



# Cracking Elements Method for Simulating Complex Crack Growth

Zizheng Sun<sup>1</sup>, Xiaoying Zhuang<sup>2,3,4</sup>, Yiming Zhang<sup>1</sup>

<sup>1</sup> School of Civil and Transportation Engineering, Hebei University of Technology  
Xiping Road 5340, 300401 Tianjin, China, Emails: Zizheng.Sun@hebut.edu.cn, Yiming.Zhang@hebut.edu.cn

<sup>2</sup> Institute of Continuum Mechanics, Leibniz Universität Hannover, Appelstraße 11, 30157 Hannover, Germany, Xiaoying.Zhuang@gmail.com

<sup>3</sup> Department of Geotechnical Engineering, Tongji University, Siping Road 1239, 200092 Shanghai, China

<sup>4</sup> State Key Laboratory for Disaster Reduction in Civil Engineering, Tongji University, Siping Road 1239, 200092 Shanghai, China

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Corresponding author: Yiming Zhang, Yiming.Zhang@hebut.edu.cn

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**Abstract.** The cracking elements method (CEM) is a novel numerical approach for simulating fracture of quasi-brittle materials. This method is built in the framework of conventional finite element method (FEM) based on standard Galerkin approximation, which models the cracks with disconnected cracking segments. The orientation of propagating cracks is determined by local criteria and no explicit or implicit representations of the cracks' topology are needed. *CEM does not need remeshing technique, cover algorithm, nodal enrichment or specific crack tracking strategies.* The crack opening is condensed in local element, greatly reducing the coding efforts and simplifying the numerical procedure. This paper presents numerical simulations with CEM regarding several benchmark tests, the results of which further indicate the capability of CEM in capturing complex crack growths referring *propagations of existed cracks as well as initiations of new cracks.*

**Keywords:** Cracking elements method, Fracture analysis, Quasi-brittle material, Complex crack growth.

## 1. Introduction

Great efforts are paid by researchers for numerically capturing discontinuity of solids, because of the hazardousness of discontinuity in engineering applications such as cracks and shear bands as well as the challengingness of mathematical modelling of it. For quasi-brittle materials, during damaging processes, emergence of macro cracks is accompanied by the unloading of surrounding undamaged regions and the width of the fracture process zone will reduce from finite width into zero [1,2]. This fact indicates the fractures of quasi-brittle materials are highly localized and anisotropic [3-8], showing so called strong discontinuity behavior.

In last decades, with the understanding of fracture processes and developing of computational mechanics, many numerical methods have been presented for numerically reproducing and simulating the fracture process, including remeshing and interface element method [9-13], multi-field mixed-mode formulation [14-16], gradient based and phase field models [17-29], numerical manifold method [30-35], Strong Discontinuity embedded Approach (SDA) [36-47], eXtended Finite Element Method (XFEM) and phantom node method [48-56], meshfree methods [57-67], cracking particles method [68-71] and peridynamics based methods [72-83], to mention a few.

All of these methods have their own advantages and disadvantages. As observed by us, the methods focusing on precise describing the crack tips such as remeshing and XFEM assume continuous crack path, which need crack tracking strategies [84-89]. Despite of the powerfulness of tracking strategies for eliminating stress locking [90], tracking strategies could be inflexible and inefficient for simulating complex crack growth. The gradient based and phase field models are good at simulating the initiation and development of damage regions, which however cannot capture the anisotropic damage properties

such as orientation and opening of the crack. The multi-field mixed-mode formulation and meshless methods are generally flexible and powerful at simulating complex crack, but they are generally inefficient comparing to classical FEM approaches. Peridynamics based methods use control equations formulated in an integral form but not partial differential form, to avoid dealing with singularity problems in fractures analysis. Nevertheless, the material parameters such as "strength" and "fracture energy" must be firstly transformed then used in peridynamics [79].

