\pm}\{n-2,1^2\}, \{a^a\}_{\mp}\rangle = 0.$$

Further, $\{2,2\}_{\pm}$ is of degree 1 therefore, $\{2,2\}_{\pm}\{2,2\}_{\pm}$ (for all four choices) and $\{2,2\}_{\pm}\{2,1^2\}$ are irreducible. The products are given by $\{2,2\}_{+}\{2,2\}_{-} = \{4\}$, $(\{2,2\}_{\pm})^2 = \{2,2\}_{\mp}$ and $\{2,2\}_{\pm}\{3,1\} = \{3,1\}$.

Proof: The part about $\{2,2\}_\pm$ was calculated with GAP. In the other part only the multiplicity of $\{a^a\}_\pm$ does not directly follow from the S_n product. We know that $[n-2,2][a^a]$ and $[n-1^2][a^a]$ are multiplicity-free. Therefore, by Lemma 15.2 we know that all constituents of the products $\{a^a\}_\pm\{n-2,2\}$ and $\{a^a\}_\pm\{n-2,1^2\}$ which are different from $\{a^a\}_\pm$ have multiplicity 1. We calculate the missing multiplicities with Lemma 17.1. Due to Proposition 5.14 we know that

$$g((a^a),(a^a),(n-2,1^2)) = (\mathrm{rem}(a^a)-1)^2 = 0.$$

Hence, the multiplicity of $\{a^a\}_\pm$ in $\{a^a\}_\pm\{n-2,1^2\}$ has to be zero, too. For the product with $\{n-2,2\}$ we calculate again m_+ and m_- from Lemma 17.1. By Proposition 5.14 we know that $g((a^a),(a^a),(n-2,2)) = 1$. The Murnaghan–Nakayama rule (Theorem 1.1) tells us that

$$[n-2,2](\sigma_{h(a^a)}) = [2,2](\sigma_{(3,1)}) = -1,$$

where $h(a^a) = (\ldots,5,3,1) \vdash n$. This tells us that $m_+ = 0$ and $m_- = 1$. With the first part of Lemma 17.1 we conclude

$$\langle \{a^a\}_\pm\{n-2,1^2\}, \{a^a\}_\pm\rangle = 0 \text{ and } \langle \{a^a\}_\pm\{n-2,1^2\}, \{a^a\}_\mp\rangle = 1.$$

This proves the lemma. □

Product of two symmetric hooks.

Lemma 17.4. *Let $\lambda = (k+1,1^k)$ be the symmetric hook of $n = 2k+1 \geq 5$, then any of the 4 products $\{\lambda\}_\pm\{\lambda\}_\pm$ is multiplicity-free.*

If $n \equiv 1 \mod 4$,

$$(\{\lambda\}_\pm)^2 = \sum_{j=0}^{\frac{k}{2}-1}\{n-2j,1^{2j}\} + \sum_{\substack{\mu\vdash n\\ d(\mu)=2}}\{\mu\} + \{\lambda\}_\pm \text{ and}$$

$$\{\lambda\}_+\{\lambda\}_- = \sum_{j=0}^{\frac{k}{2}-1}\{n-2j-1,1^{2j+1}\} + \sum_{\substack{\mu\vdash n\\ d(\mu)=2}}\{\mu\}.$$

If $n \equiv 3 \mod 4$,

$$(\{\lambda\}_\pm)^2 = \sum_{j=0}^{\frac{k-1}{2}-1}\{n-2j-1,1^{2j+1}\} + \sum_{\substack{\mu\vdash n\\ d(\mu)=2}}\{\mu\} + \{\lambda\}_\mp \text{ and}$$

$$\{\lambda\}_+\{\lambda\}_- = \sum_{j=0}^{\frac{k-1}{2}}\{n-2j,1^{2j}\} + \sum_{\substack{\mu\vdash n\\ d(\mu)=2}}\{\mu\}.$$

Proof: From [Rem89, Theorem 2.1.] and [Ros01] we know that for $\mu \vdash n$

$$g(\lambda,\lambda,\mu) = \begin{cases} 2, & \text{if } \mu \text{ is a double-hook;} \\ 1, & \text{if } \mu \text{ is a hook;} \\ 0, & \text{if } \mu \text{ is neither a hook nor a double-hook.} \end{cases}$$

If μ is not a hook or a double-hook, there is nothing to show.

If μ is a proper double-hook, we know from the Murnaghan-Nakayama rule that $[\mu](\sigma_{(n)}) = 0$, where $h(\lambda) = (n)$. By Lemma 17.1 the four inner products

$\langle \{\lambda\}_\pm \{\lambda\}_\pm, \{\mu\}\rangle$ all have the same value no matter how we choose the signs. This means that for all four choices

$$\langle \{\lambda\}_\pm \{\lambda\}_\pm, \{\mu\}\rangle = 1.$$

If $\mu \neq \lambda$ is a hook, we know that $[\mu](\sigma_{(n)}) = (-1)^{l(\mu)-1}$. If $n \equiv 1 \bmod 4$, we compute $e_\lambda = 1$. So with Lemma 17.1 we see that all the hooks of odd length are constituents of the products $(\{\lambda\}_\pm)^2$ with multiplicity 1 and all the hooks of even length are constituents of $\{\lambda\}_+\{\lambda\}_-$, again with multiplicity 1. If $n \equiv 3 \bmod 4$, $e_\lambda = -1$ and therefore, it is the other way around, meaning all the hooks of even length are constituents of the products $(\{\lambda\}_\pm)^2$ with multiplicity 1 and all the hooks of odd length are constituents of $\{\lambda\}_+\{\lambda\}_-$, again with multiplicity 1.

The last multiplicities we need to check are the ones for $\{\lambda\}_\pm$. If $n \equiv 1 \bmod 4$, we know that $\{\lambda\}_\pm(\sigma^+_{(n)}) = \frac{1}{2}(1 \pm \sqrt{n})$. So we get for the product

$$\begin{aligned}(\{\lambda\}_\pm\{\lambda\}_\pm)(\sigma^+_{(n)}) &= \frac{1}{4}(1 \pm 2\sqrt{n} + n) = \frac{1}{2}(1 \pm \sqrt{n}) + \frac{1}{4}(n-1),\\ (\{\lambda\}_+\{\lambda\}_-)(\sigma^+_{(n)}) &= \frac{1}{4}(1-n).\end{aligned}$$

Since n is odd, all the proper double-hooks are not symmetric and λ is the only symmetric hook. All possible constituents but $\{\lambda\}_\pm$ have character values from $\mathbb{Z}$ on the class $(n)^+$. Further, $g(\lambda,\lambda,\lambda) = 1$, so $\{\lambda\}_+$ and $\{\lambda\}_-$ appear exactly ones, therefore, we obtain

$$\langle \{\lambda\}_+\{\lambda\}_+, \{\lambda\}_+\rangle = \langle \{\lambda\}_-\{\lambda\}_-, \{\lambda\}_-\rangle = 1.$$

If $n \equiv 3 \bmod 4$, $\{\lambda\}_\pm(\sigma^+_{(n)}) = \frac{1}{2}(-1 \pm \sqrt{-n})$ and therefore,

$$\begin{aligned}\{\lambda\}_\pm\{\lambda\}_\pm(\sigma^+_{(n)}) &= \frac{1}{4}(1 \pm 2(-\sqrt{-n}) - n) = \frac{1}{2}(-1 \mp \sqrt{-n}) + \frac{1}{4}(3-n),\\ \{\lambda\}_+\{\lambda\}_-(\sigma^+_{(n)}) &= \frac{1}{4}(1+n).\end{aligned}$$

Applying the same argument as before it now follows:

$$\langle \{\lambda\}_+\{\lambda\}_+, \{\lambda\}_-\rangle = \langle \{\lambda\}_-\{\lambda\}_-, \{\lambda\}_+\rangle = 1.$$

This proves the decompositions of the products. □

2. Other products are not multiplicity-free

In the next step we want to show that the products which we investigated in the last section are the only multiplicity-free products involving symmetric partitions. First, we look at the case that one of the partitions is symmetric and the other one is not.