Cracking elements method (CEM) [91] is a novel numerical approach for fracture analysis based on the Statically Optimal Symmetric formulation of the SDA (SDA-SOS) [92-94]. As been pointed out in [95, 96], SDA-SOS uses the equivalent crack length, naturally supporting self-propagating cracks. The work presented in [97] solved the stress-locking problem of SDA-SOS by using elements with nonlinear interpolations for the displacement fields, paving the way for CEM's formulation. Like the other types of SDA, CEM is built in the framework of the finite element method (FEM). When modifying conventional FEM code into CEM, no remeshing technique, cover algorithm, nodal enrichment or specific crack tracking strategy are needed and the standard Galerkin method can be applied, greatly reducing the coding efforts. On the other hand, unlike the other types of SDA, CEM uses disconnected cracking segments for representing crack paths. There are no differences between elements with crack tips and elements on crack paths, then no explicit or implicit descriptions of crack's topology are needed. Hence, cracks in CEM can grow freely and naturally based on the present stress states. Local criteria are proposed for determining the initiation and orientation of the cracks, making this method very flexible for simulating complex crack growth.

In this work, we further investigate CEM in simulating complex crack growth, regarding propagations of existed cracks as well as initiations of new cracks. The remaining parts of this paper are organized as follows: In Section 2, the formulation of CEM are briefly introduced. Numerical examples are presented in Section 3 including Brazilian disk splitting test, plate with one hole and bending test with three holes. Irregular discretizations are always used for demonstrating the robustness of CEM. The paper closes with concluding remarks given in Section 4.

## 2. The formulation

### 2.1 Constitutive relation

In this paper, mixed-mode traction-separation [98-103] law is used for describing the relationship of the crack opening and the traction between two surfaces of the crack as

$$\zeta_{eq} = \sqrt{\zeta_n^2 + \zeta_t^2} \quad (1)$$

where  $\zeta_n$  and  $\zeta_t$  are the crack opening or crack width along normal and parallel directions of the crack path, with corresponding unit vector denoted as  $\mathbf{n}$  and  $\mathbf{t}$  and the traction components along  $\mathbf{n}$  and  $\mathbf{t}$ ,  $T_n$  and  $T_t$  are obtained as

$$T_n = T_{eq} \frac{\zeta_n}{\zeta_{eq}}, T_t = T_{eq} \frac{\zeta_t}{\zeta_{eq}} \quad (2a)$$

with

$$T_{eq}(\zeta_{eq}) = \begin{cases} TL(\zeta_{eq}) = f_t \exp\left(-\frac{f_t}{G_f} \zeta_{eq}\right) & \text{loading} \\ TU(\zeta_{eq}) = \frac{T_{mx}}{\zeta_{mx}} \zeta_{eq} & \text{unloading/reloading} \end{cases} \quad (2b)$$

where  $f_t$  is the uniaxial tensile strength and  $G_f$  is the fracture energy,  $\zeta_{mx}$  is the maximum crack opening the crack ever experienced, and  $T_{mx} = TL(\zeta_{mx})$  is the corresponding traction see [93, 102].

Correspondingly, the relationship between the tractions differentials  $dT_n$ ,  $dT_t$  and the crack opening differentials  $d\zeta_n$  and  $d\zeta_t$  is described by

$$\begin{bmatrix} dT_n \\ dT_t \end{bmatrix} = \mathbf{D} \begin{bmatrix} d\zeta_n \\ d\zeta_t \end{bmatrix} \quad (3)$$

with

$$\mathbf{D} = -\frac{T_{eq}}{\zeta_{eq}} \begin{bmatrix} \frac{\zeta_n^2}{\zeta_{eq}^2} + \frac{f_t \zeta_n^2}{G_f \zeta_{eq}} - 1 & \frac{\zeta_n \zeta_t}{\zeta_{eq}^2} + \frac{f_t \zeta_n \zeta_t}{G_f \zeta_{eq}} \\ \frac{\zeta_n \zeta_t}{\zeta_{eq}^2} + \frac{f_t \zeta_n \zeta_t}{G_f \zeta_{eq}} & \frac{\zeta_t^2}{\zeta_{eq}^2} + \frac{f_t \zeta_t^2}{G_f \zeta_{eq}} - 1 \end{bmatrix} \quad \text{for loading,} \quad (4a)$$

and

$$\mathbf{D} = \frac{T_{mx}}{\zeta_{mx}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{for unloading/reloading,} \quad (4b)$$



in which obviously  $\mathbf{D}$  keeps symmetric.