Product of a symmetric and a non-symmetric character.
Let $\lambda,\ \mu \vdash n$ be two partitions such that $\lambda = \lambda'$ is symmetric and $\mu \neq \mu'$ is not. If $\{\lambda\}_\pm\{\mu\}$ is multiplicity-free, we know that on the level of the symmetric group $g(\lambda,\mu,\nu) \in \{0,1\}$ for all $\nu \vdash n$ with $\nu \neq \nu'$ and $g(\lambda,\mu,\nu) \in \{0,2\}$ for all $\nu \vdash n$ with $\nu = \nu'$ and $\nu \neq \lambda$ (see Corollary 15.3). We investigated these products in the second part of this thesis and if we use these results we obtain the following theorem:

Theorem 17.5. *Let $\lambda = \lambda' \vdash n$ and $\mu \neq \mu' \vdash n$ be two partitions such that the product $\{\lambda\}_\pm\{\mu\}$ is multiplicity-free. Then λ, μ are one of the following pairs (up to conjugation of μ):*

(1) *μ is a linear partition;*
(2) *$\mu = (n-1,1)$ and λ has at most 3 removable nodes;*
(3) *$n = a^2$, $\lambda = (a^a)$ and $\mu = (n-2,2)$ or $\mu = (n-2,1^2)$;*

(4) *one of the exceptional cases:*

$((3,1^2),(3,2))$, $((3,2,1),(3,3))$, $((3^2,2),(4,4))$, $((3^3),(5,4))$, $((3^3),(6,3))$.

Proof: That products of the form $\{n\}\{\lambda\}_\pm$ are multiplicity-free is obvious. That the exceptional cases are multiplicity-free can easily be checked with GAP. For the other products we have seen in Lemma 17.2 and 17.3 that they are multiplicity-free. For the other direction we first we assume that one of the partitions λ, μ is a hook. Proposition 6.1 and 6.2 together with Corollary 15.3 tell us that the only possibilities for λ and μ (up to conjugation of μ) are the following:

(a) $\mu = (n)$;
(b) $\mu = (n-1,1)$ and $\lambda = \lambda'$ is symmetric;
(c) $n = 2k+1$ is odd, $\lambda = (k+1,1^k)$ and $\mu \neq \mu'$ is a hook;
(d) $n = a^2$, $\lambda = (a^a)$ and $\mu = (n-i,1^i)$ with $i \leq 3$;
(e) $n = a^2-1$, $\lambda = (a^{a-1}, a-1)$ and $\mu = (n-2,1^2)$;
(f) $n = 2k+1$ is odd, $\lambda = (k+1,1^k)$ and $\mu = (k+1,k)$ or $\mu = (n-2,2)$;
(g) the exceptional case $n = 6$, $\mu = (4,1^2)$ and $\lambda = (3,2,1)$.

For these we only have to check the formulas we have for the S_n products, where we can calculate $\langle \{\lambda\}_+\{\mu\}, \{\lambda\}_\pm \rangle$ with Lemma 17.1 if needed. We do this case by case. Obviously, (a) is multiplicity-free. The case $\mu^{(')} = (n-1,1)$ we have already proven with Lemma 17.2. So we assume that $\mu \neq (n-1,1)$.

In part (c) we obtain (λ, μ) from the seed $((3,1^2),(3,1^2))$ with the semigroup property (Theorem 5.8). Since $[3,2]$ is contained twice in $[3,1^2]^2$, $[\lambda][\mu]$ has a constituent $[\nu]$ with multiplicity 2 which is not a hook ($(3,2)$ has Durfee-size 2 and it does not decrease by adding boxes). Since $[\lambda][\mu]$ only contains hooks and double-hooks, we know that it contains a double-hook ν with multiplicity 2. Since n is odd, we know that ν is not symmetric and therefore, $\{\nu\}$ has multiplicity 2 in the products $\{\lambda\}_\pm\{\mu\}$.

In part (d) we assume that $i > 1$. If $i = 3$, the formula from Lemma 6.6 for the product on the symmetric group level tells us that $\{a+1, a^{a-2}, a-2, 1\}$ appears as a constituent with multiplicity 2. If $i = 2$, the product is multiplicity-free as we have seen in Lemma 17.3.

For part (e) we assume that $\lambda = (a^{a-1}, a-1)$ and $\mu = (n-2,1^2)$. Then the formula from Lemma 6.5 for the S_n product tells us that $\{a^{a-1}, a-2, 1\}$ is a constituent with multiplicity 2 of the A_n product.

To prove part (f), let $n = 2k+1$, $\lambda = (k+1,1^k)$ and $\mu = (k+1,k)$ or $\mu = (n-2,2)$. Theorem 5.12 tells us that for $n \geq 7$ $g(\lambda, \mu, (k+1,2,1^{k-2})) = 2$, and therefore, $\{k+1,2,1^{k-2}\}$ is a constituent with multiplicity 2 of the products $\{\lambda\}_\pm\{\mu\}$. That the $n = 5$ case is multiplicity-free was checked with GAP.

For (g) was checked with the GAP, that $\{4,1,1\}\{3,2,1\}_\pm$ both contain $\{4,2\}$ with multiplicity 2.

In the next step we look at the possibilities if one of the partitions is a two-row partition (for $n > 4$ this has to be μ and the case $\lambda = (2,2)$ we have already seen in Lemma 17.3). Since we already looked at the case that one of the partitions is a hook, we can assume that none of the partitions is a hook. Here Proposition 7.1 tells us that the S_n product contains a constituent with multiplicity 3 or higher which is different from λ, and therefore, the A_n product contains a constituent with multiplicity 2 or higher, if λ and μ are not from the following list:

(a) $n = a^2$, $\lambda = (a^a)$ and $\mu = (n-2,2)$ or $\mu = (n-3,3)$;
(b) $n = a^2-1$, $\lambda = (a^{a-1}, a-1)$ and $\mu = (n-2,2)$;
(c) for $n = 6$, $\lambda = (n-3,2,1) = (3,2,1)$ is symmetric, so we have to check the product with $\mu = (3,3)$;

(d) one of the exceptional cases $\lambda = (3,3,2)$ and $\mu = (4,4)$ or $\mu = (5,3)$ or $\lambda = (4,2,1^2)$ and $\mu = (4,4)$ or $\lambda = (3^3)$ and $\mu = (5,4)$ or $(6,3)$.

If $\lambda = (a^a)$ and $\mu = (n-2,2)$, the product is multiplicity-free, as we have shown in Lemma 17.3. If $\mu = (n-3,3)$, the formula for the S_n product (Lemma 7.5) tells us that $\{a^{a-1}, a-1, 1\}$ appears twice in the A_n product.

If $\lambda = (a^{a-1}, a-1)$ and $\mu = (n-2,2)$, the formula from Lemma 7.3 for the S_n product tells us that $\{a^{a-1}, a-2, 1\}$ is a constituent with multiplicity 2 of the A_n product.

We check with GAP that:

- $\{3,2,1\}_\pm\{3,3\}$ is multiplicity-free;
- $\{3^2,2\}_\pm\{5,3\}$ is not multiplicity-free, it contains $\{5,2,1\}$ twice;
- $\{3^2,2\}_\pm\{4,4\}$ is not multiplicity-free, it contains $\{4,2,1^2\}_\pm$ twice;
- $\{4,2,1^2\}_\pm\{4,4\}$ is not multiplicity-free, it contains $\{4,3,1\}$ twice;
- $\{3^3\}_\pm\{5,4\}$ and $\{3^3\}_\pm\{6,3\}$ are multiplicity-free.