### 2.2 Cracking elements formulation

Cracking elements method is built based on Statically Optimal Symmetric formulation of the SDA (SDA-SOS). Similar to other types of SDA and XFEM, the basic idea of CEM is to decompose the displacement field with regular part and the other part dedicated by the discontinuity, which is smoothed by a differentiable function  $\varphi(x)$ , see [102]. Then the strain field is corresponding decomposed by the total strain contributed by the original displacements and the enhanced strain dedicated by the displacement jumps (crack opening)  $\zeta_n$  and  $\zeta_t$ . In the FEM formulation, 8 node quadrilateral element is considered for CEM for avoiding stress locking and mesh bias [91, 97]. Hereby,  $\nabla\varphi^{(e)}$  is introduced for bridging the displacement jumps to the strain field.  $\nabla\varphi^{(e)}$  is a vector parallel to  $\mathbf{n}^{(e)}$  the length of which depends only on the shape of the element. From the classical point of view, the length of  $\nabla\varphi^{(e)}$  equals to the reciprocal of characteristic length [104]. As mentioned, our approach belongs to the SDA-SOS, which is very distinct from XFEM and the other types of SDA i.e., Kinematically Optimal Symmetric (SDA-KOS) formulation [41,106] and Statically and Kinematically Optimal Nonsymmetric (SDA-SKON) formulation [107-114], which firstly determine  $\varphi_i$  then calculate  $\nabla\varphi^{(e)} = \nabla N_i^{(e)} \cdot \varphi_i$ . In SDA-SOS, it is unnecessary to know the shape of function  $\varphi$ . Considering the equivalence of force between discrete and embedded models [91,115-117], the local crack opening is condensed and calculated by solving local balance equation. After solving, the tangent moduli [118,119] is also automatically determined. More details can also be found in [91, 97, 102].

### 2.3 Local criteria for crack propagation

In CEM, continuity of crack path is abandoned, bringing great flexibility. In local element, the orientation of crack  $\mathbf{n}^{(e)}$  keeps parallel to the maximum total strain at the center point. Hence, the orientation of the crack only depends on the regular displacement field but not the displacement jumps.

During calculation, all the non-cracked elements in the whole computation domain are separated into two subdomains as i) the propagation domain and ii) the potential root domain, the definitions of which only depend on the neighboring relationship of the cracked and non-cracked elements. The propagating domain is composed by the elements sharing at least one edge with the cracked elements and all the other non-cracked elements belong to the potential root domain. With the propagation of the cracks (or cracked elements), the propagating domain will expand, see Figure 1 [91, 102].

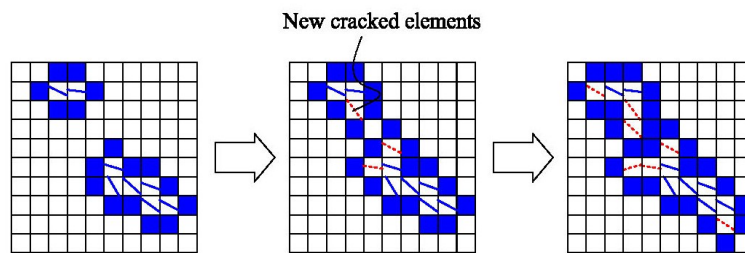


Fig. 1. The expansion of the propagation domain

### 2.4 FEM implementation and Newton iteration

Considering the static loading condition and the standard displacement-based FEM [120], following equilibrium equation is obtained