If neither λ nor μ is a hook nor a two-row partition, we just have to check the case $\{3^3\}_\pm\{4^2,1\}$ for $n = 9$ (since in $\{3^3_\pm\}\{3^3\}_\pm$ both partitions are symmetric), this follows from Theorem 5.1 and Theorem 5.2. For these products were checked with GAP that they contain $\{5,2,1^2\}$ with multiplicity 2 and therefore, we know that there are no multiplicity-free products involving exactly one symmetric factor and no hook or two-row partition. □

Products of different symmetric character.

Theorem 17.6. *Let $\lambda, \mu \vdash n$ be symmetric partitions such that $\lambda \neq \mu$ and one of the products $\{\lambda\}_\pm\{\mu\}_\pm$ is multiplicity-free, then λ and μ are $(5,1^4)$ and (3^3) and all of the products are multiplicity-free.*

Proof: We know if there is a non-symmetric partition $\nu \vdash n$ with $g(\lambda,\mu,\nu) > 2$,

$$\langle \{\lambda\}_\pm\{\mu\}_\pm, \{\nu\}\rangle = \frac{1}{2}g(\lambda,\mu,\nu) > 1.$$

Therefore, we can focus on the pairs from Theorem 5.1 and Theorem 5.2, but there are three possibilities such that λ and μ are both symmetric. One is that $\lambda = \mu$ are symmetric hooks. The second one is $\lambda = \mu = (3^3)$ but these two violate the assumption $\lambda \neq \mu$. The third pair is $(5,1^4)$ and (3^3). That the 4 products $\{5,1^4\}_\pm\{3^3\}_\pm$ are multiplicity-free was checked with GAP. □

Symmetric squares.

In this subsection we investigate the character products of the form $\{\lambda\}_\pm\{\lambda\}_\pm$ for a symmetric partition λ which is not a hook. In Lemma 17.4 we already did this for hooks.

Proposition 17.7. *Let $\lambda = \lambda'$ be a symmetric partition. If one of the products $\{\lambda\}_\pm\{\lambda\}_\pm$ is multiplicity-free, all products are and one of the following holds:*

(1) *λ is a hook;*
(2) *$n = 4$ and $\lambda = (2,2)$;*
(3) *$n = 9$ and $\lambda = (3^3)$.*

Proof: If λ is a hook we have seen the products in Lemma 17.4. Further, that the eight products $\{3^3\}_\pm\{3^3\}_\pm$, $\{2,2\}_\pm\{2,2\}_\pm$ are, indeed, all multiplicity-free was checked with GAP. What is missing is to show that the other products contain constituents with multiplicity 2 or lager. From now on let $\lambda \vdash n$ be a symmetric partition which is neither hook, $(2,2)$ nor (3^3). We use Lemma 17.1 to show that for a fitting partition $\mu \vdash n$ the multiplicities m_+ and m_- are greater or equal to 2. We use different constituents according to the number of removable nodes λ has. Therefore, we have three cases.

1st case: rem(λ) ≥ 3. Let $\mu = (n-2, 1^2)$. We know by Proposition 5.14 that

$$g(\lambda, \lambda, \mu) = (\text{rem}(\lambda) - 1)^2 \geq 4.$$

We compute character value $[\mu](\sigma_{h(\lambda)})$ with the Murnaghan–Nakayama rule (Theorem 1.1). If rem(λ) is odd, we know that $h(\lambda)$ has a part of length 1. In this case we remove all but the last two parts of $h(\lambda)$. From μ we can only remove parts from the arm since every part we remove is greater or equal to 5. If we now try to remove the second smallest part from $h(\lambda)$, we realize that this is not possible since we would have to remove a $(\tilde{n}+1)$-hook from $(\tilde{n}, 1^2)$. Therefore, $[\mu](\sigma_{h(\lambda)}) = 0$. Hence, $m_+ = m_- = \frac{1}{2} g(\lambda, \lambda, \mu) \geq 2$ if rem(λ) is odd. If rem(λ) is even, we know that all parts of $h(\lambda)$ are greater or equal to 3 and λ has at least 4 removable nodes. So if we remove every part but the smallest one, we know that we have to remove the corresponding hooks in the arm of μ. If we then remove the smallest part, we remove the two boxes in the leg and therefore, $[\mu](\sigma_{h(\lambda)}) = 1$. Since λ has at least 4 removable nodes, we know $g(\lambda, \lambda, \mu) \geq 9$ and $m_+ = \frac{1}{2}(g(\lambda, \lambda, \mu) + 1) \geq 5$ and $m_- = \frac{1}{2}(g(\lambda, \lambda, \mu) - 1) \geq 4$.

2nd case: rem(λ) $= 2$. We need two different possibilities for μ. We start with $\mu = (n-3, 3)$. We know from [Val14, Theorem 7.12] or Theorem 5.15 that

$$g(\lambda, \lambda, \mu) = h_1(h_1 - 1)(h_1 - 3) + h_2(2h_1 - 3) + h_3,$$

where h_i equals the number of i-hooks in λ. If $\min(h(\lambda)) > 3$, we conclude that $h_2 = 4$ and $h_3 \geq 2$, hence, $g(\lambda, \lambda, \mu) \geq 4$. If $\min(h(\lambda)) = 3$ we know that $n = a^2 - 1$ for an $a \in \mathbb{N}$ and $\lambda = (a^{a-1}, a-1)$. We see with GAP that the products $\{3,3,2\}_\pm \{3,3,2\}_\pm$ contain $\{5,2,1\}$ with multiplicity 2 if the signs match, and $\{4,2,2\}$ with multiplicity 2 otherwise. So we can assume that $a \geq 4$. With this, we know that $h_1 = 2$, $h_2 = 2$ and $h_3 = 3$, therefore, $g(\lambda, \lambda, \mu) = 3$. In the next step we look at the character value $[\mu](\sigma_{h(\lambda)})$. If $\min(h(\lambda)) > 3$, we apply the Murnaghan–Nakayama rule to see that $[\mu](\sigma_{h(\lambda)}) = 0$. We can remove the hooks corresponding to all but the largest part of $h(\lambda)$ only from the first row of μ, but $h(\lambda)_1 \geq 7$, therefore, we would need to remove a $h(\lambda)_1$ hook from the partition $(h(\lambda)_1 - 3, 3)$, but all the hooks contained in this partition contain at most $h(\lambda)_1 - 2$ boxes. If $\min(h(\lambda)) = 3$, we see that if we remove all but the two smallest parts (which are 5 and 3),

$$[\mu](\sigma_{h(\lambda)}) = [5,3](\sigma_{(5,3)}) = 1.$$

If $\min(h(\lambda)) > 3$, we obtain that $m_+ = m_- \geq 2$ and if $\min(h(\lambda)) = 3$, we obtain that $m_+ = 2$ and $m_- = 1$. So all that is missing is the m_- case for $\min(h(\lambda)) = 3$.

To solve this, let $\mu = (n-3, 2, 1)$. We know from [Val14, Theorem 7.12] or Theorem 5.15 that

$$g(\lambda, \lambda, \mu) = 2h_1(h_1 - 1)(h_1 - 3) + h_2(3h_1 - 4) + h_1 + h_{21},$$

where h_{21} is the number of non-linear 3 hooks in λ. Since we know that $\lambda = (a^{a-1}, a-1)$, we know that $h_1 = 2$, $h_2 = 2$ and $h_{21} = 1$, hence, $g(\lambda, \lambda, \mu) = 3$. Next, we have to calculate $[\mu](\sigma_{h(\lambda)})$. But again, if we remove all but the two smallest parts from $h(\lambda)$, we obtain

$$[\mu](\sigma_{h(\lambda)}) = [5,2,1](\sigma_{(5,3)}) = -1.$$

So we obtain $m_- = 2$, which solves this case.