$$\mathbf{K}\mathbf{U} = \mathbf{F} \tag{5}$$

where  $\mathbf{U}$  is the vectors of displacement at the nodes.  $\mathbf{F}$  is the vector of externally applied loads.  $\mathbf{K}$  is the stiffness matrix as

$$\mathbf{K} = \sum_e \int (B^{(e)})^T K^e B^{(e)} d(e) \tag{6}$$

in which  $\int d(e)$  denotes the integral of the corresponding functions in element  $e$ . In Equation 6,

$$B^{(e)} = [B_1^{(e)} \dots B_8^{(e)}], \text{ where } B_i^{(e)} = \begin{bmatrix} \frac{\partial N_i^{(e)}}{\partial x} & \\ & \frac{\partial N_i^{(e)}}{\partial y} \\ \frac{\partial N_i^{(e)}}{\partial y} & \frac{\partial N_i^{(e)}}{\partial x} \end{bmatrix} \tag{7}$$

and  $K^e$  is the tangent moduli written in matrix form with Voigt notation. Regarding the Newton-Raphson iteration at load step  $j$ , iteration step  $l$ , following equation is introduced

$$U_{j,l} = U_{j-1} + \Delta U_{j,l-1} + \Delta \Delta U_l \tag{8}$$

where  $\Delta \Delta U_l$  is the unknown, and all the parameter with subscript “ $j-1$ ” and “ $j, l-1$ ” are known. Then considering the local balance relationship, following linearized equation is obtained

$$K_s \cdot (\Delta \Delta U_l) = R_s = F - R_{j,l-1} \tag{9a}$$

where

$$R_{j,l-1} = \sum_e \int (B^{(e)})^T S_{j,l-1} d(e) \tag{9b}$$

in which  $S_{j,l-1}$  is  $\sigma$  written in vector form for cracking elements at Newton iteration step  $l-1$  and time step  $j$ . After solving Equation 9,  $\Delta U_{j,l}$  is updated as  $\Delta U_{j,l} = \Delta U_{j,l-1} + \Delta \Delta U_l$ . This iteration procedure is illustrated in ref. [102]. Since the displacement jump is condensed out at the material level, it is very easy to code and implement the approach into conventional FEM code. About the iteration procedures, the interested reader is also referred to refs. [121-130].

### 3. Numerical simulations

Plane-stress assumptions hold for all the numerical examples presented in this section.

#### 3.1 Brazilian disk Splitting test

Brazilian disk splitting test [131] is commonly used to determine the tensile strength of quasi-brittle materials such as rock and concrete. The simulated model is shown in Figure 2 with incremental displacement prescribed. The relationship between the ultimate loading per unit thickness  $F_{max}$  and tensile strength  $f_t$  is determined by

$$F_{max} = \frac{\pi D f_t}{2} \tag{10}$$

where  $D$  is the diameter of the specimen. Hence, for this example the theoretical maximum load is  $F_{max} = 471.24\text{kN}$  per unit thickness. The force-displacement curves and the results of  $F_{max}$  are shown in Figure 3, proving that CEM provides very agreeable results with ignorable mesh bias. Furthermore, the final crack paths (cracked elements) regarding different discretizations are shown in Figure 4, indicating that CEM gives disconnected but reasonable crack paths.

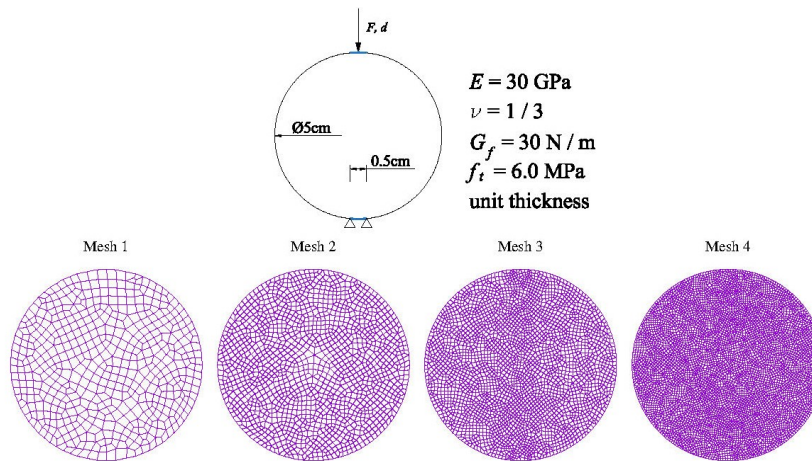


Fig. 2. Brazilian disk splitting test: model and discretizations

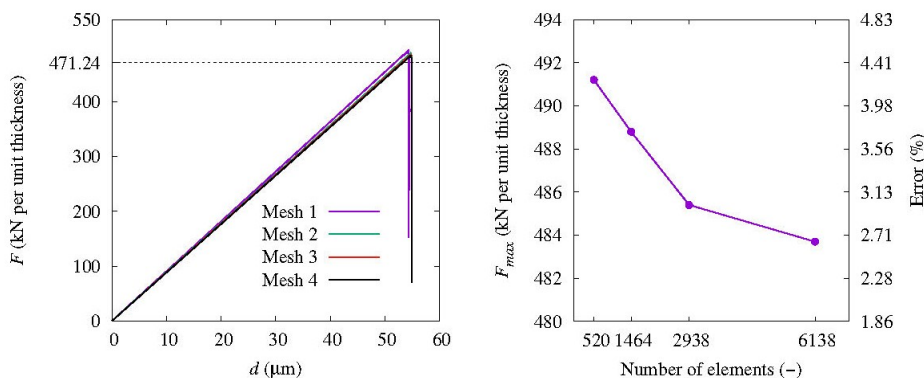


Fig. 3. Brazilian disk splitting test: force-displacement curves and the results of  $F_{max}$

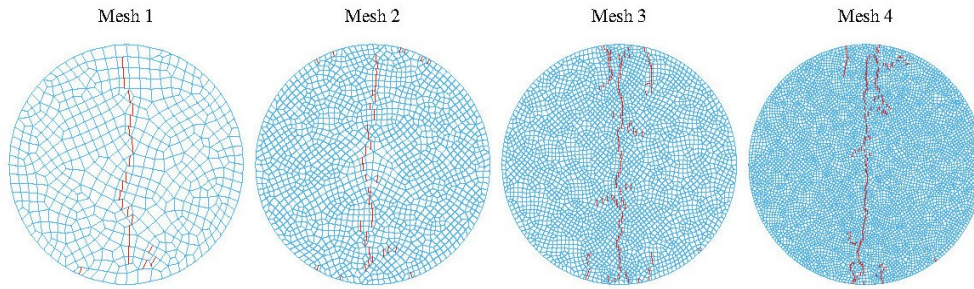


Fig. 4. Brazilian disk splitting test: cracked elements regarding different discretizations (cracked elements)

### 3.2 Plate with one hole

The second example is a plate with one hole made from brittle material, see Figure 5 for the model. There are three holes in the plate and the two holes with diameter 1 cm are used for fixing supports and transferring constant incremental displacement. The force-displacement curve is shown in Figure 6, comparing with the results published in [132,133]. The cracked elements are shown in Figure 7, which is agreeable with the experimental result shown in [133].