3rd case: rem(λ) $= 1$. We calculate the multiplicities m_+ and m_- for $\mu = (n-6, 4, 2)$. We know $n = a^2$ for some $a \geq 4$ and $\lambda = (a^a)$. First we want to show that $g(\lambda, \lambda, \mu) \geq 4$. For this we use the semigroup property (Theorem 5.8) and induction. We calculate with Sage that $g((4^4), (4^4), (10, 4, 2)) = 4$. We know that $g(1^k, 1^k, k) = 1$ for all $k \geq 1$. By induction we assume that

$$g(((a-1)^{a-1}), ((a-1)^{a-1}), (a^2 - 2a + 1 - 6, 4, 2)) \geq 4.$$

We add the triple (1^{a-1}), (1^{a-1}) and $(a-1)$ and obtain that

$$g((a^{a-1}),(a^{a-1}),(a^2-a-6,4,2)) \geq 4.$$

We conjugate the first two partitions and obtain

$$g(((a-1)^a),((a-1)^a),(a^2-a-6,4,2)) \geq 4.$$

This time we add (1^a), (1^a) and (a). This yields to

$$g((a^a),(a^a),(a^2-6,4,2)) \geq 4.$$

In the next step we want to compute $[\mu](\sigma_{h(\lambda)})$. We know exactly how $h(\lambda)$ looks like: $h(\lambda) = (2a-1, 2a-3, \dots, 5, 3, 1)$. All the hooks we can remove from the second or third row of μ consist of 5 or less boxes. So we remove all the parts from $h(\lambda)$ which are greater or equal to 9 from the first row. We end up with $[\mu](\sigma_{h(\lambda)}) = [10,4,2](\sigma_{(7,5,3,1)}) = 0$. This tells us that the multiplicity of $\{\mu\}$ as a constituent in any of the products $\{\lambda\}_\pm\{\lambda\}_\pm$ is greater or equal to 2. □

This finishes the proof of Theorem 15.1. We want to conclude this thesis by using Theorem 15.1 to look at the product of three irreducible A_n characters.

3. Product of three characters

Proposition 17.8. *The only products of three irreducible A_n characters which are multiplicity-free, such that none of the characters is the trivial one, are for $n = 4$ the products*

$\{2,2\}_\pm\{2,2\}_\pm\{2,2\}_\pm$ *and* $\{2,2\}_\pm\{2,2\}_\pm\{3,1\}$ (*in both cases all choices for* $\pm$).

Proof: We check the small cases with GAP. So we can assume that $n > 9$ to eliminate the exceptional cases. But that the products which are stated for $n = 4$ are multiplicity-free is not surprising since $\{2,2\}_\pm$ are characters of degree 1.

Let λ, μ, $\nu \vdash n > 9$ all be non-linear. If λ, μ, ν are non-symmetric and $\{\lambda\}\{\mu\}\{\nu\}$ is multiplicity-free, we know by Lemma 15.2 that also the S_n product $[\lambda][\mu][\nu]$ is multiplicity-free. Thanks to [BB17, Theorem 1.2.] we know that there is no such S_n product. Hence, there is no multiplicity-free product of three irreducible A_n characters which are labeled by non-symmetric and non-linear partitions.

Consequently we can now focus on the products of Theorem 15.1 where a character is labeled by a symmetric partition. Without loss of generality we can assume that λ is symmetric. We know that not only λ and μ are a pair from Theorem 15.1, but also all constituents of $\{\lambda\}_\pm\{\mu\}_{(\pm)}$ are labeled by partitions which are listed in Theorem 15.1 (2)-(6). But this is not possible. We check that the products from Theorem 15.1 (2), (5) and (6) contain a constituent which is labeled by a non-symmetric partition which has three removable nodes. □

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