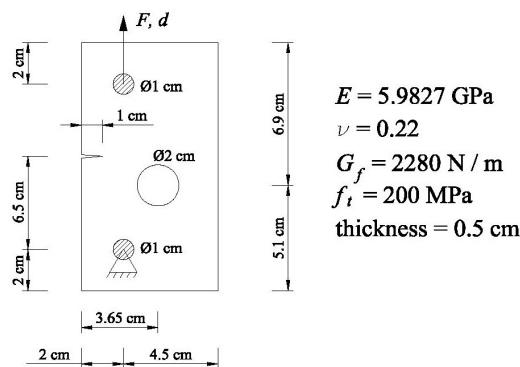


Fig. 5. Plate with one hole: model

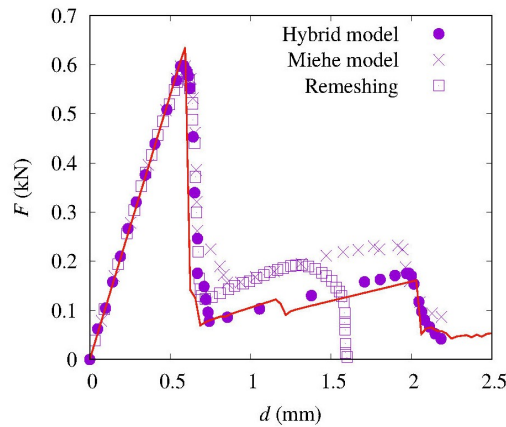


Fig. 6. Plate with one hole: force-displacement curve, comparing with the results published in [132, 133]

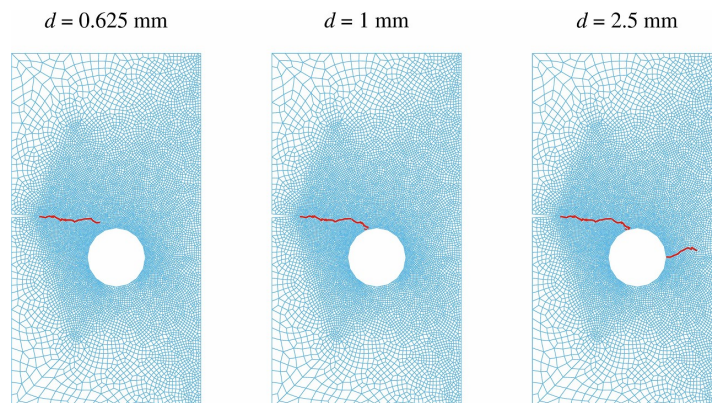


Fig. 7. Plate with one hole: cracked elements at different steps

### 3.3 Bending beam with three holes

The third example is a bending beam with three holes made from polymethylmethacrylate (PMMA), as a typical brittle material, which was investigated in [17,134-136]. The model is shown in Figure 8. Two cases are simulated regarding the position of the notch as: i)  $a=15.24$  cm,  $b=2.54$ cm; and ii)  $a=12.7$ cm,  $b=3.81$ cm. The arc-length method [137] is used for inducing a constant increment for the crack mouth opening displacement (CMOD) of the notch as  $5 \mu\text{m}$ . The force-displacement curves are shown in Figure 9. The crack paths are shown in Figures 10 and 11, indicating that CEM can capture the different crack patterns in this case. On the other hand, oscillations of force-displacement curves are found, especially in case II, which is mainly caused by the CMOD controlled arc-length method. An energy-based arc-length method [138,139] is under investigation.

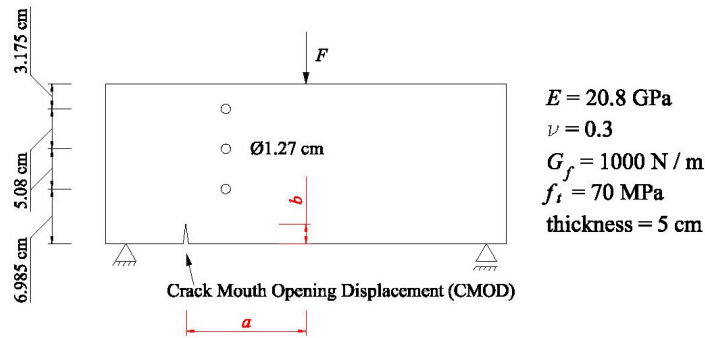


Fig. 8. Bending beam with three holes: model

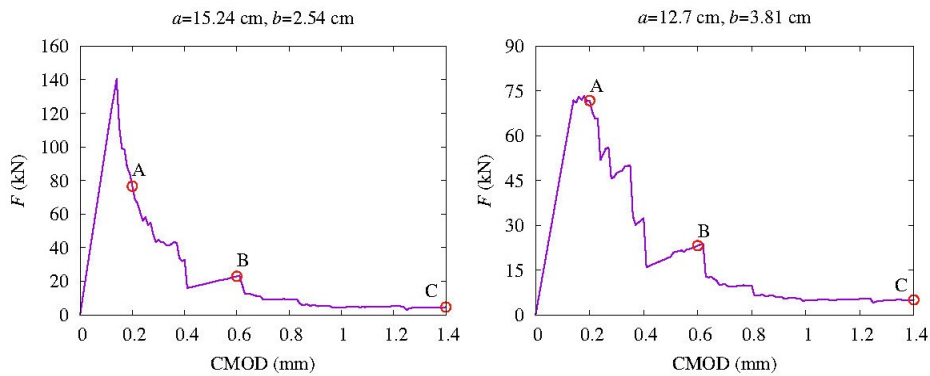


Fig. 9. Bending beam with three holes: force-displacement curves

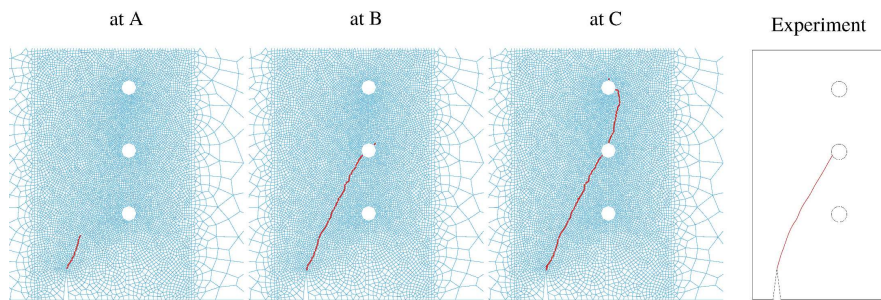


Fig. 10. Bending beam with three holes: cracked elements with  $a=15.24$  cm,  $b=2.54$  cm, comparing to the experimental result provided in [134]

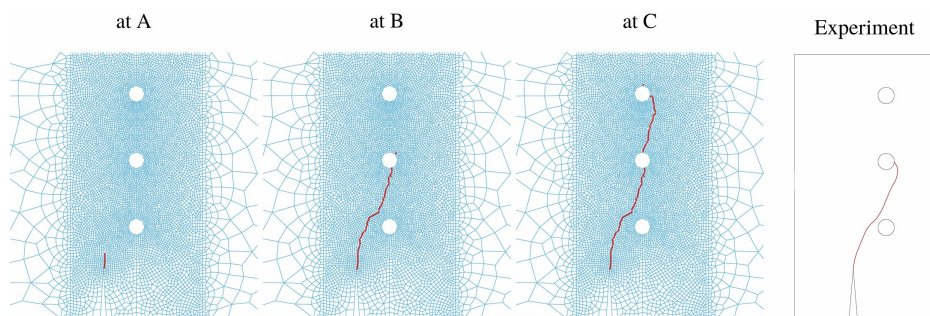


Fig. 11. Bending beam with three holes: crack paths with  $a = 12.7$  cm,  $b = 3.81$  cm, comparing to the experimental result provided in [134]

## 4. Conclusions

In this work, the numerical procedure of the CEM method was provided in detail, with its local criteria for crack growths. The advantages of CEM were shown that it does not need remeshing technique, cover algorithm, nodal enrichment or specific crack tracking strategy, which is built in the framework of conventional FEM with the standard Galerkin method. The numerical capability of CEM was investigated with several benchmark tests with or without initial imperfections, considering different discretizations and model set-ups. The results further proved the reliability and effectiveness of the CEM method, naturally capturing the initiation and propagation of cracks, which is capable of simulating random distributed cracks.

## Conflict of Interest

The author(s) declared no potential conflicts of interest with respect to the research, authorship and publication of this article.

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